ECE 6200 Homework I

Blaine Swieder

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1 Question 1

Determine using the superposition principle whether the dynamic system with input $u(t)$ and output $y(t)$ given given by the following integral-differential equation is linear or nonlinear. Determine time-variance/invariance by applying the time-shifting property. Hint: You don't necessarily need to solve this equation.

$$\frac{dy(t)}{dt} + t^2 \int_0^t y(\tau)\, d\tau = u(t)$$

1.1 Solution

Let us determine the linearity (superposition) of this system. Let

$$\mathcal{L}[y](t) = \frac{dy}{dt} + t^2 \int_0^t y(\tau)\, d\tau$$

Then it follows that,

$$\mathcal{L}[ay_1 + by_2] = a\,\mathcal{L}[y_1] + b\,\mathcal{L}[y_2]$$

this is due to the derivative and the integral being linear, and t^2 is merely a scalar function of time. Therefore, since $\mathcal{L}[y] = u$, it preserves the a, b superposition principle, and thus **linear**. Now let us apply the time-shifting property. We do this by comparing

$$\mathcal{L}[y(t-t_0)](t) = \frac{d}{dt} y(t-t_0) + t^2 \int_0^t y(\tau - t_0)\, d\tau$$

with

$$\big(\mathcal{L}[y]\big)(t-t_0) = \frac{dy}{dt}(t-t_0) + (t-t_0)^2 \int_0^{t-t_0} y(\alpha)\, d\alpha.$$

Since these two differ, notice that t^2 vs. $(t-t_0)^2$ as well as the integral limits. Hence, the system is **time-varying**. Therefore, by applying the superposition principle, we can state that dynamic system with the input $u(t)$ and the output $y(t)$ given by the integral-differential equation

$$\frac{dy(t)}{dt} + t^2 \int_0^t y(\tau)\, d\tau = u(t)$$

We can firmly state that the given system is **linear** and **time-varying**.

2 Question 2

Friction force between an object and a surface is often described to be composed of three parts: static, Coulomb, and viscous friction, which are related to the velocity. Identify by looking at the plots with friction force as the output and velocity as the input, which of the friction force relationships shown in Figs. (a) and (b) are linear. Draw a third plot (c) that shows a combined curve of Coulomb + viscous friction versus velocity. Would this new plot be linear or nonlinear?

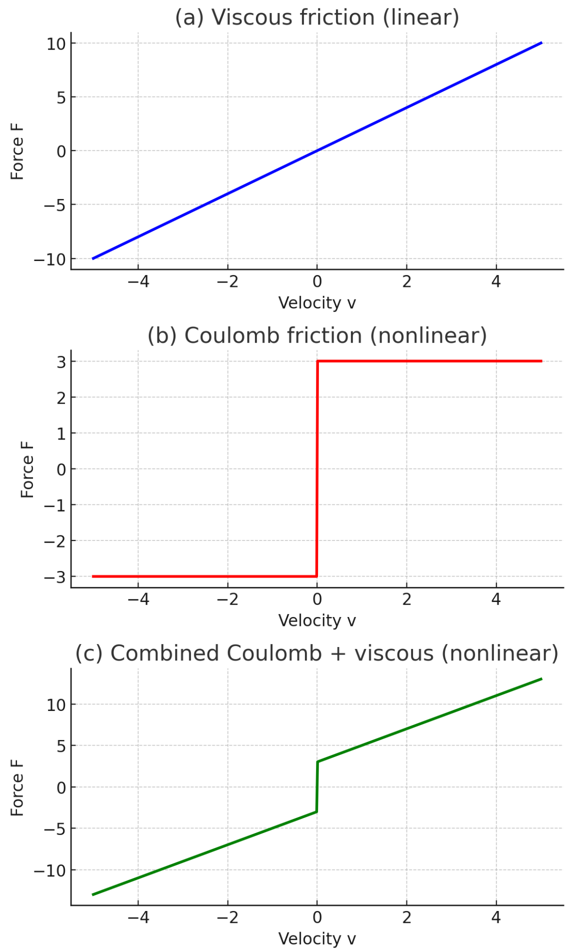


Figure 1: Friction force–velocity characteristics: (a) viscous (linear), (b) Coulomb (nonlinear), (c) combined Coulomb + viscous (nonlinear).

2.1 Solution

So we can draw the following conclusions from these diagrams and their physical definitions:

- **Viscous Friction**: The equation is $F = bv$, and notice that it is a straight line through the origin and hence is *linear*.

