ECE 6200: Linear Systems Analysis

Test 1 (100 pts.)

Due by Wednesday, October 15, 2025, 11:59 pm

Instructions:

- The test has four problems. Read all problems before starting to solve.
- The exam is open book / open lecture with MATLAB usage allowed.
- Show your steps when solving the problems to receive full credit.
- Sign and attach this page when you return the exam.

Name: ________________________

Honor Pledge: " On my honor, I have neither given nor received any aid in this test."

Signature: ______________________

Problem 1 (15 pts.):

State whether the following statements are True or False with the appropriate reasoning.

a. A linear model of a system may be a good approximation in a certain operating range.

b. Feedback cannot modify the location of poles of a transfer function.

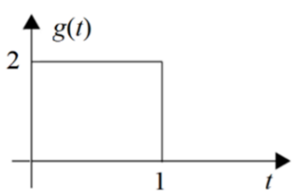
c. An impulse response is a special case of the natural response of a system.

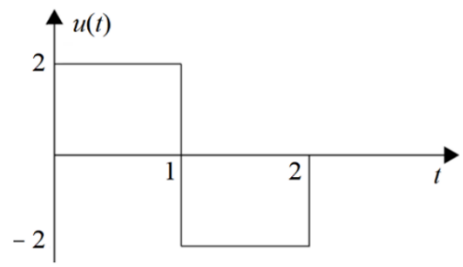
d. A discrete-time delay system is infinite dimensional.

e. The DC gain of an improper rational transfer function is infinity.

Problem 2 (35 pts.):

a. **(20 pts.)** Determine the analytical expressions of the output response $y(t)$ of the system with the following impulse response $g(t)$ and the input $u(t)$ for different time intervals. Write a MATLAB script to generate and verify the output response you obtained.





b. **(8 pts.)** Obtain a general expression for the natural output response $y[k]$ due to the initial condition $x[0]$ in terms of the system matrices for the following discrete-time LTI system. Show your steps.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$
$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

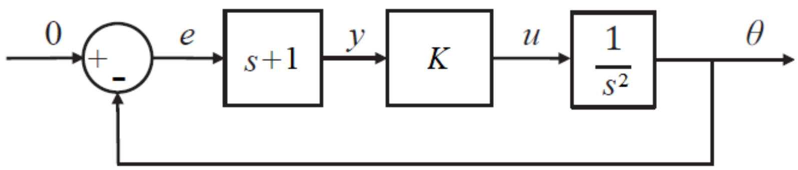
c. **(7 pts.)** Let $g(t)$ be the impulse response of an LTI system given by the convolution description $y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$. Derive a relationship between its unit-step response $\overline{y}(t)$ and the impulse response $g(t)$.

Problem 3 (25 pts.):

Consider the following satellite rotation control system in which the input-output transfer function between the satellite angle θ and the control thrust force u is given by $\dfrac{\Theta(s)}{U(s)} = \dfrac{1}{s^2}$.

Choose $x_1 = \theta, x_2 = \dot{\theta}$ as your state variables for the following problems. Hint: s is the Laplace variable and can be treated as a derivative operator, i.e, $s\Theta(s) \to \dfrac{d\theta}{dt}$.

a. (10 pts.) Find a second order state-space model using the following system block diagram.

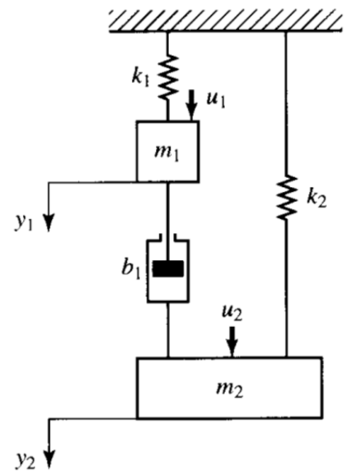


b. (10 pts.) Implement a simulation <u>of the obtained state-space model</u> in MATLAB/Simulink for $K = 1$ and the initial states $x_1(0) = 1$ and $x_2(0) = 1$. Show your state trajectory plots for x_1 and x_2.

c. (5 pts.) For a reference input $r(t)$, find an expression for the impulse response of this closed-loop system (assuming zero initial conditions).

