ECE 6200: Linear Systems Analysis

Test 2 (100 pts.)

Due by Tuesday, November 25, 2025, in class

Instructions:

- The test has three problems. Read all problems before starting to solve.
- The exam is open book / open lecture with MATLAB usage allowed.
- Show your steps when solving the problems to receive full credit.
- Sign and attach this page when you return the exam.

Name: ________________________

Honor Pledge: " On my honor, I have neither given nor received any aid in this test."

Signature: ______________________

Problem 1 (30 pts.):

a. (10 pts.) State whether the following statements are True or False. Justify with reasoning.

i. The null space of a non-singular matrix can contain non-zero vectors.

ii. Transfer functions are invariant under similarity transformation.

iii. A system that is stable in the Lyapunov sense is guaranteed to be BIBO stable.

iv. The order of the transfer function of an uncontrollable system is the same as that of its state-space representation.

v. The convolution description may be used to analyze the observability of an LTI system.

b. (10 pts.) Obtain a basis matrix $\mathbf{Q}$ for the Jordan representation of

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}.$$

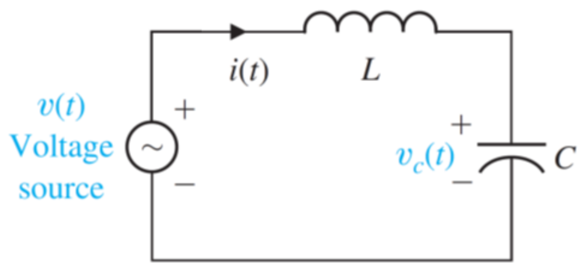
Find also the null basis and the minimal polynomial.

c. (10 pts.) Use any two methods to determine $e^{\mathbf{A}t}$ for the matrix in part (b).

Problem 2 (30 pts.):

a. **(5 pts.)** Show that $e^{\mathbf{P}^{-1}\mathbf{AP}t} = \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}$ for any t and some invertible matrix $\mathbf{P}$.

b. **(15 pts.)** Determine the BIBO stability, marginal stability, and asymptotic stability of the following circuit. Perform Lyapunov analysis with an appropriate choice of a Lyapunov function to determine the conditions that are aligned with your previous conclusions.



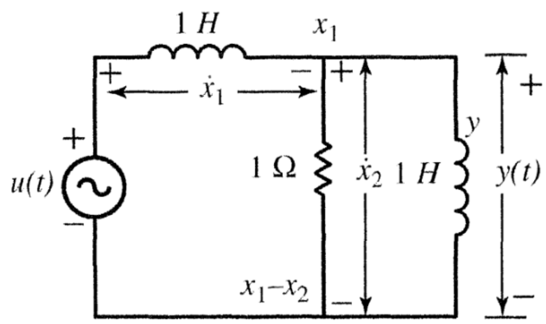
You may choose $L = 1$ H and $C = 1$ F, and the two states x_1 and x_2 as the capacitor voltage $v_C(t)$ and inductor current $i_L(t)$, respectively.

c. **(10 pts.)** Determine the range of the parameter a such that the following system is asymptotically stable. What is the value of a that achieves marginal stability and also determine the frequency at which such a system would oscillate.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2a & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}u$$

Problem 3 (40 pts.):

a. **(15 pts.)** Obtain the state-space and transfer function descriptions and determine BIBO/ asymptotic stability, controllability, and observability. Simplify the circuit to obtain a reduced-order but equivalent circuit. Is the reduced-order circuit controllable and observable?



b. **(15 pts.)** Determine the observability of the system by finding the observability Gramian of the system. Include verification using MATLAB's lyap function.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

c. **(10 pts.)** Given the observability of $(\mathbf{A}, \mathbf{C})$, determine the controllability of $(\mathbf{A}^T, \mathbf{C}^T)$.

ECE 6200 Exam II

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25 November 2025

1 Question I

1.1 Part A

Example 1.1. State whether the following statements are True or False. Justify with reasoning.

- The null space of a non-singular matrix can contain non-zero vectors.
- Transfer functions are invariant under similarity transformation.
- A system that is stable in the Lyapunov sense is guaranteed to be BIBO stable.
- The order of the transfer function of an uncontrollable system is the same as that of its state-space representation.
- The convolution description may be used to analyze the observability of an LTI system.