- **Coulomb Friction**: The equation is $F = F_c\,\mathrm{sgn}(v)$ and since this produces constant $\pm F_c$ forces that are independent of v, it is the case that it is *nonlinear*.
- **Static Friction**: We see that it exhibits a dead zone around $v = 0$ up to the breakaway force and this behavior also turns out to be **nonlinear**.

Now, let us consider combined Coulomb and Viscous. When Coulomb and viscous effects act together, it follows that the model is

$$F(v) = F_c\,\mathrm{sgn}(v) + bv.$$

And this produces slanted lines with a slope of b and a discontinuous offset at $v = 0$. It may be true that it is piecewise affine, but it does not satisfy our definition of homogeneity nor superposition, hence it is *nonlinear*.

3 Question 3

Determine if the system given by the input-output relationship with an integer time index k

$$y[k] = ky[k-1] + u[k]$$

is

1. discrete/continuous-time,
2. linear/nonlinear,
3. lumped/distributed,
4. causal/noncausal,
5. time-varying/time-invariant.

3.1 Solution

Here are my solutions to this problem:

- **Part A**: This system is *discrete-time system* as its index $k \in \mathbb{Z}$.
- **Part B**: The given system is *linear system* is that there are no products or powers of signals.
- **Part C**: The given system is a *lumped system* as it is a finite-order difference equation and there is no spatial distribution.
- **Part D**: The given system is a *casual system* as $y[k]$ depends only on $y[k-1]$ and $u[k]$.
- **Part E**: The given system is *time-varying system* as its coefficient k depends on a time index.

4 Question 4

A quarter-car vehicle suspension model is shown below. Obtain a fourth-order state-space model of this system with states being the position and velocities of the two masses. Explicitly mention all system matrices and the associated signal vectors. You may choose states x_1 and x_2 as position and velocity of mass m_1, respectively, and similarly x_3 and x_4 for the mass m_2. The outputs are the positions y_1 and y_2 of the two masses, respectively. Note that the input u in this problem is just the vertical displacement of the varying road surface. The effect of gravity is omitted as the model is relative to an equilibrium position.

4.1 Solution

First, we consider the quarter-car vehicle model with states:

$$\begin{aligned} x_1 &= z_1, & x_2 &= \dot{z}_1, \\ x_3 &= z_2, & x_4 &= \dot{z}_2, \\ u &= z_r. \end{aligned}$$

Next, the equations of motion:

$$\begin{aligned} m_1\ddot{z}_1 &= -c_s(\dot{z}_1 - \dot{z}_2) - k_s(z_1 - z_2), \\ m_2\ddot{z}_2 &= c_s(\dot{z}_1 - \dot{z}_2) + k_s(z_1 - z_2) - k_t(z_2 - u). \end{aligned}$$

Third, the **state equations**:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\tfrac{k_s}{m_1}x_1 - \tfrac{c_s}{m_1}x_2 + \tfrac{k_s}{m_1}x_3 + \tfrac{c_s}{m_1}x_4, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \tfrac{k_s}{m_2}x_1 + \tfrac{c_s}{m_2}x_2 - \tfrac{k_s+k_t}{m_2}x_3 - \tfrac{c_s}{m_2}x_4 + \tfrac{k_t}{m_2}u. \end{aligned}$$

The system can be written in standard form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \end{bmatrix}, \quad u = z_r, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Finally, the **state-space matrices**

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_s}{m_1} & -\frac{c_s}{m_1} & \frac{k_s}{m_1} & \frac{c_s}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_2} & \frac{c_s}{m_2} & -\frac{k_s+k_t}{m_2} & -\frac{c_s}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_t}{m_2} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5 Question 5

Problem 2.10. Only find state-space model (no transfer function needed).

5.1 Solution

Let us find solely the state-space model, by using the natural energy variables:

- $x_1 = v_{C_1}$. This is the voltage at node B to the ground.
- $x_2 = v_{C_2} = v_B - v_A$, left-to-right across C_2.
- $x_3 = i_L$ (inductor current, downward)
- The input u is the source voltage, its output $y = v_A$.

