discrete time Fourier transform in relation with continuous time Fourier transform

Canonical name	DiscreteTimeFourierTransformInRelationWithContinuousTimeFourierTransfo
Date of creation	2013-03-22 17:37:45
Last modified on	2013-03-22 17:37:45
Owner	fernsanz (8869)
Last modified by	fernsanz (8869)
Numerical id	10
Author	fernsanz (8869)
Entry type	Theorem
Classification	msc 94A20
Related topic	SamplingTheorem
Related topic	ApproximatingFourierIntegralsWithDiscreteFourierTransforms
Related topic	Distribution4
Related topic	SpaceOfRapidlyDecreasingFunctions
Defines	DTFT
Defines	Discrete Time Fourier Transform

Introduction

Computers are able to handle only finite number of data. Hence, if we are to study and treat real world signals (i.e. functions $\mathbb{R} \to \mathbb{C}$) in a computer, a way to characterize signals by a finite number of data has to be found.
If we sample the values of the signal $x(t)$ at periodic times $t = nT, n \in \mathbb{Z}$ we form the sequence $x_d[n] = x(nT)$. Does this sequence contain all the information relative to $x(t)$?
We know from the sampling theorem that if the signal $x(t)$ is bandlimited, the sampled sequence allows us to recover the original continuous time signal provided the sampling frequency $1/T$ is at least twice the maximum frequency of the signal. However, real signals are not of finite bandwidth as this would imply the signal to be of infinite time duration. Therefore, a problem arise of how well can we approximate the original signal $x(t)$ by the sampled sequence $x_d[n]$. In fact, we are interested in studying the spectrum of the original signal based upon the samples $x_d[n]$.
While the relation between $x_d[n]$ and the spectrum of $x(t)$ is widely used in communication and electronic engineering books, it is difficult to find a rigorous proof. We cover here the gap between engineering daily knowledge and rigorous mathematical proof of the named relations establishing under what assumptions those relations are valid.

Relation between Discrete Time Fourier Transform and Continuous time Fourier transform

Theorem 1. *Let $x : \mathbb{R} \to \mathbb{C}$ be a bounded variation (it can be, in particular, piecewise smooth) L^1 function and let $X(\omega)$ be its Fourier transform* [1]. *Let $T \in \mathbb{R}$. If $g_n(\omega) = \sum_{k=-n}^{+n} X\left(\frac{\omega-2\pi\cdot k}{T}\right)$ converges a.e as $n \to +\infty$ and is there is M such that $|g_n(\omega)| < M$ a.e. then*

$$\frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(\frac{\omega-2\pi\cdot k}{T}\right) = \sum_{k=-\infty}^{+\infty} \frac{x(kT^+)+x(kT^-)}{2} e^{-j\omega k} \quad \text{in } L^2([-\pi,\pi]) \tag{1}$$

*If additionally $\sum_{k=-\infty}^{+\infty} X\left(\frac{\omega-2\pi\cdot k}{T}\right)$ is continuous, the right hand side of (**??**) converges uniformly to the left hand side in any closed interval $[a,b] \subset [-\pi,\pi]$.*

[1] in the present entry we will take the Fourier transform of $x(t)$ to be $X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}d\omega$

Proof. By hypothesis we can form the function

$$g(\omega) = \lim_{n\to+\infty} g_n(\omega) = \lim_{n\to+\infty} \sum_{k=-n}^{+n} X\left(\frac{\omega - 2\pi \cdot k}{T}\right)$$

This function is obviously periodic of period 2π and bounded, hence it can be expanded in its Fourier series which converge in $L^2([-\pi,\pi])$; the Fourier theory shows that the convergence is uniformly in $[a,b] \subset [-\pi,\pi]$ if $g(\omega)$ is continuous.

$$g(\omega) = \sum_{k=-\infty}^{+\infty} X\left(\frac{\omega - 2\pi \cdot k}{T}\right) = \sum_{k=-\infty}^{+\infty} c_k e^{j\omega k}$$

where the coefficients c_k are given by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\omega)e^{-j\omega k}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{n\to+\infty} g_n(\omega)e^{-j\omega k}d\omega$$

As $|g_n(\omega)| < M$ we can appeal the dominated convergence theorem to write

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{n\to+\infty} g_n(\omega)e^{-j\omega k}d\omega &= \lim_{n\to+\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(\omega)e^{-j\omega k}d\omega \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int_{-\pi-2k\pi}^{\pi+2k\pi} X\left(\frac{\omega}{T}\right)e^{-j\omega k}d\omega \\
&= \frac{T}{2\pi} P.V. \int_{-\infty}^{+\infty} X(\omega)e^{j\omega(-kT)}d\omega
\end{aligned}$$

Now, as $x(t)$ is of bounded variation, the Jordan theorem on Fourier transform inversion says that

$$\frac{1}{2\pi} P.V. \int_{-\infty}^{+\infty} X(\omega)e^{j\omega(-kT)}d\omega = \frac{x(-kT^+) + x(-kT^-)}{2}$$

and the result follows.

