Numerical Methods in Engineering

Homework Report

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Contents

1 First Problem

1.1 Problem Statement

Suppose we have a functional:

$$J[u(x,y)] = \iint_B \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2u\right] dxdy \tag{1.1}$$

And boundary condition:

$$u|_{\partial B} = 0 \tag{1.2}$$

where B is a domain in $x-y$ plane enclosed by:

$$y = \pm\frac{\sqrt{3}}{3}x \quad \text{and} \quad x = b \tag{1.3}$$

Let the functional achieve the minimum value.

(1) Write the equivalent governing equation in the domain.

(2) Programming: use finite element method to solve the field u within the domain numerically. In the report, please make a brief description of your code, and illustrate the validity of your results. (Please attach the code in another file.)

1.2 Governing Equation

In $J[u(x,y)]$, the part

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

can be written as:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \nabla u \cdot \nabla u \tag{1.4}$$

thus $J[u(x,y)]$ can be written as:

$$J[u(x,y)] = \int_B (\nabla u \cdot \nabla u - 2u) d\Omega \tag{1.5}$$

considering the expression below:

$$\int_B u\nabla^2 u d\Omega$$

use integrate by part and divergence theory we will have:

$$\begin{aligned}\int_B u\nabla^2 u d\Omega &= \int_B u(\nabla \cdot \nabla u) d\Omega \\ &= \int_{\partial B} u\nabla u \cdot \vec{n} ds - \int_B \nabla u \cdot \nabla u d\Omega\end{aligned} \tag{1.6}$$

at the boundary we have $u|_{\partial B} = 0$, thus we have:

$$\int_B \nabla u \cdot \nabla u d\Omega = -\int_B u \nabla^2 u d\Omega \tag{1.7}$$

denote the operator as below:

$$(u, v) = \iint_B uvdxdy = \int_B uvd\Omega \tag{1.8}$$

thus $J[u(x, y)]$ can be written as:

$$\begin{aligned} J[u(x, y)] &= -\int_B u(\nabla^2 u + 2)d\Omega \\ &= -(\nabla^2 u + 2, u) \\ &= (-\nabla^2 u - 2, u) \end{aligned} \tag{1.9}$$

for the governing equation $Au = f$ the $J[u]$ is as below:

$$Ju = (Au - 2f, u) \tag{1.10}$$

compare eq.1.9 with eq.1.10 we get the governing equation ($A = -\nabla^2, f = 1$):

$$-\nabla^2 u = 1 \tag{1.11}$$

1.3 FEM Method in Triangle Mesh

Devide the domain B into a large number of triangle mesh cells, and use interpolation to get the value of u inside cells through the node value of each triangle cell. Now suppose we have a triangle cell called Ω, which possesses 3 nodes called $1, 2, 3$. The value of u on these nodes are called $u_j (j = 1, 2, 3)$. Thus the approximate value of u inside the cell called $\tilde{u}$ can be written as:

$$\tilde{u} = \sum_{j=1}^{3} \phi_j u_j \tag{1.12}$$

use galerkin method we have in the cell Ω we have:

$$\int_\Omega (A\tilde{u} - f)\phi_i d\Omega = 0 \quad i = 1, 2, 3 \tag{1.13}$$

in this case let $A = -\nabla^2, f = a = 1$, we have:

$$\int_\Omega (\nabla^2 \tilde{u} + a)\phi_i d\Omega = 0 \quad i = 1, 2, 3 \tag{1.14}$$

which is exactly:

$$\sum_{j=1}^{3} u_j \int_\Omega \phi_i \nabla^2 \phi_j d\Omega + a \int_\Omega \phi_i d\Omega = 0 \tag{1.15}$$

the part $\int_\Omega \phi_i \nabla^2 \phi_j d\Omega$ can be written as:

$$\int_\Omega \phi_i \nabla^2 \phi_j d\Omega = \int_{\partial\Omega} \phi_i \nabla \phi_j \cdot \vec{n} ds - \int_\Omega \nabla \phi_i \cdot \nabla \phi_j d\Omega \tag{1.16}$$

for the value $\phi_i \nabla \phi_j \cdot \vec{n}$ will be compensated by the cell opposite this edge, we can negelect this part, thus eq.1.16 turn to be:

$$\sum_{j=1}^{3} u_j \int_\Omega \nabla \phi_j \cdot \nabla \phi_i d\Omega = a \int_\Omega \phi_i d\Omega \tag{1.17}$$

in vector form that is:

$$\begin{bmatrix} \int_\Omega \nabla \phi_i \cdot \nabla \phi_1 d\Omega & \int_\Omega \nabla \phi_i \cdot \nabla \phi_2 d\Omega & \int_\Omega \nabla \phi_i \cdot \nabla \phi_3 d\Omega \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = a \int_\Omega \phi_i d\Omega \tag{1.18}$$

in triangle mesh the function ϕ_k is known, denote $2A$ as:

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \tag{1.19}$$

thus $\phi_1 \sim \phi_3$ is:

$$\phi_1 = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{2A} \quad \phi_2 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix}}{2A} \quad \phi_3 = \frac{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}}{2A} \tag{1.20}$$

here the order of node $1, 2, 3$ should be anti-clockwise. Finally we just need to add the part matrix into the general matrix to attain:

$$[K]u = b \tag{1.21}$$

the u at the boundary is known as zero, thus we need to cut the matrix $[K]$ into a smaller one. The value of u at nodes inside the domain can be calculated by the inversion of the modified $[K]$ and modified b. Denote the modified $[K]$ after removing the boundary elements as $[K]_m$ and modified b as b_m, the u_m is exactly:

