

# An Introduction to the Theory of Groups

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## Problems

### Chapter 1

#### 1.13

- (i) A permutation  $\alpha \in S_n$  is **regular** if either  $\alpha$  has no fixed points and it is the product of disjoint cycles of the same length, or  $\alpha = \mathbb{1}$ . Prove that  $\alpha$  is regular iff it is a power of an  $n$ -cycle  $\beta$ ; that is,  $\alpha = \beta^m$  for some  $m$ . (*Hint*: if  $\alpha = (a_1 a_2 \dots a_k)(b_1 b_2 \dots b_k) \dots (z_1 z_2 \dots z_k)$ , where there are  $m$  letters  $a, b, \dots, z$ , then let  $\beta = (a_1 b_1 \dots z_1 a_2 b_2 \dots z_2 \dots a_k b_k \dots z_k)$ .)

**Solution:**  $\beta^m$  takes  $a_1$  through  $b_1 \dots z_1$  to  $a_2$  as desired.  $\mathbb{1}$  can be expressed as  $\beta^n$  for  $\beta(j) = j + 1$  an  $n$ -cycle. For a general regular  $\alpha$ , disjointness of the sets  $a_j, b_j, \dots, z_j$  guaranteed that the  $\beta$  from the hint is an  $n$ -cycle. If there's some  $n$ -cycle  $\beta$  with  $n = mk$ , and we take the  $m$ th power, we also get  $m$  disjoint, length- $k$  cycles, as desired.

- (ii) If  $\alpha$  is an  $n$ -cycle, then  $\alpha^k$  is a product of  $\gcd(n, k)$  disjoint cycles, each of length  $n/\gcd(n, k)$ .

**Solution:**  $\alpha^n = \mathbb{1}$ . If  $n$  is a multiple of  $k$ , then  $\alpha^n = (\alpha^k)^{n/k}$ .  $\alpha^k$  would then be a product of  $k$   $n/k$ -cycles. In the case where  $n$  is not a multiple of  $k$ , but they have a non-trivial gcd, then starting at  $\alpha_0$ ,  $\alpha$  would take us to  $\alpha_1$ .  $\alpha^k$  will take us to  $\alpha_k$ . It takes  $\alpha_k$  to  $\alpha_{2k}$ , and so on until we get to  $\alpha_{mk} = \alpha_0$ . This happens if  $m = \frac{n}{\gcd(n, k)}$ , but I don't know how to prove that.

- (iii) If  $p$  is prime, then every power of a  $p$  cycle is either a  $p$ -cycle or  $\mathbb{1}$ .

**Solution:** This is a corollary of the last exercise, noting that  $\gcd(p, k) = 1$  if  $k \neq p$  and  $p$  if  $k = p$ .

#### 1.17 How many $\alpha \in S_n$ are there with $\alpha^2 = \mathbb{1}$ ?

**Solution:** There's  $\mathbb{1}$ , and there's disjoint unions of transpositions. In terms of single transpositions, there are  $\binom{n}{2}$  of them. If I'm going to put together a product of  $j$  transpositions, there are  $\binom{n}{2}$  ways to choose the first transposition,  $\binom{n-2}{2}$  ways to choose the second, and  $\binom{n-2j}{2}$  ways to choose the  $j$ th. Since any permutation of these transpositions is equivalent, I ought to get

$$1 + \sum_{j=1}^{n/2} \frac{1}{j!} \prod_{k=0}^j \binom{n-2k}{2}$$

or something, it's not important.

#### 1.26 A group for which $x^2 = \mathbb{1}$ for all $x$ must be Abelian.

**Solution:** We know that  $aa = aea = abba = \mathbb{1}$ , and that  $abab = \mathbb{1}$ . This implies that  $ab$  and  $ba$  must both be equal to  $b^{-1}a^{-1}$ .

