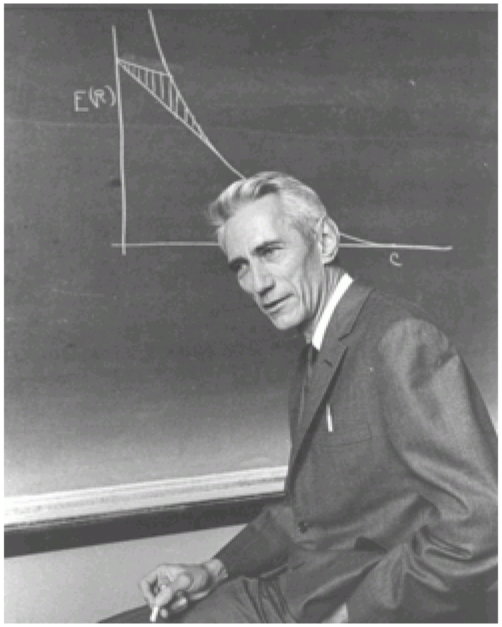
Claude E. Shannon (1916-2001),

- *A Mathematical Theory of Communication*, Bell System Technical Journal, Vol. 27, pp. 379–423, 623–656, 1948



# Channel Capacity: A Review

# Outline

- Introduction
- Capacity and Achievable Rates
- Entropy and Mutual Information
- Capacity of AWGN channels
- Capacity of Frequency-Flat Fading Channels
- Capacity of Frequency-Selective Fading Channels

# Capacity and Achievable Rates

- Before Shannon, reliable communication over noisy channel

Repetition coding

101 → 111110000011111

(info) (rate approaches zero)

- **Reliable communication**: It is possible to communicate at a *strictly positive* transmission rate with an *arbitrary small error probability*
- Maximum rate for which this is possible is called **the capacity of the channel**
- Any rate below the capacity is an **achievable rate**
- It is *impossible* to drive error probability to zero for rates *higher than capacity*

# Entropy and Mutual Information

- Consider a discrete RV with PMF $P_X(\cdot)$ taking on values from set $\mathcal{X}$
- Define **entropy**

$$H(X) = -\sum_{x\in\mathcal{X}} P_X(x)\log(P_X(x)) = -\mathbb{E}[\log(P_X(X))]$$

- Self-information of event X=x: $I(x) = -\log(P_X(x))$
- Entropy expected value of self-info: average amount of information produced per symbol by discrete source
- Also, entropy is a measure of uncertainty of RV: number of bits on average to describe RV
- Example: Bernoulli RV with prob. p

$$H(p) = -p\log(p) - (1-p)\log(1-p)$$

- <u>Note</u>: All logs base 2

# Entropy and Mutual Information

- It can be shown that $0 \leq H(X) \leq \log|\mathcal{X}|$
  - Zero when event is certain $\exists x \in \mathcal{X} \text{ s.t. } P_X(x) = 1$
  - Max when equiprobable $P_X(x) = 1/|\mathcal{X}|, \forall x \in \mathcal{X}$
- Consider now 2 RVs X and Y
- Entropy of RV Y conditioned on a given realization of X

$$H(Y|X = x_0) = -\sum_{y \in \mathcal{Y}} P_{Y|X}(y|x_0) \log(P_{Y|X}(y|x_0))$$

- Entropy of RV Y conditioned on RV X: **conditional entropy**

$$H(Y|X) = \mathbb{E}_X[H(Y|X = x)] = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_{X,Y}(x,y) \log(P_{Y|X}(y|x))$$

- Average amount of uncertainty in Y after observing X

# Entropy and Mutual Information

- The reduction of uncertainty of one RV due to another is called **mutual information**

$$I(X,Y) = I(Y,X) = H(Y) - H(Y|X) = H(X) - H(X|Y)$$

- Consider a communication channel with input X, output Y
- Output depends probabilistically on input with transition probability P(Y|X), i.e., discrete memoryless channel

$$X \rightarrow \boxed{P(Y|X)} \rightarrow Y$$

- **Capacity** of the DMC

$$C = \max_{P_X(\cdot)} I(X,Y)$$

# Entropy and Mutual Information

- Examples:

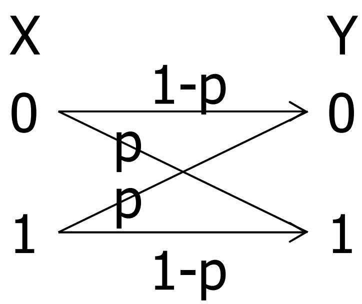
1.-Noiseless binary channel

| X | | Y |
|---|---|---|
| 0 | $\longrightarrow$ | 0 |
| 1 | $\longrightarrow$ | 1 |

$$I(X,Y) = H(Y) - \cancelto{0}{H(Y|X)}$$
$$= H(Y) \le 1$$

$C = 1$ achieved by equiprobable inputs

2.-Binary symmetric channel




$$I(X,Y) = H(Y) - H(Y|X)$$
$$= H(Y) - H(p)$$
$$\le 1 - H(p)$$

$C = 1 - H(p)$

achieved by equiprobable inputs

# Entropy and Mutual Information: Continuous sources

- How about for cts RVs with PDF $f_X(x)$ ?
- **Differential entropy**

$$h(X) = -\int f_X(u) \log(f_X(u))\, du$$

- Example: Gaussian RV $w \sim \mathcal{N}(\mu, \sigma^2)$

$$h(w) = [1/2] \cdot \log(2\pi e \sigma^2)$$

- Conditional entropy and mutual info defined similarly

$$h(Y|X) = -\int f_{X,Y}(u,v) \log(f_{Y|X}(v,u))\, dudv$$

$$I(X,Y) = h(Y) - h(Y|X)$$

## Entropy and Mutual Information: Continuous sources

- For cts RV with second moment constraint $\mathbb{E}[x^2] \leq \sigma^2$

$$h(x) \leq [1/2] \cdot \log(2\pi e \sigma^2)$$

  with equality when $x \sim \mathcal{N}(0, \sigma^2)$
- Gaussian RVs are entropy maximizers for variance constraint

## Capacity of AWGN channels

- Complex AWGN channel

$$y = hx + n$$

  - Complex input *x*, output *y*, noise $n \sim \mathcal{CN}(0, N_0)$
  - Deterministic, time-invariant channel $h = 1$
- *Power-constrained* capacity

$$C = \max_{f_X(x)} \quad I(x, y) \quad \text{s.t.} \quad \mathbb{E}[|x|^2] \leq P$$

