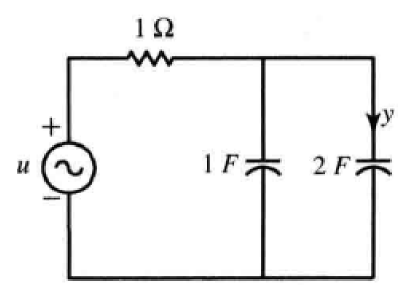
ECE 6200: Linear Systems Analysis

Homework 2 (100 pts.)

Due on Thursday, October 9, 2025, in class

Problem 1 (25 pts.):

Find two state-space models for the following circuit, one of which needs to be of second-order and the other one needs to be of first-order. Obtain also the transfer function of the circuit. Will the transfer function be the same for both the state-space models? Explain.



Problem 2 (25 pts.):

Consider the second-order ODE shown below. Use MATLAB to create a transfer function system object. Then convert it into a state-space object. Apply the 'impulse( )' command to any of the objects to plot the impulse response of the system. Also obtain the analytical expression of the impulse response. Explain why the impulse response characteristics look like that of a first-order system.

$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u$$

Problem 3 (50 pts.):

Continue building up on the Satellite Azimuth Control case study in Lecture 3. Obtain a combined transfer function from the power amplifier input $V_p(s)$ to the antenna azimuth output $\theta_o(s)$. Also obtain a combined state-space model (state and output equations) with $u(t) = v_p(t)$ as the input and $y(t) = \theta_o(t)$ as the output. You can use the existing choice of state variables as used for the subsystems in the lecture. Reproduce the MATLAB/Simulink simulations carried out in the class. Attach your MATLAB code/Simulink diagram.

ECE 6200 Homework II

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9 October 2025

1 Question 1

Find two state-space models for the following circuit, one of which needs to be of second-order and the other one needs to be of first-order. Obtain also the transfer function of the circuit. Will the transfer function be the same for both the state-space models? Explain.

1.1 Solution

So we have a voltage source $u(t)$ that powers a 1Ω series resistor that feeds a node with two shunt capacitors, $C_1 = 1\text{F}$ and $C_2 = 2\text{F}$ that are connected. The output is $y(t)$ which is the node voltage across both of these shunt capacitors.

1.1.1 Find the Minimal State-Space Model

Now, let us find the minimal or first-order state-space model. We proceed by applying KCL at the node:

Proof. Let us begin.

$$i_R = i_{C_1} + i_{C_2}$$

Then,

$$\frac{u-y}{1} = C_1\dot{y} + C_2\dot{y} = (C_1 + C_2)\dot{y}$$

From here, we substitute $C_1 + C_2 = 3$:

$$3\dot{y} + y = u$$

Next, we need to choose the single state variable $x = y$:

$$\dot{x} = -\frac{1}{3}x + \frac{1}{3}u, y = x$$

Hence, our state-space matrices are as follows:

$$A = [-\tfrac{1}{3}], \qquad B = [\tfrac{1}{3}], \qquad C = [1], \qquad D = [0].$$

and our transfer function is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{3s+1}$$

Therefore, we have our **minimal realization** of the minimal/first-order state-space model. ■

1.1.2 Find the Non-Minimal State-Space Model

Now we need to find the second-order/non-minimal state-space model.

Proof. Let the capacitor voltages be separate states:

$$x_1 = v_{C_1}, \qquad x_2 = v_{C_2} = y.$$

Since the capacitors are sharing the same node, $v_{C_1} = v_{C_2}$ and it follows that $\dot{v}_{C_1} = \dot{v}_{C_2}$. Now, by applying Kirchoff's Current Law, it follows that:

$$u - x_2 = C_1\dot{x}_1 + C_2\dot{x}_2 = 1\dot{x}_1 + 2\dot{x}_2$$

Then, by using $\dot{x}_1 = \dot{x}_2$, we arrive at:

$$\dot{x}_1 = \dot{x}_2 = \frac{u - x_2}{3}$$

Thus, we get the following cases:

$$\begin{cases} \dot{x}_1 = -\frac{1}{3}x_2 + \frac{1}{3}u, \\ \dot{x}_2 = -\frac{1}{3}x_2 + \frac{1}{3}u, \\ y = x_2 \end{cases}$$