## 1.27

- (i) Let  $G$  be a finite abelian group containing no elements  $a \neq e$  with  $a^2 = e$ . Evaluate  $a_1 * a_2 * \dots * a_n$ , where  $a_1, a_2, \dots, a_n$  is a list of all elements in  $G$  with no repetitions.

**Solution:** Just for laughs, let's invert this big element. From the result of Exercise 1.23, we get  $(a_1 * a_2 * \dots * a_n)^{-1} = a_n^{-1} * a_{n-1}^{-1} * \dots * a_1^{-1}$ . Let  $A$  be another name for this big element, so I don't have to  $\LaTeX$  it all out. The inverses of the individual group elements are unique elements of the group themselves, so the inverse of  $A$  is another product of all elements of the group.  $G$  is abelian, so  $A^{-1} = A$ , since all permutations of products are equivalent. This means  $A^2 = e$ , and the only element of  $G$  for which that holds is  $1$ .

- (ii) Prove **Wilson's Theorem**: If  $p$  is prime, then

$$(p-1)! = -1 \pmod{p}.$$

(Hint: The nonzero elements of  $\mathbb{Z}_p$  form a multiplicative group.)

**Solution:** As far as I'm concerned, this completely contradicts the last exercise, since  $-1$  is not the multiplicative identity unless  $p = 2$ . One interesting thing to note is that  $(-1)^2 = 1 = 1^2$ , violating the assumption of part (i). Now things start to get a little clearer. The inverse is unique, so, for all numbers from 2 up to  $p-2$ , the multiplicative inverse mod  $p$  is *also* in the set  $\text{range}(2, p-1)$  (ranges are taken to be Python-style). That means that  $(p-2)! = 1$ , and  $(p-1)! = p-1$ , as desired.

**1.31** Let  $G$  be a group, let  $a \in G$ , and let  $m$  and  $n$  be relatively prime integers. If  $a^m = 1$ , show that there exists a  $b$  such that  $a = b^n$ . (Hint: There are integers  $s$  and  $t$  such that  $sm + tn = 1$ .)

**Solution:** (Special thanks to Joel Klassen and Christophe Vuillot for their assistance.) We know  $a^m = 1$ , so that  $a^{m+1} = a$ . From the hint,  $a^{m+sm+tn} = a$ , and we can cancel multiples of  $m$  to obtain  $a^{tn} = a$ , so we set  $b = a^t$  and get what we were after.

**1.42** Let  $G = \{x_1, x_2, \dots, x_n\}$  be a set equipped with an operation  $*$ , let  $A = [a_{ij}]$  be its multiplication table (i.e.  $a_{ij} = x_i * x_j$ ), and assume that  $G$  has a two-sided identity  $e$ :  $e * x = x * e = x$  for all  $x \in G$ .

- (i) Show that  $*$  is commutative if and only if  $A$  is symmetric.

**Solution:** This is true by definition.

- (ii) Show that every element  $x \in G$  has a (two-sided) inverse (i.e. there is an  $x' \in G$ ) such that  $x' * x = x * x' = e$  if and only if  $A$  is a **Latin Square** (i.e. all rows and columns are permutations of  $G$ ).

**Solution:** (Thanks to Joel Klassen and Christophe Vuillot for basically doing this problem for me). If  $A$  is a Latin Square, then there exists a left inverse and a right inverse for  $x$ , since  $1$  must appear in the row and column corresponding to  $x$ .  $zx = 1$ ,  $xy = 1$ , therefore  $zxy = z$ , but  $zx = 1$ , so  $y = z$ .

Likewise, if  $A$  is not a Latin square, then there are different elements  $y$  and  $z$  such that  $xy = xz = w$ . If  $x$  has a left inverse, then  $y = z$  and we have a contradiction. If it doesn't, we've proven what we want to prove.

- (iii) Assume that  $e = x_1$  so that the first row of  $A$  has  $a_{1i} = x_i$ . Show that the first column of  $A$  has  $a_{i1} = x_i^{-1}$  for all  $i$  if and only if  $a_{ii} = 1$  for all  $i$ .

