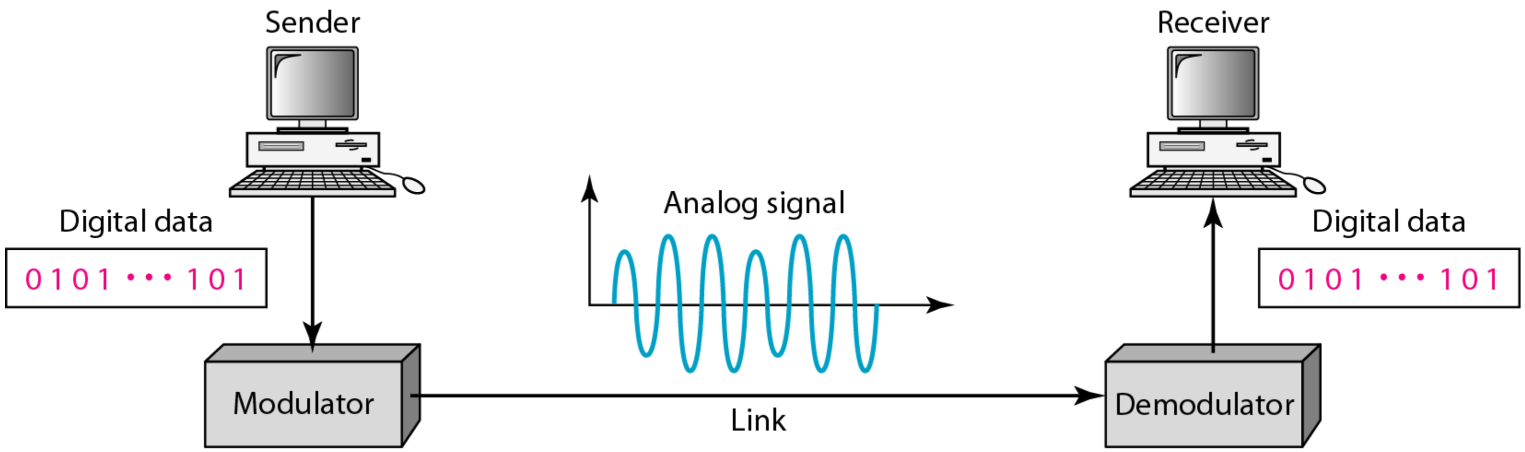
Digital Modulation Techniques: A Review

Digital Modulation ?

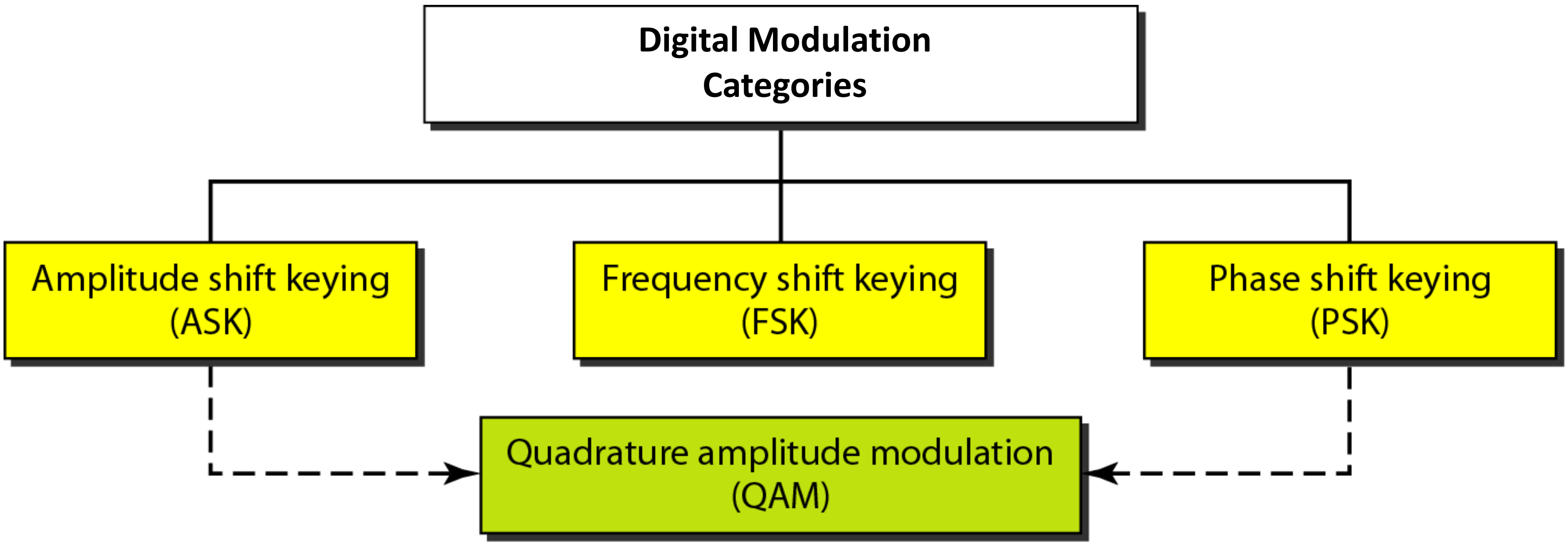
- Why digital transmission?
  - Can combine multiple information types (voice, data and video) in a single transmission channel
  - Improved security (e.g., encryption)
  - Error coding is used to detect/correct transmission errors
- Digital Modulation:
  - Mapping information bits into an analog signal for transmission over channel
- Demodulation/Detection:
  - Determining original bit sequence based on signal received over channel



How to Choose a Digital Modulation Technique?

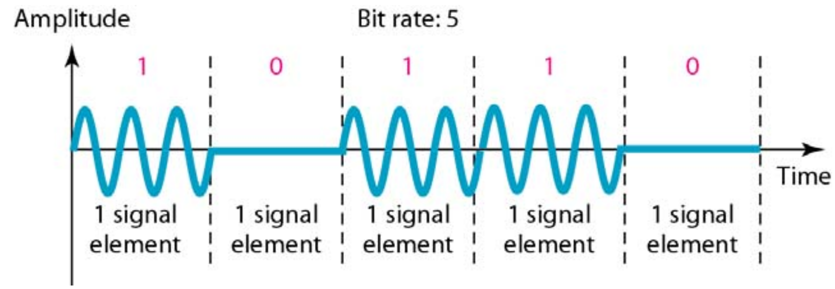
- Performance factors to consider
  - High data rate
  - High spectral efficiency (max data rate in a minimum channel bandwidth)
  - High power efficiency (minimum bit error probability for minimum transmitted power)
  - Robustness to channel impairments (minimum bit error probability)
  - Low power/cost implementation
- No existing modulation scheme simultaneously satisfies all of these requirements well
- Each one can be better in some aspects (e.g., spectral efficiency) but worse in others (e.g., power efficiency)

Main Categories of Digital Modulation

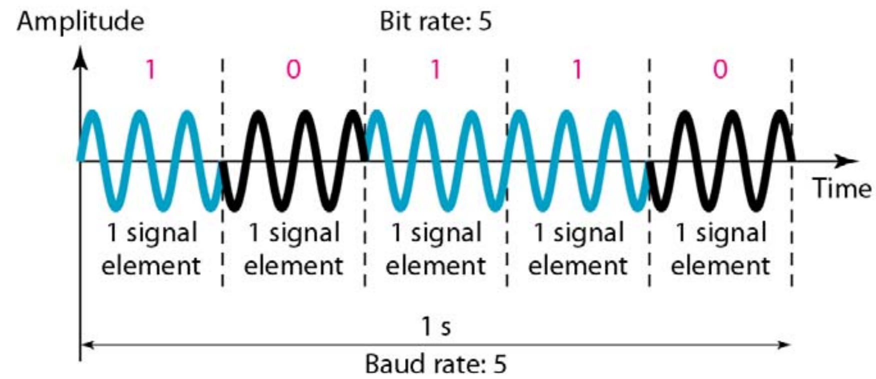


Main Categories of Digital Modulation

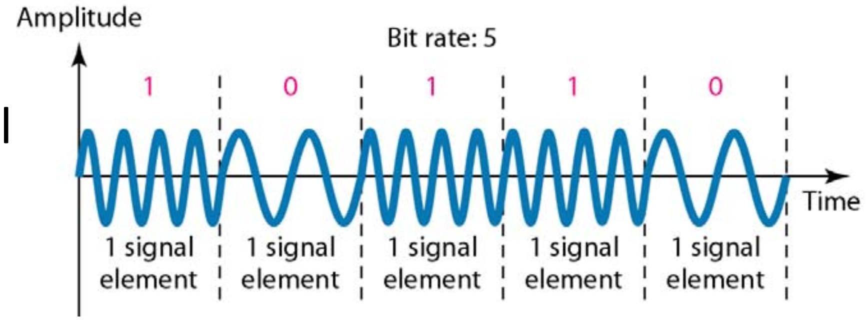
- Amplitude Shift Keying (ASK)
  - Change amplitude with each symbol
  - Spectrum-efficient



- Phase Shift Keying (PSK)
  - Change phase with each symbol
  - Spectrum-efficient

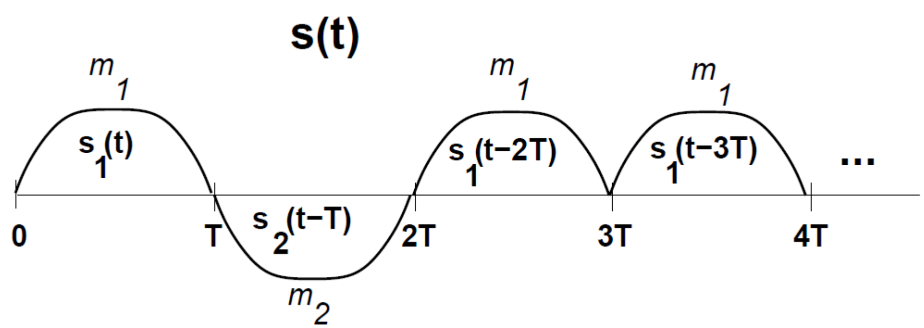


- Frequency Shift Keying (FSK)
  - Change frequency with each symbol
  - Power-efficient
  - Resistance to channel impairments

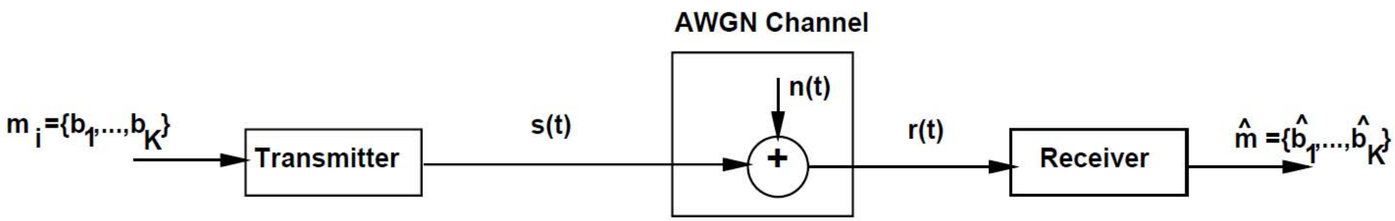


Transmitter/Modulator

- Every K bits grouped to form 1 symbol $m_i = (b_{i1}, \dots, b_{iK})$: there are 2K possible symbols, $m_i \in \boldsymbol{M} = \{m_1, \dots, m_M\}$.
- Modulation: one-to-one mapping of each symbol m_i to a distinct Tx signaling element $s_i(t)$: $m_i \to s_i(t) \in \boldsymbol{S} = \{s_1(t), \dots, s_M(t)\}$
- Time-limited signaling: $s_i(t) = 0$ for $t \notin [0, T]$, i.e., only defined in one symbol interval of T, with energy $E_i = \int_0^T |s(t)|^2 dt$
- Transmitting one randomly generated symbol in one symbol interval of T seconds:
  - Transmitted signal: $s(t) = \sum_{n=-\infty}^{+\infty} s_i(t - nT)$
  - Data rate $R = K/T$ bits per second (b/s)
- Example of a binary Tx:



Receiver Design



- Receiver
  - observes the Rx signal: $r(t) = s(t) + n(t), s(t) \in \boldsymbol{S}$, $n(t)$ is WGN, and
  - decodes/detects the Tx symbol $\hat{m} \in \boldsymbol{M}$ transmitted in each symbol interval
- Optimum receiver: minimizes the average probability of detection error:
  - Optimum criterion: $\min P_e = \sum_{i=1}^{M} Pr\{\hat{m} \neq m_i | m_i\ sent\} Pr\{m_i\ sent\}$
  - For equally probable Tx, i.e., $Pr\{m_i\ sent\} = \frac{1}{M}$

$$\rightarrow \min P_e = \sum_{i=1}^{M} Pr\{\hat{m} \neq m_i | m_i\ sent\}$$

Vector representation of signals

- Orthonormal basis functions

$$\{\phi_1(t), \cdots, \phi_N(t)\}, \ N \le M, \ \int_0^T \phi_i(t)\phi_j^*(t)\,\mathrm{d}t = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$$

- Orthonormal basis representation of each $s_i(t) \in \mathbf{S}$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \ 0 \le t < T, \ s_{ij} = \int_0^T s_i(t)\phi_j^*(t)\,\mathrm{d}t$$

- Signal constellation point $\mathbf{s}_i = (s_{i1}, \cdots, s_{iN}) \in \Re^N$

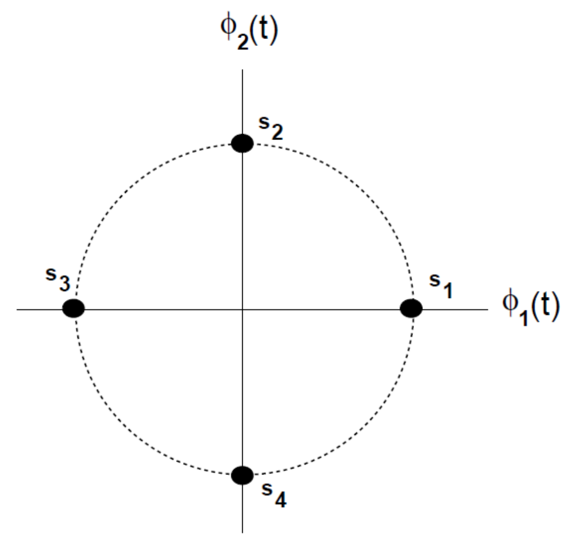
- Signal constellation $\{\mathbf{s}_1, \cdots, \mathbf{s}_M\}$

Vector Representation of Signals

- Length (power) of a signal $\|\mathbf{s}_i\| = \sqrt{\sum_{j=1}^{N} s_{ij}^2}$

- The distance between two signals

$$\|\mathbf{s}_i - \mathbf{s}_k\| = \sqrt{\sum_{j=1}^{N} \left(s_{ij} - s_{kj}\right)^2}$$
$$= \sqrt{\int_0^T \left(s_i(t) - s_k(t)\right)^2 \, \mathrm{d}t}$$

