Chapter 8
Transformation Acoustics

Steven A. Cummer

Abstract In this chapter we review the development of the concept of transformation acoustics, through which sound fields can be arbitrarily manipulated by complex acoustic materials. We describe the theory and the design equations in several different forms, and we present several explicit design examples using transformation acoustics. After briefly describing some theoretical offshoots from the original idea, we conclude with a summary of approaches for engineering composite materials with the smoothly inhomogeneous and anisotropic properties needed for many transformation acoustics devices.

8.1 Introduction

Suppose you can imagine an acoustic device that manipulates sound waves in a specific way. Let us be more specific: let us suppose you can imagine an acoustic device that manipulates sound waves by stretching or squeezing or shifting or in other ways operating on an incident sound field. Perhaps this device takes that sound field and bends it perfectly around a corner with no reflection. Or suppose this device bends incident sound energy around a central object so that it does not scatter or cast a shadow (see Fig. 8.1).

Visualizing the device is the easy part. The hard part is determining the properties of the material or materials required to realize such a device. If you are given a configuration of defined materials, there are any number of analytical or numerical techniques that will enable you to calculate how a sound wave interacts with and is operated on by these materials. But the problem we have posed is the much more challenging one in which we know the output, i.e. the desired new sound field configuration, and we want to know the acoustic medium that would have such an effect. There are very few techniques to apply to such a problem. In fact, there is no guarantee that a medium that produces the desired effect even exists.

Transformation acoustics is technique that solves this problem in a surprisingly straightforward way. More generally, it is a paradigm for the creation of sound-

S.A. Cummer (✉)
Department of Electrical and Computer Engineering, Duke University, Durham, NC 27708, USA
e-mail: cummer@ee.duke.edu

R.V. Craster, S. Guenneau (eds.), *Acoustic Metamaterials*,
Springer Series in Materials Science 166, DOI 10.1007/978-94-007-4813-2_8,

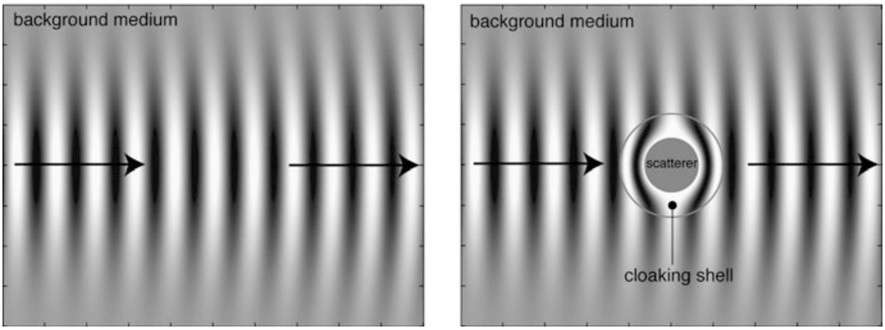

Fig. 8.1 Example of what the coordinate transformation approach can do. *Left*: A simulated field distribution for an electromagnetic beam traveling from left to right in empty space. *Right*: A simulated field distribution for an electromagnetic beam interacting with a metal cylinder surrounded by an ideal electromagnetic cloaking shell, marked by the *solid lines*. Outside the shell, the two distributions are identical, indicating zero scattering from the cloaked cylinder. Inside the shell, the fields are deformed as they would be from the coordinate transformation used to design the shell properties

manipulating materials and devices that are either difficult or impossible to derive through other theoretical approaches. It is based on the idea of a coordinate transformation of an arbitrary initial sound field. If the device you imagine can be defined in terms of a coordinate transformation, by squeezing, stretching, and/or displacing the sound field in a finite region, then transformation acoustics provides the mathematics for taking this coordinate transformation and deriving the properties of a material in that same finite region that will have exactly the same effect on the sound field as the coordinate transformation.

The theory is remarkably powerful, simple to apply, and general. However, few things in life are free, and transformation acoustics is not an exception. In most cases the materials required to physically realize these devices have very complex properties. The usual outcome of a transformation acoustics design is a material that has continuously varying properties and is also anisotropic. Such materials are not usually obtainable off-the-shelf. But many of these properties can be engineered into well-designed composite materials using the techniques from the field of acoustic metamaterials. To be sure, fabricating functional transformation acoustics devices will be a challenge. But basic ideas for how it can be done are in place, and the payoff in realizing truly novel acoustic devices is substantial.

In this chapter we review the development of the concept of transformation acoustics, through which sound fields can be arbitrarily manipulated by complex acoustic materials. We describe the theory in detail, summarize the design equations in several different forms, and briefly describe some theoretical offshoots from the original idea. We present several explicit design examples using transformation acoustics, and we conclude with a summary of approaches for engineering composite materials with the smoothly inhomogeneous and anisotropic properties needed for many transformation acoustics devices.

8.2 The Original Idea: Transformation Electromagnetics

The development of transformation acoustics actually began with the development of transformation electromagnetics, a related idea that linked coordinate transformations on electromagnetic fields to specific electromagnetic material properties. Given that all of this theoretical development occurred within five years of this writing, it is both interesting and instructive to review the sequence of the development of these profound ideas.

8.2.1 Transformation Electromagnetics Theory

Although the concept of equating coordinate transformations with physical material properties in electromagnetics has a long history [27, 38], the modern application of the idea originated quite recently [26]. It should be noted that closely related work developed a similar concept for electric current flow and applied it to electrical impedance tomography [14].

Again, the transformation electromagnetics concept is based on the finding that the specific distortion or modification of an electromagnetic field distribution by a coordinate transformation can also be created by a specific set of electromagnetic material properties in the region of the transformed fields. This transformation-material equivalence is a direct consequence of the coordinate transformation invariance of the dynamical equations describing electromagnetic fields, in this case the well-known Maxwell equations. This invariance is an integral part of general relativity and has been known for a long time. The fantastic insight of Pendry et al. was that this invariance could be interpreted in terms of physical electromagnetic materials.

At this point it will be useful to see show some of the details of how transformation electromagnetics works. Although Pendry et al. presented their version of transformation electromagnetics in terms of basis vectors and scale factors, it can be reformulated more compactly using Jacobian matrices [37]. Let us impose an arbitrary coordinate transformation on the Maxwell equations, or equivalently, on any electromagnetic field distribution that satisfies the Maxwell equations. This function F contains all of the stretching, squeezing, and displacing we might want to impose on the fields. After the transformation, the field equations are in exactly the same form but with new material parameter tensors (electric permittivity $\bar{\bar{\varepsilon}}_r$ and magnetic permeability $\bar{\bar{\mu}}_r$) expressed in matrix form (in terms of the original medium parameters $\bar{\bar{\varepsilon}}'_r$ and $\bar{\bar{\mu}}'_r$) as [37]

$$\bar{\bar{\varepsilon}}_r = \frac{A\bar{\bar{\varepsilon}}'_r A^T}{\det(A)}, \qquad \bar{\bar{\mu}}_r = \frac{A\bar{\bar{\mu}}'_r A^T}{\det(A)}, \tag{8.1}$$

where the matrix A is the Jacobian matrix of the transformation (see Sect. 8.3.1 for a more explicit description of A in the context of transformation acoustics).