$$u_m = [K]_m^{-1} b_m \tag{1.22}$$

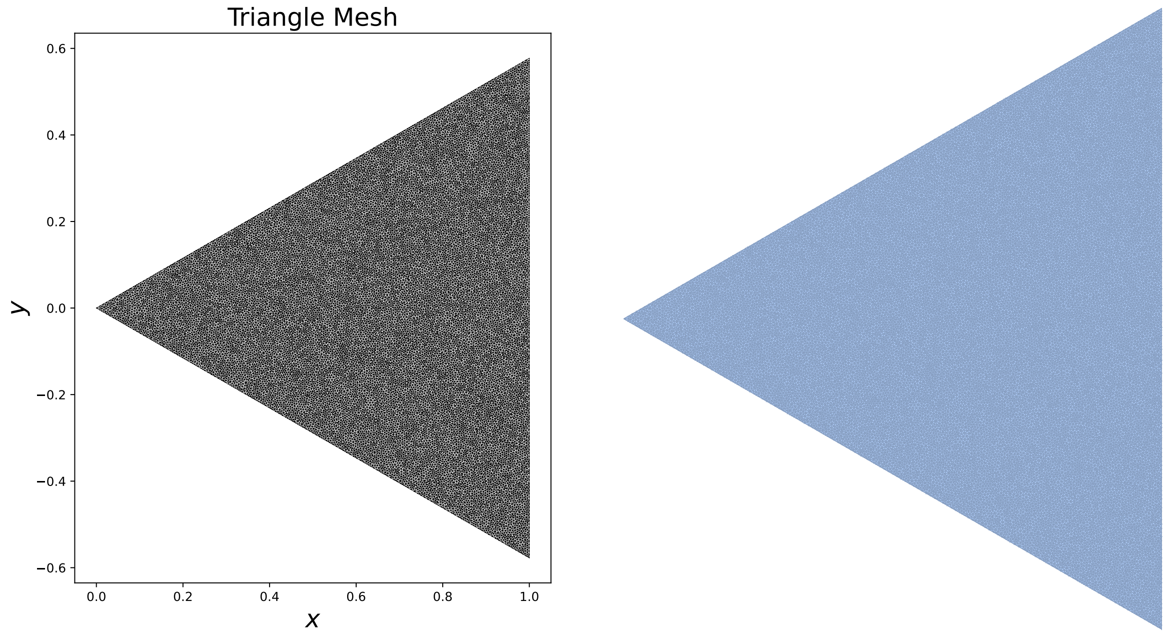
Thus through interpolation we get the distribution of u over the domain B.

1.4 Programming: Julia Code

For this work, I choose to use Julia language to finish the task. I will explain the procedure of my codes as below.

1.4.1 Mesh Part

The FEM method depends on the triangle mesh. I choose to use Mathematica software to generate of mesh. When $b = 1$, the geometry of the domain and the triangle mesh figure is shown as below in 1. This mesh contains 22689 nodes and 44895 cells.



(a) Matplotlib Plot (b) Mathematica Plot

Figure 1: Triangle mesh generated by Mathematica

In the path 'src', a julia file 'Mesh.jl' is used to read and store the mesh information. Here is the component of code in this file.

1. readfile_Vector_String: function, used to read a file as string vector;
2. readMatrix: function, used to read data from file as a matrix;
3. read_mesh_ply: function, used to read the mesh information; In addition, the mesh format should be 'ply' which generated from mma;
4. Mesh: struct, which contains:
    - nnodes: number of nodes;
    - ncells: number of triangle cells;
    - xy: the (x, y) of nodes;
    - icell: the connection information of each triangle's nodes;
5. Mesh: function, used to generate a Mesh struct from mesh file.

1.4.2 Case Part

This part is contained in file 'src//Case.jl' including:

1. Case: struct, used to set the calculation case including:
   - alpha: the angle α which is default as 30°;
   - b: the domain decided by $x = b$ which is default as 1;
   - a: the governing equation as $-\nabla^2 u = a$ which is default as 1;
   - boundary_value: the value of u on boundary which is default as zero.
2. Case: function, generate a default calculation case;
3. whether_on_boundary: function, to judge whether (x, y) is on the boundary of the calculation case.

1.4.3 Solver Part

This part is contained in file 'src//Solver.jl' used to solve the problem including:

1. generate_K_b: function, to generate $[K]$ and b from case and mesh;
2. boundary_node_ID: function, to mark the node at the boundary and store their IDs;
3. solve: solve the value u at each node.

It's noteworthy that julia has amazing speed in loop and matrix inversion. Here for the matrix $[K]$ is a sparse large scale matrix, I use a package 'SparseArrays' to store $[K]$. This saves much memory and the inversion progress is done by 'MKL', the total solution only costs about 20 seconds.

1.4.4 Main Part

This part is contained in file 'src//Main.jl' used to set the case, read the mesh, solve the problem and draw the contour of result. Here I use 'PyPlot' package to call matplotlib to draw the contour of distribution of u. For I write detailed code annotation in this file, I'm not going to expalin it here in the report.

1.4.5 The Explaination of Working Directory

The 'problem1' directory contains four sub-directories:

1. data: contains mesh file 'mesh.ply' and Mathematica notebook file 'task1.nb' which include the validation of problem 1, generation of mesh and Formula derivation. Moreover, to avoid Mathematica not available on your computer, I print the MMA notebook as a pdf;
2. image: contains the figures plot by my julia code and Mathematica code;
3. src: contains my julia FEM code;
4. draft: some test jupyter notebook files.

1.5 Result: Contour of the Julia Code Solution

With help of the api function 'tricontourf' in matplotlib, I draw the contour of u by julia code in fig.2.

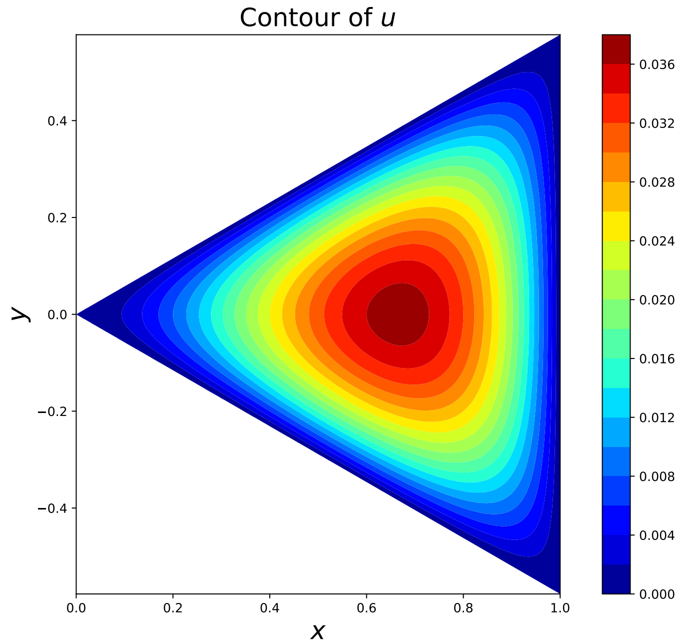


Figure 2: Contour of u of julia code solution

1.6 Validation: Comparison with Mathematica Solution

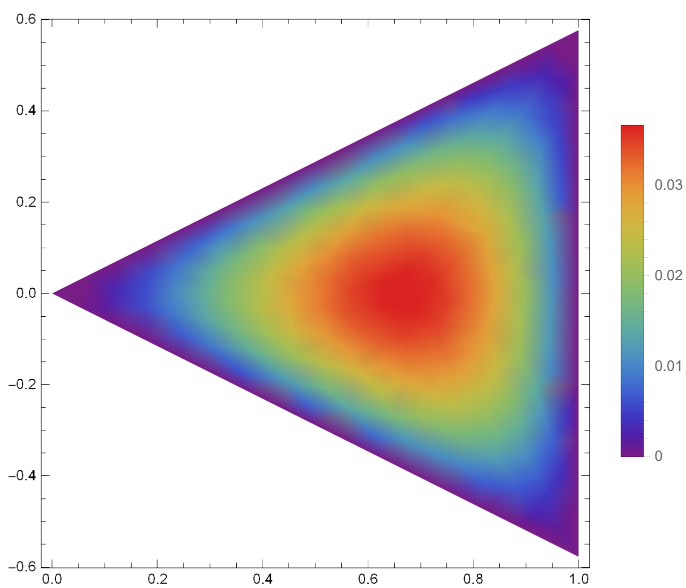


Figure 3: Contour of u of Mathematica solution

Mathematica is powerful in solveing PDE, such problem can be done within several lines of code. With MMA, I define the domain and set the governing equation as well as boundary condition, the numerical solution of the problem is shown in fig.3.

