Acoustic Waves: Spherical and Planar Waves in Metamaterials

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Difference between Spherical and Planar Waves

Let us first begin by examining the problem setup (see figure 1):

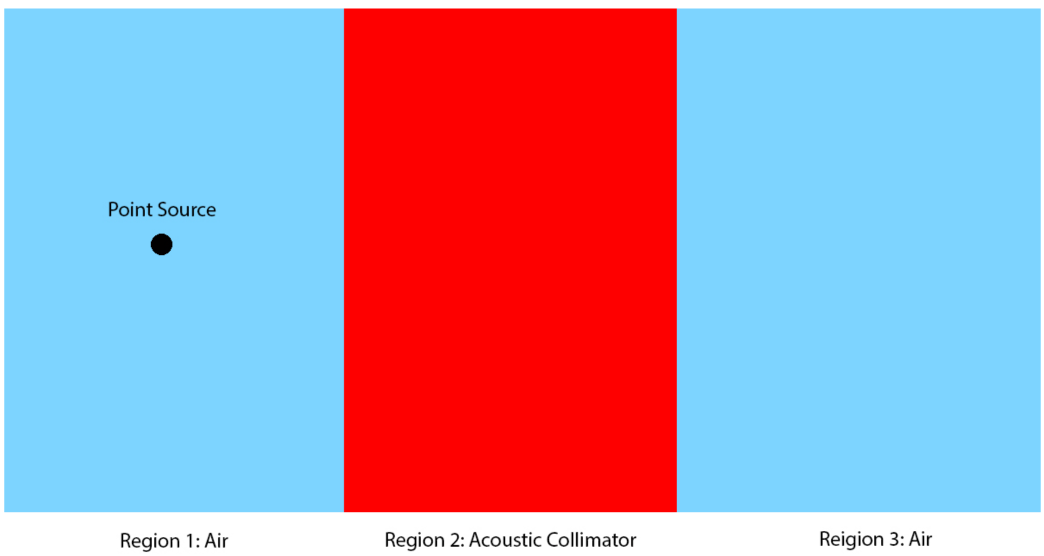

Figure 1: Experimental setup

The regions are described as follows:

1. **Region 1:** region 1 has the region medium considered to be air; thus it has a region density, $\rho_0 \approx 1.2041$. Additionally this is the location of the point source, which in this case is assumed to be a spherical wave excitation.

2. **Region 2:** region 2 is the location of the acoustic collimator. The collimator is to be assumed that it can convert a spherical wave to a planar wave with minimal losses. We shall consider this region to have an arbitrary but uniform density of density of, ρ_1.

3. **Region 3:** region 3 is following the acoustic collimator, again the medium density is assumed to be, $\rho_0 \approx 1.20414$.

Assuming that the input source is a spherical wave, we can say that the pressure (eq. 1) and velocity (eq. 2) components are described by:

$$p(t) = A\frac{e^{-i(kr-\omega t)}}{r} \quad (1)$$

$$\vec{v}(t) = \hat{n}\frac{A}{\rho c_0}\left(1 - \frac{i}{kr}\right)\frac{e^{-i(kr-\omega t)}}{r} \quad (2)$$

where A represents the amplitude of the wave, k is the wave vector, r is the distance from the source, ω is the frequency of propagation, t is the time propagating, ρ is the density of the medium of propagation, and c_0 is the speed at which the wave is propagating, in general $c_0 = 343\frac{m}{s}$ in air.
Thus we can write that the relation between pressure and velocity in spherical waves is as follows (eq. 5):

$$\vec{v}(t) = \hat{n}\frac{p(t)}{\rho c_0}\left(1 - \frac{i}{kr}\right) \quad (3)$$

Notice due to equation 3, that the phase of pressure and velocity are 90 degrees out of phase with respect to each other in spherical waves.

The goal of the acoustic collimator is to create a planar wave from a spherical wave. The definition of a planar wave is one that only changes with respect to a singular spatial coordinate (e.g. x). Thus across a singular plane by definition the entire wave should be in phase. The equations used to describe the pressure (eq. 4) and velocity (eq. 5) are as follows:

$$p(t) = Ae^{-i(\vec{k}\cdot\vec{x}-\omega t)} \quad (4)$$

$$\vec{v}(t) = \hat{n}\frac{A}{\rho c_0}e^{-i(\vec{k}\cdot\vec{x}-\omega t)} \quad (5)$$

Again A represents the amplitude of the wave, $\vec{k}$ is the wave propagation vector, $\vec{x}$ is the distance the wave has propagated, ω represents the frequency of propagation, t is the time the wave has been propagating.
Again we can synthesize these equations to describe velocity in terms of pressure (eq. 6):

$$\vec{v}(t) = \hat{n}\frac{p(t)}{\rho c_0} \quad (6)$$

Now, we are going to show given our setup that if the outcoming wave is assumed to be a planar wave then it can uniquely determined to be as such. We will do so with the following methodology:

1. Let the resultant pressure be described by p_n and the outcoming velocity v_n, where $n \in \{1, 2, \cdots x\}$ and x represents the number of outlet ports in the acoustic collimator.

