hexacode

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The hexacode is a 3-dimensional linear code of http://planetmath.org/LinearCodelength 6, defined over the field $\mathbb{F}_4$, all of whose codewords have weight 0, 4, or 6. It is uniquely determined by these properties, up to http://planetmath.org/MonomialMatrixmonomia linear transformations and in $\mathbb{F}_4$. The hexacode is crucial to the construction of the extended binary Golay code via Curtis' Miracle Octad Generator. The exposition below follows ([?], Chapter 11). Another for the hexacode is ([?], Chapter 4).

1 Construction of the hexacode

There are several constructions of the hexacode, all leading to the same result. In the following, we write the elements of $\mathbb{F}_4$ as $\{0, 1, \omega, \overline{\omega}\}$ where ω is a cube root of unity. We write elements of the hexacode as elements of $\mathbb{F}_4^6$, separated into three of two.

1. The span of the elements

$$\begin{array}{c} \omega\overline{\omega}\ \omega\overline{\omega}\ \omega\overline{\omega} \\ \omega\overline{\omega}\ \overline{\omega}\omega\ \overline{\omega}\omega \\ \overline{\omega}\omega\ \omega\overline{\omega}\ \overline{\omega}\omega \\ \overline{\omega}\omega\ \overline{\omega}\omega\ \omega\overline{\omega} \end{array}$$

2. The elements of $\mathbb{F}_4^6$ of the form

$$(a, b, c, \phi(1), \phi(\omega), \phi(\overline{\omega}))$$

where $a, b, c \in \mathbb{F}_4$ and $\phi(x) = ax^2 + bx + c$.

3. The elements $ab\ cd\ ef$ of $\mathbb{F}_4^6$ which satisfy the three rules

$$\begin{array}{c} a + b = c + d = e + f = s \\ a + c + e = a + d + f = b + c + f = b + d + e = \omega s \\ b + d + f = b + c + e = a + d + e = a + c + f = \overline{\omega} s \end{array}$$

Here s is called the of the codeword.

An element of the hexacode is called a *hexacodeword*.

2 Justification of a hexacodeword

It is not difficult to show that all of the above constructions give the same 3-dimensional linear code of 6. However, it is somewhat tedious to use one of above constructions to determine whether a given element of $\mathbb{F}_4^6$ is in the hexacode. Instead, it is possible to show that an element of $\mathbb{F}_4^6$ is a hexacodeword if and only if it satisfies the *shape* and *sign* rules below. Using these rules, one can (with some practice) quickly distinguish hexacodewords from non-hexacodewords.

The shape rule says that, up to permutations of the three blocks and flips of the elements within a block, every hexacodeword has one of the shapes

$$\begin{array}{c} 00\ 00\ 00 \\ 00\ aa\ aa \\ 0a\ 0a\ bc \\ bc\ bc\ bc \\ aa\ bb\ cc \end{array}$$

where a, b, c are $1, \omega, \overline{\omega}$ in some .

The sign rule says that in every hexacodeword, either:

- all 3 blocks have sign 0, or
- the product of the signs of the three blocks is positive.

The sign of a block is determined as follows:

- + for $0a$ or ab where $b = a\omega$
- - for $a0$ or ab where $b = a\overline{\omega}$
- 0 for 00 or aa

where $a \neq 0$.

For example,

- 00 11 $\omega\omega$ and 01 0ω $1\overline{\omega}$ are not hexacodewords because they fail the shape rule.
- 01 01 $\omega\overline{\omega}$ satisfies the shape rule, and the signs are $+++$, so it is a hexacodeword.
- 1ω $\omega 1$ 1ω satisfies the shape rule, and the signs are +-+, so it is not a hexacodeword.
- 00 11 11 satisfies the shape rule, and the signs are 0 0 0, so it is a hexacodeword.
- $\omega 0$ $\omega 0$ $\overline{\omega}1$ satisfies the shape rule, and the signs are --+, so it is a hexacodeword.

3 Completion of a partial hexacodeword

The hexacode has the property that the following two problems always have a unique solution.

1. (3-problem) Given values in any 3 of the 6 positions, complete it to a full hexacodeword.

2. (5-problem) Given values in any 5 of the 6 positions, complete it to a full hexacodeword *after possibly changing one of the given values.*

Conway says that the best method for solving these is to simply "guess the correct answer, then justify it" (using the shape and sign rules), though he also does give systematic algorithms for solving them.

Examples of 3-problems:

$$\begin{gathered} 01\ 1?\ ?? \to 01\ 10\ \overline{\omega}\omega \\ ??\ 1\overline{\omega}\ ?0 \to 0\omega\ 1\overline{\omega}\ \omega 0 \\ 1?\ \omega\omega\ ?? \to 11\ \omega\omega\ \overline{\omega\omega} \end{gathered}$$

Examples of 5-problems:

$$\begin{gathered} 00\ 11\ \omega? \to 00\ 11\ 11 \text{ (position 5 changed)} \\ 0?\ 1\overline{\omega}\ \omega\overline{\omega} \to 0\omega\ 1\overline{\omega}\ \omega 0 \text{ (position 6 changed)} \\ 1\omega\ ?1\ 1\omega \to 1\omega\ \overline{\omega}1\ 1\omega \text{ (no position changed)} \end{gathered}$$

References

[1] J. H. Conway and N. J. A. Sloane. Sphere Packings, Lattices, and Groups. Springer-Verlag, 1999.

[2] Robert L. Griess, Jr. Twelve Sporadic Groups. Springer-Verlag, 1998.