Credible occurrence probabilities for extreme geophysical events: Earthquakes, volcanic eruptions, magnetic storms

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[1] Statistical analysis is made of rare, extreme geophysical events recorded in historical data – counting the number of events k with sizes that exceed chosen thresholds during specific durations of time τ. Under transformations that stabilize data and model-parameter variances, the most likely Poisson-event occurrence rate, k/τ, applies for frequentist inference and, also, for Bayesian inference with a Jeffreys prior that ensures posterior invariance under changes of variables. Frequentist confidence intervals and Bayesian (Jeffreys) credibility intervals are approximately the same and easy to calculate: $(1/\tau)\left[(\sqrt{k}-z/2)^2, (\sqrt{k}+z/2)^2\right]$, where z is a parameter that specifies the width, $z = 1$ ($z = 2$) corresponding to 1σ, 68.3% (2σ, 95.4%). If only a few events have been observed, as is usually the case for extreme events, then these "error-bar" intervals might be considered to be relatively wide. From historical records, we estimate most likely long-term occurrence rates, 10-yr occurrence probabilities, and intervals of frequentist confidence and Bayesian credibility for large earthquakes, explosive volcanic eruptions, and magnetic storms.
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1. Introduction

[2] Large earthquakes, explosive volcanic eruptions, magnetic storms, and other extreme geophysical events are hazards for humankind, infrastructure, economies, and the activities of civilization [e.g., *Bilham*, 2009; *Self*, 2006; *Baker et al.*, 2008]. With globalization, with the urbanization of a growing world population, and with modern reliance on delicate technological systems, society is becoming increasingly vulnerable to natural hazardous events. Estimates are needed of the occurrence-rate probabilities of extreme geophysical events. To be useful, these probabilities need to be accompanied by estimates of uncertainty. But geophysical events of sufficient size to potentially cause catastrophes are rare, and historical geophysical records are often only reasonably complete and accurate for the past one or two hundred years. Therefore, inferences for occurrence probabilities rely on observations of a small number of events, and uncertainty is difficult to quantify. To address this challenge, in this analysis, we assume that the time occurrence of extreme and rare geophysical events can be described statistically in terms of an idealized Poisson model. We explore seemingly competing frequentist and Bayesian inference methods to obtain analytical estimates of long-term event occurrence rates, occurrence probabilities, and associated confidence and credibility intervals. We illustrate our results using historical counts of extreme geophysical events.

2. Events Counted Over Durations of Time

[3] According to the USGS National Earthquake Information Center (NEIC) catalog, a M9.0 megaquake occurred in Peru in the year 1868. After this date, the catalog is reasonably accurate and complete for megaquakes having magnitudes M≥9.0. Still, we know that the NEIC catalog is especially carefully developed for earthquakes after 1900 [*Engdahl and Villaseñor*, 2002]. We consider both durations of time: from 1868 to present, during which there were 6 megaquakes in the world: M9.0 Arica (1868), M9.0 Kamchatka (1952), M9.5 Chile (1960), the largest earthquake ever recorded by seismometers [*Lomnitz*, 2004], M9.2 Alaska (1964), M9.1 Sumatra (2004), and M9.0 Japan (2011); and, also, from 1900 to present, during which there were 5 megaquakes in the world. The occurrence of megaquakes can be described as a time-random process [e.g., *Michael*, 2011], therefore, we assume Poisson models for the 6 (5) megaquakes since 1868 (1900).

[4] The volcano catalog of *Newhall and Self* [1982] documents large explosive eruptions since the year 1491, including the 1815 super-colossal eruption in Tambora, Indonesia [*Stothers*, 1984], one of the largest eruptions in recorded history. With a Volcanic Explosive Index (VEI) of 7, the Tambora eruption was at least as large as the Baitoushan eruption, China-Korea border, in the year 969 [*Horn and Schmincke*, 2000]. Since 1815 there have been 4 colossal eruptions with VEI = 6: Krakatoa, Indonesia (1883), Santa Maria, Guatemala (1902), Novarupta, Alaska (1912), and Pinatubo, Philippines (1991). No VEI ≥ 6 events are listed in the catalog for years 1491–1814, possibly reflecting incompleteness or inaccuracy. For example, ice cores contain volcanic ash deposited in 1809, but its source is unknown [*Dai et al.*, 1991], and a corresponding event is not in the catalog. The occurrence of large eruptions can be described as a time-random process [e.g., *Cruz-Reyna*, 1991]; we assume Poisson models for the 2 super-colossal (5 colossal) volcanic eruptions since 969 (1815).

