Acoustic Waveguides

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Version date: January 22, 2008

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1. Introduction

1.1 Why Acoustic Waveguides?

Let us suppose that we want to know how much acoustic power can be produced by a bugle. We invite a cooperative bugle player into a room outfitted with a calibrated microphone and recording equipment (Figure 1.1). Suspecting that high notes are the most powerful, we ask her to blast away as loudly as possible in the upper range of the instrument.

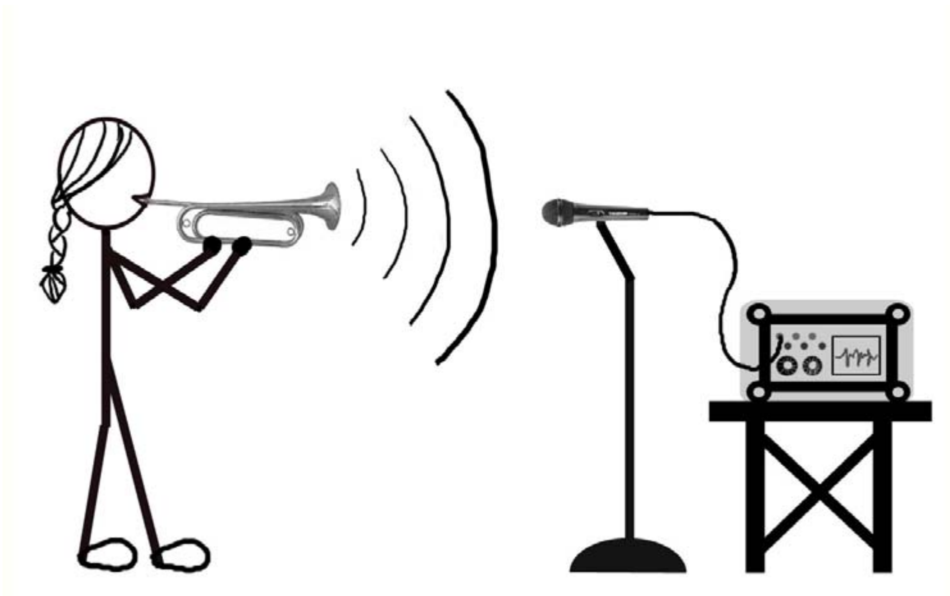

Figure 1.1. Measuring the acoustic power of a bugle.

To analyze our data we could approximate the bugle bell as a point source emitting a spherical wave, and infer the total power from the acoustic pressure measured at a known distance. A bit of elementary acoustics (see Chapter 2) shows that the relationship between the pressure amplitude p measured at a distance r from the source, and the total radiated power P is

$$P = 4\pi r^2 \frac{1}{2} \frac{p^2}{\rho v_s}, \tag{1.1}$$

where ρ is the density of air and v_s is the sound velocity, 344 m/s (at 20 °C). The frequency of the sound will be around 1000 Hz, implying a wavelength of about 30 cm.

If we actually try this experiment, we will encounter several problems. First, our results will be very sensitive to the position of the microphone and the bugle in the room. This is because, in a typical room, sound will bounce off the walls many times before it is absorbed, and each of the paths that the waves take to reach the microphone will combine coherently, producing a complicated interference pattern. We might hope that this difficulty would disappear in a very large room, since then the reflected waves would be absorbed before they reach the microphone. Unfortunately, it turns out that to suppress the effects of reflections we would need a very large room indeed, because the attenuation of 1000 Hz sound in air is only about 0.005 dB per meter (at 20 C and 30% relative humidity). Even in a large enough room we will still have to contend with floor reflections, unless we are willing to suspend our cooperative player in mid-air. A better, but costly, solution would be to build a room with sound absorbing walls and floor—an anechoic chamber.

There will be other difficulties. We do not expect the bugle to emit power equally in all directions, so for a reliable measurement of the total power we will have to extend (1.1) to allow for angular variations, and then measure the sound pressure at a range of angles relative to the bugle bell's axis of symmetry. We should also aware that our microphone scatters the sound field, so the pressure it measures is not the same as the pressure of the sound field with the microphone removed. We can deal with this by applying appropriate free-field corrections,[1] or by using a small microphone, for which the corrections will be small. If we are aware of each of these issues, can manage the effort and expense, and if our musician remains cooperative, we will then be able to make accurate measurements. For an example of a study of this kind in an anechoic chamber, see Martin.[2] In the absence of an anechoic chamber there are approximate ways to measure power in a room with reflective walls that make use of measurements of the room's reverberation time.[3]

Is there a simpler and more elegant approach? Suppose we were able to modify the bugle and install several small microphones in the wall of its tubing (as in Figure 1.2) such that they would measure, but not disturb, the pressure in the bore. With one microphone we could record the sound intensity in the bore, but would obtain no information about radiated power. However, with data from several microphones, we could separately determine the right-going and left-going wave amplitudes in the bore, and thus infer the amount of power lost to radiation when the right-going wave reaches the bell. There are two major advantages to this approach:

- We have drastically reduced the complexity of the sound field that we must measure. In a straight or slowly tapering cylindrical waveguide such as a bugle bore, there is just one propagating acoustic mode, as long as the free-space wavelength is greater than 1.71 times the inside diameter.[4] At each frequency, the entire wave field is described by just four real numbers (or by two complex amplitudes), representing the right and left-going amplitude and phase.

- We have removed the microphone bodies from the sound field so that we not longer have to apply free-field corrections.

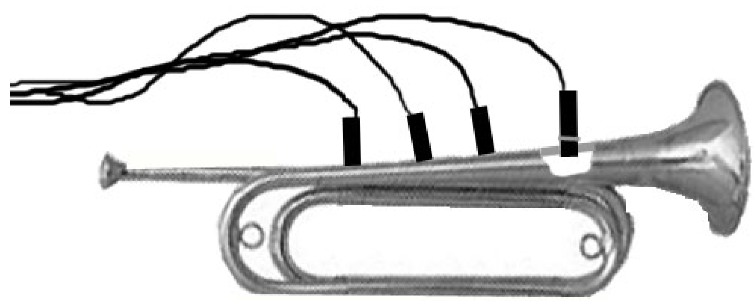

Figure 1.2. Bugle instrumented with four microphones. The microphones are inserted into holes in the bugle's tubing and sealed so that they measure the pressure inside the bore. Ideally, the measured sound pressure is the same as was present before the bugle was modified.

These advantages of waveguide measurements carry over to a wide range of studies in acoustics. The phase velocity of the lowest propagating mode in an acoustic waveguide is generally close to the free-space sound velocity,[5] so sound velocity can conveniently be measured in a waveguide as a function of gas composition, temperature, and pressure, in the presence of a flow field, and even in turbulent flows.[6] Damping of acoustic waves in waveguides is easily observed and is a function of the shear viscosity and thermal conductivity of the gas.[7] The sound absorption properties of solid materials are important in several areas of engineering, including vehicle design, concert hall acoustics, and architectural and environmental noise-control situations. Acoustic materials (such as ceiling tiles) can be characterized through their effects on room reverberation time, but it is also helpful to study samples of sound absorbing materials in acoustic waveguides.[8] Returning to the field of musical acoustics, in addition to studies of radiation from horns,[9,10] waveguide measurements on the acoustics of woodwind tone holes have been reported.[11] Woodwind instruments are self-sustaining acoustic oscillators, and like all such oscillators, they contain an amplifier. Waveguide techniques have been used to study the air-jet amplifier in flute-family instruments.[12,13] Detailed measurements of the input impedance of flutes, measured at the embouchure hole, have also been reported.[14]

Sound propagation in waveguides is of direct importance in architectural acoustics, because sound transmitted through ventilation ducts can be a major source of unwanted noise in buildings.[15] Related problems arise in the design of acoustic mufflers for engines.[16] In medicine, sound propagation in human airways has been used for diagnostic purposes.[17] Other interesting acoustic waveguide measurements are discussed or suggested in Chapter 7.

