# examples of primitive recursive encoding

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In this entry, we present three examples of primitive recursive encodings. In all the examples, the following notations are used: $\mathbb{N}$ is the set of all natural numbers (including 0), $\mathbb{N}^*$ is the set of all finite sequences over $\mathbb{N}$, and $E:\mathbb{N}^*\to\mathbb{N}$ is the encoding in question. For any sequence $a_1,\ldots,a_k$, its image under $E$ is denoted by $\langle a_1,\ldots,a_k\rangle$, and is called the sequence number corresponding to $a_1,\ldots,a_k$. Furthermore, $()$ denotes the empty sequence, and its sequence number is denoted by $\langle\rangle$. The fact that the $E$ in each of the examples below is in fact encoding is proved in `http://planetmath.org/EncodingWords`this entry.

**Example 1**. (Multiplicative encoding) $E$ is defined as follows:

$$\begin{aligned} \langle\rangle &:= 1, \\ \langle a_1,\ldots,a_k\rangle &:= p_1^{s(a_1)}\cdots p_k^{s(a_k)}, \end{aligned}$$

where $s$ is the successor function, and $p_1,\ldots,p_k$ are the first $k$ prime numbers.

To see that $E$ is primitive recursive, we verify the following:

- the predicate "$x$ is a sequence number" is primitive recursive: a number $x\in\mathbb{N}$ is a sequence number iff $x=1$ or, if $p|x$ for some prime $p$, then $q|x$ for any prime $q\le p$. The predicates

  $$\Phi_1 := \text{“}p|x \text{ for some prime } p\text{”} \equiv \text{“}\exists p\le x(P(p)\land(p|x))\text{”},$$

  $$\Phi_2 := \text{“}p \text{ is prime and } q|x \text{ for all primes } q\le p\text{”} \equiv \text{“}\forall q\le p(P(p)\land P(q)\land(q|x))\text{”}$$

  where $P(r) :=$ "$r$ is prime", are primitive recursive by bounded quantification. Thus "$x$ is a sequence number" iff "$x=1$ or $\Phi_1\to\Phi_2$" iff "$(x=1)\lor(\neg\Phi_1\lor\Phi_2)$", is primitive recursive as a result.

- $E_k(a_1,\ldots,a_k) := p_1^{s(a_1)}\cdots p_k^{s(a_k)}$ is clearly primitive recursive.

- $\mathrm{lh}(x)$ can be defined as the number of primes dividing $x$, which is primitive recursive.

- $(x)_y$ can be defined as the exponent of the $y$-th prime in $x$ (the largest power of $p_y$ dividing $x$), which is again primitive recursive.

**Example 2**. (Encoding via a pairing function) First, let $J:\mathbb{N}^2\to\mathbb{N}$ be a (primitive recursive) pairing function. For any $n\ge 2$, define

$$\begin{aligned} J_2(x_1,x_2) &:= J(x_1,x_2) \\ J_{n+1}(x_1,\ldots,x_n,x_{n+1}) &:= J(x_1,J_n(x_2,\ldots,x_{n+1})). \end{aligned}$$

Then define $E$ by

$$\begin{aligned} \langle\rangle &:= 0, \\ \langle a_1, \dots, a_k\rangle &:= J(k, J_k(a_1, \dots, a_k)). \end{aligned}$$

$E$ is primitive recursive because

- $E$ is a bijection, so the predicate "$x$ is a sequence number" is the same as "$x \in \mathbb{N}$", which is clearly primitive recursive,
- $E_k(a_1, \dots, a_k) := J(k, J_k(a_1, \dots, a_k))$ is primitive recursive since both $J$ and $J_k$ are, the latter of which can be shown to be primitive recursive by induction,
- The two functions $R, L : \mathbb{N} \to \mathbb{N}$ such that $J(L(m), R(m)) = m$ are primitive recursive. So $\mathrm{lh}(x) = L(x)$ in particular is primitive recursive.
- If $J_k(a_1, \dots, a_k) = b$, then $a_1 = L(b), a_2 = LRL(b), \dots, a_{k-1} = (LR)^{k-2}L(b)$, and $a_k = R(LR)^{k-2}L(b)$. Thus,

  $$(x)_y = \begin{cases} (LR)^y(x) & \text{if } y < L(x), \\ R(LR)^y(x) & \text{if } y = L(x), \\ 0 & \text{otherwise.} \end{cases}$$

  is primitive recursive, since each case is primitive recursive.

**Example 3**. (Digital Representation) Pick a positive integers $p > 1$. Define $E$ by

$$\begin{aligned} \langle\rangle &:= 1 \\ \langle a\rangle &:= p^{s(a)} \\ \langle a_1, \dots, a_k, a_{k+1}\rangle &:= \langle a_1\rangle(\langle a_2, \dots, a_{k+1}\rangle + 1). \end{aligned}$$

In other words,

$$\langle a_1, \dots, a_k\rangle = p^{s(a_1)} + p^{s(a_1)+s(a_2)} + \cdots p^{s(a_1)+\cdots+s(a_k)}. \tag{1}$$

To see that $E$ is primitive recursive, we first define three functions $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) := \mathrm{lo}(p, x)$, the exponent of $p$ in $x$, $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) := \mathrm{quo}(x, p^{f(x)}) \dot{-} 1$, and $h : \mathbb{N}^2 \to \mathbb{N}$ given by

$$\begin{aligned} h(x, 0) &:= x \\ h(x, n+1) &:= g(h(x, n)). \end{aligned}$$

Clearly, $f, g, h$ are all primitive recursive. Furthermore, $h$ has the property that if $h(x, n) > 0$, then $h(x, n+1) < h(x, n)$, and therefore $h(x, n) = 0$ for large enough $n$. Using $h$, we may proceed to show that $E$ is primitive recursive:

- the predicate "$x$ is a sequence number" is equivalent to the predicate

  $$\text{“}(x = 1) \vee ((x > 0) \wedge (\forall n\ p | h(x, n)))\text{”}$$

  which is equivalent to the predicate

  $$\text{“}(x = 1) \vee ((x > 0) \wedge (\forall n \leq x\ p | h(x, n)))\text{”}$$

  since $p > 1$. As the quantification is bounded, the predicate is primitive recursive.

- $E_k(a_1, \ldots, a_k) = \langle a_1, \ldots, a_k \rangle$ is primitive recursive by equation (1) above.

- $\mathrm{lh}(x)$ can be defined as the number of $n$ such that $h(x, n) \neq 0$, or

  $$\sum_{i=0}^{x} \mathrm{sgn}(h(x, i)),$$

  which is primitive recursive, because it is a bounded sum.

- If $\langle a_1, \ldots, a_k \rangle = x$, then $f(h(x, 0)) = s(a_1), \ldots, f(h(x, k-1)) = s(a_k)$. Therefore, $(x)_y$ is just $f(h(x, y \dot{-} 1)) \dot{-} 1$, which is primitive recursive.

**Remark**. In the third example, $E$ can be more generally defined so that

$$\langle a_1, \ldots, a_k, a_{k+1} \rangle := \langle a_1 \rangle (r \langle a_2, \ldots, a_{k+1} \rangle + q),$$

provided that $p, q$ are coprime. It is fairly straightforward to show that $E$ is again primitive recursive.