Problem 4 (25 pts.)

Consider the spring-mass-damper system with two masses m_1 and m_2. The inputs are the forces for u_1 and u_2, and the outputs are the displacements y_1 and y_2 of these masses, respectively.



a. **(10 pts.)** Find a state-space representation of the system with the states x_1 and x_2 being displacement and velocities of mass m_1. Similarly, x_3 and x_4 are those of mass m_2.

b. **(10 pts.)** Find an expression for the transfer function between input u_1 and output y_2.

c. **(5 pts.)** With all coefficients being 1, find the complete impulse response of this system for unit impulses $u_1(t) = u_2(t) = \delta(t)$ using MATLAB. Include the relevant script and plots.

ECE 6200 Exam I Solutions

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15 October 2025

1 Question 1

Example 1.1. State whether the following statements are True or False with the appropriate reasoning.

- A linear model of a system may be a good approximation in a certain operating range.
- Feedback cannot modify the location of poles of a transfer function.
- An impulse response is a special case of the natural response of a system.
- A discrete-time delay system is infinite dimensional.
- The DC gain of an improper rational transfer function is infinity.

1.1 My Solution

Here are my answers to the first question, along with my reasoning:

- The statement is **True**. Many nonlinear systems are well-approximated by a linear model when these systems are operated near an equilibrium or local linearization. More specifically, a local linearization is concerned with an equilibrium (essentially a Taylor Expansion) is a good approximation within a small operating region.
- This statement is **False**. It is the case that feedback DOES change the pole locations; the closed-loop poles are the roots of $1 + G(s)K(s) = 0$, which is typically different from the open-loop poles. Additionally, closing the loop will change the system that one is analyzing; however, the closed-loop poles are roots of the equation $1 + L(s) = 0$ with $L(s) = G(s)K(s)$.
- This statement is also **False**. We know that the impulse response is the zero-state response to $\delta(t)$, which is not the natural (zero-input) response due to the initial conditions.
- This statement is **False**. An integer-sample pure delay in discrete-time is said to be finite-dimensional (basically means that it can be realized with N states for the N-step delay), so infinite-dimensionality is a problem for continuous-time pure delays.
- This final statement is **False**. The DC gain is $G(0)$. The improperness (a.k.a if deg num $\geq$ deg den does not mean or rather imply that $G(0) = \infty$. However, $G(s) = s$ is improper but $G(0) = 0$. It is the case that $G(0) = \infty$ iff there is a pole that lies at the origin. So, to put this more concisely, DC gain is infinite iff there is an uncanceled pole that lies at the origin and improperness alone is not sufficient to force it.

2 Question 2

Here are my solutions to question 2.

2.1 Part A

Example 2.1. Determine the analytical expressions of the output response $y(t)$ of the system with the following impulse response $g(t)$ and the input $u(t)$ for different time intervals. Write a MATLAB script to generate and verify the output response you obtained.

From the given plots, the following equations are described:

- $g(t) = 2[u(t) - u(t-1)]$ where there is a height of 2, a width of 1 and it starts at $t = 0$.
- $u(t) = 2[u(t) - u(t-1)] - 2[u(t-1) - u(t-2)]$, where +2 on $[0, 1)$, -2 on $[1, 2)$

Now, let $p(t) = u(t) - u(t-1)$ (unit-height, width-1 pulse). Then,

$$g = 2p, \qquad u = 2p - 2p(\cdot - 1).$$

We now need to proceed to the convolution.

