University of Wisconsin-Madison
Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, NOT ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Some potentially useful identities

$$\hat{f}(s) = L[f(t)] = \int_0^\infty dt\, e^{-st} f(t)$$

$$L[\frac{t^{n-1}}{(n-1)!}] = \frac{1}{s^n}, \qquad L[\frac{2^n t^{n-1/2}}{1*3*5*...*(2n-1)\sqrt{\pi}}] = s^{-(n+1/2)}$$

$$L[e^{-at}] = \frac{1}{s+a}, \qquad L[te^{-at}] = \frac{1}{(s+a)^2}$$

$$L[\sin(at)] = \frac{a}{a^2+s^2}, \qquad L[\cos(at)] = \frac{s}{a^2+s^2}$$

$$L[\frac{e^{at}-e^{bt}}{a-b}] = \frac{1}{(s-a)(s-b)}, \qquad L[\frac{ae^{at}-be^{bt}}{a-b}] = \frac{s}{(s-a)(s-b)}$$

$$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}, \qquad L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$L[\sinh at] = \frac{a}{s^2-a^2}, \qquad L[\cosh at] = \frac{s}{s^2-a^2}$$

Student No.__________________

Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

Problem 1.

Suppose A is an $n \times n$ matrix with complex elements. Assume that A is Hermitian, which means that if we take the complex conjugate of the elements in A, and take its transpose, we get A again. Symbolically, $A = A^*$, where * denotes the complex conjugate transpose.

Prove that the eigenvalues of A are real.

Student No.________________

Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

Problem 2.

Consider the initial value problem

$$\frac{dy}{dt} = ty^3$$

with arbitrary initial point (t_o, y_o), $y_o > 0$.

a. Determine the solution. (70%)

b. Discuss the interval over which only real solutions are obtained. Assume t_o and y_o are real. (20%)

c. What is the solution if $y_o = 0$. (10%)

Student No._________________

Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

Problem 3.

a) (80%) Calculate all $z = x + iy$ in the complex domain that satisfy the following equation.

$$e^z = 2\frac{(1+i)^2}{1-i}$$

b) (20%) Sketch the location of these points in the complex plane.

Student No.________________

Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

Problem 4.

You are given 114 linear algebraic equations for 179 variables. Denoting the variables $X = (x_1, x_2, x_3, \dots x_{179})^T$, the algebraic equations can be written in the matrix form

$$AX = 0$$

The rank of matrix A is 103.

a) (25%) How many of the 114 linear equations are redundant?
b) (25%) What is the dimension of the solution space for this algebraic set of equations?
c) (25%) Suppose that in addition to satisfying $AX = 0$, a subset of the variables [e.g., $X_1 = (x_1, x_2, x_3, \dots x_{19})^T$] satisfy the matrix equation

$$BX_1 = 0$$

where B is a 19×19 matrix with rank 19. What is the dimension of the solution space of the problem $AX = 0$, $BX_1 = 0$? Assume that the equations given by $BX_1 = 0$ are distinct from the equations given by $AX = 0$; none of the equations in $BX_1 = 0$ can be obtained from linear combinations of equations in $AX = 0$.

d) (25%) What is the dimension of the solution space of the problem

$$AX = 0,$$

$$BX_1 = b,$$

where B is the same matrix of part c) and $b \neq 0$ is a column of constants.

Student No.__________________

Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

Problem 5.

Use the method of your choice to find the solution of the simultaneous equations:

$$\frac{dx}{dt} + \frac{dy}{dt} + x = -e^{-t}\ ;$$
$$\frac{dx}{dt} + 2\frac{dy}{dt} + 2x + 2y = 0\ ,$$

subject to the initial conditions: $x(0) = -1$, $y(0) = 1$.

Student No.________________

Engineering Physics Department
Fall 2011 Qualifying Exams

Mathematics

Problem 6.

Evaluate the integral: $I = \int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^3}$