In fig.2 and fig.3 we can easily find that the contour of u is the same. The red part is in the domain center where u is around 0.03, and the blue part is near the boundary where u is zero as the boundary value.

This comparison with Mathematica validate the correction of my julia code. All these files can be found in directory 'problem1'.

2 Second Problem

2.1 Problem Statement

Fig4 represents a double pendulum that is suspended from a block that moves horizontally with a prescribed motion, $x(t)$: Derive the Euler-Lagrange equations of motion of the pendulum when it oscillates in the $x - y$ plane under the action of gravity and prescribed motion.

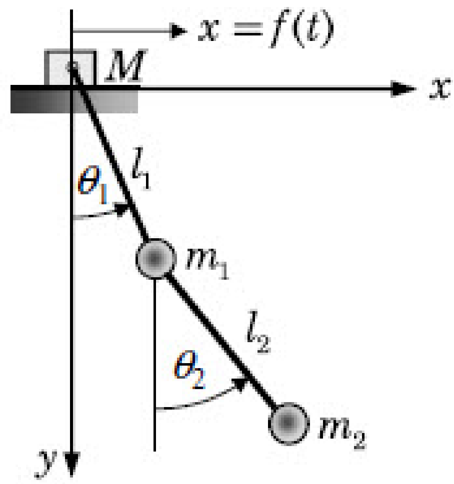


Figure 4: problem 2

2.2 Hamilton's principle

Hamilton's principle is one of the most fundamental principles in vibration analysis. It leads to the basic equations of dynamics and elasticity. It is based on the assumption that when a system moves from a state at a time t_1 to a new state at the time t_2. In a Newtonian route, then the actual route out of all the possible ones obeys stationarity. This condition leads to Hamilton's principle:

$$\delta \int_{t_1}^{t_2} (T - V)dt = \delta \int_{t_1}^{t_2} Ldt = 0 \tag{2.1}$$

where T is the kinetic energy of the system and V is the potential energy of the system. Here L represents:

$$L = L(t, q_j, \dot{q}_j) \tag{2.2}$$

where q_j represents a freedom and $\dot{q}_j = \frac{dq_j}{dt}$. Denote H as:

$$\delta H = \int_{t_1}^{t_2} \delta Ldt \tag{2.3}$$

thus we have:

$$\delta H = \int_{t_1}^{t_2} \sum_{j=1}^{N} \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt = 0 \tag{2.4}$$

here $\delta\dot{q}_j$ represents a variation of $\dot{q}_j$:

$$\int_{t_1}^{t_2} \delta\dot{q}_j dt = \delta q_j|_{t_1}^{t_2} = 0 - 0 \tag{2.5}$$

use integrate by part we will have:

$$\begin{aligned}\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_j}\delta\dot{q}_j dt &= \frac{\partial L}{\partial \dot{q}_j}\delta q_j\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j dt \\ &= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j dt\end{aligned} \tag{2.6}$$

Finally we have δH as below:

$$\delta H = -\sum_{j=1}^{N}\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j dt = 0 \tag{2.7}$$

For δq_j is arbitary, this requires that:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \quad j = 1, 2\cdots, N \tag{2.8}$$

which is so-called 'Lagrange Equation'.

2.3 Formula Derivation

In this case, denote:

$$\begin{cases} q_1 = \theta_1 \\ q_2 = \theta_2 \end{cases} \tag{2.9}$$

thus $\vec{v}_1$, the velocity of m_1 is:

$$\vec{v}_1 = (l_1\dot{\theta}_1\cos\theta_1 + \dot{f})\vec{i} + l_1\dot{\theta}_1\sin\theta_1\vec{j} \tag{2.10}$$

where $\vec{i}$ is the unit vector of x direction and $\vec{j}$ is the unit vector of y's opposite direction in fig.4. Similarly velocity of m_2 is:

$$\vec{v}_2 = (l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + \dot{f})\vec{i} + +(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2)\vec{j} \tag{2.11}$$

In this case the total the kinetic energy of the system T is:

$$T = \frac{1}{2}m_1\vec{v}_1\cdot\vec{v}_1 + \frac{1}{2}m_2\vec{v}_2\cdot\vec{v}_2 \tag{2.12}$$

Here the kinetic of M is neglected because the motion of M is prescribed as $x = f(t)$, this case has no difference with the case without M. And the potential energy of the system V is:

$$V = -m_1 g l_1\cos\theta_1 - m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \tag{2.13}$$

Thus the L is:

$$L = T - V \tag{2.14}$$

From eq.2.8 we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_j}\right) - \frac{\partial L}{\partial \theta_j} = 0 \quad j = 1, 2 \tag{2.15}$$

Use Mathematica I get the equation:

$$\begin{cases} (m_1 + m_2)(\ddot{f}\cos\theta_1 + g\sin\theta_1 + l_1\ddot{\theta}_1) + l_2 m_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + l_2 m_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) = 0 \\ \ddot{f}\cos\theta_2 + g\sin\theta_2 - l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + l_1\ddot{\theta}\cos(\theta_1 - \theta_2) + l_2\ddot{\theta}_2 = 0 \end{cases} \tag{2.16}$$

2.4 Understanding: Equivalent Gravity Field

In eq.2.16, we find a part $\ddot{f}\cos\theta_j + g\sin\theta_j$. This part actually has a pratical physical meaning: equivalent gravity field.

For a non-inertial frame, the acceleration $\vec{a}$ of this frame actually perform a function just like gravity as $-\vec{a}$. In this case, the given motion $f(t)$'s second derivative $\ddot{f}$ performs as an acceleration. In addition, this acceleration together with gravity g perform as a new equivalent gravity g' to this frame.

$$\vec{g'} = \vec{g} - \vec{a} = -\ddot{f}\vec{i} - g\vec{j} \tag{2.17}$$

thus $\ddot{f}\cos\theta_j + g\sin\theta_j$ can be seen as the projection of equivalent gravity on the direction of beam.

Fig.5 explains the vector addition.

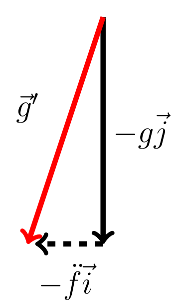


Figure 5: Sketch of equivalent gravity $\vec{g'}$

The system can be seen as a double pendulum that act under the equivalent gravity.

