Physics of Electric Propulsion

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Problems

Chapter 1

1-1 Derive Equation 1-1.

Solution: Equation 1-1 is describes the balance of forces on a rocket exerting thrust:

$$m\dot{v} = \dot{m}u_e + F_g \tag{1}$$

We can get this one from Newton's second law for an object which has variable mass:

$$F = \frac{dp}{dt} \tag{2}$$
$$F_g = \dot{m}u_e + m\dot{v} \tag{3}$$

There's probably a minus sign in there, but I'm not too bothered.

1-2 Verify the first three characteristic velocities quoted in Table 1-1. Explain why the values for gentle spiral ascent exceed those for impulsive ascent for a given Earth orbit transfer.

Solution: I'm guessing the Oberth effect is the reason for the sub-optimality of gentle spiral ascent. It could also be $g\Delta t$ loss.

1-3 For a mission characterised by a velocity increment Δv and a rocket of constant thrust level T, derive an expression for the optimum specific impulse in terms of power supply specific mass α and the initial total mass of the spacecraft.

Solution: This is covered in Section 1-4, "The Power Supply Penalty". Let's start by re-deriving the u_e which optimises the payload mass ratio for fixed Δt and T. We assume a linear relationship between plant power, plant mass, and thrust:

$$m_p = \alpha P = \frac{\alpha T u_e}{2\eta} \tag{4}$$

From kinematics, we also know

$$\Delta m = \frac{T\Delta t}{u_e} \tag{5}$$

Both of these expressions depend only on the variable of interest (u_e), and parameters which are independent of u_e, namely T, Δt (fixed by assumption), α and η (efficiencies intrinsic to their related devices).

We'd like to minimise $m_p + \Delta m$:

$$\frac{d}{du_e}(m_p + \Delta m) = \frac{d}{du_e}\left(\frac{\alpha T u_e}{2\eta} + \frac{T\Delta t}{u_e}\right) \tag{6}$$

$$= \frac{\alpha T}{2\eta} - \frac{T\Delta t}{u_e^2} = 0 \tag{7}$$

This results in a minimum at $u_e = \sqrt{\frac{2\eta\Delta t}{\alpha}}$, since the second derivative is positive at that point (being a sum of positive terms).

Now let's try the same thing, but with T and Δv fixed. Our expression for m_p still only depends on fixed quantities and u_e, but our expression for Δm depends on Δt, which in turn depends on u_e. We could also express Δm using the Tsiolkovsky equation:

$$\Delta m = m_0\left(1 - e^{-\frac{\Delta v}{u_e}}\right), \tag{8}$$

but this still contains an m_0, which depends on u_e for fixed T and Δv. Let's substitute again:

$$m_0 = (m_u + m_p)e^{\frac{\Delta v}{u_e}} \tag{9}$$

$$\Delta m = (m_u + m_p)\left(e^{\frac{\Delta v}{u_e}} - 1\right) \tag{10}$$

This leads us to the same place as if we had just written down

$$m_0 = \left(m_u + \frac{\alpha T u_e}{2\eta}\right)e^{\frac{\Delta v}{u_e}} \tag{11}$$

$$\frac{m_u}{m_0} = r = \frac{m_u e^{-\frac{\Delta v}{u_e}}}{m_u + \frac{\alpha T u_e}{2\eta}} \tag{12}$$

$$\frac{\partial r}{\partial u_e} = \frac{m_u e^{-\frac{\Delta v}{u_e}}\left(m_u\Delta v + \frac{\alpha T u_e(\Delta v - u_e)}{2\eta}\right)}{u_e^2\left(m_u + \frac{\alpha T u_e}{2\eta}\right)^2} = 0 \tag{13}$$

$$\frac{2\eta m_u \Delta v}{\alpha T} + \Delta v u_e - u_e^2 = 0 \tag{14}$$

There's only one positive root of this quadratic equation:

$$u_e = \frac{\Delta v}{2}\left(1 + \sqrt{\frac{8\eta m_u}{\alpha T\Delta v} + 1}\right) \tag{15}$$

Not too bad-looking, but who knows if it's correct?

1-4 If the mass delivered by a rocket, m_f, is separated into useful payload m_u and the power-producing system mass $m_p = \alpha P$, show that equation 1-6 takes the form

$$\frac{m_u}{m_0} = e^{-\frac{\Delta v}{u_e}} - \frac{\alpha u_e^2}{2\eta\Delta t}\left(1 - e^{-\frac{\Delta v}{u_e}}\right) \tag{16}$$

where Δt is the thrusting time and η is the conversion efficiency. What is the optimal u_e for this mission?

Solution: Let's start with the Tsiolkovsky equation (1-6 in the book)

$$\frac{m_f}{m_0} = e^{-\frac{\Delta v}{u_e}} \tag{17}$$

$$\frac{m_u + m_p}{m_0} = e^{-\frac{\Delta v}{u_e}} \tag{18}$$

$$\frac{m_u}{m_0} = e^{-\frac{\Delta v}{u_e}} - \frac{m_p}{m_0} \tag{19}$$

$$= e^{-\frac{\Delta v}{u_e}} - \frac{\alpha P}{m_0} \tag{20}$$

$$= e^{-\frac{\Delta v}{u_e}} - \frac{\alpha T u_e}{2\eta m_0} \tag{21}$$

To round this one out, we need the total impulse relation:

$$T\Delta t = u_e \Delta m \tag{22}$$

We can start again from the Tsiolkovsky equation to put Δm in the appropriate form:

$$\frac{m_0 - \Delta m}{m_0} = e^{-\frac{\Delta v}{u_e}} \tag{23}$$

$$\frac{m_0}{m_0} - \frac{m_0 - \Delta m}{m_0} = 1 - e^{-\frac{\Delta v}{u_e}} \tag{24}$$

$$\Delta m = m_0 \left(1 - e^{-\frac{\Delta v}{u_e}}\right) \tag{25}$$

$$\therefore T = \frac{m_0 u_e}{\Delta t} \left(1 - e^{-\frac{\Delta v}{u_e}}\right) \tag{26}$$

Substitution solves the problem.

Let's differentiate to find the maximum $\frac{m_u}{m_0}$:

$$\frac{d}{du_e}\left(\frac{m_u}{m_0}\right) = \frac{e^{-\frac{\Delta v}{u_e}} \Delta v}{u_e^2} - \frac{\left(1 - e^{-\frac{\Delta v}{u_e}}\right) \alpha u_e}{\eta \Delta t} + \frac{e^{-\frac{\Delta v}{u_e}} \alpha \Delta v}{2\eta \Delta t} = 0 \tag{27}$$

This equation looks pretty much unsolvable.