In other words, the Maxwell equations are coordinate transformation invariant, and the coordinate transformation manifests itself through a change in the material parameters defined above. This invariance has been long understood, but (8.1) has traditionally been interpreted as a representation of the original material parameters in a new coordinate system or space. Pendry realized this could equally well represent new material parameters in the original space. The first, and perhaps still most interesting, application of this idea was to making an object invisible by surrounding it with a cylindrical or spherical shell that bends, without reflection, any incident wave energy smoothly around the interior of the shell so that the wave energy exits the shell with exactly the same phase and amplitude that it would have had without any object present at all.

Figure 8.1 shows, for demonstration purposes, the resulting field distribution inside and outside of such a cloaking shell with electromagnetic material parameters defined from the original work in the field [26]. One can see that, outside the shell, the field distribution is exactly the same as it would be with no object at all. This means that there is no wave scattering in any direction, including the difficult-to-control forward scatter or shadow region, and any object in the interior of the shell is effectively invisible to electromagnetic waves. Note too that the resulting field distribution *inside* the shell is exactly what one would expect if the original fields in empty space in a circle are compressed into the annulus defined by the shell, which is the coordinate transformation that yielded the desired material properties.

This finding has resulted in an explosion of literature describing complex materials capable of many exotic manipulations of electromagnetic fields [33, 34] that are just beginning to be physically realized in simulation [8] and experiment [21, 36]. In general the electromagnetic parameters that result from transformation electromagnetics are anisotropic and complicated to create, but the concept of electromagnetic metamaterials [25] lends itself well to the design and fabrication of such complex materials.

8.2.2 Can Transformation Electromagnetics Be Extended to Other Waves?

An important question that followed the development of transformation electromagnetics was whether transformation electromagnetics could be extended to other wave systems. The first work to address this issue [22] analyzed the coordinate transformation invariance of the equations of elastodynamics. These equations proved not to be coordinate-transformation invariant, implying that coordinate transformations could not in general be realized through complex elastodynamic media. Importantly, this work suggested that acoustics, as a subset of elastodynamics, could not be manipulated via the transformation approach. However, as described below, it was eventually shown that transformation acoustics is valid under conditions slightly different than those considered in this work.

8.3 Transformation Acoustics

The first work to show that the concept of transformation acoustics was at least partially valid [10] found that the 2D acoustic equations with anisotropic tensor mass density take the same form as the 2D single polarization Maxwell equations with anisotropic permittivity or permeability. Thus, by analogy, any 2D transformation electromagnetics device or material could be translated directly into a 2D transformation acoustics device. Soon after, the analogy approach was applied in a slightly different form [3] and showed that the fully 3D acoustic equations with anisotropic tensor mass density take the same form as the electric conductivity equation with embedded sources, which was already known to have transformation-type solutions [14].

Transformation acoustics at this point was thus shown to be conceptually valid. Any complex manipulation of sound fields that can be described by a coordinate transformation can be realized through complex acoustic materials defined by the transformation itself. We explore and demonstrate this idea in some detailed examples below, but first we derive the basic equations in a simple form that enables us to explore the unusual material properties needed to physically realize transformation acoustics devices.

8.3.1 A Succinct Derivation of Transformation Acoustics

While the original derivations of transformation acoustics were based on analogies with other wave systems [3, 10], useful insight and understanding comes from a direct derivation of the underlying equations of transformation acoustics. Such a derivation was presented in [9], which employed the unit vector-based approach used in the original work on transformation electromagnetics [26]. A more concise derivation comes through the direct use of Jacobian matrixes, and we present such a derivation of transformation acoustics here.

Figure 8.2 articulates the basic transformation idea. We begin with the original coordinate system, sound field distribution, and material parameters (left panel) denoted with the primed ($'$) designation. This describes the virtual space that we want the wave to "see" as it propagates in the physical space that will contain the complex materials. The coordinate transformation $(x, y, z) = F(x', y', z')$ then deforms the space, and the fields and materials contained in it, as shown in the right panel of Fig. 8.2. This leads to a notation in which everything about the virtual space (i.e., coordinates, fields, and material parameters) is primed, and everything about the physical space that contains the complex materials that deform the fields as we want them deformed is unprimed.

In this case we have chosen to alter the coordinates inside a finite region (a circle of $r < 0.5$) by compressing the radius, but this is meant simply to be an illustrative example. After the transformation the right panel represents the physical space and the deformed physical sound field distribution we will obtain with the complex material parameters determined through the transformation acoustics formalism. These new materials will reside inside the region of the coordinate transformation, which

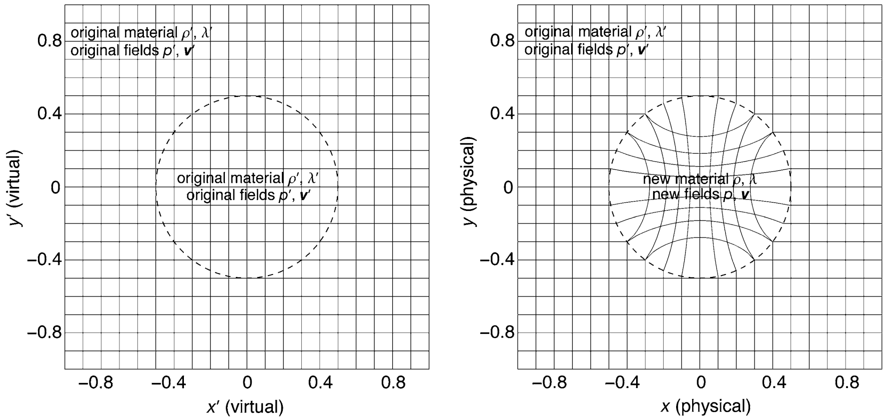

Fig. 8.2 Illustration of how transformation acoustics works. *Left*: The original, "virtual" space represented by a simple Cartesian coordinate system and filled with a simple, homogeneous fluid. *Right*: Inside the circle of $r < 0.5$, the coordinates are transformed by shrinking the radius. The sound field inside the *circle* can be deformed in exactly this way inside this transformed region by complicated acoustic material parameters derived through the transformation acoustics formalism

in this case is again a circle of $r < 0.5$. Outside this circle, the coordinates are unchanged, and thus the material is unchanged by the transformation.

We now derive expressions for the new material properties needed to physically realize the coordinate transformation. We begin the derivation with the time harmonic equations of linear acoustics in a simple inviscid fluid, namely

$$\nabla' p' = i\omega\rho' \mathbf{v}', \tag{8.2}$$

$$i\omega p' = \lambda' \nabla' \cdot \mathbf{v}', \tag{8.3}$$

where p' is pressure, $\mathbf{v}'$ is velocity, λ' is bulk modulus, and ρ' is mass density. Note that the harmonic time convention $\exp(-i\omega t)$ is employed here and throughout this chapter. These fields and material parameters are the fields and materials in virtual space (x', y', z') that usually contain simple material parameters and a simple sound field distribution.