  - Mutual information between input and output $I(x, y)$
  - Optimization over all PDFs $f_X(x)$ satisfying power constraint

$$\int |u|^2 f_X(u)\, du \leq P$$

## Capacity of AWGN channels

$$\begin{aligned} I(x,y) &= h(y) - h(y|x) \\ &= h(y) - h(n) \\ &= h(y) - \log(\pi e N_0) \\ &\leq \log(\pi e (P + N_0)) - \log(\pi e N_0) \end{aligned}$$

- Achieved with equality when y is circular symmetric Gaussian RV

$$y \sim \mathcal{CN}(0, P + N_0)$$

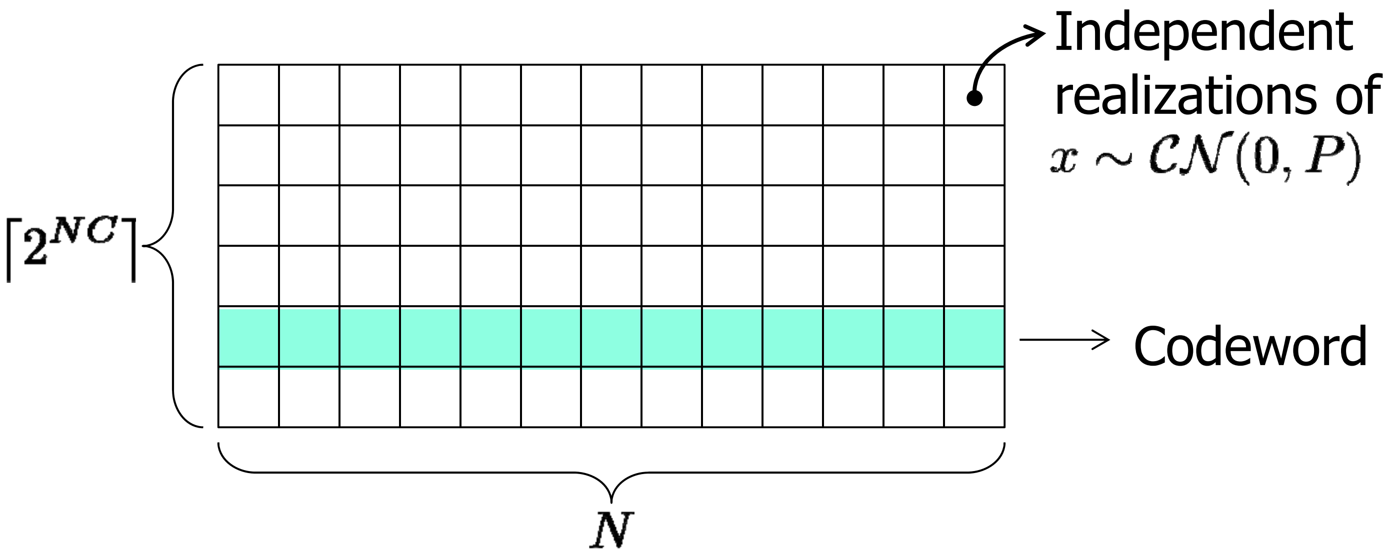
- Circular symmetric Gaussian RV maximizes differential entropy for second moment constraint $\mathbb{E}[|y|^2] \leq P + N_0$
- Therefore

$$C = \log(\pi e (P + N_0)) - \log(\pi e N_0) = \log\left(1 + \frac{P}{N_0}\right)$$

- Note: still need to prove achievability and converse

# Capacity of AWGN channels

- Given $y \sim \mathcal{CN}(0, P + N_0)$, then $x \sim \mathcal{CN}(0, P)$
- Capacity achieved by transmitting with full power a randomly selected codeword from an i.i.d. **Gaussian codebook**




- Since input x is independent, does it mean coding is not needed?
  - No, asymptotically independent for large N
  - Coding is crucial to approach capacity

# Capacity of AWGN channels

- Note: Capacity is a *channel characteristic* (independent of Tx/Rx technique or limitation)
- For practical systems with finite constellations (QPSK,QAM)
  - Need to find codes that approach the capacity
  - Codes long enough to average noise
  - Code should be easily encodable/decodable
- Advanced techniques (turbo, LDPC) perform close to capacity (capacity approaching codes)

# Capacity of AWGN Band-limited channels

- Consider cts-time AWGN with bandwidth W

$$y(t) = x(t) + n(t)$$

- Discrete-time complex baseband channel (sampled at Nyquist 1/W)

$$y_i = x_i + n_i$$

where $a_i = a(i/W), \quad n_i \sim \mathcal{CN}(0, N_0 W)$

- Capacity:

$$C = \log\left(1 + \frac{P}{WN_0}\right) \text{ [bits/symbol]}, \qquad \log(.): \log_2(.)$$

- Maximum symbol rate: W symbols/second in W [Hz]= 1 symbol/s/Hz
  - Why? Nyquist criterion for band-limited signals (no ISI)

# Capacity of AWGN Band-limited channels

- Capacity/unit time: $$C = W \log\left(1 + \frac{P}{WN_0}\right)$$ [b/s]
- Spectral efficiency: $$C/W = \log\left(1 + \frac{P}{WN_0}\right)$$ [b/s/Hz]
- Key resources: Power and bandwidth
- Power in term of SNR=P/($N_o$W)=x:

$$\log(1+x) \approx x/\ln(2), \quad x \approx 0$$
$$\log(1+x) \approx \log(x), \quad x \gg 1$$

  - For low SNR (≈0), C increases *linearly* with P (twice P → twice C)
  - For high SNR (>>1), C increases *logarithmically* (twice P → C+1)

# Capacity of AWGN Band-limited channels

- C can be shown to be increasing/concave function of W
  - When W small, SNR large: Linear increase in W compensates for log decrease in SNR. Increasing W yield rapid increase in C (**bandwidth-limited regime**)
  - When W large, SNR small: Increase in W cannot compensate for log decrease in SNR. Increasing W has small impact on C (**power-limited regime**)

$$W \log\left(1 + \frac{P}{WN_0}\right) \approx W\left(\frac{P}{WN_0 \ln(2)}\right) = \frac{P}{N_0 \ln(2)}$$

  - When W in infinite $\lim_{W\to\infty} W \log\left(1 + \frac{P}{WN_0}\right) \to \frac{P}{N_0 \ln(2)}$

# Capacity of AWGN Band-limited channels

$$y(k) = x(k) + n(k),\ n(k): AWGN$$

$$\text{Capacity: } \left(\frac{C}{B}\right) = \log_2\left(1 + \frac{E_b}{N_0}\left(\frac{C}{B}\right)\right)\ b/s/Hz,\ \frac{E_b}{N_0}\left(\frac{C}{B}\right) = \frac{P}{N_0 B} = SNR$$