From an educational standpoint, acoustic waveguides provide an inexpensive and accessible window into the basic concepts of wave propagation. These are essential to all areas of physics (classical mechanics, quantum mechanics, continuum mechanics, electromagnetism, optics, plasma physics, solid-state physics...) to nearly all areas of engineering (electrical, mechanical, civil, architectural...), and to many other basic and applied sciences (geology, oceanography, medical imaging…). At the most elementary level, wave concepts include superposition, the distinction between standing and running waves, the use of complex numbers to represent sinusoidally varying quantities, and the distinction between phase and group velocity. Slightly more advanced concepts, still common in the undergraduate curriculum, include wave impedance, termination impedance, the reflection coefficient and its relation to termination impedance, abrupt impedance discontinuities, tapered transmission lines, scattering matrices, radiation, and radiation reaction. At the advanced undergraduate and graduate levels there are more topics: propagation in periodic media and band-gaps, resonant cavities and coupling of cavities to transmission lines, the distinction between acoustic and optical propagation modes, resonant scattering, propagation in disordered media, and pulse dispersion.

1.2 Waveguide Measurement Systems

(Here we need a general introduction to types of measurement systems: multi-microphone reflectometer, VNA, impedance head, importance of multi-channel simultaneous sampling digitization.)

Historically, sensitive and stable microphones were rather large, and collecting data coherently from many channels was either impossible or prohibitively expensive. Experimenters interested in acoustic waveguide measurements developed clever schemes to obtain information from single microphones.[18-20] Standing waves in tubes could be investigated by moving a microphone along a wall with the aid of some type of sliding seal, or in a vertical tube a relatively small microphone could be hung by a wire and moved along the axis. These methods are closely analogous to slotted-line techniques used in microwave and radio-frequency electronics, which date back to Hertz. The earliest acoustic example may that of Taylor,[21] from 1913. Another approach involves creating a known acoustic flow, which, together with a pressure measurement, allows one to establish the impedance at one end of a waveguide.[22] Methods using moving microphones (or a moving plunger and a fixed microphone) are usually called standing wave tubes or impedance tubes, while those employing a known acoustic flow are referred to as impedance heads.

The use of several, fixed, coherently recorded microphones dates from the mid 1970s. Specific systems and methods of analyzing data have been described by several authors. Seybert and Ross[23] developed an influential technique involving two fixed microphones and random excitation. Systems with more microphones have also been reported,[24-26] and have become more practical as the necessary data acquisition hardware, processing power, and storage have all become more affordable. The advantage of multiple microphones can be understood by counting the number of parameters that one would like to measure. The reflection coefficient contains two real numbers, a magnitude and a phase. With two microphones one also gets two numbers, a relative phase and a magnitude ratio. (The absolute magnitude may be also measured, but the reflection coefficient is independent of it, being a ratio of amplitudes.) However, for accurate results both the phase velocity and the attenuation constant of the waves in the waveguide should be measured, requiring two more numbers, and hence at least three microphones are needed if one wants to make independent measurements at each frequency. Even more microphones are helpful to demonstrate that the assumed physical model (*i.e.*, one-dimensional waves with certain values of the attenuation constant and phase velocity) actually gives an accurate description of the sound field in the waveguide, and to provide wide frequency coverage.

Modern systems usually use laboratory-quality microphones, which are very stable and typically supplied with absolute pressure-sensitivity calibration data. However, for most measurements relative amplitude and phase calibration is more important than absolute calibration. Microphones based on inexpensive miniature electret elements can be use for accurate measurements of the reflection coefficient if relative phase and amplitude calibrations can be easily repeated, so that long-term stability is not required, and if suitable buffer electronics is provided. For accurate absolute measurements, some method of absolute calibration must also be available. In Chapter 5 we show how a convenient relative calibration method can yield reflection coefficient measurements with an accuracy of a few percent, even with inexpensive microphones.

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2. Elementary Acoustics

In this chapter we derive the basic equations governing acoustic waves, introduce plane- and spherical-wave solutions, and discuss acoustic power. Similar material can be found in many texts. For discussions at more-or-less the same level as below, see Fletcher and Rossing,[1] or Fetter and Walecka.[2] For a much more comprehensive treatment see Morse and Ingard.[3]

2.1 Helmholtz Wave Equation

Newton's law for a fluid element with velocity $\vec{u}$ and density ρ is given by

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} p,$$

where the total derivative gives the acceleration of the fluid element along its path of motion. The acceleration may instead be described in terms of the rate of change of the velocity field at a fixed location by substituting the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla},$$

which yields Euler's equation

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u}\right) = -\vec{\nabla} p. \tag{2.1}$$

To ensure that mass is conserved, we require that solutions also satisfy the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\left(\rho \vec{u}\right) = 0 \tag{2.2}$$

Acoustics is concerned with small variations δp, $\delta\rho$, and $\vec{u}$ about a quiescent state, which we take to be uniform and time-independent, with $\vec{u} = 0$ and $p = p_0, \rho = \rho_0$. We assume that the pressure variation is a smooth and local function of the density variation so that it can be expressed as

$$\delta p = \frac{\partial p}{\partial \rho} \delta\rho, \tag{2.3}$$

where the partial derivative is evaluated at the quiescent condition. The evaluation of this partial derivative depends on an equation of state and other thermodynamic variables as we shall discuss below. Linearizing (2.1) in the small quantities yields

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla}\delta p = -\frac{\partial p}{\partial \rho}\vec{\nabla}\delta\rho, \tag{2.4}$$

while (2.2) becomes

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla}\cdot\vec{u} = 0. \tag{2.5}$$

These two equations are equivalent to a wave equation, as can be seen by taking the divergence of (2.4), the time derivative of (2.5), and then eliminating the mixed derivative terms to arrive at:

$$\left(\frac{\partial^2}{\partial t^2} - v_s^2\nabla^2\right)\delta\rho = 0; \quad \left(\frac{\partial^2}{\partial t^2} - v_s^2\nabla^2\right)\delta p = 0; \quad v_s^2 = \frac{\partial p}{\partial \rho}. \tag{2.6}$$

The second wave equation for δp follows from the first because of the relation (2.3). When the wave equation for δp has been solved, (2.4) can be used to find the velocity field $\vec{u}$.

We shall usually be concerned with solutions that have sinusoidal time dependence. We use the convention that all time-dependent quantities vary as $\exp(+i\omega t)$, and take real parts to find physical quantities. (This is the same convention that is used in electronics to express voltages, currents, and impedances.) The wave equation for δp and (2.4) become:

$$\left(\omega^2 + v_s^2\nabla^2\right)\delta p = 0; \quad \vec{u} = \frac{i}{\omega\rho_0}\vec{\nabla}\delta p. \tag{2.7}$$

The first of these is the Helmholtz equation, the most common starting point for solving acoustics problems.

2.2 Sound Velocity

To express the sound velocity (2.6) in a form that is more familiar in thermodynamics, we consider an small fluid element with volume V_0 in the quiescent state that varies by δV when the density varies by $\delta\rho$:

$$\delta\rho = -\rho_0 \frac{\delta V}{V_0}$$

In terms of δV the sound velocity becomes

$$v_s^2 = -\frac{V_0}{\rho_0}\left(\frac{\partial p}{\partial V}\right). \tag{2.8}$$

Suppose that the fluid is an ideal gas at constant temperature, so that $pV = p_0 V_0$. Evaluating (2.8), we find

$$v_s^2 = \frac{p_0}{\rho_0}. \qquad \text{(ideal gas, constant temperature)} \tag{2.9}$$

On the other hand, if there is no heat flow out of the fluid element, so that the pressure change is adiabatic, an ideal gas obeys

$$pV^\gamma = p_0 V_0^\gamma; \quad \gamma = \frac{\mathrm{C_p}}{\mathrm{C_V}}, \tag{2.10}$$

where γ is the adiabatic index, equal to the ratio of the heat capacity at constant pressure to the heat capacity at constant volume.[4] The sound velocity is now

$$v_s^2 = \gamma \frac{p_0}{\rho_0}. \qquad \text{(ideal gas, adiabatic conditions)} \tag{2.11}$$

Recalling that γ=7/5 for a diatomic ideal gas, we see that (2.9) and (2.11) imply very different values for the sound velocity in air. Experiments, and more complete theories, show that adiabatic conditions apply in gasses until one reaches frequencies so high that the sound wavelength becomes comparable to the molecular mean free path.[5] The ideal gas equation of state combined with (2.11) shows that v_s varies as $\sqrt{T}$ at constant pressure, and is independent of pressure at constant temperature. For air at standard pressure and temperature, both of these predictions are borne out to high accuracy. Small variations of the sound velocity with pressure (at constant temperature) can be used to measure the second virial coefficient, *i.e.* to quantify departures from ideality.[6]

Dissipation in acoustics is an interesting and surprisingly involved subject.[7,8] Our wave equation (2.7) contains no dissipation because it was derived from Euler's equation (2.1), rather than from the more general Navier-Stokes equation, which includes dissipative bulk and shear viscosity forces. The shear viscosity is due to diffusion of momentum between adjacent regions of differing velocity $\vec{u}$, and is present even in the absence of temperature gradients.

