ECE 6200: Linear Systems Analysis

Homework 3 (100 pts.)

Due on Thursday, October 30, 2025, in class

Problem 1 (15 pts): Find the third column of the following matrix so that it becomes orthogonal.

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & ? \\ 1/\sqrt{3} & 0 & ? \\ 1/\sqrt{3} & -1/\sqrt{2} & ? \end{bmatrix}.$$

Problem 2 (25 pts): Find a Jordan representation $\mathbf{J}$ for each of the following matrices,

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Find also the basis matrix **Q** corresponding to your Jordan representation **J.** Hint: $\mathbf{A}_1$ may have complex representations and $\mathbf{A}_2$ may require generalized set of eigenvectors for **Q.**

Problem 3 (15 pts.): Even though matrix multiplication in general is not commutative, show that

$$\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}.$$

Problem 4 (20 pts.): Use any two methods to find the square root of the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$.

Problem 5 (10 pts.): Find the characteristic polynomial and the minimal polynomial for each of the following matrices,

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

Problem 6 (15 pts.): Consider a quadratic function $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}(t)\mathbf{P}\mathbf{x}(t)$

a. Show that **P** can always be written as a symmetric matrix.

b. For $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, use the chain rule of calculus to find an expression for $\dfrac{dV}{dt}$.

ECE 6200 Homework 3

Blaine Swieder

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1 Question 1

Example 1.1. Find the third column of the following matrix so that it becomes orthogonal.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & ? \\ \frac{1}{\sqrt{3}} & 0 & ? \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & ? \end{bmatrix}$$

1.1 Solution

Let the third column be $\mathbf{a}_3 = [x_1, x_2, x_3]^\top$. Since we want $\mathbf{A}$ to be orthogonal, then our columns need to be orthonormal, then:

$$\mathbf{a}_i^\top \mathbf{a}_j = 0 \quad (i \neq j), \qquad |\mathbf{a}_i| = 1.$$

Next, let:

$$\mathbf{a}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

So, now, it follows that:

$$\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{vmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}.$$

Thus,

$$\boxed{\mathbf{a}_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}}, \quad \boxed{\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}}$$

2 Question 2

Example 2.1. Find a Jordan representation $\mathbf{J}$ for each of the following matrices,

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}, \quad \text{and } \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find also the basis matrix $\mathbf{Q}$ corresponding to your Jordan representation $\mathbf{J}$. **Hint**: $\mathbf{A}_1$ may have complex representations and $\mathbf{A}_2$ may require generalized set of eigenvectors for $\mathbf{Q}$.

2.1 Solution

Let us do each matrix here individually.

2.1.1 Matrix $\mathbf{A}_1$

We are given the following matrix for $\mathbf{A}_1$:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

Now, let us find the eigenvalues for $\mathbf{A}_1$. We need to compute $\det(\lambda \mathbf{I} - \mathbf{A}_1)$:

$$\det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2 & 4 & \lambda + 3 \end{bmatrix} = \lambda(\lambda(\lambda+3) - (-1)4) - (-1)(0(\lambda+3) - (-1)2) = \lambda(\lambda^2 + 3\lambda + 4) + 2$$

It is trivial to get:

$$p_{\mathbf{A}_1}(\lambda) = \lambda^3 + 3\lambda^2 + 4\lambda + 2 = (\lambda+1)(\lambda^2 + 2\lambda + 2)$$

As a result, the roots are

$$\lambda_1 = -1, \qquad \lambda_{2,3} = -1 \pm i.$$

Thus, over the $\mathbb{C}$ field, all eigenvalues are distinct, and it follows that its Jordan form is diagonal. Let us find its eigenvectors. For $\lambda = -1$, we need to solve $(\mathbf{A}_1 + \mathbf{I})\mathbf{v} = 0$. Then,

$$\mathbf{A}_1 + \mathbf{I} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix}$$