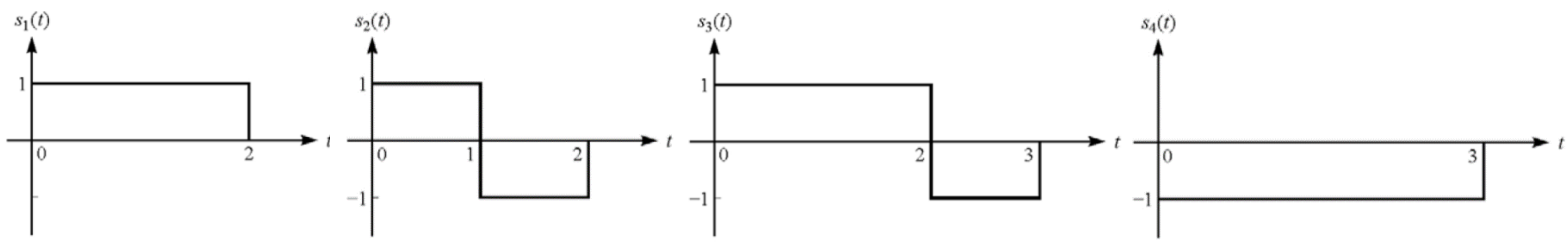


- Inner product

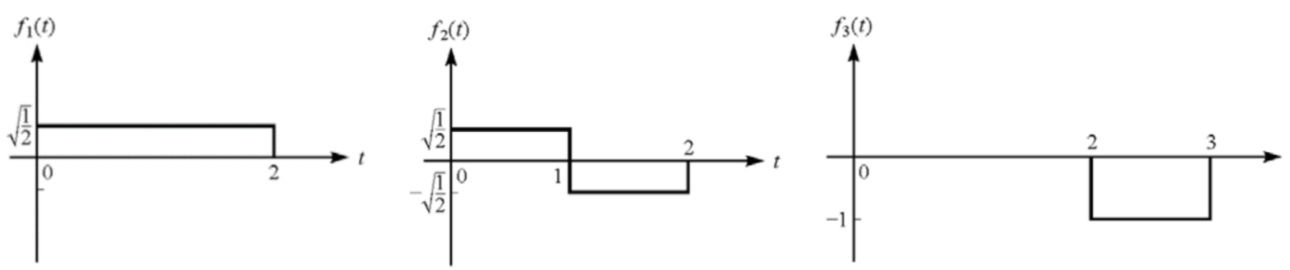
$$\langle \mathbf{s}_i, \mathbf{s}_k \rangle = \mathbf{s}_i . \mathbf{s}_k^H = \int_0^T s_i(t) s_k^*(t) \, \mathrm{d}t$$

Orthonormal basis functions: an example

- Find a complete set of orthonormal basis functions to represent the following signalling elements:

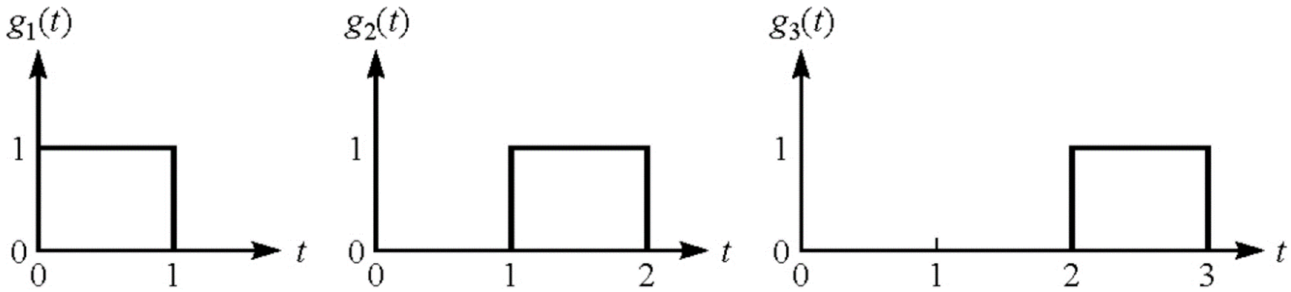


- By using the Gram-Schmidt orthogonalization procedure,

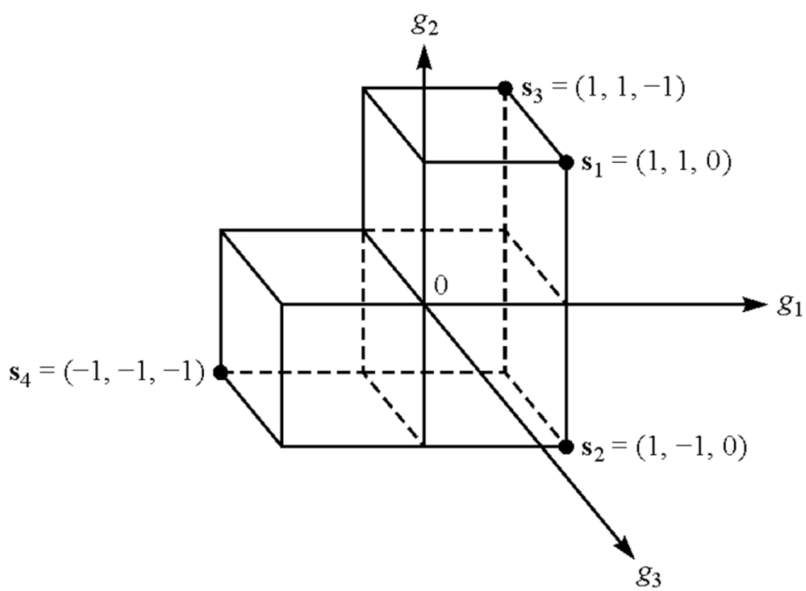


Orthonormal basis functions: an example

- another set of orthonormal basis functions:



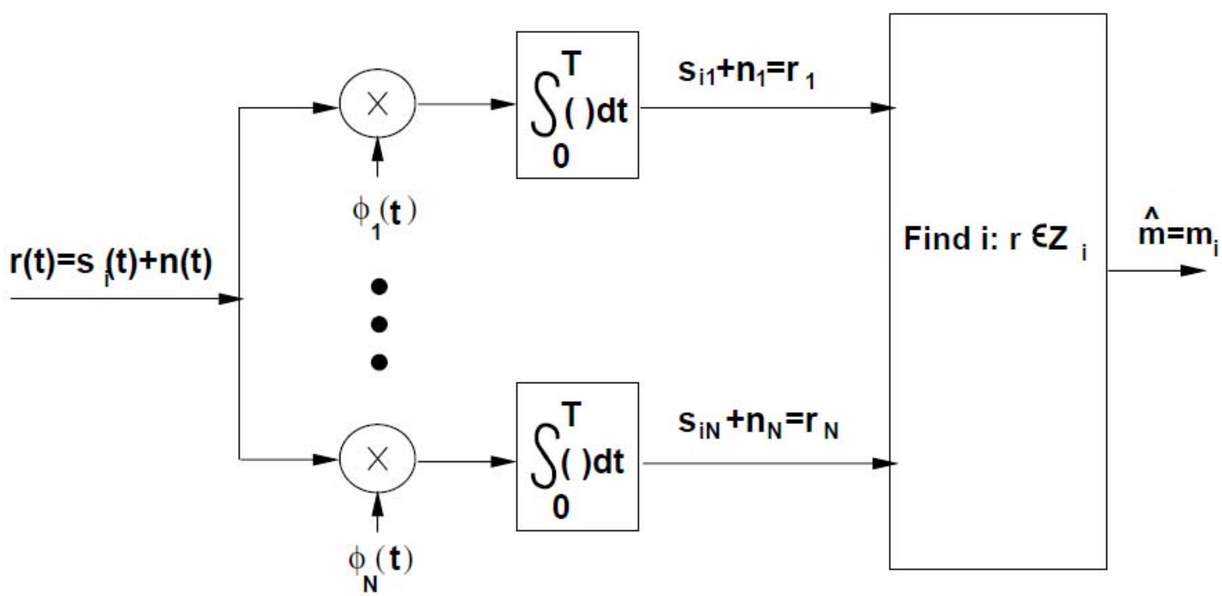
- Vector representation of 4 signalling elements:



Receiver Structure and Sufficient Statistics

In each symbol interval $r(t) = s_i(t) + n(t), 0 \le t < T$

- Signal $s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t)$
- Noise: $n(t) = \sum_{j=1}^{N} n_j\phi_j(t) + n_r(t)$ where $n_j = \langle n(t), \phi_j(t)\rangle$
- $n_r(t)$ is shown to be *irrelevant* in the detection and hence *ignored*
- $\rightarrow$ Receiver structure with N correlators: $r_j = \langle r(t), \phi_j(t)\rangle$ are *sufficient statistics*



Receiver Structure and Sufficient Statistics

- Sufficient statistics in the optimal detection

$$\mathbf{r} = (r_1, \cdots, r_N), \ \ r_j \sim N(s_{ij}, N_0/2) \text{ if } \mathbf{s}_i \text{ sent}$$

- Optimum receiver design criterion

$$\min \ P_e = \mathrm{P}(\hat{\mathbf{s}} \neq \mathbf{s}_i \mid \mathbf{s}_i \text{ sent}) = \mathrm{P}(\hat{\mathbf{s}} \neq \mathbf{s}_i \mid \mathbf{r} = \mathbf{s}_i + \mathbf{n})$$

- Maximum a posteriori probability (MAP) receiver

$$\max \ \mathrm{P}(\hat{\mathbf{s}} = \mathbf{s}_i \mid \mathbf{r} = \mathbf{s}_i + \mathbf{n})$$

- Decision regions

$$Z_i = \left(\mathbf{r} : \mathrm{P}(\mathbf{s}_i \mid \mathbf{r}) > \mathrm{P}(\mathbf{s}_j \mid \mathbf{r}), \ \ \forall j \neq i\right)$$

Maximum Likelihood Decision Criterion

Notes:

P(.): cumulative distribution function (CDF): $P_X(x)=\Pr\{X \le x\}$

p(.): probability density function (pdf): $p_X(x)=dP_X(x)/dx$

$$\mathrm{P}(\mathbf{s}_i \mid \mathbf{r}) = \frac{\mathrm{p}(\mathbf{r} \mid \mathbf{s}_i)\mathrm{P}(\mathbf{s}_i)}{\mathrm{p}(\mathbf{r})}$$

$$\underset{\mathbf{s}_i}{\operatorname{argmax}} \frac{\mathrm{p}(\mathbf{r} \mid \mathbf{s}_i)\mathrm{P}(\mathbf{s}_i)}{\mathrm{p}(\mathbf{r})} \equiv \underset{\mathbf{s}_i}{\operatorname{argmax}}\ \mathrm{p}(\mathbf{r} \mid \mathbf{s}_i)\mathrm{P}(\mathbf{s}_i)$$

- ML receiver: $\underset{\mathbf{s}_i}{\operatorname{argmax}}\ \mathrm{p}(\mathbf{r} \mid \mathbf{s}_i), \text{ if } \mathrm{P}(\mathbf{s}_i) = 1/M$

Maximum Likelihood Decision Criterion

- Since

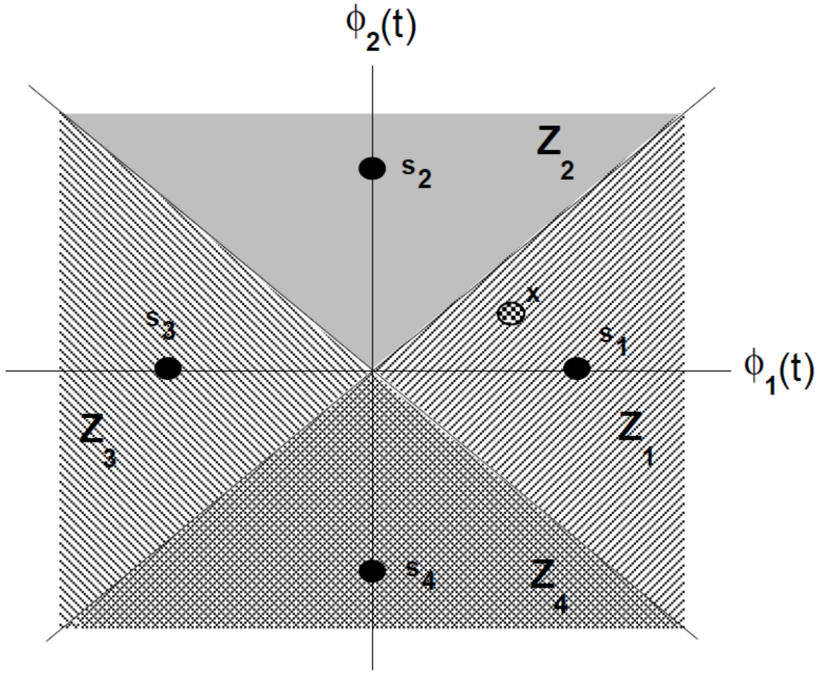
$$p(\mathbf{r} \mid \mathbf{s}_i) = \prod_{j=1}^{N} p(r_j \mid s_{ij})$$

$$= \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\frac{1}{N_0} \sum_{j=1}^{N} (r_j - s_{ij})^2 \right]$$

- ML receiver: $\boxed{\underset{\mathbf{s}_i}{\operatorname{argmin}} \; \|\mathbf{r} - \mathbf{s}_i\|^2}$

- Decision regions

$$Z_i = \left(\mathbf{r} : \|\mathbf{r} - \mathbf{s}_i\| < \|\mathbf{r} - \mathbf{s}_j\|, \forall j \neq i \right)$$

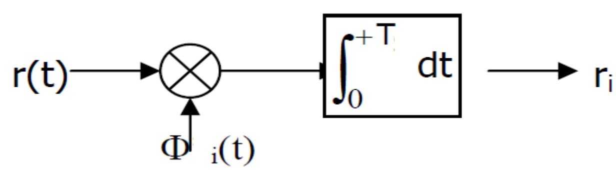


Maximum Likelihood Decision Criterion

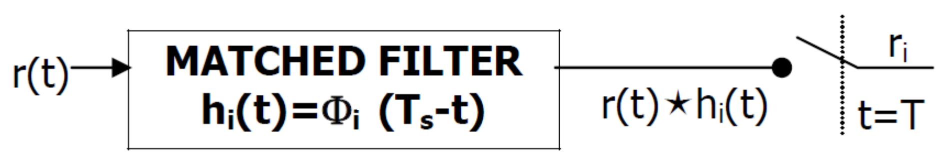
$$\left\| \mathbf{r} - \mathbf{s}_i \right\|^2 = \sum_{j=1}^{N} r_j^2 + \sum_{j=1}^{N} s_{ij}^2 - 2\sum_{j=1}^{N} r_j s_{ij}$$
$$= E_r + E_i - 2\,\mathbf{r}.\mathbf{s}_i$$