Because conditions in the bulk are nearly adiabatic, a temperature wave always accompanies the acoustic pressure wave, leading to at least some irreversible heat flow and hence additional dissipation beyond that caused by shear viscosity. Heat flow occurs along bulk temperature gradients and along gradients close to boundaries, where the gas is in contact with nearly isothermal solid walls. Significant irreversible heat flow may also occur internal to the gas, between the translational degrees of freedom and molecular vibrations or rotations. This effect contributes to bulk sound absorption in air, and explains why it is so sensitive to humidity.[9] At the end of the next chapter, will say a more about dissipation in waveguides, where the most important effects are shear viscosity and heat flow at the walls.

In this section and the previous one we have had to distinguish between the quiescent pressure and density, and the acoustic perturbations. We will have little need to do so below, so we adopt simpler notation. The quiescent density is denoted simply ρ. We use $p(t)$ and $\vec{u}(t)$ to represent the time-dependent acoustic pressure and velocity, and p and $\vec{u}$ for the time Fourier-amplitudes. In this notation, the Helmholtz equation is

$$\left(\omega^2 + v_s^2\nabla^2\right)p = 0; \quad \vec{u} = \frac{i}{\omega\rho}\vec{\nabla}p, \tag{2.12}$$

where

$$p(t) = p\exp(i\omega t); \quad \vec{u}(t) = \vec{u}\exp(i\omega t).$$

For waves, we write

$$p(t) = A\exp(-i(\vec{k}\cdot\vec{x} - \omega t)),$$

This notation helps to distinguish between the time-space wave amplitude A and the time-only amplitude p, which may depend on position. (In the software documentation, we call p the phasor, and A the wave amplitude.)

2.3 Plane Waves

The pressure plane-wave

$$p(t) = A\exp(-i(\vec{k}\cdot\vec{x}-\omega t)) \tag{2.13}$$

satisfies the Helmholtz equation if $\omega = kv_S$, showing that v_S is, in fact, the free-space phase velocity for sound waves. The corresponding longitudinal velocity wave is

$$\vec{u}(t) = \hat{n}\frac{A}{\rho v_S}\exp(-i(\vec{k}\cdot\vec{x}-\omega t)), \tag{2.14}$$

where $\hat{n}$ is a unit vector in the direction of $\vec{k}$. Note that a velocity running wave is in phase with its corresponding pressure running wave. However, the pressure standing wave

$$p(t) = A\cos(\vec{k}\cdot\vec{x})\exp(i\omega t) \tag{2.15}$$

has, according to (2.12), a corresponding velocity field that is one-quarter cycle out of phase, both in position and in time:

$$\vec{u}(t) = \hat{n}\frac{A}{\rho v_S}\sin(\vec{k}\cdot\vec{x})\exp(i(\omega t-\pi/2)). \tag{2.16}$$

The acoustic intensity of a plane wave can be found by imagining a running wave traveling in the $\hat{x}$ direction and filling the half-space x>0, and a rigid plane at x=0. Suppose the plane moves with a velocity amplitude $\vec{u} = \hat{x}u$ that matches the x=0 velocity of the plane wave. From (2.13) and (2.14), the pressure amplitude p at the plane must satisfy $p = \rho v_S u$. For general complex values of the pressure and velocity amplitudes, the time-averaged power per unit area (or intensity) carried away by the wave is

$$I = \overline{\Re(p\exp(i\omega t))\Re(u\exp(i\omega t))} = \frac{1}{2}\Re(p^* u) \tag{2.17}$$

where $\Re$ is the real part. For the plane wave we find

$$I = \frac{1}{2}\frac{|p|^2}{\rho v_s} = \frac{1}{2}\rho v_s |u|^2. \tag{2.18}$$

The wave impedance $z \equiv p/u$ of an acoustic plane wave is given by

$$z = \rho v_s \tag{2.19}$$

If (2.19) is substituted into (2.18), the results should seem familiar from the corresponding power formulae used in electronics. The factor of ½ may be unexpected; it appears because we use amplitudes rather than rms quantities.

2.4 Spherical Waves

Insight into radiation problems can be gained from examining spherical wave solutions to (2.12). An out-going spherical pressure wave may be written

$$p(t) = A\frac{\exp(-i(kr-\omega t))}{r}. \tag{2.20}$$

It is a solution to the Helmholtz equation (2.12) with $\omega = kv_s$. The corresponding velocity field is

$$\vec{u}(t) = \hat{n}\frac{A}{\rho v_s}\left(1+\frac{1}{ikr}\right)\frac{\exp(-i(kr-\omega t))}{r}, \tag{2.21}$$

where $\hat{n}$ is a unit vector in the radial direction.

To find the radiated power P, we imagine a spherical surface (the source) at the radius r that moves radially with a velocity matching the wave velocity. Using (2.17), and multiplying by the area of the source, we find:

$$P = 4\pi r^2 \frac{1}{2}\frac{|p|^2}{\rho v_s} = 4\pi r^2 \frac{1}{2}\rho v_s |u|^2 \frac{(kr)^2}{1+(kr)^2}. \tag{2.22}$$

where p and u are the pressure and velocity amplitudes at the radius r. The first result in terms of p was already noted in Chapter 1 and is essentially the same as for a plane wave. The second form in (2.22) is also similar to the plane wave result if the source is larger than a wavelength, so that $kr >> 1$. However, when the source is small compared to a wavelength, the source velocity has to be larger by a factor of $1/kr$ to produce the same power per area as a large source.

The wave impedance at the source $z \equiv p/u$ is

$$z = \rho v_s \frac{ikr}{1+ikr}, \tag{2.23}$$

the same as the plane wave result when the source is large. When the source is small, the impedance magnitude is reduced by a factor kr relative to the plane wave value, and the phase becomes imaginary and positive. In electronics, an inductor has an impedance with a positive imaginary phase. Here, the impedance is that of the inertia of a sheet of air with a thickness equal to the source radius.

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3. Acoustic Waveguides

3.1 Propagating Modes

Molecular scattering at surfaces is almost always diffuse. As a consequence, the boundary condition for a fluid at a solid surface is that all components of the velocity $\vec{u}$ must vanish.[1] The shear viscosity η tends to retard flow parallel to the surface, but it dominates the flow only in a thin region known as the viscous penetration depth

$$\delta_s = \sqrt{\frac{2\eta}{\rho\omega}}. \tag{3.1}$$

There is also a thin region near the surface where bulk adiabatic conditions transition to the nearly isothermal surface. The thermal penetration depth is given by

$$\delta_h = \sqrt{\frac{2\kappa}{\rho c_p \omega}}, \tag{3.2}$$

where κ is the thermal conductivity and c_p the heat capacity at constant pressure (per unit mass). For air under standard conditions, and for frequencies from 20 Hz to 20 kHz, δ_s varies from 0.490 to 0.0155 mm, while δ_h varies from 0.582 to 0.0184 mm. Both lengths are thus much smaller than typical waveguide dimensions. If the flow in this thin surface layer may be ignored, we can solve the Helmholtz equation using sliding boundary conditions. That is, only the perpendicular component of $\vec{u}$ must vanish at a boundary, or equivalently (according to 2.12), the normal derivative of p must be zero on the boundary. Corrections due to the boundary layers may ignored, or treated as small perturbations. See Trusler for a careful treatment of these issues.[2] In the last section of this chapter we will return to the effects of boundary layers, but until then we ignore them and use sliding boundary conditions.

If we align the direction of propagation of the plane wave (2.13) with the axis of a uniform cylindrical waveguide, the sliding boundary condition will be satisfied. We thus have found a mode of the structure that propagates with the free-space sound velocity and has a linear dispersion relation $\omega = kv_s$. The pressure distribution in this mode is uniform across a plane perpendicular to the waveguide axis.

