# ECE 6200: Linear Systems Analysis

## Final Exam (100 pts.)

Due by Tuesday, December 9, 2025, 11:59 pm, slide in the office

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**Instructions:**

- The test has five problems. Read all problems before starting to solve.
- The exam is open book / open lecture with MATLAB usage allowed.
- Show your steps when solving the problems to receive full credit.
- Sign and attach this page when you return the exam.

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**Name:** ________________________

**Honor Pledge:** "On my honor, I have neither given nor received any aid in this test."

**Signature:** ______________________

## Problem 1 (20 pts.):

a. **(10 pts.)** Determine the causality and time-invariance properties of the systems given by the following impulse responses for an impulse applied at time $\tau$.

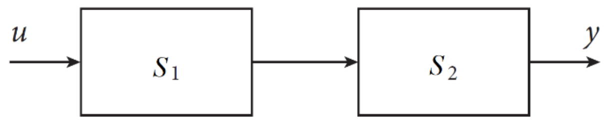
i. $g(t,\tau) = e^{-(t-\tau)^2}$

ii. $g(t,\tau) = e^{-(t-\tau)}u(t-\tau)$, where $u(t-\tau)$ is a shifted unit step

b. **(10 pts.)** Obtain a combined state-space model identifying the full set of system matrices for the system given by a cascade combination of two LTI state-space systems $S_1$ and $S_2$.

$$S_1: \quad \begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{A}_1\mathbf{x}_1(t) + \mathbf{B}_1\mathbf{u}_1(t) \\ \mathbf{y}_1(t) &= \mathbf{C}_1\mathbf{x}_1(t) + \mathbf{D}_1\mathbf{u}_1(t) \end{aligned}$$

$$S_2: \quad \begin{aligned} \dot{\mathbf{x}}_2(t) &= \mathbf{A}_2\mathbf{x}_2(t) + \mathbf{B}_2\mathbf{u}_2(t) \\ \mathbf{y}_2(t) &= \mathbf{C}_2\mathbf{x}_2(t) + \mathbf{D}_2\mathbf{u}_2(t) \end{aligned}$$



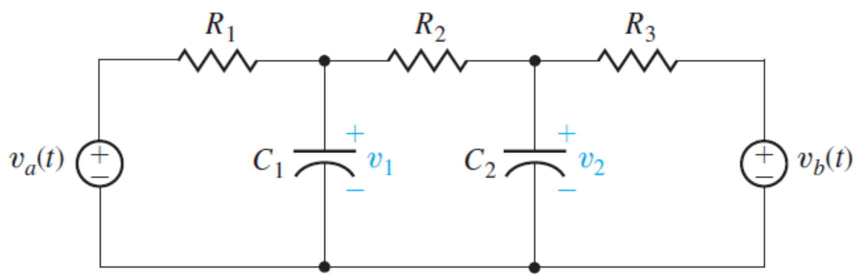

Draw a block diagram of the parallel version of this system keeping the original input and output terminals.

## Problem 2 (20 pts.):

a. **(10 pts.)** Obtain a discrete convolution input-output expression in terms of the state-space matrices for the following discrete-time LTI system.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$
$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

b. **(10 pts.)** Find a state-space model of the following circuit with the states $x_1 = v_1$ and $x_2 = v_2$. The inputs are $v_a(t)$ and $v_b(t)$. The output $y_1(t)$ is the voltage across $R_1$ and output $y_2(t)$ is the voltage across $R_2$ (any polarity is fine). Obtain the state and output equations identifying all the matrices. Plot the <u>complete</u> unit step response of both outputs (include your MATLAB script).



## Problem 3 (20 pts.):

a. **(10 pts.)** Find $\alpha$, $\beta$, and $\sin(\mathbf{A})$ given that

$$e^{\mathbf{A}t} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

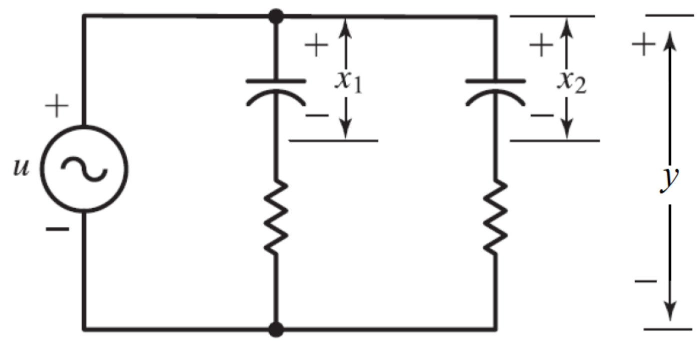
b. **(10 pts.)** Determine the asymptotic stability condition for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ for which the following Lyapunov equation holds.

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} + 2\alpha\mathbf{P} = -\mathbf{Q}$$

where $\mathbf{P} > 0$ and $\mathbf{Q} > 0$.