Now, let us define our state-space matrices:

$$A = \begin{bmatrix} 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

So our eigenvalues here are $\{0, -\frac{1}{3}\}$. Our zero eigenvalue comes from a redundant (a.k.a unobservable/uncontrollable) algebraic constraint reflecting that the two given capacitor voltages must always be equivalent. Therefore, our input-output transfer functions must always be equivalent and is defined as follows:

$$G(s) = \frac{1}{3s+1}$$

■

1.1.3 Concluding Thoughts

So, to sum up what is covered in this solution, we see that both the first-order and second-order realizations yield the same exact transfer function. The transfer function depends only on the external input-output relationship, not on the specific state-variable choices. The non-minimal model included an extraneous state that doesn't affect the output.

2 Question 2

Consider the second-order ODE shown below. Use MATLAB to create a transfer function system object. Then convert it into a state-space object. Apply the 'impulse( )' command to any of the objects to plot the impulse response of the system. Also obtain the analytical expression of the impulse response. Explain why the impulse response characteristics look like that of a first order system.

$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u$$

2.1 Solution

2.1.1 Find the Transfer Function

We need to take the Laplace Transform and assume zero-initial conditions:

$$(s^2 + 2s - 3)Y(s) = (s - 1)U(s)$$

Then,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s-1}{s^2 + 2s - 3}$$

it follows that:

$$G(s) = \frac{s-1}{(s+3)(s-1)} = \frac{1}{s+3}.$$

We essentially factored our denominator of $G(s)$ and obtained:

$$G(s) = \frac{1}{s+3}$$

Therefore, it follows that a pole-zero cancellation at $s = 1$ leaves a single pole at $s = -3$, and thus the given system is effectively *first-order*.

2.1.2 Analytical Impulse Response

Now, we are going to analyze our impulse response.

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}u(t)$$

Hence,

$$g(t) = e^{-3t}, \qquad t \geq 0.$$

2.1.3 MATLAB Implementation

The below plot implemented in MATLAB shows a single decaying exponential, e^{-3t}.

```
% Problem 2 MATLAB Implementation

num = [1 -1]; % This gives the numerator (s - 1)
den = [1 2 -3];     % The denominator (s^2 + 2s - 3)

G = tf(num, den) % Simplifies to 1/(s + 3)

sys_ss = ss(G) % The minimal state-space form

figure;
impulse(G);
grid on;
title('Impulse Response: G(s) = 1/(s + 3)');
```

From this, we get the following output in our MATLAB terminal.

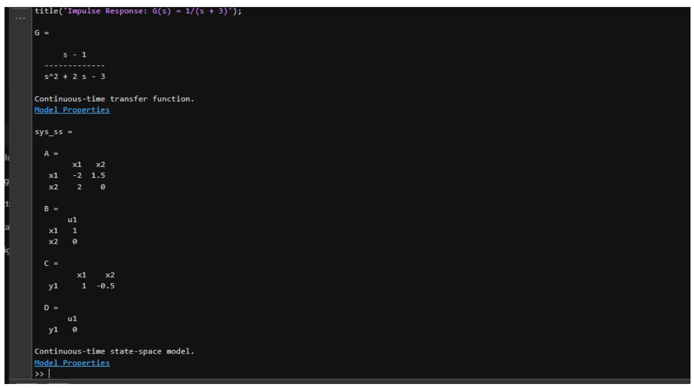


Figure 1: MATLAB Problem 2 Output

The output graph will be included the final submission.

2.1.4 Concluding Thoughts

We see that see the original differential equation is second-order; however, the exact pole-zero cancellation eliminates one dynamic mode. There is only the pole located at $s = -3$ remains, hence the system behaves similarly to a first-order system.

3 Question 3

Continue building up on the Satellite Azimuth Control case study in Lecture 3. Obtain a combined transfer function from the power amplifier input $V_p(s)$ to the antenna azimuth output $\theta_o(s)$. Also obtain a combined state-space model (state and output equations) with $u(t) = v_p(t)$ as the input and $y(t) = \theta_o(t)$ as the output. You can use the existing choice of state variables as used for the subsystems in the lecture. Reproduce the MATLAB/Simulink simulations carried out in the class. Attach your MATLAB code/Simulink diagram.