- ML receiver: $\boxed{\underset{\mathbf{s}_i}{\operatorname{argmax}} \quad \mathbf{r}.\mathbf{s}_i - \frac{E_i}{2}}$

- Correlator

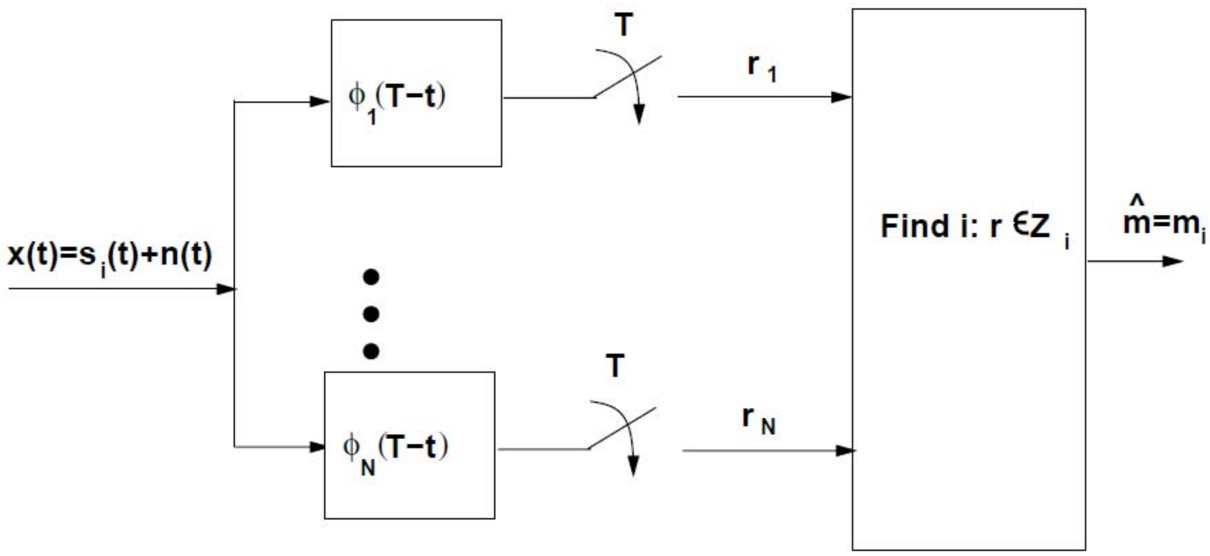


- Matched filter



Maximum Likelihood Receiver

- ML receiver structure with matched filters



Error Probability

- Error probability for ML receiver

$$P_e = \sum_{i=1}^{M} \mathrm{P}\left(\mathbf{r} \notin Z_i \mid \mathbf{s}_i\right)\mathrm{P}\left(\mathbf{s}_i\right) = \frac{1}{M}\sum_{i=1}^{M} \mathrm{P}\left(\mathbf{r} \notin Z_i \mid \mathbf{s}_i\right)$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M} \mathrm{P}\left(\mathbf{r} \in Z_i \mid \mathbf{s}_i\right) = 1 - \frac{1}{M}\sum_{i=1}^{M} \int_{Z_i} \mathrm{P}\left(\mathbf{r} = \mathbf{s}_i + \mathbf{n}\right) d\mathbf{n}$$

- Binary transmission
  - Distance between two signals: $d_{\min} = \left\|\mathbf{s}_1 - \mathbf{s}_2\right\|$
  - Average energy per bit: $E_b = \dfrac{\left\|\mathbf{s}_1\right\|^2 + \left\|\mathbf{s}_2\right\|^2}{2}$
  - Correlation coefficient between two signals:

$$\gamma = \frac{\mathbf{s}_1 . \mathbf{s}_2}{E_b}, -1 \leq \gamma \leq 1$$

Error Probability of Binary Transmission

- AWGN channel, ML receiver

$$P_b = P_e = \frac{1}{2}\left(\mathrm{P}(e \mid \mathbf{s}_1) + \mathrm{P}(e \mid \mathbf{s}_2)\right) = \mathrm{P}(e \mid \mathbf{s}_1)$$

$$= \mathrm{P}(\mathbf{r} \in Z_2 \mid \mathbf{s}_1) = \mathrm{P}(s_1 + n < 0) = \mathrm{P}\left(n < -\frac{\mathrm{d}_{\min}}{2}\right)$$

$$P_b = Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) \Rightarrow P_b = Q\left(\sqrt{\frac{E_b(1-\gamma)}{N_0}}\right)$$

- Orthogonal signaling

$$P_b = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

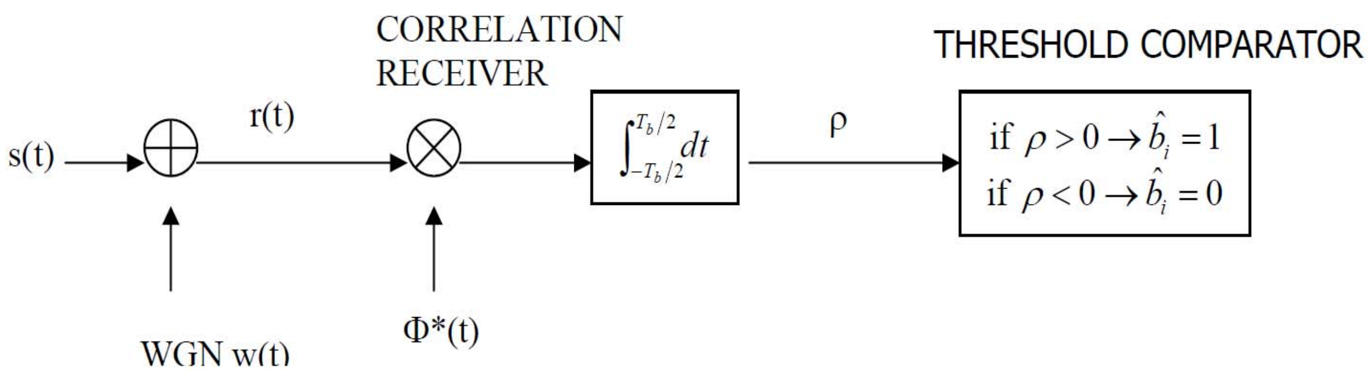
- Antipodal signaling

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Optimum Receiver for Antipodal Signaling

- ML receiver, $s_1 = -s_2$

$$\underset{\mathbf{s}_i}{\operatorname{argmax}} \; r.s_i \;\Rightarrow\; r.s_1 \underset{s_2}{\overset{s_1}{\gtrless}} r.s_2 \;\Rightarrow\; r \underset{s_2}{\overset{s_1}{\gtrless}} 0$$



Union Bound on Error Probability

- M-ary signaling scheme
- Event that r is closer to s_k than s_i

$$A_{ik} : \left\|\mathbf{r} - \mathbf{s}_k\right\| < \left\|\mathbf{r} - \mathbf{s}_i\right\|$$

- Error probability

$$\mathrm{P}\left(e \mid \mathbf{s}_i\right) = \mathrm{P}\left(\bigcup_{\substack{k=1 \\ k \neq i}}^{M} A_{ik}\right) \leq \sum_{\substack{k=1 \\ k \neq i}}^{M} \mathrm{P}\left(A_{ik}\right)$$

- Based on error probability for binary transmission

$$\mathrm{P}\left(A_{ik}\right) = Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

Union Bound on Error Probability

- Union bound

$$P_e = \frac{1}{\mathrm{M}} \sum_{\mathrm{i}=1}^{M} \mathrm{P}\left(e \mid \mathbf{s}_i\right) \leq \frac{1}{\mathrm{M}} \sum_{i=1}^{M} \sum_{\substack{k=1 \\ k \neq i}}^{M} Q\left(\frac{d_{ik}}{\sqrt{2N_0}}\right)$$

- Minimum distance

$$d_{\min} = \min_{i,k} d_{ik}$$

- Looser bound

$$P_e \leq (M-1) Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$$

Pass-Band Modulation

- Modulated signal

$$s(t) = \alpha(t)\cos(2\pi f_c t + \phi(t))$$

- In-phase and quadrature components

$$\begin{aligned} s(t) &= \alpha(t)\cos(\phi(t))\cos(2\pi f_c t) - \alpha(t)\sin(\phi(t))\sin(2\pi f_c t) \\ &= s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t) \end{aligned}$$

- Complex baseband representation

$$u(t) = s_I(t) + js_Q(t), \quad s(t) = \Re\left\{u(t)e^{j(2\pi f_c t)}\right\}$$

Amplitude and Phase Modulation

- Transmitted signal

$$s_i(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

$$s_I(t) = s_{i1} g(t) \quad \text{and} \quad s_Q(t) = s_{i2} g(t)$$

$$\phi_1(t) = g(t)\cos(2\pi f_c t) \quad \text{and} \quad \phi_2(t) = g(t)\sin(2\pi f_c t)$$

- Data rate

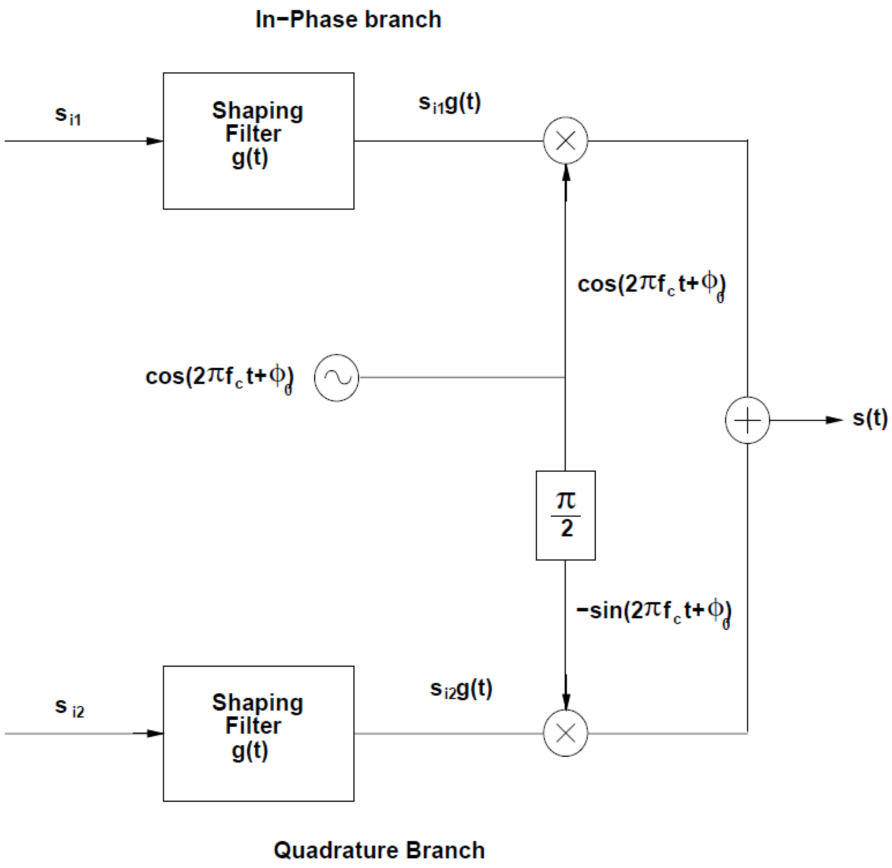
$$R = \frac{K}{T_s} = \frac{\log_2 M}{T_s}$$

Amplitude and Phase Modulation

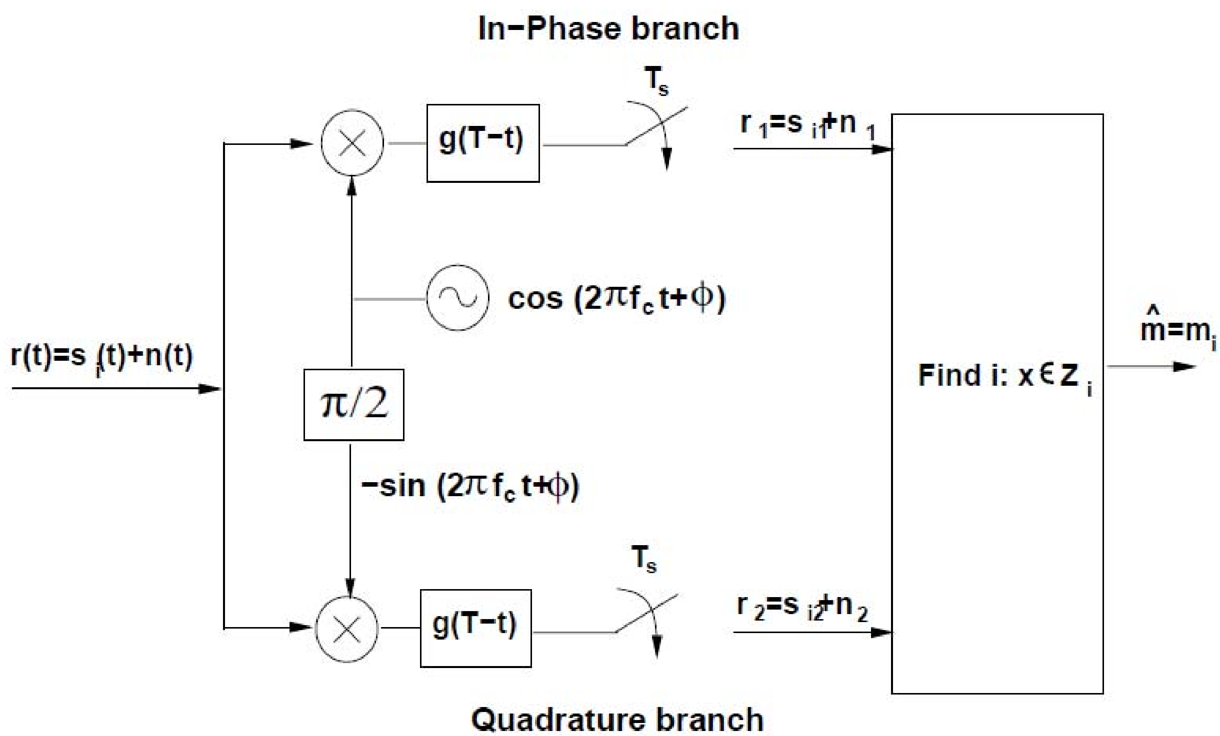
- Main categories
  - Pulse Amplitude Modulation
  - Phase Shift Keying
  - Quadrature Amplitude Modulation

- Digital modulation design
  - Number of bits per symbol
  - Signal constellation
  - Choice of shaping pulse

- Amplitude/phase modulator



Amplitude/phase demodulator



Pulse Amplitude Modulation (M-PAM)

- Linear modulation, one-dimensional, no quadrature component
- Information is encoded into the signal amplitude
- Transmitted signal

$$s_i(t) = A_i g(t) \cos(2\pi f_c t)$$
$$\phi(t) = g(t) \cos(2\pi f_c t)$$

- Time-limited signaling

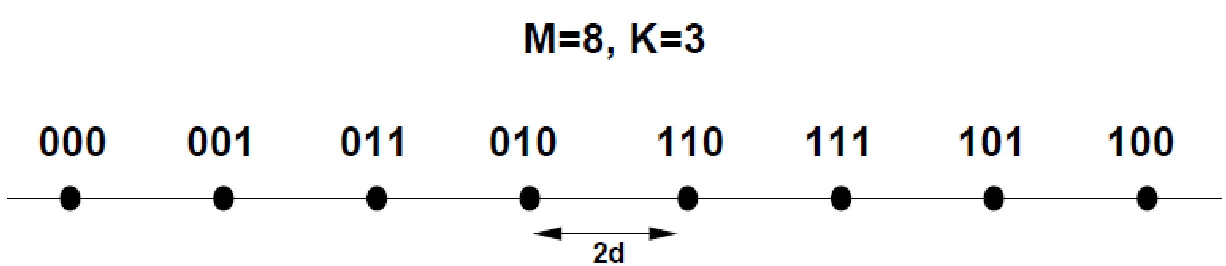
$$g(t) = \begin{cases} \sqrt{\dfrac{2}{T_s}} & 0 \le t < T_s \\ 0 & \text{elsewhere} \end{cases}$$

Pulse Amplitude Modulation (M-PAM)

- Signal constellation

$$s_{i1} = A_i, \; s_{i2} = 0, \; A_i = (2i - 1 - M)d, \; i = 1, 2, \cdots, M$$

- Constellation mapping by Gray encoding
    - Messages associated with adjacent signals differ by one bit value
    - Mistaking a symbol for an adjacent one causes only a single bit error.



