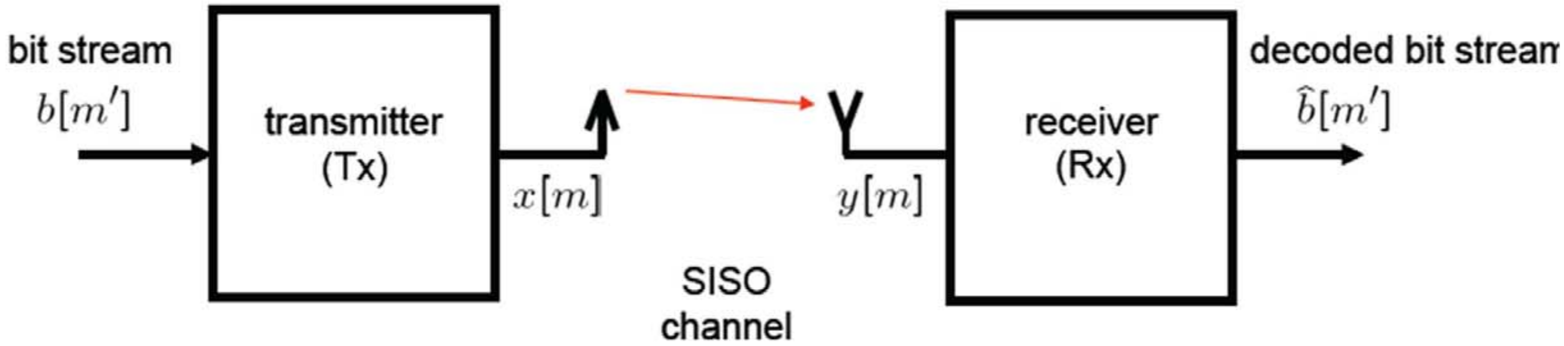
Multiple-Input-Multiple-Output (MIMO) transmission over fading channels

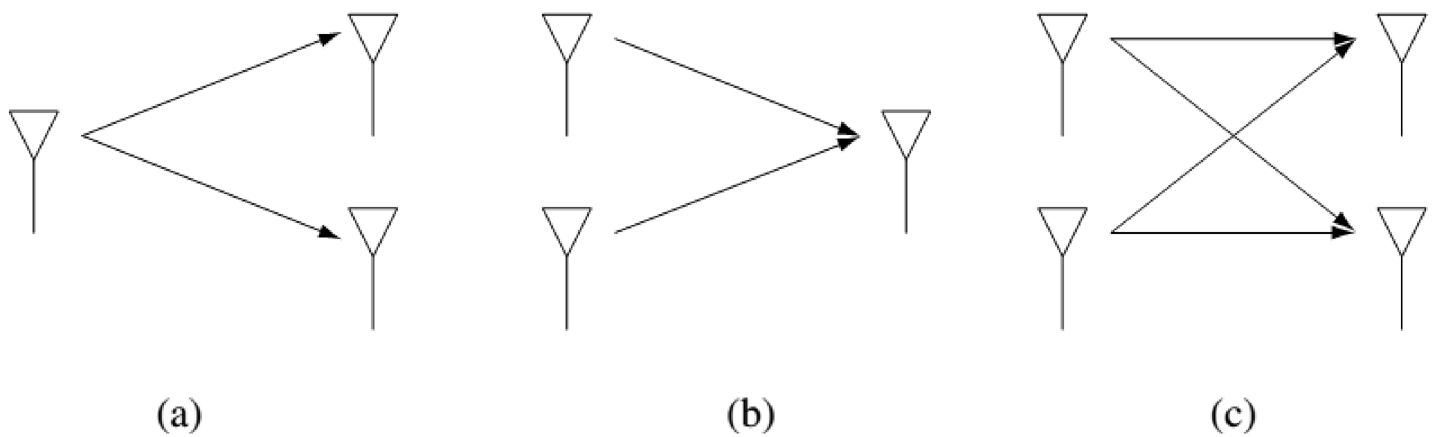
Single-Input Single-Output (SISO)



- Action of channel:
  - Block-constant flat-fading assumption:
    $$y[m] = h\,x[m] + w[m]$$
  - delay spread (frequency selectivity):
    $$y[m] = \sum_l h[l]\,x[m-l] + w[m]$$
  - delay/Doppler spread (time-frequency selectivity):
    $$y[m] = \sum_l h[m,l]\,x[m-l] + w[m]$$

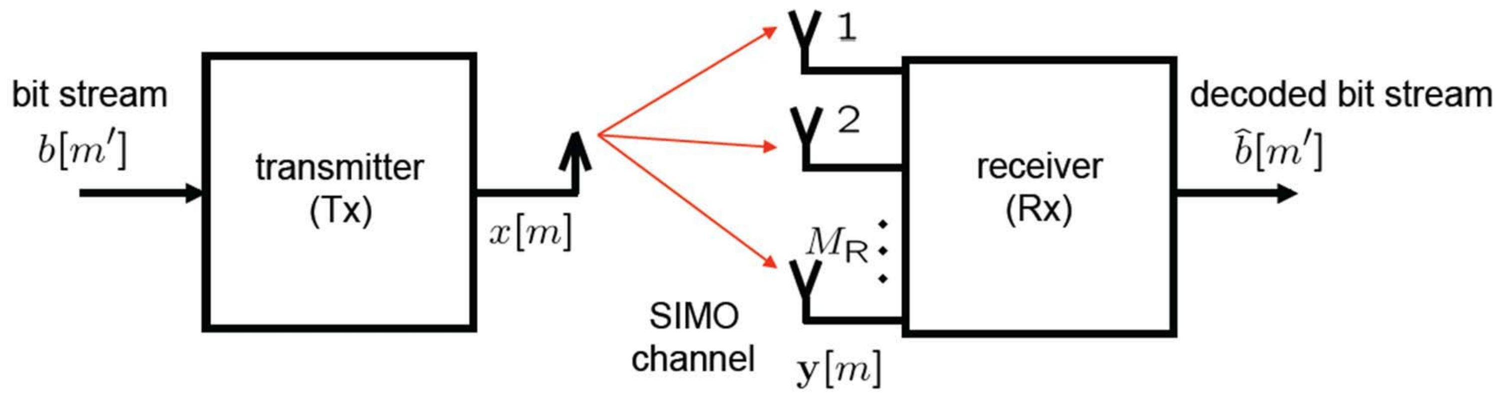
Channel causes linear distortion of input $x[m]$ and adds noise $w[m]$

Multiple-Antenna systems: Antenna diversity



a) Receive diversity (SIMO); b) Transmit diversity (MISO);
c) Transmit and Receive diversity (MIMO)

Single-Input Multiple-Output (SIMO)



- Rx beamforming possible; Rx diversity gain
- Received signal (block-constant flat fading): $\mathbf{y}[m] = \mathbf{h}\,x[m] + \mathbf{w}[m]$
- Channel described by $M_R \times 1$ vector

$$\mathbf{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_{M_R} \end{bmatrix}$$

M_R: number of receive antennas

Spatial diversity-on-receive SIMO channel

- $M_T=1$, $M_R=L$, $\mathbf{h}=[h_1,h_2,\ldots,h_L]^T$, $\mathbf{y}=\mathbf{h}x+\mathbf{w}$, $\mathbf{R_W}=\sigma_W^2\,\mathbf{I}_L$: white Gaussian noise

Maximum Likelihood (ML) decision:

- Consider equally probable transmission of x from $\mathbf{A}=\{a_k, k=1,2,\ldots,K\}$
- Receiver knows **h**, computes squared Euclidean distances: $|\mathbf{y}-\mathbf{h}a_k|^2$
- Select $x=a_k$ corresponding to the shortest Euclidean distance:

From $|\mathbf{y}-\mathbf{h}x|^2=(\mathbf{y}-\mathbf{h}x)^H(\mathbf{y}-\mathbf{h}x)=|\mathbf{h}|^2\{|\mathbf{y}|^2/|\mathbf{h}|^2+|x|^2-(x\mathbf{y}^H\mathbf{h}+x^*\mathbf{h}^H\mathbf{y})/|\mathbf{h}|^2\}$

$= |\mathbf{h}|^2\{|x-x_o|^2+|y_o|^2\}$, $x_o= \mathbf{h}^H\mathbf{y}/|\mathbf{h}|^2$: complex, $|y_o|^2 =\{|\mathbf{y}|^2/|\mathbf{h}|^2\}-|x_o|^2$

$\Rightarrow$ select $a_k\in$ **A** for min $|\mathbf{y}-\mathbf{h}a_k|^2\equiv$ select $a_k\in$ **A** for min $|x-x_o|^2$

ML Rx:

- First, with channel knowledge, Rx uses matched filter (Maximum Ratio Combiner, MRC): Rx beamforming → Rx diversity gain

$$\text{received vector: } \mathbf{y}[m]=\mathbf{h}x[m]+\mathbf{w}[m],\ E\{\|x\|^2\}=E_s,\ E\{\mathbf{w}\mathbf{w}^H\}=N_o\mathbf{I}_L$$

$$\text{matched filter: } \overset{\square}{x}_o[m]=\frac{\mathbf{h}^H}{\|\mathbf{h}\|}\mathbf{y}[m]=\|\mathbf{h}\|\,x[m]+w'[m],\ w'[m]=\frac{\mathbf{h}^H}{\|\mathbf{h}\|}\mathbf{w}[m],$$

$$E\{[\|\mathbf{h}\|\,|x|]^2\}=LE_s,\ E\{|w'|^2\}=N_o,\ SNR_{MRC}=LE_s/N_o$$

- Second, Rx selects $a_k\in\mathbf{A}$ nearest to x_o.

SIMO channel: spatial diversity performance

- In general, with ML detection, the *conditional* symbol error rate is

$$P_e(\mathbf{h}) \approx aQ\left[\sqrt{\frac{d\|\mathbf{h}\|^2 E_s}{N_0}}\right] \le a\exp\left(-\frac{d\|\mathbf{h}\|^2 E_s}{2N_0}\right),\ \|\mathbf{h}\|^2 = \sum_{l=1}^{L}|h_l|^2, \quad \text{Chernoff bound } \mathrm{Q(x)} \le \mathrm{e}^{-\mathrm{x}^2/2}$$

  a: constant related to the number of nearest neighbors

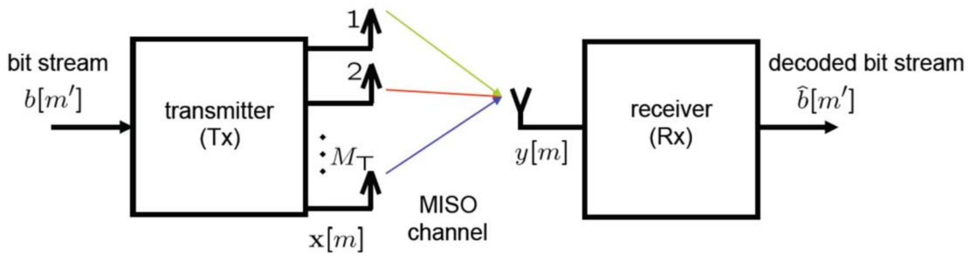
  d: constant related to the squared minimum distance (normalized to E_s)

- For Rayleigh fading channel, *average* symbol error rate

$$P_e = E_{\mathbf{h}}\left\{P_e\left(\mathbf{h}\right)\right\} \approx a\left[1+\frac{dE_s}{2N_o}\right]^{-L}$$

- **diversity order:** indicated by the exponent L.
- In general, this is an application of (L,1) repetitive coding. Diversity can be obtained in time, frequency or space. Spatial diversity costs antennas (space) but save time and bandwidth (in time or frequency diversity).

Multiple-Input Single-Output (MISO) systems



- Tx beamforming possible; Tx diversity gain and multiplexing gain
- Received signal (block-constant flat fading): $y[m] = \mathbf{h}^T \mathbf{x}[m] + w[m]$
- Channel described by $M_T \times 1$ vector

$$\mathbf{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_{M_T} \end{bmatrix}$$

M_T: number of transmit antennas

- Tx Diversity: With channel knowledge at Tx, Tx sends vector: $\mathbf{x}(m) = \frac{\mathbf{h}^*}{\|\mathbf{h}\|} x(m)$, so that the received scalar signal is:

$$y(m) = \mathbf{h}^T \mathbf{x}(m) + w(m) = \mathbf{h}^T \frac{\mathbf{h}^*}{\|\mathbf{h}\|} x(m) + w(m) = \|\mathbf{h}\| x(m) + w(m)$$

which maximizes the received SNR by in-phase addition of signals at the receiver (Tx beamforming same as Rx beamforming).

Diversity-on-transmit MISO channel: h known to Tx

- $M_T=N$, $\mathbf{h}=[h_1,h_2,\ldots,h_N]$, $P/\sigma_W^2=E_s/N_o$
- Tx use a complex weight Nx1 vector $\mathbf{p}=[p_1,p_2,\ldots,p_N]^T$ to transmit x over N antennas and keep $|\mathbf{p}|^2=1$ (for a total average Tx power of P or E_s).
- Rx signal:

$$y(m)=\mathbf{h}^T\mathbf{x}(m)+w(m)=\mathbf{h}^T\mathbf{p}x(m)+w(m), \text{ Rx } SNR=\left\|\mathbf{h}^T\mathbf{p}\right\|^2\frac{E_s}{N_o}$$

- Select **p** to maximize the received SNR:

$$\text{optimum } \mathbf{p}=\frac{\mathbf{h}^*}{\|\mathbf{h}\|}\rightarrow \text{ maximum Rx } SNR=\|\mathbf{h}\|^2\frac{E_s}{N_o},\ E\{\|\mathbf{h}\|^2\}=N$$

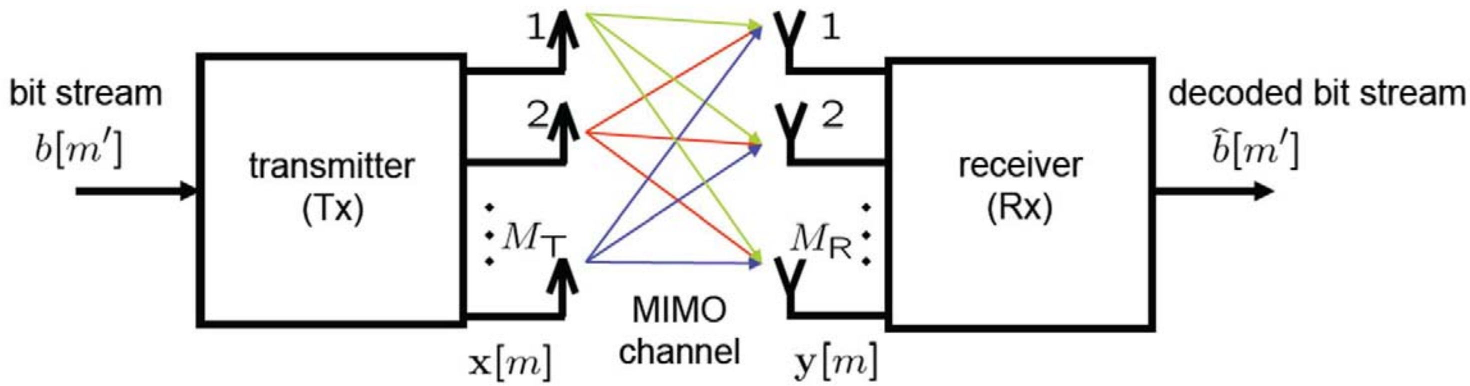
→ Rx SNR: $|E\{|\mathbf{h}|^2(E_s/N_o)\}=N(E_s/N_o)$: N time better than SISO: Tx array gain of N

- Conditional symbol error rate: $P_e(\mathbf{h})\approx aQ\left[\sqrt{\frac{|\mathbf{h}|^2 dE_s}{N_0}}\right]\leq a\exp\left(-\frac{|\mathbf{h}|^2 dE_s}{2N_0}\right)$

- For Rayleigh fading channel, average symbol error rate:

$$P_e=E_{\mathbf{h}}\left\{P_e\left(\mathbf{h}\right)\right\}\approx a\left[1+\frac{dE_s}{2N_o}\right]^{-N}$$ diversity order of N

Multiple-Input Multiple-Output (MIMO)



time-varying MIMO channel:

τ: Time delay-spread

$$\mathbf{H}(\tau, t) = \begin{bmatrix} h_{1,1}(\tau,t) & h_{1,2}(\tau,t) & \cdots & h_{1,M_T}(\tau,t) \\ h_{2,1}(\tau,t) & h_{2,2}(\tau,t) & \cdots & h_{2,M_T}(\tau,t) \\ \vdots & \vdots & \ddots & \vdots \\ h_{M_R,1}(\tau,t) & h_{M_R,2}(\tau,t) & \cdots & h_{M_R,M_T}(\tau,t) \end{bmatrix}.$$

t: time-variance

Block frequency-flat fading MIMO channel:

$$\mathbf{y}[m] = \mathbf{H}\,\mathbf{x}[m] + \mathbf{w}[m]$$

where

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & \cdots & h_{1,M_T} \\ \vdots & & \vdots \\ h_{M_R,1} & \cdots & h_{M_R,M_T} \end{bmatrix}$$

Channel state information (CSI) may be

- unknown to both Tx and Rx
- only known to Rx
- known to both Rx and Tx

zero-mean, circularly symmetric complex Gaussian (ZMCSCG)

- Consider a continuous complex Gaussian vector $\mathbf{Z}=\mathbf{Z}_I+j\mathbf{Z}_Q=[Z_1,Z_2,\ldots,Z_N]^T$, $Z_k=Z_{kI}+jZ_{kQ}$ with zero mean and NxN correlation matrix $\mathbf{R_Z}=E\{\mathbf{ZZ}^H\}$
- It can be represented by a real-valued vector $\mathbf{Z}'=[\mathbf{Z}_I,\mathbf{Z}_Q]=[Z_{1I},Z_{2I},\ldots,Z_{NI},Z_{1Q},Z_{2Q},\ldots,Z_{NQ}]^T$ with zero mean and correlation 2Nx2N matrix $\mathbf{R_{Z'}}=E\{\mathbf{Z}'\mathbf{Z}'^T\}$ $\mathbf{R}_{\mathbf{Z}'}=\begin{bmatrix}\mathbf{R}_{\mathrm{II}} & \mathbf{R}_{\mathrm{IQ}}\\ \mathbf{R}_{\mathrm{QI}} & \mathbf{R}_{\mathrm{QQ}}\end{bmatrix}$, $\mathbf{R}_{\mathrm{AB}}=E\left\{\mathbf{Z}_{\mathrm{A}}\mathbf{Z}_{\mathrm{B}}^{\mathrm{T}}\right\}$ and pdf $p_{Z'}(\mathbf{z}')=\dfrac{\exp\left(-\left[\mathbf{z}'^{\mathrm{T}}\mathbf{R}_{\mathbf{Z}'}^{-1}\mathbf{z}'\right]/2\right)}{\sqrt{(2\pi)^{2N}\det\left[\mathbf{R}_{\mathbf{Z}'}\right]}}$
- $\mathbf{R_Z}=E\{\mathbf{ZZ}^H\}=E\{(\mathbf{Z}_I+j\mathbf{Z}_Q)(\mathbf{Z}_I-j\mathbf{Z}_Q)^T\}=(\mathbf{R}_{II}+\mathbf{R}_{QQ})+j(\mathbf{R}_{QI}-\mathbf{R}_{IQ})$
- If $\mathbf{R}_{II}=\mathbf{R}_{QQ}$ and $\mathbf{R}_{QI}=-\mathbf{R}_{IQ}\Rightarrow\mathbf{R_Z}=2(\mathbf{R}_{II}-j\mathbf{R}_{IQ})$ and $E\{\mathbf{ZZ}^T\}=\mathbf{0}$ then $\mathbf{Z}$ is zero-mean, circularly symmetric complex Gaussian (ZMCSCG):
  - $\mathbf{Z}_I$, $\mathbf{Z}_Q$: zero-mean, uncorrelated since $\mathbf{R}_{IQ}=E\{\mathbf{Z}_I\mathbf{Z}_Q^T\}=E\{\mathbf{Z}_I\}[E\{\mathbf{Z}_Q\}]^T=\mathbf{0}$,
  - i.e., $\mathbf{Z}_I,\mathbf{Z}_Q$: orthogonal $\Rightarrow$ $\mathbf{Z}_I,\mathbf{Z}_Q$: statistically independent: $\mathbf{R_Z}=2\mathbf{R}_{II}$
  - If Gaussian $\mathbf{Z}_I,\mathbf{Z}_Q$: statistically independent and identically distributed:

$$p_{\mathbf{Z}}(\mathbf{z})=\frac{\exp\left(-\mathbf{z}^{\mathrm{H}}\mathbf{R}_{\mathbf{Z}}^{-1}\mathbf{z}\right)}{\pi^N\det\left[\mathbf{R}_{\mathbf{Z}}\right]}\text{ and }p_{\mathbf{Z}}(\mathbf{z})=p_{\mathbf{Z}_{\mathrm{I}}}(\mathbf{z}_{\mathrm{I}})\,p_{\mathbf{Z}_{\mathrm{Q}}}(\mathbf{z}_{\mathrm{Q}})=\left[\frac{\exp\left(-\left[\mathbf{z}_{\mathrm{I}}{}^{T}\mathbf{R}_{\mathrm{II}}^{-1}\mathbf{z}_{\mathrm{I}}\right]/2\right)}{\sqrt{(2\pi)^{N}\det\left[\mathbf{R}_{\mathrm{II}}\right]}}\right]^2$$

Eigen-decomposition and singular-value decomposition (SVD): quick review

Eigen-decomposition of a MxM complex matrix $\mathbf{A}$:

- Eigenvalue problem: $\mathbf{Ax}=\lambda\mathbf{x}$, λ: scalar, $\mathbf{x}$: Mx1 vector.
- $\mathbf{A}^H=\mathbf{A}^{T*}$, $\mathbf{A}$ is a Hermitian matrix if $\mathbf{A}=\mathbf{A}^H$
- In general, there are up to M distinct *eigenvalues* of $\mathbf{A}$, λ_n, n=1,2,…,M, which are roots of the characteristic equation: $\det[\mathbf{A}-\lambda\mathbf{I}]=0$. Corresponding to each λ_m, m=1,2,…,M, is the *eigenvector* $\mathbf{x}_m$.

