University of Wisconsin-Madison
Engineering Physics Department
Spring 2015 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, NOT ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

Some potentially useful identities

$$\hat{f}(s) = L[f(t)] = \int_0^\infty dt\, e^{-st} f(t)$$

$$L[\frac{t^{n-1}}{(n-1)!}] = \frac{1}{s^n}, \qquad L[\frac{2^n t^{n-1/2}}{1*3*5*...*(2n-1)\sqrt{\pi}}] = s^{-(n+1/2)}$$

$$L[e^{-at}] = \frac{1}{s+a}, \qquad L[te^{-at}] = \frac{1}{(s+a)^2}$$

$$L[\sin(at)] = \frac{a}{a^2+s^2}, \qquad L[\cos(at)] = \frac{s}{a^2+s^2}$$

$$L[\frac{e^{at}-e^{bt}}{a-b}] = \frac{1}{(s-a)(s-b)}, \qquad L[\frac{ae^{at}-be^{bt}}{a-b}] = \frac{s}{(s-a)(s-b)}$$

$$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}, \qquad L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$L[\sinh at] = \frac{a}{s^2-a^2}, \qquad L[\cosh at] = \frac{s}{s^2-a^2}$$

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Problem 1.

Consider the integral

$$I = \int_0^{1+i} \frac{dz}{z-i}$$

a) (50%) What is I if the integration path is along Γ_1 where Γ_1 is parameterized by $z = u\,(1+i)$, $0 \le u \le 1$?

b) (50%) What is I if the integration path is the sequence along $\Gamma_2 + \Gamma_3$ where Γ_2 is parameterized by $z = v(-1 + 3i)$, $0 \le v \le 1$ and Γ_3 is parameterized by $z = w(2-2i) - 1 + 3i$, $0 \le w \le 1$?

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Problem 2.

Use any technique you'd like to calculate *x(t)* and *y(t)* from the following set of coupled differential equations.

$$\frac{dx}{dt} + 4x + 3y = 12e^{t},$$
$$\frac{dy}{dt} + x + 2y = 12e^{t}.$$

subject to the initial conditions

$$x(0) = 0, \qquad y(0) = 0.$$

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Problem 3.

Find the solution to the following differential equation

$$\frac{d^2x}{dt^2} = -\frac{x}{2} - \frac{1}{x}\left(\frac{dx}{dt}\right)^2$$

subject to the initial conditions

$$x(0) = 2 \qquad \frac{dx}{dt}(0) = -2$$

Hint: Consider the variable substitution, $s = x^2$.

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Problem 4.

Let A and B be $n \times n$ real symmetric matrices. Assume that B has eigenvalues ≥ 0. The trace of a matrix is defined as the sum of the terms on its diagonal. Show that the trace of the matrix product AB is bounded by the values in the expression

$$\lambda_{\min}(A)tr(B) \leq tr(AB) \leq \lambda_{\max}(A)tr(B)$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the minimum and maximum eigenvalues of A, respectively, and $tr(B)$ is the trace of B.

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Problem 5. Let $f(z) = (x - y)^2 + 2i(x + y)$.

(a) (30%) Where in the complex plane are the Cauchy-Riemann (C-R) equations satisfied? Is $f(z)$ analytic?

(b) (60%) Obtain an expression for $f'(z)$ by considering a finite increment, $\Delta z = \Delta x + i\Delta y$, and first constructing a general expression for $\Delta f/\Delta z$. For the conditions of part (a) where the C-R equations are satisfied, find an expression for $f'(z)$ by letting Δz go to zero.

(c) (10%) Demonstrate that the result of part (b) is consistent with an expression for $f'(z)$ having real and imaginary parts determined by the C-R terms.

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Problem 6. The tetrahedral element shown below has six edges all of length a, making each face an equilateral triangle. The face OBC lies in the xz-plane. Point O is the origin of the coordinate system and the coordinates of points A, B and C are:

$$A : a\left(\frac{1}{2},\ \frac{\sqrt{2}}{\sqrt{3}},\ \frac{1}{2\sqrt{3}}\right) \qquad B : a\left(\frac{1}{2},\ 0,\ \frac{\sqrt{3}}{2}\right) \qquad C : a(1,\ 0,\ 0)$$

(a) (20%) Demonstrate that line segments OA and BC are perpendicular to each other.
(b) (40%) Find a unit vector normal to both OA and BC, choosing the unit normal with positive **j**-component.
(c) (40%) The projection of any relative position vector from one line to the other onto the unit normal from part (b) gives the minimum distance between the lines. Evaluate this distance using relative position vectors BA and CO and verify the results are identical.