proof of Gaussian maximizes entropy for given covariance

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Let f, K, ϕ be as in the http://planetmath.org/GaussianMaximizesEntropyForGivenCovari entry.

The proof uses the nonnegativity of relative entropy $D(f||\phi)$, and an interesting property of quadratic forms. If A is a quadratic form and p, q are probability distributions each with mean $\mathbf{0}$ and covariance matrix $\mathbf{K}$, we have

$$\int p\ x_i x_j\ dx_i dx_j = K_{ij} = \int q\ x_i x_j\ dx_i dx_j \tag{1}$$

and thus

$$\int Ap = \int Aq \tag{2}$$

Now note that since

$$\phi(\mathbf{x}) = ((2\pi)^n |\mathbf{K}|)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{x}\right), \tag{3}$$

we see that $\log\phi$ is a quadratic form plus a constant.

$$\begin{aligned}
\mathbf{0} &\leq D(f||\phi) \\
&= \textstyle\int f \log \frac{f}{\phi} \\
&= \textstyle\int f \log f - \int f \log \phi \\
&= -h(f) - \textstyle\int f \log \phi \\
&= -h(f) - \textstyle\int \phi \log \phi \qquad \text{by the quadratic form property above} \\
&= -h(f) + h(\phi)
\end{aligned}$$

and thus $h(\phi) \geq h(f)$.