From KCL at node B:

$$\frac{u - x_1}{R} = C_1\dot{x}_1 + C_2\dot{x}_2$$

Through C_2 into L:

$$i_{C_2} = C_2\dot{x}_2 = i_L = x_3$$

and our Inductor is:

$$v_A = L\dot{x}_3$$

and also

$$v_A = v_B - v_{C_2} = x_1 - x_2$$

Now, for the given circuit our state equations and its output are:

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{C_1}\left(\frac{u - x_1}{R} - x_3\right) = -\frac{1}{RC_1}x_1 - \frac{1}{C_1}x_3 + \frac{1}{RC_1}u,\\
\dot{x}_2 &= \frac{1}{C_2}x_3,\\
\dot{x}_3 &= \frac{1}{L}(x_1 - x_2),\\
y &= x_1 - x_2.
\end{aligned}$$

and Matrix form ($x = [x_1\ x_2\ x_3]^T$):

$$\dot{x} = \underbrace{\begin{bmatrix} -\frac{1}{RC_1} & 0 & -\frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} \frac{1}{RC_1} \\ 0 \\ 0 \end{bmatrix}}_{B} u, \qquad y = \underbrace{\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}}_{C} x, \quad \underbrace{0}_{D}.$$

6 Question 6

Problem 2.17. Also write MATLAB code using your solved problem to generate and verify the output responses shown in the problem. Include your code.

6.1 MATLAB Code

Here is my code, and I have included the figures as well.

```
clear; clc; close all;

N  = 8;         % time index goes from 0..8
k  = 0:N;       % time axis
y0 = 0;         % initial condition
pad = @(z) [0, z];   % pad with zero to shift first jump to t=1

%% (a) Positive feedback, a = 1
a = 1.0; s = +1;
y = zeros(1,N); y_prev = y0;
for n = 1:N
    y(n) = a*(1 + s*y_prev);
    y_prev = y(n);
end
figure('Color','w');
stairs(k, pad(y), 'LineWidth', 2);
xlim([0 8]); ylim([0 6.1]);
xticks(0:1:8); yticks(0:1:6);
xlabel('t'); ylabel('y(t)'); title('(a)  a = 1');
box off;
exportgraphics(gcf,'Question_2_17_Figure 1.png','Resolution',300);

%% (b) Positive feedback, a = 0.5
a = 0.5; s = +1;
y = zeros(1,N); y_prev = y0;
for n = 1:N
    y(n) = a*(1 + s*y_prev);
    y_prev = y(n);
end
figure('Color','w');
stairs(k, pad(y), 'LineWidth', 2);
xlim([0 8]); ylim([0 1.05]);
xticks(0:1:8); yticks([0 0.5 0.75 1]);
xlabel('t'); ylabel('y(t)'); title('(b)  a = 0.5');
box off;
exportgraphics(gcf,'Question_2_17_Figure 2.png','Resolution',300);

%% (c) Negative feedback, a = 1
a = 1.0; s = -1;
y = zeros(1,N); y_prev = y0;
for n = 1:N
    y(n) = a*(1 + s*y_prev);
    y_prev = y(n);
end
figure('Color','w');
stairs(k, pad(y), 'LineWidth', 2);
xlim([0 8]); ylim([-1.05 1.05]);
xticks(0:1:8); yticks([-1 0 1]);
xlabel('t'); ylabel('y(t)'); title('(c)  a = 1');
box off;
exportgraphics(gcf,'Question_2_17_Figure 3.png','Resolution',300);

%% (d) Negative feedback, a = 0.5
a = 0.5; s = -1;
```

```
y = zeros(1,N); y_prev = y0;
for n = 1:N
    y(n) = a*(1 + s*y_prev);
    y_prev = y(n);
end
figure('Color','w');
stairs(k, pad(y), 'LineWidth', 2);
xlim([0 8]); ylim([0 0.52]);
xticks(0:1:8); yticks([0 0.25 0.5]);
xlabel('t'); ylabel('y(t)'); title('(d)  a = 0.5');
box off;
exportgraphics(gcf,'Question_2_17_Figure 4.png','Resolution',300);
```

6.2 Solution

Here is the figure for Part A.

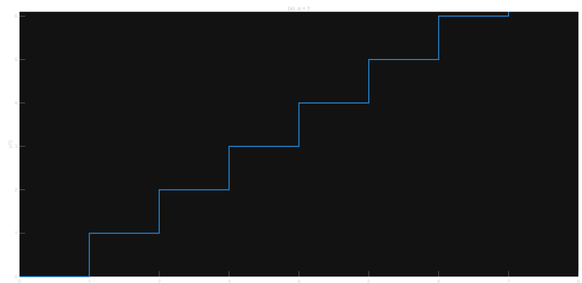


Figure 2: MATLAB Plot for Part A

Here is the figure for Part B.

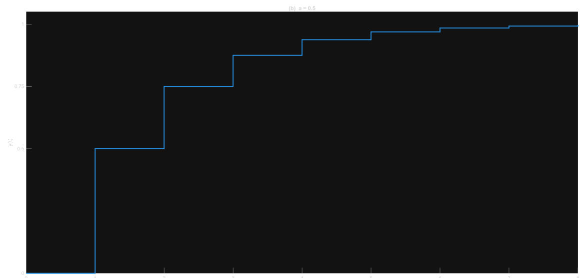


Figure 3: MATLAB Plot for Part B

Here is the figure for Part C.

