Metamaterial Design for Acoustic Collimator

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1 Effective Density and Bulk Modulus

Provided a metamaterial structure, we need to compute the effective density and bulk modulus. We begin the derivation with the time harmonic wave equations of linear acoustics in a inviscid fluid, namely,

$$\begin{aligned} \nabla p &= -i\omega\rho \boldsymbol{v}, \\ \rho c^2 \nabla \cdot \boldsymbol{v} &= -i\omega p \end{aligned} \tag{1}$$

where p and $\boldsymbol{v}$ are the acoustic pressure and velocity, respectively, ω is the angular velocity, ρ is the mass density, and c is the speed of sound. Here $\lambda = \rho c^2$ is defined as the bulk modulus.

1.1 Planar Wave Equation

The planar wave equation is defined in time-harmonic form as

$$p(\boldsymbol{x}) = Ae^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2}$$

where $\boldsymbol{k} = k\boldsymbol{n}$. Here k being the wave number satisfying $\omega = kc$, and $\boldsymbol{n}$ is a normalized vector pointing to the traveling direction of the planar wave. Using the time-harmonic wave equation, we obtain the acoustic velocity, expressed as

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{k}\frac{A}{\rho\omega}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \boldsymbol{n}\frac{A}{\rho c}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

1.2 Infinite Plane Transmission Matrix

We have the transmission matrix T that relates acoustic pressures and velocities along the vertical direction. In particular, (p_1, v_1) and (p_2, v_2) in Figure 1 satisfy

$$\begin{pmatrix} p_2 \\ v_2 \end{pmatrix} = \mathsf{T} \begin{pmatrix} p_1 \\ v_1 \end{pmatrix}. \tag{3}$$

Here v_1 and v_2 are the acoustic velocity scalars at the bottom and top ports of the metamaterial cell. They are positive value if the acoustic velocity points toward the

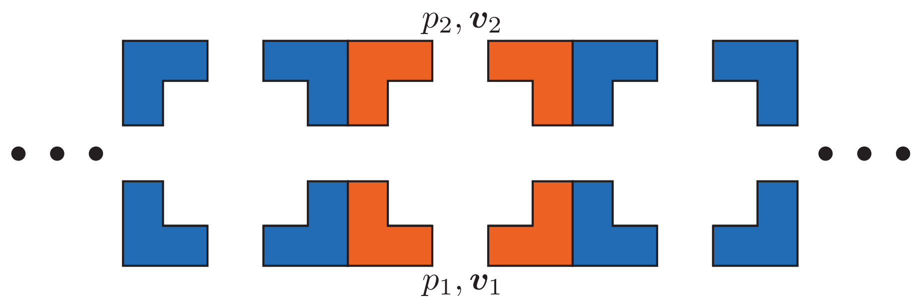

Figure 1: Assemble the metamaterial units into a infinite plane.

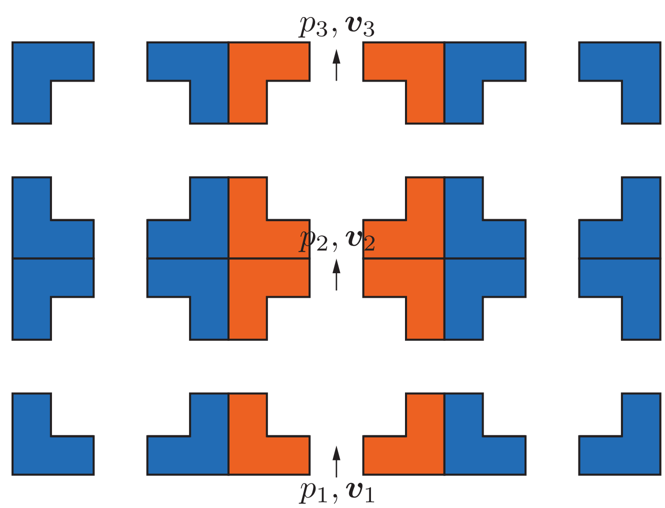

Figure 2: Two layers of acoustic metamaterials.

outside of the cell. Eq. (3) implies that

$$\begin{pmatrix} p_1 \\ v_1 \end{pmatrix} = \mathsf{T}^{-1} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix}.$$

Meanwhile, because of the symmetry with respect to the x-axis, one can switch (p_1, v_1) and (p_2, v_2) and obtain

$$\begin{pmatrix} p_1 \\ v_1 \end{pmatrix} = \mathsf{T} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix}.$$

Combining the above two expression indicates that $\mathsf{T} = \mathsf{T}^{-1}$.

Now, consider two layers of metamaterials, as illustrated in Figure 2. We obtain the following relationships,

$$\begin{pmatrix} p_3 \\ v_3 \end{pmatrix} = \mathsf{T} \begin{pmatrix} p_2 \\ -v_2 \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} p_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} T_{11} & -T_{12} \\ T_{21} & -T_{22} \end{pmatrix} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix}.$$

$$\begin{pmatrix} p_2 \\ v_2 \end{pmatrix} = \mathsf{T} \begin{pmatrix} p_1 \\ -v_1 \end{pmatrix} \implies \begin{pmatrix} p_1 \\ -v_1 \end{pmatrix} = \mathsf{T} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix} \implies \begin{pmatrix} p_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ -T_{21} & -T_{22} \end{pmatrix} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix}.$$

Then, the difference between (p_3, v_3) and (p_1, v_1) is

$$\begin{pmatrix} p_3 - p_1 \\ v_3 - v_1 \end{pmatrix} = \begin{pmatrix} 0 & -2T_{12} \\ 2T_{21} & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ v_2 \end{pmatrix}.$$

Now, recall the wave equation (1), we can compute the effective mass and bulk modulus,

$$\begin{aligned} \nabla p &\approx \frac{p_3 - p_1}{2L} = -\frac{T_{12}}{L} v_2 \approx -i\omega\rho v, \\ \nabla v &\approx \frac{v_3 - v_1}{2L} = \frac{T_{21}}{L} p_2 \approx -\frac{i\omega}{\rho c^2} p \end{aligned} \tag{4}$$

The effective density and effective bulk modulus are

$$\rho' = \frac{1}{L\omega} \mathsf{imag}(T_{12}), \text{ and } \lambda' = \rho' c'^2 = -\frac{L\omega}{\mathsf{imag}(T_{21})}.$$