## Problem 4 (15 pts.):

a. **(10 pts.)** Obtain a state-space model of this system and determine the conditions in terms of the circuit time constants $\tau = RC$ of each branch to make this system controllable (if possible). Justify if it would be possible to observe this system for any $R$ and $C$.




b. **(5 pts.)** Is it possible to have a state-space model that is not controllable but observable, and not asymptotically stable but BIBO stable. Provide some numerical example to prove or disprove.

## Problem 5 (25 pts.)

Consider the following system:

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{u}$$
$$\mathbf{y} = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}$$

**a.** **(8 pts.)** Determine the controllability and observability of the system. If the system is not controllable and/or observable then determine which eigenvalues are uncontrollable and/or unobservable.

**b.** **(10 pts.)** Is it possible to design a state feedback controller and a state estimator that guarantee closed-loop stability? If so, find the value of the state feedback gain $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ such that both closed-loop eigenvalues of $\mathbf{A} - \mathbf{BK}$ are at –2. Similarly, find the value of the estimator gain $\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ such that both closed-loop eigenvalues of $\mathbf{A} - \mathbf{LC}$ are at –2. Is your choice of $\mathbf{K}$ and $\mathbf{L}$ unique or the said pair of closed-loop eigenvalues of $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$ can be achieved by another choice of $\mathbf{K}$ and $\mathbf{L}$, respectively?

**c.** **(7 pts.)** Simulate this system in MATLAB/Simulink starting with initial state $\mathbf{x}(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and show results of the stabilization of the closed-loop system with state estimator. Include your MATLAB code / Simulink diagram.

# ECE 6200 Final Exam

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9 December 2025

# 1 Question 1

## 1.1 Part A

**Example 1.1.** Determine the causality and time-invariance properties of the systems given by the following impulse responses for an impulse applied at time $\tau$.

1. $g(t,\tau) = e^{-(t-\tau)^2}$
2. $g(t,\tau) = e^{-(t-\tau)}u(t-\tau)$, where $u(t-\tau)$ is a shifted unit step.

### 1.1.1 Part 1

Let us consider $g(t,\tau) = e^{-(t-\tau)^2}$. For $t < \tau$, it is the case that $(t-\tau)^2 > 0 \Rightarrow e^{-(t-\tau)^2} > 0$. Then, $g(t,\tau) \neq 0$ and it follows that it is **not causal**. Additionally, since $g(t,\tau)$ depends only on $t-\tau$ which follows from the fact that it is a function of $(t-\tau)^2$ only, thus it is **time-invariant**.

### 1.1.2 Part 2

Now, we consider $g(t,\tau) = e^{-(t-\tau)}u(t-\tau)$. First, if the unit step is $u(t-\tau) = 0$ for $t < \tau$, then it follows that $g(t,\tau) = 0$ for all $t < \tau$ and thus is *causal*. Again, $g(t,\tau)$ is a function only of $t-\tau$ where both the exponential and step term depend on $(t-\tau)$ and as a result is *time-invariant*. Therefore, this system is **causal** and **time-invariant**.

## 1.2 Part B

**Example 1.2.** Obtain a combined state-space model identifying the full set of system matrices for the system given by a cascade combination of two LTI state-space systems $S_1$ and $S_2$. Draw a block diagram of the parallel version of this system keeping the original input and output terminals.

For Part B, we need to combine the state-space model for cascade $S_1 \to S_2$. We are given:

$$\begin{aligned} S_1 : \quad & \dot{x}_1(t) = A_1x_1(t) + B_1u_1(t), \\ & y_1(t) = C_1x_1(t) + D_1u_1(t), \end{aligned}$$

$$\begin{aligned} S_2 : \quad & \dot{x}_2(t) = A_2x_2(t) + B_2u_2(t), \\ & y_2(t) = C_2x_2(t) + D_2u_2(t). \end{aligned}$$

Cascade Interconnections:

- Overall Input: $u(t)$
- $u_1(t) = u(t)$
- $u_2(t) = y_1(t) = C_1 x_1(t) + D_1 u(t)$
- Overall output: $y(t) = y_2(t)$

Next, let us define a combined state vector:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Our state-equations are:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u, \\ \dot{x}_2 &= A_2 x_2 + B_2 u_2 = A_2 x_2 + B_2(C_1 x_1 + D_1 u) \\ &= B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u. \end{aligned}$$

Then, in block matrix form, we have:

$$\dot{x}(t) = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u(t).$$

Thus, it follows that:

$$A = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix}.$$

Now, our output equations are:

$$y = y_2 = C_2 x_2 + D_2 u_2 = C_2 x_2 + D_2(C_1 x_1 + D_1 u) = D_2 C_1 x_1 + C_2 x_2 + D_2 D_1 u.$$

So,

$$C = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}, \qquad D = D_2 D_1.$$