Pulse Amplitude Modulation (M-PAM)

- Signal energy

$$E_{s_i} = \int_0^{T_s} s_i^2(t)\, dt = \int_0^{T_s} A_i^2 g^2(t) \cos^2(2\pi f_c t) dt = A_i^2$$

- Average energy per symbol

$$E_s = \frac{1}{M}\sum_{i=1}^{M} E_{s_i} = \frac{1}{M}\sum_{i=1}^{M} A_i^2$$

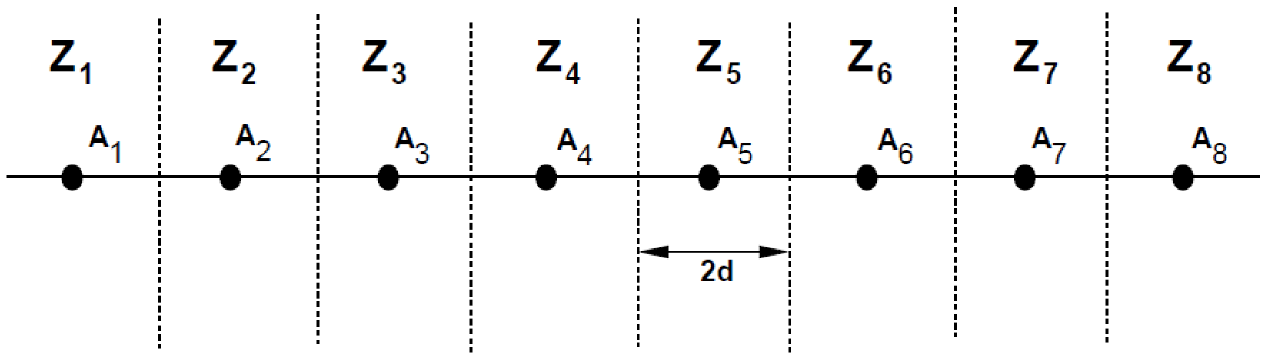
$$= \frac{1}{M}\sum_{i=1}^{M} (2i - 1 - M)^2 d^2 = \frac{1}{3}\left(M^2 - 1\right) d^2$$

- Minimum distance

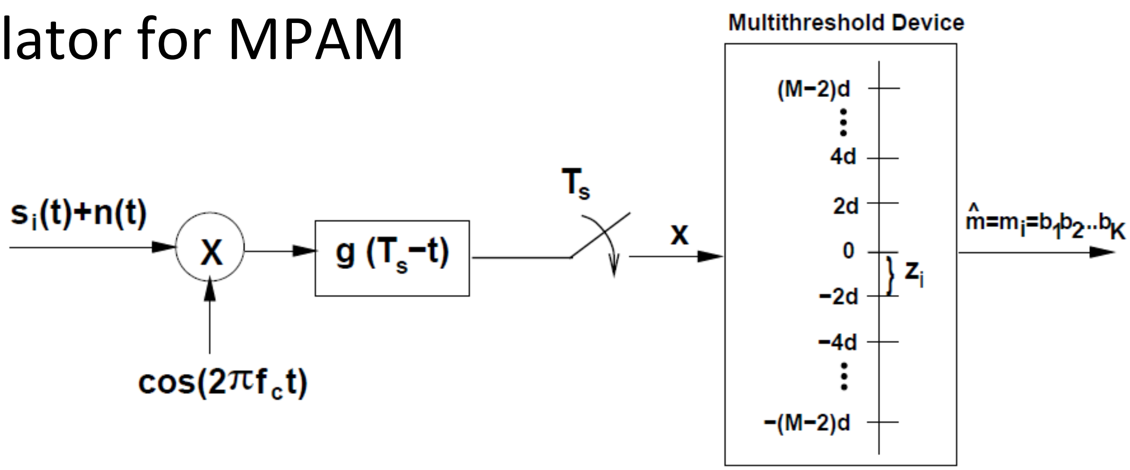
$$d_{\min} = \min_{i,j} \left| A_i - A_j \right| = 2d$$

Pulse Amplitude Modulation (M-PAM)

- Decision regions



- Demodulator for MPAM



Pulse Amplitude Modulation (M-PAM)

- Symbol error probability

$$P(e \mid \mathbf{s}_i) = \mathrm{P}\left(|n| > d\right) = 2Q\left(\sqrt{\frac{2d^2}{N_0}}\right), \quad i = 2, \cdots, M-1$$

$$P(e \mid \mathbf{s}_i) = \mathrm{P}\left(n > d\right) = Q\left(\sqrt{\frac{2d^2}{N_0}}\right), \quad i = 1, M$$

$$P_s = \frac{1}{M}\sum_{i=1}^{M} P(e \mid \mathbf{s}_i) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2d^2}{N_0}}\right)$$

$$= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6}{M^2-1}\frac{E_s}{N_0}}\right)$$

Phase Shift Keying (M-PSK)

- Linear modulation, two-dimensional
- Information is encoded into the signal phase
- Transmitted signal (with $a = 0$ or 1)

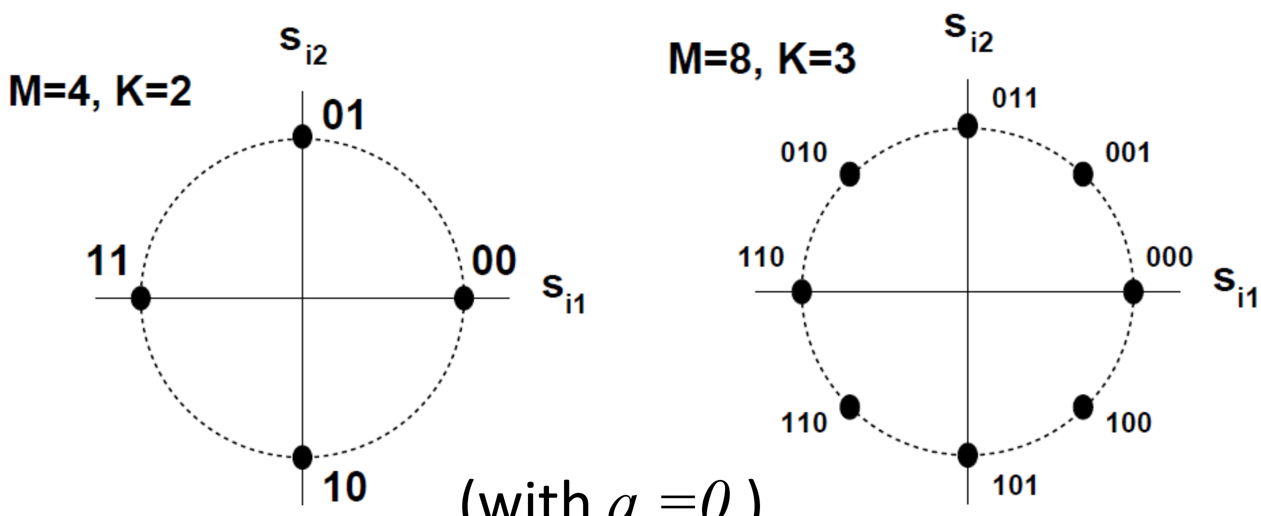
$$s_i(t) = Ag(t)\cos\left[\frac{2\pi(i-a)}{M}\right]\cos(2\pi f_c t) - Ag(t)\sin\left[\frac{2\pi(i-a)}{M}\right]\sin(2\pi f_c t)$$

$$\phi_1(t) = g(t)\cos(2\pi f_c t), \qquad \phi_2(t) = -g(t)\sin(2\pi f_c t)$$

$$s_{i1} = A\cos\left[\frac{2\pi(i-a)}{M}\right], \quad s_{i2} = A\sin\left[\frac{2\pi(i-a)}{M}\right]$$

Phase Shift Keying (M-PSK)

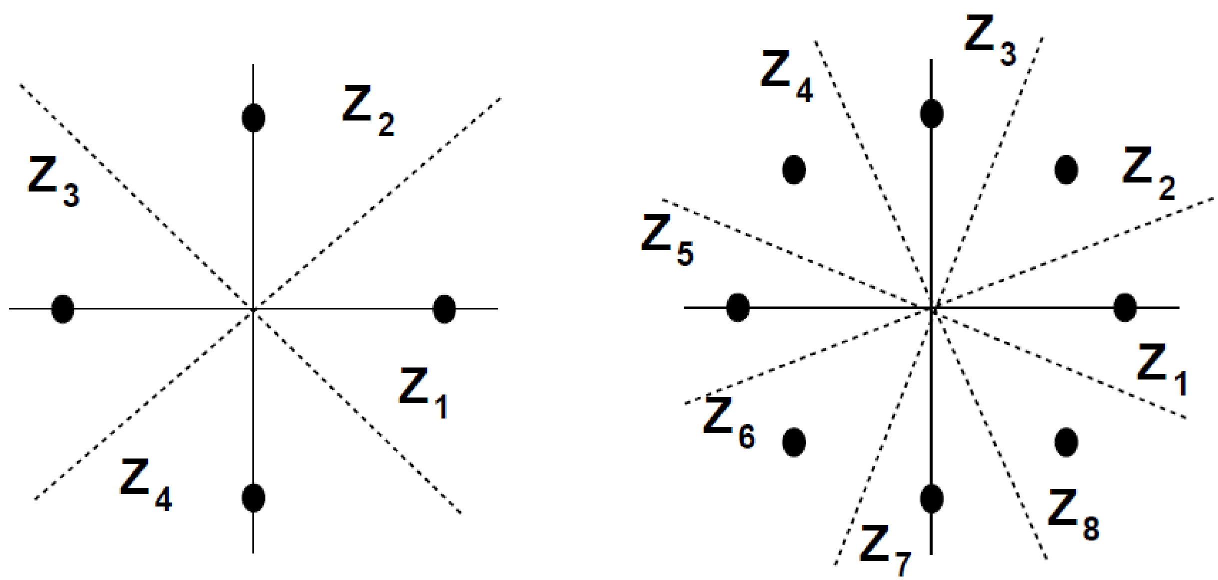
- Equal energy $E_{s_i} = E_s = \int_0^{T_s} s_i^2(t)\,\mathrm{d}t = A^2$
- Minimum distance $d_{\min} = 2A\sin(\pi/M)$
- Constellation mapping by Gray encoding



(with $a = 0$)

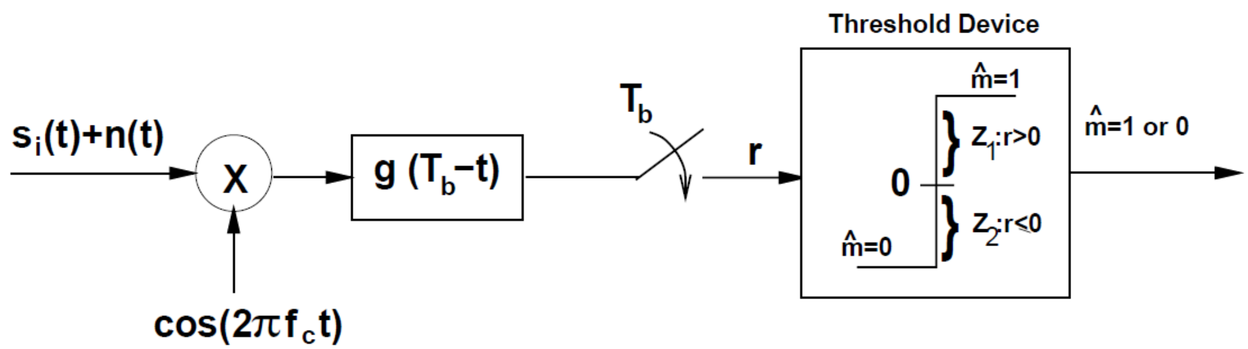
Phase Shift Keying (M-PSK): Decision regions

$$Z_i = \left\{ re^{j\theta} : 2\pi(i-0.5)/M \le \theta < 2\pi(i+0.5)/M \right\}$$



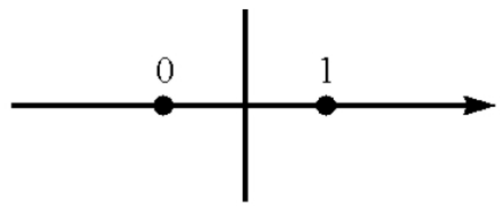
(with $a = 0$)

