公式推导与真解

$$
\begin{aligned}
I_n &= \int_0^1 \frac{x}{x+5} x^{n-1} dx \\
&= \int_0^1 x^{n-1} dx - 5 \int_0^1 \frac{x_{n-1}}{x+5} dx \\
&= \frac{1}{n} - 5 I_{n-1}
\end{aligned} \tag{1}
$$

因此对于推导而言，存在如上的递推关系。为了考量比较真实值和递推值，我们可以先使用 `SymPy` 推导出真正的解。

```python
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
import matplotlib as mpl
mpl.rcParams['text.usetex'] = True
```

让我们定义一些基本量，来进行计算:

```python
n = 8
n_range = np.array(range(0, n+1), dtype=int)
x_sym = sp.symbols("x")
expr = sp.Matrix([[x_sym**n_range[i] / (x_sym+5) for i in range(len(n_range))]])
expr.T
```

$$
\left[\begin{matrix}\frac{1}{x+5}\\ \frac{x}{x+5}\\ \frac{x^{2}}{x+5}\\ \frac{x^{3}}{x+5}\\ \frac{x^{4}}{x+5}\\ \frac{x^{5}}{x+5}\\ \frac{x^{6}}{x+5}\\ \frac{x^{7}}{x+5}\\ \frac{x^{8}}{x+5}\end{matrix}\right]
$$

因此看可以导出积分的精确解析表达式:

```python
expr_integrate = sp.integrate(expr, x_sym)
expr_integrate.T
```

$$
\left[\begin{matrix}\log(x+5)\\ x - 5\log(x+5)\\ \frac{x^2}{2} - 5x + 25\log(x+5)\\ \frac{x^3}{3} - \frac{5x^2}{2} + 25x - 125\log(x+5)\\ \frac{x^4}{4} - \frac{5x^3}{3} + \frac{25x^2}{2} - 125x + 625\log(x+5)\\ \frac{x^5}{5} - \frac{5x^4}{4} + \frac{25x^3}{3} - \frac{125x^2}{2} + 625x - 3125\log(x+5)\\ \frac{x^6}{6} - x^5 + \frac{25x^4}{4} - \frac{125x^3}{3} + \frac{625x^2}{2} - 3125x + 15625\log(x+5)\\ \frac{x^7}{7} - \frac{5x^6}{6} + 5x^5 - \frac{125x^4}{4} + \frac{625x^3}{3} - \frac{3125x^2}{2} + 15625x - 78125\log(x+5)\\ \frac{x^8}{8} - \frac{5x^7}{7} + \frac{25x^6}{6} - 25x^5 + \frac{625x^4}{4} - \frac{3125x^3}{3} + \frac{15625x^2}{2} - 78125x + 390625\log(x+5)\end{matrix}\right]
$$

进而看可以求出 I_n 的真实值如下（取小数点后 6 位）

```python
In_real_expr = expr_integrate.subs(x_sym, 1) - expr_integrate.subs(x_sym, 0)
In_real_expr.T
```

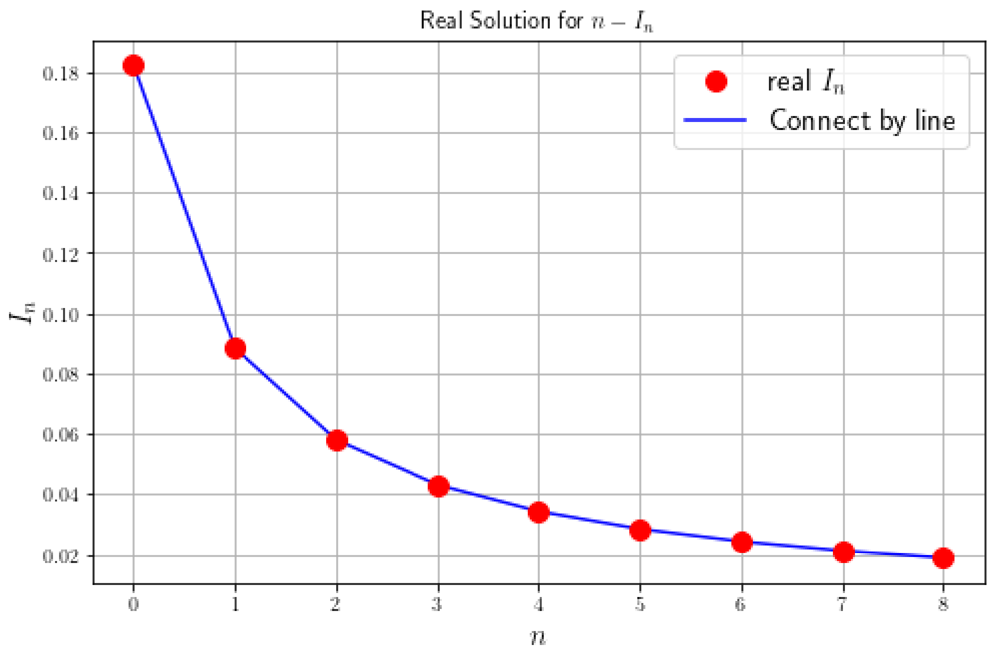
$$
\left[\begin{matrix}-\log(5) + \log(6)\\ -5\log(6) + 1 + 5\log(5)\\ -25\log(5) - \frac{9}{2} + 25\log(6)\\ -125\log(6) + \frac{137}{6} + 125\log(5)\\ -625\log(5) - \frac{1367}{12} + 625\log(6)\\ -3125\log(6) + \frac{34187}{60} + 3125\log(5)\\ -15625\log(5) - \frac{11395}{4} + 15625\log(6)\\ -78125\log(6) + \frac{398829}{28} + 78125\log(5)\\ -390625\log(5) - \frac{3988283}{56} + 390625\log(6)\end{matrix}\right]
$$

```python
dec = 6
np.set_printoptions(formatter={'float': ('{: 0.' + str(dec) + 'f}').format})
In_real = np.round(np.array(In_real_expr, dtype=float), dec)[0]
print(In_real)
```

```
[ 0.182322  0.088392  0.058039  0.043139  0.034306  0.028468  0.024325
  0.021233  0.018837]
```