We wish to impose a coordinate change described by new curvilinear coordinates x, y, and z on these equations. The Jacobian matrix A of the transformation from (x', y', z') to (x, y, z) is given by

$$A = \begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial z'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial z'} \\ \frac{\partial z}{\partial x'} & \frac{\partial z}{\partial y'} & \frac{\partial z}{\partial z'} \end{bmatrix}, \tag{8.4}$$

where it is implicitly assumed that each new coordinate (e.g., x) is a function of the three original coordinates (x', y', z'). In terms of A, the gradient operation can be expressed in terms of the new unprimed coordinates as (for example, [15])

$$\nabla' p' = A^T \nabla p' = A^T \nabla p, \tag{8.5}$$

where ∇ denotes the gradient operator in the new coordinate system, and the pressure in the new coordinate system p is simply the original pressure p' translated or dragged to the new coordinates. In other words, the new pressure p reflects the coordinate transformation as in the right panel of Fig. 8.2, but it is not rescaled by the transformation.

Similarly, the divergence operation can be expressed in the new coordinates as

$$\nabla' \cdot \mathbf{v}' = \det(A) \nabla \cdot \frac{A}{\det(A)} \mathbf{v}' = \det(A) \nabla \cdot \mathbf{v}, \tag{8.6}$$

where $\nabla\cdot$ denotes the divergence operator in the new coordinates, and the velocity vector after the transformation is given by

$$\mathbf{v} = \frac{A}{\det(A)} \mathbf{v}'. \tag{8.7}$$

Thus, in contrast to the scalar pressure, the vector velocity is translated and rescaled by the coordinate transformation. In the most general case, this rescaling can involve both a rotation and a magnitude change determined by the matrix A.

With these expressions the original system of equations in (8.2)–(8.3) can be written in the new coordinates as

$$\nabla p = i\omega \left[\det(A) \left(A^T\right)^{-1} \rho \left(A^{-1}\right) \right] \mathbf{v}, \tag{8.8}$$

$$i\omega p = \left[\lambda \det(A) \right] \nabla \cdot \mathbf{v}. \tag{8.9}$$

Note that this set of equations is in exactly the same fundamental form as the original acoustic equations in (8.2)–(8.3)but with new material parameters

$$\bar{\bar{\rho}} = \det(A) \left(A^T\right)^{-1} \rho' \left(A^{-1}\right), \tag{8.10}$$

$$\lambda = \lambda' \det(A). \tag{8.11}$$

That the fundamental structure of the equations remains unchanged after this spatial coordinate transformation is the essence of transformation acoustics. This means that if the original pressure p' and velocity $\mathbf{v}'$ fields are solutions to the acoustic equations in a medium defined by the original mass density ρ' and bulk modulus λ', then the transformed p (which has been dragged through space according to the transformation but is otherwise unmodified) and v (which has been both dragged and rescaled according to (8.7) are *also* solutions to the acoustic equations in a new medium defined by the transformed and anisotropic mass density $\bar{\bar{\rho}}$ and bulk modulus λ given above.

Transformation acoustics is thus a very powerful tool for problems of acoustic design or synthesis. While there are plenty of techniques that can tell you how sound waves interact with a certain material distribution, there are very few (if any) general purpose tools for determining the material properties and spatial distribution required to execute a specific operation on sound waves. Transformation acoustics is just such a tool—it tells you the precise material properties needed to manipulate sound waves in a certain way, provided that manipulation can be described in terms of a coordinate transformation. Section 8.4 below demonstrates this concept in two concrete examples.

8.3.2 *Some Initial Comments on the Materials Needed for Transformation Acoustics*

Equations (8.10) and (8.11) describe the materials needed to physically realize a transformation acoustics device. These materials are not, in general, simple, and a few comments about their properties might be helpful at this point. How one can realize these kinds of materials is described in more detail in Sect. 8.5, although it should be noted that figuring out how to create such materials is a very active area of research at the time of this writing.

For smooth transformations, which are commonly used in practice, $\det(A)$ will be a smooth and continuous function of position defined by the new coordinates (x, y, z). According to (8.11), this means that the bulk modulus of the needed medium must vary smoothly with position. Importantly, the resulting material is still described by a scalar bulk modulus, and is thus fundamentally still an inviscid fluid. Although smoothly-varying bulk modulus is not a property common in real fluids, one can imagine that composite materials can be designed with spatially varying composition that mimics this property. For example, it has long been known that under some conditions the effective bulk modulus of a mixture of fluids and solids is simply the volume-weighted average of the bulk modulus of the components [41]. This suggests that a fluid with solid inclusions of continuously increasing concentration can behave like a material with smoothly-varying bulk modulus.

In contrast, (8.10) indicates that a general transformation acoustics material must exhibit some properties that are dramatically different from the original fluid. In most cases the original fluid filling the original, virtual space will have a simple, isotropic mass density. The physical material required to mimic the coordinate transformation, however, is described by a mass density tensor or matrix. This means that the fluid needs to behave for acoustic waves as if it has a different mass density for oscillatory fluid motion in different directions. In other words, the effective dynamic mass density of the fluid must be anisotropic.

This is not as crazy a concept as it might sound at first. Obviously a parcel of fluid by itself will not exhibit mass anisotropy. If you grab that parcel and apply a steady force to it, it will accelerate according to Newton's second law of motion or $F = ma$. And if you apply a steady force in a different direction, say perpendicular to that original direction, it will still accelerate as if it has the same mass m. However, oscillatory motion is not the same as steady motion, and for manipulating sound waves we definitely wish to control the response to oscillatory forces.

A relatively simple conceptual model for a material that exhibits anisotropic dynamic mass has been described in the literature [22], namely a hollow shell containing a spring-loaded mass in which the springs attaching the mass to the shell have different spring constants in different directions. The mass inside the shell will exhibit a resonance at a specific frequency that, because of the different spring constants, varies with the direction of the motion. If one applies an oscillatory force to the shell at a frequency far from a resonance frequency, then the dynamic effective mass of the shell will be close to its total static mass. However, if an oscillatory force is applied close to the resonance frequency, the resonant motion of the interior mass will strongly change the dynamic effective mass of the shell. The different resonance frequencies for different directions of motion means that this object exhibits anisotropic dynamic effective mass.

Another possibility for creating anisotropic effective mass density is a composite material made of alternating thin layers two or more different background fluids. It has been shown analytically [35] that sound waves traveling normal to these layers, provided they are sufficiently thin compared to a wavelength, experience a different effective mass than do sound waves traveling parallel to these layers. Simulations have shown that transformation acoustics devices can in principle be realized physically by these alternating thin fluid layers [5, 40]. These conceptual models are primarily meant to show how anisotropic effective dynamic mass density is not an inconceivable concept. More details of how transformation acoustic materials can be physically realized are described in Sect. 8.5, particularly with structures that are simpler to physically construct.

8.4 Application Examples

Through the equations above, transformation acoustics enables any modification of sound fields that can be described by a coordinate transformation to be physically realized with a specific, and admittedly often quite complex, medium. At this stage it will be helpful to demonstrate several concrete examples of applying transformation acoustics to derive the material parameters required for specific applications. Obviously the universe of possible transformations and thus possible transformation acoustics devices is enormous, and these analytically tractable examples are meant to demonstrate these possibilities and illustrate the mathematical mechanics of applying transformation acoustics.

