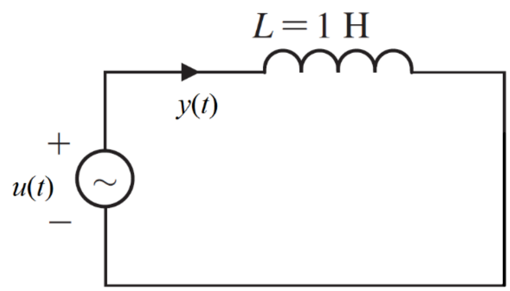
ECE 6200: Linear Systems Analysis

Homework 4 (100 pts.)

Due on Thursday, November 20, 2025, in class

Problem 1 (30 pts.):

Consider the circuit with the current through the inductor ($L = 1$ H) as the state $x(t)$ (also the output $y(t)$) and the voltage source as the input $u(t)$.



a. Use any two theorem results to determine BIBO stability.
b. Determine internal stability and show whether the response is bounded or not for any arbitrary finite initial condition.
c. Determine the boundedness of the output due to step and sinusoidal inputs.

Problem 2 (30 pts.): Determine whether the LTI and LTV systems are marginally stable or asymptotically stable or neither.

a. $\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}$ b. $\mathbf{x}[k+1] = \begin{bmatrix} 0.9 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}[k]$ c. $\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} \mathbf{x},\ (t_0 \geq 0)$

Problem 3 (20 pts.): Determine system stability for the LTV impulse responses.

a. $g(t,\tau) = e^{-2|t|-|\tau|},\ t \geq \tau$ b. $g(t,\tau) = \sin t e^{-(t-\tau)} \cos\tau,\ t \geq \tau$

Problem 4 (20 pts.): Choose a Lyapunov function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}(t)\mathbf{P}\mathbf{x}(t)$ and determine the conditions under which $\dot{V}(\mathbf{x}) < 0$ for the autonomous system $\dot{\mathbf{x}} = \begin{bmatrix} 0 & \alpha \\ 2 & -1 \end{bmatrix} \mathbf{x}$. Do the conditions suffice for BIBO stability for any arbitrary bounded input matrix **B**?

ECE 6200 Homework IV

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1 Question 1

Example 1.1. Consider the circuit with the current through the inductor ($L = 1H$) as the state $x(t)$(also the output $y(t)$) and the voltage source as the input $u(t)$.

- **Part A**: Use any two theorem results to determine BIBO stability.
- **Part B**: Determine internal stability and show whether the response is bounded or not for any arbitrary finite initial condition.
- **Part C**: Determine the boundedness of the output due to step and sinusoidal inputs.

We are given an inductor with $L = 1\,\text{H}$, an input voltage $u(t)$, and the state/output is the current $x(t) = y(t)$. Recall the inductor law:

$$u(t) = L\frac{di}{dt} = \frac{dx}{dt} \quad \text{with} \quad L = 1.$$

So,

$$\dot{x}(t) = u(t), \quad y(t) = x(t).$$

The given state-space matrices were and are $\mathbf{A} = 0, \mathbf{B} = 1, \mathbf{C} = 1, \mathbf{D} = 0$ and the transfer function is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s}$$

1.1 Part A

Recall the first theorem:

Theorem 1.1. *An LTI system is BIBO stable iff its impulse response $h(t)$ is absolutely integrable:*

$$\int_0^\infty |h(t)|\,dt < \infty.$$

Here, the impulse response is the output when $u(t) = \delta(t)$:

$$\dot{x}(t) = \delta(t) \Rightarrow x(t) = 1(t) \Rightarrow h(t) = 1(t).$$

Then,

$$\int_0^\infty |h(t)|\,dt = \int_0^\infty 1\,dt = \infty.$$

Therefore, the system is NOT BIBO stable. Now, let us also recall the second theorem to review:

Theorem 1.2. *A proper rational LTI system is BIBO stable iff all poles of $G(s)$ lie strictly in the open left half-plane.*

Here,

$$G(s) = \frac{1}{s}$$

has a pole at $s = 0$. This is on the imaginary axis, not on the left half-plane. Therefore, again, the system is NOT BIBO stable.

1.2 Part B

For the zero-input dynamics, set $u(t) = 0$:

$$\dot{x} = 0 \Rightarrow x(t) = x(0) = x_0, \quad y(t) = x_0.$$

The state matrix $\mathbf{A} = 0$ has an eigenvalue $\lambda = 0$, so the system is **marginally (Lyapunov) stable** which means the state neither grows nor decays, and thus staying constant. For any finite initial condition x_0, the zero-input response $x(t) = x_0$ is *bounded* for all t. Therefore, the system is stable (in the Lyapunov Sense), but not asymptotically stable.

