hw4-task1

- P167，4.9 习题，1,2,3,4,5

```
dec = 6 # 设置每一步计算保留小数点后位数（精度，可以自己调整）
import numpy as np
np.set_printoptions(formatter={'float': ('{: 0.' + str(dec) + 'f}').format})
import matplotlib.pyplot as plt
import matplotlib as mpl
mpl.rcParams['text.usetex'] = True
import sympy as sp
```

习题4.1

四个点插值3次方，是没有误差的，因此插值结果就是原函数：

$$f(x) = 56x^3 + 24x^2 + 5$$

习题4.2

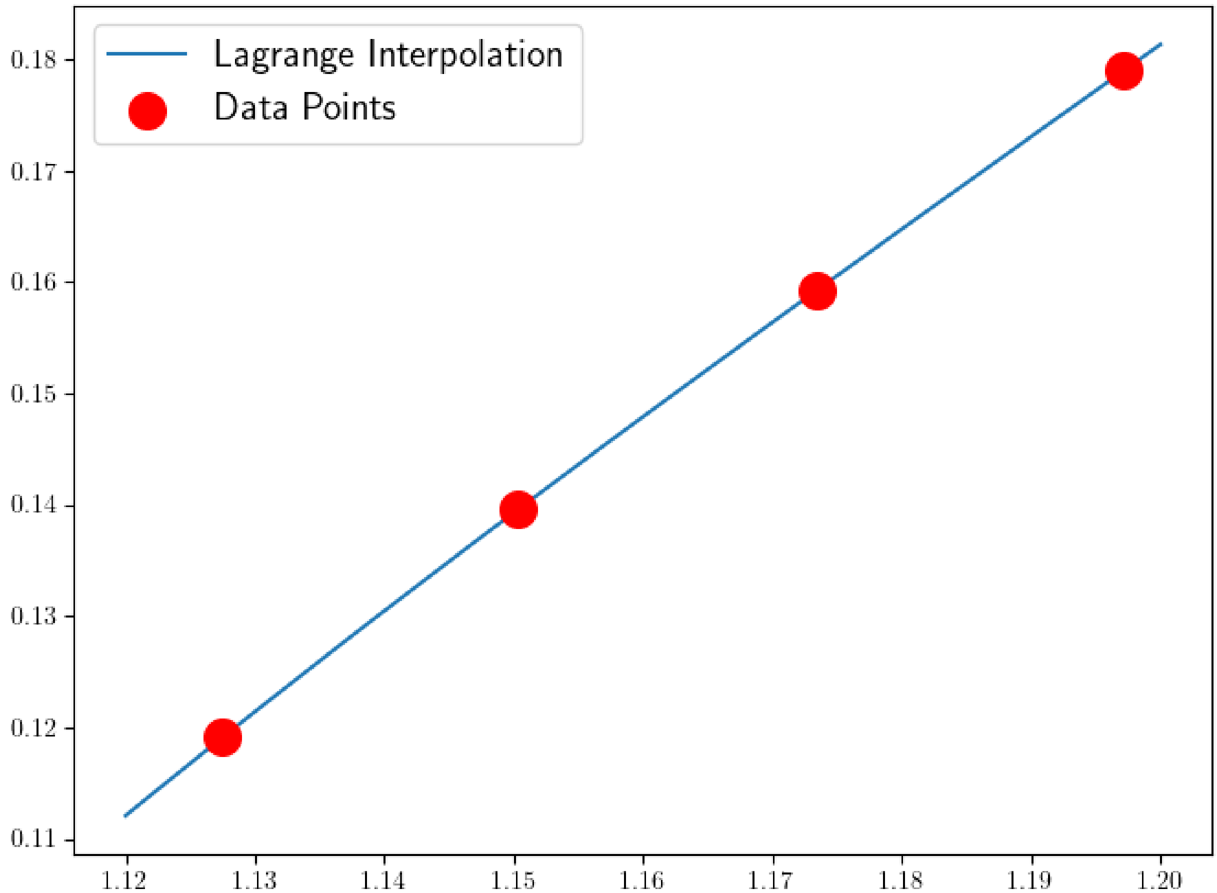
```
# 简单定义一个 lagrange 插值类型
class LagInterpolation:

    def __init__(self, x, y):
        self.x = x
        self.y = y
        self.N = len(x)
        mat = np.zeros([self.N, self.N])
        for i in range(self.N):
            mat[i] = self.x**i
            pass
        mat = mat.T
        self.coeff = np.linalg.solve(mat, self.y)
        pass

    def value(self, x):
```

```python
        X = np.array([x**i for i in range(self.N)])
        return self.coeff @ X
        pass

    def value_array(self, x):
        y = np.zeros(len(x))
        for i in range(len(x)):
            y[i] = self.value(x[i])
            pass
        return y
        pass

    pass
```

```python
x0 = np.array([
    1.1275, 1.1503, 1.1735, 1.1972
])
y0 = np.array([
    0.1191, 0.13954, 0.15932, 0.17903
])
ip = LagInterpolation(x0, y0)
print("多项式系数：", ip.coeff)
```

```
多项式系数： [-13.221562  32.002945 -26.106933   7.287827]
```

```python
xx = np.linspace(1.12, 1.2, 100)
yy = ip.value_array(xx)
plt.figure(figsize=(8, 6))
plt.plot(xx, yy, label="Lagrange Interpolation")
plt.scatter(x0, y0, label="Data Points", color="red", s=200, zorder=100)
plt.legend(fontsize=15)
plt.show()
```




```
print("value at x=1.1300:")
print(ip.value(1.1300))
```

```
value at x=1.1300:
0.12140575555034161
```