  $\Rightarrow\mathbf{Ax}_m=\lambda_m\mathbf{x}_m\Rightarrow\mathbf{A}[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_M]=[\lambda_1\mathbf{x}_1,\lambda_2\mathbf{x}_2,\ldots,\lambda_M\mathbf{x}_M]$

  or $\mathbf{AU}=\mathbf{U}\Lambda\Rightarrow\mathbf{A}=\mathbf{U}\Lambda\mathbf{U}^H$ (spectral decomposition theorem) where

  - $\mathbf{U}=[\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_M]$: *unitary* matrix,i.e., $\mathbf{UU}^H=\mathbf{U}^H\mathbf{U}=\mathbf{I}$, $\mathbf{U}^H=\mathbf{U}^{-1}$, and
  - $\Lambda=\mathrm{diag}[\lambda_1,\lambda_2,\ldots,\lambda_M]$

Singular-value decomposition (SVD) of a complex-valued LxN matrix $\mathbf{G}$:

$$\mathbf{U}^H\mathbf{G}\mathbf{V}=\begin{bmatrix}\mathbf{D}_{LxL} & \mathbf{0}_{Lx(N-L)}\end{bmatrix}\text{for L<N, or }\mathbf{U}^H\mathbf{G}\mathbf{V}=\begin{bmatrix}\mathbf{D}_{NxN}\\ \mathbf{0}_{(L-N)xN}\end{bmatrix}\text{ for L}\geq\text{N,}$$

where LxL unitary $\mathbf{U}^H=\mathbf{U}^{-1}$, NxN unitary $\mathbf{V}^H=\mathbf{V}^{-1}$, $\mathbf{D}=diag(d_1,d_2,\ldots,d_{\min(L,N)})$,

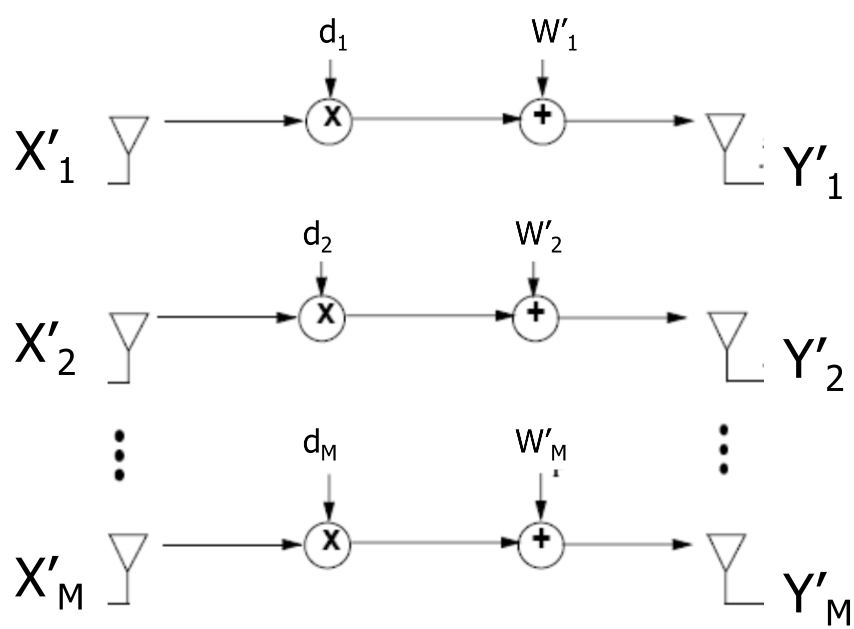
$d_1,d_2,\ldots,d_{\min(L,N)}$: singular values of $\mathbf{G}$, square-roots of the eigenvalues of $\mathbf{G}^H\mathbf{G}$ if L>N, or of $\mathbf{G}\mathbf{G}^H$ if L<N

Parallel Decomposition of a MIMO channel **y=Gx+w**

- M_T=N, M_R=L, LxN Channel Matrix: **G**=[g_{ln}] with E{$|g_{ln}|^2$}=1 and (squared) Frobenius norm $$|\mathbf{G}|^2 = \sum_{l=1,n=1}^{L,N} |g_{\ln}|^2 = tr\left\{\mathbf{G}\mathbf{G}^H\right\} \text{ or } tr\left\{\mathbf{G}^H\mathbf{G}\right\} = \sum_{m}^{\min(L,N)} \lambda_m,\ \lambda_m = d_m^2$$
  where d_m is the m[th] singular values of **G**
- Tx : $\mathbf{X}=[X_1,X_2,\ldots,X_N]^T$, Rx: $\mathbf{Y}=[Y_1,Y_2,\ldots,Y_L]^T$ noise: $\mathbf{W}=[W_1,W_2,\ldots,W_L]^T$
- From **y=Gx+w,** using singular-value decomposition (SVD), we can obtain
  - $\Rightarrow$ $\mathbf{y}'=\mathbf{U}^H\mathbf{y}=\mathbf{U}^H\mathbf{G}\mathbf{V}\mathbf{V}^H\mathbf{x}+\mathbf{U}^H\mathbf{w} = \mathbf{D}_{\min(L,N)}\,\mathbf{x}'+\mathbf{w}'$ where
  - $\mathbf{y}'=\mathbf{U}^H\mathbf{y}$, $\mathbf{w}'=\mathbf{U}^H\mathbf{w}$, $\mathbf{x}'=\mathbf{V}^H\mathbf{x}$ are column vectors of M=min(L,N) elements.

- The SVD of **G** transform the MIMO channel into M=min(L,N) parallel virtual SISO channels:

  $Y'_m=d_mX'_m+W'_m$ where
  - Y'_m, X'_m, W'_m are the m[th] components of **Y'**, **X'**, **W'**, respectively, and
  - d_m is the m[th] singular values of **G**.



MIMO channel with Gaussian noise y=Gx+w: conditional mutual information for a given G

- **y=Gx+w**
- Additive Gaussian noise: $\mathbf{W}=[W_1,W_2,\ldots,W_L]^T$: ZMCSCG with LxL correlation matrix $\mathbf{R_W}=E\{\mathbf{WW}^H\}$, $\mathbf{R_W}=\sigma_W^2\,\mathbf{I}_L$ for white Gaussian noise ($\sigma_W^2=N_o/T_s$)
- Continuous source $\mathbf{X}=[X_1,X_2,\ldots,X_N]^T$, with correlation NxN matrix $\mathbf{R_X}=E\{\mathbf{XX}^H\}$, and Tx power $P=tr[\mathbf{R_X}]$: trace of $\mathbf{R_X}$: sum of N diagonal elements of $\mathbf{R_X}$
- **X**, **W**: independent, continuous Rx $\mathbf{Y}=[Y_1,Y_2,\ldots,Y_L]^T$ with LxL correlation matrix $\mathbf{R_Y}$

$$\mathbf{R}_Y = E\left\{(\mathbf{GX}+\mathbf{W})(\mathbf{GX}+\mathbf{W})^H\right\} = E\left\{\mathbf{GXX}^H\mathbf{G}^H + \mathbf{GXW}^H + \mathbf{WX}^H\mathbf{G}^H + \mathbf{WW}^H\right\} = \mathbf{GR}_X\mathbf{G}^H + \mathbf{R}_W$$

- Conditional entropy $H(\mathbf{Y}|\mathbf{X},\mathbf{G})= H(\mathbf{W})= [L\log(\pi e)+\log(\det[\mathbf{R_W}])]$
- conditional mutual information for a given **G:** $I(\mathbf{X},\mathbf{Y}|\mathbf{G})=H(\mathbf{Y})-H(\mathbf{Y}|\mathbf{X})$
- Capacity $C= \max_{p(x)} I(\mathbf{X},\mathbf{Y}|\mathbf{G})=\max_{p(x)} [H(\mathbf{Y})-H(\mathbf{Y}|\mathbf{X})]$ equivalent to $\max_{p(x)} H(\mathbf{Y})$ when **X** is ZMCSCG
  - **X**, **W**: independent, $\Rightarrow$**Y**: Gaussian, zero-mean,
  - $H(\mathbf{Y})=E\{-\log[p_{\mathbf{Y}}(\mathbf{y})]\}=[L\log(\pi e)+\log(\det[\mathbf{R_Y}])]$
  - $I(\mathbf{X},\mathbf{Y}|\mathbf{G})=H(\mathbf{Y})-H(\mathbf{Y}|\mathbf{X})= \log(\det[\mathbf{R_W}+\mathbf{GR_XG}^H]/\det[\mathbf{R_W}])$

capacity of MIMO channel G with Gaussian noise:
G: known at the transmitter

- C= log(det[$\mathbf{R_W}$+$\mathbf{GR_XG}^H$]/det[$\mathbf{R_W}$]) with $\max_{\mathbf{R}_x} tr\left[\mathbf{R}_x\right] \le P$: constant Tx power
  where P: total power, symbol energy: $E_s = PT_s$
- $\mathbf{R_X}$: non-negative definite, to be designed
- For $\mathbf{R_W}=\sigma_W^2\,\mathbf{I}_L$: white Gaussian noise ($\sigma_W^2=N_o/T_s$), C= log(det[$\mathbf{I}_L$+$\mathbf{GR_XG}^H/\sigma_W^2$])
- From det[$\mathbf{I}$+$\mathbf{AB}$]=det[$\mathbf{I}$+$\mathbf{BA}$] $\Rightarrow$det[$\mathbf{I}+\mathbf{GR_XG}^H/\sigma_W^2$]=det[$\mathbf{I}+\mathbf{R_XG}^H\mathbf{G}/\sigma_W^2$]
- Eigen-Decomposition of a Hermitian matrix: $\mathbf{G}^H\mathbf{G}=\mathbf{U}\Lambda\mathbf{U}^H$
- $\Rightarrow$ det[$\mathbf{I}+\mathbf{R_XG}^H\mathbf{G}/\sigma_W^2$]=det[$\mathbf{I}+\mathbf{R_XU}\Lambda\mathbf{U}^H/\sigma_W^2$]=det[$\mathbf{I}+\Lambda\mathbf{U}^H\mathbf{R_XU}/\sigma_W^2$]
- tr{$\mathbf{U}^H\mathbf{R_XU}$}= tr{$\mathbf{U}^H\mathbf{UR_X}$}=tr{$\mathbf{R_X}$} and $\mathbf{U}^H\mathbf{R_XU}$: also non-negative definite
- Recall: any non-negative definite NxN $\mathbf{A}$ satisfies the Hadamard inequality:

$$\det\left(\mathbf{A}\right) \le \prod_{n=1}^{N} a_{nn}$$ where a_{nn}: diagonal elements of $\mathbf{A}$

$$\Rightarrow \det[\mathbf{I}+\Lambda\mathbf{U}^H\mathbf{R_XU}/\sigma_W^2] \le \prod_{n=1}^{N}\left[1+\lambda_n P_{nn}/\sigma_W^2\right] = \prod_{n=1}^{N}\lambda_n\left[\left(1/\lambda_n\right)+P_{nn}/\sigma_W^2\right]$$

where $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_N$: eigenvalues of $\mathbf{G}^H\mathbf{G}$ and P_{nn}: diagonal elements of $\mathbf{U}^H\mathbf{R_XU}$. The equality holds only when $\mathbf{U}^H\mathbf{R_XU}$ is a diagonal matrix.

$$\Rightarrow \quad \max_{P_{nn}} C = \sum_{n=1}^{N}\log\left[1+\lambda_n P_{nn}/\sigma_W^2\right] \text{ subject to the power constraint } \sum_{n=1}^{N} P_{nn} \le P$$

capacity of MIMO channel G with Gaussian noise & G known at the transmitter: water-filling procedure

- As log(.) is monotonously increasing with its argument, to maximize C, we select a set of $\{P_{nn}\}$ to meet the total power constraint *P*, and maximize

$$\prod_{n=1}^{N} \lambda_n \left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right]$$

- Depending on *P*, some $P_{nn}=0$ selected to be corresponding to the smallest λ_n. Let $N^+ \le N$ such that $P_{nn}>0$ for $n \le N^+$ and $\Lambda^+ = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_{N+}]$. Then

$$\prod_{n=1}^{N} \lambda_n \left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right] = \prod_{n=1}^{N} \lambda_n \prod_{n=1}^{N^+} \left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right] \prod_{n=N^++1}^{N} \left[1/\lambda_n \right]$$

$$= \left[\prod_{n=1}^{N^+} \lambda_n \right] \prod_{n=1}^{N^+} \left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right] \le \left[\prod_{n=1}^{N^+} \lambda_n \right] \left[\left(\sum_{n=1}^{N^+} \left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right] \right) / N^+ \right]^{N^+}$$

- The inequality follows the relation between the *arithmetic* and *geometric means*. The equality holds only if

$$\left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right] = \left(\sum_{n=1}^{N^+} \left[(1/\lambda_n) + P_{nn}/\sigma_W^2 \right] \right) / N^+ = \left[Tr\left(\left[\Lambda^+ \right]^{-1} \right) + P/\sigma_W^2 \right] / N^+$$

- water-filling procedure:

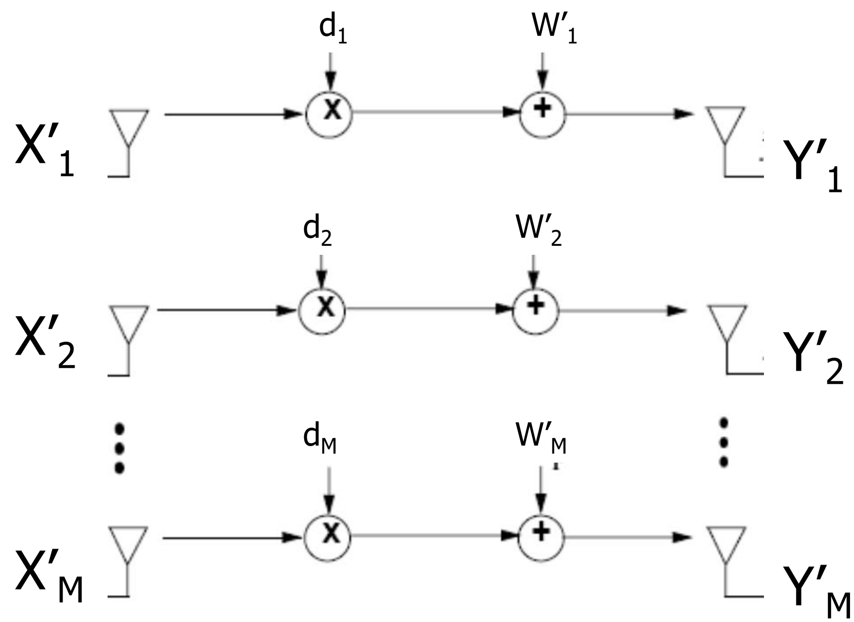
Fill power (water) $P_{nn} = \left(\rho - \sigma_W^2 / \lambda_n \right)$ where "water-fill level" $\rho = \left[\sigma_W^2 Tr\left(\left[\Lambda^+ \right]^{-1} \right) + P \right] / N^+$

capacity of MIMO channel G with Gaussian noise & G known at the transmitter & using water-filling procedure

- Transmit more signal power in the better channels (larger λ_n). N^+ can be implicitly found from the above water-filling procedure: Start with $N^+=N$, if $\left(\rho \le \sigma_W^2 / \lambda_{N^+}\right)$ set $N^+ = N^+-1$ and continue until $\left(\rho > \sigma_W^2 / \lambda_{N^+}\right)$.
- The capacity of the channel **G** when **G** is known to the transmitter is

$$C_T(\mathbf{G}) = \log\left(\det\left(\mathbf{\Lambda}^+\right) \left(\left[Tr\left(\left[\mathbf{\Lambda}^+\right]^{-1} \right) + P / \sigma_W^2 \right] / N^+ \right)^{N^+} \right)$$

- i.e., From $\mathbf{y=Gx+w}$
$\Rightarrow \mathbf{y'}=\mathbf{U}^H\mathbf{y}=\mathbf{U}^H\{\mathbf{GVx'+w}\}$, $\mathbf{x'}=\mathbf{V}^H\mathbf{x}$, the SVD of **G** transform the MIMO channel into rank $M \le \min(L,N)$ virtual SISO channels: $Y'_m=[\lambda_m]^{1/2}X'_m+W'_m$ where Y'_m, X'_m, W'_m are the m^{th} components of $\mathbf{Y'}$, $\mathbf{X'}$, $\mathbf{W'}=\mathbf{U}^H\mathbf{w}$,
- optimum $\mathbf{R_X}$ satisfies
$\mathbf{U}^H\mathbf{R_X}\mathbf{U}=\mathrm{diag}\{P_{nn}, n=1,2,\ldots,N^+\}$



Ergodic capacity of MIMO flat-fading Gaussian channel: G known to the receiver but **unknown to the transmitter**

- G: is Zero Mean Spatially White (channel elements can be assumed to be i.i.d random variables), stationary and ergodic (*time* averages of a sample function are equal to the corresponding *ensemble average* or *expectation* at a particular time instant).
- Without knowledge of G at Tx, Tx *equally* distributes power over Tx antennas, i.e., $R_X=\sigma_X^2 I$, $\sigma_X^2=P/N$,
- $I(X,Y|G)=H(Y)–H(Y|X)= \log(\det[R_W+GR_XG^H]/\det[R_W])$, AWGN: $R_W=\sigma_W^2 I$
- Ergodic capacity: $C=E_G\{C_{UT}(G)\}$, $C_{UT}(G)=\log(\det[R_W+GR_XG^H]/\det[R_W])$ subject to power constraint (i.e., total Tx power of P).
- $C_{UT}(G)=\log(\det[I_L +(P/N\sigma_W^2) GG^H]$ for L<N for GG^H to be of full rank
- $C_{UT}(G)=\log(\det[I_N +(P/L\sigma_W^2) G^HG]$ for L>N for G^HG to be of full rank
- eigen-decomposition of a Hermitian matrix: $GG^H=U\Lambda U^H$ for L<N or $G^HG=U^H\Lambda U$ for L>N.
- From $\det[I+AB]=\det[I+BA] \Rightarrow C_{UT}(G)=\log(\det[I_L+(P/N\sigma_W^2)U\Lambda U^H]$
- $C_{UT}(G)=\log(\det[I_L+(P/N\sigma_W^2)\Lambda U^HU]$ for L<N, or $=\log(\det[I_N+(P/L\sigma_W^2)\Lambda UU^H]$ for L>N

$$\Rightarrow \quad C_{UT}(\mathbf{G}) = \log\left[\prod_{m=1}^{\min(L,N)}\left(1+P\lambda_m/\sigma_W^2 \max(L,N)\right)\right] = \sum_{m=1}^{\min(L,N)} \log\left[1+P\lambda_m/\sigma_W^2 \max(L,N)\right]$$

- MIMO ergodic capacity = sum of capacities of min(L,N) virtual SISO channels defined by the spatial eigenmodes of G^HG or GG^H.
- For N=L, as $N\to\infty$, $\log_a\left[1+P\lambda_m/N\sigma_W^2\right] \to \dfrac{P\lambda_m}{N\sigma_W^2 \log_e(a)}$ (Taylor's series) and

$$C=\sum_{m=1}^{N} E\left\{\log_a\left[1+P\lambda_m/N\sigma_W^2\right]\right\} \to \frac{P\lambda_{avg}}{\sigma_W^2 \log_e(a)} \text{ where } \lambda_{avg}=\frac{1}{N}\sum_{m=1}^{N} E\{\lambda_m\}$$

i.e., C is *linearly* (rather than *logarithmically*) increased with SNR.