**Solution:** This is mad trivial.

- (iv) With the multiplication table as shown in (iii), show that  $*$  is associative if and only if  $a_{ij}a_{jk} = a_{ik}$ .

**Solution:**

**If:** The trick here is that, if the matrix is arranged such that  $\mathbb{1}$  is on the diagonal, then  $a_{ij} = x_i x_j^{-1}$ . If multiplication is associative,  $a_{ij}a_{jk} = x_i x_j^{-1} x_j x_k^{-1} = x_i x_k^{-1} = a_{ik}$ .  $\square$

**Only If:** Every  $x_k$  can be expressed as  $x_i x_j^{-1}$  for fixed  $x_i$ , since the multiplication table is a latin square. This implies that the product of three elements  $x_k x_l x_m = x_i x_j^{-1} x_j x_n^{-1} x_n x_o^{-1} = a_{ij}a_{jn}a_{no}$ . We can evaluate this product in either order, using the assumed product:  $a_{ij}a_{jn}a_{no} = a_{in}a_{no} = a_{ij}a_{jo} = a_{io}$ .  $\square$

**2.3** The set-theoretic union of two subgroups is a subgroup if and only if one is contained in the other. Is this true if we use three subgroups?

**Solution:** No, see Bruckheimer et al, 1970: <https://www.jstor.org/stable/pdf/2316854.pdf>.

**2.4** Let  $S$  be a proper subgroup of  $G$ . If  $G - S$  is the complement of  $S$ , prove that  $\langle G - S \rangle = G$ .

**Solution:** We know that  $\langle G - S \rangle$  contains  $G - S$ , so all we have to prove is that the elements of  $S$  can be generated by multiplying together two things in  $G - S$ . Pick an element of  $G - S$   $g$ .  $g^{-1} \notin S$ , since that would contradict the inclusion of the inverse. Also,  $sg^{-1} \notin S$  for any  $s \in S$ , since that would contradict closure. However,  $sg^{-1} \cdot g = s$ , so we can find two elements of  $G - S$  that generate  $s$  under multiplication.

**2.5** Let  $f : G \rightarrow H$  and  $g : G \rightarrow H$  be homomorphisms, and let

$$K = \{a \in G : f(a) = g(a)\}.$$

Must  $K$  be a subgroup of  $G$ ?

**Solution:** We need:

- the identity to be in the set. This is guaranteed by Theorem 1.13 –  $f(\mathbb{1}) = \mathbb{1}'$ , so  $\mathbb{1} \in K$ .
- closure under the inverse. This is also guaranteed by Theorem 1.13 –  $f(a^{-1}) = f(a)^{-1} = g(a)^{-1} = g(a^{-1})$ .
- closure under multiplication. This is guaranteed by the defining property of the homomorphism –  $f(ab) = f(a)f(b) = g(a)g(b) = g(ab)$ .

$K$  is a subgroup.  $\square$

**2.7** If  $n > 2$ , then  $A_n$  is generated by all the 3-cycles. (*Hint:*  $(ij)(jk) = (ijk)$  and  $(ij)(kl) = (ijk)(jkl)$ ).

**Solution:** Parity is defined as the number of transpositions in a decomposition of a permutation. For each adjacent pair of transpositions in such a decomposition, we can apply one of the two formulae in the hint to express it as a product of 3-cycles.

**2.8** Imbed  $S_n$  as a subgroup of  $A_{n+2}$ , but show that, for  $n \geq 2$ ,  $S_n$  cannot be imbedded in  $A_{n+1}$ .

