University of Wisconsin-Madison
Engineering Physics Department
Fall 2012 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, NOT ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Some potentially useful identities

$$\hat{f}(s) = L[f(t)] = \int_0^\infty dt\, e^{-st} f(t)$$

$$L[\frac{t^{n-1}}{(n-1)!}] = \frac{1}{s^n}, \qquad L[\frac{2^n t^{n-1/2}}{1*3*5*...*(2n-1)\sqrt{\pi}}] = s^{-(n+1/2)}$$

$$L[e^{-at}] = \frac{1}{s+a}, \qquad L[te^{-at}] = \frac{1}{(s+a)^2}$$

$$L[\sin(at)] = \frac{a}{a^2+s^2}, \qquad L[\cos(at)] = \frac{s}{a^2+s^2}$$

$$L[\frac{e^{at}-e^{bt}}{a-b}] = \frac{1}{(s-a)(s-b)}, \qquad L[\frac{ae^{at}-be^{bt}}{a-b}] = \frac{s}{(s-a)(s-b)}$$

$$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}, \qquad L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$L[\sinh at] = \frac{a}{s^2-a^2}, \qquad L[\cosh at] = \frac{s}{s^2-a^2}$$

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Problem 1.

(50%) (a) Consider the nonlinear first-order equation:

$$\frac{dy}{dx} + P(x)y + Q(x)y^2 = R(x)$$

Show that the substitution, $y(x) = \dfrac{u'}{Q(x)u}$, where $u' = \dfrac{du}{dx}$, converts the *nonlinear* first order equation into a *linear* second order equation,

$$\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} + b(x)u(x) = 0$$

Find expressions for $a(x)$ and $b(x)$.

(50%) (b) Consider now the specific equation:

$$x^2\frac{dy}{dx} + xy + x^2y^2 = 1$$

Find a solution for $y(x)$ based on your results from the part **(a)**.

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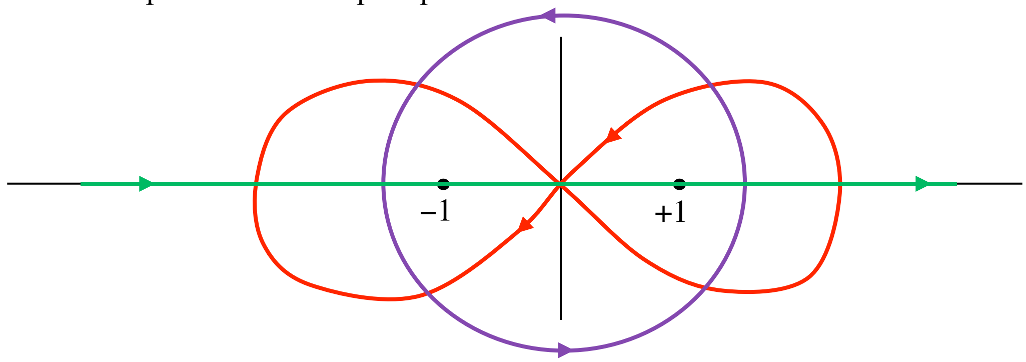
Problem 2.

Evaluate the following integral:

$$I = \int_C \frac{dz}{z^2 - 1}$$

for each of the three paths in the complex plane shown below.



(40%) (a) The path C is the closed, figure-8 like path.
(30%) (b) The path C is the closed, circular path.
(30%) (c) The path C is the real axis, extending from $-\infty$ to $+\infty$.

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Problem 3.

Consider the eigenvalue problem

$$\lambda BX = CX$$

where λ is the eigenvalue with B and C given by

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 3 & 0 \\ 4 & 4 & 4 \\ 4 & 8 & 5 \end{pmatrix}$$

One of the eigenvectors is given by $E = (4/5, 0, -3/5)^T$. Determine all of the eigenvalues and eigenvectors.

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Problem 4.

Evaluate the following integral in the complex plane. The integration path is the unit circle encompassing $z = 0$.

$$I = \oint_{|z|=1} \frac{dz}{z^m (i - \frac{z}{\pi})}.$$

Here, $m > 0$ is an integer.

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Problem 5.

Column matrices A, B, and C represent vectors in three-dimensional Euclidean space R^3. If the vectors are given by

$$A = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} \qquad B = \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} \qquad C = \begin{Bmatrix} -1 \\ 0 \\ 2 \end{Bmatrix}$$

(50%) (a) Use a matrix based approach to show that they span all of R^3.

(50%) (b) Use a vector based approach to compute the angle between vector C and the plane containing vectors A and B.

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Problem 6.

Find the function $y(x)$ that satisfies

$$y^2 \frac{d^2 y}{dx^2} + 1 = 0$$

$$y(1) = 1$$

$$\frac{dy}{dx}(1) = \sqrt{2}$$