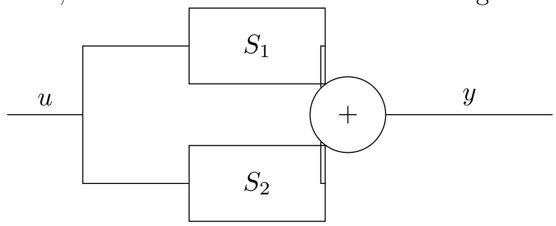
Now, we need to consider the combined state space model (cascade $S_1$ then $S_2$):

$$\boxed{\begin{aligned} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{aligned}}$$

with

$$A = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix}, \quad C = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}, \quad D = D_2 D_1.$$

Now, we need to construct the block diagram:



# 2 Question 2

## 2.1 Part A

**Example 2.1.** Obtain a discrete convolution input–output expression in terms of the state-space matrices for the following discrete-time LTI system:

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k],\\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k].\end{aligned}$$

Given the discrete-time LTI system

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k],\\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k],\end{aligned}$$

the state at time $k$ starting from $\mathbf{x}[0]$ is

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i}\mathbf{B}\,\mathbf{u}[i].$$

Thus,

$$\begin{aligned}\mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]\\ &= \mathbf{C}\mathbf{A}^k\mathbf{x}[0] + \mathbf{C}\sum_{i=0}^{k-1} \mathbf{A}^{k-1-i}\mathbf{B}\,\mathbf{u}[i] + \mathbf{D}\mathbf{u}[k].\end{aligned}$$

For zero initial conditions ($\mathbf{x}[0] = \mathbf{0}$), this becomes

$$\mathbf{y}[k] = \sum_{i=0}^{k-1} \mathbf{C}\mathbf{A}^{k-1-i}\mathbf{B}\,\mathbf{u}[i] + \mathbf{D}\mathbf{u}[k].$$

We can rewrite this in standard convolution form. Let $j = k - i$. Then for $i = 0, \dots, k-1$, we have $j = k, \dots, 1$, and

$$\mathbf{C}\mathbf{A}^{k-1-i}\mathbf{B} = \mathbf{C}\mathbf{A}^{j-1}\mathbf{B}.$$

Define the impulse response sequence $\mathbf{h}[k]$ by

$$\mathbf{h}[0] = \mathbf{D}, \qquad \mathbf{h}[j] = \mathbf{C}\mathbf{A}^{j-1}\mathbf{B}, \quad j \geq 1.$$

Then the input–output relation can be written as the discrete-time convolution

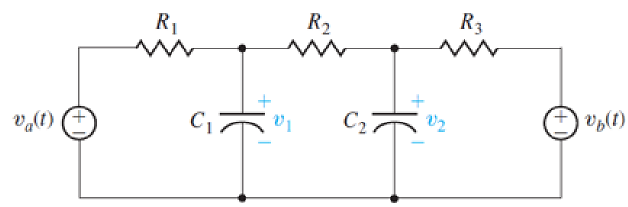
$$\mathbf{y}[k] = \sum_{j=0}^{k} \mathbf{h}[j]\,\mathbf{u}[k-j],$$

with

$$\mathbf{h}[0] = \mathbf{D}, \qquad \mathbf{h}[j] = \mathbf{C}\mathbf{A}^{j-1}\mathbf{B}, \quad j \geq 1.$$

## 2.2 Part B

**Example 2.2.** Find a state-space model of the following circuit with the states $x_1 = v_1$ and $x_2 = v_2$. The inputs are $v_a(t)$ and $v_b(t)$. The output $y_1(t)$ is the voltage across $R_1$ and the output $y_2(t)$ is the voltage across $R_2$ (any polarity is fine). Obtain the state and output equations identifying all the matrices. Plot the **complete** unit step response of both outputs (include your MATLAB script).




Here, we have the following states:

$$x_1 = v_1 \quad \text{(voltage across } C_1), \quad x_2 = v_2 \quad \text{(voltage across } C_2).$$

And we have the inputs:

$$u_1 = v_a(t), \qquad u_2 = v_b(t).$$

With the outputs with the following polarities:

- $y_1$: voltage across $R_1$, take $(y_1 = v_a - v_1)$.
- $y_2$: voltage across $R_2$, take $(y_2 = v_1 - v_2)$.