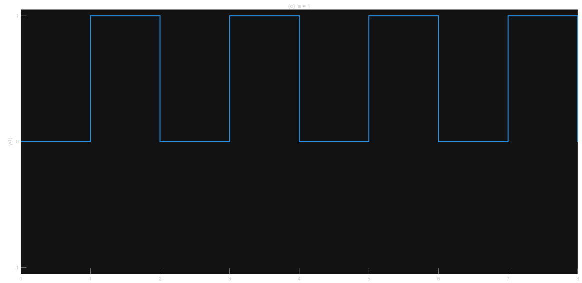


Figure 4: MATLAB Plot for Part C

Here is the figure for Part D.

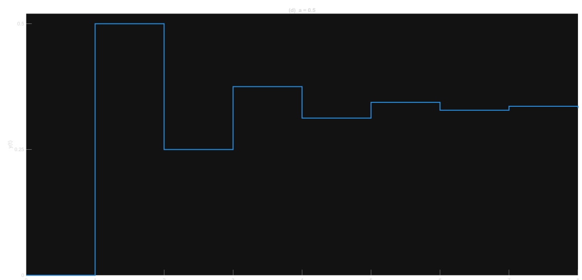


Figure 5: MATLAB Plot for Part D

2.16 Consider a two-input and two-output system described by

$$D_{11}(p)y_1(t) + D_{12}(p)y_2(t) = N_{11}(p)u_1(t) + N_{12}(p)u_2(t)$$
$$D_{21}(p)y_1(t) + D_{22}(p)y_2(t) = N_{21}(p)u_1(t) + N_{22}(p)u_2(t)$$

where N_{ij} and D_{ij} are polynomials of $p := d/dt$. What is the transfer matrix of the system?

2.17 Consider the feedback systems shown in Fig. 2.5. Show that the unit-step responses of the positive-feedback system are as shown in Fig. 2.23(a) for $a = 1$ and in Fig. 2.23(b) for $a = 0.5$. Show also that the unit-step response of the negative-feedback system are as shown in Figs. 2.21(c) and 2.21(d), respectively, for $a = 1$ and $a = 0.5$.

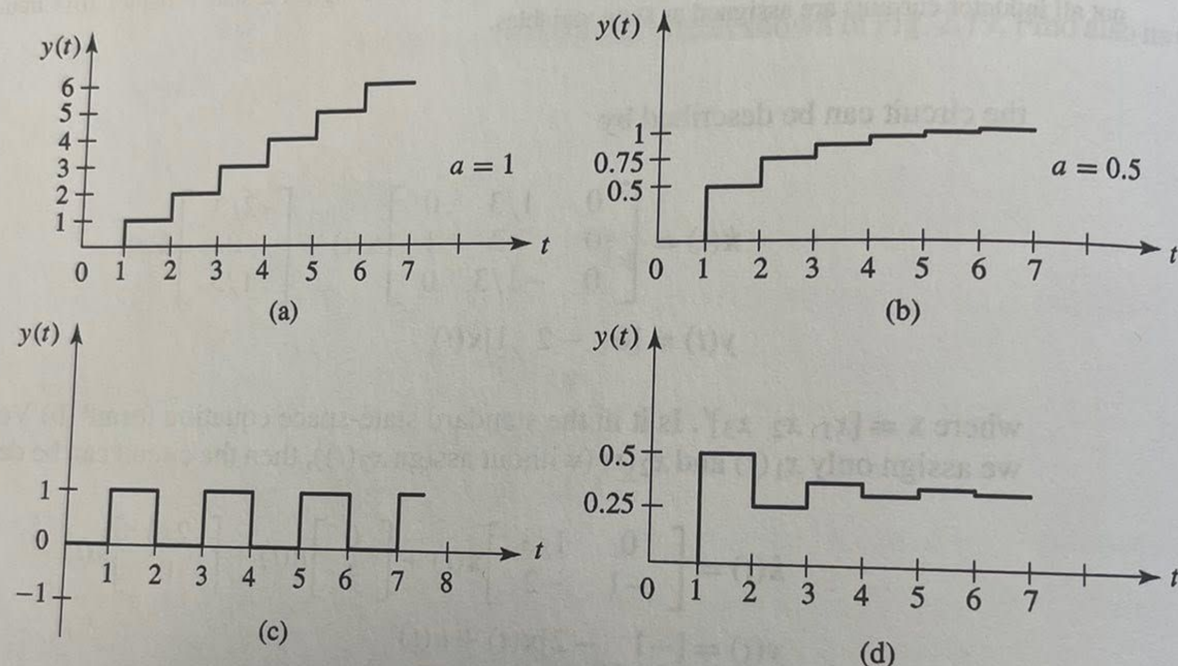


FIGURE 2.23

2.18 Find state-space equations to describe the pendulum systems in Fig. 2.24. The systems are useful to model one- or two-link robotic manipulators. If θ, θ_1, and θ_2 are very small, can you consider the two systems as linear?

