University of Wisconsin-Madison
Engineering Physics Department
Spring 2010 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, NOT ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

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Problem 1.

Using the method of "variation of parameters," construct the general solution to the differential equation:

$$x^2 y'' - 4xy' + 6y = x^4 \sin x$$

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Problem 2.

Consider the integral

$$I = \oint_C \frac{dz}{z}$$

where C is a closed path in the complex plane.

(a) (25%) What is the value of the integral if C corresponds to the unit circle in the complex plane?
(b) (25%) How does the result from part (a) differ for any other closed path encircling the origin?
(c) (25%) Sketch the closed path in the complex plane given by the polar equation

$$r(\theta) = 2 - \sin^2 \frac{\theta}{4}$$

(d) (25%) Determine the value of the integral I if C is the closed path defined in part c).

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Problem 3.

Let A be an $n \times n$ matrix with n distinct eigenvalues λ_i that satisfy the characteristic equation

$$\lambda_i^n + c_{n-1}\lambda_i^{n-1} + c_{n-2}\lambda_i^{n-2} + \ldots c_0 = 0$$

Show that the matrix A itself satisfies the characteristic equation

$$A^n + c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \ldots c_0 I_n = 0$$

where I_n is the identity matrix.

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Problem 4.

Derive the first three non-zero terms of the power series solution near $x = 0$ for the differential equation

$$y'' + \left(1 + x^3\right) y' + \left(1 + 2x^2\right) y = 0$$

subject to the boundary conditions:

$$y(0) = 0 \qquad y'(0) = \frac{1}{3}$$

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Problem 5.

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in the complex plane at $z = x + iy$ show that the function

$$\Phi = e^{-f}$$

is analytic at z by an explicit demonstration that the Cauchy-Riemann equations for Φ are satisfied.

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Problem 6.

The following matrix equation $Ax = b$ contains two constants p and q.

$$\begin{pmatrix} 1 & -1 & 2 \\ 1 & p & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} q \\ 2 \\ 1 \end{pmatrix}$$

Determine conditions on the constants p and q such that:

a) (40%) A unique solution for x exits. If a unique solution exists, find x as a function of p and q.
b) (30%) No solution for x exists.
c) (30%) An infinite number of solutions for x exist. Provide a form for the solution in terms of a sum of column vectors.