By using KCL, we obtain the following node equations: At node $v_1$

$$C_1\dot{v}_1 + \frac{v_1 - v_a}{R_1} + \frac{v_1 - v_2}{R_2} = 0$$

$$\Rightarrow \dot{v}_1 = \frac{1}{C_1}\left(\frac{v_a - v_1}{R_1} + \frac{v_2 - v_1}{R_2}\right) = \frac{1}{C_1}\Big[-\Big(\frac{1}{R_1} + \frac{1}{R_2}\Big)v_1 + \frac{1}{R_2}v_2 + \frac{1}{R_1}v_a\Big].$$

And at node $v_2$:

$$C_2\dot{v}_2 + \frac{v_2 - v_1}{R_2} + \frac{v_2 - v_b}{R_3} = 0$$

$$\Rightarrow \dot{v}_2 = \frac{1}{C_2}\left(\frac{v_1 - v_2}{R_2} + \frac{v_b - v_2}{R_3}\right) = \frac{1}{C_2}\Big[\frac{1}{R_2}v_1 - \Big(\frac{1}{R_2} + \frac{1}{R_3}\Big)v_2 + \frac{1}{R_3}v_b\Big].$$

And by using the above matrices, we obtain the following state-space matrices. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad u = \begin{bmatrix} v_a \\ v_b \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

and then it follows that,

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

where

$$A = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3}\right) \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{C_1 R_1} & 0 \\ 0 & \frac{1}{C_2 R_3} \end{bmatrix},$$

(outputs are chosen) and

$$C = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

#### 2.2.1 MATLAB Modeling

```matlab
% Problem 2(b): State-Space and Unit-Step Response

clear; clc; close all;


% ----------------------------------------------------
% Parameters (Can be adjusted to specific values)
% Assumption: R1 = R2 = R3 = 1 ohm, C1 = C2 = 1 F


R1 = 1;
R2 = 1;
R3 = 1;
C1 = 1;
C2 = 1;

% ----------------------------------------------------
% State-Space Matrices

A = [ -(1/R1 + 1/R2)/C1, 1/(R2*C1); 1/(R2*C2), -(1/R2 + 1/R3)/C2 ];
B = [ 1/(R1*C1), 0; 0, 1/(R3*C2) ];
C = [-1, 0; 1, -1];
D = [1, 0; 0, 0];

% ---------------------------------------------------
% Build Continuous-Time State-Space System

sys = ss(A,B,C,D);


% ---------------------------------------------------
% Step Responses

figure;
step(sys);
grid on;
title('Step responses of y_1 and y_2 to unit steps in v_a and v_b');

% Both Inputs step to 1 at t = 0 simultaneously
```

```
t = linspace(0,0.1,1000);           % adjust tfinal as necessary
u = ones(length(t),2);              % both v_a and v_b are unit steps
[y,t_out] = lsim(sys,u,t);

figure;
plot(t_out,y);
grid on;
xlabel('Time (s)');
ylabel('Outputs');
legend('y_1 (across R_1)','y_2 (across R_2)');
title('Outputs to simultaneous unit steps in v_a and v_b');
```

The given code produces the following figures:

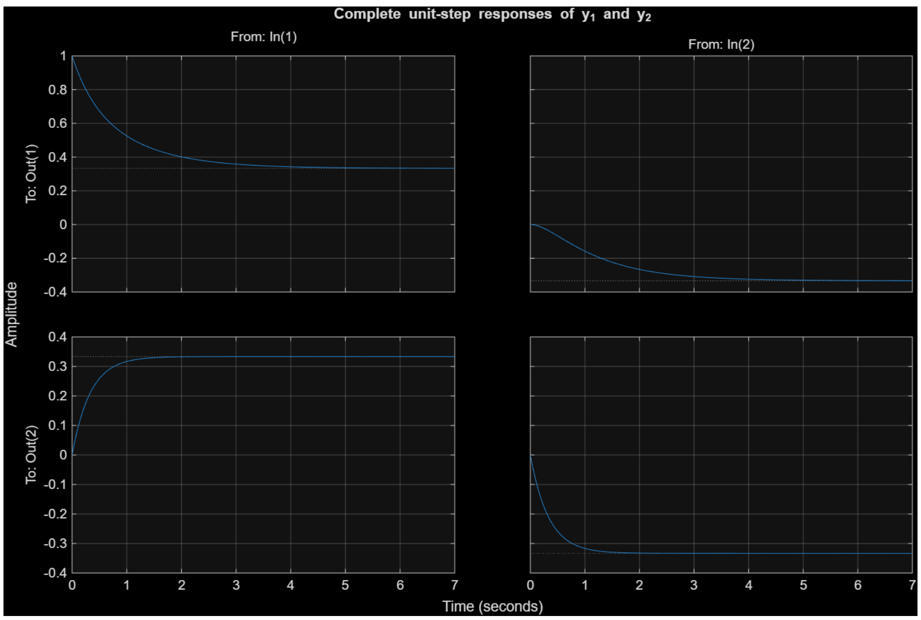



Figure 1: Simulated time response of the outputs $y_1$ (voltage across $R_1$) and $y_2$ (voltage across $R_2$) to simultaneous unit-step inputs in $v_a, v_b$ using the assumed values $R_1 = R_2 = R_3 = 1\ \Omega$ and $C_1 = C_2 = 1$ F