2.5 A Case: Solution From Mathematica and Discussion

Here gives a simple case, where $f(t)$ is as below:

$$f(t) = \frac{3}{2}t^2 + t \tag{2.18}$$

which means the acceleration of the frame is:

$$a = \ddot{f} = 3m/s^2 \tag{2.19}$$

Other parameters are shown as follow:

- $m_1 = 2kg$, $m_2 = 1kg$;
- $g = 9.8m/s^2$;
- $l_1 = 50cm$, $l_2 = 30cm$.

and set the initial condition:

$$\theta_1(0) = 10^\circ \quad \theta_2(0) = 20^\circ \quad \dot{\theta}_1(0) = 2rad/s \quad \dot{\theta}_2(0) = 1rad/s \tag{2.20}$$

Using Mathematica's 'NDSolve' function to solve this ODE numerically. And the simulation time $t = 0 \sim 10s$. $\theta_1(t), \theta_2(t)$'s variation is plotted as follow:

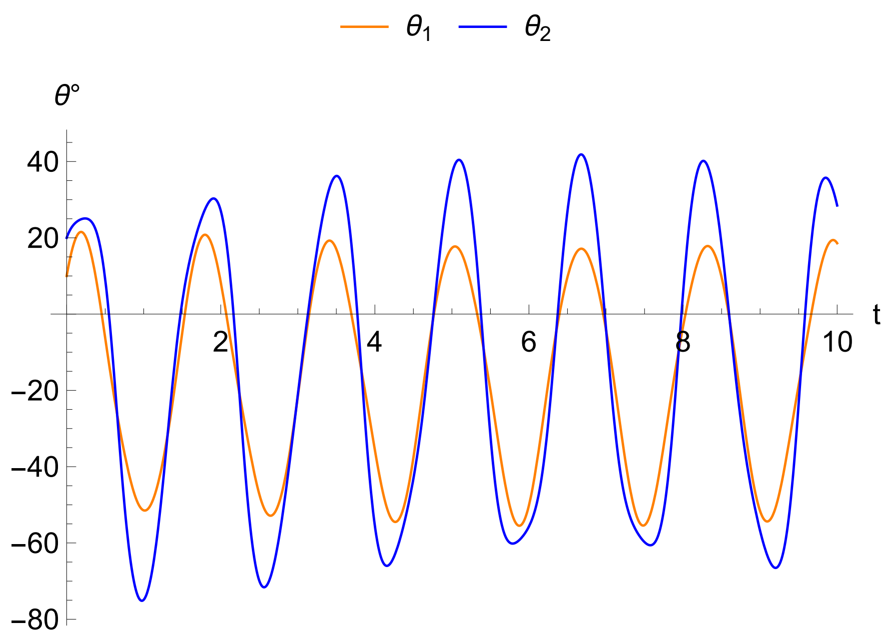
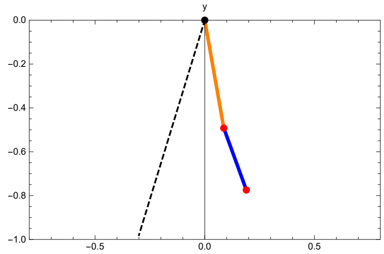


Figure 6: The variation θ_1 and θ_2 within time

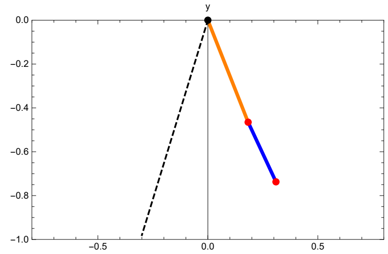
The lines of θ_1 and θ_2 fluctuate around $-15 \sim -20$, which is exactly:

$$\alpha = -\arctan\frac{\ddot{f}}{g} = -\arctan\frac{3}{9.8} \approx -17.02^\circ \tag{2.21}$$

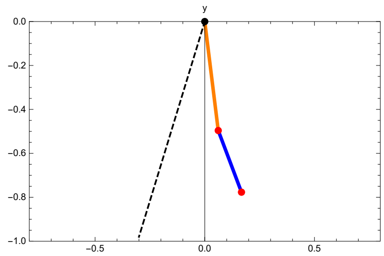
Below is the figure at different time, of which the dashed line means the direction of equivalent gravity. These series figures illustrate that the prescribed motion $x = f(t)$ actually adds an extra acceleration to the two pendulum system. You can find all these pictures , vivid gif and Mathematica calculation notebook in path 'problem2'.



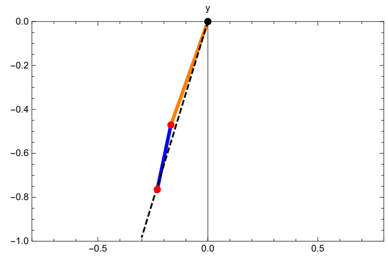
(a) $t = 0.0s$



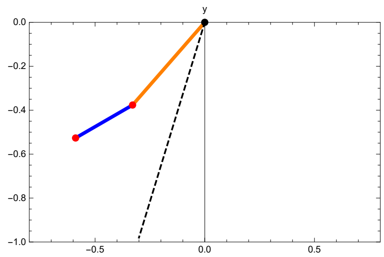
(b) $t = 0.2s$



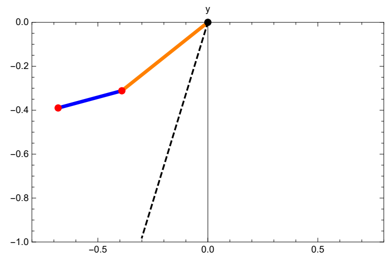
(c) $t = 0.4s$



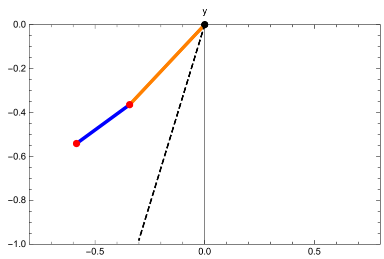
(d) $t = 0.6s$



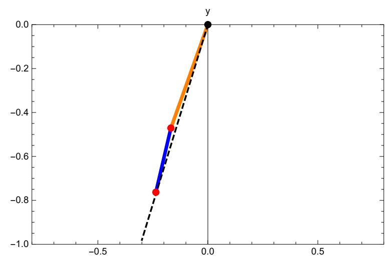
(e) $t = 0.8s$



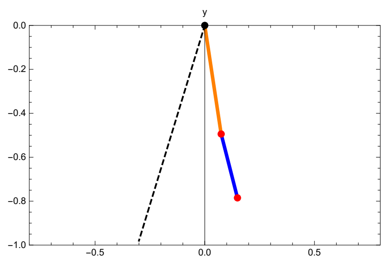
(f) $t = 1.0s$



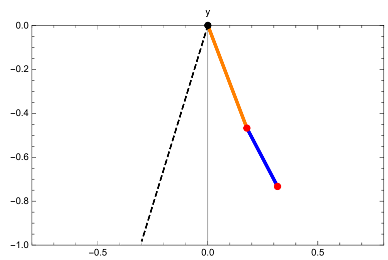
(g) $t = 1.2s$



(h) $t = 1.4s$



(i) $t = 1.6s$



(j) $t = 1.8s$

Figure 7: Position of ths system at different time

3 Third Problem

3.1 Problem Statement

Use Bernoulli-Euler beam elements to analyze (i.e., determine deflections and slopes at the nodes, and reactions at the supports of) the beam structures shown in Fig.8. In the report, please make a brief description of your code, and illustrate the validity of your results. (Please attach the code in another file.)