**Solution:** The first part is trivial. If we have elements  $n + 1$  and  $n + 2$ , we can decide whether to multiply a permutation by  $(n + 1 \ n + 2)$  on its way into the subgroup, and everything becomes even. The second part is tricky. We can easily prove that  $S_2$  can't be imbedded in  $A_3$ , since  $A_3$  is generated by the 3-cycle  $(1\ 2\ 3)$  which is order 3 (as are all of its subgroups), and  $S_2$  is order 2. Jonas Helsen says that, in order to be imbedded, the order of the large group has to be an integer multiple of the order of the small group.  $|S_n| = n!$  and  $|A_{n+1}| = (n+1)!/2$ , so the ratio is  $n+1/2$ , which is not an integer if  $n$  is even. We could also (try to) use the fact that, for  $n + 1 \geq 5$ ,  $A_{n+1}$  is simple, and

$S_n$  is not, since it has  $A_n$  as a normal subgroup. This would leave us with the  $n = 3$  case, which we can allegedly show by seeing that  $A_4$  has no subgroups of order 6. I'm not quite happy with this solution.

## 2.9

- (i) Prove that  $S_n$  can be generated by  $\{(1\ k), k \in \mathbb{Z}_n\}$

**Solution:** We know that every element of  $S_n$  can be expressed as a product of transpositions, and  $(j\ k) = (1\ k)(1\ j)(1\ k)$ , so we're good.  $\square$

- (ii) Prove that  $S_n$  can be generated by  $\{(j\ j+1 \pmod n), j \in \mathbb{Z}_n\}$ .

**Solution:** If we can generate the generators from the generating set in the last part, we're good. Let's use recursion, because this is *mathematics*.  $(1\ k) = (1\ k-1)(k-1\ k)(1\ k-1)$ .

- (iii) Prove that  $S_n$  can be generated by the two elements  $(1\ 2)$  and  $C = (1\ 2 \cdots n)$ .

**Solution:** First, if I take the  $n$ th power of  $C$ , I get  $\mathbb{1}$ . This means that  $C^{-1}$  is the  $n-1$ th power, and I can use it to generate other stuff. Let's look at  $C^k(1\ 2)C^{-k}$ . This operation takes  $k+1 \rightarrow 1 \rightarrow 2 \rightarrow k+2$  and  $k+2 \rightarrow 2 \rightarrow 1 \rightarrow k+1$ , so I can generate the generators from the generating set in the last part.

- (iv) Prove that  $S_4$  is not generated by  $A = (1\ 3)$  and  $C = (1\ 2\ 3\ 4)$ .

**Solution:** I can write down an arbitrary product of these operators as  $C^{i_0}AC^{i_1}AC^{i_2}\dots AC^{i_m}$ . Knowing that  $C^{-1}$  is generated, I express the product as

$$C^{i_0}AC^{-i_0}C^{i_0+i_1}AC^{-(i_0+i_1)}\dots C^{\sum_{j=0}^{m-1} i_j}AC^{-\sum_{j=0}^{m-1} i_j}C^{-\sum_{j=0}^m i_j}$$

The only two values of  $C^iAC^{-i}$  are  $(1\ 3)$  and  $(2\ 4)$ , and the only 3 non- $\mathbb{1}$  values of  $C^i$  are  $(1\ 2\ 3\ 4)$ ,  $(1\ 3)(2\ 4)$ , and  $(1\ 4\ 3\ 2)$ . Brute force enumeration of the words of this set reveals that  $(1\ 2)$  isn't generated.  $\square$

## 2.12

- (i) Prove that every group  $G$  of order 4 is isomorphic either to  $\mathbb{Z}_4$  or the 4-group  $\mathbf{V}$ .

**Solution:** Without loss of generality, the elements of a group of order 4 are  $\mathbb{1}$ ,  $a$ ,  $b$ , and  $ab$ . Each of these elements needs an inverse, and the inverse of the inverse is the element itself, so that if  $a^{-1} = b$ , then  $b^{-1} = a$  also. Up to permutation of labels, there are two possible scenarios, either  $a^{-1} = a$ ,  $b^{-1} = b$ , or  $a^{-1} = b$ . Writing out the multiplication tables then either gives us  $\mathbb{Z}_4$  or  $\mathbf{V}$ .