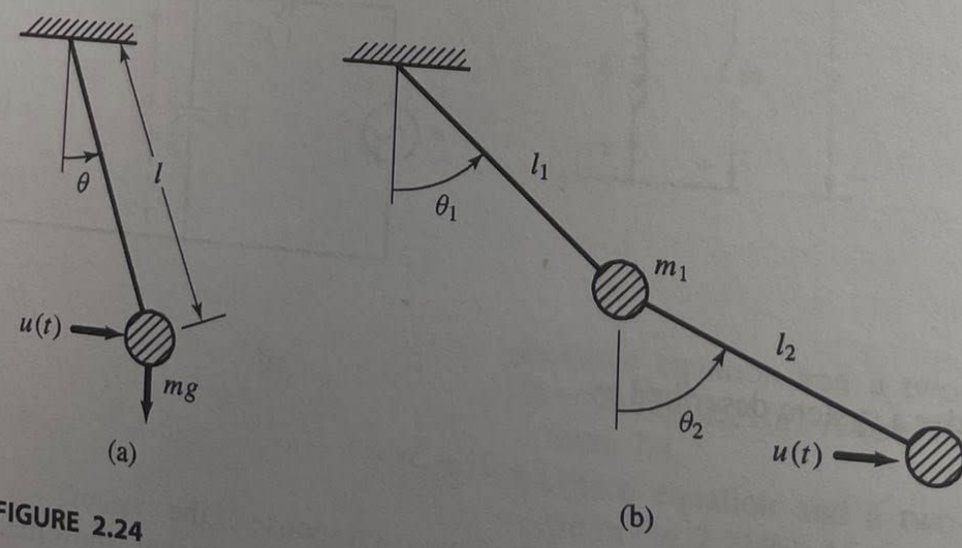