3.1 Solution

3.1.1 Combine the Subsystems

From $V_p(s)$ or the power amplifier input, through motor and load to $\theta_o(s)$, we get:

$$G(s) = G_{PA}(s) \cdot G_{ML}(s)$$

Next, we will substitute:

$$G(s) = \frac{K_A}{s + a_A} \cdot \frac{K_m}{s(s + a_m)}$$

Then, let's combine our constants to get:

$$G(s) = \frac{K}{s(s + a_A)(s + a_m)}, \qquad \text{where} K = K_A K_m$$

We can observe from this transfer function that it is a *third-order transfer function* with one integrator and two real poles. Now, our combined transfer function is defined as:

$$\frac{\theta_o(s)}{V_p(s)} = \frac{K_A K_m}{s(s + a_A)(s + a_m)}$$

Now, if we include the preamplifier or K_p and the feedback potentiometer K_{pot} later in the closed-loop analysis, they will multiply in series:

$$G_{open}(s) = K_p K_{pot} \frac{K_A K_m}{s(s + a_A)(s + a_m)}$$

We do not need this, so we will stop before K_p.

3.1.2 Choose the State Variables

So we will use our familiar notation used in Lecture 3. Let $x_1 = \theta_o$ be our antenna azimuth angle, $x_2 = \dot{\theta}_o$ be our angular velocity, and x_3 be our power-amp internal state (armature current / internal voltage). We will derive a cascade realization.

From our transfer function:

$$G(s) = \frac{K}{s(s + a_A)(s + a_m)}$$

we will then multiply both sides by our denominator and obtain:

$$s(s + a_A)(s + a_m)\Theta_o(s) = KV_p(s)$$

By expanding, we obtain:

$$(s^3 + (a_m + a_A)s^2 + a_m a_A s)\Theta_o(s) = KV_p(s)$$

Finally, we apply the inverse Laplace Transform (time-domain):

$$\theta_o^{(3)} + (a_m + a_A)\ddot{\theta}_o + a_m a_A \dot{\theta}_o = Kv_p(t)$$

Therefore, we have now obtained our state variables.

3.1.3 Obtain State Equations

Let

$$x_1 = \theta_o, \quad x_2 = \dot{\theta}_o, \quad x_3 = \ddot{\theta}_o,$$

Then, it will follow that:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(a_m + a_A)x_3 - a_m a_A x_2 + K, u \end{cases}$$

with $u(t) = v_p(t)$, $y = x_1$. Next, we need to obtain our state-space matrices for this problem:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_m a_A & -(a_m + a_A) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Then, it must be the case that:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_m a_A & -(a_m + a_A) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

Therefore, we have now effectively obtained our state-space model.

3.1.4 Example Numerical Substitution

Let us consider from Lecture 3, the following constants:

$$a_m = 25, \quad a_A = 100, \quad K_A K_m = 250$$

It then follows that:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2500 & -125 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 250 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0$$

Now, we will implement this into MATLAB.

```
% Problem 3: Antenna Azimuth Contol -  MATLAB Implementation

a_m = 25;
a_A = 100;
K = 250; % K = K_A * K_m

A = [0 1 0;
     0 0 1;
     0 -a_m * a_A -(a_m + a_A)];
B = [0; 0; K];
C = [1 0 0];
D = 0;

sys = ss(A, B, C, D);
G = tf(sys)
```

```
disp('Combined Transfer Function G(s) = theta_o(s)/V_p(s):');
G

% Impulse and Step Responses
figure; impulse(sys); grid on; title('Impulse Response: Antenna Azimuth System');
figure; step(sys); grid on; title('Step Response: Antenna Azimuth System');
```

The result is a 3rd-order response.
This is our output in the terminal:

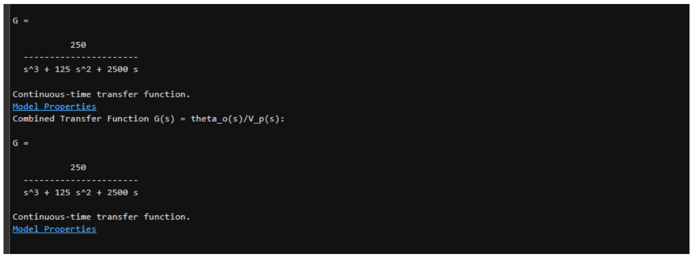
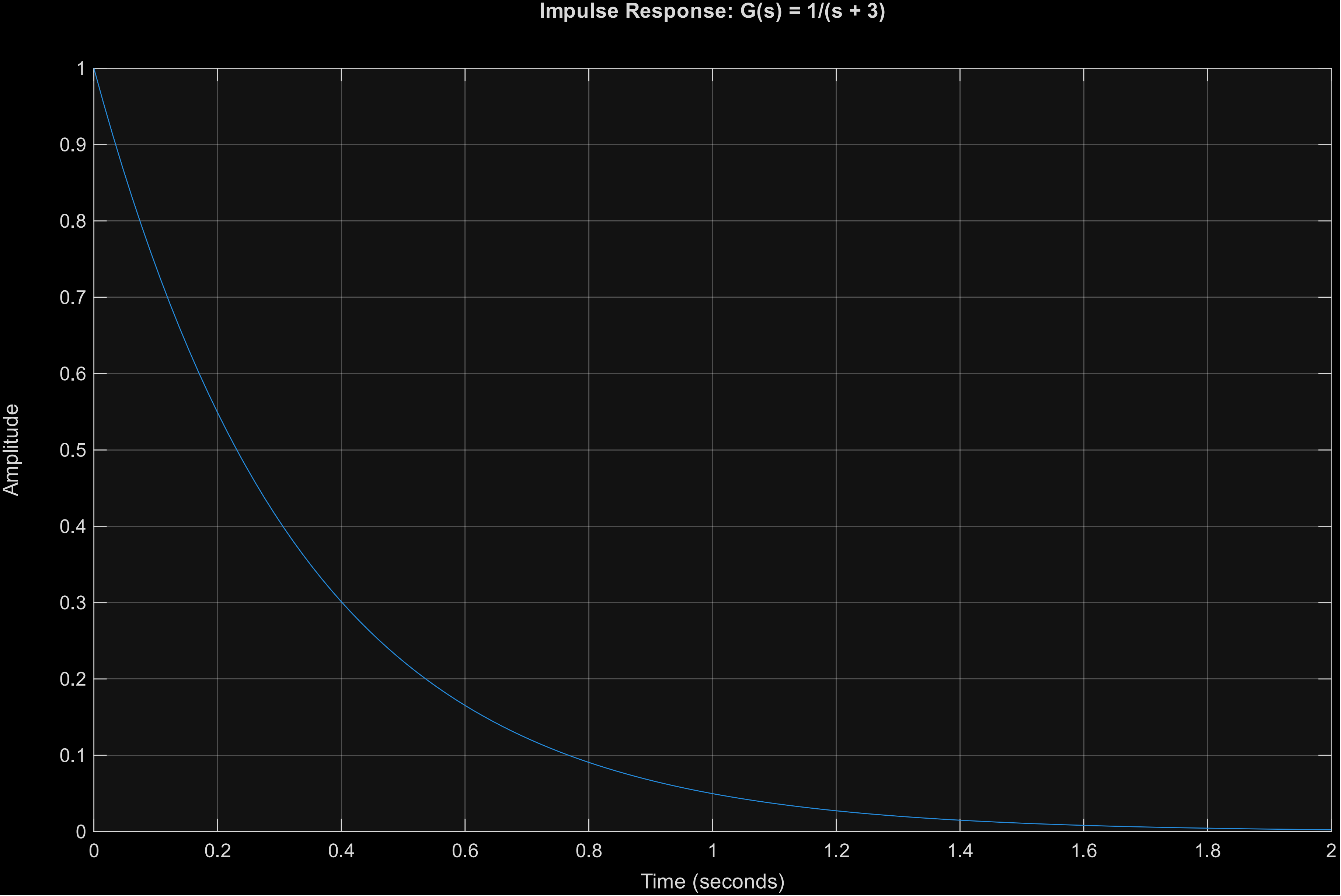