$$y = g * u = 2p * (2p - 2p(\cdot - 1)) = 4(p * p) - 4(p * p)(\cdot - 1).$$

Recall that, $p * p = \Lambda(t)$ where we define $\Lambda(t)$ as the unit triangular pulse supported on $[0, 2]$ is as follows:

$$\Lambda(t) = \begin{cases} t, & 0 \le t < 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$y(t) = 4\Lambda(t) - 4\Lambda(t-1).$$

Now, the piecewise form is as follows:

$$y(t) = \begin{cases} 0, & t < 0, \\ 4t, & 0 \le t < 1, \\ 12 - 8t, & 1 \le t < 2, \\ -12 + 4t, & 2 \le t < 3, \\ 0, & t \ge 3. \end{cases}$$

2.1.1 MATLAB Script

```
% The Minimal MATLAB to show Discrete-Time Convolution

% Time Grid

dt = 1e-3;
t = -0.5:dt:3.5;

```

```
% Unit Step

U = @(x) double(x>=0);

% Define g(t) and u(t)

g = 2*(U(t)-U(t-1));                    % 2 on [0,1)
u = 2*(U(t)-U(t-1)) - 2 * (U(t - 1) - U(t - 2));  % +2 on [0,1), -2 on [1,2)

% The Numeric convolution (scale by dt)
y_num = conv(g,u)*dt;
t_y = (t(1)+t(1)) : dt : (t(end)+t(end));

% analytic y(t) for comparison
Lambda = @(x) (x>=0 & x<1).*(x) + (x>=1 & x<2).*(2-x);
y_ana = 4*Lambda(t) - 4*Lambda(t-1);

% plot
figure; hold on
plot(t, g, 'DisplayName','g(t)')
plot(t, u, 'DisplayName','u(t)')
plot(t, y_ana, 'LineWidth',1.5, 'DisplayName','y_{analytic}(t)')
plot(t_y, y_num, '--', 'DisplayName','y_{numeric}(t)');
xlim([-0.1 3.1]); grid on; legend; xlabel('t'); ylabel('amplitude')
title('The Check for Convolution')
```

And the following MATLAB Code gives the following when you compile it in the terminal:

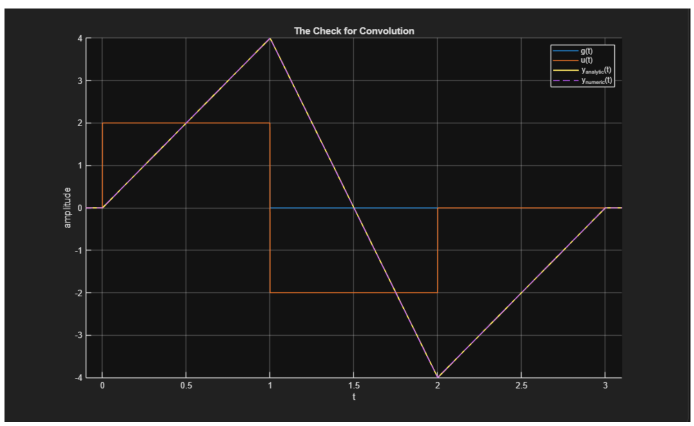

Figure 1: MATLAB Problem 2 Output

2.2 Part B

Example 2.2. Obtain a general expression for the natural output response $y[k]$ due to the initial condition $x[0]$ in terms of the system matrices for the following discrete-time LTI system. Show your steps.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$
$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

2.2.1 My Solution

Proposition 2.0.1. *For the natural (zero-input) response of the discrete-time LTI system*

$$\mathbf{x}[k+1] = \mathbf{A}\,\mathbf{x}[k], \qquad \mathbf{y}[k] = \mathbf{C}\,\mathbf{x}[k],$$

with given $\mathbf{x}[0]$*, we have* $\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0]$ *and hence* $\mathbf{y}[k] = \mathbf{C}\,\mathbf{A}^k\mathbf{x}[0]$ *for all* $k \geq 0$*.*