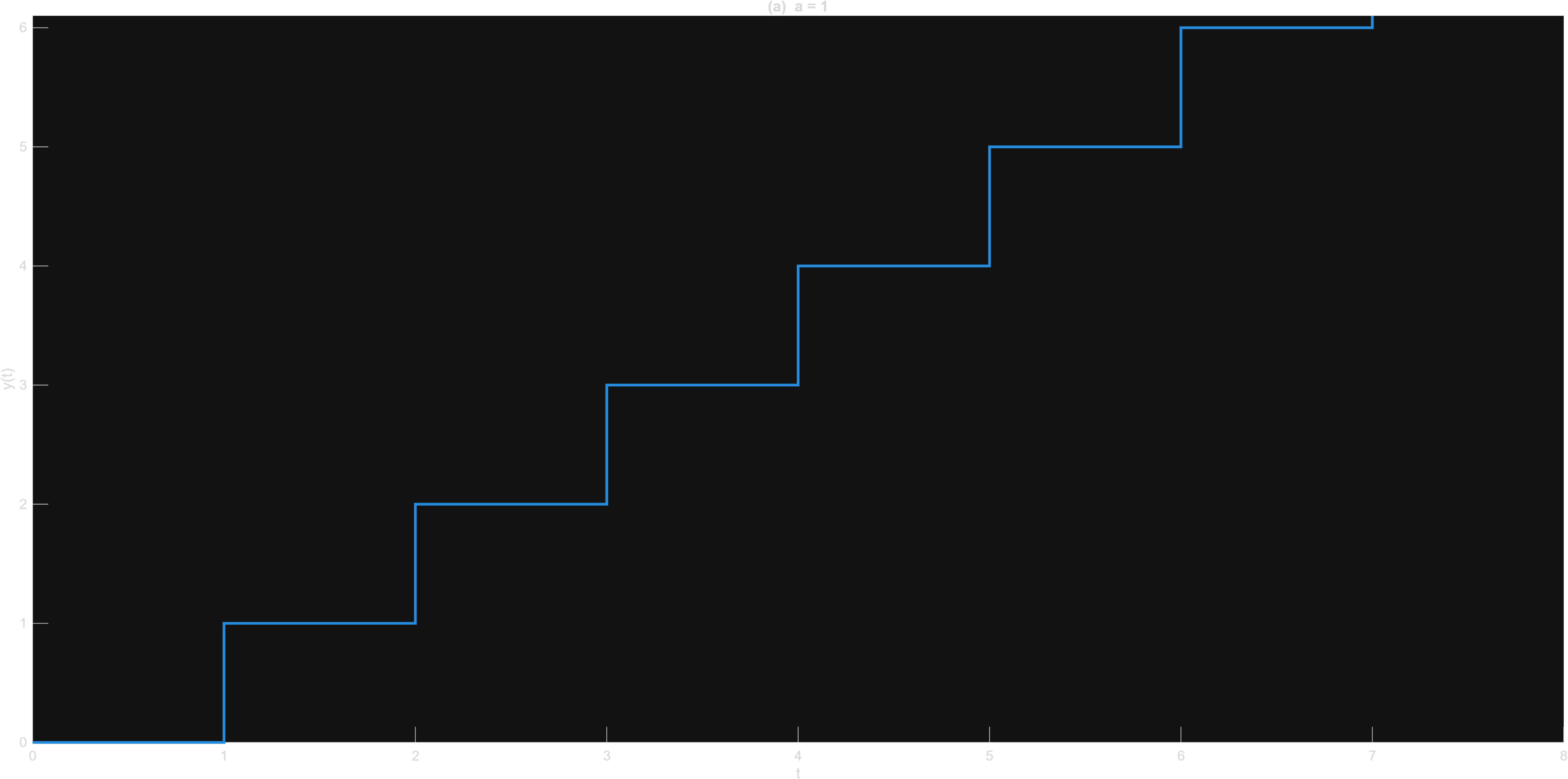
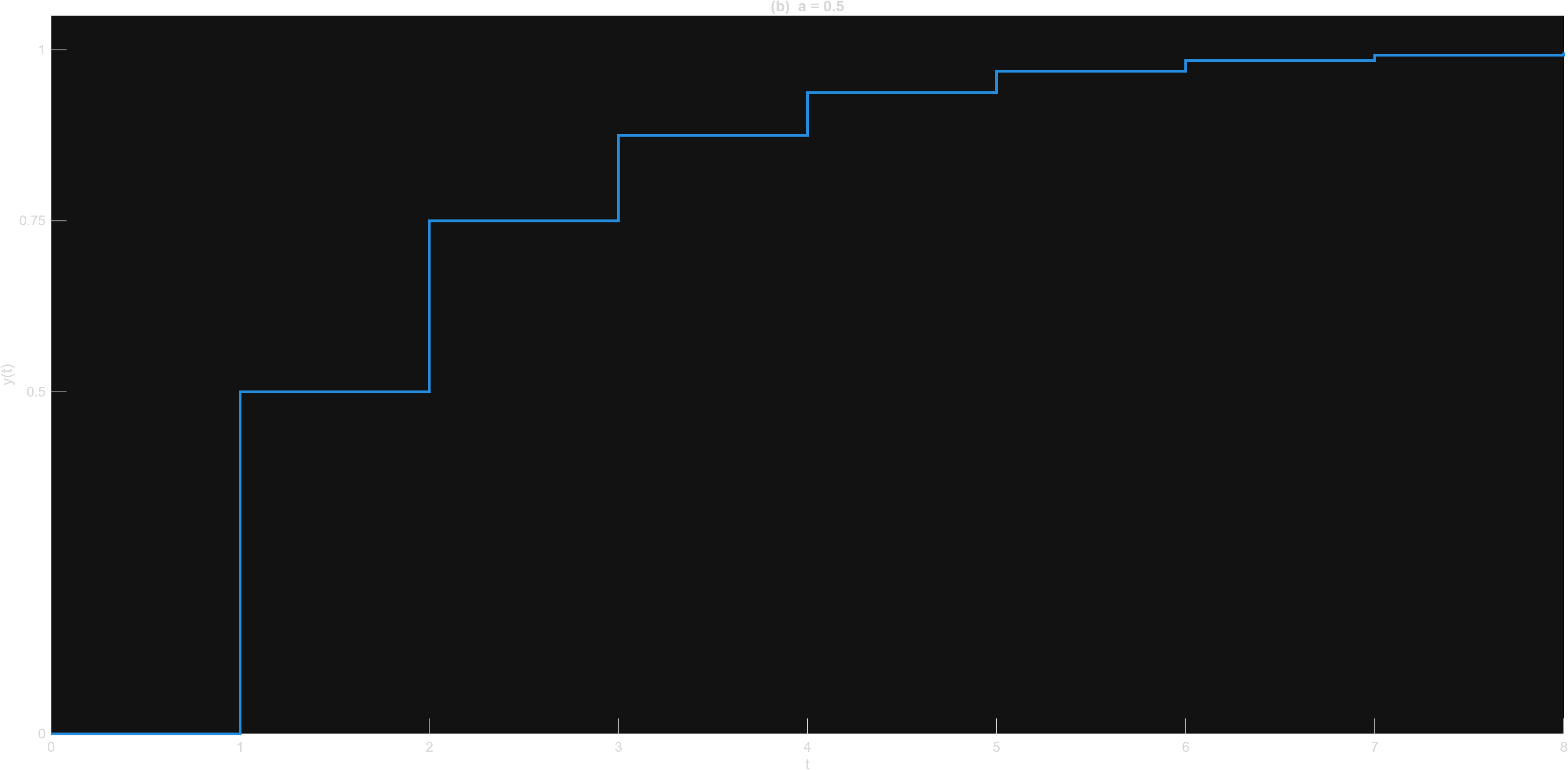