There are also higher modes that have nodal surfaces parallel to the axis, as shown in Figure 3.1. All modes may be labeled by two integers (m,n), where m is the number of nodal planes and n is the number of nodal cylinders parallel to the axis. All modes with $m \neq 0$ are doubly degenerate. Only the (0,0) mode propagates at all frequencies and has a linear dispersion relation. The other modes only propagate above certain cut-off frequencies f_c. The free-space wavelengths $\lambda_c = v_s / f_c$ corresponding to the lowest three cut-off frequencies are 1.7063, 1.0286, or 0.8199 times the waveguide inside diameter, for modes (1,0), (2,0), and (0,1) respectively. For 25 mm diameter waveguide filled with air under standard conditions, the cut-off frequencies for these three modes are 8064 Hz, 13,380 Hz and 16,780 Hz. Waveguide measurement devices are generally designed to operate with only the (0,0) mode propagating. The onset of propagation of the (1,0) mode thus marks the useful the upper frequency limit. Similar results are found for uniform waveguides with other cross-sectional shapes.[3]

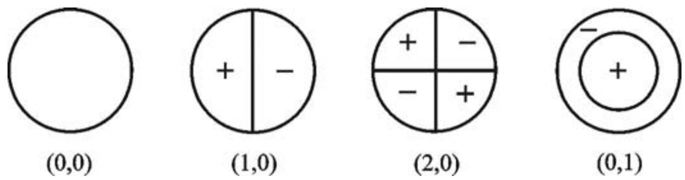

Figure 3.1. Pattern of nodal planes and cylinders for the uniform mode and the three modes with lowest cut-off frequencies. The relative sign of the pressure on opposite sides of the nodal surfaces is indicated.

In other fields, modes with linear dispersion relations and zero frequency at zero wave number are called either acoustic modes or, by analogy to photons, massless modes. Modes with a cut-off frequency are called either optical modes, after phonons that couple strongly to light, or massive modes, after the dispersion relation for a massive particle.

3.2 Acoustic Circuit Elements

A circuit is a description of a dynamical system in terms of circuit elements obeying Kirchoff's laws.[4] Each circuit element (Figure 3.2) has two or more terminals at which are defined two kinds of scalar variables: 'through' and 'across'. The product of a through variable and an across variable has the dimensions of power. Terminals are connected together by wires to form networks, and there may be junctions at which three or more wires come together. Kirchhoff's laws are: 1) The through variable is strictly conserved through junctions, wires, and circuit elements at every instant. That is, at every instant the amount of through variable flowing into one end of a wire is the same as the amount flowing out the other end; the total through variable flowing into a junction is equal to the total amount flowing out; and the total through variable flowing into a circuit element is equal to the total amount flowing out. 2) The across variable is a constant across wires and junctions and is single-valued throughout the circuit. As a consequence, the sum of changes of the across variable around any closed path (which may traverse wires, junctions, and circuit elements) is zero. Each circuit element has a constitutive relation that connects its through and across variables. These relations may only on depend upon differences of the across variables.

Electronics, with current and voltage as the through and across variables, is , of course, the prototypical circuit theory from which the general idea has been abstracted. When an accurate circuit representation of a dynamical system can be given, it may represent an enormous simplification, since the equations of motion become a set of ordinary integro-differential equations in time, rather than one or more partial differential equations. It is not required that the constitutive relations be either linear or time-invariant, but when this is the case, and the system is transformed to frequency-domain, the description simplifies further to a set of linear algebraic equations. In many cases, even when there are non-linear elements (such as transistors) the equations can still be linearized for small variations of the variables about a steady state, and again the problem becomes one of linear algebra.

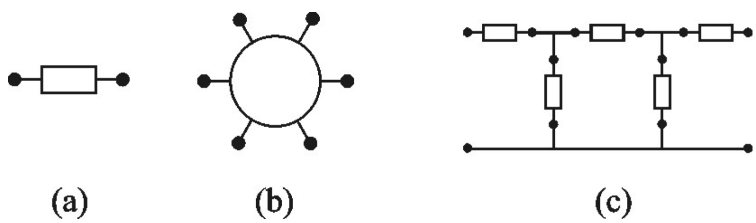

Figure 3.2. (a) A 2-terminal circuit element. (b) A 5-terminal circuit element. (c) A network containing 5 2-terminal elements and 4 junctions. This network has two ports, or terminal pairs.

For the reader familiar with electronics, translating an acoustic problem into the analogous electronic problem can be a great aid to intuition. Even if one does not have this advantage, the electronic language provides access to a repertoire of problem solving techniques, and to powerful software tools devoted to circuit problems.* For this last reason particularly, we adopt electronic terminology, sign conventions, and schematic symbols in this manual.

We shall define our terminals at planes perpendicular to the axis of a cylindrical waveguide and in regions where the flow is uniform and directed along the axis. It is not necessary to be so restrictive, but this will give us a useful starting point. Consider first the short section of waveguide shown in Figure 3.3(a). For the across variables we use the acoustic pressures p at each end. The through variable is the volume velocity $U = uS$, where S is the cross-sectional area of the terminal and u is the fluid velocity. The product pU has the dimensions of power.

* For example, SPICE and its derivatives. Search ‘LTspice’ on the internet for an excellent free version.

This circuit element will require a network containing a pair of two-terminal elements for its description. To isolate the effects of the elements one at a time, we consider two idealizations. First, we suppose that the fluid is incompressible. Then U will be conserved; the same volume velocity (volume per unit time) will flow in one end as flows out the other. Thus in Figure 3.3(a) we have $U_1 = U_2 = U$. This will also be true of the diameter transition shown in 3.3(b), while for the branched pipe 3.3(c) we have $U_1 = U_2 + U_3$, so U is again conserved. One might suppose that the mass current $uS\rho$ would be a suitable through variable. However, for a section of waveguide with a density discontinuity (see Figure 3.3(d)) this choice would not yield a conserved flow.

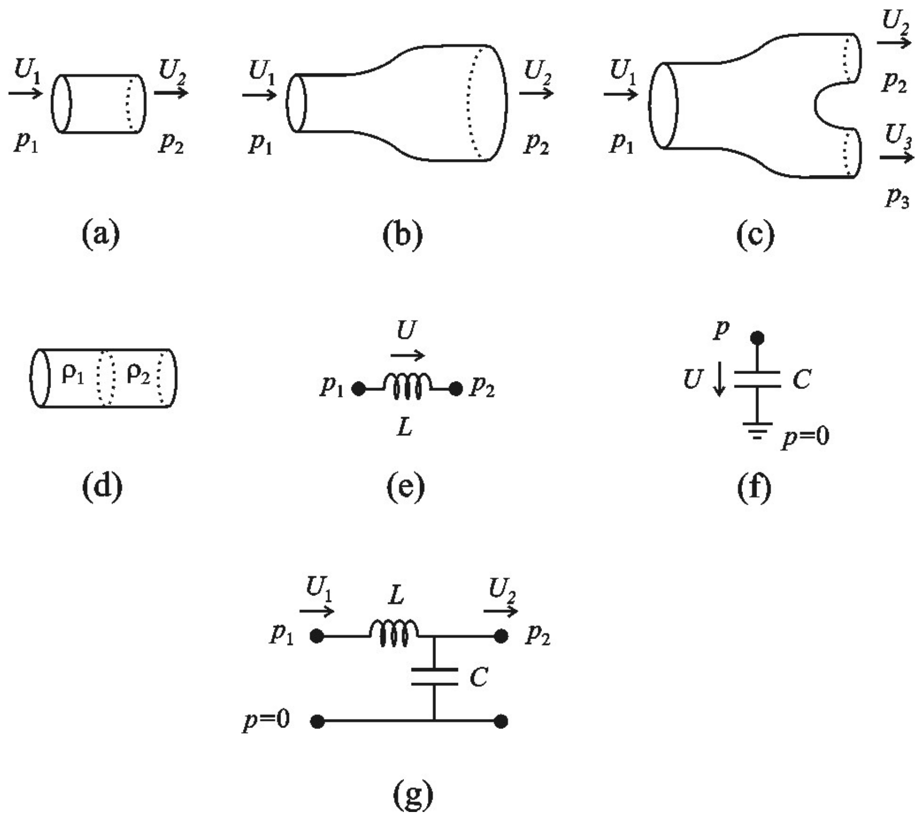

Figure 3.3. (a) Circuit element for a short section of waveguide. (b) Waveguide section with area change. (c) Branched waveguide. (d) Section with density discontinuity. (e) Inertance circuit element. (f) Compliance circuit element. (g) Lumped element model for a waveguide section.