Proof. We proceed by induction on k. Let us proceed with the base case, and by definition it follows that $\mathbf{A}^0 = \mathbf{I}$. For $k = 0$, $\mathbf{A}^0 = \mathbf{I}$ so $\mathbf{x}[0] = \mathbf{A}^0\mathbf{x}[0]$. Now, assume $\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0]$. Then it follows that:

$$\mathbf{x}[k+1] = \mathbf{A}\,\mathbf{x}[k] = \mathbf{A}\left(\mathbf{A}^k\mathbf{x}[0]\right) = \mathbf{A}^{k+1}\mathbf{x}[0].$$

Then, $\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0]$ for all $k \geq 0$ and when we substitute $\mathbf{y}[k] = \mathbf{C}\,\mathbf{x}[k]$, it follows that $\mathbf{y}[k] = \mathbf{C}\,\mathbf{A}^k\mathbf{x}[0]$.

∎

2.3 Part C

Example 2.3. Let $g(t)$ be the impulse response of an LTI system given by the convolution description

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,dt.$$

Derive a relationship between its unit-step response $\overline{y}(t)$ and the impulse response $g(t)$.

2.3.1 My Solution

For this Linear Time-Invariant System, the following is true:

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau$$

Note, if the input is *unit-step*, then it follows that $u(\tau)$ for $\tau \geq 0$. Thus, the **unit-step response** is:

$$\overline{y}(t) = \int_0^t g(t-\tau)\,d\tau.$$

Let $\sigma = t - \tau$ such that $d\sigma = -d\tau$. By changing our limits of integration, we get:

$$\overline{y}(t) = \int_0^t g(\sigma)\,d\sigma$$

Therefore, the unit-step response is the integral of the impulse response. If we take the derivative of both sides with respect to t we get:

$$g(t) = \frac{d}{dt}, \overline{y}(t)$$

So this can be summarized as:

$$\overline{y}(t) = \int_0^t g(\tau), d\tau \quad \text{and hence} \quad g(t) = \frac{d}{dt}\overline{y}(t).$$

3 Question 3

Example 3.1. Consider the following satellite rotation control system in which the input-output transfer function between the satellite angle θ and the control thrust force u is given by $\frac{\Theta(s)}{U(s)} = \frac{1}{s^2}$. Choose $x_1 = \theta$, $x_2 = \dot{\theta}$ as your variables for the following problems. **Hint**: s is the Laplace variable and can be treated as a derivative operator, i.e., $s\Theta(s) = \dfrac{d\theta}{dt}$.

3.1 Part A

Example 3.2. Find a second order state-space model using the following system block diagram.

3.1.1 My Solution

We are required to use $x_1 = \theta, x_2 = \dot{\theta}$. For the following diagram, we get the following:

- From the given diagram: $e = 0 - \theta = -\theta$.
- The controller block, $s + 1$, gives us $y = \dot{e} + e = -(\dot{\theta} + \theta) = -(x_1 + x_2)$
- Then $u = K$, $y = -K(x_1 + x_2)$.
- Plant $\frac{1}{s^2}$ implies that $\ddot{\theta} = u \Rightarrow \dot{x_1} = x_2, \dot{x}_2 = u$.

So, it follows that

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -K(x_1 + x_2), \qquad \text{or} \qquad \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -K & -K \end{bmatrix} \mathbf{x} \\ y &= \theta = x_1 \end{aligned}$$

where we have an output of $\theta = x_1$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -K & -K \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0$$

There is no external input here since our reference is fixed at 0.

3.2 Part B

Example 3.3. Implement a simulation of the obtained state-space model in MATLAB/Simulink for $K = 1$ and the initial states $x_1(0) = 1$ and $x_2(0) = 1$. Show your state trajectory plots for x_1 and x_2.

3.2.1 My Solution

Note: I only used $C = I$ was used solely for visualization, while I physically considered $y = \theta = x_1$.