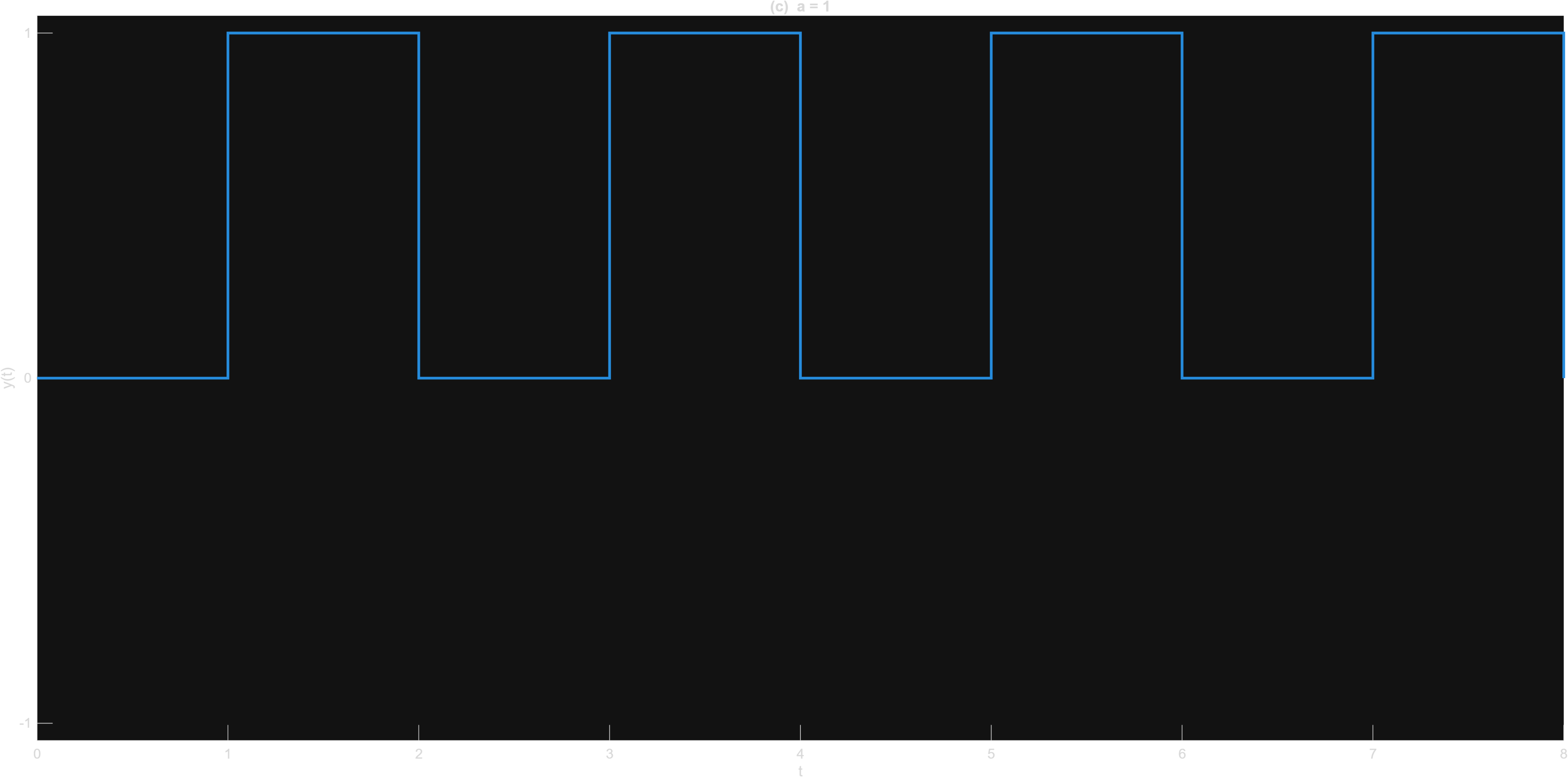