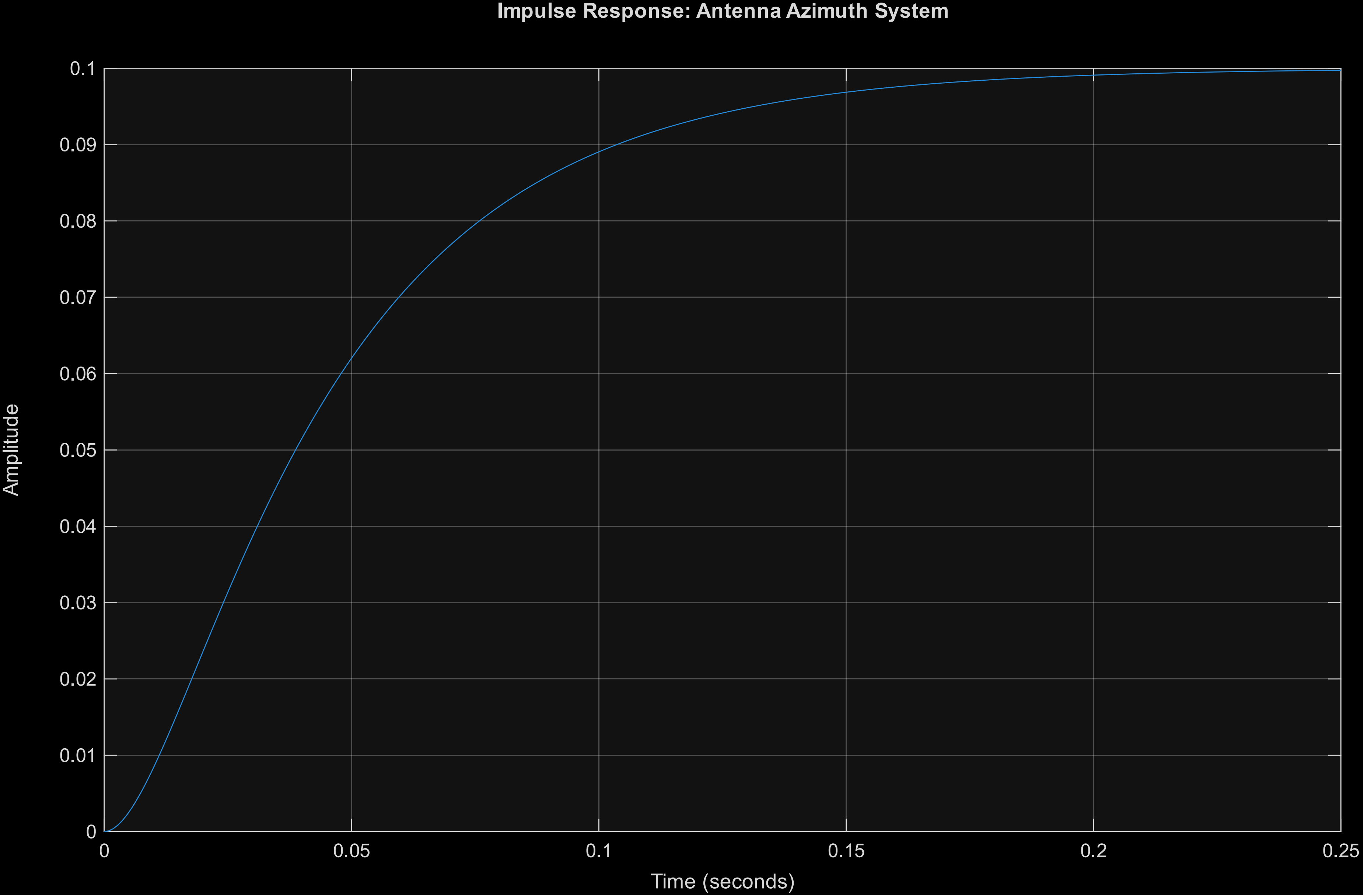