Capacity comparison

$$\frac{C_T(\mathbf{G})}{C_{UT}(\mathbf{G})}=\frac{\log\left(\left[\prod_{m=1}^{\min(L,N)}\lambda_m\right]\left(\left[\left(\sum_{m=1}^{\min(L,N)}1/\lambda_m\right)+P/\sigma_W^2\right]/N^+\right)^{N^+}\right)}{\log\left[\prod_{m=1}^{\min(L,N)}\left(1+P\lambda_m/\sigma_W^2\max(L,N)\right)\right]}$$

- FOR LARGE SNR: $P/\sigma_W^2\to\infty$

$$\frac{C_T(\mathbf{G})}{C_{UT}(\mathbf{G})}\to\frac{\log\left(\left[\prod_{m=1}^{\min(L,N)}\lambda_m\right]\left(P/N^+\sigma_W^2\right)^{N^+}\right)}{\log\left(\prod_{m=1}^{\min(L,N)}\left(P\lambda_m/\sigma_W^2\max(L,N)\right)\right)}=\frac{\log\left(\left[\prod_{m=1}^{\min(L,N)}\lambda_m\right]\left(P/N^+\sigma_W^2\right)^{N^+}\right)}{\log\left(\left[\prod_{m=1}^{\min(L,N)}\lambda_m\right]\left[P/\max(L,N)\sigma_W^2\right]^{\min(L,N)}\right)}\to 1$$

- FOR SMALL SNR: In $C_T(\mathbf{G})$, all Tx power allocated to the dominant eigenvector, corresponding to λ_1, of $\mathbf{GG}^H$ and the channel is essentially reduced to a SISO channel with gain λ_1.

$$\Rightarrow C_T(\mathbf{G})\approx\log_a\left(1+\left[P/\sigma_W^2\right]\lambda_{\max}\right)\approx\left[P/\sigma_W^2\right]\lambda_{\max}/\log_e(a)$$

$$C_{UT}(\mathbf{G})=\sum_{m=1}^{\min(L,N)}\log_a\left(1+P\lambda_m/\sigma_W^2\max(L,N)\right)\approx\sum_{m=1}^{\min(L,N)}P\lambda_m/\sigma_W^2\max(L,N)\log_e(a)$$

$$\Rightarrow\frac{C_T(\mathbf{G})}{C_{UT}(\mathbf{G})}\approx\max(L,N)\frac{\lambda_{\max}}{\sum_{m=1}^{\min(L,N)}\lambda_m}\Rightarrow\frac{C_T(\mathbf{G})}{C_{UT}(\mathbf{G})}\approx N\ \text{for}\ L=1\ \text{or}\ L\ \text{for}\ N=1$$

- $\Rightarrow$ CSI at Tx is more important for low SNR

SIMO, MISO: capacity

DIVERSITY-ON-RECEIVE SIMO CHANNEL:

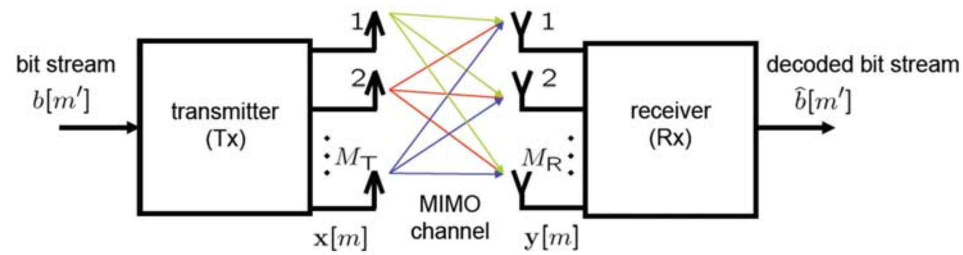
- $M_T=1$, $M_R=L$, $\mathbf{h}=[h_1,h_2,\ldots,h_L]^T$, $\mathbf{y=h}x+\mathbf{w}$, $\mathbf{R_W}=\sigma_W^2\,\mathbf{I}_L$: white Gaussian noise
- $C=E_{\mathbf{h}}\{\log[1+(P/\sigma_W^2)\mathrm{tr}\{\mathbf{h}^H\mathbf{h}\}]\}=E_{\mathbf{h}}\{\log[1+(P|\mathbf{h}|^2/\sigma_W^2)]\}$, $P/\sigma_W^2=E_s/N_o$
- Additional Rx antennas provides only a log-increase in capacity.
- Channel knowledge at Tx for SIMO channels has no capacity benefit.
- $\mathrm{tr}\{\mathbf{hh}^H\}=|h_1|^2+|h_2|^2+|h_3|^2+\ldots+|h_L|^2=|\mathbf{h}|^2$: sum of eigen values of $\mathbf{h}^H\mathbf{h}$
- The result confirms the maximal-ratio combining (MRC) principle.

DIVERSITY-ON-TRANSMIT MISO CHANNEL:

- $M_T=N$, $\mathbf{h}=[h_1,h_2,\ldots,h_N]^T$, $P/\sigma_W^2=E_s/N_o$
- **h** *unknown* at Tx: equal power, P/N, over N Tx antennas

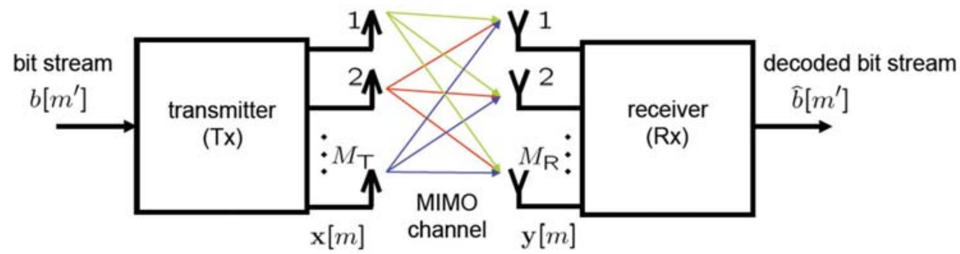
  $C=E_{\mathbf{h}}\{\log[1+(P|\mathbf{h}|^2/N\sigma_W^2)]\}$: no improvement in capacity over a SISO channel.
- **h** *known* at Tx: $C=E_{\mathbf{h}}\{\log[1+(P|\mathbf{h}|^2/\sigma_W^2)]\}$, same as in SIMO

Diversity in MIMO:



- Diversity techniques rely on transmitting the signal over multiple (ideally) independently fading paths (in time/frequency/space).
- Spatial (or antenna) diversity is preferred over time/frequency diversity as it does not incur an expenditure in transmission time or bandwidth.
- Each of $M_R M_T$ pairs of Tx-Rx antennas provides a signal path from transmitter to receiver.
- By sending the SAME information through different paths, *d* independently-faded replicas of the Tx signal can be obtained and combined at the receiver such that the resultant signal exhibits considerably reduced amplitude variability in comparison to a SISO link and we have a diversity order *d*, where $1 \leq d \leq M_R M_T$
- **Diversity gain** (order) *d* implies that in the high SNR region, $P_e \sim 1/SNR^d$ as opposed to 1/SNR for a SISO system: The higher the diversity gain, the lower the P_e
- Extracting spatial diversity gain in the absence of channel knowledge at the transmitter is possible using suitably designed transmit signals, e.g., space–time coding techniques

MIMO: Diversity & Multiplexing gains



Spatial Diversity:

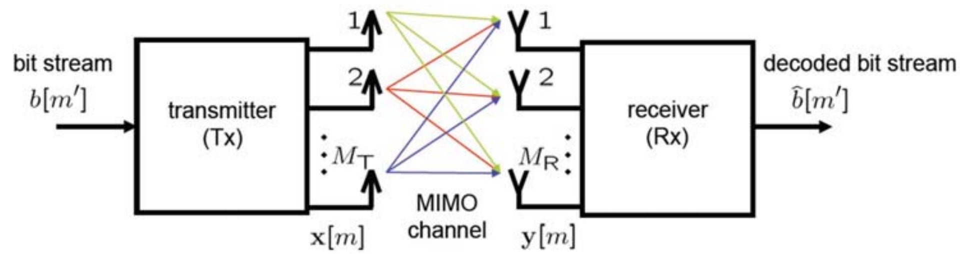
- Sending the SAME information through different paths (selected from the total of $M_R M_T$ pairs) to obtain a diversity of d to reduce P_e ($\sim 1/SNR^d$): Go for Reliability, Counter fading

Spatial Multiplexing :

- We can convert a general MIMO model to an equivalent diagonal model of some rank $M \leq \min(M_T, M_R)$ and send DIFFERENT information over different links (selected from M) to increase the transmission rate, i.e., go for Multiplexing (optimistic approach using rich scattering/fading to advantage)

$\Rightarrow$ **Diversity/Multiplexing Tradeoff in MIMO**

MIMO: Array & Diversity gains



Array Gain:

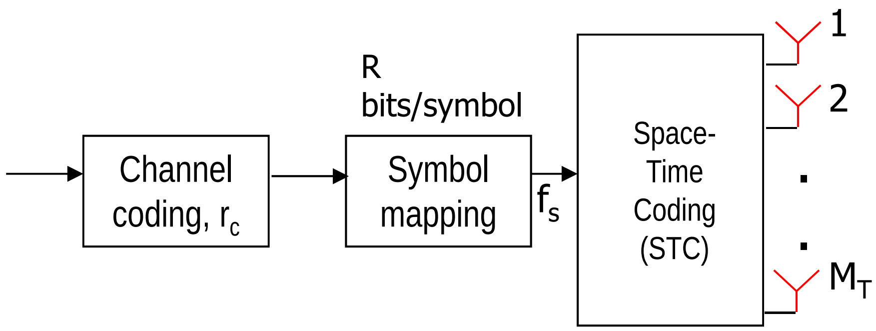
- Array gain can be made available through processing at the transmitter and the receiver and results in an increase in average receive SNR due to a coherent combining effect.
- Transmit/receive array gain requires channel knowledge in the transmitter and receiver, respectively, and depends on the number of transmit and receive antennas.
- Channel knowledge in the receiver is typically available whereas channel state information in the transmitter is in general more difficult to maintain.

Array Gain (a_{max}) & Diversity Order (d_{max}):

- channel knowledge availability: U (at Rx only) , K (at both Tx and Rx)
- Channel: SIMO ($1, M_R$), MISO (M_T,1), MIMO (M_T, M_R)

	SIMO-U	SIMO-K	MISO-U	MISO-K	MIMO-U	MIMO-K
a_{max}	M_R	M_R	1	M_T	M_R	$E\{\lambda_{max}\}$
d_{max}	M_R	M_R	M_T	M_T	$M_T M_R$	$M_T M_R$

Space-Time Coding (STC)

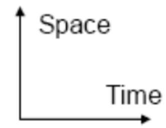


- Channel coding (FEC): with redundancy in time with coding rate $r_c < 1$
- Space-Time Coding (STC): with rate $r_s = N/N_T$ where N is number of *different* symbols transmitted in N_T *symbol intervals* using a bandwidth B
- Assume $B=f_s$ (symbol rate=1/symbol interval)
- r_s represents spatial multiplexing gain:
  - $r_s \leq 1$: diversity mode
  - $r_s = M_T > 1$: spatial multiplexing mode (max transmission rate)
- Spectral efficiency $\varepsilon_{BW}=(f_s R r_c r_s)/B= R r_c r_s$ in bits/s/Hz

Alamouti 2x2 Orthogonal Space-Time Block Code for 2 Tx antennas

S: transmission matrix: orthogonal in both the space and time

$$\mathbf{S} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}$$



- Unitarity=Complex orthogonality

$$\mathbf{S} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} \rightarrow \mathbf{S}\mathbf{S}^H = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} \begin{bmatrix} s_1^* & s_2^* \\ -s_2 & s_1 \end{bmatrix} = \left(|s_1|^2 + |s_2|^2\right)\mathbf{I}_{2x2} \rightarrow \mathbf{S}^{-1} = \mathbf{S}^H / \left(|s_1|^2 + |s_2|^2\right)$$

- linearity: $\mathbf{S} = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} = s_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s_1^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - s_2^* \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(Siavash M. Alamouti, "A Simple Transmit Diversity Technique for Wireless Communications", IEEE JSAC, Oct.1998, pp.1451-1458)

Decoding Alamouti OSTBC

- For 1x2 complex channel $\mathbf{G}=[G_1,G_2]$, and Tx vector $\mathbf{s}=[s_1,s_2]^T$,
- Rx signals in 2 consecutive time slots, Y_1, Y_2 are

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} G_1 s_1 + G_2 s_2 \\ -G_1 s_2^* + G_2 s_1^* \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \rightarrow \begin{bmatrix} Y_1 \\ Y_2^* \end{bmatrix} = \begin{bmatrix} G_1 s_1 + G_2 s_2 \\ -G_1^* s_2 + G_2^* s_1 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2^* \end{bmatrix} = \begin{bmatrix} G_1 & G_2 \\ G_2^* & -G_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2^* \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} G_1 & G_2 \\ G_2^* & -G_1^* \end{bmatrix}^H \begin{bmatrix} Y_1 \\ Y_2^* \end{bmatrix} = \begin{bmatrix} G_1 & G_2 \\ G_2^* & -G_1^* \end{bmatrix}^H \begin{bmatrix} G_1 & G_2 \\ G_2^* & -G_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} G_1 & G_2 \\ G_2^* & -G_1^* \end{bmatrix}^H \begin{bmatrix} W_1 \\ W_2^* \end{bmatrix}$$

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \left(|G_1|^2 + |G_2|^2 \right) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} G_1^* W_1 + G_2 W_2^* \\ G_2^* W_1 - G_1 W_2^* \end{bmatrix}$$

For iid zero-mean Gaussian noise components, W_1, W_2 with variance σ_W^2, $G_1^* W_1 + G_2 W_2^*$ and $G_2^* W_1 - G_1 W_2^*$ are Gaussian components with $\mathrm{E}\{G_1^* W_1 + G_2 W_2^*\} = \mathrm{E}\{G_2^* W_1 - G_1 W_2^*\} = 0$ and

$$\mathrm{E}\left\{ \left| G_1^* W_1 + G_2 W_2^* \right|^2 \right\} = \mathrm{E}\left\{ G_1^* W_1 G_1 W_1^* + G_2 W_2^* G_1 W_1^* + G_1^* W_1 G_2^* W_2 + G_2 W_2^* G_2^* W_2 \right\} = \left(|G_1|^2 + |G_2|^2 \right) \sigma_W^2$$

$$\mathrm{E}\left\{ \left| G_2^* W_1 - G_1 W_2^* \right|^2 \right\} = \left(|G_1|^2 + |G_2|^2 \right) \sigma_W^2$$

- Z_1, Z_2 can be independently decoded for s_1, s_2, respectively

Decoding Alamouti OSTBC: ML decision

- Equally probable Tx of s_1, s_2 from $\mathbf{A}=\{a_k, k=1,2,\ldots,K\}$
- Receiver knows $\mathbf{G}$, computes Euclidean distances: $|Z_j - (|G_1|^2+|G_2|^2)a_k|^2$
- ML decision rule: select a_k for the shortest Euclidean distance.
- Rx SNR: $(|\mathbf{G}|^4 E_s/2)/(|\mathbf{G}|^2 N_o)=(|\mathbf{G}|^2/2)/(E_s/N_o) \Rightarrow E\{(|\mathbf{G}|^2/2)(E_s/N_o)\}=(E_s/N_o)$
- Absence of channel knowledge does not provide array gain

$$P_e(\mathbf{G}) \approx aQ\left[\sqrt{\frac{|\mathbf{G}|^2 dE_s}{2N_0}}\right] \leq a\exp\left(-\frac{|\mathbf{G}|^2 dE_s}{4N_0}\right)$$

- For Rayleigh fading channel, avg. symbol error rate diversity order of 2 and no array gain
- Full diversity, full-rate complex STBC, i.e., code rate=1.
- For $M_T=2$, $M_R=1$, the Alamouti code is the only optimal STBC that satisfies the capacity formula (3dB worse than MRC diversity-on-receive channel with $M_T=1$, $M_R=2$)
- It is the only orthogonal STBC that achieves rate 1, i.e., can achieve its full diversity gain without needing to sacrifice its data rate.

ST coding in MIMO frequency flat fading channels

- no channel knowledge at Tx
- over N Tx antennas, L Rx antennas, J symbol time intervals.
- Symbol set $\mathbf{A}=\{a_i, i=1,2,\ldots, 2^q\}$
- Block of qk information bits encoded to qn bits.
- Received LxJ matrix $\mathbf{Y}=[\mathbf{y}(1), \mathbf{y}(2),\ldots, \mathbf{y}(J)]=\mathbf{GS}+\mathbf{W}$,
- Transmitted NxJ matrix $\mathbf{S}=[\mathbf{s}(1), \mathbf{s}(2),\ldots, \mathbf{s}(J)]$, $E\{\mathbf{s}\}=E_s/N=PT_s/N$
- LxJ noise matrix: $\mathbf{W}=[\mathbf{w}(1), \mathbf{w}(2),\ldots, \mathbf{w}(J)]$
- $\mathbf{y}(j)=\mathbf{G}\mathbf{s}(j)+\mathbf{w}(j)$, $j=1, 2,\ldots,J$; $\mathbf{w}(j)$: zero-mean WGN with $N_o\mathbf{I}_L$
- $\mathbf{G}$: LxN channel matrix with $E\{g_{ln}\}=0$, $E\{|g_{ln}|^2\}=1$
- **Rx ML decoding:** Rx knows **G** $\mathbf{Y}=\arg\min_{\mathbf{S}} |\mathbf{Y}-\mathbf{GS}|^2=\arg\min_{\mathbf{S}} \sum_{j=1}^{J} |\mathbf{y}(j)-\mathbf{Gs}(j)|^2$

PEP: pairwise error probability:

$$\Pr\left\{\mathbf{S}\rightarrow\mathbf{S'}\middle|\mathbf{G}\right\}=Q\left(\sqrt{\left\|\mathbf{G}\left[\mathbf{S}-\mathbf{S'}\right]\right\|^2/2N_o}\right)\ \le e^{\left(-|\boldsymbol{\gamma}\boldsymbol{\Sigma}|^2/4N_o\right)},$$

$$\text{where } \left\|\mathbf{G}\left[\mathbf{S}-\mathbf{S'}\right]\right\|^2=\sum_{l=1}^{L}\mathbf{g}_l\left[\mathbf{S}-\mathbf{S'}\right]\left[\mathbf{S}-\mathbf{S'}\right]^H\mathbf{g}_l^H=\left|\mathbf{G}\left[\mathbf{S}-\mathbf{S'}\right]\right|^2=\left|\boldsymbol{\gamma}\boldsymbol{\Sigma}\right|^2=\mathrm{tr}\left[\boldsymbol{\Sigma}^H\boldsymbol{\gamma}^H\boldsymbol{\gamma}\boldsymbol{\Sigma}\right]$$

$$\text{where } \boldsymbol{\gamma}=\left[vec(\mathbf{G}^T)\right]^T \text{ and } \boldsymbol{\Sigma}=\mathbf{I}_L\otimes\left[\mathbf{S}-\mathbf{S'}\right]$$

Notes: Kronecker product of mxn matrix $\mathbf{A}$ and m'xn' matrix $\mathbf{B}$ $\Rightarrow$ mm'xnn' matrix

$$\mathbf{C}=\mathbf{A}\otimes\mathbf{B}=\begin{pmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B}\end{pmatrix},\quad \mathbf{A}=\left[\mathbf{a}_1,\mathbf{a}_2,\cdots\mathbf{a}_n\right]$$

$$vec(\mathbf{A})=\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \vdots\\ \mathbf{a}_n\end{bmatrix}:\text{stack the mxn matrix } \mathbf{A} \text{ into a mnx1 vector columwise}$$

$$\Rightarrow \Pr\left\{\mathbf{S}\rightarrow\mathbf{S'}\right\}=E_G\left\{\Pr\left\{\mathbf{S}\rightarrow\mathbf{S'}\middle|\mathbf{G}\right\}\right\}\le\left(\det\left[I_{NL}+E_G\left\{\boldsymbol{\Sigma}^H\boldsymbol{\gamma}^H\boldsymbol{\gamma}\boldsymbol{\Sigma}\right\}/4N_o\right]\right)^{-L}$$

- for **G**, γ: Gaussian

ST code design criteria:

- For iid spatial white channel **G**= $\mathbf{G}_W$, i.e., $E\{g_{ln}g_{ij}^*\}=\delta(l-i)\delta(n-j)$

$$\Rightarrow E_G\left\{\boldsymbol{\Sigma}^H\boldsymbol{\gamma}^H\boldsymbol{\gamma}\boldsymbol{\Sigma}\right\}=\boldsymbol{\Sigma}^H\boldsymbol{\Sigma}$$

$$\Rightarrow \Pr\{\mathbf{S}\to\mathbf{S}'\}\le\left(\det\left[I_{NL}+\boldsymbol{\Sigma}^H\boldsymbol{\Sigma}/4N_o\right]\right)^{-L}=\left(\prod_{i=1}^{I(\mathbf{S},\mathbf{S}')}\left[1+\left(\lambda_i(\mathbf{S},\mathbf{S}')/4N\right)\left(E_s/N_o\right)\right]\right)^{-L}$$

where $\lambda_i(\mathbf{S},\mathbf{S}')$ is the i$^{\text{th}}$ non-zero eigenvalue of $\left(E_s/N\right)^{-1}[\mathbf{S}-\mathbf{S}'][\mathbf{S}-\mathbf{S}']^H$ for i=1,2,...,$I(\mathbf{S},\mathbf{S}')$

$$\Rightarrow \text{ for high } \left(E_s/N_o\right)>>1,\ \Pr\{\mathbf{S}\to\mathbf{S}'\}\le\left(\prod_{i=1}^{I(\mathbf{S},\mathbf{S}')}\lambda_i(\mathbf{S},\mathbf{S}')\right)^{-L}\left[\left(E_s/N_o\right)/\left(4N\right)\right]^{-LI(\mathbf{S},\mathbf{S}')}$$

- **determinant criterion:** coding gain depends on $\prod_{i=1}^{I(\mathbf{S},\mathbf{S}')}\lambda_i(\mathbf{S},\mathbf{S}')$; therefore to maximize the coding gain, we maximize the minimum of the determinant of $[\mathbf{S}-\mathbf{S}'][\mathbf{S}-\mathbf{S}']^H$
- **rank criterion:** spatial diversity gain is indicated by LI(**S**,**S'**), with I(**S**,**S'**)≤N; therefore, design code for difference matrix of any pair (**S**,**S'**) to be full-rank, i.e., I(**S**,**S'**)=N

V. Tarokh, N. Seshadri, A. R. Calderbank, "Space–Time Codes for High Data Rate Wireless Communication: Performance Criterion and Code Construction", *IEEE Transactions on Information Theory*, Vol. 44, No. 2, March 1998, pp. 744-765

Orthogonal STBC Designs

- Alamouti code: 2x2 orthogonal STBC, spatial rate=1

$$\mathbf{S}=\begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} \rightarrow [\mathbf{S}-\mathbf{S}'][\mathbf{S}-\mathbf{S}']^H=\left(|\Delta s_1|^2+|\Delta s_2|^2\right)\mathbf{I}_{2x2} \rightarrow [\mathbf{S}-\mathbf{S}']:\text{orthogonal}$$

($\mathbf{S}-\mathbf{S}'$: codeword different matrix)

  - effective channel is rendered orthogonal regardless of the channel realization $\Rightarrow$ Rx: simple scalar ML detection
  - achieve full-diversity without wasting bandwidth
  - achieve full-rate (two symbols over two time slots)

  Extension to the case of more than 2 antennas?