What is the equation of motion for the element in Figure 3.3(a) in the incompressible case? Using Newton's law, we have

$$p_1(t) - p_2(t) = p(t) = \frac{\rho l}{S}\frac{dU(t)}{dt},$$

where we have defined the pressure drop *p(t)*. The fluid density is ρ, and l is the length of the waveguide section. This is a linear and time-invariant relation which may be Fourier-transformed (using $\exp(+i\omega t)$ time dependence) to yield

$$p = Z_L U, \quad Z_L = i\omega L, \quad L = \frac{l\rho}{S}. \tag{3.3}$$

The description of an inductor in electronics is exactly analogous; there L is called the inductance and Z_L is the impedance of the inductor with inductance L. For acoustics we will retain the symbol L but call it inertance instead of inductance.

Now suppose that the fluid is nearly massless, but the adiabatic compressibility (fractional change in volume per change in pressure, with no heat flow) has the value of the actual fluid of interest. Then there are no inertia forces so there will be no pressure drop across the element. However, as the pressure changes the fluid will compress and there must be net flow into the element. The total volume velocity into the element (from both sides) is

$$U = \frac{V}{\rho \frac{\partial p}{\partial \rho}}\frac{dp}{dt} = \frac{V}{\rho v_s^2}\frac{dp}{dt},$$

where V is the volume of the element. (The first fraction in this equation is the adiabatic compressibility.) If we Fourier transform as before we find

$$p = Z_C U, \quad Z_C = 1/i\omega C, \quad C = V/\rho v_s^2. \tag{3.4}$$

The quantity C is analogous to the capacitance in electronics; in acoustics we call it the compliance, and Z_c is the impedance of the compliance. The schematic symbol for the compliance is shown in Figure 3.3(e). The ground symbol at the bottom terminal indicates that the pressure there is zero, *i.e.* the pressure drop across the compliance is measured relative to the quiescent or ambient pressure. In a sense this is a cheat; circuit element constitutive relations are only allowed to depend on differences of the across variable, but the fluid density depends on the common value of the pressures p_1 and p_2, not their difference, so we have had to introduce another terminal connected to zero pressure. In electronics it makes sense to have either end of a capacitor connected to time-varying potentials, but in acoustics it does not.

Returning to realistic values of both the mass density and the compressibility, we may represent all effects with the circuit of Figure 3.3(f), provided that the waveguide section being modeled is much shorter than a wavelength of sound at the frequency of interest. This restriction is required simply because the model only contains two volume velocity variables and two pressure variables, and thus it can only include the effects of linear variations of p and U with position. Circuit elements like L and C that describe objects small compared to a wavelength are called lumped. More general elements that can model continuous variations of the field variables are called distributed.

The impedance for any (linear and time-invariant) two-terminal element is $Z = p/U$. For multi-terminal elements with one terminal at a reference potential, the impedance is defined $Z_{ij} = p_i U_j$ where i, j label the terminals, and pressures are measured relative to a reference terminal. When i = j we speak of a driving-point impedance, otherwise a trans-impedance. Similar definitions are used for the admittance, $Y = U/p$. The impedance may be separated into real and imaginary parts $Z = R + iX$ and likewise the admittance $Y = G + iB$. R is called resistance, X reactance, G conductance, and B susceptance.

There is an important sign convention for two-terminal elements buried in the definitions above: if the pressure drop is defined as $p = p_1 - p_2$, then a positive value of U means that flow enters on the p_1 side of the element. If this is reversed, all impedances and admittances will have unconventional signs.

Figure 3.4(a) shows a coaxial electrical transmission line, which, like our acoustic line, has an 'acoustic' mode for which the across and through variables (voltage and current) have no dependence on coordinates perpendicular to the axis. Figure 3.4(d) shows an exactly analogous acoustic line. The lumped element model in Figure 3.4(b) approximates either system. Consider first the straight section on the left side of the figure that is modeled by three identical LC sections. (Many more sections might be needed depending on the wavelengths and accuracy of interest.)

It is straightforward to show that this model reproduces the Helmholtz wave equation in the limit where many LC sections are used. Let the section length l become Δx, the flow difference across an element be $\Delta U = U_2 - U_1$ and the pressure difference be $\Delta p = p_2 - p_1$. Then the acoustic circuit equations for one element are

$$\frac{\Delta U}{\Delta x} = -i\omega p \frac{S}{\rho v_s^2}, \quad \frac{\Delta p}{\Delta x} = -i\omega U \frac{p}{S},$$

where p is the pressure at the top of the compliance and U is the current through the inertance. Taking the limit $\Delta x \to 0$, and substituting the second equation into the first, leads to the Helmholtz equation in one dimension:

$$\left(\omega^2 + v_s^2 \frac{\partial^2}{\partial x^2}\right) p = 0, \quad U = i \frac{S}{\omega\rho} \frac{\partial p}{\partial x}. \tag{3.5}$$

A one-dimensional running wave moving in the positive x direction and its corresponding current are written

$$p = A\exp(-ikx), \quad U = A\frac{S}{\rho v_s}\exp(-ikx) = A\sqrt{\frac{C}{L}}\exp(-ikx).$$

The ratio

$$p / U = Z_0 = \rho v_s / S = \sqrt{L / C} \tag{3.6}$$

is know as the characteristic impedance Z_0 of the line and is equal to the wave impedance z (defined in Section 2.3) divided by the waveguide area S. The units of the acoustic impedance Z are (Pa s)/m^3, a combination that is sometimes called acoustic ohms; we denote it Ωa. For air at standard conditions and a 25 mm diameter waveguide, $Z_0 = 0.84\ \mathrm{M\Omega a}$. This sets the scale for impedance in waveguide problems. Notice that reducing the diameter of an acoustic line increases its impedance, although it decreases the impedance of a coaxial line.

Combining the above results for the inertance and the compliance one finds

$$v_s = \frac{l}{\sqrt{LC}}. \tag{3.7}$$

If the line length l is moved to the denominator of this expression we see that the sound velocity depends on the inertance and compliance per unit length of line.

Figure 3.4 shows several other features that can easily be handled with circuit elements. The system is driven at the left by an electrical current source, or its acoustic analog, a moving piston which establishes a volume velocity U. Near the middle of the line there is a diameter transition. If it is much smaller than a wavelength, it can be modeled using a compliance given by (3.4) and an inertance

$$L = \frac{1}{|U|^2} \int \rho |\vec{u}|^2 \, dV, \tag{3.8}$$

where the integral is over the volume of the element and $\vec{u}$ is the fluid velocity, now a function of the coordinates. The calculation is done using the flow field in the static limit; that is, a solution of (2.12) in the limit $\omega \to 0$. Equation (3.8) follows from the fact that $\frac{1}{2}L|U|^2$ is the kinetic energy.

The coaxial line in Figure 3.4 has a resistive leak from the center conductor to the shield. The constitutive relation for an acoustic resistor is

$$p = UR, \tag{3.9}$$

where, as in (3.3) and (3.4), p is the pressure drop across the two-terminal element. The acoustic resistor corresponds to a leak to ambient pressure. A given internal pressure creates a flow to the outside that is proportional to the pressure and varies inversely with the flow resistance. One can also have resistors in series with the inertances that will create additional pressure drops along the line.

The electrical line in Figure 3.4 is terminated at the right end by an electrically-small loop antenna. This means that it is much smaller than a wavelength of electromagnetic radiation at the frequency of interest. A good lumped-element model for such an antenna is an inductor in series with a resistance. The resistance represents radiation reaction; it models the power lost to radiation. The open end shown in Figure 3.4(d) is a close acoustic analog to the electrically-small antenna, assuming that the waveguide cross-section is much small than an acoustic wavelength, as is normally the case. For a rough understanding, we can suppose that every point of the terminating plane at the open end of the waveguide (of radius a) is subject to the same wave impedance $z = p/u$ as every point on the surface of a radiating sphere of radius a. (The radiating sphere was discussed in Section 2.3.) Expanding (2.23) for small ka and dividing by the waveguide area S, we find for $Z = z/S$:

$$Z = Z_0(ika + (ka)^2 + \ldots) = i\omega \frac{\rho a}{S} + Z_0(ka)^2 + \ldots \; . \tag{3.10}$$

The first term is the impedance of an inertance of length a and area S. It acts just like an extension of the waveguide with a length equal to the waveguide radius. Detailed calculations show that the correct 'end correction' is 0.62a for an un-flanged open end and 0.86a for a flanged end.[5] The second term in (3.10) gives the value of the radiation resistance of the termination. It is $(ka)^2$ less than the characteristic impedance, and thus poorly matched to the line. (Detailed calculations give half this value for a flanged pipe, and one fourth for an un-flanged open.) We will see in the next section that the mismatch implies that most of the power incident on the termination will be reflected.