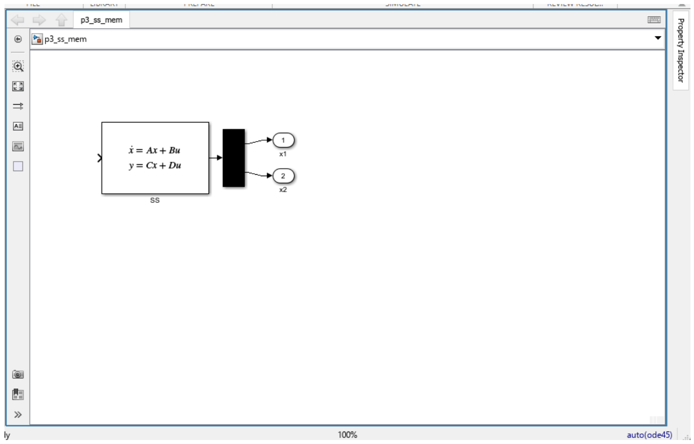

Figure 2: MATLAB Problem 3B Output

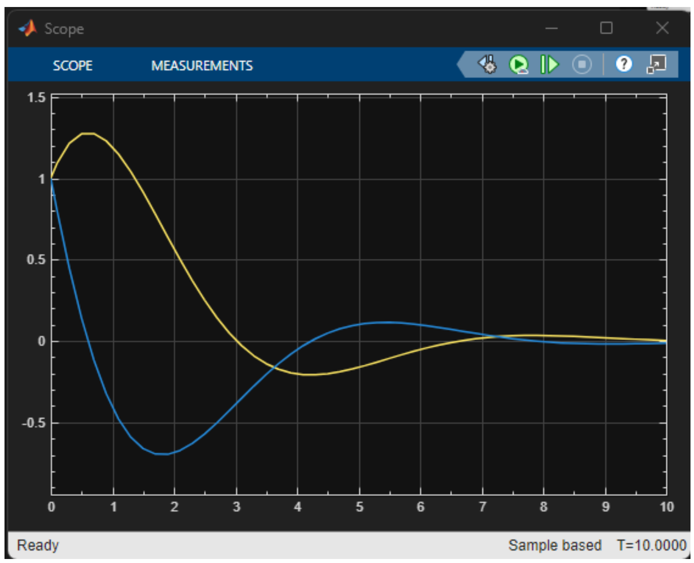

Figure 3: MATLAB Problem 3B Output

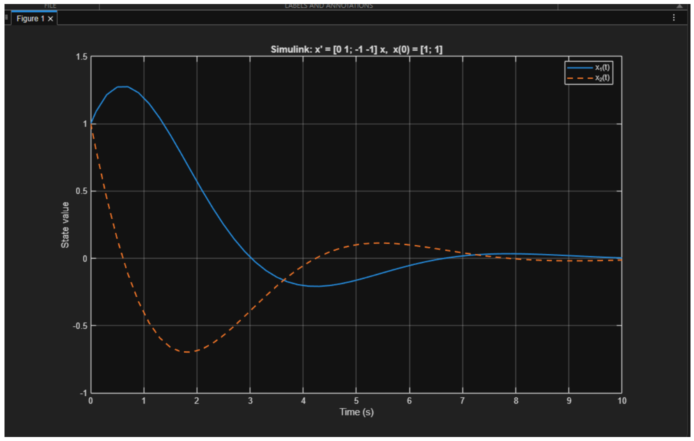

Figure 4: MATLAB Problem 3B Output

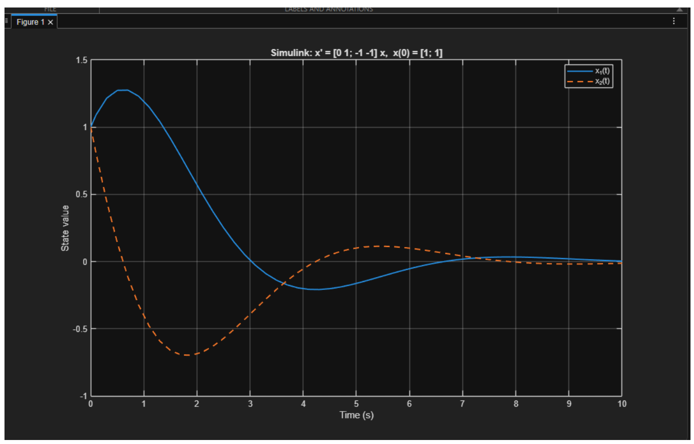

Figure 5: MATLAB Problem 3B Parameters over 10s

So in this problem, our objective was to simulate the closed-loop state-space model from part a with $k = 1$ from the following:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x, \quad y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

As we can see, the state-space block output or a two-element vector:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

That was sent directly to a scope with elements as channels. The trajectories that are show are decaying oscillations to zero, which consistent with the poles $-0.5 \pm j0.866$. As a result, this satisfies the goal of implementing and simulating the obtained state-space model in MATLAB/Simulink and to show the state trajectories for x_1 and x_2.

3.3 Part C

Example 3.4. For a reference input $r(t)$, find an expression for the impulse response of this closed-loop system (assuming zero initial conditions).

3.3.1 My Solution

From our given diagram, we see that the forward path is $K(s+1)$ and the plant is $\frac{1}{s^2}$ with unity negative feedback. Hence, the closed-loop transfer is:

$$H(s) = \frac{\Theta(s)}{R(s)} = \frac{K(s+1)}{s^2 + Ks + K}.$$

Now, for an impulse input $r(t) = \delta(t)$ with zero initial conditions, the impulse response is:

$$h(t) = \mathcal{L}^{-1}\left\{\frac{K(s+1)}{s^2 + Ks + K}\right\}, \quad t \geq 0$$

Let the poles be as follows:

$$p_{1,2} = \frac{-K \pm \sqrt{K^2 - 4K}}{2}.$$