- (ii) If  $G$  is a group with  $|G| \leq 5$ , then  $G$  is abelian.

**Solution:** Order 2 is trivial, there's only one non-trivial group element. 3 is prime, so the group has to be cyclic, and therefore abelian. Order 4 only permits the two groups we saw earlier, and 5 is prime.

## 2.16 If $H \leq G$ has index 2, then $a^2 \in H$ for every $a \in G$

**Solution:** (Christophe Vuillot knocked this out in 60 seconds.) If  $a$  is in  $H$ , we're done. If not, it can be expressed as  $gh$  for some  $h$  in  $H$ . Let's assume that the square is not in the subgroup. This implies  $ghgh = gh'$ ,  $\therefore hgh = h'$ ,  $\therefore gh = h^{-1}h'$ , which is a contradiction, since  $h^{-1}h'$  is in  $H$ .

## 2.17

- (i) If  $a, b \in G$  commute and if  $a^m = 1 = b^n$ , then  $(ab)^k = 1$ , where  $k = \text{lcm}(m, n)$ .

**Solution:** They commute so  $(ab)^k = a^k b^k$ .

- (ii) Let  $A$  and  $B$  be

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Show that  $A^4 = B^3 = 1$ , but  $AB$  has infinite order.

**Solution:** Let's take a look at the effect of  $AB$  acting from the left on a matrix.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

This preserves  $c$  and  $d$ , so that if they were 0 and 1 on the way in, it'll be the same on the way out.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Note that this matrix doesn't generate the Fibonacci sequence like I thought it would, which is why I did this trivial exercise.

**Note** There comes a point in every yellow book where the physicist reading it will begin to skip non-trivial exercises, simply because the physicist in question is uninterested in the answers to those questions. This is that point.

**2.30** If  $S \leq G$  and  $S = 2$ , then  $S \triangleleft G$ .

**Solution:** We'd like to show that  $gsg^{-1} \in S$ . There are, just as in exercise 2.16, two cases. If  $g \in S$ , in which case  $g^{-1} \in S$  and  $gsg^{-1}$  is just a product of three subgroup elements. If  $g \notin S$ , then  $g = as_0$  with  $s_0 \in S$ . Let's assume that  $gsg^{-1} \notin S$ . This implies that  $as_0ss_0^{-1}a^{-1} = as_1$ , which implies  $s_0ss_0^{-1}a^{-1} = s_1$ . Inverting both sides, we obtain a contradiction:  $as_0s^{-1}s_0^{-1}$  must be in the coset, but  $s_1^{-1}$  must be in the subgroup.

**2.32** If  $H \leq G$  then  $H \triangleleft G$  if and only if for all  $x, y \in G$ ,  $xy \in H$  if and only if  $yx \in H$ .

**Solution:** If  $H \triangleleft G$  and  $xy \in H$ , then  $gxyg^{-1} \in H$  for all  $g$ . Setting  $g = x^{-1}$  gets us  $yx \in H$ . If  $xy \in H$  implies  $yx \in H$ , then  $xy(yx)^{-1} = xyx^{-1}y^{-1}$  is in  $H$ . This implies that, if  $y^{-1} \in H$ , then  $xyx^{-1} \in H$  for all  $x$ .  $y^{-1} \in H$  also implies  $y \in H$ . □

**2.38** If  $H$  and  $K$  are normal subgroups of  $G$ , show that  $H \vee K \triangleleft G$ .

**Solution:**  $ghkg^{-1} = ghg^{-1}gkg^{-1} = h'k' \in H \vee K$ . □

**2.39** Prove that if a normal subgroup  $H$  of  $G$  has index  $n$ , then  $g^n \in H$  for all  $g \in G$ .

**Solution:**