University of Wisconsin-Madison
Engineering Physics Department
Fall 2014 Qualifying Exams

Mathematics

You must solve 4 out of the 6 problems.
Start each problem on a new page.

SHOW ALL YOUR WORK.
WRITE ONLY ON THE FRONT PAGES OF THE WORKSHEETS, NOT ON THE EXAM PAGES

Grading is based on both the final answer and work done in reaching your answer. All problems receive an equal number of points.

Clearly indicate which problems you want graded. If you do not indicate which problems are to be graded, the first four solutions you provide will be graded.

1. _____
2. _____
3. _____
4. _____
5. _____
6. _____

Some potentially useful identities

$$\hat{f}(s) = L[f(t)] = \int_0^\infty dt\, e^{-st} f(t)$$

$$L[\frac{t^{n-1}}{(n-1)!}] = \frac{1}{s^n}, \qquad L[\frac{2^n t^{n-1/2}}{1*3*5*...*(2n-1)\sqrt{\pi}}] = s^{-(n+1/2)}$$

$$L[e^{-at}] = \frac{1}{s+a}, \qquad L[te^{-at}] = \frac{1}{(s+a)^2}$$

$$L[\sin(at)] = \frac{a}{a^2+s^2}, \qquad L[\cos(at)] = \frac{s}{a^2+s^2}$$

$$L[\frac{e^{at}-e^{bt}}{a-b}] = \frac{1}{(s-a)(s-b)}, \qquad L[\frac{ae^{at}-be^{bt}}{a-b}] = \frac{s}{(s-a)(s-b)}$$

$$L[e^{at}\cos bt] = \frac{s-a}{(s-a)^2+b^2}, \qquad L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$L[\sinh at] = \frac{a}{s^2-a^2}, \qquad L[\cosh at] = \frac{s}{s^2-a^2}$$

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Problem 1.

Use residue calculus to evaluate the following integral:

$$I = \int_0^\infty \frac{dx}{(x^2 + a^2)^4} \qquad a \text{ real and positive}$$

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Problem 2. Find the solution *y(x)* to the following differential equation subject to the condition that $y(1) = 0$.

$$x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$$

Hint: The following integral identity may be helpful: $\int \frac{du}{\sqrt{1+u^2}} = \ln[u + \sqrt{1+u^2}\,]$

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Problem 3. Assume you are given the following matrix differential equation

$$M\ddot{x} + Kx = 0$$

where M and K are real symmetric *2 X 2* matrices. The matrix of eigenvectors is given by

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}$$

where ϕ_1 and ϕ_2 are column vectors. The eigenvectors are normalized with respect to matrix M, such that

$$\phi^T M \phi = I$$

in which I is a 2x2 identity matrix.

a) (50%) Derive an expression for the inverse of M written only in terms of the matrix ϕ.

b) (50%) Derive an expression for a 2x2 matrix P_1 such that ϕ_1 is an eigenvector of P_1 with eigenvalue $\lambda_1 = 1$, and ϕ_2 is an eigenvector of P_1 with eigenvalue $\lambda_2 = 0$. Prove your result.

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Problem 4. Find the solution to the following differential equation

$$t^3 \frac{d^2 y}{dt^2} - t\frac{dy}{dt} + y = 0$$

subject to the initial conditions

$$y(1) = 1 \qquad \frac{dy}{dt}(1) = 2$$

Hint: Consider using reduction of order.

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Problem 5.

a) (8 points) Calculate all $z = x+iy$ in the complex domain that satisfy

$$sinh^2(z^2) = -1$$

b) (2 points) Sketch the location of these points in the complex plane.

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Problem 6.

Consider the following set of algebraic equations for x,y,z

$$x + 2y + z = 1$$

$$x + Py + z = 3$$

$$-Px + y + 2z = Q$$

where P and Q are parameters.

For which values of P and Q does

a) (3 points) a unique solution exists
b) (4 points) no solutions exists.
c) (3 points) an infinite number of solutions exist. Find a representation for all solutions in this case as sums of column vectors.