derivation of Hartley function

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We want to show that the Hartley function $\log_2(n)$ is the only function mapping natural numbers to real numbers that

1. $H(mn) = H(m) + H(n)$ (),
2. $H(m) \leq H(m+1)$ (monotonicity), and
3. $H(2) = 1$ (normalization).

Let f be a function on positive integers that satisfies the above three properties. Using the additive property, it is easy to see that the value of $f(1)$ must be zero. So we want to show that $f(n) = \log_2(n)$ for all integers $n \geq 2$.

From the additive property, we can show that for any integer n and k,

$$f(n^k) = kf(n). \tag{1}$$

Let $a > 2$ be an integer. Let t be any positive integer. There is a unique integer s determined by

$$a^s \leq 2^t < a^{s+1}.$$

Therefore,

$$s \log_2 a \leq t < (s+1) \log_2 a$$

and

$$\frac{s}{t} \leq \frac{1}{\log_2 a} < \frac{s+1}{t}.$$

On the other hand, by monotonicity,

$$f(a^s) \leq f(2^t) \leq f(a^{s+1}).$$

Using Equation (1) and $f(2) = 1$, we get

$$sf(a) \leq t \leq (s+1)f(a),$$

and

$$\frac{s}{t} \leq \frac{1}{f(a)} \leq \frac{s+1}{t}.$$

Hence,

$$\left| \frac{1}{f(a)} - \frac{1}{\log_2(a)} \right| \leq \frac{1}{t}.$$

Since t can be arbitrarily large, the difference on the left hand of the above inequality must be zero,

$$f(a) = \log_2(a).$$