8.4.1 An Acoustic Beam Shifter

We first describe in detail a beam shifter, or a material that laterally shifts an acoustic beam traveling through it. The material properties for such a device were first described for electromagnetics in [32] and for acoustics in [9]. Although not necessarily the most exciting device, it is described naturally in Cartesian coordinates,

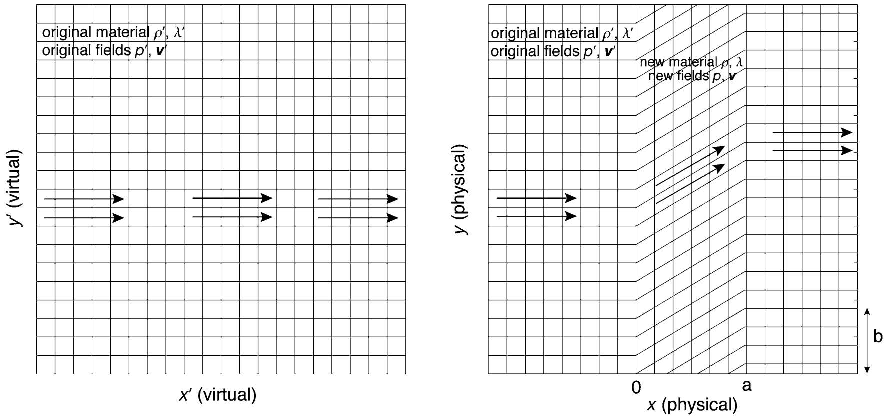

Fig. 8.3 Illustration of the design of an acoustic beam shifter. *Left*: The original, virtual space represented by a simple Cartesian coordinate system and filled with a simple, homogeneous fluid. *Right*: For $0 < x < a$, the coordinates are transformed by linearly shifting the y coordinate with distance such that the total vertical shift is b. The sound field inside the *slab* can be deformed in exactly this way inside this transformed region by acoustic material parameters derived here. The result is that a sound beam that travels from left to right in the virtual space, as in the *arrows* in the *left panel*, will follow the deformed path marked by the *arrows* in the *right panel*

and the Jacobian contains off-diagonal elements that make it useful in illustrating the transformation acoustics concept.

Figure 8.3 illustrates the original virtual space (x', y') and the transformed physical space (x, y). For $x = [0, a]$, the y coordinate is shifted as a linear function of x, yielding the distorted grid in the right panel of the figure. If the space $x = [0, a]$ is filled with material defined by (8.10)–(8.11), then the resulting sounds fields will be distorted in exactly the way shown by the distorted grid. An acoustic beam propagating along the x direction will be laterally shifted as it propagates through this medium, as illustrated by the arrows in the figure, so that it exits the $x = [0, a]$ region at a different y position than it entered.

In the $x = [0, a]$ domain of the coordinate transformation, the transformation itself is given by

$$x = x', \qquad y = y' + \frac{b}{a}x', \qquad z = z'. \tag{8.12}$$

According to (8.10)–(8.11), the key matrixes are the inverse of the Jacobian A^{-1} and $(A^T)^{-1}$, which according to (8.4) are given by

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \left(-\frac{b}{a}\right) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.13}$$

and

$$\left(A^T\right)^{-1} = \begin{bmatrix} 1 & \left(-\frac{b}{a}\right) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.14}$$

Since $\det(A) = (\det(A))^{-1} = 1$, (8.11) says directly that

$$\lambda = \lambda', \tag{8.15}$$

or the fluid modulus in the transformed region required for the beam shifter is the same as the background fluid. The effective mass density required is more complicated. Applying (8.10) gives

$$\bar{\bar{\rho}} = \det(A)\left(A^T\right)^{-1}\rho'\left(A^{-1}\right) = \begin{bmatrix} 1+\left(\frac{b}{a}\right)^2 & \left(-\frac{b}{a}\right) & 0 \\ \left(-\frac{b}{a}\right) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\rho'. \tag{8.16}$$

Thus the effective mass density in the transformed region needed to make the beam shifter is strongly anisotropic with off-diagonal elements in a Cartesian coordinate system. Note, however, that this matrix can be diagonalized and thus there is a rotated (x, y) coordinate system in which the mass density has no off-diagonal elements.

And this concludes the transformation acoustics design procedure. We have determined the properties of the fluid in the region $0 < x < a$ that are required to yield the device illustrated in Fig. 8.3, in which a sound beam traveling in the x direction is deflected, without any reflections, to exit the material at $x = a$ with its position in y shifted by an amount b. Note that the more aggressive the beam shifting (denoted by a larger (b/a)), the more anisotropic is the required material. This is a general characteristic of transformation acoustics design: the more extreme the manipulation of the fields (via bending, shifting, squeezing, etc.), the more extreme the material parameters needed to manipulate those fields.

It would be satisfying to confirm via numerical simulations that the derived material parameters in (8.15)–(8.16) really do manipulate sound waves as expected from the original coordinate transformation. Doing so obviously requires the simulation of acoustic waves in materials with anisotropic mass density. Such a capability is not common in commercial simulation tools. Thus, to test the idea, we invoke the equivalence of acoustic and electromagnetic waves in two dimensions [10]. We convert the acoustic material parameters to electromagnetic material parameters, and use the commercial software package COMSOL Multiphysics to compute what is the equivalent of the pressure field distribution in such a material.

Figure 8.4 shows the results of these calculations. After mapping the solution back to the acoustic fields, the left panel shows the resulting pressure field distribution for a beam normally incident on the beam-shifting material of (8.15)–(8.16) with $b/a = 1$. It is clear that the sound field behaves exactly as predicted by the

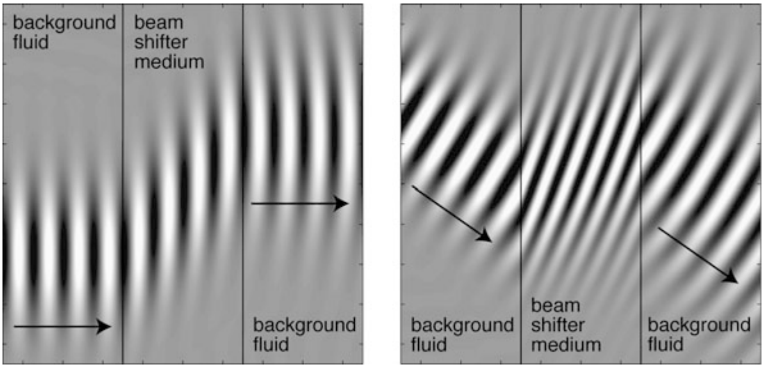

Fig. 8.4 Simulated performance of an acoustic beam shifter. *Left*: A time snapshot of the pressure field resulting from an acoustic beam normally incident on the beam shifting material. The beam is manipulated in precisely the way that is expected from the original transformation. *Right*: Even for off-normal incidence, an incident beam is shifted by the anisotropic acoustic material and exits the slab offset by the expected amount and at the same angle it was incident. The beam propagation direction outside the beam shifting medium is marked by *arrows* in both images

original transformation and illustrated in Fig. 8.3. It is interesting to note that the lines of constant phase of the incident wave are not distorted by the material. This means that although the acoustic wave power is traveling along a direction different from x, the phase fronts and wave normal vector k do point in the x direction. This is a consequence of the anisotropy of the material.