To obtain these vectors, we applied Gaussian Elimination. So, it follows that one eigenvector for $\lambda = -1$ is:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda = -1 - i$, another eigenvector is:

$$\mathbf{v}_2 = \begin{bmatrix} -i \\ -1+i \\ 2 \end{bmatrix}$$

And for $\lambda = -1 + i$, the final eigenvector is:

$$\mathbf{v}_3 = \begin{bmatrix} i \\ -1-i \\ 2 \end{bmatrix}.$$

Now, we need to take the Jordan form and basis. So,

$$\mathbf{Q}_1 = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = \begin{bmatrix} 1 & -i & i \\ -1 & -1+i & -1-i \\ 1 & 2 & 2 \end{bmatrix}.$$

Then, it follows that:

$$\mathbf{Q}_1^{-1}\mathbf{A}_1\mathbf{Q}_1 = \mathbf{J}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -1+i \end{bmatrix}.$$

Thus, the Jordan representation for $\mathbf{A}_1$ over $\mathbb{C}$ is:

$$\mathbf{J}_1 = \mathrm{diag}(-1,\ -1-i,\ -1+i)$$

with the basis matrix $\mathbf{Q}_1$ given above. Or the Real Jordan Form, then the conjugate pair $(-1 \pm i)$ becomes a 2×2 real block:

$$\mathbf{J}_{1,\mathbb{R}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

2.1.2 Matrix $\mathbf{A}_2$

We are given the following matrix for $\mathbf{A}_2$:

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, the matrix is upper triangular, and it follows that its eigenvalues lie on its diagonal:

$$\lambda = 1 \text{ (mult. 2)}, \quad \lambda = 2 \text{ (mult. 1)}.$$

Now, for the eigenvalue $\lambda = 1$, we need to solve $(\mathbf{A}_2 - \mathbf{I})\mathbf{x} = 0$ So,

$$\mathbf{A}_2 - \mathbf{I} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By applying Gaussian Elimination to the matrix created by $(\mathbf{A}_2 - \mathbf{I})\mathbf{x} = 0$, we get the following eigenvectors. As a result, we get two independent eigenvectors, it follows that:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence the geometric multiplicity of $\lambda = 1$ is 2, and it follows that no Jordan chain is necessary for this eigenvalue. And for eigenvalue $\lambda = 2$, we need to solve $(\mathbf{A}_2 - 2\mathbf{I})\mathbf{x} = 0$:

$$\mathbf{A}_2 - 2\mathbf{I} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

And when we apply Gaussian Elimination to the matrix, we get:

$$\mathbf{w}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Now, for our Jordan form and basis, we take:

$$\mathbf{Q}_2 = [\mathbf{w}_1 \ \mathbf{w}_2 \ \mathbf{w}_3]$$

So,

$$\mathbf{Q}_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And it follows now that:

$$\mathbf{Q}_2^{-1}\mathbf{A}_2\mathbf{Q}_2 = \mathbf{J}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore, the Jordan Representation for $\mathbf{A}_2$ already diagonal:

$$\mathbf{J}_2 = \text{diag}(1, 1, 2)$$

and with a basis matrix

$$\mathbf{Q}_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3 Question 3

Example 3.1. Even though matrix multiplication in general is not commutative, show that

$$\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

3.1 Solution

Proposition 3.0.1. *Show that* $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$.

Proof. We need to show that $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$. First, let us recall the definition of the matrix exponential, which is defined as:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{(\mathbf{A}t)^2}{2!} + \frac{(\mathbf{A}t)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}$$

Next, if we multiply the left side by $\mathbf{A}$, it follows that:

$$\mathbf{A}e^{\mathbf{A}t} = \mathbf{A}\sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{A}(\mathbf{A}t)^k}{k!}$$

Since t is a scalar and is commutes with $\mathbf{A}$, we are able to write $(\mathbf{A}t)^k = \mathbf{A}^k t^k$. From this expression, it follows that:

$$\mathbf{A}(\mathbf{A}t)^k = \mathbf{A}\mathbf{A}^k t^k = \mathbf{A}^{k+1} t^k$$