[5] The largest magnetic storm ever recorded by magnetometers is the Carrington event of September 1859, for which the ring-current index $-Dst$ reached a maximum

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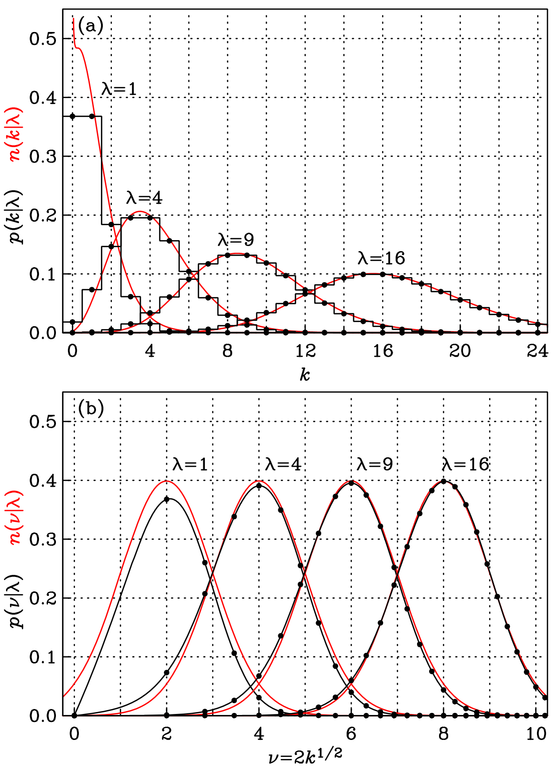

Figure 1. Probability densities as functions of (a) event count k and (b) transformed event count v, each for various Poisson parameters λ: (black) Poisson density functions $p(k|\lambda)$ and $p(\nu|\lambda)$, (red) normal density functions $n(k|\lambda)$ and $n(\nu|\lambda)$.

value of 1760 nT [*Tsurutani et al.*, 2003]. The second largest recorded storm is that of May 1921, maximum $-Dst$ = 900 nT [*Kappenman*, 2006]. The third largest is possibly the March 1989 storm, maximum $-Dst$ = 589 nT, which caused widespread disruption to technological systems [*Allen et al.*, 1989].A storm in September 1909 might have been larger than the 1989 storm, but records required to estimate Dst are (to our knowledge) incomplete; for now, we count 3 superstorms (1859, 1921, 1989) for the 153 yr duration of time since and including the Carrington event. The occurrence of large magnetic storms can be described as a time random process [e.g., *Tsubouchi and Omura*, 2007], with inter-event wait times that are long compared to the solar-cycle modulation of lower levels of geomagnetic activity. We assume that the occurrence of superstorms can be analyzed in terms of a stationary Poisson model, although we acknowledge that, in comparison to megaquakes and colossal eruptions, this assumption might be questioned.

3. Poisson Frequentist Statistics

[6] A time-stationary Poisson model [e.g., *Cox and Lewis*, 1966] can be used to describe the random occurrence of discrete events. Denoting the long-term average rate of occurrence as ρ, then the probability that a count of k events will be realized over a duration of time τ is given by the density function

$$p(k|\lambda) = \frac{\lambda^k}{k!}\exp(-\lambda). \tag{1}$$

The characteristic parameter of the Poisson model is

$$\lambda = \rho\tau, \tag{2}$$

and under a frequentist interpretation of statistics [e.g., *Stuart et al.*, 1999], λ has a fixed value. When it is specified, the probability of future data is predicted. In our case, for earthquakes and volcanoes, the value of λ is a function of the geology and geophysics of the Earth system; for magnetic storms, it is a function of the physics of the combined Earth-Sun system. For an infinite set of independent realizations of a Poisson process, where events are counted in independent periods of time of equal duration, the mean number of events is $\mu = \lambda$, and the event-number variance is $\sigma^2 = \lambda$. In our analysis of extreme geophysical events, we have, for each event type, only a single small-count datum, k, for a single historical duration of time, τ. By assuming, from the start, that extreme geophysical events occur in time as a Poisson process, then we can make an optimal estimate of λ, and we can place "error bars" on that estimate. Details of derivations that follow are given in the auxiliary material.[1]