The right panel of Fig. 8.4 shows what happens when an acoustic beam is obliquely incident on the same material. The end result is exactly the same, i.e. the beam exits the material at a location that is offset in y from where it entered the material by a distance equal to the thickness of the material (since we chose $b/a = 1$), and it does so without reflection at either material interface because of the continuity of the coordinate transformation. Note that in this case the wave front behavior is a bit more complicated because they are not parallel to a coordinate direction. Nevertheless, the device performance is exactly what is expected from the original transformation. As the theory says, the designed material manipulates the sound fields as specified by the original coordinate transformation. This is the essence of transformation acoustics.

8.4.2 An Acoustic Cloaking Shell

The most interesting device thus far described in the transformation acoustics literature is an acoustic cloaking shell, i.e. a material shell that can surround an object and render the composite object (the shell plus the object) completely free of acoustic scattering. Below we derive the properties of and demonstrate via simulation the performance of this device.

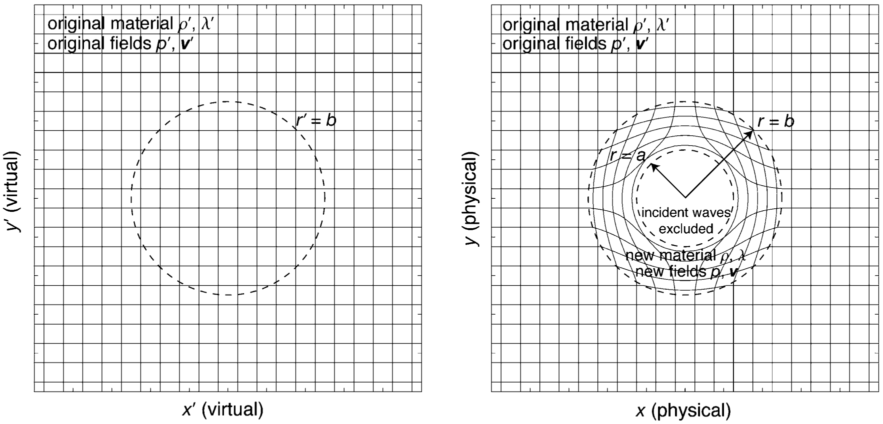

Fig. 8.5 Illustration of the design of an acoustic cloaking shell. *Left*: The original, virtual space represented by a simple Cartesian coordinate system and filled with a simple, homogeneous fluid. *Right*: The transformation opens a hole in space to map the original disk ($r' < b$) to an annulus ($a < r < b$). This squeezes the original fields into that annulus and also creates a region ($r < a$) in the interior of the annulus that does not map to virtual space. The result is that any incident sound beam is smoothly bent, without scattering or reflections, around the interior region, and any object hidden in that interior does not interact with any wave incident from the outside

Figure 8.5 illustrates the idea behind an acoustic cloaking shell. A two-dimensional cloaking transformation is the mapping of a disk in virtual space, defined by $0 < r' < b$, to an annulus in physical space, defined by $a < r < b$ [26]. This transformation squeezes the original sound fields that filled the disk into the annulus and effectively removes the interior of the annulus from the domain of the problem in physical space because there are no fields in the virtual space that map to the center of the annulus. This isolation means that, ideally, no fields are able to enter the interior, and thus any object placed in the interior cannot interact with fields incident from the outside. Equally importantly, the continuity of the transformation at $r = b$ ensures that the fields outside the annulus ($r > b$) are completely undisturbed. Consequently there is no wave scattering of any kind from the annulus, and its presence is (again ideally) undetectable from any wave measurements made outside of it.

The Jacobian-based approach for transformation acoustics derived in Sect. 8.3.1 can be used to derive the parameters of the acoustic cloaking shell. However, it is tricky (at least in the opinion of the author) to use cylindrical or spherical coordinates with this approach. Consequently we derive here the cloaking material parameters using the unit vector-scale parameter approach that was used in the original derivation of transformation electromagnetics [26] and has been outlined for transformation acoustics in several papers [4, 9] that we follow here.

The cloaking transformation is most naturally dealt with in cylindrical coordinates, and is given by

$$r = \frac{(b-a)}{b} r' + a, \qquad \phi = \phi', \qquad z = z'. \tag{8.17}$$

Note that this is not the only way to map $0 < r' < b$ to $a < r < b$, and other functional forms for the transformation result in different material parameters that, ideally, result in identical cloaking performance. In practice, different mapping functions can result in material parameters that are harder or easier to realize in practice, and the exploration of these design space degrees of freedom is a topic of active research at the time of this writing. For illustration purposes we focus only on this particular transformation.

This transformation is particularly simple in that it only squeezes r in an r' dependent way. In this case the material parameters can be more easily derived (in the opinion of the author) through the scale function approach [4, 9, 26] than by a direct calculation of the Jacobian matrix in cylindrical coordinates. These scale functions describe how much the transformation alters distances in each coordinate direction, and are defined by

$$Q_r = \frac{dr'}{dr}, \qquad Q_\phi = \frac{r'\,d\phi'}{r\,d\phi}, \qquad Q_z = \frac{dz'}{dz}. \tag{8.18}$$

For the transformation in (8.17) the specific expressions in terms of the physical (unprimed) coordinates are

$$Q_r = \frac{b}{b-a}, \qquad Q_\phi = \frac{b}{(b-a)}\frac{(r-a)}{r}, \qquad Q_z = 1. \tag{8.19}$$

Since we are effectively transforming from cylindrical to cylindrical coordinates, there is no change in the basis vectors and the ensuing complications that result from changed basis vectors [9, 26] are not needed.

Following [4, 9], and assuming the initial (primed) material is a uniform isotropic fluid with density ρ_0' and bulk modulus λ_0', the material parameters (relative to those of the background medium) that result from this transformation are, in terms of these scale functions,

$$\rho^{-1} = \left(\rho_0'\right)^{-1} Q_r Q_\phi Q_z \begin{bmatrix} Q_r^{-2} & 0 & 0 \\ 0 & Q_\phi^{-2} & 0 \\ 0 & 0 & Q_z^{-2} \end{bmatrix} \tag{8.20}$$

and

$$\lambda = \lambda_0' (Q_r Q_\phi Q_z)^{-1}. \tag{8.21}$$

Plugging in the specific values from (8.19) and rearranging (8.20), we find that

$$\rho_r = \rho_0' \frac{r}{r-a}, \qquad \rho_\phi = \rho_0' \frac{r-a}{r}, \qquad \rho_z = \rho_0' \frac{(b-a)^2}{b^2}\frac{r}{r-a} \tag{8.22}$$

and

$$\lambda = \lambda_0' \frac{(b-a)^2}{b^2}\frac{r}{r-a}. \tag{8.23}$$

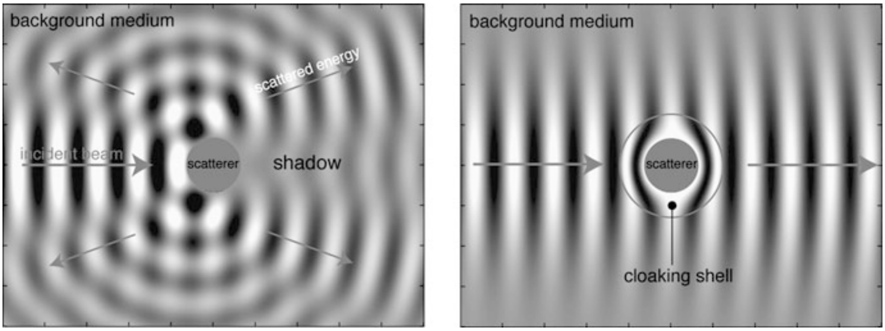

Fig. 8.6 Simulated performance of a 2D acoustic cloak. *Left*: A time snapshot of the pressure field resulting from an acoustic beam normally incident on a rigid scatterer. This object produces scattering in most directions and strong scattering in the forward direction (*shadow*) and several other directions. *Right*: When the same object is surrounded by the acoustic cloaking shell, the acoustic scattering is essentially eliminated. The pressure field outside the object is basically identical to that if there were no object at all (see the left panel of Fig. 8.1), indicating no scattering, and the field inside the cloaking shell is deformed as prescribed by the coordinate transformation used to design the material (see the right panel of Fig. 8.5)