Hence, it follows that:

$$\mathbf{A}e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^{k+1} t^k}{k!}$$

Likewise, if we multiply $e^{\mathbf{A}t}$ by $\mathbf{A}$ on the right hand side, then it must be the case that:

$$e^{\mathbf{A}t}\mathbf{A} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k\mathbf{A}}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k\mathbf{A}t^k}{k!} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^{k+1} t^k}{k!}$$

Since both expressions are equivalent on both sides, it follows that $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$. Hence, $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$, is commutative. Therefore, while it is the case that matrix multiplication is not commutative, it follows that $\mathbf{A}$ computes with all of its powers and since our given statement has an exponential series that solely involves powers of $\mathbf{A}$, the equality holds, as desired. ■

4 Question 4

Example 4.1. Use any two methods to find the square root of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

4.1 Solution

4.1.1 Method I: Solve ($\mathbf{X}^2 = \mathbf{A}$) directly

Assume that

$$\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$\mathbf{X}^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

So, we get the following system:

$$a^2 + bc = 0, \tag{1}$$
$$b(a+d) = 1, \tag{2}$$
$$c(a+d) = -1, \tag{3}$$
$$bc + d^2 = -2. \tag{4}$$

From (2) and (3), assuming $a + d \neq 0$,

$$\frac{c(a+d)}{b(a+d)} = \frac{-1}{1} \quad \Longrightarrow \quad \frac{c}{b} = -1 \quad \Longrightarrow \quad c = -b.$$

Substitute $c = -b$ into (1):

$$a^2 + b(-b) = 0 \quad \Longrightarrow \quad a^2 - b^2 = 0 \quad \Longrightarrow \quad a = \pm b.$$

Case 1: $a = b$**.** Then it follows that (2) becomes:

$$b(b+d) = 1 \quad \Longrightarrow \quad d = \frac{1}{b} - b.$$

Equation (4) becomes

$$-b^2 + d^2 = -2 \quad \Longrightarrow \quad d^2 = b^2 - 2.$$

But with $d = \frac{1}{b} - b$,

$$\left(\frac{1}{b} - b\right)^2 = \frac{1}{b^2} - 2 + b^2.$$

And by setting this equal to $b^2 - 2$ would force $\frac{1}{b^2} = 0$, which is impossible. Hence, it follows that $a = b$ has no solution.

Case 2: $a = -b$**.** Since, with $c = -b$ and $a = -b$, equation (2) becomes

$$b(-b+d) = 1 \quad \Longrightarrow \quad d - b = \frac{1}{b} \quad \Longrightarrow \quad d = b + \frac{1}{b}.$$

Equation (4) becomes

$$-b^2 + d^2 = -2 \quad \Longrightarrow \quad d^2 = b^2 - 2.$$

If we substitute $d = b + \frac{1}{b}$:

$$\left(b + \frac{1}{b}\right)^2 = b^2 - 2.$$

Now, if we expand the left-hand side:

$$b^2 + 2 + \frac{1}{b^2} = b^2 - 2 \quad \Longrightarrow \quad 2 + \frac{1}{b^2} = -2 \quad \Longrightarrow \quad \frac{1}{b^2} = -4 \quad \Longrightarrow \quad b^2 = -\frac{1}{4} \quad \Longrightarrow \quad b = \pm\frac{i}{2}.$$

Take first $b = \frac{i}{2}$. Then

$$a = -b = -\frac{i}{2}, \qquad c = -b = -\frac{i}{2}, \qquad d = b + \frac{1}{b} = \frac{i}{2} + \frac{2}{i} = \frac{i}{2} - 2i = -\frac{3i}{2}.$$

Now, we need to verify that $\mathbf{X}^2 = \mathbf{A}$, we calculate:

$$\mathbf{X}^2 = \left(\begin{bmatrix} \frac{i}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{3i}{2} \end{bmatrix}\right)^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \mathbf{A},$$

and the other matrix from Case II also squares to $\mathbf{A}$ too, it is just the negative of this root. Therefore, one square root is:

$$\mathbf{X}_1 = \frac{i}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix},$$

and the other is

$$\mathbf{X}_2 = -\mathbf{X}_1 = -\frac{i}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