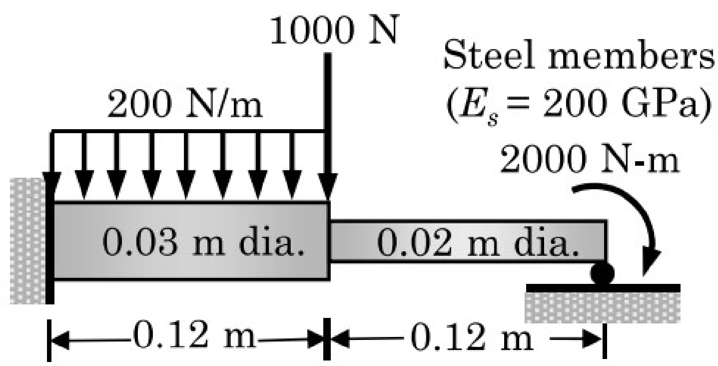


Figure 8: problem 3

3.2 Governing Equation

Let's see how the governing equation is prescribed. We first denote the deflection of beam as w while angle as θ, which satisfies:

$$\theta = \frac{dw}{dx} \tag{3.1}$$

The relation of curvature and radius of curvature (ROC) ρ is:

$$\frac{1}{\rho} = \frac{d\theta}{dx} \tag{3.2}$$

Fig.9 shows a simple draft of the bending of beam.

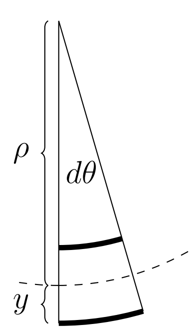


Figure 9: Bending of the beam

In fig.9 $\rho d\theta = dx$ which means there's no change on neutral layer. Thus the strain $\varepsilon(y)$ at position y is:

$$\begin{aligned} \varepsilon(y) &= \frac{dl}{dx} \\ &= \frac{(\rho + y)d\theta - \rho d\theta}{\rho d\theta} \\ &= \frac{y}{\rho} \end{aligned} \tag{3.3}$$

By virture of Hooke's law we have the tensor $\sigma(y)$:

$$\sigma(y) = E\varepsilon(y) = E\frac{y}{\rho} \tag{3.4}$$

The bending moment on the cross section of the beam is equal to the bending moment produced by sum of $\sigma(y)$:

$$\begin{aligned} M &= \int_A \sigma(y) y dA \\ &= \frac{E}{\rho}\int_A y^2 dA \end{aligned} \tag{3.5}$$

For a circular section it's obvious that:

$$\int_A y^2 dA = \int_A z^2 dA \tag{3.6}$$

and use integrate in polar coordinates we have:

$$\begin{aligned} \int_A y^2 dA + \int_A z^2 dA &= \int_A R^2 dA \\ &= \int_0^{2\pi} d\theta \int_0^r R^3 dR \\ &= \frac{\pi}{2} r^4 \\ &= \frac{\pi}{32} d^4 \end{aligned} \tag{3.7}$$

Denote J_z be $\int_A y^2 dA$ we will have its value in the case of circular section that:

$$J_z = J_y = \frac{\pi}{64} d^4 \tag{3.8}$$

Thus the relation of distribution over the beam $M(x)$ with $w(x)$ can be written as:

$$\frac{M}{EJ} = \frac{dw^2}{dx^2} \tag{3.9}$$

For the distribution of F and q is the first and second derivative of $M(x)$, finally we have the governing equation of the beam:

$$\frac{d^4 w}{dx^4} = \frac{q}{EJ} \tag{3.10}$$

3.3 Analytical Solution: Classical Mechanics of Materials

By virture of classical theory of mechanics of materials we will have the analytical solution. Superposition principle reveals that the external forces can be applied on this beam system separately.

Denote that:

$$q = 200N/m \quad F = 1000N \quad M = 2000N \cdot m \quad l = 0.12m \tag{3.11}$$

and J_1 represents the moment of inertia of thet left hand side's beam and J_2 for the right hand side. Denote the ground reaction force of the right point is N. The effect of q, F, M, N to this cantilever beam system (hang on by the left side) is to make the right side of the beam's displacement $w_r = 0$.

Use superposition principle, let's consider the effect of q on the right hand's displacement $w_{r.q}$ without other external force:

$$w_{r.q} = \frac{ql^3}{6EJ_1} \cdot l + \frac{ql^4}{8EJ_1} \tag{3.12}$$

where the first part means the angle at the interface of the two beams and the second part means the displacement at the interface. Similarly we have $w_{r.F}$:

$$w_{r.F} = \frac{Fl^2}{2EJ_1} \cdot l + \frac{Fl^3}{3EJ_1} \tag{3.13}$$

and $w_{r.M}$ equals to:

$$w_{r.M} = \frac{Ml}{EJ_1} \cdot l + \frac{Ml^2}{2EJ_1} + \frac{Ml^2}{2EJ_2} \tag{3.14}$$

where the first and second part represent the displacement caused by a equivalent force system of M to the beam system's left hand. Meanwhile the third part represents the M's effect on the right hand of the beam system. Similarly we have the $w_{r.N}$ as below:

$$w_{r.N} = -\left[\left(\frac{Nl^2}{2EJ_1} + \frac{(Nl) \cdot l}{EJ_1}\right) \cdot l + \frac{Nl^3}{3EJ_1} + \frac{(Nl) \cdot l^2}{2EJ_1} + \frac{Nl^3}{3EJ_2}\right] \tag{3.15}$$

