1. Deduction and Programming: consider the boundary value problem for the function $u(\boldsymbol{x})$ in two dimensional domain D enclosed by the surface S, governed by the Poisson's equation and the boundary conditions as follows.

$$\nabla \bullet \nabla u(\boldsymbol{x}) + \bar{f}(\boldsymbol{x}) = 0 \quad \text{for} \quad \boldsymbol{x} \in D \tag{1}$$

$$u(\boldsymbol{x}) = \bar{u}(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in S_u \tag{2}$$

$$\boldsymbol{n} \bullet \nabla u(\boldsymbol{x}) = \bar{t}(\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{x} \in S_t = S \setminus S_u \tag{3}$$

where $\bar{f}$ and $\bar{t}$ are given values, and $\boldsymbol{n}$ is the normal vector to the surface S_t.

(1) Show that eqs. (1) to (3) can be obtained by minimizing the total potential energy U defined by:

$$U[u] = \int_D \left[\frac{1}{2} \left\{ \left(\frac{\partial u(\boldsymbol{x})}{\partial x_1} \right)^2 + \left(\frac{\partial u(\boldsymbol{x})}{\partial x_2} \right)^2 \right\} - u(\boldsymbol{x}) \bar{f}(\boldsymbol{x}) \right] dV - \int_{S_t} u(\boldsymbol{x}) \bar{t}(\boldsymbol{x}) dS \tag{4}$$

with the boundary conditions given by eqn (2).

(2) Let $\tilde{u}$ be the solution for the following weak form:

$$\int_D \left[\nabla w(\boldsymbol{x}) \bullet \nabla \tilde{u}(\boldsymbol{x}) - w(\boldsymbol{x}) \bar{f}(\boldsymbol{x}) \right] dV - \int_{S_t} w(\boldsymbol{x}) \bar{t}(\boldsymbol{x}) dS = 0 \tag{5}$$

where w is an arbitrary weight function. Show that if $w(\boldsymbol{x}) = 0$ and $\tilde{u}(\boldsymbol{x}) = \bar{u}(\boldsymbol{x})$ on S_u, $\tilde{u}$ in eqn (5) coincides with the solution u in eqs. (1) to (3).

(3) Programming: As shown by Fig. 1, let the domain D be the triangular region, in which u satisfies eqn (1) subjected to no body force ($\bar{f} = 0$) and the following boundary conditions:

$$u(\boldsymbol{x}) = 0 \quad \text{on} \quad \boldsymbol{x} \in S_{BC} \tag{6}$$

$$\boldsymbol{n} \bullet \nabla u(\boldsymbol{x}) = 1 \quad \text{on} \quad \boldsymbol{x} \in S_{AB} \tag{7}$$

$$\boldsymbol{n} \bullet \nabla u(\boldsymbol{x}) = 0 \quad \text{on} \quad \boldsymbol{x} \in S_{AC} \tag{8}$$

Use finite element method to determine the distribution of u within domain D and show the validity of your result together with FEM codes.

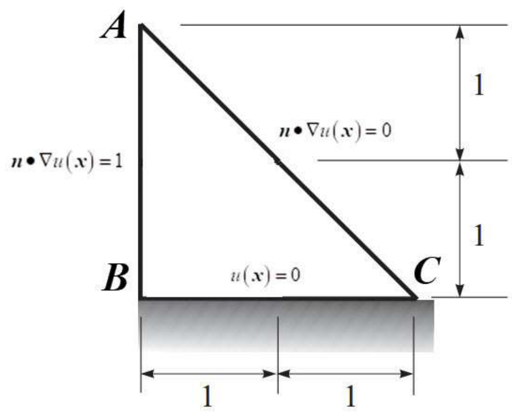

Fig. 1 Problem 1

2. Programming: Use the minimum possible number of Bernoulli-Euler beam elements, unless stated otherwise, to analyze at the beam structures shown in Fig. 2. Show your results together with FEM codes.
(Note: dia. means diameter.)

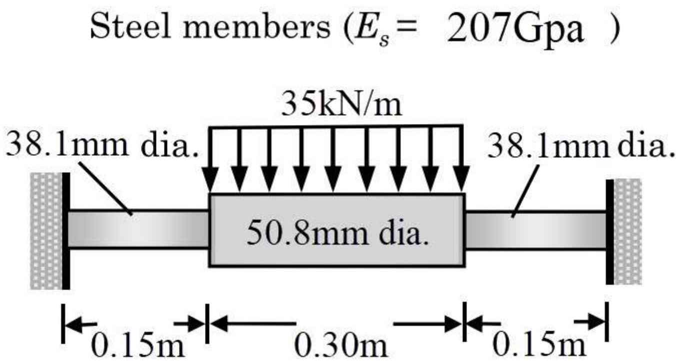

Fig. 2 Problem 2