- For N=4: 4x4 orthogonal STBC, spatial rate=1 (full-rate)

$$\text{For N=4, } \mathbf{S}=\begin{bmatrix} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & s_4 & -s_3 \\ s_3 & -s_4 & s_1 & s_2 \\ s_4 & s_3 & -s_2 & s_1 \end{bmatrix}, s_i:\text{real-valued} \Rightarrow \mathbf{S}-\mathbf{S}'=\text{orthogonal},$$

$$I(\mathbf{S},\mathbf{S}')=N=4, \lambda_i(\mathbf{S},\mathbf{S}')=\left[\left(E_s/N\right)^{-1}|\mathbf{S}-\mathbf{S}'|^2\right]/N \Rightarrow \text{ for high } \left(E_s/N_o\right)>>1,$$

$$\Pr\{\mathbf{S}\rightarrow\mathbf{S}'\}\le\left(\left[\left(E_s/N\right)^{-1}|\mathbf{S}-\mathbf{S}'|^2\right]/N\right)^{-LN}\left[\left(E_s/N_o\right)/(4N)\right]^{-LN}=\left[\left(|\mathbf{S}-\mathbf{S}'|^2/N_o\right)/(4N)\right]^{-LN}, N=4$$

V. Tarokh, H. Jafarkhani, A. R. Calderbank, “Space–Time Block Codes from Orthogonal Designs”, *IEEE Trans on Info. The.*, July 1999, pp. 1456-1467

orthogonal STBC with spatial rate<1:

- orthogonal STBC with complex constellation and N>2
- does not exist for spatial rate=1,
- exist for spatial rate ≤1/2

spatial rate=1/2, N=3

$$\mathbf{S} = \begin{bmatrix} s_1 & -s_2 & -s_3 & -s_4 & s_1^* & -s_2^* & -s_3^* & -s_4^* \\ s_2 & s_1 & s_4 & -s_3 & s_2^* & -s_1^* & -s_4^* & -s_3^* \\ s_3 & -s_4 & s_1 & s_2 & s_3^* & -s_4^* & s_1^* & s_2^* \end{bmatrix}$$

spatial rate=3/4, N=3

$$\mathbf{S} = \begin{bmatrix} s_1 & s_2^* & s_3^*/\sqrt{2} & s_3^*/\sqrt{2} \\ s_2 & s_1^* & s_3^*/\sqrt{2} & -s_3^*/\sqrt{2} \\ s_3/\sqrt{2} & s_3/\sqrt{2} & \left[-s_1 - s_1^* + s_2 - s_2^*\right]/2 & \left[s_1 - s_1^* + s_2 - s_2^*\right]/2 \end{bmatrix}$$

V. Tarokh, H. Jafarkhani, A. R. Calderbank, "Space–Time Block Codes from Orthogonal Designs", *IEEE Transactions on Information Theory*, Vol. 45, No. 5, July 1999, pp. 1456-1467

Orthogonal STBC Designs (cont.)

- **For N = 8**: 8x4 STBC, complex-value, spatial rate=1/2

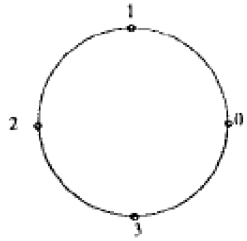
$$S = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3 & s_4 & s_1 & s_2 \\ -s_4 & -s_3 & s_2 & s_1 \\ s_1^* & s_2^* & s_3^* & s_4^* \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & s_4^* & s_1^* & -s_2^* \\ -s_4^* & -s_3^* & s_2^* & s_1^* \end{bmatrix}$$

- It is not always possible to find full-rate, full-diversity STBC using (square) orthogonal designs.
- For real constellation (PAM), full-rate, full-diversity designs exist for N = 2, 4, 8
- For complex constellation (e.g. QAM, 8-PSK…), full-rate designs exist if and only if N = 2 (Alamouti scheme)

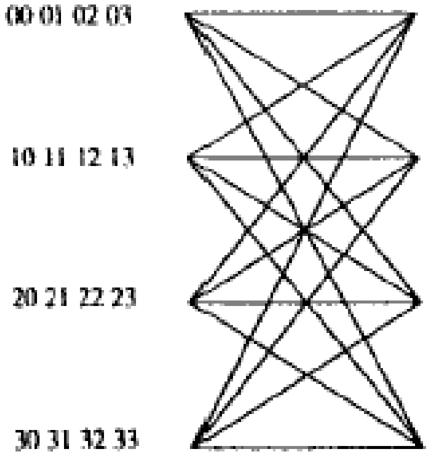
Space-Time Trellis Codes (STTC)

- Trellis codes applied to MIMO
- A basic trellis structure determining the coded symbols to be transmitted from different antennas
- Memorized from one block to another
- Coding advantage vs. Decoding complexity

Space-Time Trellis Codes (STTC) design: 2b/s/Hz, with 4PSK and N=2:

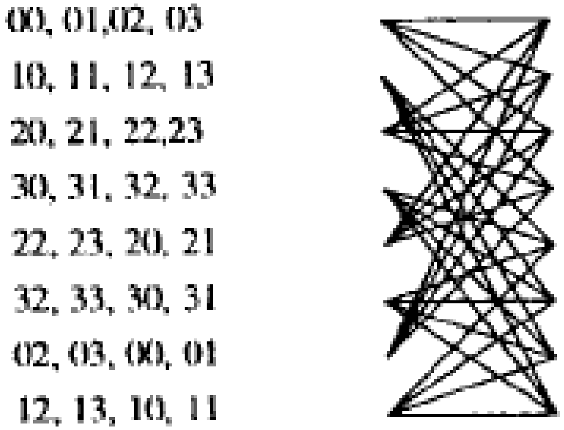


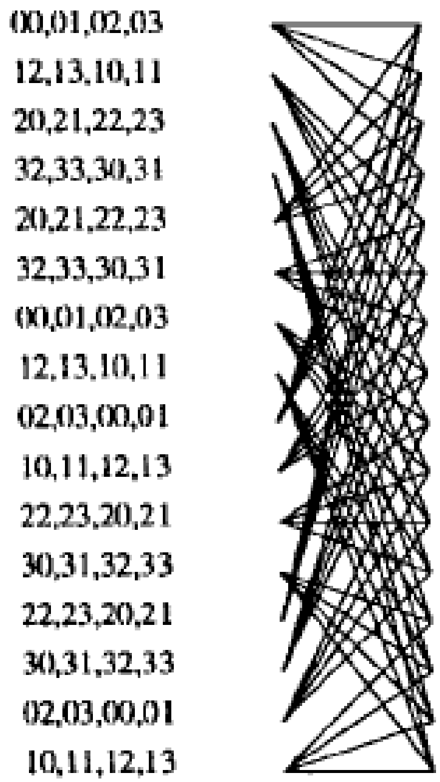
INPUT SIGNAL:	0	1	2	3
OUTPUT SIGNALS:	$S_{0a}S_{0b}$	$S_{1a}S_{1b}$	$S_{2a}S_{2b}$	$S_{3a}S_{3b}$



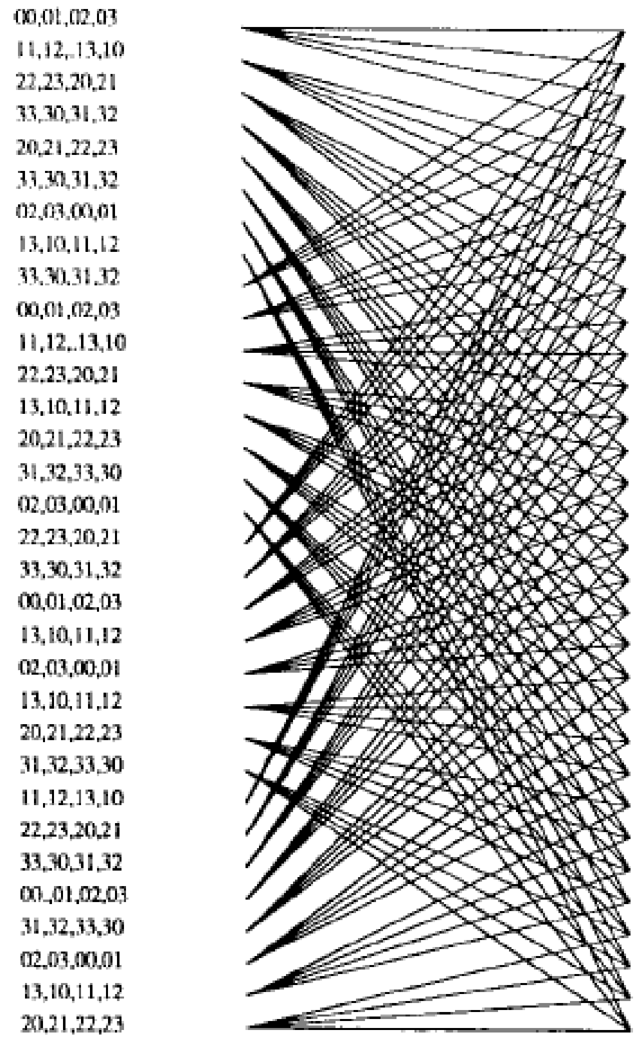
4-STATE

8-STATE





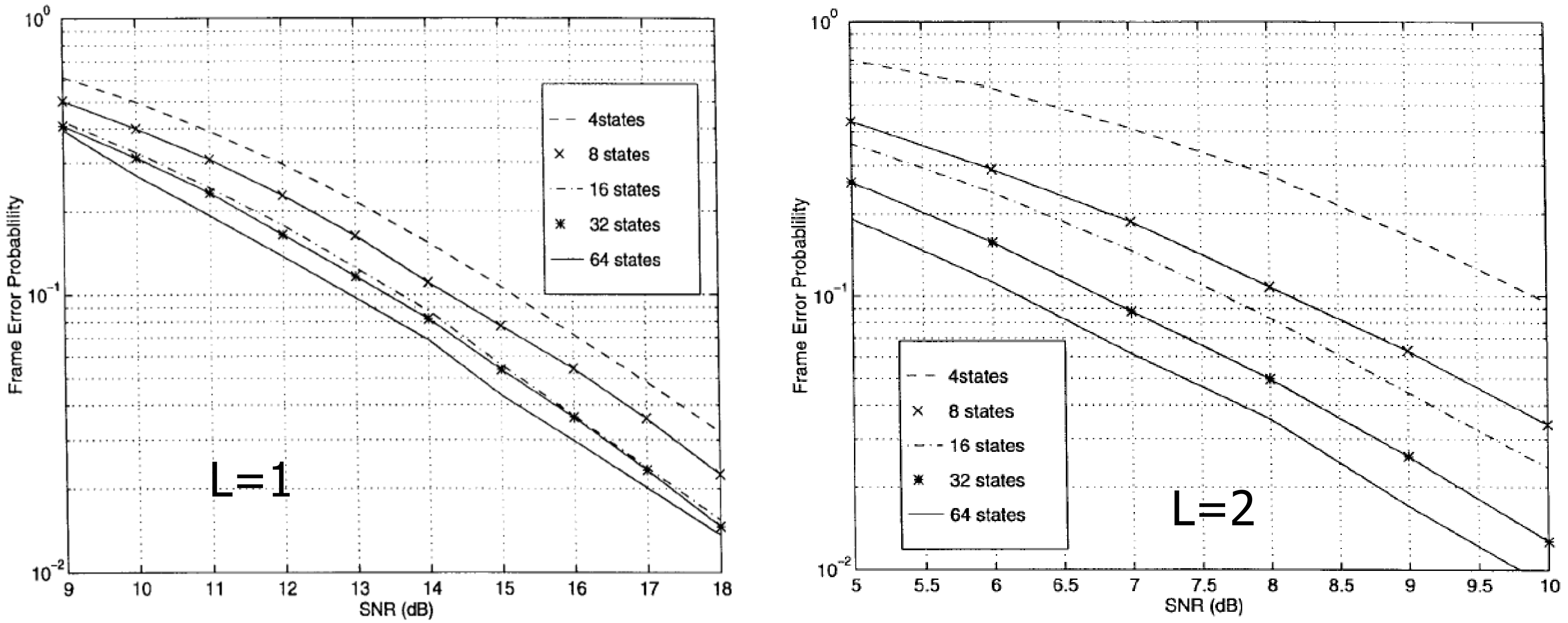
16-STATE



32-STATE

V. Tarokh, N. Seshadri, A. R. Calderbank, "Space–Time Codes for High Data Rate Wireless Communication: Performance Criterion and Code Construction", *IEEE Transactions on Information Theory*, Vol. 44, No. 2, March 1998, pp. 744-765

STTC 2b/s/Hz, with 4PSK and N=2: performance



- Performance: simulation results
- Frame of 130 transmissions out of each transmit antenna

V. Tarokh, N. Seshadri, A. R. Calderbank, "Space–Time Codes for High Data Rate Wireless Communication: Performance Criterion and Code Construction", *IEEE Transactions on Information Theory*, Vol. 44, No. 2, March 1998, pp. 744-765

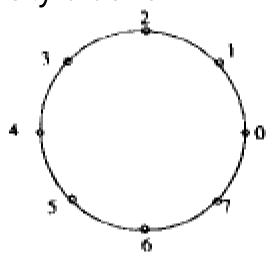
DELAY DIVERSITY CODING: L=1, N=2

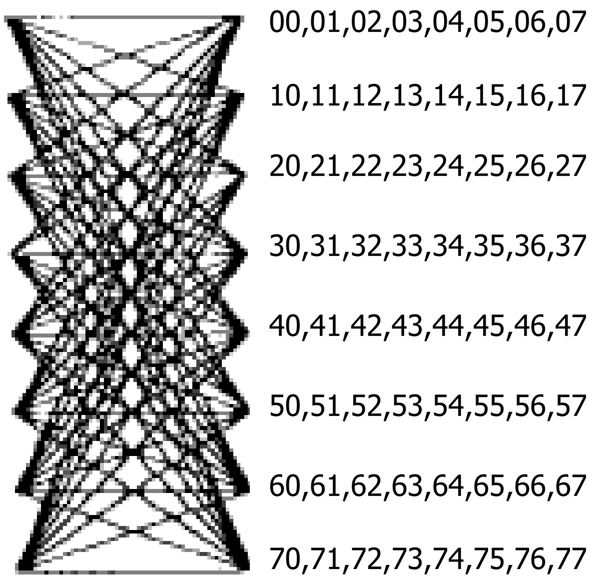
Transmitting the symbol stream over 1st antenna and its one-symbol delayed version over the 2nd antenna.

$$\mathbf{S} = \begin{bmatrix} s_1 & s_2 & s_3 & \dots & s_J & 0 \\ 0 & s_1 & s_2 & s_3 & \dots & s_J \end{bmatrix}$$

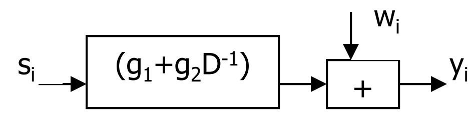
[S-S'] has rank 2 $\Rightarrow$ diversity order of 2L=2

STTC representation: e.g., 8PSK

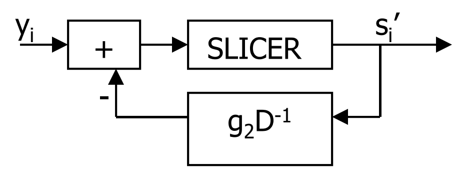




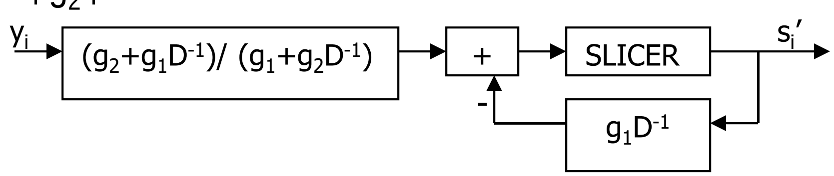
- Rx: y=GS+w, G=[g_1,g_2], ML decoding: find S for min $|y\text{-}GS|^2$: complex
- $y=[y_1,y_2,\dots]$, $y_1=g_1s_1+w_1$, $y_{i+1}=g_2s_i+g_1s_{i+1}+w_{i+1}$
- Channel model:



- ZF-DFE (Zero-Forcing Decision-Feedback Equalizer) using Symbol ML decoding and iterative canceling:
  - select s_1 for min $|y_1\text{-}g_1s_1|^2$: SLICER (symbol MLD)
  - using previously detected s_i' to compute $z_{i+1}=y_{i+1}\text{-}g_2s_i$ (DFE)
  - and find s_{i+1} for min $|z_{i+1}\text{-}g_1s_i|^2$ (SLICER)
- If $|g_1|>|g_2|$: minimum-phase channel: good. Otherwise, error propagation, use all-pass filter $(g_2+g_1D^{-1})/(g_1+g_2D^{-1})$
- Case: $|g_1|>|g_2|$



- Case: $|g_1|<|g_2|$



ST coding for spatial multiplexing

- objective of spatial multiplexing: to maximize transmission rate.
- MIMO channels offer a linear increase in capacity for no additional power or bandwidth expenditure, Rate (SNR) = r log SNR.
- The gain, r, referred to as spatial multiplexing gain (spatial rate), is realized by transmitting independent data signals from the individual antennas, r≤N.
- Under conducive channel conditions, such as rich scattering, the receiver can separate the different streams, yielding a linear increase in capacity.
- Rx performs ML decoding of each Rx signal (codeword)

$$\Rightarrow \text{only one } \lambda(\mathbf{S},\mathbf{S}')\text{: non-zero eigenvalue of } \left(E_s/N\right)^{-1}\left[\mathbf{S}-\mathbf{S}'\right]\left[\mathbf{S}-\mathbf{S}'\right]^H \text{ for i=}I(\mathbf{S},\mathbf{S}')=1$$

$$\Rightarrow \text{ for high } \left(E_s/N_o\right)>>1,\ \Pr\{\mathbf{S}\to\mathbf{S}'\}\le\left(\lambda(\mathbf{S},\mathbf{S}')\right)^{-L}\left[\left(E_s/N_o\right)/\left(4N\right)\right]^{-L}=\left[\left|\mathbf{S}-\mathbf{S}'\right|^2/4N_o\right]^{-L}$$

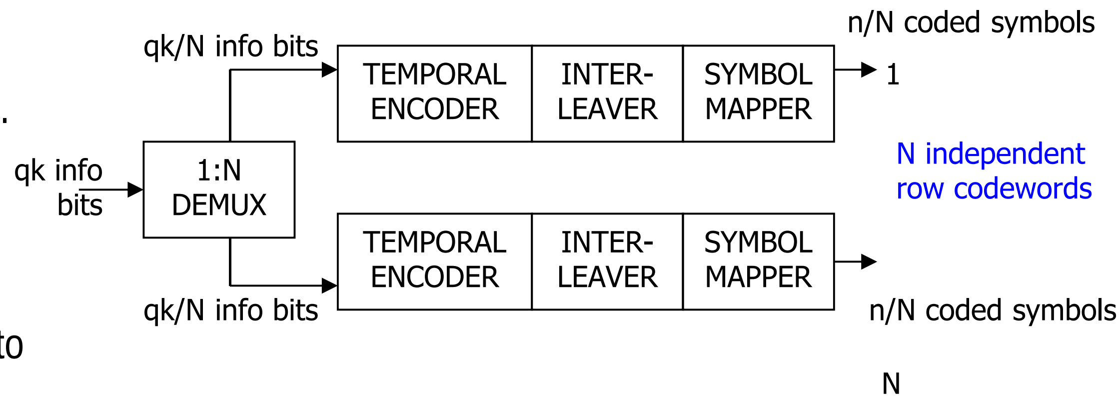
- ⇒ J=1, Nx1 codeword difference matrix **S-S′**, [**S-S′**] [**S-S′**]^H: rank 1
- ⇒ no coding gain, diversity order of L

Vertical Bell Labs Layered Space-Time (V-BLAST) Coding

- **Encoding** with symbol set

$A=\{a_i, i=1,2,\ldots, 2^q\}$ and block of qk info bits.