Demodulator for BPSK



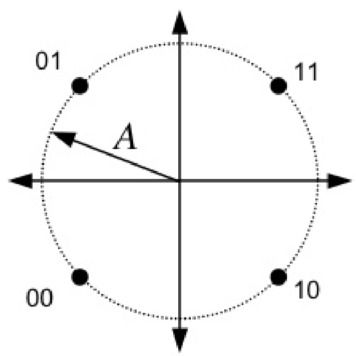
- Bit error probability

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$



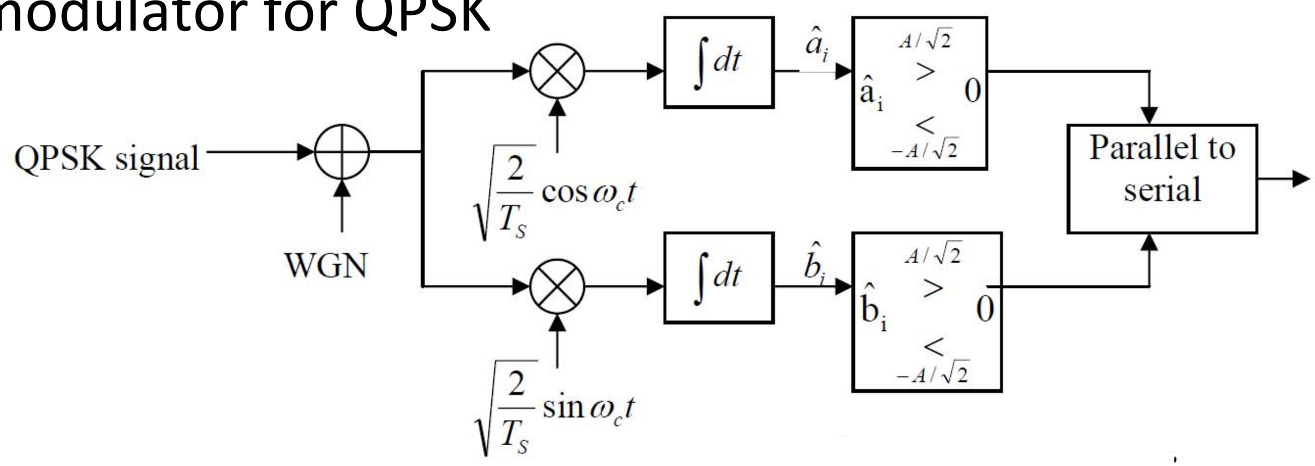
Quadrature Phase Shift Keying (QPSK)

- Signal constellation(with $a = 0$)



θ_i	a_i	b_i
$\pi/4$	$A/\sqrt{2}$	$A/\sqrt{2}$
$3\pi/4$	$-A/\sqrt{2}$	$A/\sqrt{2}$
$5\pi/4$	$A/\sqrt{2}$	$-A/\sqrt{2}$
$7\pi/4$	$-A/\sqrt{2}$	$-A/\sqrt{2}$

- Demodulator for QPSK



QPSK: Error probability

- Bit error probability on each branch is the same as for BPSK

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \Rightarrow P_s = 1 - (1 - P_b)^2, \quad E_s = 2E_b$$

$$\Rightarrow P_s = 1 - \left(1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right)^2 \approx 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

Error probability for MPSK

- Exact value
  - Error probability is the same for each signal by symmetry

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} \mathrm{P}(\theta)$$

$$= 1 - \int_{-\pi/M}^{\pi/M} \frac{1}{\pi} e^{-2\frac{E_s}{N_0}\sin^2(\theta)} \int_0^{\infty} z \exp\left[-\left(z - \sqrt{2\frac{E_s}{N_0}} \cos(\theta) \right)^2 \right] \mathrm{d}z$$

- Nearest neighbor approximation

$$P_s \approx M_{d_{\min}} Q\left(\frac{d_{\min}}{\sqrt{2N_0}} \right) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin(\pi/M) \right)$$

Quadrature Amplitude Modulation (M-QAM)

- Linear modulation, two-dimensional
- Information is encoded into both the amplitude and phase
  - Two degrees of freedom
  - More spectrally-efficient than MPAM and MPSK
  - Encode most number of bits per symbol for a given average energy
- Transmitted signal

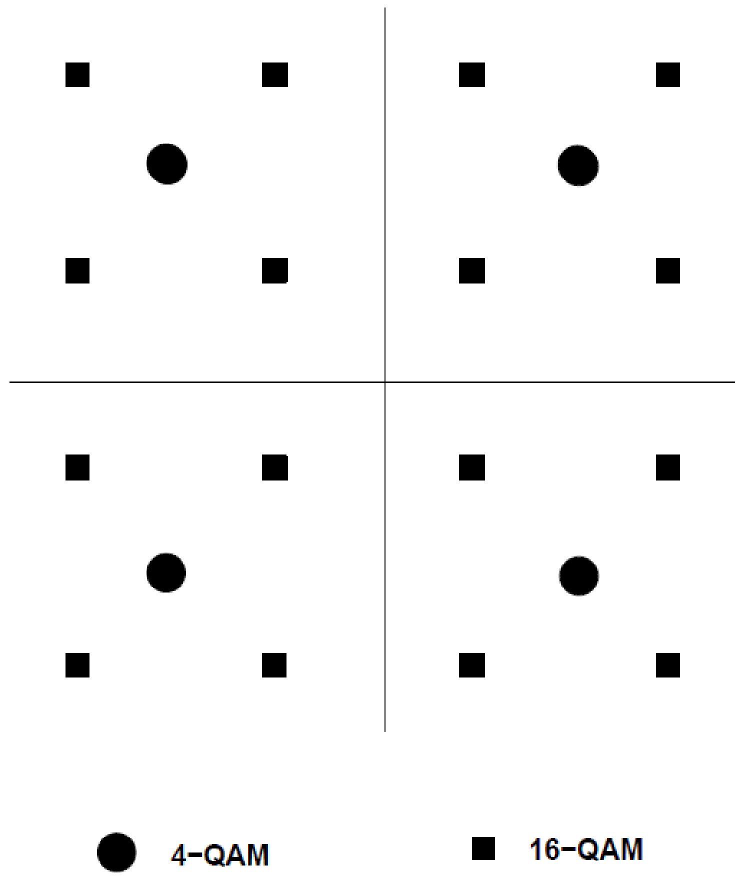
$$s_i(t) = A_i \cos(\theta_i) g(t) \cos(2\pi f_c t) - A_i \sin(\theta_i) g(t) \sin(2\pi f_c t)$$

$$\phi_1(t) = g(t)\cos(2\pi f_c t), \quad \phi_2(t) = -g(t)\sin(2\pi f_c t)$$

$$s_{i1} = A_i \cos(\theta_i), \qquad s_{i2} = A_i \sin(\theta_i)$$

Quadrature Amplitude Modulation (M-QAM)

- Square constellation



Quadrature Amplitude Modulation (M-QAM)

- Square constellation

$$M = L^2 = 2^{2l}, \quad s_{i1}, s_{i2} \in \{(2i - 1 - M)d, \; i = 1, 2, \cdots, M\}$$

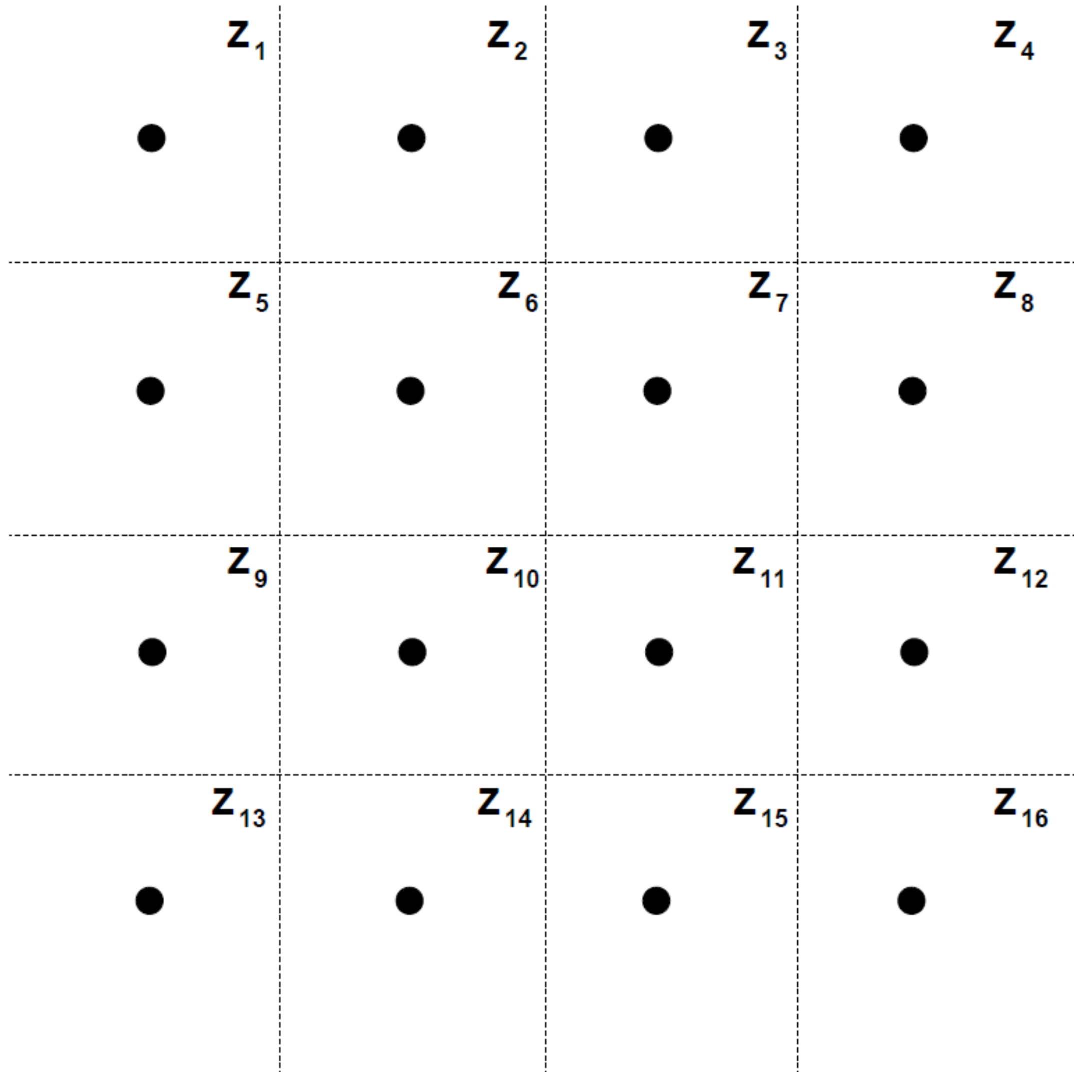
- Minimum distance $d_{\min} = 2d$

- Equivalent to PAM with size L on each of the in-phase and quadrature signal components

- Error probability

$$P_{s-QAM}(M) = 1 - (1 - P_{s-PAM}(L))^2$$
$$\approx 2P_{s-PAM}(L) \approx 4\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{E_s}{N_0}}\right)$$

16-QAM: Decision regions



Error Probability Approximation (Coherent demodulation)

$$P_s(\gamma_s) \approx \alpha_M Q\left(\sqrt{\beta_M \gamma_s}\right), \quad \gamma_s = \frac{E_s}{N_0}$$

Modulation	$P_s(\gamma_s)$	$P_b(\gamma_b)$
BFSK:		$P_b = Q\left(\sqrt{\gamma_b}\right)$
BPSK:		$P_b = Q\left(\sqrt{2\gamma_b}\right)$
QPSK,4QAM:	$P_s \approx 2\,Q\left(\sqrt{\gamma_s}\right)$	$P_b \approx Q\left(\sqrt{2\gamma_b}\right)$
MPAM:	$P_s \approx \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\overline{\gamma}_s}{M^2-1}}\right)$	$P_b \approx \frac{2(M-1)}{M\log_2 M} Q\left(\sqrt{\frac{6\overline{\gamma}_b \log_2 M}{(M^2-1)}}\right)$
MPSK:	$P_s \approx 2Q\left(\sqrt{2\gamma_s}\sin(\pi/M)\right)$	$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M}\sin(\pi/M)\right)$
Rectangular MQAM:	$P_s \approx \frac{4(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4(\sqrt{M}-1)}{\sqrt{M}\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{(M-1)}}\right)$
Nonrectangular MQAM:	$P_s \approx 4Q\left(\sqrt{\frac{3\overline{\gamma}_s}{M-1}}\right)$	$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\overline{\gamma}_b \log_2 M}{(M-1)}}\right)$

Performance of M-ary Digital Modulation in an AWGN Channel

binary, antipodal signaling: M = 2, $P_b = P_e = \frac{1}{2} erfc\left[\sqrt{\frac{E_b}{N_0}}\right]$

$E_S = (\log_2 M) E_b$

Union bound: $P_e \le \frac{1}{2}(M-1) erfc\left[\frac{d}{2\sqrt{N_0}}\right]$

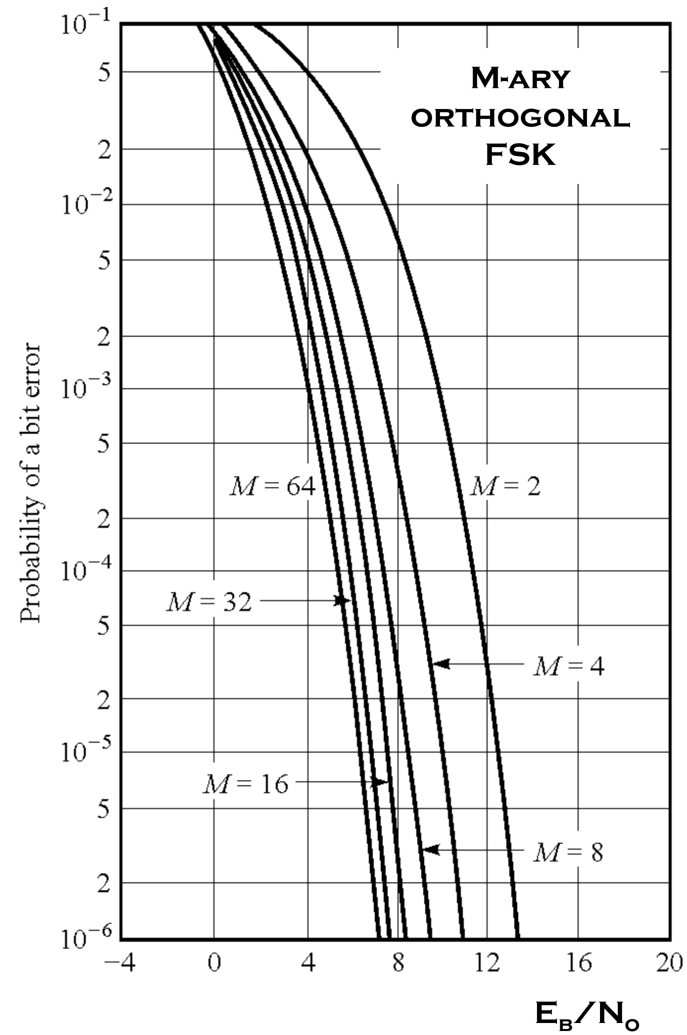
M-ary ASK: $P_e \approx erfc\sqrt{\frac{3}{M^2-1}\frac{E_S}{N_0}}, \, d = \sqrt{\frac{12}{(M^2-1)}E_S}$