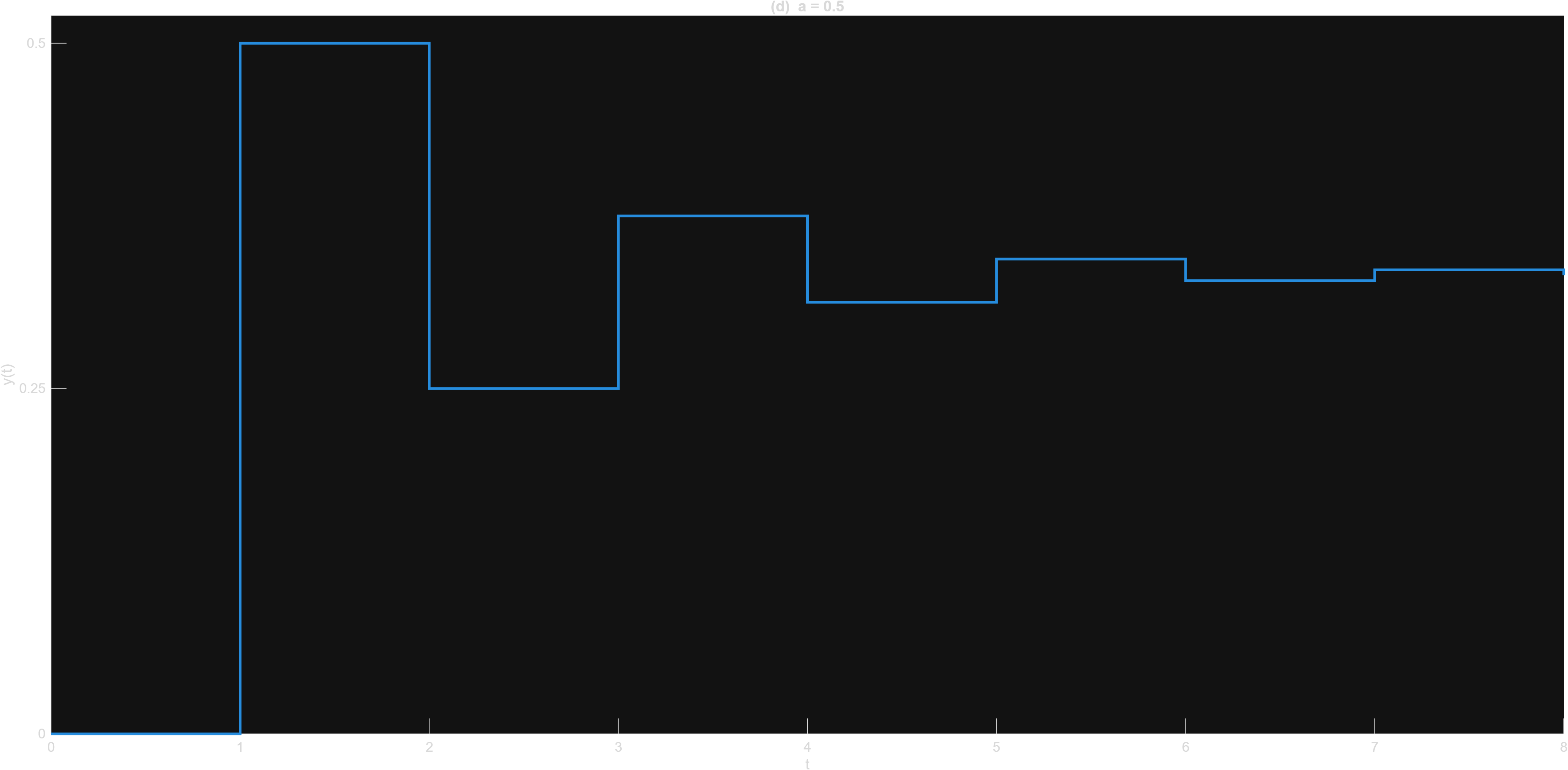
FIGURE 2.24



(a) a = 1
y(t)
t
0
1
2
3
4
5
6
7
8


(b) a = 0.5
y(t)
1
0.75
0.5
0
0
1
2
3
4
5
6
7
8
t


(c) a = 1
1
0
-1
y(t)
0
1
2
3
4
5
6
7
8
t


(d) a = 0.5
y(t)
0.5
0.25
0
0
1
2
3
4
5
6
7
8
t