The '-' represents that the N is opposite to the direction of F which make the displacement w_r be negative. Thus we get the value of N:

$$N = \frac{7J_2l^2q + 20J_2lF + (12J_1 + 36J_2)M}{(8J_1 + 56J_2)L} \tag{3.16}$$

Set the origin point $(0, 0)$ at the left side of the beam. Let's first calculate the distribution of $M(x)$. $M(0)$ is equal to:

$$M_0 = M(0) = -\frac{1}{2}ql^2 - Fl - 2Nl - M \tag{3.17}$$

as well as $F(0)$ is equal to:

$$F_0 = F(0) = ql + F - N \tag{3.18}$$

Thus the distribution of $M(x)$ over the beam is:

$$M(x) = \begin{cases} -\dfrac{1}{2}qx^2 + F_0x + M_0 & 0 < x < l \\ -ql\left(x - l + \dfrac{l}{2}\right) - F(x-l) + F_0x + M_0 & l \le x < 2l \end{cases} \tag{3.19}$$

Denote $EJ(x)$ as:

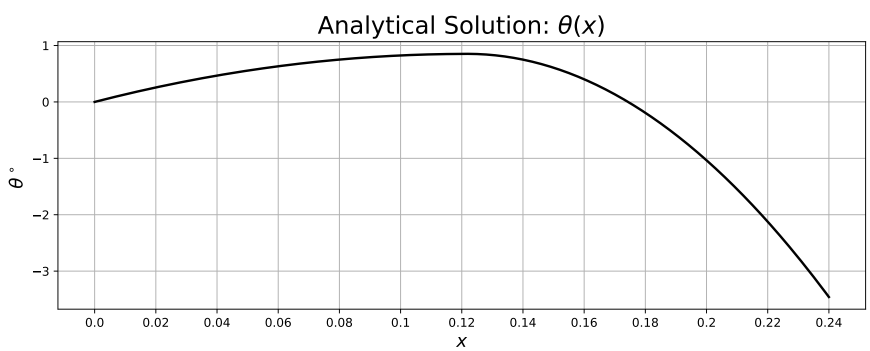
$$EJ(x) = \begin{cases} E_sJ_1 & 0 < x < l \\ E_sJ_2 & l < x < 2l \end{cases} \tag{3.20}$$

use eq.3.9 we will have the analytical solution of $\theta(x)$ and $w(x)$. For the symbolic equation is complex, I just write down the equation numerically:

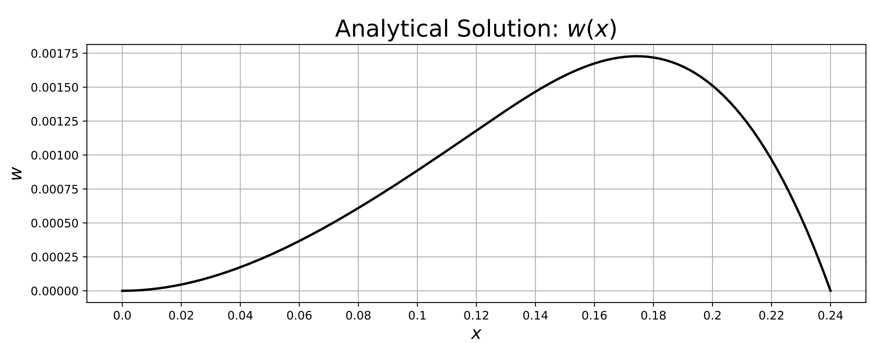
$$\theta(x) = \begin{cases} -0.00419174x^3 - 0.999404x^2 + 0.243843x & 0 \le x < 0.12 \\ -5.38543x^2 + 1.31177x - 0.0649994 & 0.12 \le x \le 0.24 \end{cases} \tag{3.21}$$

$$w(x) = \begin{cases} -0.00104793x^4 - 0.333135x^3 + 0.121922x^2 & 0 \le x < 0.12 \\ -1.79514x^3 + 0.655884x^2 - 0.0649994x + 0.00263701 & 0.12 \le x \le 0.24 \end{cases} \tag{3.22}$$

Draw the analytical solution of $\theta(x)$ and $w(x)$ in fig.10.



(a) analytical solution of $\theta(x)$



(b) analytical solution of $w(x)$

Figure 10: Analytical solution

This part is done by Mathematica, you may found it in directory 'problem3//data//task3_part1.nb'.

3.4 FEM Method: Hermite Interpolation

Consider a 1D element (cell) of which the left node's position is x_1 and the right node's position x_2, denote the length of the cell is $h = x_2 - x_1$, which is shown in fig.11.

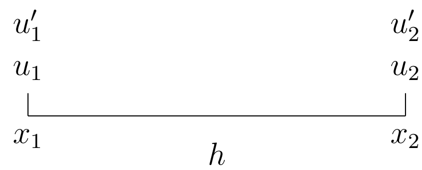


Figure 11: Hermite interpolation cell

Assuming that a distribution of parameter u over the cell $u(x)$ satisfies a cubic function:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \tag{3.23}$$

Assume $u(x)$ at the boundary is u_1 and u_2, as well as $\frac{du(x)}{dx}$'s value u_1', u_2'. Thus we have:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{bmatrix} \tag{3.24}$$

Thus $u(x)$ can be written as:

$$u(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_1' \\ u_2 \\ u_2' \end{bmatrix} \tag{3.25}$$

Denote that:

$$\begin{bmatrix} \phi_{10}(x) & \phi_{11}(x) & \phi_{20}(x) & \phi_{21}(x) \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{bmatrix}^{-1} \tag{3.26}$$

thus the $u(x)$ can be written as a linear combination of the value on the node. For convenience, denote:

$$\begin{gathered} \mu_1 = u_1 \quad \mu_2 = u_1' \quad \mu_3 = u_2 \quad \mu_4 = u_2' \\ \phi_1(x) = \phi_{10}(x) \quad \phi_2(x) = \phi_{11}(x) \quad \phi_3(x) = \phi_{20}(x) \quad \phi_4(x) = \phi_{10}(x) \end{gathered} \tag{3.27}$$

thus:

$$u(x) = \sum_{j=1}^{4} \phi_j(x)\mu_j \tag{3.28}$$

The governing equation is given as:

$$Au = f \tag{3.29}$$

use galerkin method we will have:

$$\int_{x_1}^{x_2} (Au - f)\phi_i(x)dx = 0 \quad i = 1, 2, 3, 4 \tag{3.30}$$

In the beam's case, governing equation is:

$$\frac{d^4 w(x)}{dx^4} = \frac{q(x)}{EJ(x)} = f(x) \tag{3.31}$$

From eq.3.30 we have:

$$\sum_{j=1}^{4} \mu_j \int_{x_1}^{x_2} \phi_i \frac{d^4\phi_j}{dx^4} dx = \int_{x_1}^{x_2} f\phi_i dx \tag{3.32}$$

use integrate by part we will have:

$$\begin{aligned} \int_{x_1}^{x_2} \phi_i \frac{d^4\phi_j}{dx^4} dx &= \phi_i \frac{d^3\phi_j}{dx^3}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d\phi_i}{dx}\frac{d^3\phi_j}{dx^3} dx \\ &= \phi_i \frac{d^3\phi_j}{dx^3}\bigg|_{x_1}^{x_2} - \frac{d\phi_i}{dx}\frac{d^2\phi_j}{dx^2}\bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2} dx \end{aligned} \tag{3.33}$$

neglect the part $H(\phi_i, \phi_j)|_{x_1}^{x_2}$ we will have:

$$\sum_{j=1}^{4} \mu_j \int_{x_1}^{x_2} \frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2} dx = \int_{x_1}^{x_2} f\phi_i dx \tag{3.34}$$