M-ary PSK: $P_e \approx erfc\left[\sin\frac{\pi}{M}\sqrt{\frac{E_S}{N_0}}\right], d_{min} = \sqrt{E_S}.\sin \pi/M$

squared M-ary QAM: $P_{e,M-aryQAM} \approx 2P_{eASK} \approx 2\left(1-\frac{1}{\sqrt{M}}\right) erfc\left(\sqrt{\frac{3}{2(M-1)}\frac{E_S}{N_0}}\right)$

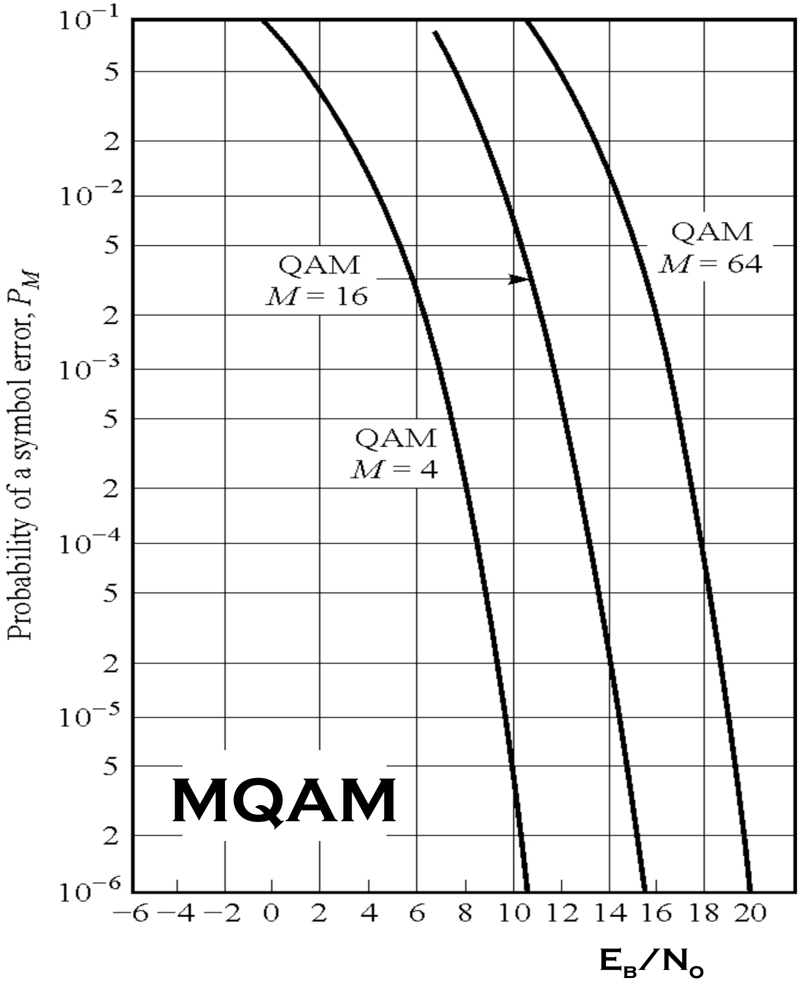
Orthogonal FSK: $P_e \le \frac{1}{2}(M-1) erfc\sqrt{\frac{E_S}{2N_0}}$

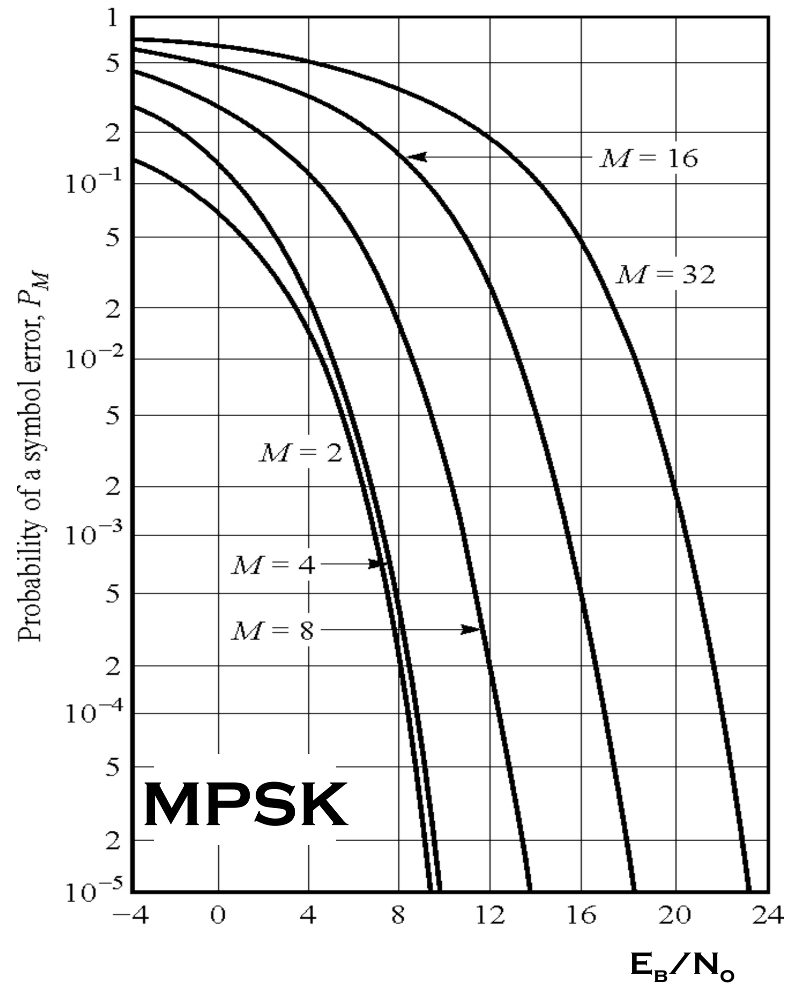
M-ary orthogonal FSK signaling schemes are power-efficient but not bandwidth-efficient.



Performance in AWGN: PROBABILITY OF SYMBOL ERROR

error probability decays exponentially in SNR in the AWGN channel



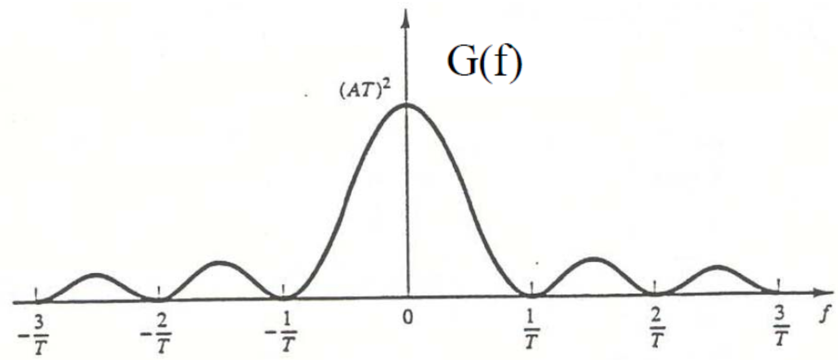


M-QAM, M-PSK: BW-efficient but not power-efficient
For M>8, M-QAM outperforms M-PSK

Time-limited Signaling

- Time-limited signaling ⇨ infinite bandwidth

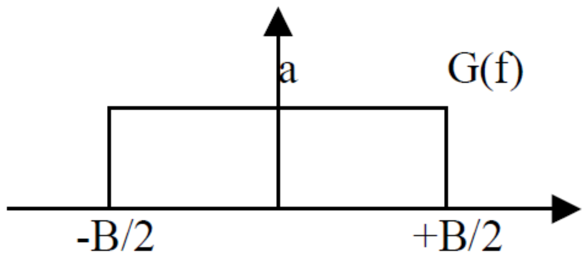
$$g(t) = \begin{cases} A & 0 \le t < T \\ 0 & \text{elsewhere} \end{cases}$$



- Band-limited transmission ⇨ limited bandwidth

⇨ $g(t)$ is **no longer** time-limited

$$g(t) = \frac{aB \sin(\pi B t)}{\pi B t}$$
$$= aB \operatorname{sinc}(\pi B t)$$



Power Spectra of Linearly Modulated Signals

- Passband signal $$s(t) = \Re\left\{u(t)e^{j(2\pi f_c t)}\right\}$$
- Complex transmitted signal for linear modulation

$$u(t) = \sum_{n=-\infty}^{+\infty} I_n g(t - nT)$$

- I_n : Random variable that represents transmitted symbol at n^{th} period
- I_n is real for M-PAM and complex for M-PAM, M-PSK, M-QAM

Power Spectra of Linearly Modulated Signals

- Mean of I_n $\quad E\{I_n\} = \mu_i$
- Autocorrelation function of I_n $\quad \Phi_{II}(m) = \frac{1}{2} E\{I_n^* I_{n+m}\}$
- Autocorrelation function of $u(t)$

$$\Phi_{uu}(t+\tau, t) = \frac{1}{2} E\{u^*(t) u(t+\tau)\}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} E\{I_n^* I_{m+n}\} g^*(t - nT) g(t + \tau - mT - nT)$$

$$= \sum_{m=-\infty}^{+\infty} \Phi_{II}(m) . \left[\sum_{n=-\infty}^{+\infty} g^*(t - nT) g(t + \tau - mT - nT) \right]$$

Power Spectra of Linearly Modulated Signals

- Autocorrelation function $\Phi_{uu}(t+\tau, t)$ is periodic in t with period T
- Mean of $u(t)$ is also periodic in t with period T

$$E\{u(t)\} = \sum_{n=-\infty}^{+\infty} E\{I_n\} g(t-nT) = \mu_i \sum_{n=-\infty}^{+\infty} g(t-nT)$$

$$\Downarrow$$

- $u(t)$ is **cyclostationary** (periodically stationary in wide sense)

Power Spectra of Linearly Modulated Signals

- Averaging over a single period to remove t

$$\overline{\Phi}_{uu}(\tau\tau = \frac{1}{T}\int_0^T \Phi_{uu}\left(t+\tau,t\right)dt$$

$$= \sum_{m=-\infty}^{+\infty} \Phi_{II}(m).\ \frac{1}{T}\sum_{n=-\infty}^{+\infty}\int_{-nT}^{-(n-1)T} g^*(u)g(u+\tau-mT)\,du$$

$$= \frac{1}{T}\sum_{m=-\infty}^{+\infty} \Phi_{II}(m)\int_{-\infty}^{+\infty} g^*(u)g(u+\tau-mT)du$$

- Time autocorrelation function of $g(t)$

$$\Phi_{gg}(\tau) = g(\tau) * g^*(-\tau) = \int_{-\infty}^{+\infty} g^*(\mathrm{t})\mathrm{g}(\mathrm{t}+\tau)\mathrm{d}\,\tau$$

$$\Phi_{gg}(f) = \left|G(f)\right|^2$$

Power Spectra of Linearly Modulated Signals

$$\overline{\Phi}_{uu}(\tau) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} \Phi_{II}(m)\Phi_{gg}(\tau - mT)$$

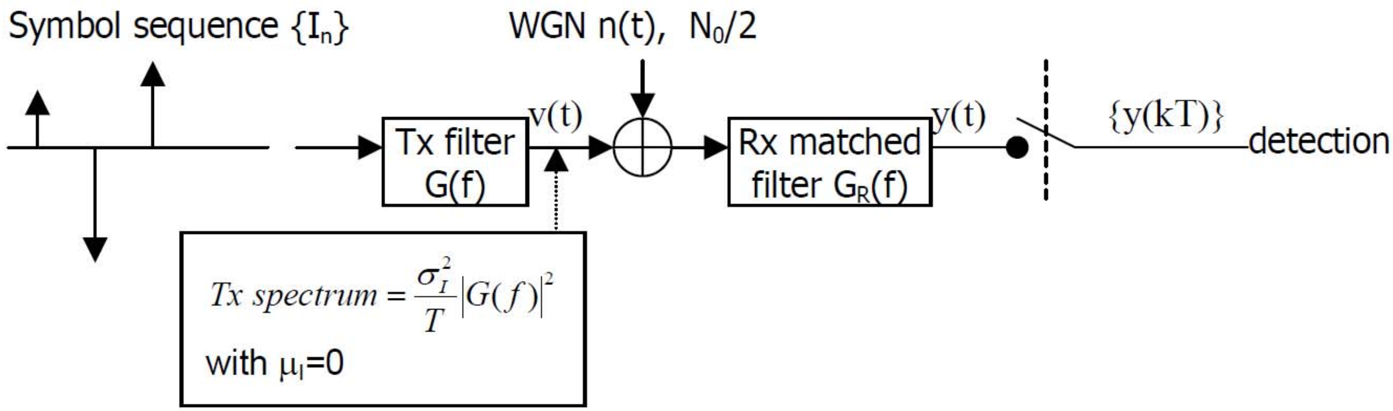
$$\Downarrow$$

$$\boxed{S_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2 \sum_{m=-\infty}^{+\infty} \Phi_{II}(m) e^{-j2\pi fmT}}$$

- For i.i.d. $\{I_n\}$, $\quad \Phi_{II}(m) = \begin{cases} \mu_i^2 & m \neq 0 \\ \mu_i^2 + \sigma_i^2 & m = 0 \end{cases}$

$$\boxed{S_{uu}(f) = \frac{\sigma_i^2}{T}\left|G(f)\right|^2 + \frac{\mu_i^2}{T^2}\sum_{m=-\infty}^{+\infty} \left|G\left(\frac{m}{T}\right)\right|^2 \delta\left(f - \frac{m}{T}\right)}$$

Signal Design for Band-limited Transmission



- Tx average signal power $P_{avg} = \mathrm{E}\left\{|u(t)|^2\right\} = \dfrac{\sigma_i^2}{T}\int_{-\infty}^{+\infty}|G(f)|^2\,\mathrm{d}f$
- Tx average symbol energy $E_s = T\,P_{avg} = \sigma_i^2\int_{-\infty}^{+\infty}|G(f)|^2\,\mathrm{d}f$
- Transmitted signal is convolved with channel impulse response and matched filter $h(t) = g(t) * c(t) * g_R(t)$
    - AWGN channel $h(t) = g(t) * g_R(t)$

Inter Symbol Interference (ISI)

- Rx filter output

$$y(t) = [u(t) + n(t)] * g_R(t) = \sum_{n=-\infty}^{+\infty} I_n h(t - nT) + n(t) * g_R(t)$$

$$y(kT) = \sum_{n=-\infty}^{+\infty} I_n h[(k-n)T] + n_k$$

$$= \underbrace{I_k h(0)}_{\text{main component}} + \boxed{\underbrace{\sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} I_{k-m} h(mT)}_{\text{ISI}}} + \underbrace{n_k}_{N(0,\sigma_n^2)}$$