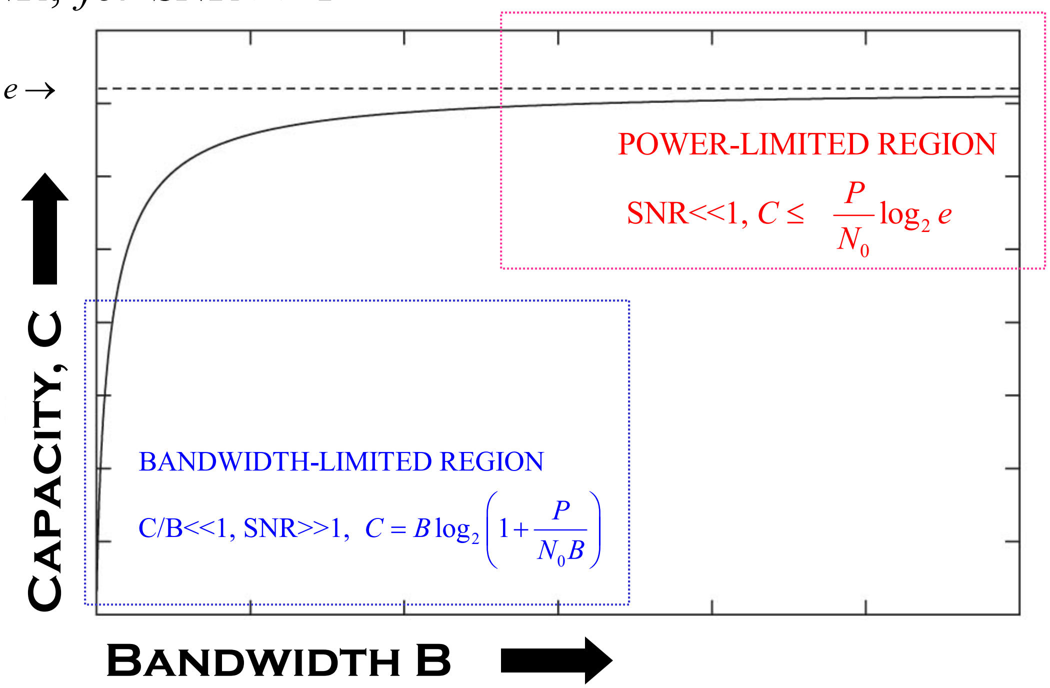
$$\Rightarrow C = B\log_2(1+SNR) \approx \begin{cases} \dfrac{P}{N_0}\log_2 e, & for\ SNR \approx 0 \\ B\log_2 SNR, & for\ SNR >> 1 \end{cases}$$

$$C \le \frac{P}{N_0}\log_2 e \rightarrow$$

CAPACITY IN

- POWER-LIMITED REGION:
  - linear in power,
  - insensitive to bandwidth: Capacity is finite even if W is not

- BANDWIDTH-LIMITED REGION:
  - logarithmic in power,
  - approximately linear in bandwidth



# Capacity of AWGN Band-limited channels

- How about minimizing energy per bit $E_b = P/C$
- Operate at most power-efficient regime
- Minimum bit SNR required for reliable communications

$$\left(\frac{E_b}{N_0}\right)_{\min} = \lim_{P \to 0} \frac{P}{C N_0} = \ln(2) = -1.59 \text{ dB}$$

# Capacity of Frequency-Flat Fading Channels

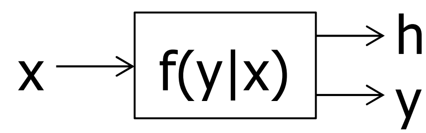
- Consider flat-fading channel

$$y = hx + n$$

  where noise $n \sim \mathcal{CN}(0, N_0)$
- Without loss of generality, fading process $\mathbb{E}[|h|^2] = 1$
- Capacity depends on how fast channel changes
  - **Fast fading**: Codeword spans several coherence periods
  - **Slow fading**: Codeword spans a single coherence interval
- Depends also on that is known about the channel (**CSI**)
  - Channel distribution known at Tx/Rx (hard)
  - Rx/Tx know distribution, and Rx knows channel gain (**CSIR**)
  - Rx/Tx know distribution and channel gain (**CSIT**)

# Capacity of Frequency-Flat Fading Channels:
## Fast Fading – CSIR

- Block fading: h constant for $T_c$ symbol periods and changes after
- **CSIR**: Rx tracks channel value
- Channel can be modeled as having two outputs




- Capacity

$$C = \max_{f_X(x)} \quad I(x, \{y, h\}) \quad \text{s.t.} \quad \mathbb{E}[|x|^2] \le P$$

- From chain rule of mutual info and from independence of x and h

$$I(x, \{y, h\}) = I(x, h) + I(x, y|h) = I(x, y|h)$$

where conditional mutual info defined similar to conditional entropy

$$I(x, y|h) = h(y|h) - h(y|x, h) = \mathbb{E}_h[\underbrace{I(x, y|h = \mathrm{h})}_{\text{Mutual info btw input \& output conditioned on realization of h}}]$$

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIR

- Condition on a realization of h, channel is AWGN
  - Gaussian inputs with full power optimal

$$I(x, y | h = \mathrm{h}) = \log\left(1 + \frac{P|\mathrm{h}|^2}{N_0}\right)$$

- CSIR capacity

$$C = \mathbb{E}_h\left[\log\left(1 + \frac{P|h|^2}{N_0}\right)\right]$$

- This is known as **ergodic capacity**: codeword must be long enough to capture ergodicity of channel

# Fast Flat-Fading Channel: Fading Known at Receiver only

$y(k) = hx(k) + n(k), n(k): AWGN,\quad a = |h|^2$ : power fading with pdf $p(a)$ , $E\{a\} = 1$

instantaneous: $P_s(a) \approx A_M Q\left(\sqrt{B_M^2(aSNR)}\right), C(a) = \log_2(1 + aSNR)$ b/s/Hz,

For **Fast Fading**, $T_{symbol} \approx T_{coherence}$ or $> T_{coherence}$

- introduced random phase can remove correlation between symbol phases, and hence leads to an irreducible error floor for differential modulation/demodulation.
- For coherent demodulation, if fading is known by receiver only, taking average over many independent fades $h$, we have

Average (symbol) error probability: $\overline{P}_s = \int_0^\infty P_s(a)p(a)da$

$$\text{ergodic:} C = E\{C(a)\} = \int_0^\infty \log_2(1 + aSNR)\,p(a)da \le \log_2(1 + SNR)$$

i.e., If **Fading Known at Receiver Only,** at best, ergodic C approaches $C_{AWGN}$

(using Jensen's inequality, i.e., $E\{f(u)]\} \le f(E\{u\})$ for a strictly concave $f(u)$)

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIR

- Define average received

$$\mathrm{SNR} = \mathbb{E}_h[P|h|^2/N_0] = P/N_0$$

- How to achieve this capacity?

