convolution, associativity of

Canonical name	ConvolutionAssociativityOf
Date of creation	2013-03-22 16:56:36
Last modified on	2013-03-22 16:56:36
Owner	mps (409)
Last modified by	mps (409)
Numerical id	5
Author	mps (409)
Entry type	Derivation
Classification	msc 94A12
Classification	msc 44A35

Proposition. *Convolution is associative.*

Proof. Let f, g, and h be measurable functions on the reals, and suppose the convolutions $(f*g)*h$ and $f*(g*h)$ exist. We must show that $(f*g)*h = f*(g*h)$. By the definition of convolution,

$$\begin{aligned}((f*g)*h)(u) &= \int_{\mathbb{R}} (f*g)(x)h(u-x)\,dx \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(y)g(x-y)\,dy\right] h(u-x)\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)h(u-x)\,dy\,dx.\end{aligned}$$

By Fubini's theorem we can switch the order of integration. Thus

$$\begin{aligned}((f*g)*h)(u) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)g(x-y)h(u-x)\,dx\,dy \\ &= \int_{\mathbb{R}} f(y)\left[\int_{\mathbb{R}} g(x-y)h(u-x)\,dx\right]\,dy.\end{aligned}$$

Now let us look at the inner integral. By translation invariance,

$$\begin{aligned}\int_{\mathbb{R}} g(x-y)h(u-x)\,dx &= \int_{\mathbb{R}} g((x+y)-y)h(u-(x+y))\,dx \\ &= \int_{\mathbb{R}} g(x)h((u-y)-x)\,dx \\ &= (g*h)(u-y).\end{aligned}$$

So we have shown that

$$((f*g)*h)(u) = \int_{\mathbb{R}} f(y)(g*h)(u-y)\,dy,$$

which by definition is $(f*(g*h))(u)$. Hence convolution is associative. $\square$