- Transmitter: Split data into $M_T=N$ streams
- Rx: Y=GS+W
- ML decoding: select S corresponding to the min | Y-GS $|^2$: very complex



- If G can be GS-decomposed, G=QB, then $Z=Q^HY=BS+Q^HW$
- Gram-Schmidt (GS) decomposition: An mxn matrix A with rank n can be factored uniquely as A =QB where Q is an mxn matrix whose columns are unit length and orthogonal and span the range of A, and $Q^HQ=I$.

$$\mathbf{A}=[\mathbf{a}_1,\mathbf{a}_2,\cdots,\mathbf{a}_n]=\mathbf{QB}$$

$$=[\mathbf{q}_1,\mathbf{q}_2,\cdots,\mathbf{q}_n]\begin{bmatrix} b_{11} & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{(n-1)1} & b_{(n-1)2} & \vdots & b_{(n-1)(n-1)} & 0 \\ b_{n1} & b_{n2} & \cdots & \cdots & b_{nn} \end{bmatrix}$$

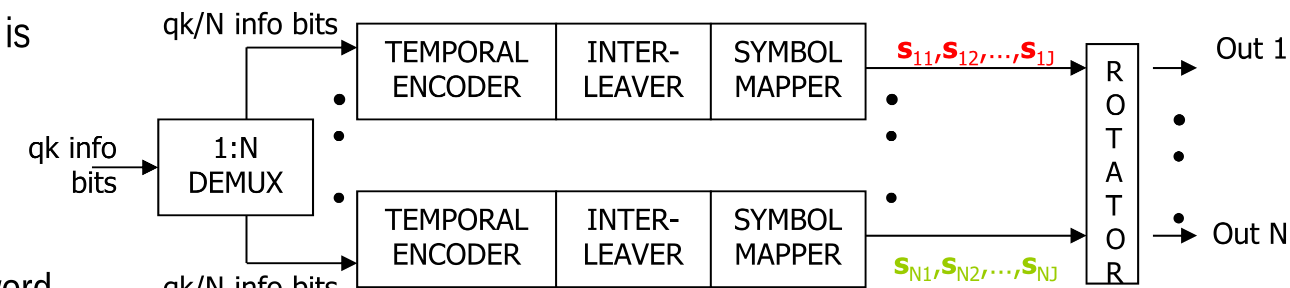
- Layered decoding:
- 1^{st} row: $z_1=b_{11}s_1+n_1$ can be independently decoded for s_1'
- 2^{nd} row: removing s_1 (using the previously decoded s_1') from z_2
- $\Rightarrow z_2-b_{21}s_1=b_{22}s_2+n_2$ can be decoded for s_2' , …
- i^{th} row: removing $s_1,s_2,\ldots,s_{i-1}$ (using their previously decoded values $s_1',s_2',\ldots,s_{i-1}'$) from z_i
- $\Rightarrow z_i-(b_{i1}s_1+ b_{i2}s_2+\ldots+b_{i(i-1)}s_{i-1})=b_{ii}s_i+n_i$ can be decoded for s_i'
- A small value of b_{ii} could yield unreliable decoded s_i', which will create a chain effect in decoding the subsequent rows.

diagonal D-BLAST

- G. Foschini, "Layered space-time architecture for wireless communication in a fading environment when using multi-element antennas",*Bell Labs Tech. J., pp.41-59, 1996*

- In V-BLAST, there is no coding across these sub-channels: outage therefore occurs whenever one of these sub-channels is in a deep fade and cannot support the rate of the stream using that sub-channel.
- D-BLAST: achievable diversity order of ML for optimum temporal coding and rotation with infinite block-size Gaussian code books,
- coding gain from temporal code; achievable array gain of L

- The coded stream from Tx#n is subdivided into J blocks of several symbols s_{nj}.



- The rotator fills the ST codeword from N coded streams in a block-diagonal fashion as shown in the following example with J=8, N=4.
- ROTATOR OUTPUTS with N=4:

Out 1:	s_{11}	s_{21}	s_{31}	s_{41}	s_{15}	s_{25}	s_{35}	s_{45}	0	0	0
Out 2:	0	s_{12}	s_{22}	s_{32}	s_{42}	s_{16}	s_{26}	s_{36}	s_{46}	0	0
Out 3:	0	0	s_{13}	s_{23}	s_{33}	s_{43}	s_{17}	s_{27}	s_{37}	s_{47}	0
Out 4:	0	0	0	s_{14}	s_{24}	s_{34}	s_{44}	s_{18}	s_{28}	s_{38}	s_{48}

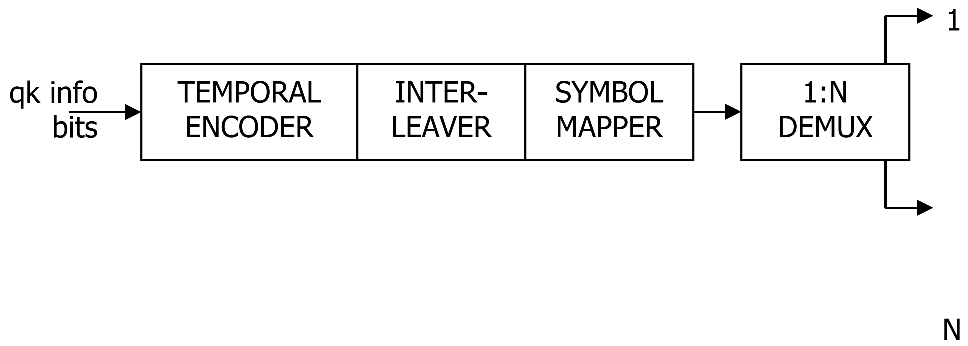
Rx: Diagonal-Layer decoding: 2 ways:

- column by column tentative decisions and cancellation, and then decoding the complete stream, or
- (more complex, better performance) detecting the diagonally layered stream, and applying cancellation of previously detected streams

- By coding across the sub-channels, D-BLAST can average over the randomness of the individual sub-channels and get better outage performance. The D-BLAST scheme suffers from a rate loss because in the initialization phase some of the antennas have to be kept silent.

horizontal encoding

- achievable diversity order >L ,
- coding gain from temporal code; achievable array gain of L
- can reach optimality since each info bit can potentially be spread across all antennas, but requires complex joint decoding of the sub-streams



STC for frequency-selective fading:

$g_{ln}(t)$: equivalent baseband impulse response of the channel from Tx antenna n to Rx antenna l: $g_{ln}(t)=0$ for $t<0$ and $t>DT_s$

$x_n(t)$: signal transmitted from Tx antenna n,

signal received at Rx antenna l: $y_l(t)=\sum_{n=1}^{N}\int_0^{DT_s} g_{ln}(\tau)x_n(t-\tau)d\tau + w_l(t), \quad l=1,2,\ldots,L$

Discrete-time representation: $y_l(t)=\sum_{n=1}^{N}\mathbf{g}_{ln}(i)\mathbf{x}_n(t-i)+w_l(t), \quad l=1,2,\ldots,L$

$column-vector\ \mathbf{x}_n(t)=\left[x_n(t-D), x_n(t-[D-1]),\cdots,x_n(t)\right]^T$

$\mathbf{g}_{ln}=\left[g_{ln}(D), g_{ln}(D-1),\cdots,g_{ln}(0)\right]$: symbol-spaced samples of the equivalent baseband impulse response of the channel from Tx antenna n to Rx antenna l: assumed to be zero-mean complex circular Gaussian.

At time t, for a block of T contiguous symbol intervals

$$\mathbf{Y}=\left[\mathbf{y}(t),\mathbf{y}(t+1),\cdots,\mathbf{y}(t+T-1)\right]=\mathbf{GS}+\mathbf{W}, \quad \mathbf{G}=\begin{pmatrix}\mathbf{g}_{11} & \cdots & \mathbf{g}_{1N}\\ \vdots & \ddots & \vdots\\ \mathbf{g}_{L1} & \cdots & \mathbf{g}_{LN}\end{pmatrix}, \quad \mathbf{S}=\begin{bmatrix}\mathbf{S}_1\\ \vdots\\ \mathbf{S}_N\end{bmatrix}$$

$$\mathbf{S}_n=\begin{bmatrix} x_n(t-D) & x_n(t-D+1) & \cdots & x_n(t-D+T-1)\\ \vdots & \vdots & \cdots & \vdots\\ x_n(t-1) & x_n(t) & \cdots & x_n(t+T-2)\\ x_n(t) & x_n(t+1) & \cdots & x_n(t+T-1)\end{bmatrix}$$: signal block from Tx antenna n

PEP performance:

Rx uses ML decoding with the PEP:

$$\Pr\left\{\bar{\mathbf{S}} \to \bar{\mathbf{S}}' \middle| \bar{\mathbf{G}}\right\} = Q\left(\sqrt{\left|\bar{\mathbf{G}}\left[\bar{\mathbf{S}} - \bar{\mathbf{S}}'\right]\right|^2 / 2N_o}\right) \le e^{\left(-\left|\bar{\mathbf{G}}\left[\bar{\mathbf{S}}-\bar{\mathbf{S}}'\right]\right|^2 / 4N_o\right)}, \text{ for spatially and temporally white channel}$$

$$\Rightarrow \Pr\left\{\bar{\mathbf{S}} \to \bar{\mathbf{S}}'\right\} \le \left(\det\left[I_{NL} + \left[\bar{\mathbf{S}} - \bar{\mathbf{S}}'\right]\left[\bar{\mathbf{S}} - \bar{\mathbf{S}}'\right]^H / 4N_o\right]\right)^{-L} = \left(\prod_{i=1}^{I(\bar{\mathbf{S}},\bar{\mathbf{S}}')}\left[1 + \left(\lambda_i(\bar{\mathbf{S}},\bar{\mathbf{S}}')/4N\right)\left(E_s/N_o\right)\right]\right)^{-L}$$

where $\lambda_i(\bar{\mathbf{S}},\bar{\mathbf{S}}')$ is the i^{th} non-zero eigenvalue of $\left(E_s/N\right)^{-1}\left[\bar{\mathbf{S}} - \bar{\mathbf{S}}'\right]\left[\bar{\mathbf{S}} - \bar{\mathbf{S}}'\right]^H$ for i=1,2,...,$I(\bar{\mathbf{S}},\bar{\mathbf{S}}')$

$$\Rightarrow \text{ for high } \left(E_s/N_o\right) >> 1,\ \Pr\left\{\bar{\mathbf{S}} \to \bar{\mathbf{S}}'\right\} \le \left(\prod_{i=1}^{I(\bar{\mathbf{S}},\bar{\mathbf{S}}')}\lambda_i(\bar{\mathbf{S}},\bar{\mathbf{S}}')\right)^{-L}\left[\left(E_s/N_o\right)/\left(4N\right)\right]^{-LI(\bar{\mathbf{S}},\bar{\mathbf{S}}')}$$

$$\left[\bar{\mathbf{S}} - \bar{\mathbf{S}}'\right] : N(D+1)\text{x}T \text{ matrix} \Rightarrow I(\bar{\mathbf{S}},\bar{\mathbf{S}}') \le N(D+1) \Rightarrow \text{ achievable diversity order of } NL(D+1)$$

Delay diversity codes for frequency-selective fading:

standard delay diversity code: Transmitting the symbol stream over 1st antenna, its 1-symbol delayed version over the 2nd antenna, and its 2-symbol delayed version over the 3rd antenna, and so on.

e.g., N=2, L=1, D=1, T=4

antenna 1: $x_1(t), x_1(t+1), x_1(t+2), x_1(t+3), x_1(t+4), 0 = s_1, s_2, s_3, s_4, 0, 0$

antenna 2: $x_2(t+d) = x_1(t+d-1) \rightarrow x_2(t), x_2(t+1), x_2(t+2), x_2(t+3), x_2(t+4), 0 = 0, s_1, s_2, s_3, s_4, 0$

Note: additional 0 at the end for guard of D=1

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} 0 & s_1 & s_2 & s_3 & s_4 & 0 \\ s_1 & s_2 & s_3 & s_4 & 0 & 0 \\ 0 & 0 & s_1 & s_2 & s_3 & s_4 \\ 0 & s_1 & s_2 & s_3 & s_4 & 0 \end{bmatrix} \text{: row 1= row 4, rank of 3 (low)}$$

generalized delay diversity code: Transmitting the symbol stream over 1st antenna, its (D+1)-symbol delayed version over the 2nd antenna, and its 2(D+1)-symbol delayed version over the 3rd antenna, and so on.

e.g., N=2, L=1, D=1

antenna 1: $s_1(t), s_1(t+1), s_1(t+2), s_1(t+3), s_1(t+4), s_1(t+5), 0 = s_1, s_2, s_3, s_4, 0, 0, 0$

antenna 2: $s_2(t), s_2(t+1), s_2(t+2), s_2(t+3), s_2(t+4), s_1(t+5), 0 = 0, 0, s_1, s_2, s_3, s_4, 0$

Note: additional 0 at the end for guard of $D=1$

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{bmatrix} = \begin{bmatrix} 0 & s_1 & s_2 & s_3 & s_4 & 0 & 0 \\ s_1 & s_2 & s_3 & s_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 & s_2 & s_3 & s_4 \\ 0 & 0 & s_1 & s_2 & s_3 & s_4 & 0 \end{bmatrix} \text{: full rank of 4=}(D+1)LN$$

MIMO-OFDM:SFBC

MIMO-OFDM: N Tx, L Rx, with F tones

For tone f, f=1,2,…,F

In frequency domain, $\mathbf{y}(f)=\mathbf{H}(f)\mathbf{s}(f)+\mathbf{v}(f)$

$$\Rightarrow LF\text{x}1\ \ \mathbf{y}=\begin{bmatrix}\mathbf{y}(1)\\ \mathbf{y}(2)\\ \vdots\\ \mathbf{y}(F)\end{bmatrix}=\mathbf{H}\mathbf{s}+\mathbf{w},\ \ NF\text{x}1\ \ \mathbf{s}=\begin{bmatrix}\mathbf{s}(1)\\ \mathbf{s}(2)\\ \vdots\\ \mathbf{s}(F)\end{bmatrix}\ \ LF\text{x}1\ \ \text{ZMCSCG}\ \ \mathbf{w}=\begin{bmatrix}\mathbf{w}(1)\\ \mathbf{w}(2)\\ \vdots\\ \mathbf{w}(F)\end{bmatrix}\ \ LF\text{x}NF\ \ \mathbf{H}=\begin{pmatrix}\mathbf{H}(1) & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \mathbf{H}(F)\end{pmatrix}$$

spatial diversity coding: e.g., Alamouti code with tone index, N=2, L=1

Tx: $\begin{bmatrix}s(1,f) & s(1,f+1)\\ s(2,f) & s(2,f+1)\end{bmatrix}=\begin{bmatrix}s_1 & -s_2^*\\ s_2 & s_1^*\end{bmatrix}\rightarrow\mathbf{S}(f)=\begin{bmatrix}s_1\\ s_2\end{bmatrix},\mathbf{S}(f+1)=\begin{bmatrix}-s_2^*\\ s_1^*\end{bmatrix}$

then IFFT+CPI for 2 Tx

Rx: removes CP and FFT, and processes

$$\mathbf{y}_f=\begin{bmatrix}y(f)\\ y^*(f+1)\end{bmatrix}=\begin{bmatrix}\mathbf{H}(f)\mathbf{s}(f)+w(f)\\ \mathbf{H}^*(f+1)\mathbf{s}^*(f+1)+w^*(f+1)\end{bmatrix}=\begin{bmatrix}H_1(f) & H_2(f)\\ H_2^*(f+1) & -H_1^*(f+1)\end{bmatrix}\begin{bmatrix}s_1\\ s_2\end{bmatrix}+\begin{bmatrix}w(f)\\ w^*(f+1)\end{bmatrix}$$

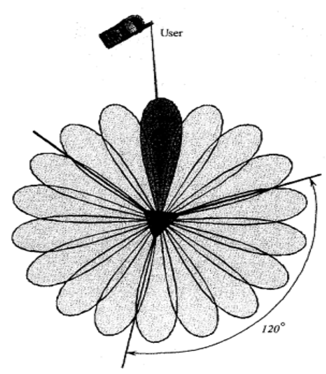
$$\Rightarrow\begin{bmatrix}H_1(f) & H_2(f)\\ H_2^*(f+1) & -H_1^*(f+1)\end{bmatrix}^H\mathbf{y}_f=|\mathbf{H}|^2(f)\begin{bmatrix}s_1\\ s_2\end{bmatrix}+\begin{bmatrix}H_1^*w(f)+H_2w^*(f+1)\\ H_2^*w(f)-H_1w^*(f+1)\end{bmatrix}\ \ \text{if}\ \mathbf{H}(f)=\mathbf{H}(f+1)$$

the noise components are iid ZMCSCG with variance of $|\mathbf{H}|^2(f)N_o$

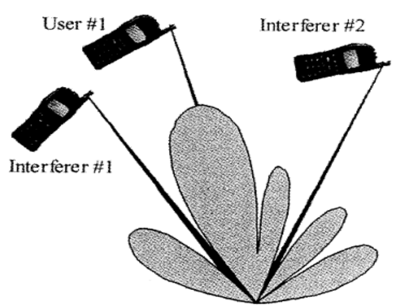
Beamforming (BF)

- BF techniques are used to create a certain required antenna directive pattern to give the required performance under the given conditions, e.g., using phased-array systems.
- Switched BF requires the overall system to determine the direction of arrival of the incoming signal and then switch in the most appropriate beam:
  - Finite number of fixed predefined patterns
  - the fixed beam is unlikely to exactly match the required direction.
- Adaptive BF:
  - large number of patterns adjusted to the scenario in real time.
  - are able to direct the beam in the exact direction needed, and also move the beam in real time - this is a particular advantage for moving systems - mobile telecommunications. The cost is the considerable extra complexity

Switched Beamforming

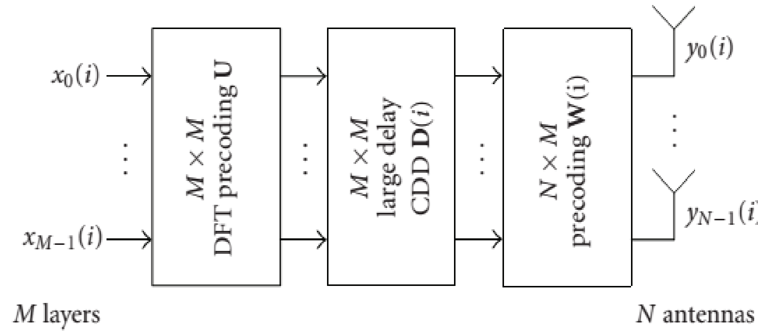


Adaptive Beamforming

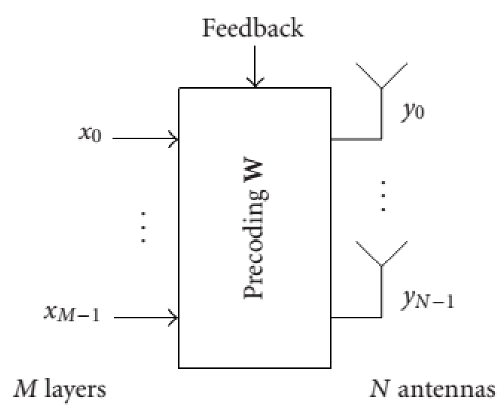


Open-loop MIMO vs. Closed-loop MIMO

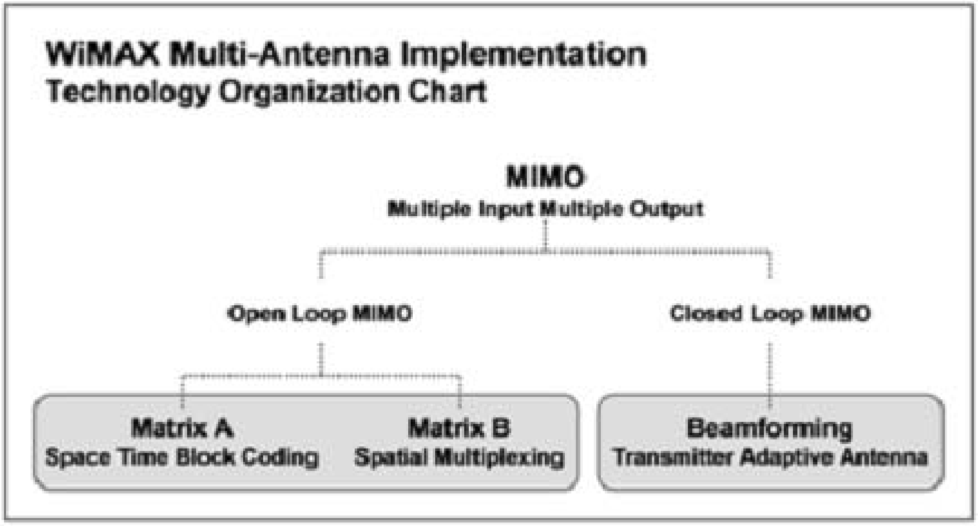
- Open-loop systems do not require knowledge of the channel at the transmitter. Open loop operations occur when the access network does not have information or feedback from the user to do coding adjustment.