  1.- Use **single-rate** Gaussian codebook with rate=C over multiple h

  2.- Use **multi-rate** Gaussian codebook: for each fading state,

  rate= $\log(1+|h|^2\mathrm{SNR})$ (need CSIT, can do better)

- How does it compare to AWGN?
  - From Jensen's inequality $\mathbb{E}[f(u)] \leq f(\mathbb{E}[u])$ for strictly concave f

$$C = \mathbb{E}_h[\log(1+|h|^2\mathrm{SNR})] \leq \log(1+\mathrm{SNR}) = C_{\mathbf{awgn}}$$

  - Equality when h is deterministic
  - Peaks cannot compensate for valleys

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIR

- Low SNR: fading approaches AWGN

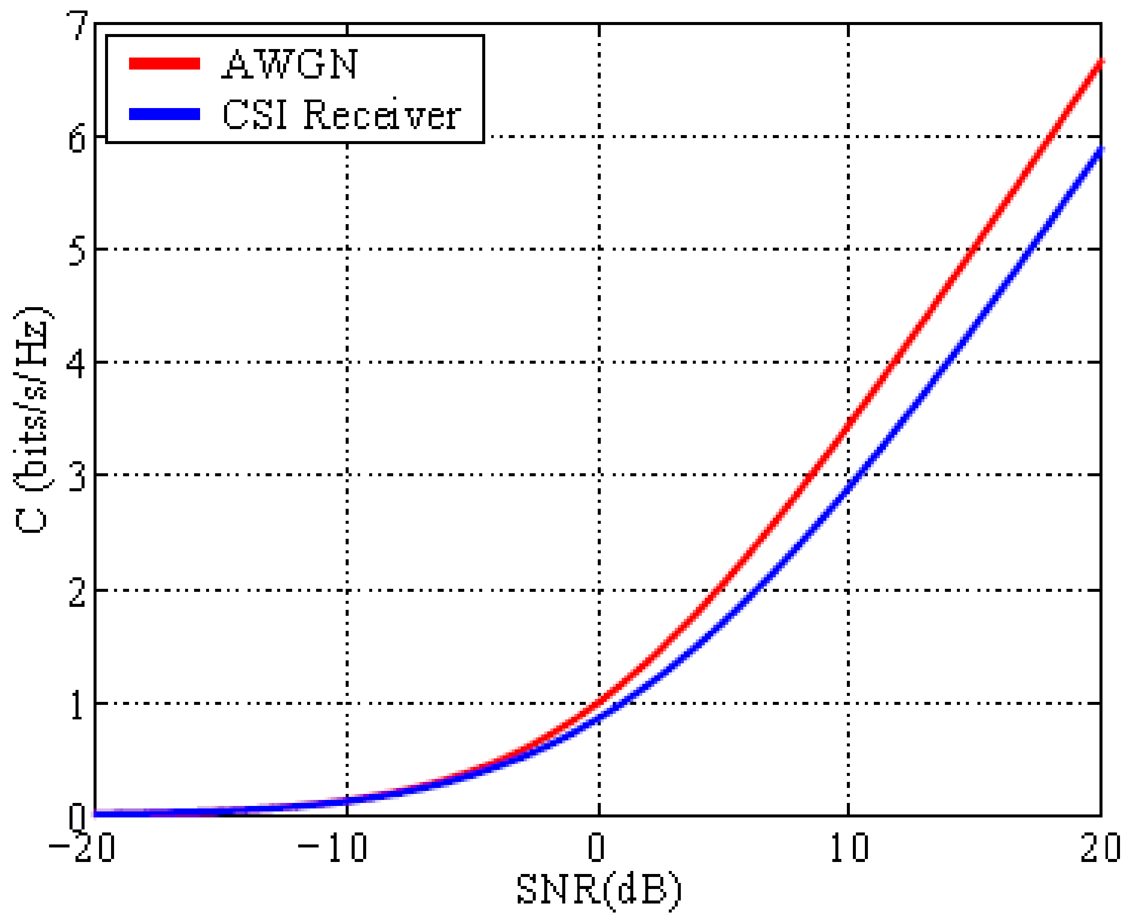
$$C = \mathbb{E}[\log(1 + |h|^2 \mathrm{SNR})] \approx \mathbb{E}\left[\frac{|h|^2 \mathrm{SNR}}{\ln(2)}\right] = \frac{\mathrm{SNR}}{\ln(2)} \approx C_{\mathbf{awgn}}$$

- High SNR: constant difference

$$\begin{aligned} C &\approx \mathbb{E}[\log(|h|^2 \mathrm{SNR})] \\ &= \log(\mathrm{SNR}) + \mathbb{E}[\log(|h|^2)] \\ &\approx C_{\mathbf{awgn}} + \mathbb{E}[\log(|h|^2)] \end{aligned}$$

  - For Rayleigh, $-0.83\ \mathrm{b/s/Hz}$ (around 2.5dB)

# Fast Fading: Capacity if Channel is Known at Receiver only




- If fading is known at receiver only, at best, $C_{CSIR}$ approaches $C_{AWGN}$

# Capacity of Flat-Fading Channels

- If fading is also known at transmitter, can we do better?
- Consider a family of codes, one for each possible fading state *a,* and build a *variable-rate* coding scheme that *adaptively* selects code with appropriate rate depending on *a*:
  - We achieve log(1 + SNR*a*) for *each state* with fixed SNR
  - Therefore, at least we can achieve C = E[log(1+SNR*a*)].
  - Can we do better?
- We can actually do better by adapting the transmitted power for each state A(a)SNR to achieve

$$\text{instantaneous: } C\left[A(a)\right] = \log_2\left[1 + aA(a)SNR\right]$$

$$\Rightarrow \text{ergodic:} C_A = \int_0^\infty \log_2\left[1 + aA(a)SNR\right] p(a)da$$

- can have a capacity greater than that of the AWGN channel, i.e., fading can provide more opportunities for performance enhancement in an opportunistic communication approach.
- How to choose A(a)?