A closed end in acoustics has no similar corrections; it is simply a point where $U=0$, corresponding to an open circuit. In electronics, an open circuited end has corrections due to stray capacitance, while a shorted end can be made nearly ideal.

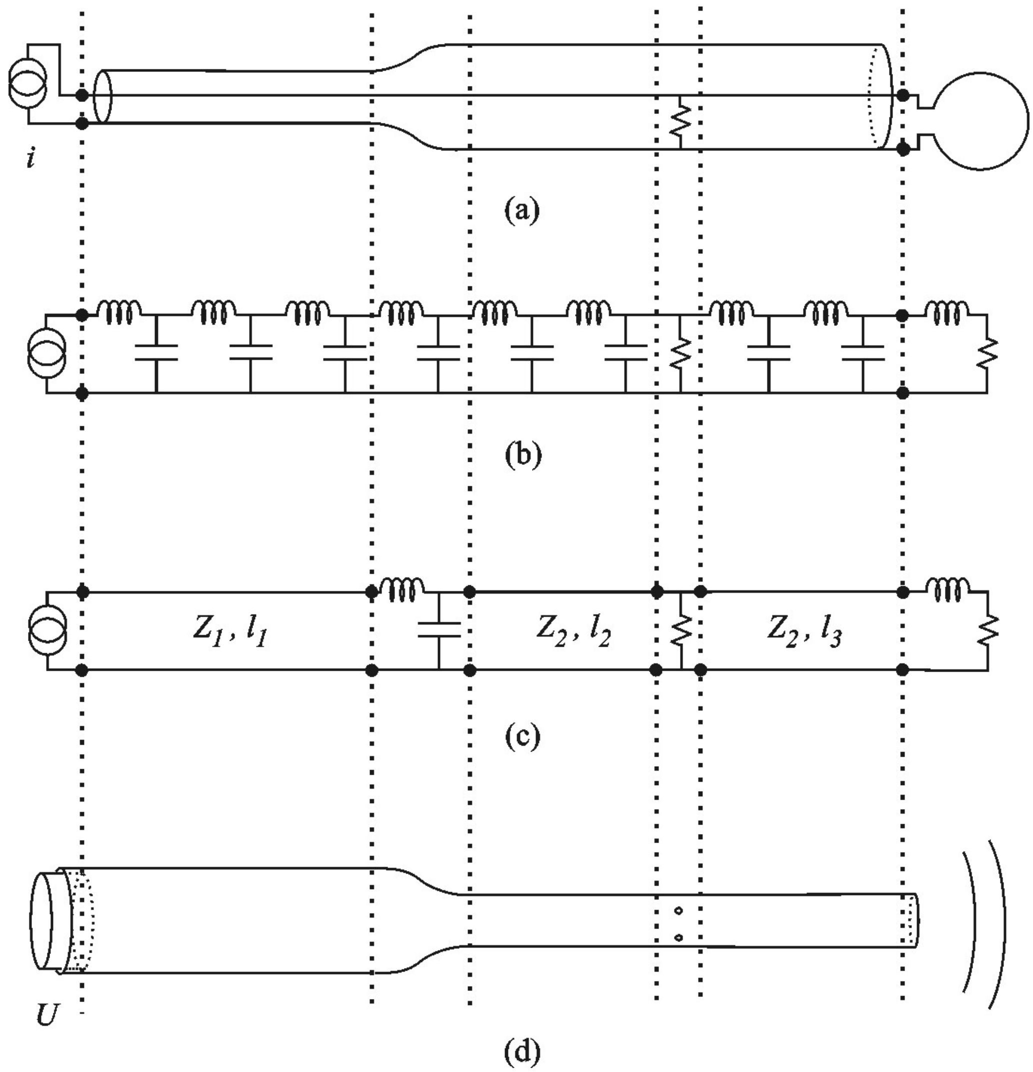

Figure 3.4. (a) An electromagnetic coaxial transmission line with a diameter change, a resistive leak, and a loop antenna load. The line is driven by a current source. (b) A lumped-element approximation to the coaxial line. (c) A mixed representation of the coaxial line with lumped elements and distributed transmission-line sections. The line sections are labeled by their impedances Z and lengths l. (d) An analogous acoustic waveguide driven by a piston, with a diameter change, a leak, and an open termination.

Instead of the lumped-element model of Figure 3.3(b), it is often convenient to use a combination of lumped and distributed elements, as in Figure 3.3(c). Each straight section is modeled by a two-port element (a port is a pair of terminals) with exact constitutive relations parameterized by characteristic impedances and lengths. (The diagram conventionally used for these elements may cause confusion. The lower line is a wire at pressure $p = 0$. The upper line is not a wire in the sense we have been using, it just represents the wave-carrying medium.) Line-section circuit elements like these are very useful and are commonly available in circuit simulation programs. We will develop concepts and techniques for describing their behavior in the next two sections.

3.3 Reflection Coefficient

Consider the distributed line section in Figure 3.5, with characteristic impedance Z_0. At the right end we place a termination or load with impedance *Z*. We suppose that somewhere to the left there is a sinusoidal generator that can inject waves into the line. The coordinate *x* has an origin at some arbitrary location, and in terms of this coordinate the location of the load is *l*. (The wires to the right of the terminals do not extend the length of the line.) The most general pressure and volume-velocity waveforms that can be present (at a single frequency) are

$$p = A_R \exp(-ikx) + A_L \exp(+ikx), \quad U = Z_0^{-1}(A_R \exp(-ikx) - A_L \exp(+ikx)), \quad (3.11)$$

where A_R, A_L are the right-going and left-going wave amplitudes. We have used (3.5) for *U*, and $\omega = kv_s$ to express the first factor in terms of Z_0. We define the reflection coefficient

$$S(x) = \frac{A_L \exp(+ikx)}{A_R \exp(-ikx)}, \quad (3.12)$$

as the ratio of the pressure amplitude reflected off the termination to the incident amplitude. In reflectometry, the position *x* at which the reflection coefficient is measured is known as the reference plane.

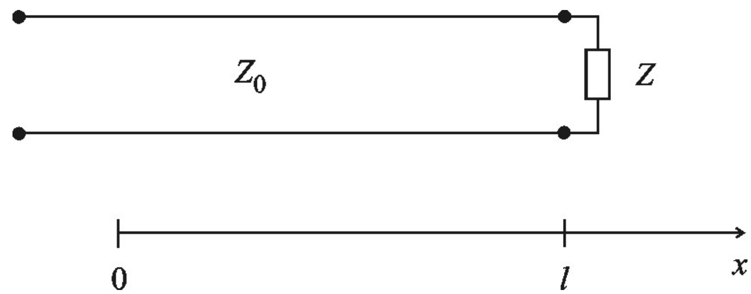

Figure 3.5. Transmission line section with characteristic impedance Z_0 and termination impedance Z. The distance from the reference plane to the termination is l.