In the beam's case we have:

$$\begin{bmatrix} \frac{12}{h^3} & \frac{6}{h^2} & -\frac{12}{h^3} & \frac{6}{h^2} \\ \frac{6}{h^2} & \frac{4}{h} & -\frac{6}{h^2} & \frac{2}{h} \\ -\frac{12}{h^3} & -\frac{6}{h^2} & \frac{12}{h^3} & -\frac{6}{h^2} \\ \frac{6}{h^2} & \frac{2}{h} & -\frac{6}{h^2} & \frac{4}{h} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \int_{x_1}^{x_2} \frac{q}{EJ}\phi_1 dx \\ \int_{x_1}^{x_2} \frac{q}{EJ}\phi_2 dx \\ \int_{x_1}^{x_2} \frac{q}{EJ}\phi_3 dx \\ \int_{x_1}^{x_2} \frac{q}{EJ}\phi_4 dx \end{bmatrix} \tag{3.35}$$

Here, if $\frac{q}{EJ}$ is constant, we will have:

$$\begin{bmatrix} \frac{12}{h^3} & \frac{6}{h^2} & -\frac{12}{h^3} & \frac{6}{h^2} \\ \frac{6}{h^2} & \frac{4}{h} & -\frac{6}{h^2} & \frac{2}{h} \\ -\frac{12}{h^3} & -\frac{6}{h^2} & \frac{12}{h^3} & -\frac{6}{h^2} \\ \frac{6}{h^2} & \frac{2}{h} & -\frac{6}{h^2} & \frac{4}{h} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \frac{q}{EJ} \begin{bmatrix} \frac{h}{2} \\ \frac{h^2}{12} \\ \frac{h}{2} \\ -\frac{h^2}{12} \end{bmatrix} \tag{3.36}$$

If there's only a concentrated load F_{ex} at x_2 on this cell, which prescribed that:

$$\lim_{\varepsilon \to 0} [F(x_2) - F(x_2 - \varepsilon)] = F_{ex} \tag{3.37}$$

denote that $F_1 = F(x_1), F_2 = F(x_2)$, obviously that:

$$F(x) = \begin{cases} F_1 & x_1 \le x < x_2 \\ F_2 = F_1 + F_{ex} & x = x_2 \end{cases} \tag{3.38}$$

we can calculate that:

$$\begin{aligned} \int_{x_1}^{x_2} \frac{q}{EJ}\phi_i dx &= \frac{F}{EJ}\phi_i\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{F}{EJ}\frac{d\phi_i}{dx}dx \\ &= \frac{F_2}{EJ}\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{F_1}{EJ}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} - \frac{F_1}{EJ}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} \\ &= \frac{1}{EJ}\begin{bmatrix}0\\-F_1\\F_{ex}\\F_2\end{bmatrix} \end{aligned} \tag{3.39}$$

For F_1, F_2 will be compensated by the cell's neighbourhood, this part can be written as:

$$\int_{x_1}^{x_2} \frac{q}{EJ}\phi_i dx = \frac{1}{EJ}\begin{bmatrix}0\\0\\F_{ex}\\0\end{bmatrix} \tag{3.40}$$

Similarly, suppose we only have a concentrated bending moment M_{ex} at x_2 and use the same symbolic mark:

$$M(x) = \begin{cases} M_1 & x_1 \le x < x_2 \\ M_2 = M_1 + M_{ex} & x = x_2 \end{cases} \tag{3.41}$$

thus:

$$\begin{aligned} \int_{x_1}^{x_2} \frac{q}{EJ}\phi_i dx &= \frac{F}{EJ}\phi_i\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{F}{EJ}\frac{d\phi_i}{dx}dx \\ &= \frac{F}{EJ}\phi_i\Big|_{x_1}^{x_2} - \frac{M}{EJ}\frac{d\phi_i}{dx}\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{M}{EJ}\frac{d^2\phi_i}{dx^2}dx \\ &= \frac{M_1}{EJ}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} - \frac{M_2}{EJ}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} + \frac{M_1}{EJ}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \\ &= \frac{1}{EJ}\begin{bmatrix}0\\0\\0\\-M_{ex}\end{bmatrix} \end{aligned} \tag{3.42}$$

To conclude, we have the relation on this cell part as the format $[K]u = b$:

$$EJ \begin{bmatrix} \frac{12}{h^3} & \frac{6}{h^2} & -\frac{12}{h^3} & \frac{6}{h^2} \\ \frac{6}{h^2} & \frac{4}{h} & -\frac{6}{h^2} & \frac{2}{h} \\ -\frac{12}{h^3} & -\frac{6}{h^2} & \frac{12}{h^3} & -\frac{6}{h^2} \\ \frac{6}{h^2} & \frac{2}{h} & -\frac{6}{h^2} & \frac{4}{h} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = q_{ex} \begin{bmatrix} \frac{h}{2} \\ \frac{h^2}{12} \\ \frac{h}{2} \\ -\frac{h^2}{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_{ex} \\ -M_{ex} \end{bmatrix} \tag{3.43}$$

And the application of q_{ex}, F_{ex}, M_{ex} is shown in fig.12.

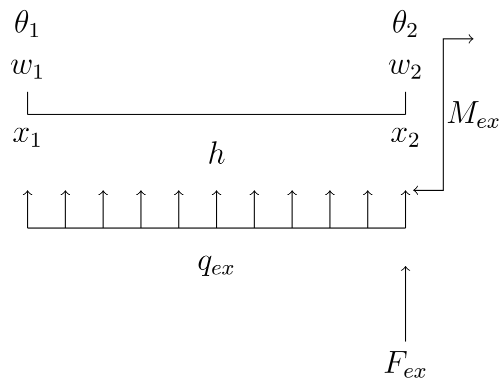


Figure 12: Bernoulli-Euler beam element

You may find all these formula derivation progress in 'problem3/data/task3_part2.nb' Mathematica notebook. In case you have no Mathematica on your computer, I print this file as 'task3_part2.pdf' in which the detailed derivation is included.

3.5 Programming: Julia Code

For convenience, uniform 1-D mesh is considered. Also I choose Julia to finish this project. You may find my codes under directory 'problem3/src/'. Here's a brief description of the julia codes.