1.1.1 My Response

- **False**, since a non-singular matrix $\mathbf{A}$ is invertible, so the only solution of $\mathbf{Ax} = 0$ is $\mathbf{x} = 0$. Therefore, its null space contains solely the zero vector.
- **True**, for a realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, the transfer function is

  $$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

  Under a similarity transform $\mathbf{x} = \mathbf{T}\overline{\mathbf{x}}$:

  $$\overline{\mathbf{A}} = \mathbf{TAT}^{-1}, \quad \overline{\mathbf{B}} = \mathbf{TB}, \quad \overline{\mathbf{C}} = \mathbf{CT}^{-1}, \quad \overline{\mathbf{D}} = \mathbf{D}$$

  Then,

  $$\overline{G}(s) = \overline{\mathbf{C}}(s\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{B}} + \overline{\mathbf{D}} = \mathbf{CT}^{-1}(s\mathbf{I} - \mathbf{TAT}^{-1})^{-1}\mathbf{TB} + \mathbf{D} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = G(s).$$

  Therefore, the transfer function is invariant under similarity.
- **False**, the Lyapunov stability refers to the zero-input system $\dot{\mathbf{x}}$, where all trajectories that start near the origin are bounded. BIBO stability is about input–output behavior. For example,

  – Transfer function $G(s) = \frac{1}{s}$ (integrator)
  – The state matrix has eigenvalue 0, so the zero-input system is (marginally) Lyapunov stable.
  – But a bounded input (e.g. unit step) gives unbounded output, so the system is not BIBO stable.

  Therefore, Lyapunov stability does not guarantee BIBO stability.

- **False**, for an uncontrollable or unobservable realization, some internal modes do not appear in the input–output behavior; they cancel in the transfer function. Thus, the transfer function order equals the order of a minimal realization, which can be lower than the state-space order.
- **False**, the convolution description:

$$y(t) = (h * u)(t)$$

  describes only the input–output relationship (via impulse response $h(t)$) with zero initial conditions. Observability is about recovering the state from outputs (and inputs). Different realizations with different observability properties can share the same $h(t)$.Therefore, convolution alone cannot be used to analyze observability.

1.2 Part B

Example 1.2. Obtain a basis matrix $\mathbf{Q}$ for the Jordan representation of

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

Find also the null basis and the minimal polynomial.

1.2.1 My Response

First, we need to find the eigenvalues of the given matrix:

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -4 & -3 \\ 0 & \lambda - 20 & -16 \\ 0 & 25 & \lambda + 20 \end{bmatrix}$$

Now, for the determinant calculation:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda \det \begin{bmatrix} \lambda - 20 & -16 \\ 25 & \lambda + 20 \end{bmatrix} = \lambda^3$$

Thus, the only eigenvalue is $\lambda = 0$ with a multiplicity 3. Next, we need to find and solve the null basis. Solve $\mathbf{Av} = 0$ and by using the y, z equations, we get:

$$\begin{bmatrix} 4 & 3 \\ 20 & 16 \\ -25 & -20 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{4} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow y = z = 0.$$

Then x is free, and $\mathcal{N}(\mathbf{A}) = \text{span}\{(1,0,0)^\top\}$. Hence, the null basis is $\{(1,0,0)^\top\}$. Now, we need to find the minimal polynomial. We have,

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}^3 = 0.$$

Hence, $\mathbf{A}^3 = 0$ but $\mathbf{A}^2 \neq 0$, so

$$m_{\mathbf{A}}(\lambda) = \lambda^3.$$

Finally, we need to find the Jordan Basis and $\mathbf{Q}$. We require a length-3 Jordan chain for $\lambda = 0$:

$$\mathbf{A}v_1 = 0, \quad \mathbf{A}v_2 = v_1, \quad \mathbf{A}v_3 = v_2.$$

Now, let's take $v_1 = \{(1,0,0)^\top\}$. For $v_2 = \{(a,b,c)^\top\}$, $\mathbf{A}v_2 = v_1$ gives:

$$\begin{bmatrix} 4 & 3 \\ 20 & 16 \\ -25 & -20 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow (b,c) = (4,-5).$$