不妨绘制出这里的值：

```python
plt.figure(figsize=(8, 5), facecolor="white")
plt.scatter(n_range, In_real, s=100, color="red", zorder=10, label="real $I_n$")
plt.plot(n_range, In_real, color="blue", label="Connect by line")
plt.legend(loc="upper right", fontsize=15)
plt.xlabel("$n$", fontsize=15)
plt.ylabel("$I_n$", fontsize=15)
plt.grid(True)
plt.title("Real Solution for $n-I_n$")
plt.show()
```



从 0 开始的递推

采用递推公式：

$$I_n = \frac{1}{n} - 5I_{n-1}$$

每一步计算都保留小数点后 6 位：

```
In_from0 = np.zeros(n+1)
I0 = np.round(np.log(6/5), dec)
In_from0[0] = I0
I0
```

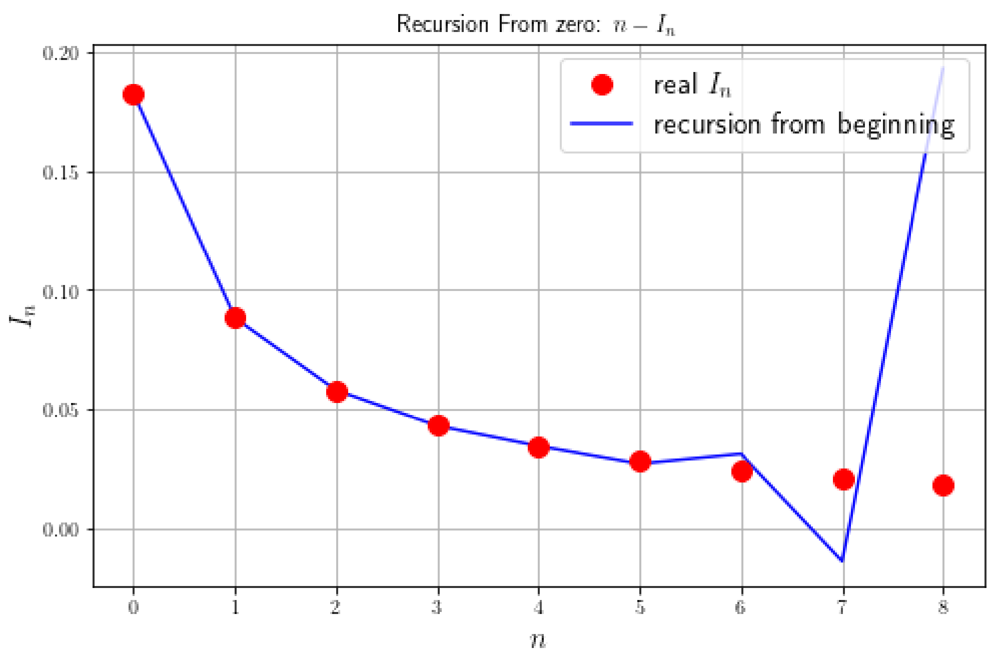
```
0.182322
```

```
for i in range(1, len(In_from0)):
    In_from0[i] = np.round(1/i - 5*In_from0[i-1], dec)
    pass
print(In_from0)
```

```
[ 0.182322  0.088390  0.058050  0.043083  0.034585  0.027075  0.031292
 -0.013603  0.193015]
```