## Adaptive Channel Inversion ?

- $P$: average Tx power, $B$: Tx bandwidth, E{$a$}=1 for avg Rx SNR= E{$a$}$P/BN_o$=$P/BN_o$

**Channel Inversion:** With $a$ know, find an appropriate constant $\alpha$ and set $A(a)=\alpha/a$ to keep the instantaneous Rx SNR=$A(a)[aP/(BN_o)]$= $\mathrm{SNR}_{ZO}$ (=$\alpha P/(BN_o)$: constant) for zero outage under average Tx power constraint.

instantaneous Tx power: $A(a)P=\alpha P/a \;\rightarrow$ zero-outage:$C_{ZO}=C[A(a)]=\log_2\left[1+SNR_{ZO}\right]$

but with Tx power constraint:$\int_0^{\infty}[\alpha P/a]p(a)da = P \rightarrow \alpha = \left[E\{1/a\}\right]^{-1}$

$\Rightarrow$ ergodic, zero-outage:$C_{ZO}=\log_2\left[1+SNR_{ZO}\right]$ with $SNR_{ZO}=\dfrac{P/(BN_o)}{E\{1/a\}}$

for Rayleigh fading channel: $E\{a^{-1}\}=\int_{a_0}^{\infty} a^{-1}e^{-a}da \approx e^{-a_0}\ln(1+a_0^{-1}) \rightarrow \infty$ as $a_0 \rightarrow 0 \Rightarrow C_{ZO}=0$

- simplifies design (i.e., fixed rate at all channel states) but is power-inefficient since for very small $a$, *$A(a)=\alpha/a$ is very large.*
- achieves a delay-limited capacity, but with a greatly reduced capacity.

**Truncated inversion:** $A(a)=\alpha/\max(a,a^*)$ only if a is above cutoff fade depth $a^*$

- to maintain constant SNR (and hence fixed rate) above cutoff $a^*$
- to increase capacity with appropriate choice of cutoff $a^*$: Close to optimal

## Capacity of Frequency-Flat Fading Channels

### Fast Fading – CSIT: Opportunistic transmission

- **CSIT**: Tx/Rx track channel
  - How Tx knows channel? Feedback or channel reciprocity in TDD
- **Opportunistic transmission**: Tx can adapt to channel conditions
- Assume codeword spans L coherence intervals
- modelled as a **parallel channel**

$$y_l = h_l x_l + n_l \quad (l = 1, \ldots, L)$$

- As shown before, for given channel, Gaussian optimal
- Average rate

$$\frac{1}{L}\sum_{l=1}^{L} I(x_l, y_l | h = h_l) = \frac{1}{L}\sum_{l=1}^{L} \log\left(1 + \frac{P_l |h_l|^2}{N_0}\right)$$

power allocated to sub-channel $l$: $P_l$

## Capacity of Frequency-Flat Fading Channels
### Fast Fading – CSIT

- Capacity

$$\max_{P_l} \frac{1}{L}\sum_{l=1}^{L} \log\left(1+\frac{P_l|h_l|^2}{N_0}\right) \quad \text{s.t.} \quad \frac{1}{L}\sum_{l=1}^{L} P_l \le P$$

- Optimal solution (using KKT): **waterfilling**

$$P_l^\star = \left(\frac{1}{\lambda} - \frac{N_0}{|h_l|^2}\right)^+$$

where lambda satisfies full power

$$\frac{1}{L}\sum_{l=1}^{L}\left(\frac{1}{\lambda} - \frac{N_0}{|h_l|^2}\right) = P$$

- Problem?

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIT

- However, when $L \to \infty$

$$\frac{1}{L}\sum_{l=1}^{L}\left(\frac{1}{\lambda}-\frac{N_0}{|h_l|^2}\right) \to \mathbb{E}\left[\left(\frac{1}{\lambda}-\frac{N_0}{|h|^2}\right)^+\right] = P$$

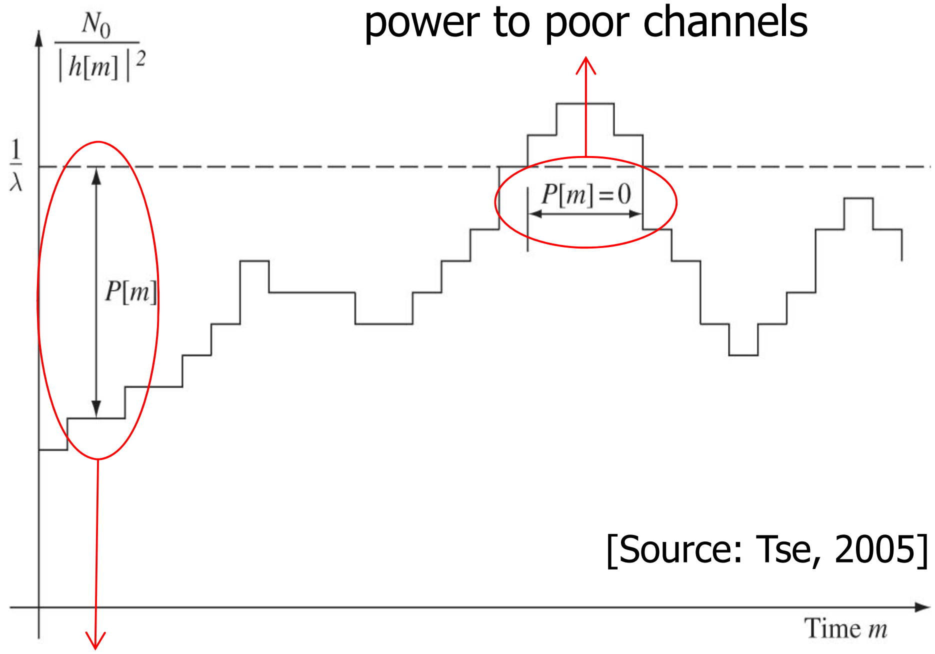
$$\frac{1}{L}\sum_{l=1}^{L}\log\left(1+\frac{P_l^\star |h_l|^2}{N_0}\right) \to \mathbb{E}\left[\log\left(1+\frac{P^\star(h)|h|^2}{N_0}\right)\right]$$

- Capacity

$$C = \mathbb{E}\left[\log\left(1+\frac{P^\star(h)|h|^2}{N_0}\right)\right] \qquad \underbrace{P^\star(h) = \left(\frac{1}{\lambda}-\frac{N_0}{|h|^2}\right)^+}_{\text{Depends on present value of h and statistics}}$$

# Waterfilling

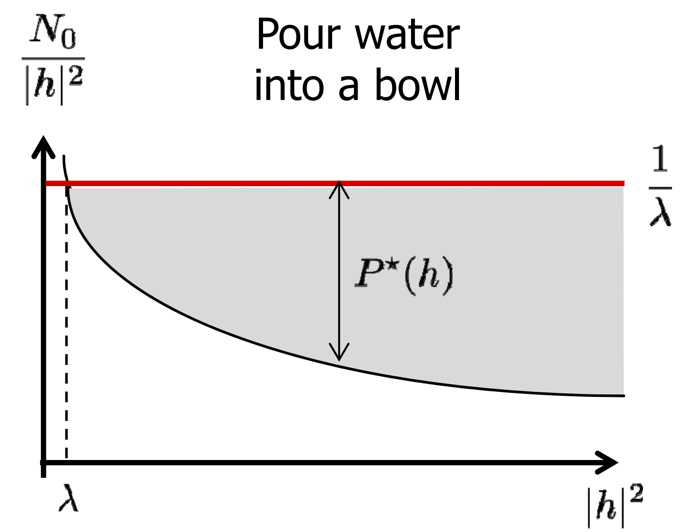
Tx allocates less or no power to poor channels