- Time-limited signaling ⇨ no ISI

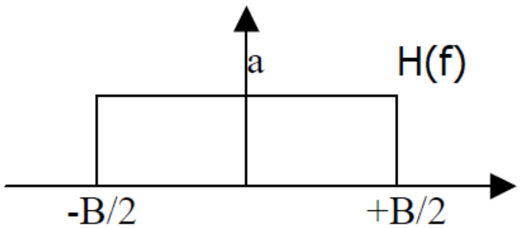
$$g_R(t) = g^*(-t) \quad \Rightarrow \quad h(t) = 0 \ \text{ for } \ t \notin (-T, T)$$

Band-limited Signaling with No ISI

- To have zero ISI,

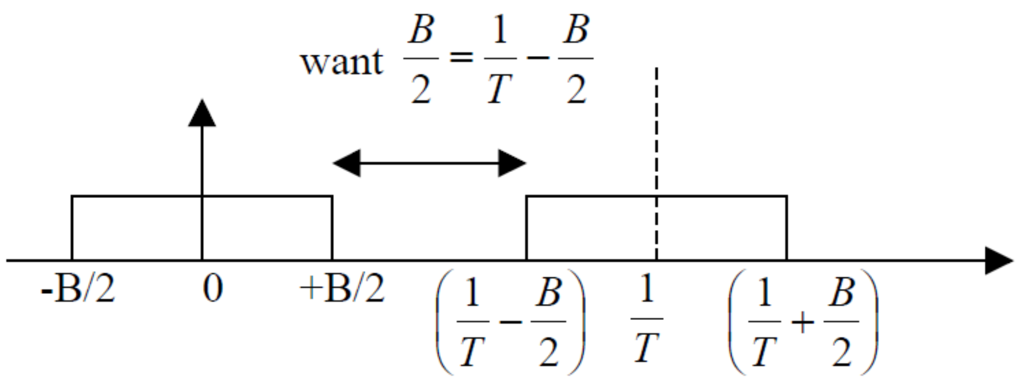
$$h(mT) = \begin{cases} h_0 & m = 0 \\ 0 & m \neq 0 \end{cases} \quad \Rightarrow \quad \sum_{k=-\infty}^{+\infty} H\left(f + \frac{k}{T}\right) = Th_0$$

- Ideal, strictly band-limited filter



- **Minimum BW** for zero ISI

$$\boxed{B = \frac{1}{T}}$$

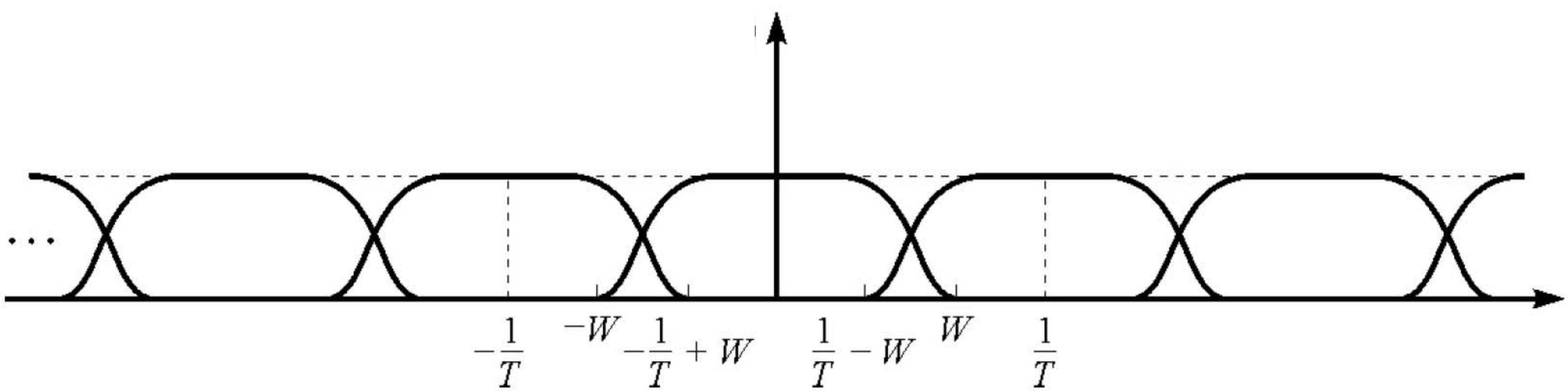


Band-limited Signaling with No ISI

- Ideal "brick-wall" filter with rectangular frequency response is not physically realizable

- For a physically realizable $H(f)$ with single-sided bandwidth

  W, **minimum required bandwidth** ⇨ $\boxed{B = 2W > \frac{1}{T}}$



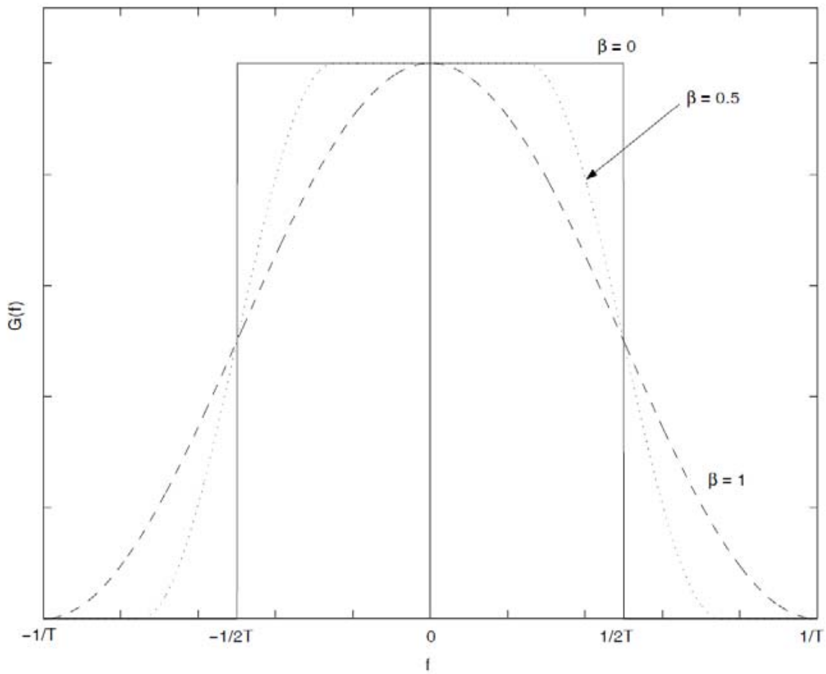
Band-limited Signaling with No ISI

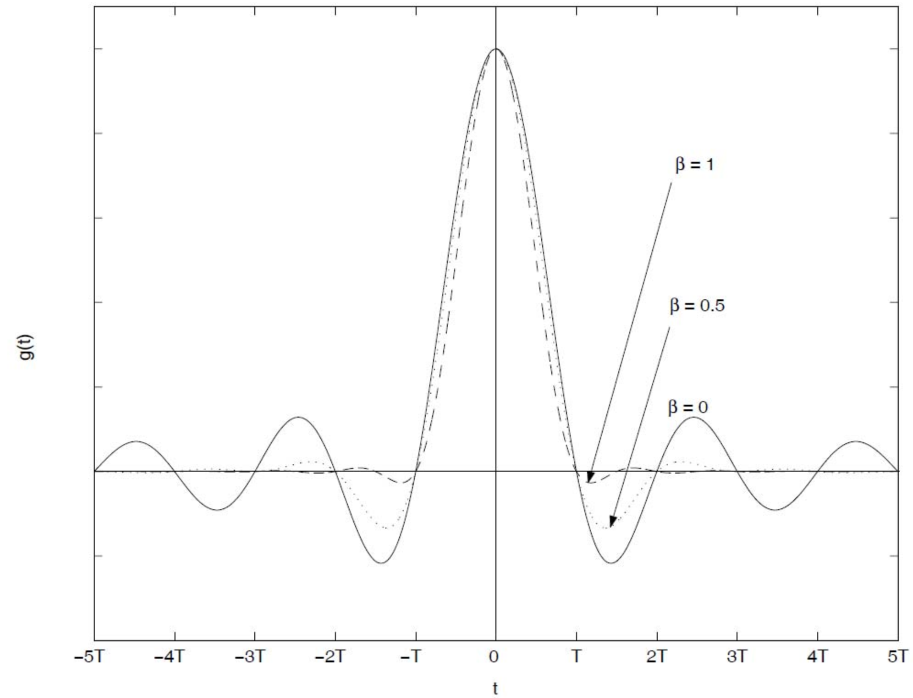
- Raised-cosine (RC) filter (widely used in practice)

$$R(f,\beta)=\begin{cases} T & 0\le |f|\le \dfrac{1-\beta}{2T} \\ \dfrac{T}{2}\left\{1+\cos\left[\dfrac{\pi T}{\beta}\left(|f|-\dfrac{1-\beta}{2T}\right)\right]\right\} & \dfrac{1-\beta}{2T}\le |f|\le \dfrac{1+\beta}{2T} \\ 0 & |f|>\dfrac{1+\beta}{2T} \end{cases}$$

- β is the roll-off factor, which determines rate of spectral roll-off
- In time domain, $r(t,\beta)=\dfrac{\cos(\pi\beta t/T)}{1-(2\beta t/T)^2}\,\mathrm{sinc}(\pi t/T)$

Raised-cosine (RC) filter





Design of Tx and Rx Filters: RRC

- For zero ISI, $h(t) = g(t) * g_R(t) \;\Rightarrow\; H(f) = G(f)G_R(f) = R(f,\beta)$

- Tx/Rx use square-root raised-cosine (RRC) filter:

$$\boxed{G(f) = \sqrt{R(f,\beta)} \quad \text{and} \quad G_R(f) = \sqrt{R(f,\beta)}}$$

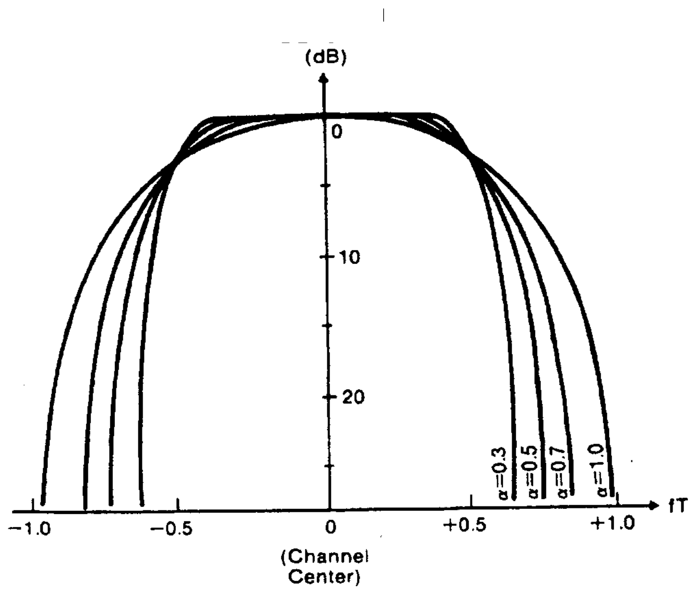
- Tx average signal power

$$P_{avg} = \frac{\sigma_i^2}{T}\int_{-\infty}^{+\infty} |G(f)|^2 \,\mathrm{d}f = \frac{\sigma_i^2}{T}\int_{-\infty}^{+\infty} R(f,\beta)\,\mathrm{d}f = \frac{\sigma_i^2}{T}$$

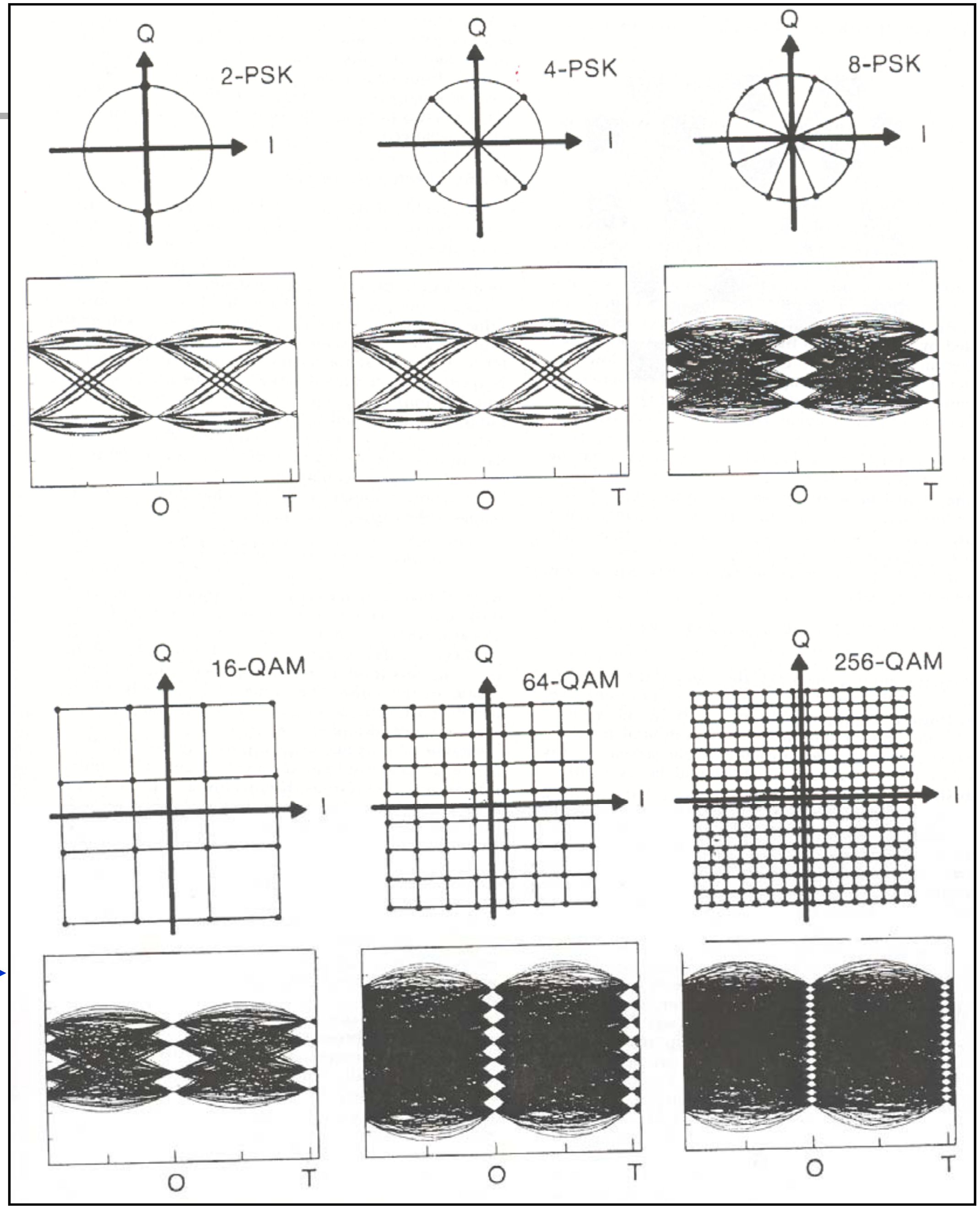
- Noise power

$$\sigma_n^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |G_R(f)|^2 \,\mathrm{d}f = \frac{N_0}{2}\int_{-\infty}^{+\infty} R(f,\beta)\,\mathrm{d}f = \frac{N_0}{2}$$