[7] In Figure 1a (black) we show the density function $p(k|\lambda)$ for various values of λ and as a function of event count k; note that the asymmetry of the variance (skew) for small λ. Useful symmetry can be obtained by making a "variance stabilizing" transformation [e.g., *Stuart et al.*, 1999, chap. 32.38–40] for the Poisson-event data,

$$\nu = 2\sqrt{k} \quad \text{and} \quad k = \tfrac{1}{4}\nu^2. \tag{3}$$

Changing variables gives a gamma function,

$$p(\nu|\lambda) = f(\lambda)\frac{\nu}{2}\frac{\lambda^{\frac{1}{4}\nu^2}}{\Gamma\left(\frac{1}{4}\nu^2+1\right)}\exp(-\lambda). \tag{4}$$

The function $f(\lambda)$ is an adjustment for our having made a transformation from a discrete density function (1) to a function (4) that we are treating as continuous; $f(\lambda)$ can be used to ensure that equation (4) is a properly normalized probability density function. For this, $f(\lambda) = 1 + \frac{1}{4}\exp(-\lambda)$ works well. We note that $f(\lambda) \to 1$ for $\lambda > 1$, and, therefore, we simply take $f(\lambda) = 1$. In Figure 1b (black), we see that the transformed Poisson function $p(\nu|\lambda)$ is nearly symmetrical and invariant for different values of λ, except for translation along the v axis. Expanding (4) in a Taylor series for small $\epsilon = \nu - 2\sqrt{\lambda}$ and using Stirling's approximation of the gamma function, we find that the transformed Poisson function can be approximated by

$$n(\nu|\mu) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(\nu-\mu)^2\right]. \tag{5}$$

This is a normal density function, $n(\nu|\mu, \sigma^2) = n(\nu|\mu)$, for the statistical realization of transformed event count v, with a mean that is a transformation of the Poisson parameter $\mu = 2\sqrt{\lambda}$ and variance $\sigma^2 = 1$; Figure 1 (red).

[1] Auxiliary materials are available in the HTML. doi:10.1029/2012GL051431.

[8] If we interpret equation (5) in terms of likelihood, then we can use it to estimate μ given ν. The most likely value of the transformed Poisson parameter is

$$\mu_{ML} = \nu. \tag{6}$$

After transforming back to the standard parameterization, the most likely Poisson parameter is what we might have expected it to be,

$$\lambda_{ML} = k. \tag{7}$$

With (2) we can estimate the most likely occurrence rate,

$$\rho_{ML} = \frac{k}{\tau}. \tag{8}$$

Uncertainty is often expressed in terms of the lower and upper values of confidence intervals, C = [Lower, Upper]. From equation (5), we choose confidence intervals for the transformed parameter μ that are centered on the most likely estimate,

$$C_z(\mu|\nu) = [(\nu - z), (\nu + z)], \tag{9}$$

where, for example, the 1σ, 68.3% (2σ, 95.4%) confidence interval corresponds to setting the width parameter to $z = 1$ ($z = 2$). With inverse transformation, the confidence intervals of the standard Poisson parameter λ are

$$C_z(\lambda|k) = \left[\left(\sqrt{k} - \tfrac{1}{2}z\right)^2, \left(\sqrt{k} + \tfrac{1}{2}z\right)^2\right] \tag{10}$$

[e.g., *Davison*, 2003, p. 59], and where, to ensure positivity, it is required that $z < 2\sqrt{k}$ for the lower limit. With (2), the confidence intervals on the rate parameter ρ are

$$C_z(\rho|k) = \frac{1}{\tau}\left[\left(\sqrt{k} - \tfrac{1}{2}z\right)^2, \left(\sqrt{k} + \tfrac{1}{2}z\right)^2\right]. \tag{11}$$

Under a frequentist interpretation of statistics [e.g., *Stuart et al.*, 1999, chap. 19], for each Poisson datum k there is a corresponding confidence interval. And, just as each datum is a random realization from a distribution, each confidence interval is also a random realization from a distribution. A set of confidence intervals "covers" the true unknown λ with a certain specified probability. But this is not a useful concept for the case considered here, where we have just one confidence interval obtained from one small-count Poisson datum k. And since the events we analyze are rare, we would probably have to wait very patiently to acquire any more data! Our situation invites a Bayesian analysis.

4. Poisson Bayesian Statistics

[9] With a Bayesian approach to statistical inference [e.g., *O'Hagan and Forster*, 2004], the Poisson parameter λ is treated as a random realization from a "posterior" distribution having a probability density function $g(\lambda|k)$ that is constructed, for event-count data k, from the equation

$$g(\lambda|k) \propto p(k|\lambda) \times \pi(\lambda). \tag{12}$$

The "prior" function $\pi(\lambda)$ describes knowledge, belief, or prejudice that is held before an inference is made. Almost always, there is little by way of specific prior knowledge, and in such circumstances, it is appealing to invoke something like Laplace's principle of indifference. Guided by the observed data, an unbiased parameter estimate is sought from the space of all possible values. In this context, the Jeffreys "least-informative" prior is often used in Bayesian analyses.