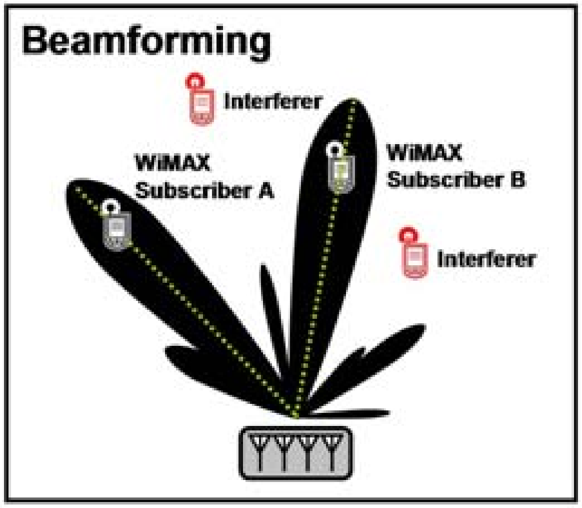


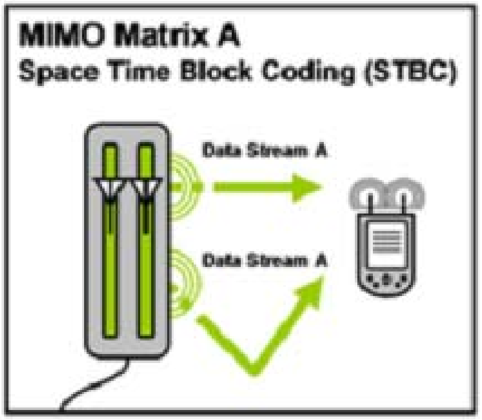
- Closed-loop systems require channel knowledge at the transmitter, thus necessitating either channel reciprocity—same uplink and downlink channel, possible in TDD—or more commonly a feedback channel from the receiver to the transmitter.

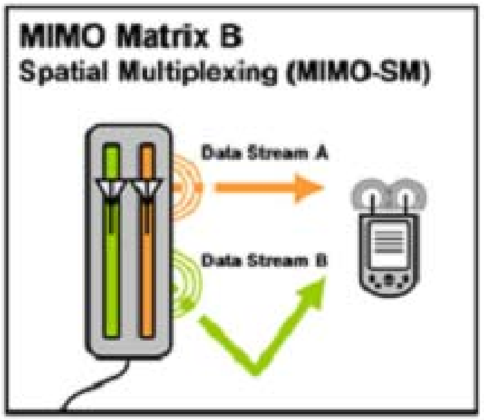


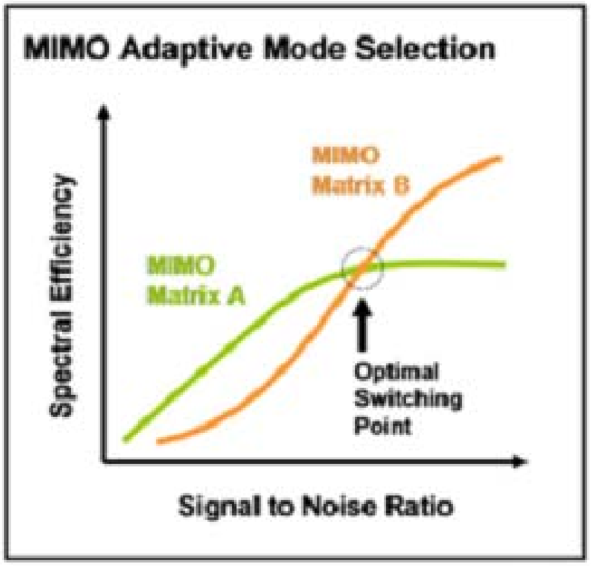
MIMO in WiMAX





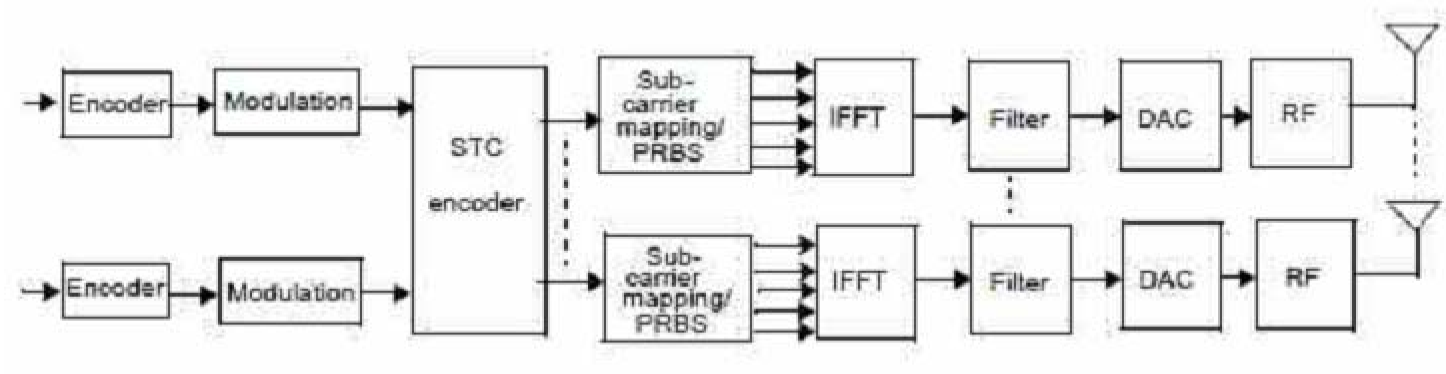






MIMO in WiMAX (cont.)

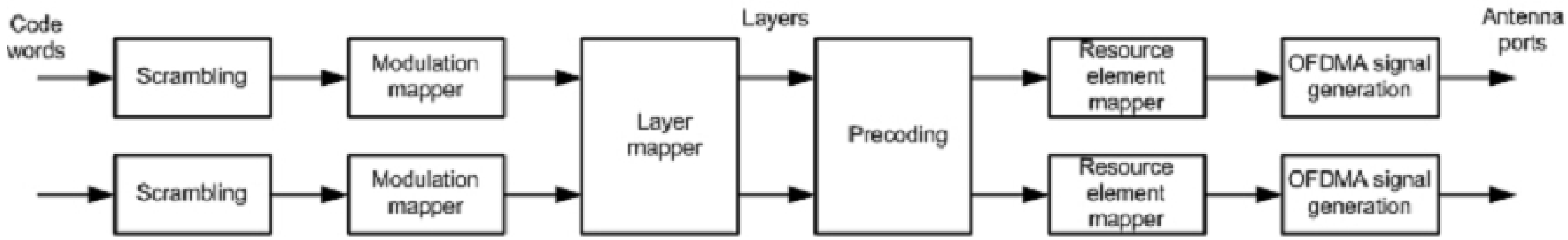
- WiMAX promises a peak data rate of 74 Mbps at a bandwidth of up to 20 MHz.
- Modulation types are QPSK, 16QAM, and 64QAM.
- Downlink:
  - The WiMAX 802.16e-2005 standard specifies MIMO in wireless MAN-OFDMA mode.
  - This standard defines a large number of different matrices for coding and distributing to antennas. In principle, two, three or four TX antennas are possible.



$$A = \begin{bmatrix} S_1 & -S^*_2 \\ S_2 & S^*_1 \end{bmatrix} \qquad B = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

MIMO in LTE

- Downlink
  - Single antenna transmission, no MIMO
  - Transmit diversity
  - Open-loop spatial multiplexing, no UE feedback required
  - Closed-loop spatial multiplexing, UE feedback required
  - Multi-user MIMO (more than one user is assigned to the same resource block)
  - N Beamforming

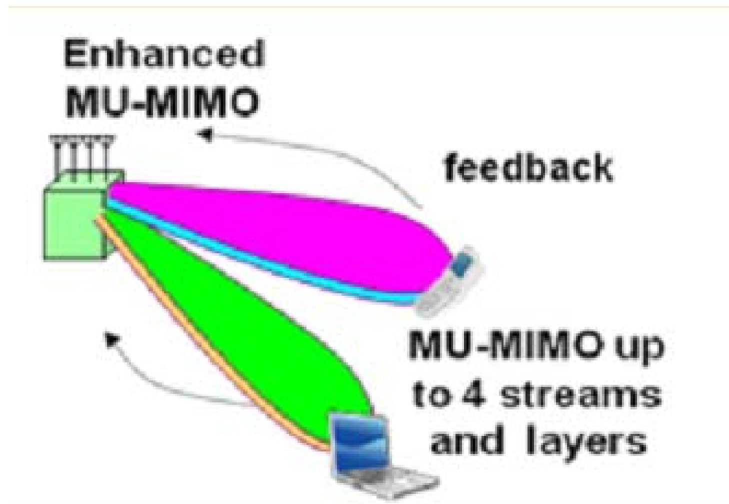


- Uplink
  - In order to keep the complexity low at the UE end, MU-MIMO is used in the uplink. To do this, multiple UEs, each with only one Tx antenna, use the same channel.

SU-MIMO vs. MU-MIMO

- Single - user MIMO
  - The data rate is to increased for a single user
  - In the 3GPP release 8 (LTE) standard, allows us to achieve 300Mbps for DL and 75Mbps for UL
  - Using the Spatial Multiplexing to combine the bandwidth of each antennas.

- Multi- user MIMO: the individual streams are assigned to various users.

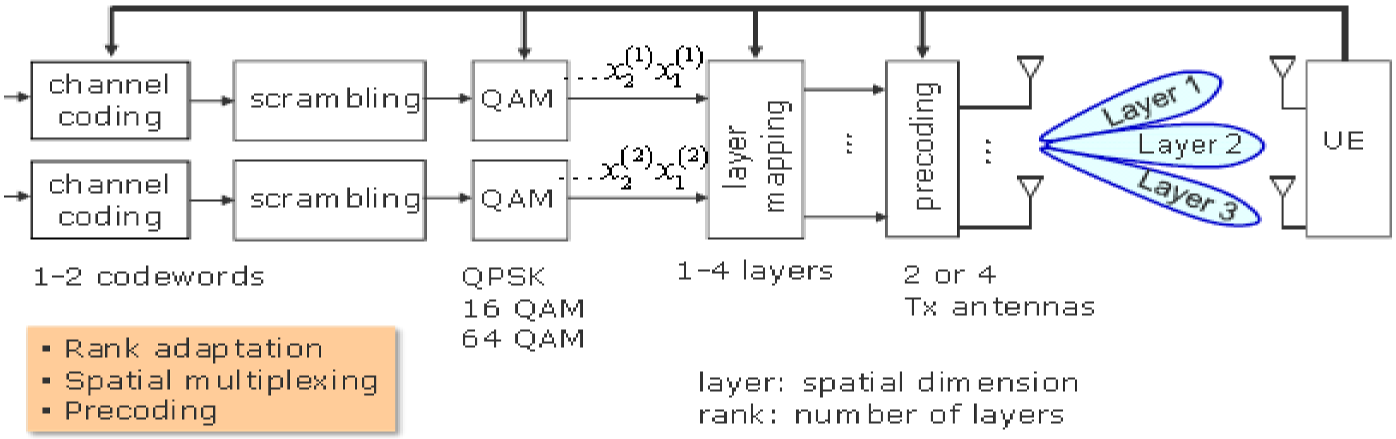


4G Systems: SU-MIMO vs. MU-MIMO

Key downlink MIMO techniques		802.16m	3GPP E-UTRA		
			LTE		LTE-A
			Release 8	Release 9	Release 10
DL	SU-MIMO	Up to 8 streams	Up to 4 streams	Up to 4 streams	Up to 8 streams
	MU-MIMO	Up to 4 users (non-unitary precoding)	Up to 2 users (unitary precoding)	Up to 4 users (non-unitary precoding)*	Under development
UL	SU-MIMO	Up to 4 streams	1 stream	1 stream	Up to 4 streams
	MU-MIMO	Up to 4 users	Up to 8 users	Up to 8 users	Under development

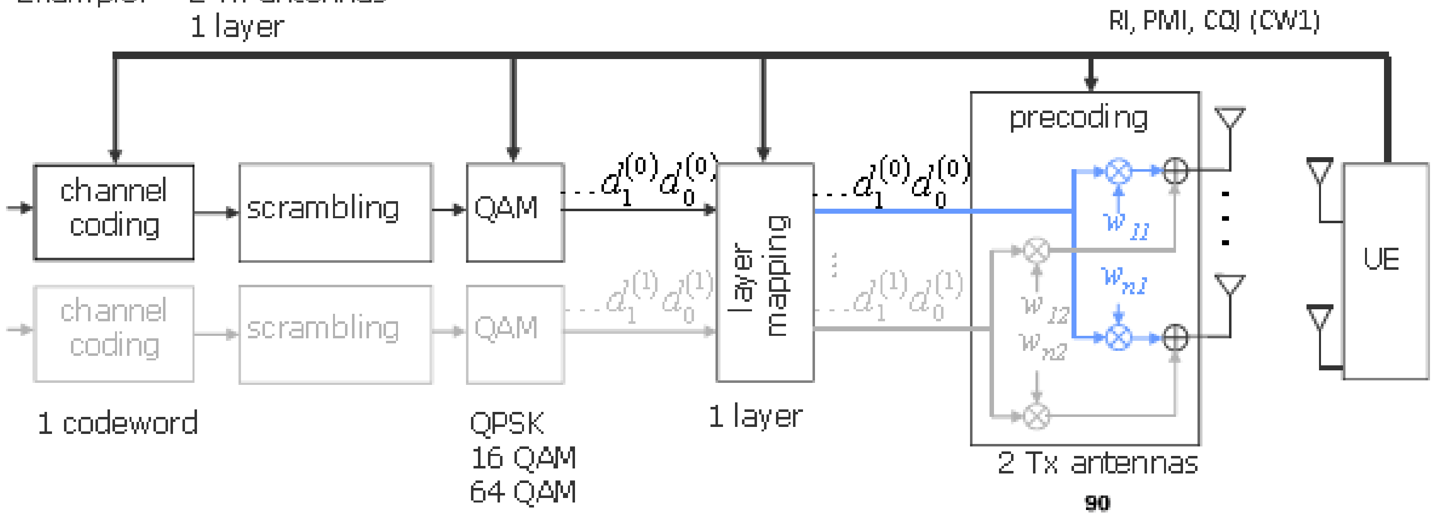
3GPP-LTE: General Structure for Downlink Physical Channels

feedback	2 Tx	4 Tx
rank indicator (RI):	1 bit	2 bit
precoder matrix indicator (PMI)	2 bit	4 bit
channel quality indicator (CQI) (modulation and coding scheme (MCS))	1st codeword: 4 bit 2nd codeword: 3 bit (differential)	



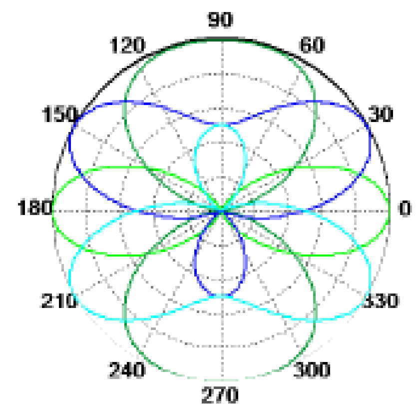
Closed-Loop-MIMO
Precoded Spatial Multiplexing: Rank 1

Example: 2 Tx antennas
1 layer



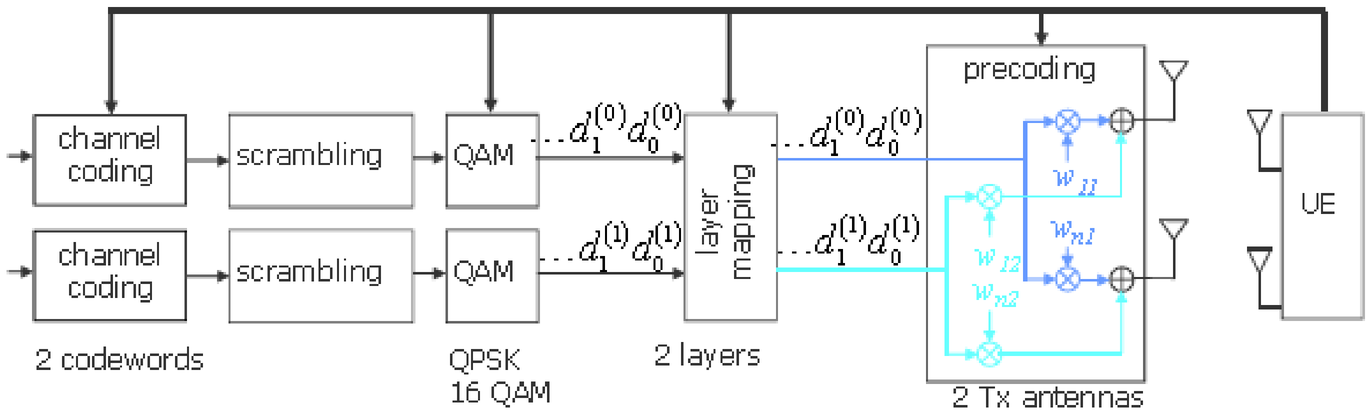
2 Tx antennas:
DFT-based precoder
codebook

index	1 layer	2 layers
0	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$
1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & 1\\j & -j\end{bmatrix}$
2	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$	
3	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	



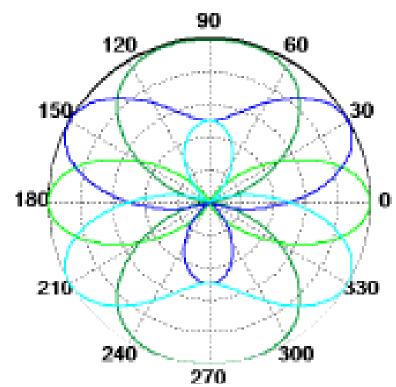
Closed-Loop-MIMO
Precoded Spatial Multiplexing: Rank 2