显然发现这里计算出现了负数值，计算发生了不稳定现象，这里绘图比较一下：

```python
plt.figure(figsize=(8, 5), facecolor="white")
plt.scatter(n_range, In_real, s=100, color="red", zorder=10, label="real $I_n$")
# plt.plot(n_range, In_real, color="blue", label="Connect by line")
plt.plot(n_range, In_from0, color="blue", label="recursion from beginning")
plt.legend(loc="upper right", fontsize=15)
plt.xlabel("$n$", fontsize=15)
plt.ylabel("$I_n$", fontsize=15)
plt.grid(True)
plt.title("Recursion From zero: $n-I_n$")
plt.show()
```



从尾开始的递推

反过来使用公式如下:

$$I_{n-1} = \frac{1}{5}\left(\frac{1}{n} - I_n\right)$$

我们可以估计一下 I_8 的值如下:

$$I_8 = \int_0^1 \frac{x^8}{x+5} dx$$

因此:

$$\frac{1}{6}\int_0^1 x^8 dx \leq I_8 \leq \frac{1}{5}\int_0^1 x^8 dx$$

显然可以估计范围如下:

$$I_8 \in \left(\frac{1}{54}, \frac{1}{45}\right)$$

给出其估计值如下:

$$I_8 \approx \frac{1}{2}\left(\frac{1}{45}+\frac{1}{54}\right)=\frac{11}{18\times 5\times 6}$$

```
I8 = np.round(11/18/5/6, dec)
print(I8)
```

```
0.02037
```

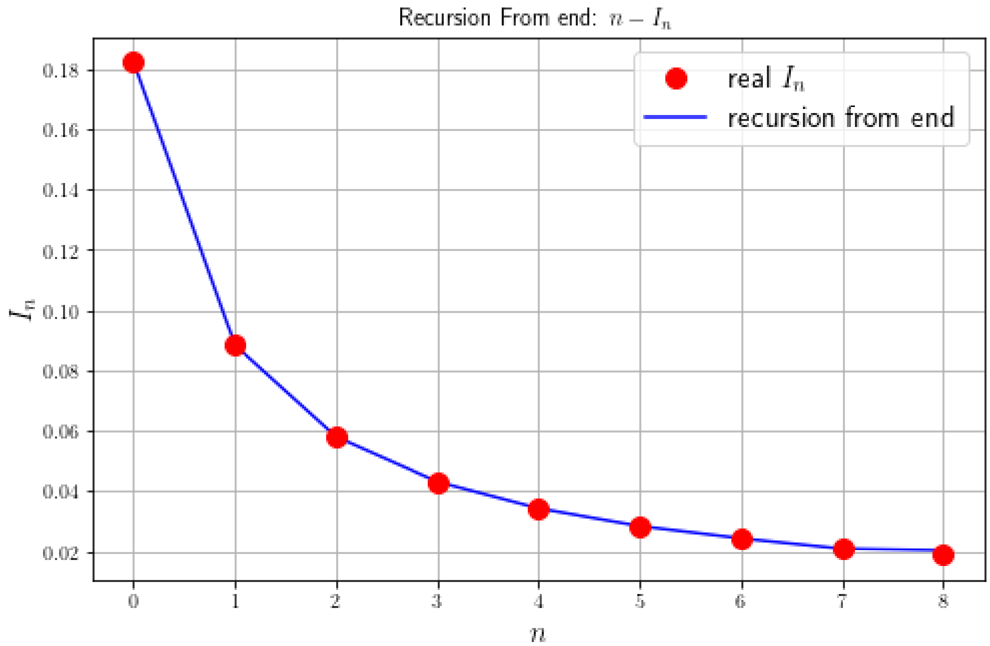
接下来从尾巴处进行递推如下:

```
In_frome = np.zeros(n+1)
In_frome[-1] = I8
for i in range(1, len(In_frome)):
    In_frome[n-i] = np.round((1/(n-i+1) - In_frome[n-i+1])/5, dec)
    pass
print(In_frome)
```

```
[ 0.182322  0.088392  0.058039  0.043138  0.034309  0.028456  0.024386
  0.020926  0.020370]
```

不难看出，似乎和真实值差的值不算大。。。因此画一下如下

```
plt.figure(figsize=(8, 5), facecolor="white")
plt.scatter(n_range, In_real, s=100, color="red", zorder=10, label="real $I_n$")
# plt.plot(n_range, In_real, color="blue", label="Connect by line")
plt.plot(n_range, In_frome, color="blue", label="recursion from end")
plt.legend(loc="upper right", fontsize=15)
plt.xlabel("$n$", fontsize=15)
plt.ylabel("$I_n$", fontsize=15)
plt.grid(True)
plt.title("Recursion From end: $n-I_n$")
plt.show()
```



因此上述的计算表明一件事情，那就是采用误差不会放大的递推计算式子才能保证问题计算的精度，负责只会让误差放大到离谱的地步。与兴趣的同学可以更改程序中 `dec` 值，会发现从 0 开始的递推很大程度上取决于 `dec` 的值。