4.1.2 Method II: Jordan form / Function of a Matrix

Let us first find the characteristic polynomial of $\mathbf{A}$ that is given by:

$$\chi_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix} = \lambda(\lambda + 2) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2.$$

Then it follows that $\mathbf{A}$ has a single eigenvalue $\lambda = -1$ with an algebraic multiplicity of 2. Now, we need to find an eigenvector, so we solve $(\mathbf{A} + \mathbf{I})\mathbf{v} = 0$:

$$\mathbf{A} + \mathbf{I} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad (\mathbf{A} + \mathbf{I})\begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \Longrightarrow \quad x + y = 0.$$

Next, we take:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

And since the eigenvalue is repeated, we need to find a generalized eigenvector $\mathbf{v}_2$ from

$$(\mathbf{A} + \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1.$$

That is,

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the first row gives us $x + y = 1$, we can choose:

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now, let us form:

$$\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = -\mathbf{I} + \mathbf{N}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; \mathbf{N}^2 = 0.$$

Next, we need to find an $\mathbf{S}$ such that $\mathbf{S}^2 = \mathbf{J}$. Since $\mathbf{J} = -\mathbf{I} + \mathbf{N}$ and $\mathbf{N}^2 = 0$, we can try:

$$\mathbf{S} = \alpha\mathbf{I} + \beta\mathbf{N}.$$

Then,

$$\mathbf{S}^2 = \alpha^2\mathbf{I} + 2\alpha\beta\mathbf{N}.$$

Now, we need:

$$\alpha^2 = -1 \quad \text{and} \quad 2\alpha\beta = 1.$$

Let us choose $\alpha = i$. Then,

$$\beta = \frac{1}{2\alpha} = \frac{1}{2i} = -\frac{i}{2},$$

so it follows that:

$$\mathbf{S} = i\mathbf{I} - \frac{i}{2}\mathbf{N} = \begin{bmatrix} i & -\frac{i}{2} \\ 0 & i \end{bmatrix}.$$

Finally, we get:

$$\mathbf{X} = \mathbf{P}\mathbf{S}\mathbf{P}^{-1}.$$

And when we carry out the multiplication, we get:

$$\mathbf{X} = \frac{i}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix},$$

and thus, it is the case that:

$$\mathbf{X}^2 = \begin{bmatrix} \frac{i}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{3i}{2} \end{bmatrix} = \mathbf{A},$$

as desired. This follows from the simple fact that $(-\mathbf{X})^2 = \mathbf{X}^2$.

4.2 Final Answer

Therefore, two complex square roots of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

are

$$\boxed{\mathbf{X}_1 = \frac{i}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{X}_2 = -\frac{i}{2}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}}$$

5 Question 5

Example 5.1. Find the characteristic polynomial and the minimal polynomial for each of the following matrices,

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \text{and } \mathbf{A}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

5.1 Solution

Now, let us compute the characteristic polynomial for $\mathbf{A}_1$. So,

$$\det(\mathbf{A}_1 - \lambda \mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 & 0 & 0 \\ 0 & -1-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 0 & 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)^3(-1-\lambda) = (-1-\lambda)^4.$$

Hence, the characteristic polynomial of $\mathbf{A}_1$ can be written as:

$$\boxed{p_{A_1}(\lambda) = (\lambda+1)^4}.$$

Now, let us determine the minimal polynomial of $\mathbf{A}_1$. So,

$$\mathbf{A}_1 = -\mathbf{I} + \mathbf{N}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{N}^2 = 0.$$

Now, the minimal polynomial for a Jordan block with an eigenvalue -1 and a single superdiagonal entry 1 is as follows:

$$\boxed{m_{\mathbf{A}_1}(\lambda) = (\lambda+1)^2}$$

Therefore, the characteristic polynomial of $\mathbf{A}_1$ is $(\lambda+1)^4$ and its minimal polynomial is $(\lambda+1)^2$.

