University of Wisconsin-Madison
Engineering Physics Department
Fall 2007 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, <u>NOT</u> ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. ______
2. ______
3. ______
4. ______
5. ______
6. ______

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Problem 1.

Determine the general solution *y(x)* of:

$$\frac{d^3y}{dx^3} - \frac{dy^2}{dx^2} - \frac{dy}{dx} + y = 0$$

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Problem 2.

Evaluate the following real integral using residue calculus:

$$\int_{-\infty}^{+\infty} \frac{x^2 dx}{(x^2+1)(x^2-2x\cos\omega+1)}$$

The parameter ω is real and non-zero and lies in the interval $0 < \omega < +\pi$.

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Problem 3.

Find a functional relationship between y and x from the following ordinary differential equation. Your solution should not involve any unevaluated integrals.

$$\frac{dy}{dx} = \frac{1}{y^2 + 3x + 5}$$

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Problem 4.
Consider the following linear equation $Ax = b$:

$$\begin{pmatrix} 3 & 1 & 0 & -1 \\ 2 & 1 & -4 & 1 \\ -1 & -1 & 8 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ c \end{pmatrix}.$$

Employ matrix theory to determine for which values of the constant c this equation has:

(a) no solutions (30%);
(b) a unique solution if it exists (find the solution) (30%);
(c) an infinite number of solutions (find a representation for all solutions as a sum of column vectors) (40%).

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Problem 5.

Let E be the eigenvector associated with the eigenvalue λ for the problem

$$ME = \lambda E$$

The matrix M can be written as the sum of a Hermitian matrix and a skew-Hermitian matrix.

$$M = H + S$$

$$\overline{H} = H^T \qquad \overline{S} = -S^T$$

a) Write the real and imaginary parts of the eigenvalue λ using quadratic forms involving H, S and E. (40%)

b) For the special case

$$M = \begin{pmatrix} a & b \\ 0 & a + ic \end{pmatrix}$$

where a, b, are c are real numbers, determine the matrices H and S. (30%)

c) One of the (unnormalized) eigenvectors is

$$E = \begin{pmatrix} b \\ ic \end{pmatrix}.$$

Use this and the formulae derived in part a) to determine the real and imaginary parts of the associated eigenvalue. (30 %)

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Problem 6.

The function $f(z)$ has the following Laurent series in the vicinity of the point $z = z_0$ in the complex plane for $0 < |z - z_0| < R$

$$f(z) = \sum_{n=-2}^{n=0} a_n (z - z_0)^n .$$

Consider the four contours in the complex plane: C_1 (starts at $z = z_2$, ends at $z = z_1$), C_2 (starts at $z = z_3$, ends at $z = z_2$), C_3 (starts at $z = z_3$, ends at $z = z_1$), C_4 (starts at $z = z_3$, ends at $z = z_1$). All points on all of the contours lie in the annulus $0 < |z - z_0| < R$

$$z_1 = z_0 + \frac{1+i}{\sqrt{2}}, \qquad z_2 = z_0 + \frac{-1+i}{\sqrt{2}}, \qquad z_3 = z_0 - \frac{1+i}{\sqrt{2}}$$

Defining the quantities

$$I_1 = \int_{C_1} dz f(z), \quad I_2 = \int_{C_2} dz f(z), \quad I_3 = \int_{C_3} dz f(z), \quad I_4 = \int_{C_4} dz f(z)$$

Calculate the following.

a) $I_1 + I_2 + I_3$ (33%)
b) $I_1 + I_2 - I_3$ (33%)
c) $I_3 - I_4$ (33%)