From this equation, it follows that a compact time-domain form is:

$$h(t) = K\left[\frac{p_1 + 1}{p_1 - p_2}e^{p_1 t} + \frac{p_2 + 1}{p_2 - p_1}e^{p_2 t}\right]u(t)$$

Therefore, if you utilize the value that is used in part b, $k = 1$ it follows that:

$$h(t) = e^{-t/2}\left[\cos\left(\tfrac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right],\ t \geq 0.$$

4 Question 4

Example 4.1. Consider the spring mass-damper system with two masses m_1 and m_2. The inputs are the forces for u_1 and u_2 , and the outputs are the displacements y_1 and y_2 of these masses, respectively.

4.1 Part A

Example 4.2. Find a state-space representation of the system with the states x_1 and x_2 being the displacement and velocities of mass m_1. Similarly, x_3 and x_4 are those of mass m_2.

4.1.1 My Solution

So here are the elements: The spring k_1 from m_1 to ground, damper b_1 between m_1 and m_2, spring k_2 from m_2 to ground. And the inputs u_1, u_2 are external forces on m_1, m_2 (downward).

And our states here are:

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} y_1 & \dot{y}_1 & y_2 & \dot{y}_2 \end{bmatrix}^T.$$

So,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ m_1\dot{x}_2 &= -k_1x_1 + b_1(x_4 - x_2) + u_1 \\ \dot{x}_3 &= x_4 \\ m_2\dot{x}_4 &= -k_2x_3 - b_1(x_4 - x_2) + u_2 \end{aligned}$$

Thus, it follows that:

$$\dot{x} = \mathbf{Ax} + \mathbf{Bu}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{Cx} + \mathbf{Du}$$

Our state-space matrices are as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{b_1}{m_1} & 0 & \frac{b_1}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{b_1}{m_2} & -\frac{k_2}{m_2} & -\frac{b_1}{m_2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note it is downward positive, $x_1 = y_1$ and $x_3 = y_2$.