Let us first find S with the reference plane at the location of the load, or with $x = l$. From the definition of the impedance and (3.11) we have

$$Z = \frac{p(l)}{U(l)} = Z_0 \frac{A_R \exp(-ikl) + A_L \exp(+ikl)}{A_R \exp(-ikl) - A_L \exp(+ikl)} = Z_0 \frac{1 + S(l)}{1 - S(l)}.$$

Expressing $S(l)$ in terms of Z we find

$$S(l) = \frac{Z - Z_0}{Z + Z_0}. \tag{3.13}$$

Using (3.12) we can write S at an arbitrary location x in terms of $S(l)$:

$$S(x) = \frac{A_L \exp(ikl)}{A_R \exp(-ikl)} \frac{\exp(ik(x-l))}{\exp(-ik(x-l))} = S(l)\exp(2ik(x-l)). \tag{3.14}$$

In words, this equation says that moving the reference plane towards the load a distance Δx multiples S by a phase $\exp(2ik\Delta x)$, and moving it towards the source multiples S by a phase $\exp(-2ik\Delta x)$. Substituting (3.13) into (3.14) gives

$$S(x) = \frac{Z - Z_0}{Z + Z_0} \exp(2ik(x-l)). \tag{3.14b}$$

The inverse of this is also useful:

$$Z = Z_0 \frac{1 + S(x)\exp(-2ik(x-l))}{1 - S(x)\exp(-2ik(x-l))}. \tag{3.15}$$

Another basic formula for reflectometry answers the following question: Suppose we have, as in Figure 3.5, an impedance Z terminating a line at a the position l. What impedance $Z(x)$ is this equivalent to at a general position x? To answer this question we switch x and l in (3.15) so that is gives us $Z(x)$ for a given $S(l)$, and then substitute in $S(l)$ from (3.13). The result is

$$Z(x) = Z_0 \frac{Z\cos(k(x-l)) - iZ_0\sin(k(x-l))}{Z_0\cos(k(x-l)) - iZ\sin(k(x-l))}. \tag{3.16}$$

In Figure 3.6 we map the reflection coefficient S on the complex plane, with $x = l$. From (3.13), zero reflection occurs for a matched termination $Z = Z_0$. We find $S = +1$ for $Z \to \infty$ and $S = -1$ for $Z = 0$, corresponding in acoustics to ideal closed and open ends (but in circuit language to open and shorted terminations, respectively). Our amplitudes are defined in terms of pressure, so $S = +1$ means that the reflected pressure wave is in phase with the incident wave and thus there is a pressure anti-node at a closed end, as expected, and a node at an ideal open end.

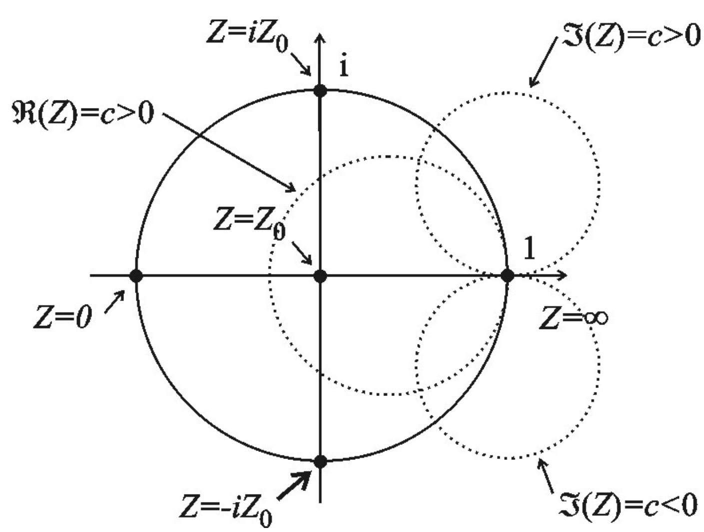

Figure 3.6. The reflection coefficient S for various values of the termination impedance Z, with the reference plane at the termination. The dotted lines show loci for fixed values of the real constant c.

Analysis of (3.13) shows that the locus of points with any fixed value of $\Re(Z)$ is a circle passing through +1 tangent to the line $\Re(S) = 1$. Likewise, the locus of points with any fixed $\Im(Z)$ is a circle passing through +1 tangent to the line $\Im(S) = 0$. All S values for $\Re(Z) > 0$ satisfy $|S| < 1$. In other words, any termination with positive resistance produces a reflected wave with a smaller amplitude than the incident wave. Such terminations are called passive. A lossless termination ($\Re(Z) = 0$) will have a reflection coefficient on the unit circle. The blowing end of a woodwind instrument is an active termination (though perhaps not a very linear one) with $\Re(Z) < 0$ and S outside the unit circle. Beautiful and accurate paper maps of the reflectance plane, known as Smith charts, were in heavy use before computer plotting made them less essential. To see these, and examples of their use, consult a textbook on microwave circuits.[5]

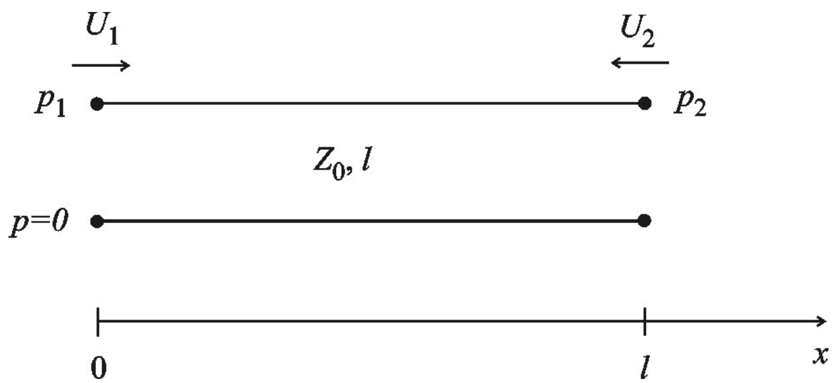

Figure 3.6b. Terminal variables for a line section.

As an application of these ideas, let us find the constitutive relations for a line section with characteristic impedance Z_0 and length *l*. The terminal variables are defined in Figure 3.6b. The direction shown for positive U_2 is commonly chosen but not universal. The impedance matrix elements are defined as

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{3.17}$$

If we connect an infinite impedance to port 2 so that $U_2 = 0$, then

$$Z_{11} = \frac{p_1}{U_1}, \quad Z_{21} = \frac{p_2}{U_1}.$$

In the limit $Z \to \infty$, (3.14) gives

$$S = \frac{A_L}{A_R} = \exp(-2ikl) \tag{3.18}$$

for the reflection coefficient into port 1. On the other hand, from (3.11)

$$Z_{11} = \frac{p_1}{U_1} = Z_0 \frac{A_R + A_L}{A_R - A_L}, \quad Z_{21} = \frac{p_2}{U_1} = Z_0 \frac{A_R \exp(-ikl) + A_L \exp(ikl)}{A_R - A_L},$$

which may be combined with (3.18) to yield

$$Z_{ij} = -iZ_o \begin{pmatrix} \cot(kl) & 1/\sin(kl) \\ 1/\sin(kl) & \cot(kl) \end{pmatrix}. \tag{3.19}$$

We have used $Z_{22} = Z_{11}$ and $Z_{12} = Z_{21}$, which follow from symmetry under exchange of the ports.

3.4 Waveguide Scattering Matrix

The scattering or *S*-matrix is a generalization of the reflection coefficient *S* for systems with more than one port. We use the notation and coordinates shown in Figure 3.7(a). The x-coordinates are defined so the positive direction is in-going on each side, and the ports are distances l_1 and l_2 from the origins. General pressure waves on each side are

$$p = A_{R1} \exp(-ikx_1) + A_{L1} \exp(+ikx_1), \quad p = A_{R2} \exp(-ikx_2) + A_{L2} \exp(+ikx_2) \tag{3.20}$$

Note that we now use the subscript *R* for in-going wave amplitudes regardless of the orientation of the line. We suppose that the line impedance Z_0 is the same on both sides, although this restriction can be removed with little complication. The ports need not be co-linear and we can readily generalize to any number of ports, and/or to more than one propagating mode in each waveguide. Our S-matrix is the same as the object discussed in free-space scattering theory; only the choice of modes is different.[6]

The *S*-matrix is defined

$$\begin{pmatrix} A_{L1} \exp(+ikx_1) \\ A_{L2} \exp(+ikx_2) \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_{R1} \exp(-ikx_1) \\ A_{R2} \exp(-ikx_2) \end{pmatrix}, \tag{3.21}$$

so that it linearly transforms a vector of in-going amplitudes into a vector of out-going amplitudes. If one port is driven and all others are connected to matched terminations Z_0, so the only incident wave is from the driven side, then

$$S_{11} = \frac{A_{L1}\exp(+ikx_1)}{A_{R1}\exp(-ikx_1)}, \quad S_{22} = \frac{A_{L2}\exp(+ikx_2)}{A_{R2}\exp(-ikx_2)},$$
$$S_{21} = \frac{A_{L2}\exp(+ikx_2)}{A_{R1}\exp(-ikx_1)}, \quad S_{12} = \frac{A_{L1}\exp(+ikx_1)}{A_{R2}\exp(-ikx_2)}. \tag{3.22}$$