1.3 Part C

Let us take an arbitrary finite $x(0) = x_0$. Now, for the step input. Let $u(t) = U_0 1(t)$ (bounded step). So,

$$\dot{x}(t) = U_0 \Rightarrow x(t) = x_0 + \int_0^t U_0\,d\tau = x_0 + U_0 t.$$

As $t \to \infty$, $|x(t)| \to \infty$ if $U_0 \neq 0$. So for a bounded step input, the output $y(t) = x(t)$ is unbounded. Now, for the sinusodial input. Let $u(t) = U_0 \sin(\omega t)$. Then,

$$x(t) = x_0 + \int_0^t U_0 \sin(\omega\tau)\,d\tau = x_0 + \frac{U_0}{\omega}(1 - \cos(\omega t)).$$

Here $|1 - \cos(\omega t)| \leq 2$, so $x(t)$ and hence $y(t)$ remains bounded for all t.

2 Question 2

Example 2.1. Determine whether the LTI and LTV systems are marginally stable or asymptotically stable or neither.

- $$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

- $$\mathbf{x}[k+1] = \begin{bmatrix} 0.9 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}[k]$$

- $$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} \mathbf{x} \qquad (t_0 \geq 0)$$

2.1 My Solution

2.1.1 Part A

Let us consider the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, let us write the state equations:

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = 0.$$

From ($\dot{x}_3 = 0 \Rightarrow x_3(t) = x_3(0)$) (a constant) Then,

$$\dot{x}_2 = x_3(0) \Rightarrow x_2(t) = x_2(0) + x_3(0)t,$$

which then grows unbounded when $x_3(0) \neq 0$. Since some trajectories are unbounded, the origin is not stable. Therefore, the given system is neither marginally nor asymptotically stable (it is unstable).

2.1.2 Part B

Now, let us consider $\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k]$ given by:

$$\mathbf{A}_d = \begin{bmatrix} 0.9 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have scalar recursions:

$$x_1[k+1] = 0.9x_1[k] + 2x_3[k], \quad x_2[k+1] = x_2[k], \quad x_3[k+1] = x_3[k].$$

Hence,

$$x_2[k] = x_2[0], \quad x_3[k] = x_3[0].$$

For x_1:

$$x_1[k+1] - 0.9x_1[k] = 2x_3[0].$$

And its solution:

$$x_1[k] = 0.9^k\big(x_1[0] - 20x_3[0]\big) + 20x_3[0],$$

is bounded $\forall k$ since $|0.9| < 1$. Hence, every trajectory is bounded, but if for instance suppose that if $x_2[0] \neq 0$ then $x_2[k] = x_2[0] \not\to 0$. Therefore, the origin is stable but not asymptotically stable which means it follows that the given system is **marginally stable**.

2.1.3 Part C

Consider the system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ and

$$\mathbf{A}(t) = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix}$$

We get the following equations:

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = -e^{-3t}x_1.$$

First,

$$x_1(t) = x_1(0)e^{-t} \xrightarrow[t\to\infty]{} 0.$$

Then,

$$\dot{x}_2 = -e^{-3t}x_1(0)e^{-t} = -x_1(0)e^{-4t}.$$

Now, we need to integrate (take $(t_0 = 0)$ w.l.o.g.):

$$x_2(t) = x_2(0) - x_1(0)\int_0^t e^{-4\tau}\,d\tau = x_2(0) - \frac{x_1(0)}{4}(1 - e^{-4t}).$$

Thus,

$$x_2(t) = \left(x_2(0) - \tfrac{x_1(0)}{4}\right) + \tfrac{x_1(0)}{4}e^{-4t},$$

and it is the case that $x_2(t)$ is bounded and

$$x_2(t) \xrightarrow[t\to\infty]{} x_2(0) - \tfrac{x_1(0)}{4},$$

which is usually not zero unless the initial condition lies on the line $(x_2(0) = x_1(0)/4)$. Thus, all trajectories remain bounded, but they do not all converge to the origin. Therefore, the system is stable but not asymptotically stable, which implies that the given system is **marginally stable**.

3 Question 3

Example 3.1. Determine system stability for the LTV impulse responses.

- $g(t,\tau) = e^{-2|t|-|\tau|}$, $t \geq \tau$
- $g(t,\tau) = \sin t e^{-(t-\tau)}\cos\tau$, $t \geq \tau$

For an LTV system with impulse response $(g(t,\tau))$ (and $(t \geq \tau)$), the BIBO stability test is

$$\sup_t \int_{-\infty}^t |g(t,\tau)|\,d\tau < \infty$$

And it should be noted that if the given supremum is finite, then the system is BIBO stable.

3.1 Part A

Example 3.2. Determine system stability for the LTV impulse response for $g(t,\tau) = e^{-2|t|-|\tau|}$, $t \geq \tau$.