2. If the following relations hold (eq. 7, 8):

$$\frac{Re(p_n)}{Re(v_n)} = \frac{Im(p_n)}{Im(v_n)} \quad (7)$$

$$\frac{Re(p_n)}{Re(v_n)} = \frac{Im(p_n)}{Im(v_n)} = \frac{Re(p_{n+1})}{Re(v_{n+1})} = \frac{Im(p_{n+1})}{Im(v_{n+1})} \quad (8)$$

   for all such n; then from equation 7 we know the pressure and velocity waves are in phase at single port, and from 8 we know across all such ports the waves are in phase. As a consequence, the output wave must be considered to be a planar wave, since the acoustic collimator can be considered to be a uniform medium then the collimator's medium performs a uniform transformation on the wave, consequently the wave will be propagating at the same velocity after the collimator; thus if the pressure and velocities of the resulting medium are in phase and the wave is traveling at the same speed, then the would should be considered a planar wave.

Let us begin by showing that these relations hold for planar waves:

Lemma 1: Let us first choose an arbitrary exit port from the collimator, assuming that the wave from the exit port from the collimator is a planar wave then we can model the pressure and velocity with the following equations (eq. 9-10):

$$p(t) = Ae^{-i(\vec{k}\cdot\vec{x}-\omega t)} \quad (9) \qquad \vec{v}(t) = \hat{n}\frac{p(t)}{\rho c_0} \quad (10)$$

In order to satisfy equation 7, we must satisfy:

$$\frac{Re(p(t))}{Re(v(t))} = \frac{Im(p(t))}{Im(v(t))}$$

Plugging in the associated values after applying Euler's formula for the left side yields:

$$\frac{Acos(\vec{k}\cdot\vec{x}-\omega t)}{\frac{Acos(\vec{k}\cdot\vec{x}-\omega t)}{\rho c_0}} = \rho c_0$$

Similarly the right side yields:

$$\frac{-Aisin(\vec{k}\cdot\vec{x}-\omega t)}{\frac{-Aisin(\vec{k}\cdot\vec{x}-\omega t)}{\rho c_0}} = \rho c_0$$

Thus we can say that:

$$\frac{Acos(\vec{k}\cdot\vec{x}-\omega t)}{\frac{Acos(\vec{k}\cdot\vec{x}-\omega t)}{\rho c_0}} = \rho c_0 = \frac{-Aisin(\vec{k}\cdot\vec{x}-\omega t)}{\frac{-Aisin(\vec{k}\cdot\vec{x}-\omega t)}{\rho c_0}}$$

meaning that equation 7 is satisfied for a planar wave at a singular port.

Lemma 2: Let us first choose two arbitrary distinct ports from the collimator and enumerate them as port 1 and port 2; the corresponding pressures and velocities are p_1, v_1 and p_2, v_2 respectively. Now assume that the output wave at each port is a planar wave; this allows us to model the outputs of the waves with the following equations (eq. 11-14):

$$p_1(t) = Ae^{-i(\vec{k}\cdot\vec{x}-\omega t)} \quad (11) \qquad \vec{v_1}(t) = \hat{n}\frac{p(t)}{\rho c_0} \quad (12)$$

$$p_2(t) = A'e^{-i(\vec{k'}\cdot\vec{x'}-\omega' t')} \quad (13) \qquad \vec{v_2}(t) = \hat{n}\frac{p_2(t)}{\rho' c_0} \quad (14)$$

In order to satisfy equation 8, we must satisfy:

$$\frac{Re(p_1)}{Re(v_1)} = \frac{Im(p_1)}{Im(v_1)} = \frac{Re(p_2)}{Re(v_2)} = \frac{Im(p_2)}{Im(v_2)}$$

From Lemma 1, since we are still looking at an acoustic collimator and at individual ports we know that:

$$\frac{Re(p_1)}{Re(v_1)} = \frac{Im(p_1)}{Im(v_1)} \text{ and } \frac{Re(p_2)}{Re(v_2)} = \frac{Im(p_2)}{Im(v_2))}$$