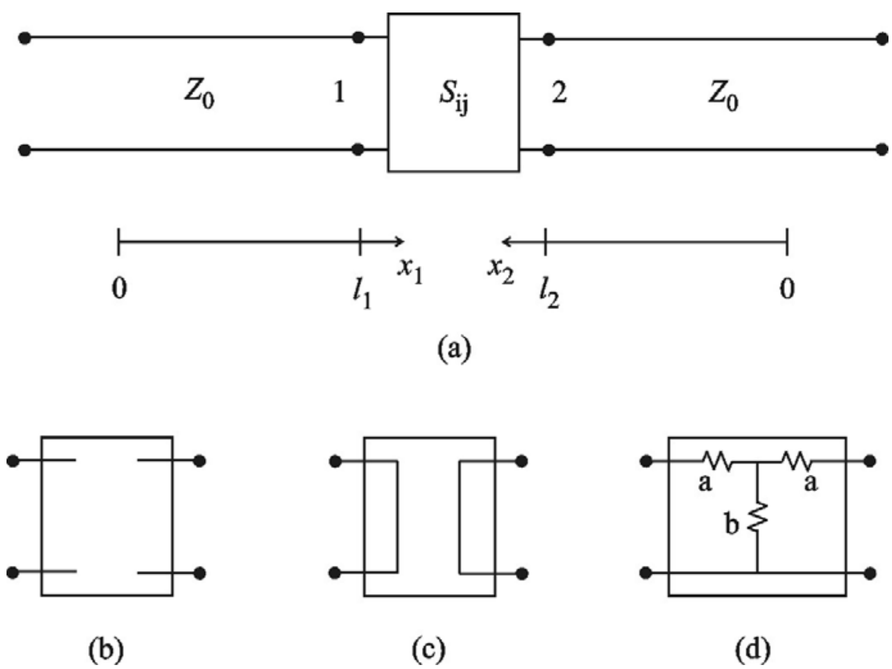

Figure 3.7. (a) Coordinates and port definitions for the scattering matrix. Reference planes are located at the coordinate origins. (b) Idealized zero-length blocked line. (c) Idealized zero length open ends. (d) Matched-T attenuator.

Figure 3.7(b) shows a two-port with open circuits at both ports, corresponding to a zero-length blockage in an acoustic waveguide, while Figure 3.7(c) shows circuit shorts, corresponding to idealized zero-length open ends. The S-matrices are (with both reference planes located at the ports),

.

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (block)}, \quad S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ (open ends)} \tag{3.23}$$

In other words, the reflection coefficients are ± 1 at both ports and there is no transmission.

Figure 3.7(d) shows a resistive attenuator. If the resistor values a and b are chosen to be:

$$a = \frac{1-\alpha}{1+\alpha}, \quad b = \frac{2\alpha}{1-\alpha^2}, \quad 0 < \alpha < 1, \tag{3.24}$$

then we have a matched attenuator, with *S* matrix

$$S = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \text{ (matched attenuator).} \tag{3.25}$$

To demonstrate this, first find the driving-point impedance of port 1 with a matched termination at port 2. The result will be Z_0, showing that there will be no reflected wave, and $S_{11} = 0$. Next, show that the ratio of the pressure at port 2 to the pressure at port 1 is α, when port 1 is driven and port 2 is connected to a matched termination. With the reference planes at the ports, this implies $S_{21} = \alpha$. The other elements are given by symmetry. In electronics, broadband matched resistive attenuators are commonplace. Similar components for acoustic waveguides can be imagined, but as far as we are aware they have not been demonstrated.

The resistive attenuator with $\alpha = 1$ is simply a wire, or zero-length line. A line of length *l* and characteristic impedance Z_0 has the *S*-matrix

$$S = \begin{pmatrix} 0 & \exp(-ikl) \\ \exp(-ikl) & 0 \end{pmatrix} \text{ (matched line).} \tag{3.26}$$

corresponding to no reflection and a phase shift of $\exp(-ikl)$. This S-matrix fully describes the line section and contains the same information as the impedance matrix (3.19), but in a simpler form. However, it would more complicated for a line segment with Z_0' different from the feed line impedance Z_0, whereas in (3.19) we can handle that case with the replacement $Z_0 \to Z_0'$.

A unitary *S*-matrix describes a two-port in which no power is dissipated. If we adopt the notations $\underline{S}$, A_R, and A_L for the S-matrix and the in-going and out-going wave amplitude vectors, we can write (3.21) as

$$A_L = \underline{S} A_R. \tag{3.25}$$

Multiplying on the left by the hermetian conjugate gives

$$A_L^\dagger A_L = A_R^\dagger \underline{S}^\dagger \underline{S} A_R. \tag{3.26}$$

Thus if $\underline{S}$ is unitary ($\underline{S}^{\dagger}\underline{S} = \underline{1}$) we find that the norm of the wave amplitude vector is conserved, which implies that the out-going power is equal to the in-going power. The *S*-matrices given above for the block, the open end, and the line segment are all unitary, but of course the attenuator is not, except when $\alpha = 1$.

Much more about waveguide circuits, including questions of synthesis (how to construct an element with a given *S*-matrix or other specifications) and realizability (what specifications are possible), can be found in text books on microwave electronics and circuit theory.[7]

3.5 Attenuation in Waveguides

As was mentioned at the start of this chapter, use of the Helmholtz equation with sliding boundary conditions is an approximation that neglects both the viscous boundary layer with thickness δ_s and the thermal boundary layer with thickness δ_h. Both effects are dissipative and cause damping of propagating modes. Detailed calculations are somewhat involved, though they date back to 19th century work by Helmholtz and Kirchhoff. Here we only quote the results, following the treatment given by Trusler,[2] who references both the original calculations and precise experimental tests.

In the presence of attenuation the most general pressure wave becomes

$$p = A_R \exp(-ikx)\exp(-\alpha x) + A_L \exp(+ikx)\exp(\alpha x), \tag{3.27}$$

where the attenuation constant α is given by

$$\alpha = \frac{\omega}{2v_s a}\left(\delta_s + (\gamma-1)\delta_h\right) \propto \omega^{1/2}a^{-1}. \tag{3.28}$$

Here a is the waveguide radius, and γ is the adiabatic index. Evaluating this expression for air under standard conditions, and for 25 mm diameter waveguide, we find that the attenuation length $1/\alpha$ varies from 92 m to 3.0 m as the frequency varies from 20 Hz to 20 kHz. It is essential to include this effect in many cases.

The boundary layers also reduce slightly the phase velocity so that its value v is no longer equal to the free-space velocity v_S. The fractional change of the phase velocity is

$$\frac{v - v_s}{v_s} = \frac{-1}{2a}\left(\delta_s + (\gamma-1)\delta_h\right) \propto \omega^{-1/2}a^{-1}. \tag{3.29}$$

For the same conditions, this varies from -2.9% to -0.091% from 20 Hz to 20 kHz.

The results derived in this chapter concerning the reflection coefficient and *S*-matrix can now be adapted with the substitution

$$k \to \tilde{k} = \frac{\omega}{v} - i\alpha \tag{3.30}$$

To be complete, one should also correct the characteristic impedance Z_0 so that it has a small imaginary part, but we can ignore this in most situations. It is possible to add small resistors to our lumped-element waveguide section to reproduce all of these results,[8] but in most cases it is sufficient to rely on lossless calculations and then introduce the attenuation as in (3.27), with α either calculated from (3.28) or fit to data as a phenomenological parameter. Fitting may be necessary, since the theory for α can only be relied on when the surface roughness of the walls is much less than the boundary layer thicknesses.

References for Chapter 3

1. basic fluid mechanics book, Landau or F&W.
2. J.P.M. Trusler, *Physical Acoustics and Metrology of Fluids*, (Taylor and Francis, New York, 1991).
3. papers on analogs and lumped elements, if any
4. calculations of end corrections, for example Fletcher
5. microwave book with Smith charts, the one in my office
6. scattering theory, probably Shankar
7. David M. Pozar, *Microwave Engineering*, (John Wiley & Sons, New York, 1997); L. Weinberg, *Network Analysis and Synthesis*, (McGraw-Hill, New York, 1962); H. Baher, *Synthesis of Electrical Networks* (Wiley, New York, 1984).
8. A.H. Benade, J. Acoust. Soc. Am., **22**, 616-623 (1968).