PSD & eye diagrams with RRC filters



Tx Power Spectrum with RRC



Eye diagrams with RC:

(from T. Noguchi, Y. Daido, J.A. Nossek, "Modulation Techniques for Microwave Digital Radio", *IEEE Communications Magazine*, October 1986, pp. 21-30)

Performance in a Frequency-Flat Fading Channel

- Received signal power randomly varies in a (frequency-flat) fading environment
- Instantaneous received SNR, γ_s, is a random variable with pdf $p_{\gamma_s}(\gamma) \rightarrow$ Error probability, $P_s(\gamma_s)$, is also *random*
- For *fast* fading with relatively short coherence time, $T_c \approx T_s$, *each* symbol is assumed to experience *iid* fading, **average error probability** can be averaged over γ_s.
- *Interleaving* can be used to achieve the *iid* fading assumption at the cost of long delay. Forward error coding can be applied to improve the average error probability performance.
- However, for *slow* fading with $T_c \gg T_s$, *low* γ_s can last for a *long period* $\rightarrow$ **outage probability**, the probability that γ_s falls below a threshold γ_0

Performance in Fading Channels

Outage probability: $P_{out} = \mathrm{P}(\gamma_s < \gamma_0) = \int_0^{\gamma_0} p_{\gamma_s}(\gamma)\, d\gamma$

where γ_0: required minimum SNR to achieve a target *highest allowable error probability*

- In **Rayleigh** fading, $P_{out} = \int_0^{\gamma_0} \frac{1}{\bar{\gamma}_s} e^{-\frac{\gamma_s}{\bar{\gamma}_s}} d\gamma_s = 1 - e^{-\frac{\gamma_0}{\bar{\gamma}_s}}$

Average error probability: $\overline{P_s} = \int_0^{\infty} P_s(\gamma)\, p_{\gamma_s}(\gamma)\, d\gamma$

- In **Rayleigh** fading,
  - BPSK: bit=symbol→ γ_b=γ_s $\rightarrow P_b = Q(2\gamma_b) \quad \Rightarrow \quad \overline{P_b} = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_b}{1+\bar{\gamma}_b}}\right] \approx \frac{1}{4\bar{\gamma}_b}$
  - Generally, if the *instantaneous* symbol error is:

$$P_s \approx \alpha_M Q(\beta_M \gamma) \Rightarrow \overline{P_s} = \frac{\alpha_M}{2}\left[1 - \sqrt{\frac{.5\beta_M \bar{\gamma}_s}{1 + .5\beta_M \bar{\gamma}_s}}\right] \approx \frac{\alpha_M}{2\beta_M \bar{\gamma}_b}$$

EXAMPLE OF BPSK PERFORMANCE

AWGN channel:

$r[k] = x[k] + w[k]$ where $x[m] = \pm a$ with prob. of 1/2,

$E_b = a^2 T_b$ and $w[m]$: Gaussian $(0, N_o/2)$

$$P_e = Q\left(\sqrt{2E_b/N_o}\right) \approx 0.5e^{-E_b/N_o}$$

- Error probability decays exponentially with SNR

Rayleigh Flat-Fading Channel:

$r[k] = |h[k]| x[k] + w[k]$

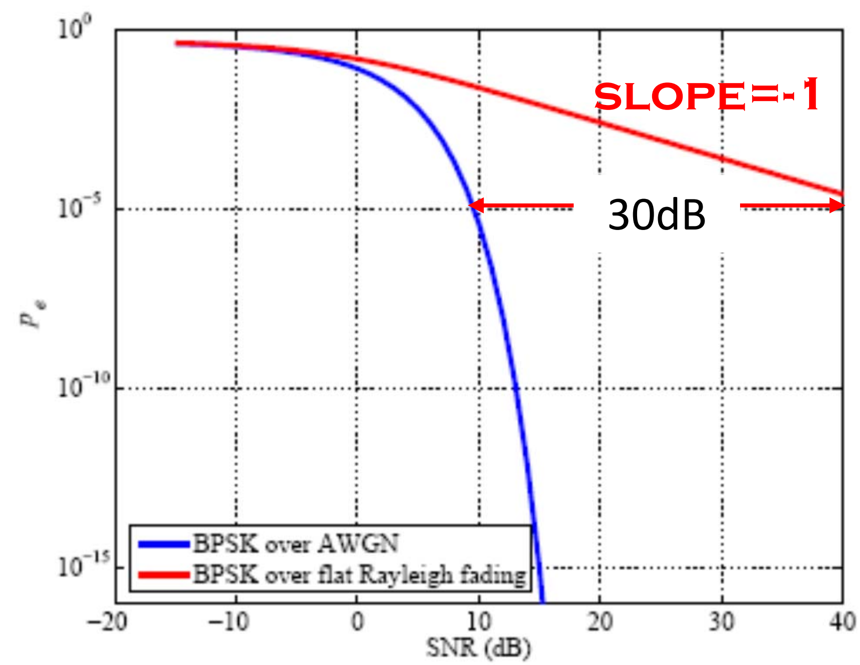
$|h[k]|$: Rayleigh with $\sigma^2 = 1$,

$$P_{s\|h[k]\|} = Q\left(|h[k]|\sqrt{2E_b/N_o}\right)$$

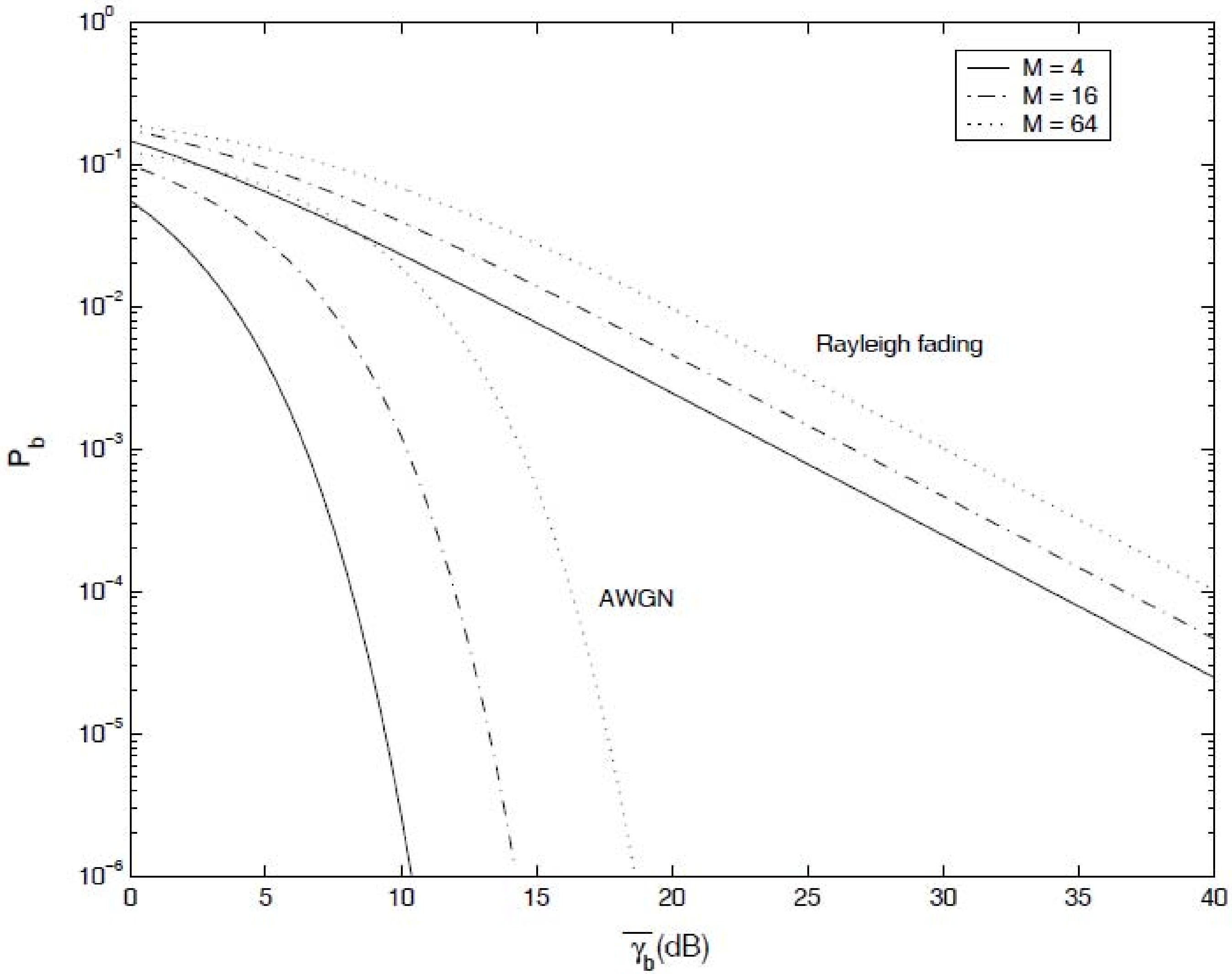
$$\overline{P}_s = \frac{1}{2}\left(1 - \sqrt{\frac{[E_b/N_o]}{1+[E_b/N_o]}}\right) \approx \frac{1}{4[E_b/N_o]}$$

Fading channel can be in deep fade: high probability |h[k]| is small

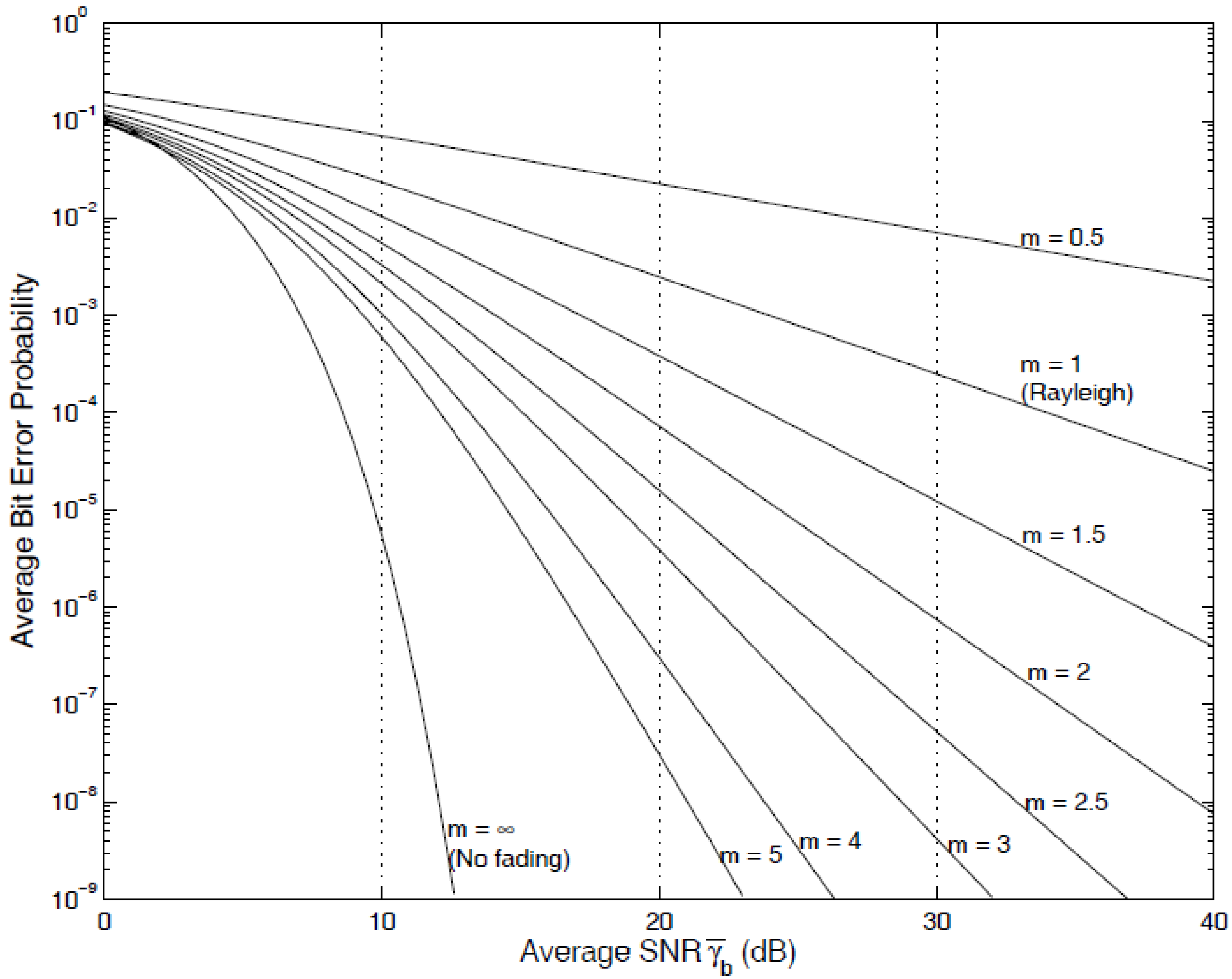
- average error probability decays only inversely with SNR



Average bit error probability for MQAM in Rayleigh fading



Average bit error probability for BPSK in Nakagami fading



poor performance over fading channels: why?

- poor performance over fading channels mainly due to the randomness of channel gain $|h|^2$:
  - When the *instantaneous* received SNR $|h|^2$SNR » 1, P_e is very small (good), since the tail of Q-function decays rapidly (exponential).
  - However, when $|h|^2$SNR « 1 (i.e., channel in deep fades), *separation* between signal points is of the same order as the standard deviation of noise, P_e becomes *significantly large* (i.e., bad and dominant)
- The probability of this deep fade (i.e., $|h|^2 < 1/SNR$) event:

$$Pr\{|h|^2 < 1/SNR\} = \int_0^{1/SNR} e^{-x}dx = \frac{1}{SNR} + O(\frac{1}{SNR^2}) \approx \frac{1}{SNR}$$

- At high (average) SNR, error events occur most often because the channel is in *deep fade*, and *not* because of *large* noise.

  → increasing (average) SNR is *not* effective

  → reducing *deep fade* by using *diversity* is more effective.

References

- A. Goldsmith, *Wireless Communications,* Cambridge University Press, 2005, Chapters 5 and 6
- J.G. Proakis, Digital Communications, 4th Ed, McGraw-Hill, 2001
- Materials from various sources