[Source: Tse, 2005]

Tx allocates more power to good channels

Pour water into a bowl



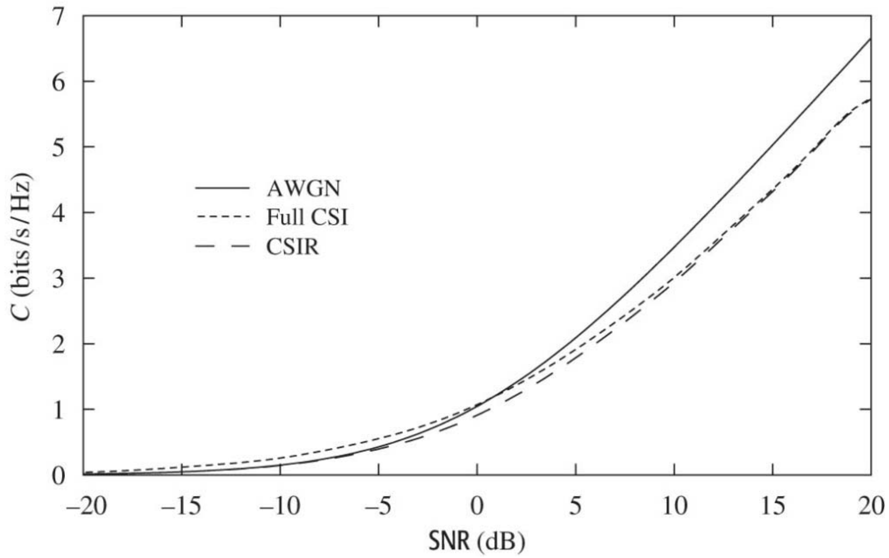
## Capacity of Frequency-Flat Fading Channels

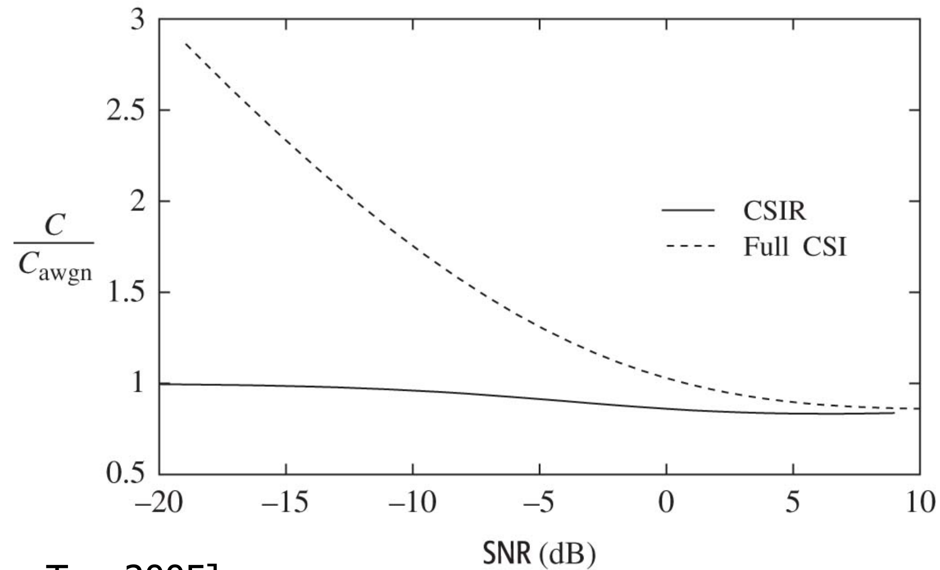
### Fast Fading – CSIT

- How to achieve?
  1.- **Multi-rate variable-power** Gaussian scheme: for each fading state, adapt rate and power to $\log(1+P^{\star}(h)|h|^2/N_0)$
  2.- **Single-rate variable-power** Gaussian scheme: adapt power but keep constant rate $\mathbb{E}[\log(1+P^{\star}(h)|h|^2/N_0)]$
- 1st strategy simpler (no need to code across channel states)
- How does it compare to AWGN or CSIR?