4.2 Part B

Example 4.3. Find an expression for the transfer function between input u_1 and output y_2.

4.2.1 My Solution

So from the two coupled equations in the Laplace domain, we get the following two equations:

$$\begin{aligned} (m_1s^2 + b_1s + k_1)X_1 - b_1sX_3 &= U_1, \\ -b_1sX_1 + (m_2s^2 + b_1s + k_2)X_3 &= U_2, \end{aligned}$$

.

Now, set $U_2 = 0$ and solve for $\frac{X_3}{U_1}$, it follows that:

$$\Delta(s) = (m_1s^2 + b_1s + k_1)(m_2s^2 + b_1s + k_2) - (b_1s)^2,$$

Hence, we get the following:

$$G_{y_2,u_1}(s) = \frac{X_3(s)}{U_1(s)} = \frac{b_1s}{\Delta(s)}.$$

Therefore, we have obtained our transfer function.

4.3 Part C

Example 4.4. With all the coefficients being 1, find the complete impulse response of this system for unit impulses $u_1(t) = u_2(t) = \delta(t)$ using MATLAB. Include the relevant script and plots.

4.3.1 My Solution

In the below section, we will implement the impulse response of this system.

4.3.2 MATLAB Script

```
% Our Parameters

m1 = 1;
m2 = 1;
b1 = 1;
k1 = 1;
k2 = 1;

A = [0 1 0 0; -k1/m1 -b1/m1 0 b1/m1; 0 0 0 1; 0 b1/m2 -k2/m2 -b1/m2];
B = [0 0; 1/m1 0; 0 0; 0 1/m2];
C = [1 0 0 0; 0 0 1 0];
D = zeros(2, 2);

sys = ss(A,B,C,D);

% Impulse Response from both inputs applied simultaneously

t = linspace(0, 20, 4001);
[y,t] = impulse(sys,t);

% Sum responses from u1 and u2 impulses because the system is LTI

y_combined = y(:,:,1) + y(:,:,2); % column 1 = y1, column 2 = y2

% Plot our figures

figure; plot(t, y_combined(:,1), 'LineWidth',1.5);
xlabel('Time (s)'); ylabel('y_1(t)'); title('Impulse response y_1 for u_1=\delta,
    u_2=\delta'); grid on;

figure; plot(t, y_combined(:,2), 'LineWidth',1.5);
xlabel('Time (s)'); ylabel('y_2(t)'); title('Impulse response y_2 for u_1=\delta,
    u_2=\delta'); grid on;

% Show each input contribution seperately

figure; plot(t, y(:,1,1), t, y(:,1,2)); legend('y1<-u1','y1<-u2'); grid on;
figure; plot(t, y(:,2,1), t, y(:,2,2)); legend('y2<-u1','y2<-u2'); grid on;
```