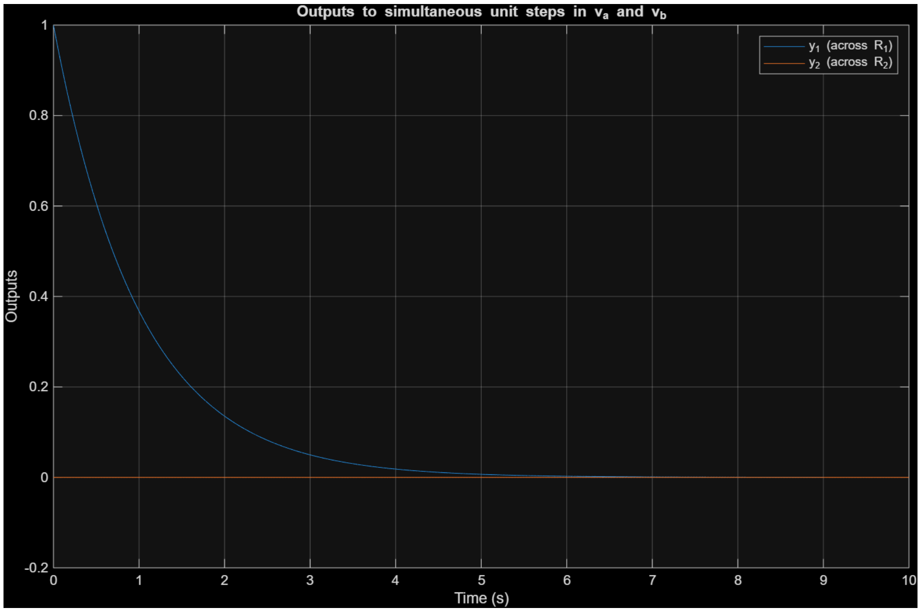



Figure 2: Complete unit-step responses of outputs $y_1$and $y_2$ obtained with MATLAB's 'step' command. Each subplot shows the SISO response from one input $v_a$ or $v_b$ to one output (voltage across $R_1$ or $R_2$, for ($R_1 = R_2 = R_3 = 1\ \Omega$) and ($C_1 = C_2 = 1$ F).

# 3 Question 3

## 3.1 Part A

**Example 3.1.** Find $\alpha, \beta$, and $\sin(\mathbf{A})$ given that

$$e^{\mathbf{A}T} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Find $\alpha, \beta, \sin(A)$. We are given:

$$e^{At} = \begin{bmatrix} -e^{-t} + \alpha e^{-2t} & -e^{-t} + \beta e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}.$$

First, we need to use $e^{A\cdot 0} = I$ to get $\alpha, \beta$. We need to set $t = 0$:

$$e^{A\cdot 0} = \begin{bmatrix} \alpha - 1 & \beta - 1 \\ 0 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From here, it follows that:

$$\alpha - 1 = 1 \Rightarrow \boxed{\alpha = 2}, \qquad \beta - 1 = 0 \Rightarrow \boxed{\beta = 1}.$$

Second, we need to find $\mathbf{A}$. By definition, $A = \left.\frac{d}{dt}e^{At}\right|_{t=0}$. We differentiate the given $e^{At}$ plug $t = 0$, and use $\alpha = 2, \beta = 1$:

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1, \lambda_2 = -2$. Finally, we compute $\sin A$. For a $2 \times 2$ matrix with distinct eigenvalues, any analytic function $f(A)$ can be written as:

$$f(A) = f(\lambda_1)\frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} + f(\lambda_2)\frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}.$$

With $f(\lambda) = \sin\lambda, \lambda_1 = -1, \lambda_2 = -2$:

$$\sin(A) = \sin(-1)\frac{A + 2I}{-1 + 2} + \sin(-2)\frac{A + I}{-2 + 1}.$$

Using $\sin(-x) = -\sin x$ and through simplification, we obtain:

$$\boxed{\sin(A) = \begin{bmatrix} \sin(1) - 2\sin(2) & \sin(1) - \sin(2) \\ 2\big(\sin(2) - \sin(1)\big) & \sin(2) - 2\sin(1) \end{bmatrix}}.$$

## 3.2 Part B

**Example 3.2.** Determine the asymptotic stability condition for the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ for which the following Lyapunov Equation holds.

$$\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} + 2\alpha\mathbf{P} = -\mathbf{Q}$$

where $\mathbf{P} > 0$ and $\mathbf{Q} > 0$.