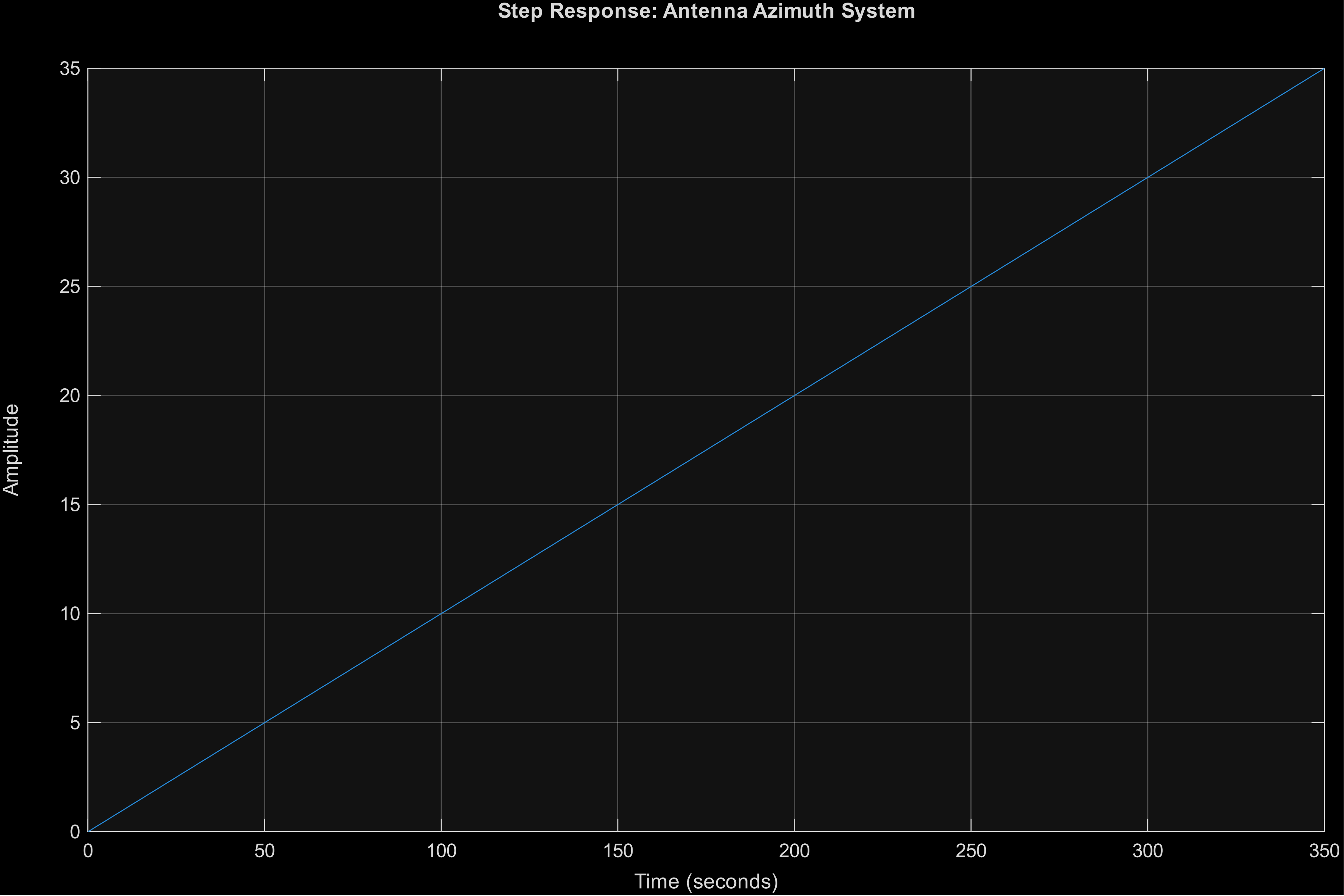
Figure 2: MATLAB Problem 3 Terminal Output



Impulse Response: G(s) = 1/(s + 3)
Amplitude
Time (seconds)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



Impulse Response: Antenna Azimuth System
Amplitude
Time (seconds)
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.1
0
0.05
0.1
0.15
0.2
0.25


Step Response: Antenna Azimuth System
Amplitude
Time (seconds)
0
5
10
15
20
25
30
35
0
50
100
150
200
250
300
350