1. Case.jl: to set a case for this calculation. There's one struct and two functions in it:
   - Case: struct, use to store the calculation parameters;
   - moment_of_inertia: function, calculate the moment of inertia on circular section when diameter d is given;
   - Case: function, to generate a default Case struct. It is noteworthy that to show the advance of Hermite interpolation, I choose to devide the beam into only devide = 10 cells on each side.
2. HermiteFEM.jl: to solve the problem including:
   - generate_K_b: function, to generate ths $[K]u = b$'s $[K]$ and b from the case;
   - solve: function, to solve the given case;

- hermite_interpolation: function, to generate a interpolation function from the calculation result;
- interpolation_minor: function, refine the value inside the cells. Here 'default_minor=4' which means there will be 4 more nodes' values on each cell generated by Hermite interpolation.

3. Main.jl: the main code to solve the case and draw the picture.

3.6 Julia Code Result: Hermite Interpolation inside the Cell

Denote that cell k at position $[x_k, x_{k+1}]$, and $t = x - x_1, h = x_2 - x_1$. We will have the $w(x), \theta(x)$'s value inside the cell:

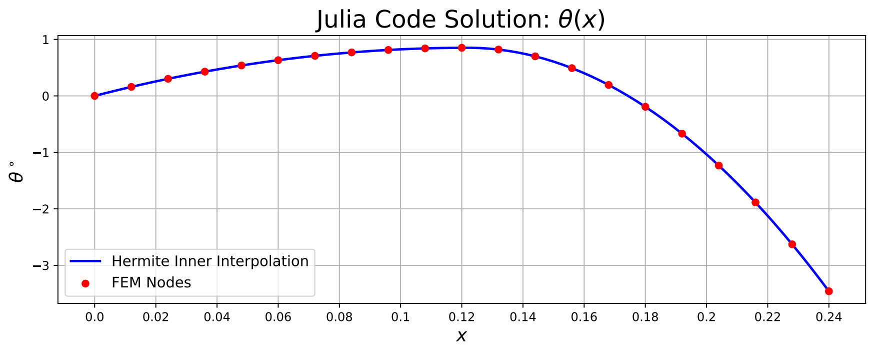
$$w(x) = \frac{(h-t^2)(h+2t)}{h^3}w_k + \frac{t(h-t)^2}{h^2}\theta_k + \frac{t^2(3h-2t)}{h^3}w_{k+1} + \frac{t^2(t-h)}{h^2}\theta_{k+1} \quad x_1 \leq x < x_2 \tag{3.44}$$

$$\theta(x) = \frac{6t(t-h)}{h^3}w_k + \frac{(h-3t)(h-t)}{h^2}\theta_k + \frac{6t(h-t)}{h^3}w_{k+1} + \frac{t(3t-2h)}{h^2}\theta_{k+1} \quad x_1 \leq x < x_2 \tag{3.45}$$

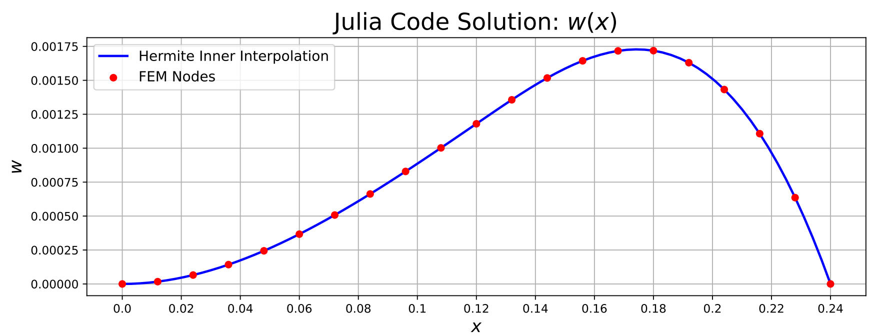
Such derivation is also included in file 'problem3//data//task3_part2.nb' Mathematica notebook.

Within Julia code, I separate the 1-D domain into 21 nodes and 20 cells, in which 10 for the left side's beam and 10 for the right side's beam. By virture of package 'SparseArrays.jl', it's possible to devide the domain into large cells. I've tested the case of 10001 nodes and the total run time of solution is only 0.24 seconds on my 8th Intel CPU 8750H. However, to show the advance of Bernoulli-Euler beam elements, only 21 nodes are given. You may change the number of nodes by modify the value 'devide' in given struct case.

Fig.13 shows the julia code solution, of which the red points are the interpolation nodes and the blue curve is generated by inner hermite interpolation.



(a) julia code solution of $\theta(x)$

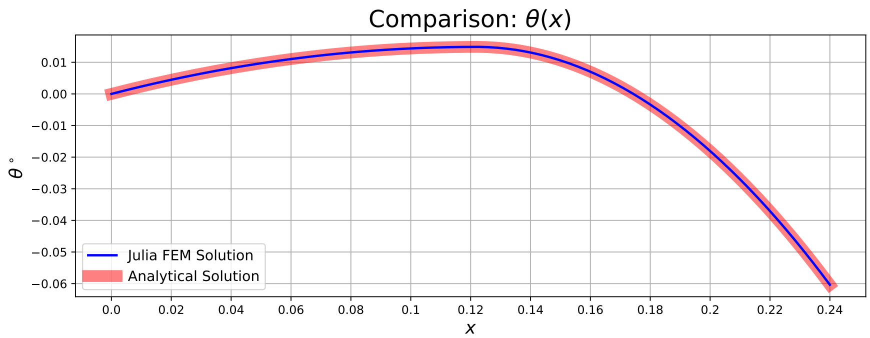


(b) julia code solution of $w(x)$

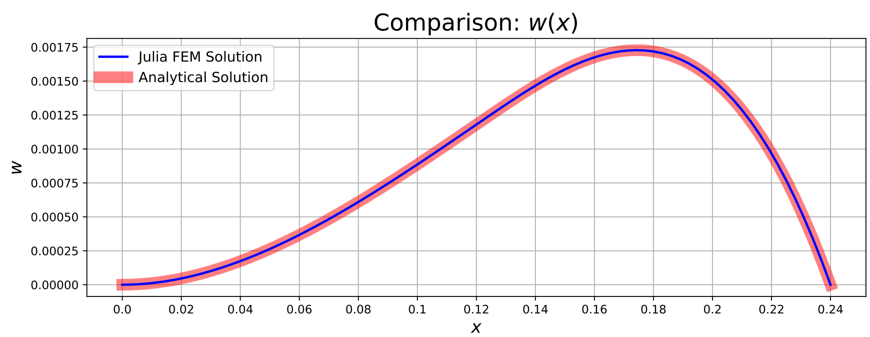
Figure 13: Julia code solution

3.7 Validation: Comparison with Analytical Solution

Draw the analytical curve and julia FEM method code's curve in the same figure, we will have the comparison in fig.14.



(a) julia code solution of $\theta(x)$



(b) julia code solution of $w(x)$

Figure 14: Comparison with analytical solution