We are given the Lyapunov equation:

$$\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A} + 2\alpha\mathbf{P} = -\mathbf{Q}, \quad \mathbf{P} > 0, \mathbf{Q} > 0.$$

And we can rewrite this equation as:

$$(\mathbf{A} + \alpha\mathbf{I})^\top\mathbf{P} + \mathbf{P}(\mathbf{A} + \alpha\mathbf{I}) = -\mathbf{Q}.$$

From this, we get the exact Lyapunov equation for the matrix $\mathbf{A} + \alpha\mathbf{I}$. By the Lyapunov Theorem, it follows that there exists $\mathbf{P} > 0$ for any $\mathbf{Q} > 0$ iff all eigenvalues of $\mathbf{A} + \alpha\mathbf{I}$ have negative real parts. That is,

$$\Re(\lambda_i(A + \alpha I)) < 0 \quad \Longleftrightarrow \quad \Re(\lambda_i(A)) < -\alpha, \forall i.$$

Therefore, the asymptotic stability condition for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ under the given Lyapunov Equation is

$$\boxed{\text{All eigenvalues of } A \text{ must satisfy } \Re(\lambda_i(A)) < -\alpha.}$$

# 4 Question 4

## 4.1 Part A

**Example 4.1.** Obtain a state-space model of this system and determine the conditions in terms of the circuit time constants $\tau = RC$ of each branch to make this system controllable (if possible). Justify if it would be possible to observe this system for any $R$ and $C$.

First, let us consider our system. The circuit provided consists of two parallel RC branches, with each branch having a resistor in series with a capacitor. Both branches are driven by the same source voltage $u(t)$. We have the following states:

$$x_1 = \text{voltage across } C_1, \quad x_2 = \text{voltage across } C_2.$$

The output here is the source that drives the top node, so

$$\boxed{y(t) = u(t)}.$$

Next, onto the branch dynamics for each branch $i = 1, 2$:

- **KVL**: $u = x_i + v_{R_i}$.
- **Capacitor Current**: $i_i = C_i \dot{x}_i$.
- **Resistor Current**: $i_i = \frac{v_{R_i}}{R_i}$.

And when equate currents, we get:

$$v_{R_i} = R_i C_i \dot{x}_i.$$

Next, we subsitute into KVL:

$$u = x_i + R_i C_i \dot{x}_i \quad \Rightarrow \quad \dot{x}_i = -\frac{1}{\tau_i} x_i + \frac{1}{\tau_i} u,$$

where

$$\tau_i = R_i C_i.$$

Second, we move onto our state-space model. We define:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then it follows that

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{\tau_1} & 0 \\ 0 & -\frac{1}{\tau_2} \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \frac{1}{\tau_1} \\ \frac{1}{\tau_2} \end{bmatrix}}_{B} u.$$

$$y = \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{C} x + \underbrace{1}_{D} u.$$

Next, we need to determine controllability. The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau_1} & -\frac{1}{\tau_1^2} \\ \frac{1}{\tau_2} & -\frac{1}{\tau_2^2} \end{bmatrix}.$$

And its determinant is:

$$\det(\mathcal{C}) = \frac{\tau_2 - \tau_1}{\tau_1^2 \tau_2^2}.$$

Thus,

$$\boxed{\text{System is controllable} \iff \tau_1 \neq \tau_2 \quad (R_1 C_1 \neq R_2 C_2).}$$

Hence, if $\tau_1 = \tau_2$, the two branches behave identically which implies one state becomes uncontrollable. Next, let us determine observability. Since,

$$C = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

it follows that the observability matrix is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The rank is 0 which is less than 2 which means that it is never observable, regardless of the component values. So,

$$\boxed{\text{The system is not observable for any } R_1, R_2, C_1, C_2.}$$

Therefore, we can interpret this to mean that its output is simply the source $u$, so it reveals nothing about the capacitor voltages.

## 4.2 Part B

**Example 4.2.** Is it possible to have a state-space model that is not controllable but observable, and not asymptotically stable but BIBO stable. Provide some numerical example to prove or disprove.

We need to have a state-space model that is not controllable but observable, and not asymptotically stable but BIBO stable. Yes, it is possible for such a system to exist. Let's consider:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0.$$

Let us determine controllability. From the matrix:

$$\mathcal{C} = [B; AB]$$

So,

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad \text{rank}(\mathcal{C}) = 1 < 2 \Rightarrow \text{not controllable}.$$

Next, we need to determine observability. Then,

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \det(\mathcal{O}) = -2 \neq 0 \Rightarrow \text{observable.}$$

Thirdly, we need to consider asymptotic stability. The eigenvalues of $\mathbf{A}$ are $1, -1$, so the system matrix is NOT asymptotically stable as there is one unstable eigenvalue. Next, we move onto to BIBO stability. We have the following transfer function:

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s+1}.$$

Hence, we see that all the poles of $G(s)$ are in the open left half-plane, and as a result, the input-output system is **BIBO stable**, even though $A$ has an unstable mode. Therefore the system is not controllable but is observable and is not asymptotically stable (eigenvalue 1) and is BIBO stable (pole at -1 only).