Now, for $\mathbf{A}_2$ we need to do the same. For the characteristic polynomial,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (-1-\lambda)^2(-2-\lambda)(-3-\lambda)$$

and is defined as:

$$\boxed{p_{\mathbf{A}_2}(\lambda) = (\lambda+1)^2(\lambda+2)(\lambda+3)}.$$

Next, we need a minimal polynomial for $\mathbf{A}_2$. All of the eigenvalues here correspond to distinct diagonal entries, but it is true that -1 appears twice and has two linearly independent eigenvectors. Hence, the minimal polynomial contains each distinct linear factor once:

$$\boxed{m_{A_2}(\lambda) = (\lambda+1)(\lambda+2)(\lambda+3)}$$

6 Question 6

Example 6.1. Consider a quadratic function $V(\mathbf{x}) = \mathbf{x}^\top(t)\mathbf{P}\mathbf{x}(t)$.

- Show that $\mathbf{P}$ can always be written as a symmetric matrix.
- For $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, use the chain rule of calculus to find an expression for $\dfrac{dV}{dt}$.

6.1 My Response

6.1.1 Part A

Proposition 6.0.1. *Show that* $\mathbf{P}$ *can always be written as a symmetric matrix.*

Proof. Let us consider the quadratic form, $\mathbf{x}^\top\mathbf{P}\mathbf{x}$. Since it was a scalar, then it follows that it is equivalent to its own transpose:

$$(\mathbf{x}^\top\mathbf{P}\mathbf{x})^\top = \mathbf{x}^\top\mathbf{P}^\top\mathbf{x} = \mathbf{x}^\top\mathbf{P}\mathbf{x}.$$

So,

$$\mathbf{x}^\top(\mathbf{P} - \mathbf{P}^\top)\mathbf{x} = 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^n.$$

It then follows that the term $\mathbf{P} - \mathbf{P}^\top$ is skew-symmetric, and thus does not affect the value of the quadratic form. As a result, we can replace $\mathbf{P}$ with its symmetric part,

$$\mathbf{P}_s = \frac{1}{2}(\mathbf{P} + \mathbf{P}^\top),$$

and since,

$$\mathbf{x}^\top\mathbf{P}\mathbf{x} = \mathbf{x}^\top\mathbf{P}_s\mathbf{x}.$$

Thus every quadratic form that is defined by $\mathbf{P}$ can be equivalently represented by a symmetric matrix, $\mathbf{P}_s$. Therefore, $\mathbf{P}$ can always be written as a symmetric matrix, as desired. ■

So, what is important to take away from here is that any quadratic form $\mathbf{x}^\top\mathbf{P}\mathbf{x}$ can be expressed with a symmetric matrix $\mathbf{P}$.

6.1.2 Part B

Proposition 6.0.2. *For* $\dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$*, use the chain rule of calculus to find an expression for* $\dfrac{dV}{dt}$.

Proof. For the differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, we need to use the chain rule to compute $\dfrac{dV}{dt}$. If we differentiate $V(x) = \mathbf{x}^\top\mathbf{P}\mathbf{x}$ with respect to time, then:

$$\frac{dV}{dt} = \dot{\mathbf{x}}^\top\mathbf{P}\mathbf{x} + \mathbf{x}^\top\mathbf{P}\dot{\mathbf{x}}.$$

Now, if we substitute in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, then:

$$\frac{dV}{dt} = (\mathbf{A}\mathbf{x})^\top\mathbf{P}\mathbf{x} + \mathbf{x}^\top\mathbf{P}(\mathbf{A}\mathbf{x}) = \mathbf{x}^\top(\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}.$$

Therefore, the time derivative of V is:

$$\frac{dV}{dt} = \mathbf{x}^\top(\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}.$$

$\blacksquare$

Likewise for this problem, if $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, then it must be the case that:

$$\boxed{\frac{dV}{dt} = \mathbf{x}^\top(\mathbf{A}^\top\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x}}$$