So in order to satisfy equation 8 all that we need to show is that:

$$\frac{Re(p_1)}{Re(v_1)} = \frac{Re(p_2)}{Re(v_2)}$$

Again from Euler's Formula and some expansion we can say that:

$$\frac{Re(p_1)}{Re(v_1)} = \frac{Acos(\vec{k}\cdot\vec{x}-\omega t)}{\frac{Acos(\vec{k}\cdot\vec{x}-\omega t)}{\rho c_0}} = \rho c_0$$

and

$$\frac{Re(p_2)}{Re(v_2)} = \frac{A' cos(\vec{k'} \cdot \vec{x'} - \omega t)}{\frac{A' cos(\vec{k'} \cdot \vec{x'} - \omega t)}{\rho c_0}} = \rho c_0$$

Thus we can say that:

$$\frac{Re(p_1)}{Re(v_1)} = \frac{Re(p_2)}{Re(v_2)}$$

which means that:

$$\frac{Re(p_1)}{Re(v_1)} = \frac{Im(p_1)}{Im(v_1)} = \frac{Re(p_2)}{Re(v_2)} = \frac{Im(p_2)}{Im(v_2)}$$

consequently, equation 8 holds for planar waves.

Now we are going to show that although equation 7 sometimes holds, if the wave is spherical then equation 8 cannot hold.

Lemma 3: Let us first choose an arbitrary exit port from the collimator, assuming that the output wave from the collimator can be modeled as a spherical wave, then we can model the pressure and velocity at it's boundary conditions with the following equations (eq. 15, 16):

$$p(t) = A\frac{e^{-i(kr-\omega t)}}{r} \quad (15) \qquad \vec{v}(t) = \hat{n}\frac{p(t)}{\rho c_0}\left(1 - \frac{i}{kr}\right) \quad (16)$$

Now if equation 7 held then:

$$\frac{Re(p(t))}{Re(v(t))} = \frac{Im(p(t))}{Im(v(t))}$$

Expanding this equation yields:

$$\frac{\frac{A}{r}cos(kr-\omega t)}{\frac{A}{\rho r c_0}\left(cos(kr-\omega t) + \frac{sin(kr-\omega t)}{kr}\right)} = \frac{\frac{A}{r} - isin(kr-\omega t)}{\frac{A}{\rho r c_0}\left(-isin(kr-\omega t) - \frac{icos(kr-\omega t)}{kr}\right)}$$

Simplifying this yields:

$$\frac{cos(kr-\omega t)}{cos(kr-\omega t) + \frac{sin(kr-\omega t)}{kr}} = \frac{sin(kr-\omega t)}{sin(kr-\omega t) + \frac{cos(kr-\omega t)}{kr}}$$

Cross multiplying the equation yields:

$$sin(kr-\omega t)cos(kr-\omega t) + \frac{cos(kr-\omega t)^2}{kr} = sin(kr-\omega t)cos(kr-\omega t) + \frac{sin(kr-\omega t)^2}{kr}$$

This can be simplified to:

$$sin(kr-\omega t)^2 = cos(kr-\omega t)^2$$

This means that the only solutions to this equation are when:

$$kr - \omega t \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \cdots \frac{\pi}{4} + \frac{n\pi}{2} | n \in \mathbb{Z}\right\}$$

Lemma 4: Let us first choose two arbitrary distinct ports from the collimator and enumerate them as port 1 and port 2; the corresponding pressures and velocities are p_1, v_1 and p_2, v_2 respectively. Now from lemma 3, if the relation would continue to hold then for every such port would need to have it's phase component, $kr - \omega t$, to be in some form of $\frac{\pi}{4} + \frac{n\pi}{2} | n \in \mathbb{Z}$. However, given our simulation setup, as we are finding boundary conditions t will be consistent for all such ports; similarly since we are looking at identical frequencies across the ports for every calculation ω will be constant.

This leaves us to deal with k and r. Now, as k is the wavenumber and it's magnitude is described by, $\frac{2\pi}{\lambda}$, this means k varies between 109.91 and 457.958 for the frequency range of 25 - 60kHz. However, as the differences in the distances x is are orders of magnitude smaller (4-5) it is impossible to get a consistent phase shift across all ports such that the phase component appears to be a planar wave but is in fact a spherical wave.
From lemma's 1 and 2, we have shown that our methodology is consistent and holds for planar waves, and through lemma's 3 and 4 we have shown that there is a very hard to reach special case that will apply to spherical waves.