3.1.1 My Response

Here $(g(t,\tau) \geq 0)$, so

$$\int_{-\infty}^{t} |g(t,\tau)|\, d\tau = e^{-2|t|} \int_{-\infty}^{t} e^{-|\tau|}\, d\tau \leq e^{-2|t|} \int_{-\infty}^{\infty} e^{-|\tau|}\, d\tau = 2e^{-2|t|} \leq 2.$$

Hence, it follows that:

$$\sup_t \int_{-\infty}^{t} |g(t,\tau)|\, d\tau \leq 2 < \infty.$$

Therefore, the given system is **BIBO stable**.

3.2 Part B

Example 3.3. Determine system stability for the LTV impulse response $g(t,\tau) = \sin t e^{-(t-\tau)} \cos \tau$, $t \geq \tau$.

3.2.1 My Response

We have $(|\sin t| \leq 1)$ and $(|\cos \tau| \leq 1)$, then

$$|g(t,\tau)| = |\sin t| e^{-(t-\tau)} |\cos \tau| \leq e^{-(t-\tau)}.$$

So, for any t, it follows that:

$$\int_{-\infty}^{t} |g(t,\tau)|\, d\tau \leq \int_{-\infty}^{t} e^{-(t-\tau)}\, d\tau = \int_{0}^{\infty} e^{-s}\, ds = 1.$$

Thus,

$$\sup_t \int_{-\infty}^{t} |g(t,\tau)|\, d\tau \leq 1 < \infty,$$

and therefore this system is also **BIBO stable**.

4 Question 4

Example 4.1. Choose a Lyapunov function $V(\mathbf{x}) = \mathbf{x}^\top(t)\mathbf{P}\mathbf{x}(t)$ and determine the conditions under which $\dot{V}(\mathbf{x}) < 0$ for the autonomous system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \alpha \\ 2 & -1 \end{bmatrix} \mathbf{x}$$

Do the conditions suffice for BIBO stability for any arbitrary bounded input matrix $\mathbf{B}$?

4.1 My Response

For this problem, we are given the following autonomous system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \begin{bmatrix} 0 & \alpha \\ 2 & -1 \end{bmatrix}$$

Now, let us take a quadratic Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}^\top \mathbf{P}\mathbf{x}, \qquad \mathbf{P} = \mathbf{P}^\top > 0.$$

Then

$$\dot{V}(\mathbf{x}) = \mathbf{x}^\top(\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}.$$

and it follows that we need $\mathbf{P} > 0$ and

$$\mathbf{S} := \mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} < 0 \qquad \text{(negative definite)}$$

for asymptotic stability. For an LTI system, this will only happen iff all eigenvalues of $\mathbf{A}$ are in the open left half-plane (note this is the standard Lyapunov Theorem). Thus, it's the case that we instead need to find the condition on α such that $\mathbf{A}$ is Hurwitz. Now, let us determine the characteristic polynomial:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda & -\alpha \\ -2 & \lambda + 1 \end{bmatrix} = \lambda^2 + \lambda - 2\alpha.$$

For a 2nd-order polynomial $\lambda^2 + a_1\lambda + a_0$, we know that stability requires $a_1 > 0$ and $a_0 > 0$. Here,

$$a_1 = 1 > 0, \qquad a_0 = -2\alpha > 0 \Longrightarrow \alpha < 0.$$

So, for $\alpha < 0$, the matrix $\mathbf{A}$ is Hurwitz, and there exists a symmetric $\mathbf{P} > 0$ which solves the Lyapunov Equation:

$$\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}, \mathbf{Q} = \mathbf{Q}^\top > 0,$$

such that,

$$\dot{V}(\mathbf{x}) = -\mathbf{x}^\top\mathbf{Q}\mathbf{x} < 0, \ \forall \mathbf{x} \neq 0.$$

One explicit choice with $\mathbf{Q} = \mathbf{I}$ is

$$\mathbf{P} = \begin{bmatrix} \frac{2\alpha-5}{4\alpha} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} - \frac{\alpha}{4} \end{bmatrix}.$$

Hence, it follows that the condition is $\dot{V}(\mathbf{x}) < 0$: $\alpha < 0$. Now, we need to consider the BIBO Stability Question. Consider the corresponding input system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x},$$

with any constant matrix ($\mathbf{B}$ with the entries bounded) and bounded input $u(t)$. If $\mathbf{A}$ is Hurwitz and $\alpha < 0$, then

$$|e^{\mathbf{A}t}| \leq Me^{-\gamma t},$$

and so for bounded (u),

$$x(t) = e^{\mathbf{A}t}x(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau), d\tau$$

remains bounded, and hence $(y(t))$ is bounded. This holds for any finite $(\mathbf{B})$.

Therefore, the condition $\alpha < 0$ (which guarantees the existence of a Lyapunov function V with $\dot{V} < 0$ and hence that $\mathbf{A}$ is Hurwitz) does suffice for BIBO stability for any constant input matrix $\mathbf{B}$ with finite entries (and any finite output matrix $\mathbf{C}$).