The above expressions define the effective mass density tensor components and the bulk modulus of a 2D acoustic cloaking shell. The resulting medium has a strongly anisotropic mass density (although it is diagonal in cylindrical coordinates), and both the bulk modulus and the density tensor components are smoothly inhomogeneous in radius (but uniform in polar angle ϕ). These are not simple material parameters to realize physically, but if these complex material properties can be created, then so can an acoustic cloaking shell.

To confirm that these material properties yield the desired wave behavior, we use the same simulation approach used in Sect. 8.4.1, namely exploiting the equivalence of acoustic and electromagnetic waves in two dimensions [10] and using the electromagnetic simulation capabilities of COMSOL Multiphysics. Figure 8.6 shows a fixed time snapshot of the pressure field that results when a rigid scattering object is illuminated from the left by a time-harmonic acoustic wave beam, with and without a shell of material described by (8.22)–(8.23). Without the shell, as shown in the left panel, the wave scattering from the object is substantial, as expected. The object casts a strong shadow, which indicates the expected strong forward scattering, and also produces strong backscattering and specular scattering in several different directions. Clearly this object would be acoustically visible from measurements of the scattered sound field.

But when the same rigid object is surrounded by the acoustic cloaking shell, as shown in the right panel, the pressure field outside the composite object made of the shell and scatterer is essentially identical to the field of the beam in empty space (refer back to the left panel of Fig. 8.1. This means that all scattering, including the difficult-to-eliminate forward scattering (i.e. the shadow) is effectively eliminated, and this cloaked object would not be detectable from measurements of the scattered sound field. Note also that the pressure field inside the shell is deformed exactly as prescribed by the coordinate transformation used to design the material, as illustrated in the right panel of Fig. 8.5.

8.5 Realizing Materials for Transformation Acoustics

As evident from the examples above, the material properties that result from transformation acoustics device design are complex. Equations (8.10)–(8.11) show that, for a general design, the bulk modulus needs to be smoothly inhomogeneous, while the effective mass density not only needs to be smoothly inhomogeneous but also anisotropic, i.e. exhibiting different effective mass densities in different directions. In Sect. 8.3.2 above, we briefly described several simple approaches from the literature that have shown how one might engineer a material that exhibits an inhomogeneous anisotropic effective mass. Neither of these approaches, however, yield designs that are easy to fabricate.

Here we will briefly describe simulations and measurements of composite structures that can exhibit anisotropic mass but that are easier to fabricate, and thus are some initial steps that show that at least some transformation acoustics devices can be physically realized and their performance tested. It should be emphasized that, at the time of this writing, developing approaches for designing and fabricating materials for transformation acoustics design is a very active area of research that is still in its very early stages. Because of this, few practical constraints such as mass, mechanical robustness, visual appearance, etc., are being addressed. The goal is simply to design a structure that achieves the needed effective material parameters and that can be measured under laboratory conditions. We will focus on passive structures that do not contain resonant components and thus exhibit broadband effective properties [29, 39, 45].

It should be mentioned that the design of composite materials that achieve desired complex acoustic properties, which is to be sure an old field, has been given the modern name of acoustic metamaterials. The field of acoustic metamaterials encompasses an effort to achieve a wide variety of effective material properties in composite materials, including negative dynamic density, in which net material motion opposes the applied force [17, 19], and negative bulk modulus, in which the net material strain is opposite the applied stress [11, 18]. These negative properties do arise in some rather exotic applications of transformation electromagnetics and transformation acoustics, such as making one object appear to be another [16], but in most cases the transformation approach generates the need for materials with positive properties.

One approach for physically realizing composite materials with smoothly inhomogeneous properties and/or anisotropic mass involves layers of solid material with gaps embedded in a host fluid or, alternately, arrangements of solid scatterers in a host fluid [29, 39, 45], as illustrated in Fig. 8.7. The gaps serve two purposes. They ensure that the background fluid permeates the entire metamaterial, and the gap dimensions can easily be tuned from point to point to tune the effective properties

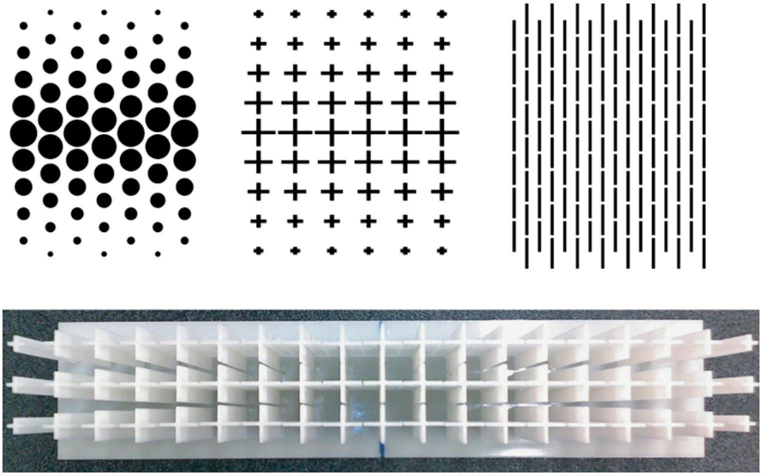

Fig. 8.7 Examples of acoustic metamaterial structures. *Top left*: An array of solid circular cylinders of varying size surrounded by a host fluid can be used to create an isotropic and inhomogeneous material (adapted from [6]). *Top middle*: An array of more complex shapes of varying size can also be used to create a similar inhomogeneous material with less mass and a better impedance match to the background fluid (adapted from [44]). *Top right*: An array of thin solid plates, or alternately perforated thin solid sheets, can be used to create an anisotropic acoustic metamaterial (adapted from [45]). *Bottom*: A photograph of a fabricated acoustic metamaterial sample based on the structure shown in the *top middle*, and experimentally demonstrated in [44]

of the medium. This approach has been applied to tune the effective properties of isotropic acoustic metamaterials to create smoothly inhomogeneous materials for devices such as gradient index sonic lenses [6, 29, 44].

Rotationally symmetric solid inclusions yield isotropic material properties that do not vary with direction. In contrast, rotational asymmetry can yield anisotropic material properties. Asymmetry in the lattice of symmetric inclusions can be used to create acoustic metamaterials with modestly anisotropic effective mass [39]. In contrast, asymmetry in the inclusions themselves can be used to create strongly anisotropic acoustic metamaterials [29]. This concept has been demonstrated experimentally through reflection and transmission measurements of the same metamaterial under two different orientations [45]. At the time of this writing, the most sophisticated device fabricated using this general approach is a two-dimensional acoustic cloak in which a water background permeates an aluminum structure [43].