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIT



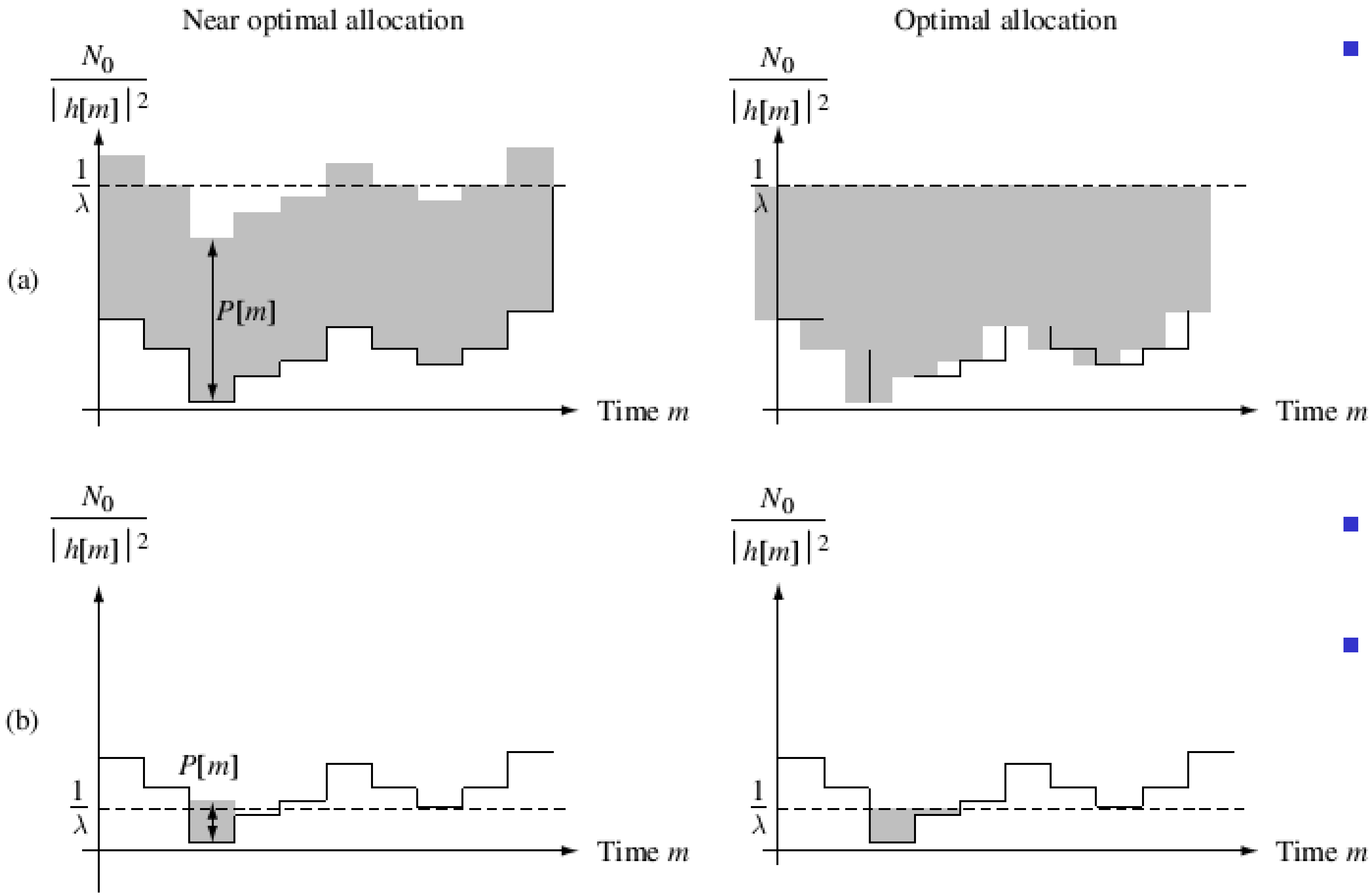




[Source: Tse, 2005]

- Low SNR: CSIT better than CSIR, and even *better than AWGN!*
- High SNR: CSIT and CSIR comparable performance (AWGN best)
  - Still good since it simplifies coding
- Capacity very sensitive to dynamic power allocation in low SNRs but insensitive in high SNRs

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIT




[Source: Tse, 2005]

- High SNR: uniform close to optimal

- Low SNR: give all power to best
- Take advantage of peaks (**opportunistic**)

# Capacity of Frequency-Flat Fading Channels

## Fast Fading – CSIT: Channel inversion

- suboptimal solution
- Invert fading: Convert channel into AWGN $P(h) = \sigma / |h|^2$

$$\mathbb{E}[P(h)] = \mathbb{E}\left[\frac{\sigma}{|h|^2}\right] = P \ \Rightarrow \ \sigma = \frac{P}{\mathbb{E}[1/|h|^2]}$$

$$C = \log\left(1 + \frac{P}{N_0 \mathbb{E}[1/|h|^2]}\right)$$

- Exact opposite to waterfilling
- Simplifies design (fixed rates for all channel states, **zero outage**)
- Power inefficient (hard to invert poor channels)
- In Rayleigh, $\mathbb{E}[1/|h|^2] \to \infty$ so capacity is zero

## Capacity of Frequency-Flat Fading Channels

### Fast Fading – CSIT: Truncated channel inversion

- Invert fading only when channel above cut-off fade

$$P(h) = \begin{cases} \sigma/|h|^2, & |h|^2 \geq |h_0|^2 \\ 0, & |h|^2 < |h_0|^2 \end{cases}$$

$$\sigma = \frac{P}{\mathbb{E}_{h_0}[1/|h|^2]}, \quad \mathbb{E}_{h_0}[1/|h|^2] = \int_{|h_0|^2}^{\infty} \frac{1}{|h|^2} f(|h|^2)\, d|h|^2$$

$$C = \log\left(1 + \frac{P}{N_0 \mathbb{E}_{h_0}[1/|h|^2]}\right) \cdot \mathbb{P}(|h|^2 \geq |h_0|^2)$$

- Constant rate for good channels, no rate for poor ones (**outage**)
- Still need to find optimal value of cut-off

# Capacity of Frequency-Flat Fading Channels
## Slow Fading – CSIR

- In slow fading, codeword spans single realization
- Conditioned on realization, maximum rate is $\log(1+|h|^2\mathrm{SNR})$
  - Max rate is random
- Regardless of selected rate, there is a probability that

$$\log(1+|h|^2\mathrm{SNR}) < R$$

  - Error rate cannot be made arbitrarily small
- Ergodic capacity theoretically zero
- System is said to be in **outage**

$$P_{\mathrm{out}}(R) = \mathbb{P}\{\log(1+|h|^2\mathrm{SNR}) < R\}$$

  - Reliable communication possible as long as not in outage (deep fade)

## Capacity of Frequency-Flat Fading Channels

### Slow Fading – CSIR

- Example: Rayleigh fading

$$P_{\text{out}}(R) = 1 - \exp\left(\frac{-(2^R - 1)}{\text{SNR}}\right)$$

  - At high SNR

$$P_{\text{out}}(R) \approx \frac{(2^R - 1)}{\text{SNR}}$$

  - Decays as 1/SNR: coding cannot improve significantly error performance over slow fading
- ε-**outage capacity**: alternative performance measure
- Largest rate such that $P_{\text{out}}(R) < \epsilon$

$$C_\epsilon = \log(1 + F^{-1}(1-\epsilon)\text{SNR})$$

where complementary CDF $F(x) = \mathbb{P}\{|h|^2 > x\}$

# Capacity of Frequency-Flat Fading Channels
## Slow Fading – CSIR

- How does it compare to AWGN?
  - At high SNRs: relative loss gets smaller at high SNRs (constant difference)