Example: 2 Tx antennas
2 layers

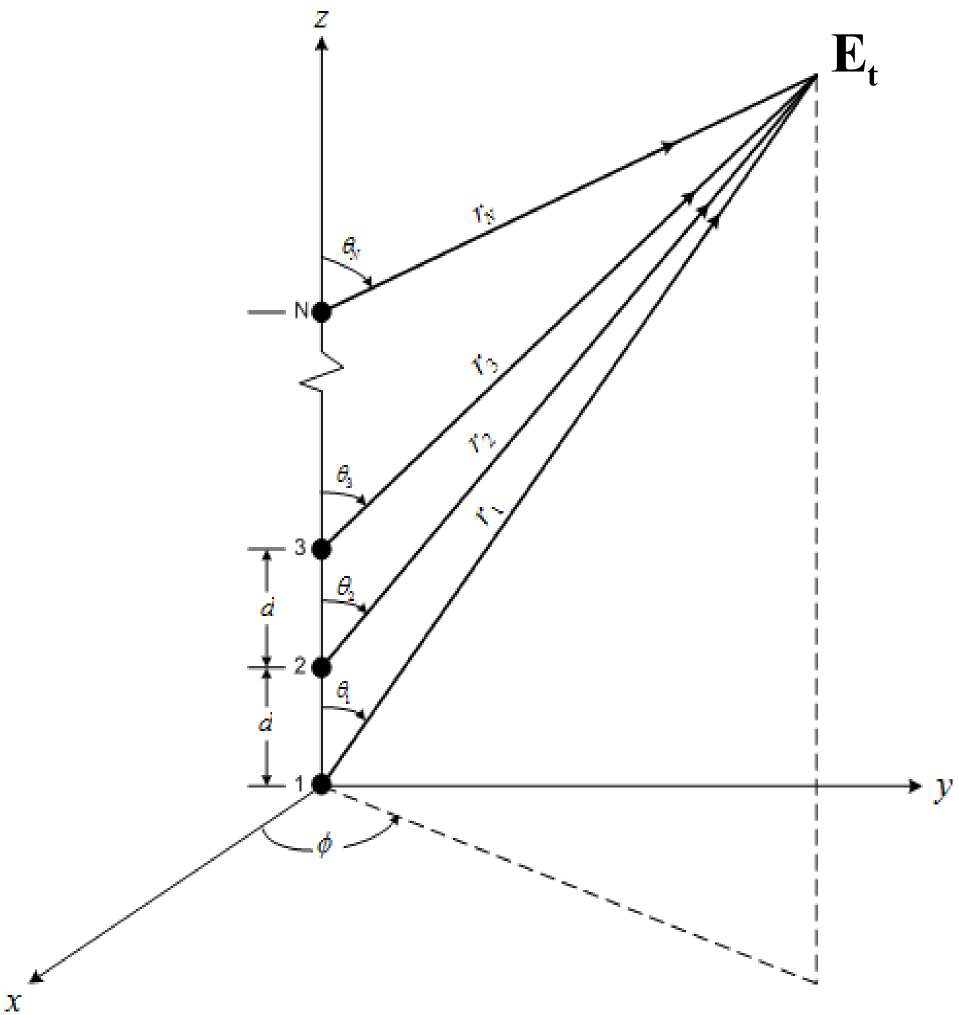


2 Tx antennas:
DFT-based precoder
codebook

index	1 layer	2 layers
0	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$
1	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1 & 1\\j & -j\end{bmatrix}$
2	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$	
3	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$	



Far-field Analysis of Linear Array



$$\mathbf{E_t} = \mathbf{E_1} + \mathbf{E_2} + \mathbf{E_3} + \ldots + \mathbf{E_N}$$

$$\mathbf{E_t} = w_1 \frac{e^{-jkr_1}}{r_1}\mathbf{E_0} + w_2 \frac{e^{-jkr_2}}{r_2}\mathbf{E}_0 + w_3 \frac{e^{-jkr_3}}{r_3}\mathbf{E}_0 + \ldots + w_N \frac{e^{-jkr_N}}{r_N}\mathbf{E}_0$$

In far field region, r_i>> array dimension:

$$\theta_1 \approx \theta_2 \approx \theta_3 \approx \ldots \approx \theta_N \approx \theta$$

$$\left.\begin{aligned} r_1 &= r \\ r_2 &\approx r - d\cos\theta \\ r_3 &\approx r - 2d\cos\theta \\ r_N &\approx r - (N-1)d\cos\theta \end{aligned}\right\}\text{for phase variations}$$

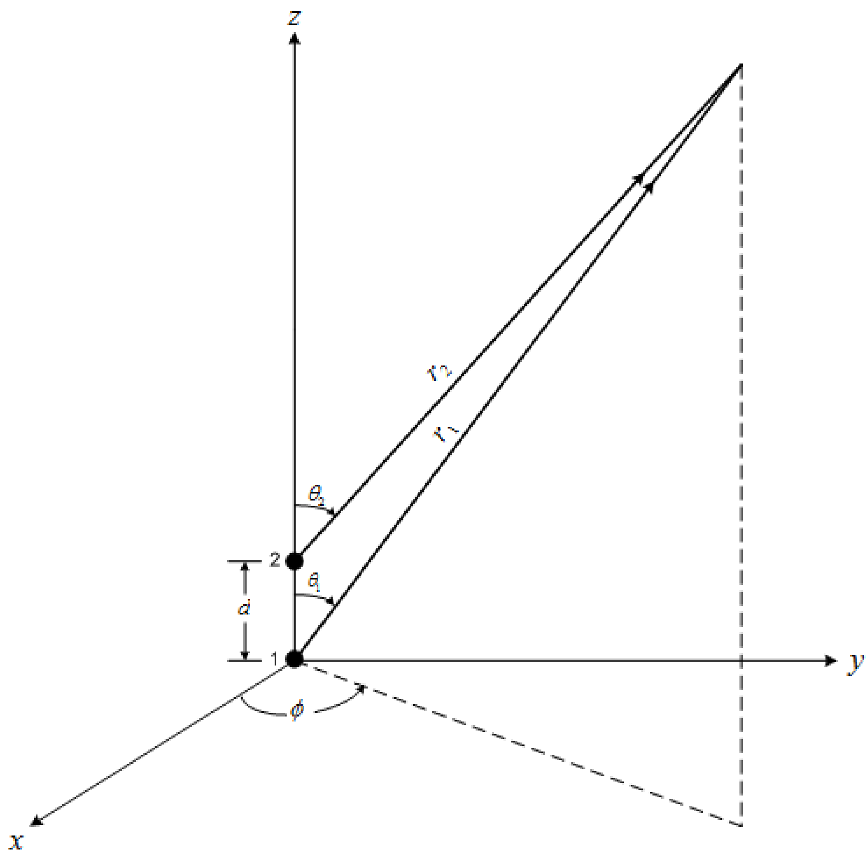
$$r_1 \approx r_2 \approx r_3 \approx \ldots \approx r_N \approx r \quad \text{for amplitude variations}$$

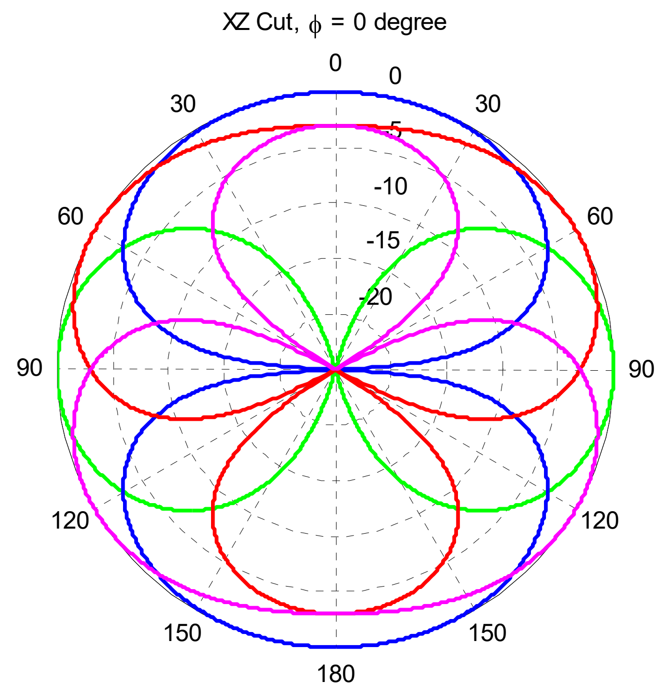
$$\mathbf{E_t} = \underbrace{\frac{e^{-jkr}}{r}\mathbf{E_0}}_{\substack{\mathbf{E}\text{(single element} \\ \text{at reference point)}}} \underbrace{\left(w_1 + w_2 e^{jkd\cos\theta} + w_3 e^{jk2d\cos\theta} + \ldots + w_N e^{jk(N-1)d\cos\theta}\right)}_{\text{Array Factor (AF)}}$$

$$\boxed{\mathbf{E}(\text{total}) = [\mathbf{E}(\text{single element at reference point})] \times [\text{Array Factor}]}$$

Reference: C.A. Balanis, Antenna Theory Analysis and Design, 3rd edition, John Wiley & Son, Inc, 2005

Two-Element Array



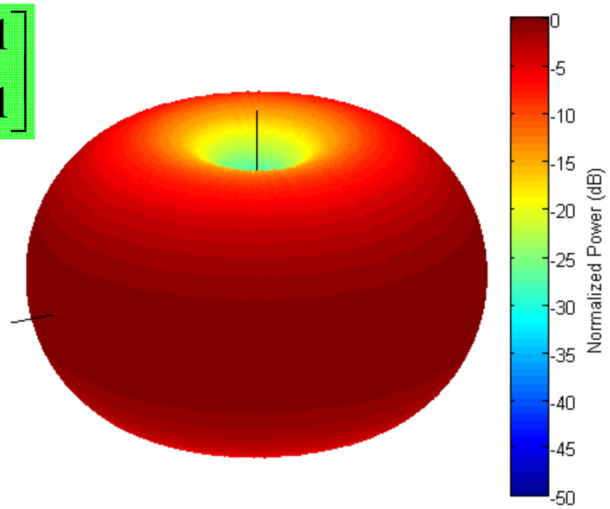


Element spacing : $d = \dfrac{\lambda}{2}$

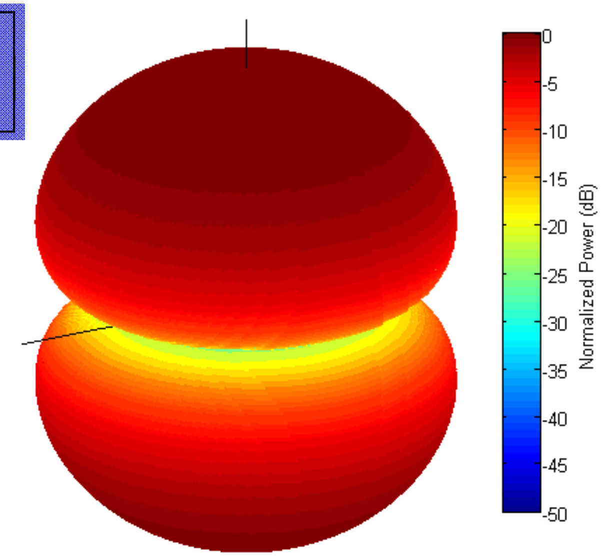
$w = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ $w = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ $w = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$ $w = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$

3D Radiation Patterns of 2 Element Array

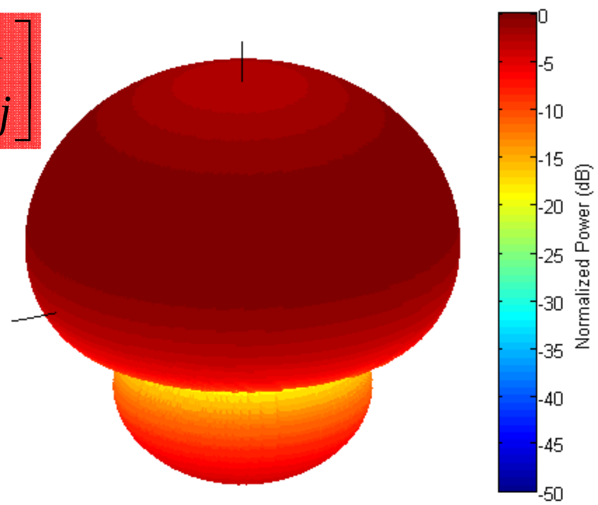
$$w = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$



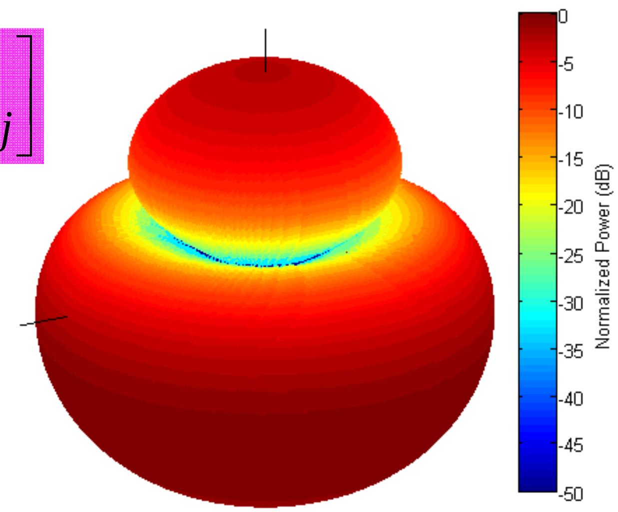
$$w = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$



$$w = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\j\end{bmatrix}$$

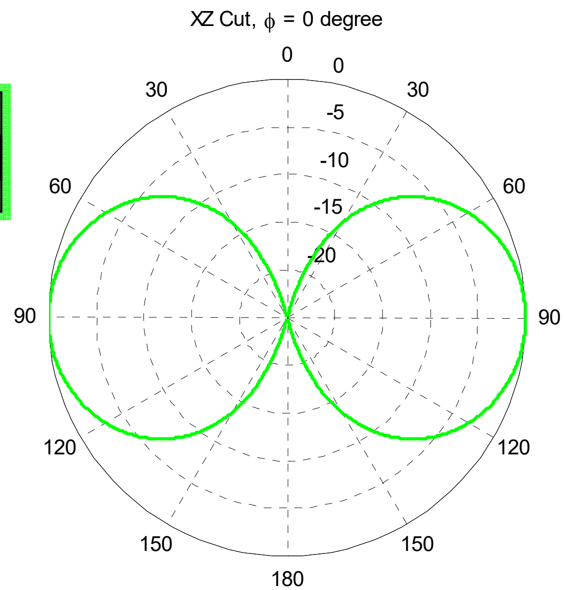


$$w = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-j\end{bmatrix}$$

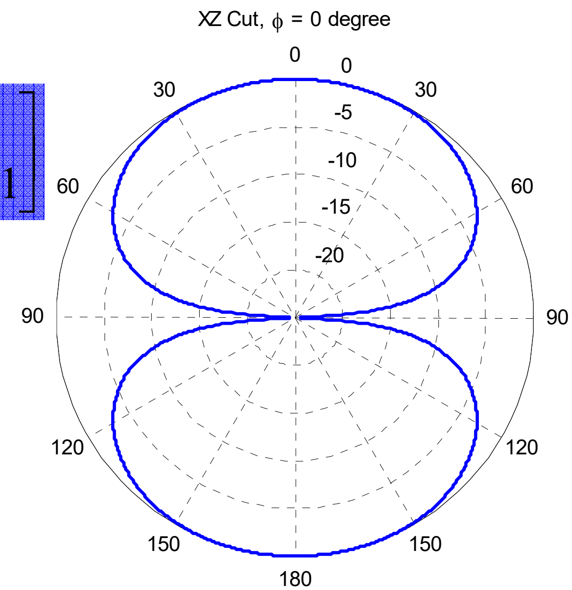


2D Radiation Patterns of 2 Element Array

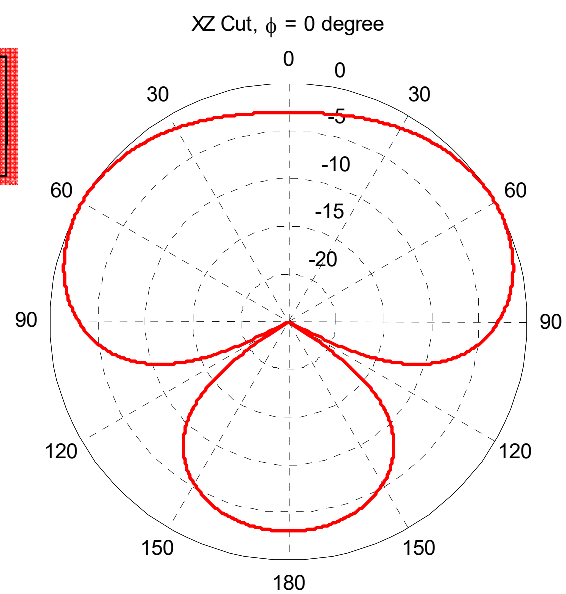
$$w = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



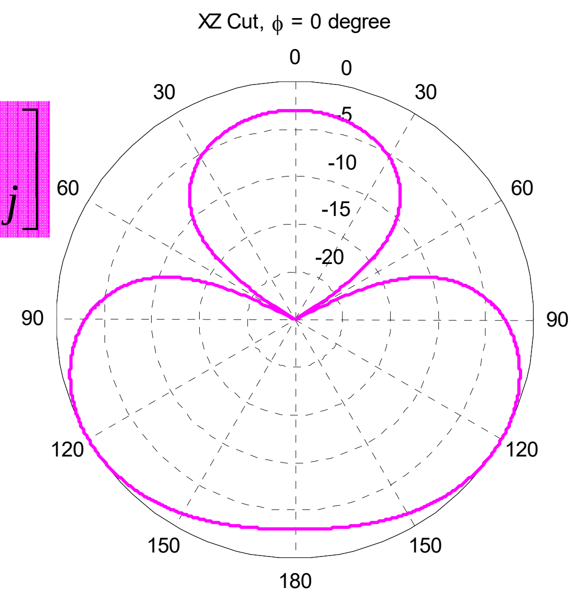
$$w = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



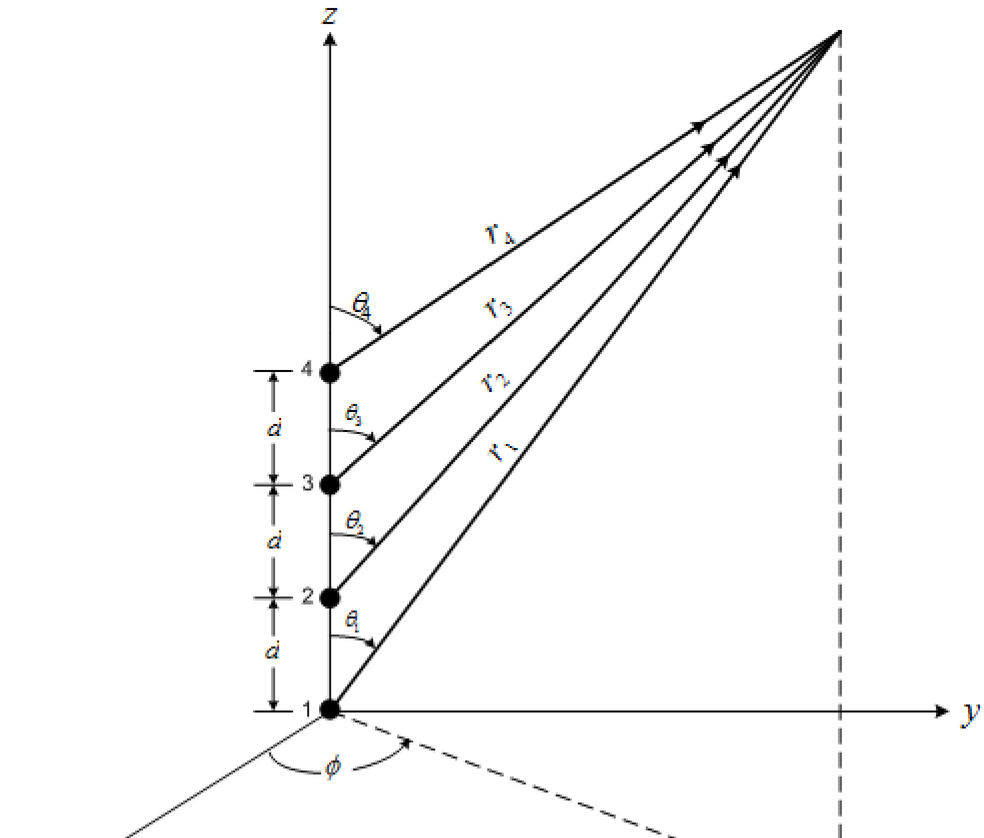
$$w = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$$



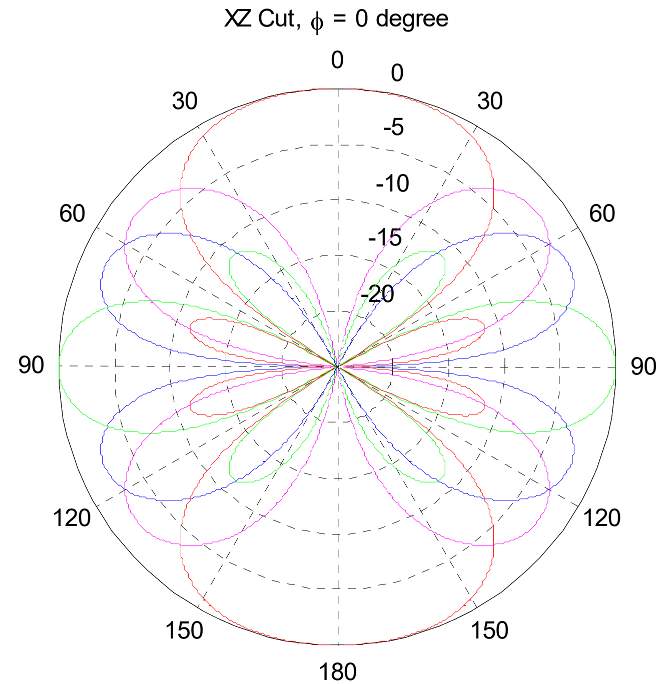
$$w = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -j \end{bmatrix}$$