Conceptually it is fairly easy to understand how this kind of composite medium results in a material with an anisotropic effective mass. For sound waves traveling parallel to the thin plates, there is minimal interaction between the sound pressure and the plate, and the wave travels as if it is in a material with properties close to the background fluid. But sound waves that travel perpendicular to the plates interact strongly with the plates and can result in waves that behave as if they are traveling in a fluid with properties very different to that of the background fluid. This anisotropy turns out to be identical to that produced by a fluid with anisotropic effective mass

[29], which is not surprising because the fluid permeating the entire volume ensures that the material has an isotropic and fluid-like stiffness.

Most applications of transformation acoustics envisioned at this point involve acoustic metamaterials either in air or in water. There are some interesting differences between these two host media in the kinds of effective material properties that can be easily realized that are worth mentioning. Air is obviously a very low density and very compressible material compared to almost all materials one might use as metamaterial inclusions. This means that large effective mass (relative to that of air) can be created in such a material, and very high levels of mass anisotropy can be achieved. However, for the same reasons, it is difficult to create an acoustic metamaterial that is lighter or more compressible than air. This limits the range of acoustic metamaterial properties that can be realized in air.

In contrast, water is only a little bit less dense and more compressible than most solid inclusions. This means that it is difficult to achieve high effective mass and high effective stiffness relative to the water background. However, it is possible to use gas-filled materials that exhibit masses less than that of the water background. One possibility that has been identified in the literature is closed-cell metal foam [30].

These differences between water and air mean that the achievable effective parameters are fundamentally different for air and water applications. For example, the challenges of trying to create a material that exhibits the acoustic cloak parameters of (8.10)–(8.11) are different for these two host fluids. The ideal parameters contain one density component (ρ_r) that is very large at the inner edge of the shell and one component (ρ_ϕ) that is very small. In air, the former is straightforward to realize but the latter is difficult, while in water, large values of ρ_r are the bigger challenge.

In light of these limitations, an active area of research is designing approximations of devices that still perform reasonably well but can be fabricated with material parameters that are physically realizable for a given host fluid. This idea has been more fully explored in transformation electromagnetics than transformation acoustics, but the concepts from one are immediately applicable to the other. Concepts that have been demonstrated, at least theoretically, include using different coordinate transformations that yield different material properties [2, 7], and numerical optimization of device performance given material property constraints [1, 28].

8.6 Recent Experimental Results in Transformation Acoustics

At the time of this writing, the authors' research group had just completed and published [31] an experimental effort to demonstrate one version of acoustic cloaking in air using transformation acoustics design concepts. The goal of this effort was to design, fabricate, and test a material shell that is capable of hiding from acoustic scattering an object in the vicinity or on top of a reflecting plane. This kind of cloak is referred to as a "carpet cloak", and also a ground-plane cloak or a reflecting plane

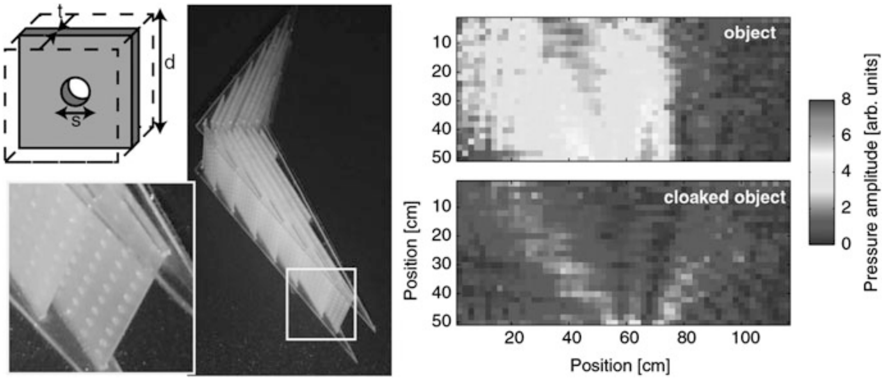

Fig. 8.8 Summary of experimentally demonstrated acoustic cloaking in air reported in [31]. *Left panels*: Photographs of the fabricated acoustic cloaking shell based on perforated sheets of thin plastic stacked with thin air gaps in between. This material provides the effective mass anisotropy needed for the device to function. *Right panels*: Measurements of the acoustic scattering from the uncloaked and cloaked object. The sound reflections from the object by itself are dramatically reduced when it is covered with the shell, rendering it hidden from sound waves. Figures adapted from [31]

cloak. This problem was first considered in the context of electromagnetics [20], and has become a common test problem in transformation electromagnetics because the material properties needed to realize the cloaking shell are much simpler to realize than for a free space cloaking shell.

Applying a relatively straightforward transformation that effectively opens a hole underneath the reflecting plane in which an object can hide results in material parameters not dramatically different from the background fluid, which in this case was air. The effective mass density of the cloaking shell is anisotropic, with a mass density about 5 times higher in one direction than the other [31]. One way such a material can be physically realized is with thin perforated plates, stacked with air gaps in between, as shown in the left panels of Fig. 8.8. The thickness of these plates and the size of the circular perforations control the effective mass density and enable the needed values to be realized with high precision.

The acoustic scattering from a triangular scatterer on a flat surface was then measured with and without the cloaking shell. The measured scattered field, i.e. the sound field after subtracting the incident field and the reflection from the flat surface, for both of these cases is shown in the right panels of Fig. 8.8. With the object on the surface, significant scattering is observed in two directions. One of these scattered beams is the specular reflection from the tilted surface of the object, and the other is forward scattering from the lack of a reflected signal in the specular direction for a flat plate. With the object and cloaking shell on the flat surface, both of these scattered field components are significantly reduced, meaning that sound energy reflects from the cloaked object on the reflecting surface essentially as if there were only a flat surface. Thus, this object is hidden from acoustic scattering.

8.7 Related Theories and Implications

Transformation acoustics theory also has some interesting consequences for other fields. The equations of linear acoustics, (8.2)–(8.3), can be reduced to the well-known scalar Helmholtz equation, which means that the transformation concept can be applied to any other wave system that can be reduced to the scalar Helmholtz equation. This includes surface water waves [12], waves on thin plates [13], and even quantum mechanical matter waves [42]. The latter is particularly interesting and suggests that if one can control the effective mass of a particle (which has to be anisotropic) and the potential field in which the particle moves, then one can arbitrarily manipulate the particle wavefunction and thus the statistical particle position. Physically realizing this latter phenomenon will be a challenge but suggests the very far-out possibility of cloaking matter.

It should also be noted that the above sections all focus on a formulation of transformation acoustics that builds out of the initial work in the field [3, 9, 10] and that maintains the basic fluid nature of the required materials. It turns out, however, that this is not the only way in which transformation acoustics can be made to work. Theoretical work first reported by Norris [23] and then later expanded and elucidated [24] has shown that there is a non-fluid class of materials that can also emerge from the fundamental transformation acoustics theory. These materials are solid materials with an anisotropic stiffness but isotropic mass. In some ways these materials are fluid-like, for instance their inability to support a shear stress, but they are fundamentally different from fluids because of the anisotropic stiffness. How to design and construct this type of material is not well-described in the literature at the time of this writing, but it is a very active area of research given the interesting devices that could be constructed from this approach.