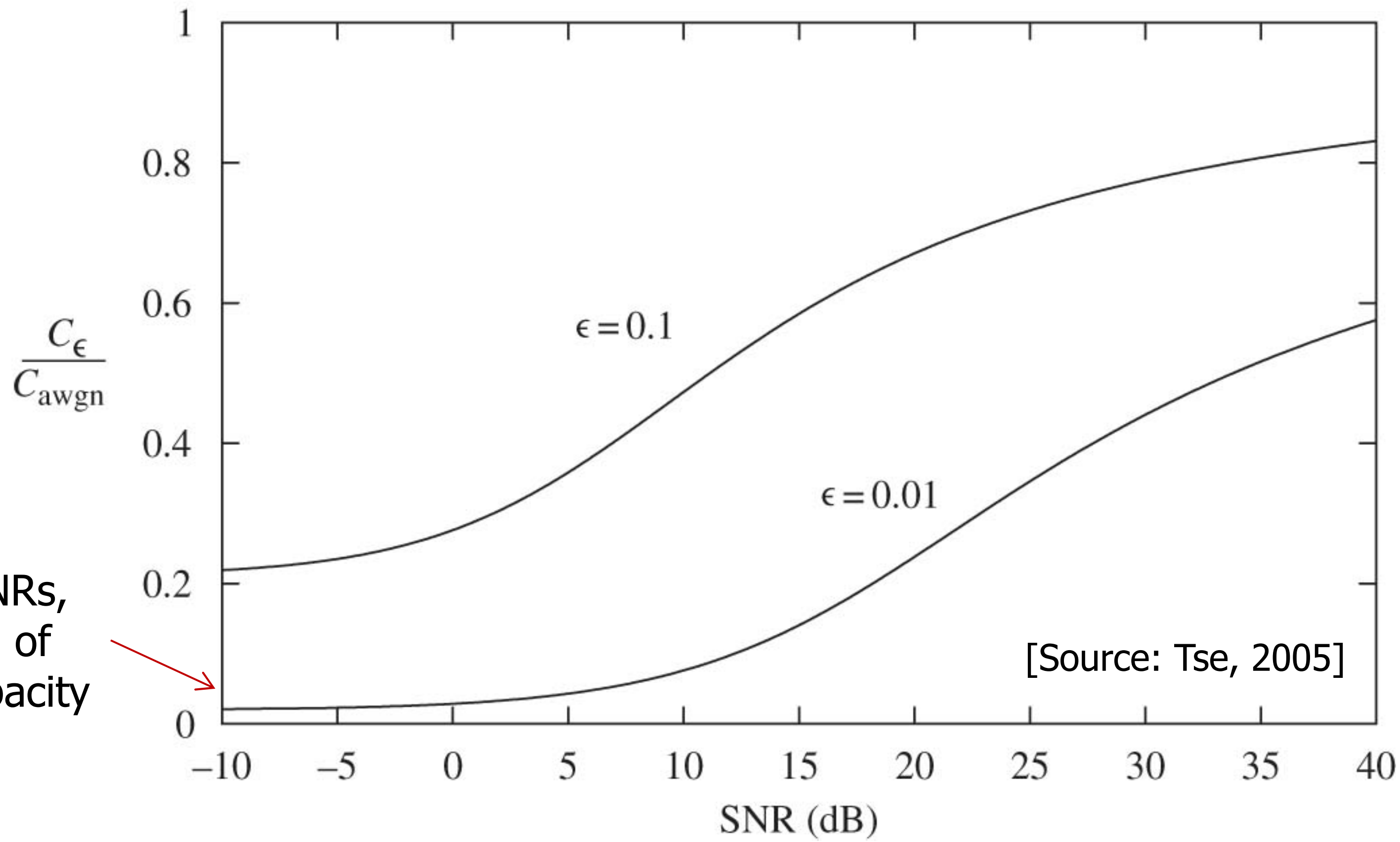
$$\begin{aligned} C_\epsilon &\approx \log(\mathrm{SNR}) + \log(F^{-1}(1-\epsilon)) \\ &= C_{\mathrm{awgn}} - \log(1/F^{-1}(1-\epsilon)) \end{aligned}$$

  - At low SNRs: only a small fraction of AWGN

$$\begin{aligned} C_\epsilon &\approx F^{-1}(1-\epsilon) \cdot \mathrm{SNR}/\ln(2) \\ &= F^{-1}(1-\epsilon) \cdot C_{\mathrm{awgn}} \end{aligned}$$

  - For Rayleigh with small ε: $F^{-1}(1-\epsilon) \approx \epsilon$

# Capacity of Frequency-Flat Fading Channels
## Slow Fading – CSIR



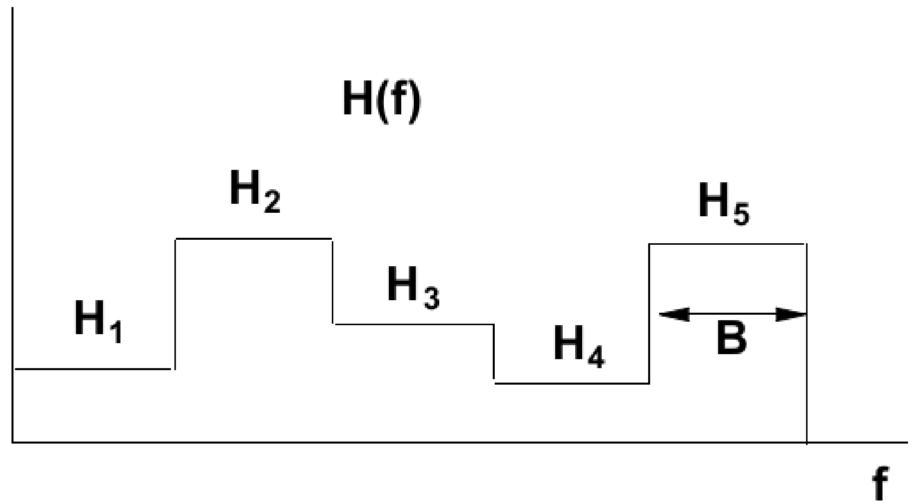

At low SNRs, only 1% of AWGN capacity

[Source: Tse, 2005]

# Capacity of Frequency-Selective Channels

## Time-invariant

- Consider time-invariant channel with frequency response H(f)
- Assume H(f) is block fading with N subcarriers B=W/N




- This can be modeled as a parallel channel with

$$\mathrm{SNR} = \frac{|H_n|^2 P_n}{N_0 B}$$

# Capacity of Frequency-Selective Channels

## Time-invariant

- Capacity

$$\max_{P_n} \sum_{n=1}^{N} \log\left(1 + \frac{P_n |H_n|^2}{N_0 B}\right) \quad \text{s.t.} \quad \frac{1}{N}\sum_{n=1}^{N} P_n \leq P$$

  - Similar to parallel channel before
  - Waterfilling over frequency optimal

$$P_n^\star = \left(\frac{1}{\lambda} - \frac{N_0 B}{|H_n|^2}\right)^+ \qquad \frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{\lambda} - \frac{N_0 B}{|H_n|^2}\right) = P$$

- How to achieve?
  - Separate coding/power for each subcarrier

## Capacity of Frequency-Selective Channels

### Time-invariant

- As N approaches infinity, for cts H(f)

$$P^\star(f) = \left(\frac{1}{\lambda} - \frac{N_0}{|H(f)|^2}\right)^+ \qquad \int P^\star(f)\,df = P$$

$$C = \int \log\left(1 + \frac{P^\star(f)|H(f)|^2}{N_0}\right) df$$

# References


- A. Goldsmith, *Wireless Communications,* Cambridge University Press, 2005, Chapter 4D.
- Tse, P. Viswanath, *Fundamentals of Wireless Communication,* Cambridge University Press, 2005, Chapter 5

- and materials from various sources