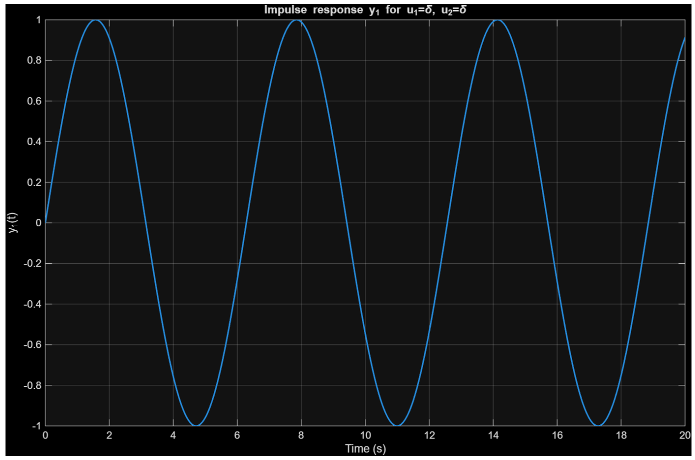

Figure 6: Impulse Response y_1 for $u_1 = \delta$ and $u_2 = \delta$

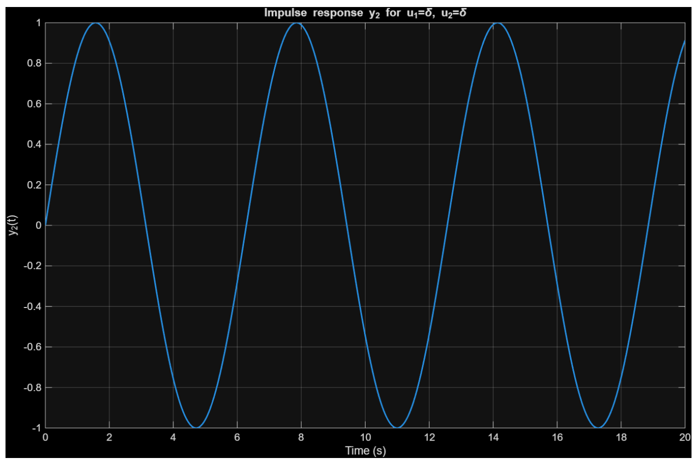

Figure 7: Impulse Response y_2 for $u_1 = \delta$ and $u_2 = \delta$

4.3.3 MATLAB Figures

Since $m_1 = m_2 = b_1 = k_1 = k_2 = 1$ and that $u_1 = u_2 = \delta$, the initial velocities are equivalent. It then follows that the damper force $b_1(\dot{y}_2 - \dot{y}_1) = 0$. The masses move in common mode with

$$\omega_n = \sqrt{\frac{k}{m}} = 1,$$

which follows that $y_1(t) = y_2(t) = \sin t$. Thus, the undamped sinusoid that is seen in the plots is given above.

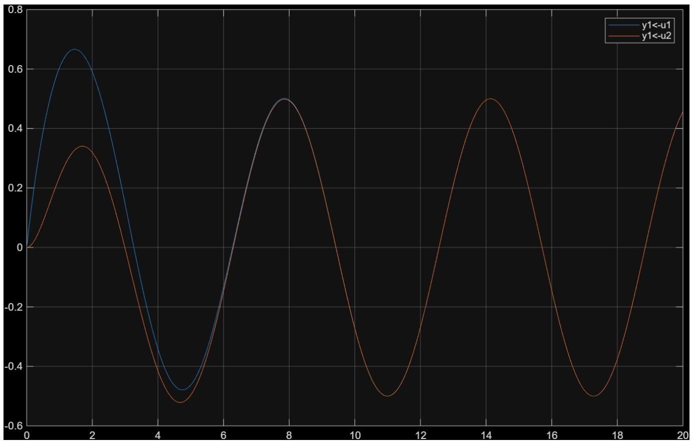

Figure 8: Figure 3

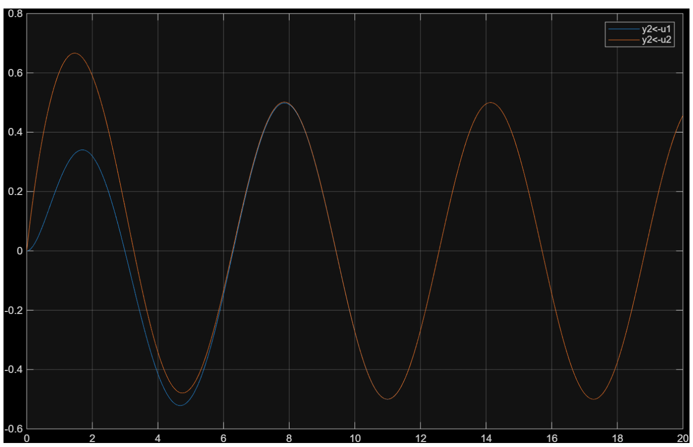

Figure 9: Figure 4