习题4.3

第一问

记 $f(x)=x^j, j\leq n$，于是根据据定义有显然其 n 阶拉格朗日插值函数为其本身，因此满足：

$$\sum_{k=0}^{n} x_k^j l_k(x) = x^j \quad 0\leq j\leq n$$

第二问

记函数：

$$f(x) = (x-t)^j$$

同前所述，$f(x)$ 可以用插值函数表示（被一组基函数的线性组合所表达）：

$$f(x) = (x-t)^j = \sum_{k=0}^{n}(x_k - t)^j l_k(x)$$

计算 $f(t) = 0$ 则：

$$0 = \sum_{k=0}^{n}(x_k - t)^j l_k(t)$$

得证。

习题4.4

第一问

记：$f(x) = x^5$，其可以被6个点的`Lagrange`插值精确得到：

$$x^5 = \sum_{j=0}^{5} x_j^5 l_j(x)$$

因此：

$$0 = \sum_{j=0}^{5} x_j^5 l_j(0)$$

第二问

同4.3，为0。

第三问

如第一问所述，对于 $j \in \{0,1,2,3,4,5\}$ 有：

$$x^j = \sum_{i=0}^{5} x_i^j l_i(x)$$

因此：

$$\text{原式} = x^5 + 2x^4 + x^3 + 1$$

让我们来测试一下，先创建 6 个点

```python
n = 5
x = sp.symbols('x')
xk = sp.symbols('x_0:'+str(n+1))
X = sp.Matrix(xk)
X
```

$$\left[\begin{matrix}x_{0}\\x_{1}\\x_{2}\\x_{3}\\x_{4}\\x_{5}\end{matrix}\right]$$

再得到 6 个插值基函数

```python
def getLagrangeFunction(j, n):
    L = 1
    for i in range(n+1):
        if i != j:
            L *= (x - xk[i]) / (xk[j] - xk[i])
            pass
        pass
    return L
    pass
lX = sp.zeros(X.shape[0], X.shape[1])
for i in range(n+1):
    lX[i] = getLagrangeFunction(i, n)
    pass
lX
```

$$\left[\begin{matrix}\frac{(x-x_{1})(x-x_{2})(x-x_{3})(x-x_{4})(x-x_{5})}{(x_{0}-x_{1})(x_{0}-x_{2})(x_{0}-x_{3})(x_{0}-x_{4})(x_{0}-x_{5})}\\\frac{(x-x_{0})(x-x_{2})(x-x_{3})(x-x_{4})(x-x_{5})}{(-x_{0}+x_{1})(x_{1}-x_{2})(x_{1}-x_{3})(x_{1}-x_{4})(x_{1}-x_{5})}\\\frac{(x-x_{0})(x-x_{1})(x-x_{3})(x-x_{4})(x-x_{5})}{(-x_{0}+x_{2})(-x_{1}+x_{2})(x_{2}-x_{3})(x_{2}-x_{4})(x_{2}-x_{5})}\\\frac{(x-x_{0})(x-x_{1})(x-x_{2})(x-x_{4})(x-x_{5})}{(-x_{0}+x_{3})(-x_{1}+x_{3})(-x_{2}+x_{3})(x_{3}-x_{4})(x_{3}-x_{5})}\\\frac{(x-x_{0})(x-x_{1})(x-x_{2})(x-x_{3})(x-x_{5})}{(-x_{0}+x_{4})(-x_{1}+x_{4})(-x_{2}+x_{4})(-x_{3}+x_{4})(x_{4}-x_{5})}\\\frac{(x-x_{0})(x-x_{1})(x-x_{2})(x-x_{3})(x-x_{4})}{(-x_{0}+x_{5})(-x_{1}+x_{5})(-x_{2}+x_{5})(-x_{3}+x_{5})(-x_{4}+x_{5})}\end{matrix}\right]$$

测试 $x^5+2x^4+x^3+1$（没问题）：

```
s = 0
for i in range(n+1):
    s += x[i]**5 * lx[i]
    s += 2*x[i]**4 * lx[i]
    s += x[i]**3 * lx[i]
    s += lx[i]
    pass
sp.simplify(s)
```

$x^5 + 2x^4 + x^3 + 1$

习题4.5

$$\ln \omega(x) = \sum_{i=0}^{n} \ln(x - x_i)$$

两侧对 x 求导:

$$\frac{\omega'(x)}{\omega(x)} = \sum_{i=0}^{n} \frac{1}{x - x_i}$$

因此:

$$\omega'(x) = \sum_{i=0}^{n} \prod_{j \neq i}^{n} (x - x_j)$$

因此:

$$\omega'(x_k) = \sum_{i=0}^{n} \prod_{j \neq i}^{n} (x_k - x_j)$$

当 $k \neq i$ 时，会发现 $\prod_{j \neq i}^{n} (x_k - x_j) = 0$ ，因此上式化简为:

$$\omega'(x_k) = \prod_{j \neq k}^{n} (x_k - x_j)$$

```

```