# 5 Question 5

Consider the following system:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ \mathbf{y} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}. \end{aligned}$$

## 5.1 Part A

**Example 5.1.** Determine the controllability and observability of the system. If the system is not controllable and/or observable then determine which eigenvalues are uncontrollable and/or unobservable.

Let

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

For controllability, we find that

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{rank}(\mathcal{C}) = 1 < 2.$$

Since the rank is less than 2, then the system is **not controllable**. Regarding its eigenvalues, those are as follows:

$$\lambda_1 = -2, v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 0, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To determine which eigenvalue is uncontrollable, we check:

$$\text{rank} \begin{bmatrix} \lambda I - A & B \end{bmatrix}$$

So for $\lambda = -2$, its rank is equivalent to 1 which is less than 2, so it is *uncontrollable*. But for $\lambda = 0$, the rank is 2 so it is *controllable*. Now, let us determine its observability. So,

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{rank}(\mathcal{O}) = 1 < 2$$

So the system is not observable. Now by using the dual test, we get that for $\lambda = -2$ has a rank of $1 < 2$, which means that it is unobservable, and for $\lambda = 0$ it is observable since its rank equals 2.

## 5.2 Part B

**Example 5.2.** Is it possible to design a state feedback controller and a state estimator that guarantee closed-loop stability? If so, find the value of the state feedback gain

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

such that both closed-loop eigenvalues of $\mathbf{A} - \mathbf{BK}$ are at $-2$. Similarly, find the value of the estimator gain

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

such that both closed-loop eigenvalues of $\mathbf{A} - \mathbf{LC}$ are at $-2$. Is your choice of $\mathbf{K}$ and $\mathbf{L}$ unique or the said pair of closed-loop eigenvalues of $\mathbf{A} - \mathbf{BK}$ and $\mathbf{A} - \mathbf{LC}$ can be achieved by another choice of $\mathbf{K}$ and $\mathbf{L}$, respectively.

Let $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Then,

$$A_{cl} = A - BK = \begin{bmatrix} -1 - k_1 & 1 - k_2 \\ 1 - k_1 & -1 - k_2 \end{bmatrix}$$

So, the characteristic polynomial here is:

$$\det(\lambda I - A_{cl}) = (\lambda + 2)(\lambda + k_1 + k_2)$$

Then, it follows that our closed-loop eigenvalues are $\{-2, -(k_1 + k_2)\}$. So we need both eigenvalues at -2, so:

$$-(k_1 + k_2) = -2 \Rightarrow k_1 + k_2 = 2.$$

Hence, it follows that any $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ that satisfies $k_1 + k_2 = 2$ works. Consider, for example,

$$\mathbf{K} = \begin{bmatrix} 2 & 0 \end{bmatrix} \Rightarrow \text{eig}(A - BK) = \{-2, -2\}$$

Now, let us move onto the estimator:

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

and

$$A_{obs} = A - LC = \begin{bmatrix} -1 - l_1 & 1 - l_1 \\ 1 - l_2 & -1 - l_2 \end{bmatrix}$$

We get the following characteristic polynomial:

$$\det(\lambda I - A_{obs}) = (\lambda + 2)(\lambda + l_1 + l_2),$$

and its eigenvalues are $\{-2, -(l_1 + l_2)\}$. If we require both to be at $-2$, it follows that $l_1 + l_2 = 2$. So any $L$ on that line works here.

Thus,

$$L = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \text{eig}(A - LC) = \{-2, -2\}$$

Now what about uniqueness? Well the uncontrollable/unobservable eigenvalue $-2$ is fixed and cannot be moved, and only the controllable/observable eigenvalue (originally 0) can be assigned, and it depends only upon the sum $k_1 + k_2$ or $l_1 + l_2$. Therefore, $K$ is not unique since any $K$ with $k_1 + k_2 = 2$ gives the same pair of closed-loop eigenvalues. Likewise, $L$ is not unique, so for any $L$ with $l_1 + l_2 = 2$ gives the same eigenvalues for $A - LC$.

## 5.3 Part C

**Example 5.3.** Simulate this system in MATLAB/Simulink starting with initial state

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

and show results of the stabilization of the closed-loop system with state estimator. Include your MATLAB code / Simulink diagram.