Four-Element Array, W0



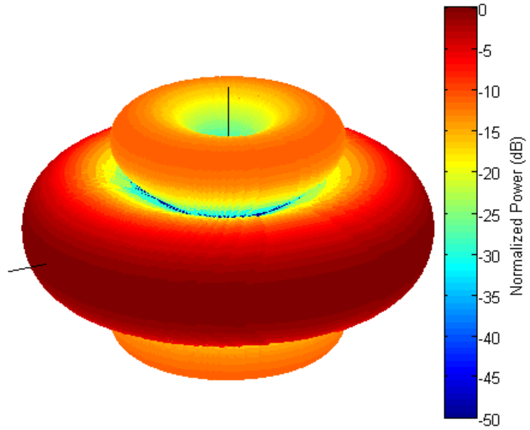
Element spacing : $d = \frac{\lambda}{2}$



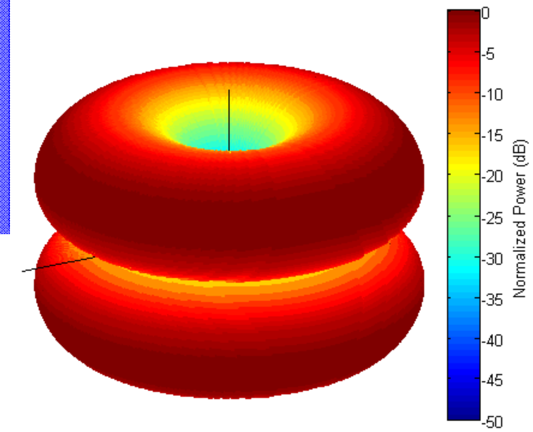
$$w_0 = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

3D Radiation Patterns of 4 Element Array

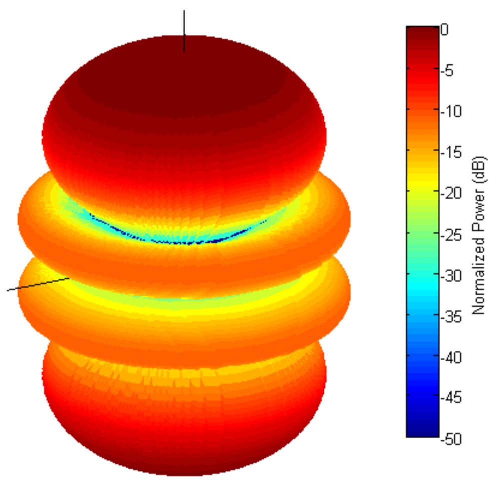
$$w_{00} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$



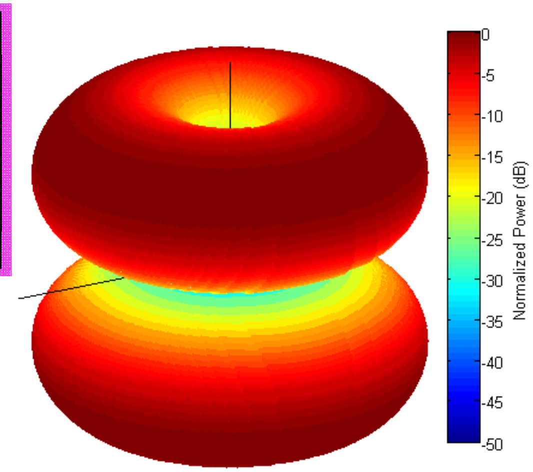
$$w_{01} = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$



$$w_{02} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

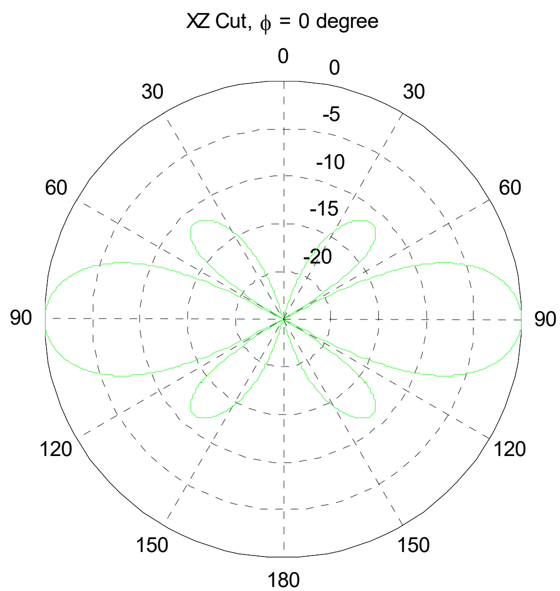


$$w_{03} = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$$

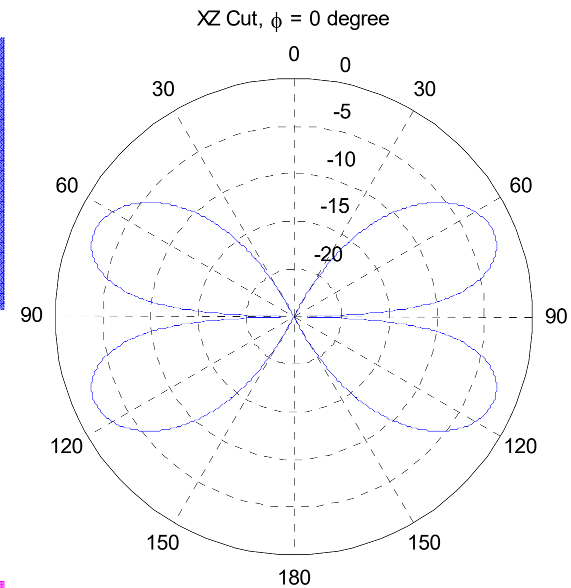


2D Patterns of 4 Element Array, W0

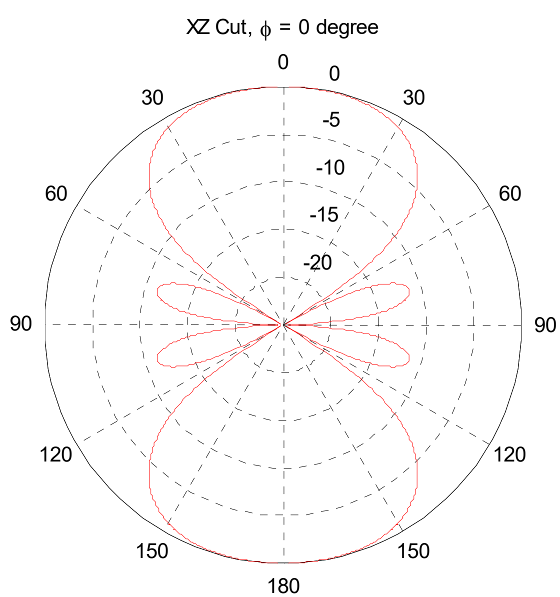
$$w_{00} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$



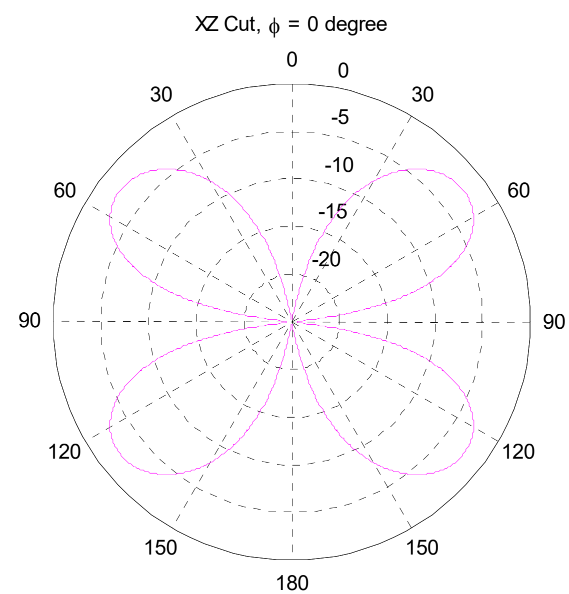
$$w_{01} = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$



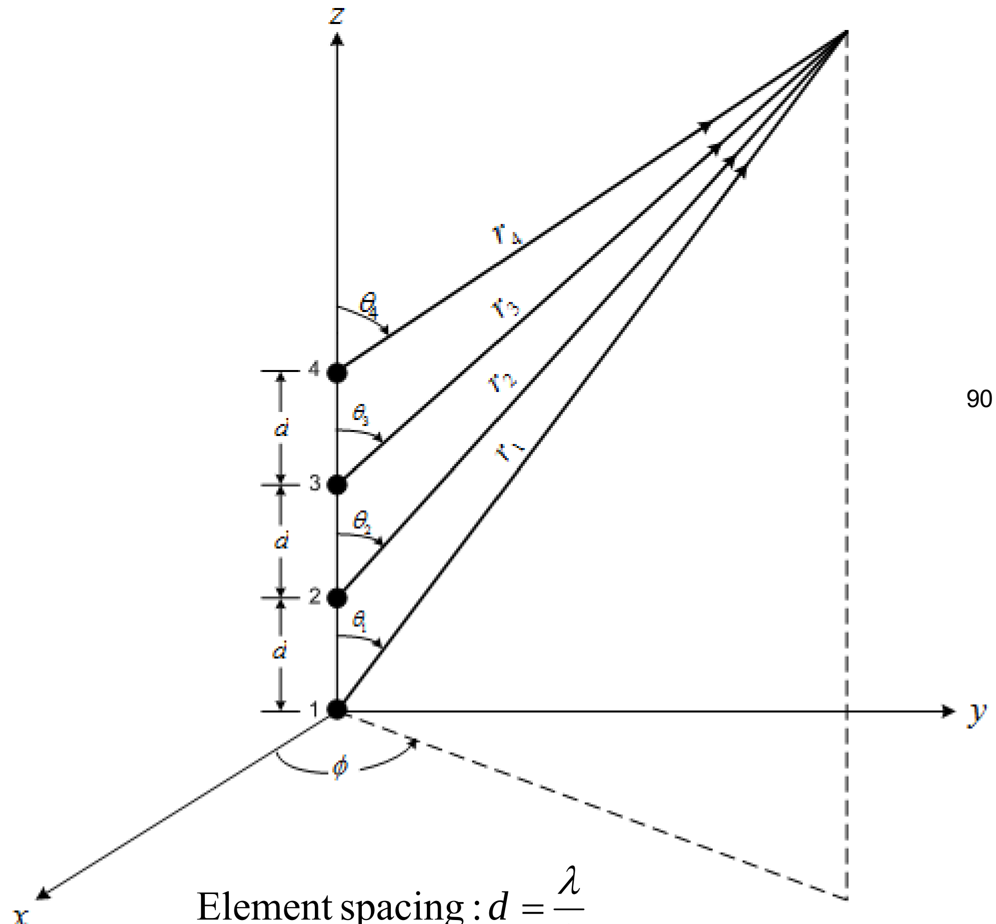
$$w_{02} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$



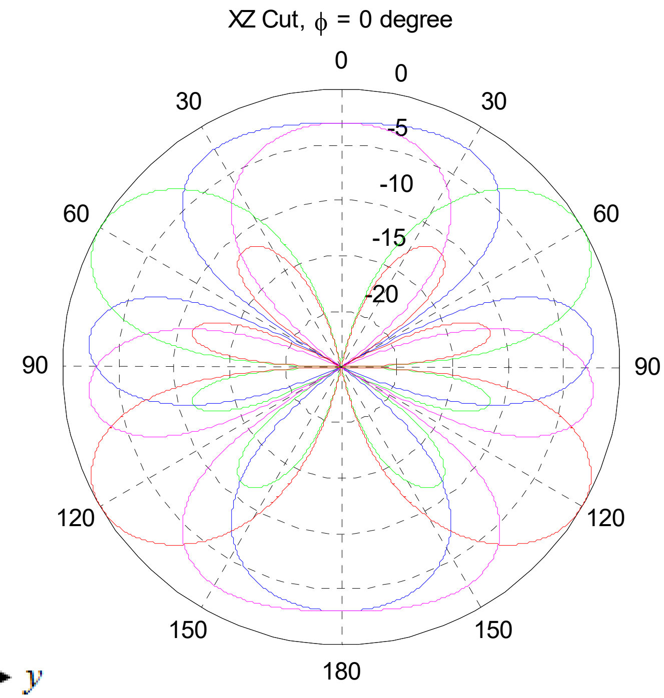
$$w_{03} = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix}$$



Four-Element Array, W1



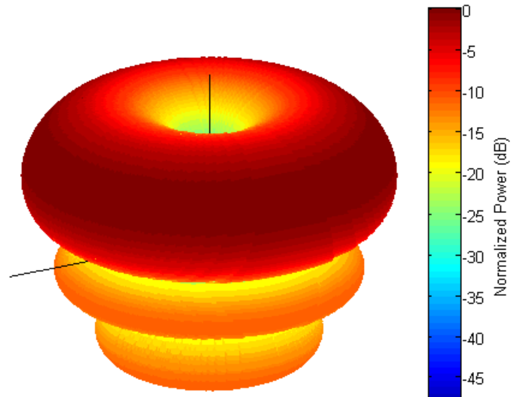
Element spacing : $d = \frac{\lambda}{2}$



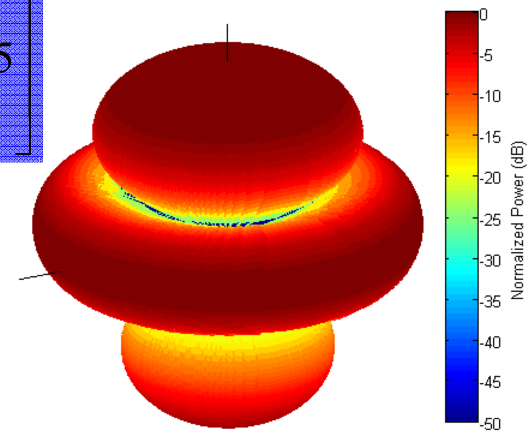
$$w_1 = \begin{bmatrix} 0.5 & -j0.5 & -0.5 & j0.5 \\ j0.5 & 0.5 & j0.5 & 0.5 \\ -0.5 & -j0.5 & 0.5 & j0.5 \\ -j0.5 & 0.5 & -j0.5 & 0.5 \end{bmatrix}$$

3D Radiation Patterns of 4 Element Array

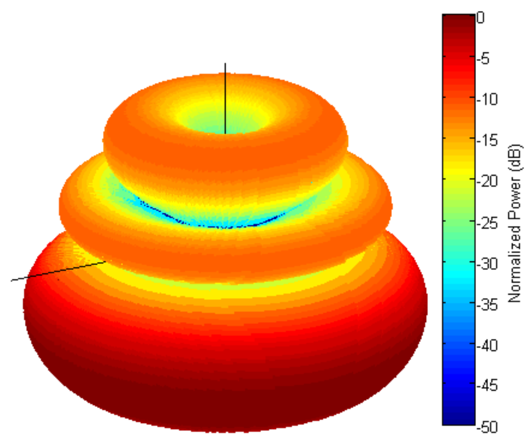
$$w_{10} = \begin{bmatrix} 0.5 \\ j0.5 \\ -0.5 \\ -j0.5 \end{bmatrix}$$



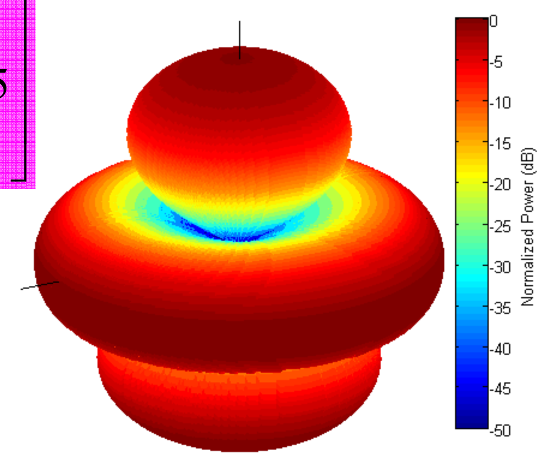
$$w_{11} = \begin{bmatrix} -j0.5 \\ 0.5 \\ -j0.5 \\ 0.5 \end{bmatrix}$$



$$w_{12} = \begin{bmatrix} -0.5 \\ j0.5 \\ 0.5 \\ -j0.5 \end{bmatrix}$$

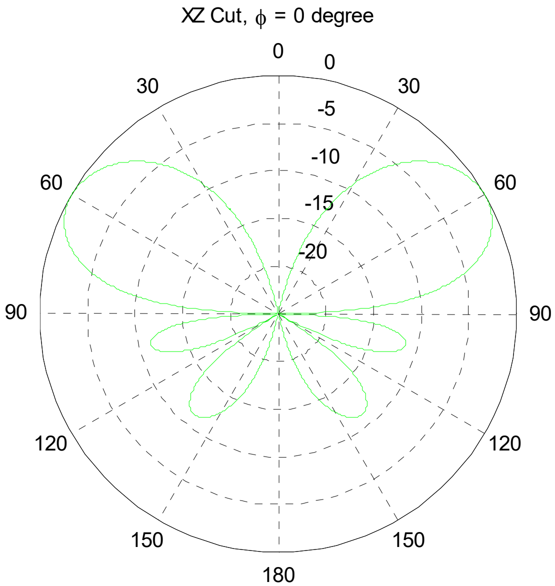


$$w_{13} = \begin{bmatrix} j0.5 \\ 0.5 \\ j0.5 \\ 0.5 \end{bmatrix}$$

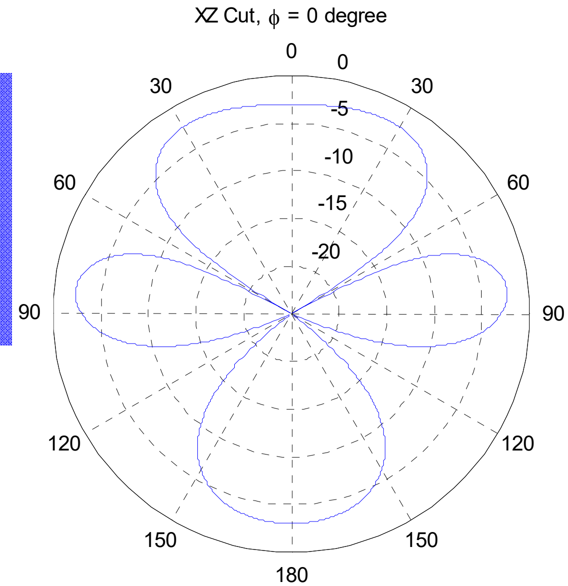


2D Patterns of 4 Element Array, W1

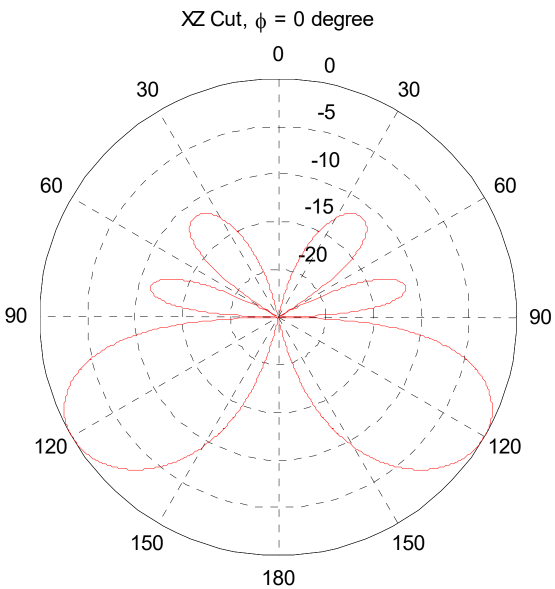
$$w_{10} = \begin{bmatrix} 0.5 \\ j0.5 \\ -0.5 \\ -j0.5 \end{bmatrix}$$



$$w_{11} = \begin{bmatrix} -j0.5 \\ 0.5 \\ -j0.5 \\ 0.5 \end{bmatrix}$$



$$w_{12} = \begin{bmatrix} -0.5 \\ j0.5 \\ 0.5 \\ -j0.5 \end{bmatrix}$$



$$w_{13} = \begin{bmatrix} j0.5 \\ 0.5 \\ j0.5 \\ 0.5 \end{bmatrix}$$