8.8 Summary

In this chapter we have presented a concise summary of the development of the concept of transformation acoustics. First conceived only 4 years before the time of this writing, this design paradigm offers the possibility of creating novel acoustic materials and devices that are difficult if not impossible to design by other techniques. The main challenge for the field for the field at present is devising techniques to fabricate composite materials with the complex acoustic properties needed to realize transformation acoustic devices. Initial work in this area has developed several approaches that appear promising, and full experimental demonstrations can probably be expected soon.

References

1. Andkjaer, J., Sigmund, O.: Topology optimized low-contrast all-dielectric optical cloak. Appl. Phys. Lett. **98**, 021112 (2011)

2. Cai, W., et al.: Designs for optical cloaking with high-order transformations. Opt. Express **16**(8), 5444–5452 (2008)
3. Chen, H., Chan, C.T.: Acoustic cloaking in three dimensions using acoustic metamaterials. Appl. Phys. Lett. **91**, 183518 (2007)
4. Chen, H., Chan, C.T.: Acoustic cloaking and transformation acoustics. J. Phys. D **43**, 113001 (2010)
5. Cheng, Y., et al.: A multilayer structured acoustic cloak with homogeneous isotropic materials. Appl. Phys. Lett. **92**, 151913 (2008)
6. Climente, A., et al.: Sound focusing by gradient index sonic lenses. Appl. Phys. Lett. **97**, 104103 (2010)
7. Cummer, S.A., et al.: A rigorous and nonsingular two dimensional cloaking coordinate transformation. J. Appl. Phys. **105**, 056102 (2009)
8. Cummer, S.A., et al.: Full-wave simulations of electromagnetic cloaking structures. Phys. Rev. E **74**(3), 036621 (2006)
9. Cummer, S.A., et al.: Material parameters and vector scaling in transformation acoustics. New J. Phys. **10**, 115025 (2008)
10. Cummer, S.A., Schurig, D.: One path to acoustic cloaking. New J. Phys. **9**, 45 (2007)
11. Fang, N., et al.: Ultrasonic metamaterials with negative modulus. Nat. Mater. **5**, 452–456 (2006)
12. Farhat, M., et al.: Broadband cylindrical acoustic cloak for linear surface waves in a fluid. Phys. Rev. Lett. **101**, 134501 (2008)
13. Farhat, M., et al.: Ultrabroadband elastic cloaking in thin plates. Phys. Rev. Lett. **103**, 024301 (2009)
14. Greenleaf, A., et al.: Anisotropic conductivities that cannot be detected by EIT. Physiol. Meas. **24**, 413–419 (2003)
15. Knupp, P., Steinberg, S.: Fundamentals of Grid Generation. CRC Press, Boca Raton (1994)
16. Lai, Y., et al.: Illusion optics: the optical transformation of an object into another object. Phys. Rev. Lett. **102**, 253902 (2009)
17. Lee, S.H., et al.: Acoustic metamaterial with negative density. Phys. Lett. A **373**, 4464–4469 (2009)
18. Lee, S.H., et al.: Acoustic metamaterial with negative modulus. J. Phys. Condens. Matter **21**, 175704 (2009)
19. Li, J., Chan, C.T.: Double-negative acoustic metamaterial. Phys. Rev. E **70**(5), 055602 (2004)
20. Li, J., Pendry, J.B.: Hiding under the carpet: A new strategy for cloaking. Phys. Rev. Lett. **101**, 203901 (2008)
21. Liu, R., et al.: Broadband ground-plane cloak. Science **323**, 366 (2009)
22. Milton, G.W., et al.: On cloaking for elasticity and physical equations with a transformation invariant form. New J. Phys. **8**, 248 (2006)
23. Norris, A.N.: Acoustic cloaking theory. Proc. R. Soc. A **464**, 2411–2434 (2008)
24. Norris, A.N.: Acoustic metafluids. J. Acoust. Soc. Am. **464**, 839–849 (2008)
25. Padilla, W.J., et al.: Negative refractive index metamaterials. Mater. Today **9**, 28 (2006)
26. Pendry, J.B., et al.: Controlling electromagnetic fields. Science **312**, 1780–1782 (2006)
27. Plebanski, J.: Electromagnetic waves in gravitational fields. Phys. Rev. **118**, 1396–1408 (1960)
28. Popa, B.-I., Cummer, S.A.: Cloaking with optimized homogeneous anisotropic layers. Phys. Rev. A **79**, 023806 (2009)
29. Popa, B.-I., Cummer, S.A.: Design and characterization of broadband acoustic composite metamaterials. Phys. Rev. B **80**, 174303 (2009)
30. Popa, B.-I., Cummer, S.A.: Homogeneous and compact acoustic ground cloaks. Phys. Rev. B. In review (2011)
31. Popa, B.-I., et al.: Experimental acoustic ground cloak in air. Phys. Rev. Lett. **106**, 253901 (2011)
32. Rahm, M., et al.: Optical design of reflectionless complex media by finite embedded coordinate transformations. Phys. Rev. Lett. **100**, 063903 (2008)

33. Rahm, M., et al.: Transformation-optical design of adaptive beam bends and beam expanders. Opt. Express **16**, 11555 (2008)
34. Rahm, M., et al.: Design of electromagnetic cloaks and concentrators using form-invariant coordinate transformations of Maxwell's equations. Photonics Nanostruct. **6**, 87–95 (2008)
35. Schoenberg, M., Sen, P.N.: Properties of a periodically stratified acoustic half-space and its relation to a Biot fluid. J. Acoust. Soc. Am. **73**, 61–67 (1983)
36. Schurig, D., et al.: Metamaterial electromagnetic cloak at microwave frequencies. Science **314**, 977–980 (2006)
37. Schurig, D., et al.: Calculation of material properties and ray tracing in transformation media. Opt. Express **14**, 9794–9804 (2006)
38. Tamm, I.Y.: Electrodynamics of an anisotropic medium and the special theory of relativity. J. Russ. Phys.-Chem. Soc. **56**, 248 (1924)
39. Torrent, D., Sanchez-Dehesa, J.: Acoustic metamaterials for new two-dimensional sonic devices. New J. Phys. **9**, 323 (2007)
40. Torrent, D., Sanchez-Dehesa, J.: Acoustic cloaking in two dimensions: A feasible approach. New J. Phys. **10**, 063015 (2008)
41. Wood, A.B.: A Textbook of Sound. Macmillan, New York (1955)
42. Zhang, S., et al.: Cloaking of matter waves. Phys. Rev. Lett. **100**, 123002 (2008)
43. Zhang, S., et al.: Broadband acoustic cloak for ultrasound waves. Phys. Rev. Lett. **106**, 024301 (2011)
44. Zigoneanu, L., et al.: Design and measurements of a broadband 2D acoustic lens. Phys. Rev. B **84**, 024305 (2011)
45. Zigoneanu, L., et al.: Design and measurements of a broadband 2D acoustic metamaterial with anisotropic effective mass density. J. Appl. Phys. **109**, 054906 (2011)