Now, let us choose $a = 0$, $v_2 = \{(0,4,-5)^\top\}$. For $v_3 = \{(a,b,c)^\top\}$, $\mathbf{A}v_3 = v_2$ gives us:

$$\begin{bmatrix} 4 & 3 \\ 20 & 16 \\ -25 & -20 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{4} & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow (b,c) = (-3,4).$$

Choose $a = 0$, $v_3 = \{(0,-3,4)^\top\}$. Finally, a Jordan Basis is $\{v_1, v_2, v_3\}$ and its basis matrix is:

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix}$$

with

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

1.3 Part C

Example 1.3. Use any two methods to determine $e^{\mathbf{A}t}$ for the matrix in part (b).

1.4 My Response

1.4.1 Method I: Power Series and Nilpotency

Recall the power series definition:

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots$$

However, the series truncates, so it follows that:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2}$$

Using the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We get the following:

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 4t + \frac{5}{2}t^2 & 3t + 2t^2 \\ 0 & 1 + 20t & 16t \\ 0 & -25t & 1 - 20t \end{bmatrix}$$

1.4.2 Method II: Jordan Form

From part (b), $\mathbf{A} = \mathbf{Q}\mathbf{J}\mathbf{Q}^{-1}$, with

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{J}^3 = 0.$$

Thus,

$$e^{\mathbf{A}t} = \mathbf{Q}e^{\mathbf{J}t}\mathbf{Q}^{-1}, \quad e^{\mathbf{J}t} = \mathbf{I} + t\mathbf{J} + \frac{t^2}{2}\mathbf{J}^2 = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, by multiplying $\mathbf{Q}e^{\mathbf{J}t}\mathbf{Q}^{-1}$ it gives the same exact matrix as above.

2 Question 2

2.1 Part A

Example 2.1. Show that $e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}t} = \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}$ for all t and every invertible matrix $\mathbf{P}$.

2.1.1 My Response

Proof. Recall the power-series definition

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}.$$

We first show that, for any invertible $\mathbf{P}$,

$$(\mathbf{P}^{-1}\mathbf{AP})^k = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} \quad \text{for all } k \geq 0.$$

For $k = 0$ this is $\mathbf{I} = \mathbf{P}^{-1}\mathbf{IP}$. Suppose it holds for some $k \geq 0$; then

$$(\mathbf{P}^{-1}\mathbf{AP})^{k+1} = (\mathbf{P}^{-1}\mathbf{AP})^k(\mathbf{P}^{-1}\mathbf{AP}) = \mathbf{P}^{-1}\mathbf{A}^k\mathbf{P}\,\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{A}^{k+1}\mathbf{P},$$

so the claim holds for all k by induction.

Then,

$$\begin{aligned} e^{\mathbf{P}^{-1}\mathbf{AP}t} &= \sum_{k=0}^{\infty} \frac{(\mathbf{P}^{-1}\mathbf{AP}t)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(\mathbf{P}^{-1}\mathbf{AP})^k \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{P}^{-1}\mathbf{A}^k\mathbf{P} = \mathbf{P}^{-1}\left(\sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}\right)\mathbf{P} \\ &= \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}. \end{aligned}$$

Thus, $e^{\mathbf{P}^{-1}\mathbf{AP}t} = \mathbf{P}^{-1}e^{\mathbf{A}t}\mathbf{P}$ for all t. ∎

2.2 Part B

Example 2.2. Determine the BIBO stability, marginal stability, and asymptotic stability of the following circuit. Perform Lyapunov analysis with an appropriate choice of a Lyapunov function to determine the conditions that are aligned with your previous conclusions. You may choose $L = 1H$

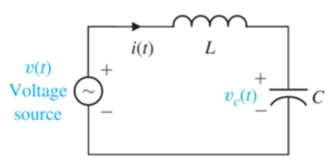

Figure 1: 2B Diagram

and $C = 1F$, and the two states x_1 and x_2 as the capacitor voltage $v_C(t)$ and inductor current $i_L(t)$, respectively.