□

There are, however, very which do not satisfy the conditions required in the theorem above. Such is the case of the *pulse* function

$$rect(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2}, \\ 0 & \text{if } |t| \geq \frac{1}{2}. \end{cases}$$

whose Fourier transform is $\frac{sin(x/2)}{x/2}$ -with the definition made in footnote 1-. $sinc(x)$ behaves as $\frac{1}{x}$, so $\sum_{k=-\infty}^{+\infty} sinc\left(\frac{\omega-2\pi\cdot k}{T}\right)$ will not converge in general [2]; it will converge for those T that make the series an `http://planetmath.org/AlternatingSeries`alte one, but not for the rest values of T. Therefore we need another result which somehow relates both sides of eq **??**.

As we have pointed out, the problem with the pulse function is that its Fourier transform does not decay rapid enough for the series to converge. So we will try smoothing the signal out so that its Fourier transform will decay faster and, hopefully, the series converges. We wish the smoothed version of $x(t)$ to resemble the original signal, so uniform approximation seems reasonable. But, as we will see, for an infinite number of samples $\{nT, n \in \mathbb{Z}\}$ each of these might require a different degree of approximation and it could be impossible to find an uniform approximation for all the samples. So, we will focus on time limited signal, for which we have the following result.

Notation. $\mathcal{D}(\mathbb{R})$ will denote the set of test functions on $\mathbb{R}$ and $\mathcal{S}$ the set of rapidly decreasing functions on $\mathbb{R}$ -the Schwartz space-. The symbol $\hat{}$ above a function will denote its Fourier transform. We know that $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}$ and that the Fourier transform is an isomorphism of $\mathcal{S}$ onto itself. The product of two functions X and ϕ at ω will be denoted by $X\phi(\omega)$. Convolution will be denoted by $*$.

Theorem 2. *Let $\phi \in \mathcal{D}(\mathbb{R})$ be a test function and $\{\phi_j(t) = j \cdot \phi(j \cdot t),\ j = 1, 2, \ldots\}$ be an approximate identity. Let $x(t)$ be a time limited bounded variation signal. If $\sum_{k=-\infty}^{+\infty} X\hat{\phi}_j\left(\frac{\omega-2\pi\cdot k}{T}\right)$ converges for all $j = 1, 2, \ldots$ to a continuous function, then*

$$\lim_{j\to\infty} \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\hat{\phi}_j\left(\frac{\omega - 2\pi\cdot k}{T}\right) = \sum_{k=-\infty}^{+\infty} \frac{x(kT^+) + x(kT^-)}{2} e^{-j\omega k} \qquad (2)$$

Proof. Take the signal $(x * \phi_j)(t)$ whose Fourier transform is $X\hat{\phi}(\omega)$. This signal satisfies, by hypothesis, the conditions of Theorem 1, so we can write

$$\frac{1}{T} \sum_{k=-\infty}^{+\infty} X\hat{\phi}_j\left(\frac{\omega - 2\pi\cdot k}{T}\right) = \sum_{k=-\infty}^{+\infty} \frac{(x*\phi_j)(kT^+) + (x*\phi_j)(kT^-)}{2} e^{-j\omega k}$$

[2]for monotone decreasing functions "series behaves as integrals", that is, if $\sum_{k=0}^{+\infty} f(x - k\cdot M)$ converges or diverges so does $\int_0^{+\infty} f$

The right hand side is actually a sum of a finite number of terms since the signal $(x * \phi_j)(t)$ is time limited -being the convolution of two compact supported functions-. This makes the Fourier series a continuous function, which, together with the hypothesis that $\frac{1}{T}\sum_{k=-\infty}^{+\infty} X\hat{\phi}_j\left(\frac{\omega-2\pi\cdot k}{T}\right)$ is continuous, shows that equality in the above equation is pointwise.
Now let $j \to \infty$ and use the fact that $\{\phi_j(t)\}$ is an approximate identity to obtain equation (**??**). $\square$

The rationale behind choosing test functions in the last theorem is that, in most cases,$\sum_{k=-\infty}^{+\infty} X\hat{\phi}_j\left(\frac{\omega-2\pi\cdot k}{T}\right)$ will converge even though $\sum_{k=-\infty}^{+\infty} X\left(\frac{\omega-2\pi\cdot k}{T}\right)$ do not. This is because rapidly decreasing functions decay faster than $\frac{1}{x^\alpha}$ for any α. So, for example, the Fourier transform of the pulse function has been tamed enough to make the series converge.

Remark 1. When the signal $x(t)$ is continuous, the right hand side of eqs. (**??**) or (**??**) reads

$$\sum_{k=-\infty}^{+\infty} x_d[k]e^{-j\omega k}$$

where $x_d[k] = x(kT)$. This is defined as the *(Discrete Time) Fourier Transform, DTFT* of the sequence $x_d[n] = x(nT)$