Below is the MATLAB Code that correctly models this:

```
% Problem 5c: Closed-Loop System with State-Estimator

% -------------------------------------------------------------
% System Matrices

A = [-1 1; 1 -1];
B = [1; 1];
C = [1 1];
D = 0;

% State-Feedback and Observer Gains

K = [2 0];
L = [2; 0];

A_cl = [A -B*K; L*C A-L*C - B*K];
B_cl = zeros(4,1);
C_cl = eye(4);
D_cl = zeros(4,1);

% -----------------------------------------------------------
% State-Space Model of Closed-Loop System with Estimator

sys_cl = ss(A_cl, B_cl, C_cl, D_cl);

% Initial Conditions

```

```
x0 = [1;2];
xhat0 = [0;0];
z0 = [x0; xhat0];

% Time Vector

t = 0:0.01:10;

% --------------------------------------------------------------
% Simulating and Graphing It

[y, t, x] = initial(sys_cl, z0, t);

% Plot actual states

figure;
plot(t, x(:,1), t, x(:,2), 'LineWidth', 1.5); grid on;
xlabel('Time (s)');
ylabel('States');
legend('x_1','x_2');
title('Closed-loop states with estimator');

% Plot estimated states
figure;
plot(t, x(:,3), t, x(:,4), 'LineWidth', 1.5); grid on;
xlabel('Time (s)');
ylabel('Estimated states');
legend('\hat{x}_1','\hat{x}_2');
title('Estimated states');

% --------------------------------------------------------------
```

We get the following figures.

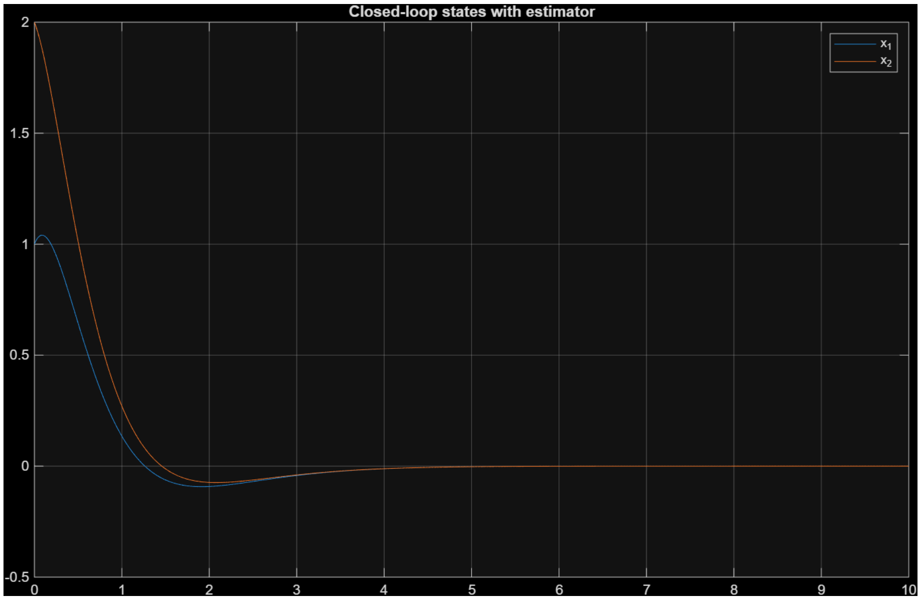



Figure 3: This plot shows that the true states $x_1$ and $x_2$ exponentially converge to zero under the designed state-feedback controller.

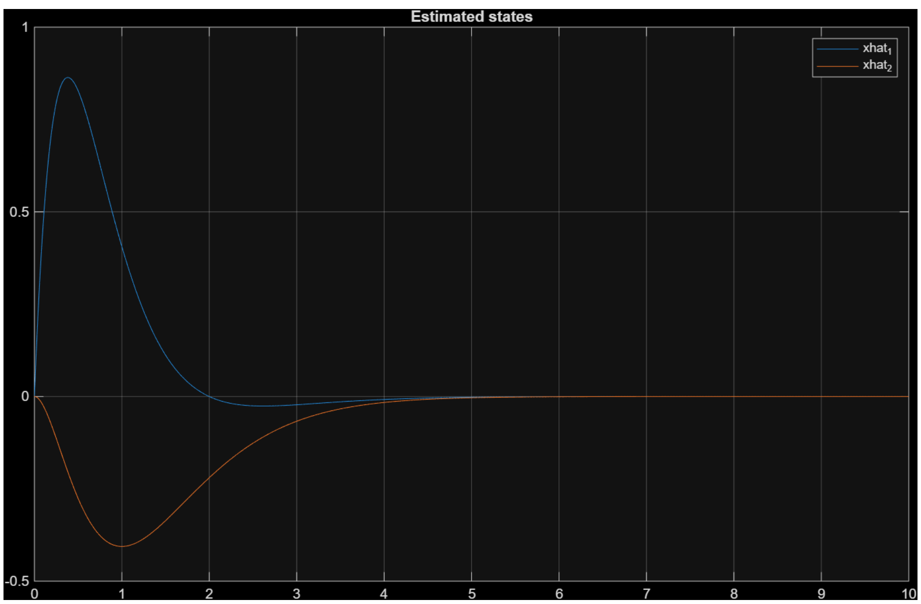



Figure 4: This plot shows that the estimator states $\hat{x}_1$ and $\hat{x}_2$ converge to the true states, indicating that the observer correctly reconstructs the system dynamics.