2.2.1 My Response

Choose the states $x_1(t) = v_C(t)$ and $x_2(t) = i_L(t)$, with input $u(t) = v_s(t)$ and $L = C = 1$. Using $i_C = C\dot{v}_C$, $v_L = L\dot{i}_L$, and KVL $u = v_L + v_C$, we obtain

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + u(t).$$

For zero input ($u \equiv 0$) this is

$$\dot{x} = Ax, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0 \;\Rightarrow\; \lambda = \pm j.$$

No eigenvalue has positive real part, but a conjugate pair lies on the imaginary axis, so the origin is Lyapunov (marginally) stable but not asymptotically stable; solutions are undamped sinusoids.

Take the stored energy as a Lyapunov function:

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2,$$

which is positive definite. For $u(t) = 0$,

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2 + x_2(-x_1) = 0.$$

Thus V is constant along trajectories, so trajectories stay on closed curves around the origin. This proves stability of the equilibrium but not asymptotic convergence.

Now consider BIBO stability with output $y(t) = x_1(t) = v_C(t)$. With

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0,$$

the transfer function is

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^2 + 1}.$$

The impulse response is

$$h(t) = \mathcal{L}^{-1}\{G(s)\} = \sin t, \quad t \geq 0.$$

Since

$$\int_0^\infty |h(t)|\, dt = \int_0^\infty |\sin t|\, dt = \infty,$$

we have $h \notin L^1[0, \infty)$ and the system is not BIBO stable. Equivalently, with $u(t) = \sin t$ the output $y(t)$ grows on the order of t due to resonance.

Therefore, the system is Lyapunov (marginally) stable, not asymptotically stable, and BIBO unstable.

2.3 Part C

Example 2.3. Determine the range of the parameter a such that the following system is asymptotically stable. What is the value of a that achieves marginal stability and also determine the frequency at which such a system would oscillate.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2a & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} u$$

2.3.1 My Response

Now, let us use the given system:

$$\dot{\mathbf{x}} = A(a)\mathbf{x} + Bu, \quad \mathbf{A}(a) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2a & -3 \end{bmatrix}.$$

where the stability depends only on $A(a)$. Now, we need to determine the characteristic polynomial and Routh-Hurwitz stability. Then,

$$\det(\lambda I - A) = \lambda^3 + 3\lambda^2 + 2a\lambda + 3 = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

with

$$a_1 = 3, \quad a_2 = 2a, \quad a_3 = 3.$$

For a third-order continuous-time system, asymptotic stability requires:

- $a_1 > 0$,
- $a_2 > 0$,
- $a_3 > 0$,
- $a_1 a_2 > a_3$.

So let us apply this below:

- $a_1 = 3 > 0$, which is fine for all $a_1 > 0$.
- If $a_2 = 2a > 0$, then $a > 0$.
- If $a_3 = 3 > 0$, then it satisfies all $a_1 > 0$.
- If $a_1 a_2 = 3 \cdot 2a = 6a > a_3 = 3$, then $a > \frac{1}{2}$.

Now, the system is *asymptotically stable* for:

$$\boxed{a > \tfrac{1}{2}.}$$

Next, determine the marginal stability and oscillation frequency. The marginal stability occurs at the boundary:

$$a_1 a_2 = a_3 \quad \Rightarrow \quad 6a = 3 \Rightarrow a = \tfrac{1}{2}.$$

Then, for $a = \frac{1}{2}$, the characteristic polynomial becomes:

$$\lambda^3 + 3\lambda^2 + \lambda + 3 = (\lambda + 3)(\lambda^2 + 1).$$

Finally, the eigenvalues are $\lambda = -3$ and $\lambda = \pm j$. Therefore, at $a = \frac{1}{2}$, the system is marginally stable with the oscillation frequency:

$$\boxed{\omega = 1 \text{ rad/s}},$$

that corresponds to the purely imaginary poles at $\pm j$.

3 Question 3

3.1 Part A

Example 3.1. Obtain the state-space and transfer function descriptions and determine BIBO/ asymptotic stability, controllability, and observability. Simplify the circuit to obtain a reduced order but equivalent circuit. Is the reduced-order circuit controllable and observable?

