ECE 6200 Lecture 10/21

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1 Introduction

All of these notes are adaptive from *Linear System Theory and Design* 3rd edition by Chen.

Systems are designed to be stable, so we need to have *stability* so that a given system remains stable. Systems, in addition to stability, need to satisfy other things such as to track the desired signals and to surpass noise in order to be useful. The response of linear systems can always be broken into zero-state response and zero-input response. Now, we will introduce the BIBO stability for the zero-state response and marginal and asymptotic stabilities for the zero-input response.

2 Input-Output Stability of LTI Systems

Now, let us consider a SISO Linear Time-Invariant System that is described by:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^\tau g(\tau)u(t-\tau)d\tau$$

where $g(t)$ is the impulse response at $t = 0$.

Remark. We need to consider the following:

- So in order to be desirable by above, a given system must be **linear**, **time-invariant**, and **casual**.
- The system must also be relaxed at $t = 0$.

Definition 2.1. *An input $u(t)$ is said to be* ***bounded*** *if $u(t)$ does not grow to positive or negative infinity, or similarly, there exists a constant u_m such that:*

$$|u(t)| \leq u_m < \infty, \quad \forall t \geq 0$$

Definition 2.2. *A system is said to be* ***BIBO stable*** *if every bounded input excites a bounded output.*

This stability is defined for the zero-state response and only works for an initially relaxed system.

Theorem 2.1. *A SISO-System described by the following equation,*

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^\tau g(\tau)u(t-\tau)d\tau,$$

is said to be BIBO stable if and only if $g(t)$ is absolutely integrable in $[0,\infty)$, or

$$\int_0^\infty |g(t)|\, dt \leq M < \infty$$

for some constant M.

Proof. First, we need to show that if $g(t)$ is *absolutely integrable*, then every bounded input excites a bounded output. Let $u(t)$ be an arbitrary input with $|u(t)| \leq u_m < \infty$ for all $t \geq 0$. Then

$$\big|y(t)\big| = \left| \int_0^t g(\tau)u(t-\tau)d\tau \right| \leq \int_0^t |g(\tau)||u(t-\tau)|d\tau \leq u_m \int_0^\infty |g(\tau)|d\tau \leq u_m M$$

Thus, the input is bounded. Next, we need to show that if $g(t)$ is not absolutely integrable, then the system is not BIBO stable. If $g(t)$ is not absolutely integrable; then for any arbitrarily large N, there exists a t_1 such that

$$\int_0^{t_1} |g(\tau)|d\tau \geq N$$

Now, let us choose

$$u(t_1 - \tau) = \begin{cases} 1, & \text{if } g(\tau) \geq 0, \\ -1, & \text{if } g(\tau) < 0. \end{cases}$$

It is trivial that u is bounded. Hence, the output excited by this input equals

$$y(t_1) = \int_0^{t_1} g(\tau)u(t_1-\tau)d\tau = \int_0^{t_1} |g(\tau)|d\tau \geq N$$

Since $y(t_1)$ can be arbitrarily large, it follows that a similar bounded input can excite an unbounded output.

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Theorem 2.2. *If a system with impulse response $g(t)$ is* ***BIBO stable****, then as $t \to \infty$:*

1. *For constant input $u(t) = a$:*

$$y(t) \to \hat{g}(0)a$$

2. *For sinusoidal input $u(t) = \sin(\omega_0 t)$:*

$$y(t) \to |\hat{g}(j\omega_0)| \sin(\omega_0 t + \angle\hat{g}(j\omega_0))$$

where

$$\hat{g}(s) = \int_0^\infty g(\tau)e^{-s\tau}d\tau$$

is the Laplace Transform of $g(t)$.

Proof. If $u(t) = a$ for all $t \geq 0$, then

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\,\tau = a\int_0^t g(\tau)d\,\tau$$

and it follows that

$$y(t) \to a\int_0^\infty g(\tau)d\,\tau = a\hat{g}(0) \quad \text{as } t \to \infty$$

where we applied

$$\hat{g}(s) = \int_0^\infty g(\tau)e^{-s\tau}d\,\tau$$

with $s = 0$. Hence, we proved part (i) of the theorem. Now, if $u(t) = \sin(\omega_0 t)$ it becomes

$$y(t) = \int_0^t g(\tau)\sin\omega_0(t-\tau)d\,\tau$$

$$= \int_0^t g(\tau)\big[\sin(\omega_0 t)\cos(\omega_0\tau) - \cos(\omega_0 t)\sin(\omega_0 t)\big]d\,\tau$$

$$\sin(\omega_0 t)\int_0^t g(\tau)\cos(\omega_0\tau)d\tau - \cos(\omega_0 t)\int_0^t g(\tau)\sin(\omega_0\tau)d\,\tau$$

Hence, it follows that as $t \to \infty$,

$$y(t) \to \sin(\omega_0 t)\int_0^\infty g(\tau)\cos(\omega_0\tau)d\tau - \cos(\omega_0 t)\int_0^\infty g(\tau)sin(\omega_0\tau)d\tau$$

The rest of the proof is left to the reader as an exercise. ■

Book gets a bit confusing with notation, as there is a misprint.