University of Wisconsin-Madison
Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, NOT ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Potentially useful identities:

$$\hat{f}(p) = L[f(t)] = \int_0^\infty dt\, e^{-pt} f(t)$$

$$L[\frac{t^{n-1}}{(n-1)!}] = \frac{1}{p^n}, \qquad L[\frac{2^n t^{n-1/2}}{1*3*5*...*(2n-1)\sqrt{\pi}}] = p^{-(n+1/2)}$$

$$L[e^{-at}] = \frac{1}{p+a}, \qquad L[te^{-at}] = \frac{1}{(p+a)^2}$$

$$L[\sin(at)] = \frac{a}{a^2+p^2}, \qquad L[\cos(at)] = \frac{p}{a^2+p^2}$$

Student No.________________

Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

Problem 1.

Consider the following differential equation:

$$\frac{d^4 y}{dt^4} + 4\Omega^4 y = 1$$

Here Ω is a real positive constant. Find the general solution and write the solutions in terms of real functions.

Student No.________________

Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

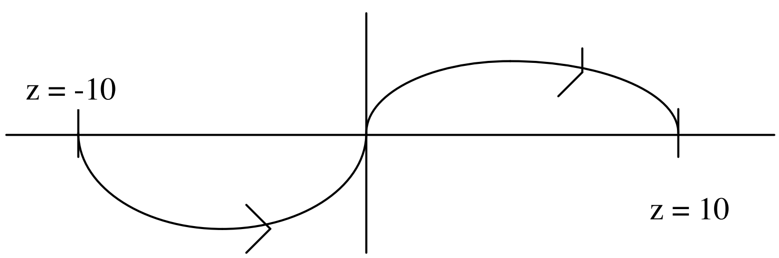
Problem 2.

Consider the integral in the complex plane

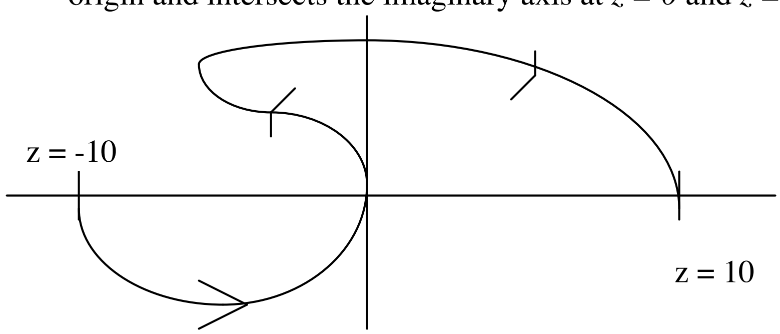
$$I = \int_{-10}^{10} dz \frac{\sin(z)}{1+z^2}$$

Determine the integral if the integration path is:

(a) (30%) Along the real axis $z = x; =-10 \leq x \leq 10$.
(b) (30%) Along the integration path shown below. This contour passes through the origin and only passes the imaginary axis one time.



(c) (40%) Along the integration path shown below. This contour passes through the origin and intersects the imaginary axis at $z = 0$ and $z = 5i$.



Student No.________________

Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

Problem 3.

Find the general solution $y(x)$ for the ordinary differential equation

$$\frac{dy}{dx} = \frac{2x^3 + y^3}{3xy^2}$$

Student No._________________

Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

Problem 4.

Compute the matrix $\Phi(t)$ for the first order matrix differential equation $\dot{x} = Ax$ such that the solution is given by $x = \Phi x_o$ where x_o is the vector of initial conditions. Matrix A is given by

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Student No.________________

Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

Problem 5.

Consider the function, $f(z) = \dfrac{1}{(z^2+1)^2}$.

(a) (50%) What is $I = \oint_C f(z)dz$ **,** where C is the unit circle in the complex plane centered at $z = 0$?

(b) (50%) What is $I = \int_{-\infty}^{+\infty} f(x)dx$, along the real axis?

Student No.________________

Engineering Physics Department
Fall 2010 Qualifying Exams

Mathematics

Problem 6.

Finite element models of continuous structural media undergoing vibration without damping take the form:

$$\mathbf{KD} = \omega^2 \mathbf{MD}$$

Here **K** and **M** are square matrices, **M** is not singular, and **D** is a displacement vector, and ω is the natural frequency of vibration. For a two-element model of a bar with one end fixed undergoing longitudinal vibration, there are two common representations for this system. If one uses the consistent mass representation, the equation looks like this:

$$\alpha \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 2/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

If one uses the lumped mass representation, the equation looks like this:

$$\alpha \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In these equations, α is a known parameter incorporating the stiffness and mass of the individual elements.

a) (90%) Find the natural frequencies for each representation as a multiple of $\sqrt{\alpha}$.
b) (10%) Comparing the two representations, what is the difference between the fundamental natural frequency values of each as a percentage of the average of the two results?