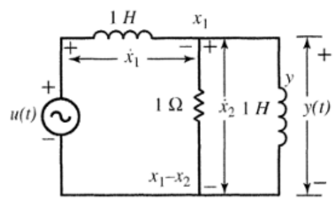

Figure 2: 3A Diagram

3.1.1 My Response

Choice of states and signals. For this problem, let's choose the following states:

- $x_1(t)$ is the current through the left inductor (top branch, $1\,\mathrm{H}$)
- $x_2(t)$ is the current through the right inductor (vertical, $1\,\mathrm{H}$)
- The input is $u(t)$, the source voltage.
- The output is $y(t)$, the voltage at the right node (across the right inductor and resistor).

The node at the top of the $R - L_2$ parallel is at voltage $y(t)$ w.r.t. ground. Next, we will find the inductor and node equations.

Inductor and node equations. For the inductors (with $L = 1$ H so $v_L = \dot{i}_L$):

$$v_{L_1} = \dot{x}_1 = u(t) - y(t), \qquad v_{L_2} = \dot{x}_2 = y(t).$$

If we apply Kirchhoff's Current Law at the right node, we get:

$$\text{current in from left inductor } x_1 = \text{current out through } R + \text{current out through } L_2 = y + x_2,$$

so it follows that:

$$x_1 = y + x_2 \quad \Rightarrow \quad y = x_1 - x_2.$$

Now, let us find the state-space model.

State–space model. Using $y = x_1 - x_2$:

$$\dot{x}_1 = u - y = u - (x_1 - x_2) = -x_1 + x_2 + u,$$
$$\dot{x}_2 = y = x_1 - x_2.$$

In vector form, with $x = [x_1\ x_2]^\top$,

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{A}} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{B}} u, \qquad y = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{\mathbf{C}} x, \quad \mathbf{D} = 0.$$

Transfer function. We begin by trying to solve:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

We need to compute:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+1 & -1 \\ -1 & s+1 \end{bmatrix}, \quad (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)^2 - 1} \begin{bmatrix} s+1 & 1 \\ 1 & s+1 \end{bmatrix} = \frac{1}{s(s+2)} \begin{bmatrix} s+1 & 1 \\ 1 & s+1 \end{bmatrix}.$$

Thus, it follows that:

$$G(s) = \begin{bmatrix} 1 & -1 \end{bmatrix} \frac{1}{s(s+2)} \begin{bmatrix} s+1 \\ 1 \end{bmatrix} = \frac{s}{s(s+2)} = \boxed{\frac{1}{s+2}}.$$

Stability. To determine stability, we need to find the eigenvalues of $\mathbf{A}$:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda+1 & -1 \\ -1 & \lambda+1 \end{vmatrix} = \lambda(\lambda+2) \quad \Rightarrow \quad \lambda_1 = 0,\ \lambda_2 = -2.$$

If one eigenvalue is zero, then the state does not converge to the origin for all initial conditions. Hence, the system is not asymptotically stable, but only marginally stable.

Moreover, the transfer function has only one pole at $s = -2$ (the pole at 0 is unobservable and thus cancels), so all poles of $G(s)$ have negative real part. Therefore, the input–output map is **BIBO stable**.

Controllability. Let us now determine controllability:

$$\mathcal{C} = [B\ AB] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \det \mathcal{C} = 1 \neq 0.$$

Since $\text{rank}(\mathcal{C}) = 2$, the system is controllable.

Observability. Next, let us determine its observability:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

Note that if the rows are linearly dependent ($\text{rank}\,\mathcal{O} = 1 < 2$), then the system is not observable. Thus, the mode associated with $\lambda = 0$ is *unobservable*.

Reduced-order (minimal) realization. Finally, we need to determine its reduced-order or minimal realization. Only the combination

$$z = x_1 - x_2$$

affects the output (since $y = z$). Hence, its dynamics are as follows:

$$\dot{z} = \dot{x}_1 - \dot{x}_2 = (u - z) - z = u - 2z.$$

Thus, a minimal realization is:

$$\boxed{\dot{z} = -2z + u, \qquad y = z,}$$

which is first order and has $G(s) = \frac{1}{s+2}$, identical to the original circuit. Therefore, this reduced system (scalar $A = -2$, $B = 1$, $C = 1$) is both controllable and observable, and it is asymptotically/BIBO stable.

3.2 Part B

Example 3.2. Determine the observability of the system by finding the observability Gramian of the system. Include verification using MATLAB's lyap function.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x}, \quad \mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

3.2.1 My Response

We are given:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}.$$

By computing the characteristic polynomial of $\mathbf{A}$, we find that the eigenvalues of $\mathbf{A}$ are $-1, -2, -3$ (all strictly negative), so the system is *asymptotically stable* and the observability Gramian is well-defined.

Observability Gramian. The observability Gramian $\mathbf{W_o}$ is defined as the unique symmetric solution of the Lyapunov equation

$$\mathbf{A}^\top \mathbf{W_o} + \mathbf{W_o} \mathbf{A} = -\mathbf{C}^\top \mathbf{C}.$$

So, let us assume that:

$$\mathbf{W_o} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{12} & w_{22} & w_{23} \\ w_{13} & w_{23} & w_{33} \end{bmatrix}.$$

Then, by substituting into $\mathbf{A}^\top \mathbf{W_o} + \mathbf{W_o} \mathbf{A} = -\mathbf{C}^\top \mathbf{C}$ and solving for w_{ij} yields

$$\mathbf{W_o} = \begin{bmatrix} \frac{73}{60} & \frac{11}{20} & \frac{1}{12} \\ \frac{11}{20} & \frac{37}{120} & \frac{1}{20} \\ \frac{1}{12} & \frac{1}{20} & \frac{1}{120} \end{bmatrix}.$$

Positive definiteness. Now, we need to determine its positive definiteness. We do this by checking the leading principal minors:

$$w_{11} = \frac{73}{60} > 0,$$

$$\det \begin{bmatrix} \frac{73}{60} & \frac{11}{20} \\ \frac{11}{20} & \frac{37}{120} \end{bmatrix} = \frac{523}{7200} > 0,$$

$$\det W_o = \frac{1}{172800} > 0.$$

Since it follows that all leading principal minors are positive, then $\mathbf{W_o}$ is positive definite. Therefore, the system is **observable**.

MATLAB verification. Now, let us confirm this by implementing it in MATLAB.

```
A = [0 1 0; 0 0 1; -6 -11 -6];
C = [1 0 0];

Wo = lyap(A', C'*C); % This is the observability Gramian
eig(Wo) % All positive
rank(Wo) % Should output 3
```

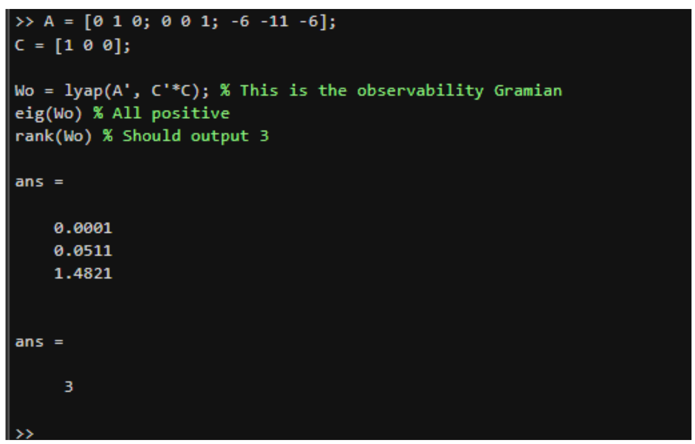

Figure 3: MATLAB Terminal Output

Therefore, the positive eigenvalues and full rank of $\mathbf{W_o}$ confirm observability.

3.3 Part C

Example 3.3. Given the observability of $(\mathbf{A}, \mathbf{C})$, determine the controllability of $(\mathbf{A}^\top, \mathbf{C}^\top)$

3.3.1 My Response

By duality, $(\mathbf{A}, \mathbf{C})$ is observable if and only if $(\mathbf{A}^\top, \mathbf{C}^\top)$ is controllable. Since Part 3(b) showed that $(\mathbf{A}, \mathbf{C})$ is observable, we conclude that $(\mathbf{A}^\top, \mathbf{C}^\top)$ is controllable.