[10] According to the philosophy of *Jeffreys* [1961, chap. 3.10], the prior for λ should ensure conservation of posterior probability under arbitrary changes of variables. In our case, this is accomplished by the prior

$$\pi(\lambda) \propto \sqrt{|I(\lambda)|}, \tag{13}$$

where the *Fisher* [1922] information is given by

$$I(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2}\log\{p(k|\lambda)\}\middle|\lambda\right], \tag{14}$$

and where the expectation E is calculated relative to all possible event counts k. Fisher information is a measure of curvature; where it is high (low), probability density is also high (low); therefore, it is qualitatively reasonable that $I(\lambda)$ would lead to the invariance property advocated by Jeffreys.

[11] After performing the needed calculations for the Poisson model $p(k|\lambda)$, the prior function is found to be

$$\pi(\lambda) \propto \frac{1}{\sqrt{\lambda}}. \tag{15}$$

As with many priors, this is not a proper probability density function; its integral over the λ domain $[0, \infty)$ is infinite, but when it is combined with the likelihood, via Bayes's relation (12), the posterior is proper [*Jeffreys*, 1961, chap. 3.10]. After normalization, the posterior is found to be a gamma density function,

$$g(\lambda|k) = \frac{\lambda^{k-\frac{1}{2}}}{\Gamma\left(k + \frac{1}{2}\right)}\exp(-\lambda). \tag{16}$$

In Figure 2a (black) we show the posterior $g(\lambda|k)$ for various event counts k and as a function of the Poisson parameter λ. In this standard parameterization, the maximum likelihood value is biased, $\lambda_{ML} = k - 1/2$, and variance is not symmetrically distributed; note the skew for small k.

[12] With the variance stabilizing transformation,

$$\phi = 2\sqrt{\lambda} \quad \text{and} \quad \lambda = \tfrac{1}{4}\phi^2, \tag{17}$$

the posterior density function (16) becomes

$$g(\phi|k) = \frac{1}{2^{2k}}\frac{\phi^{2k}}{\Gamma\left(k + \frac{1}{2}\right)}\exp\left(-\tfrac{1}{4}\phi^2\right). \tag{18}$$

In Figure 2b (black), we see that the transformed posterior density function $g(\phi|k)$ is nearly symmetrical and invariant for different values of k, except for translation along the ϕ axis. This property, which even holds for $k = 1$, is what *Box and Tiao* [1973, chap. 1.3.4] call "data translation". Since we want to let the data speak for themselves, this appears to be the most natural formulation for making unbiased parameter estimates.

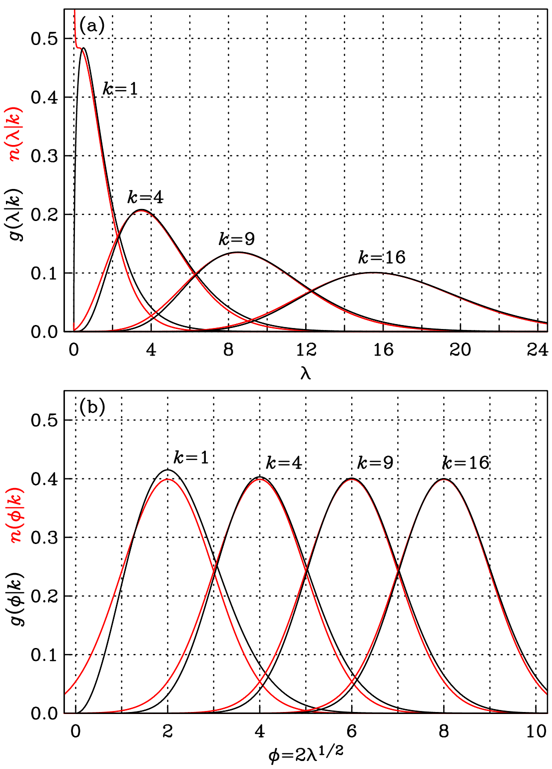

Figure 2. Posterior probability densities as functions of (a) Poisson parameter λ and (b) transformed parameter ϕ, each for various event counts k: (black) posterior density functions $g(\lambda|k)$ and $g(\phi|k)$, (red) normal density functions $n(\lambda|k)$ and $n(\phi|k)$.

[13] After expanding (18) in a Taylor series for small $\epsilon = \phi - 2\sqrt{k}$ and using Stirling's approximation, we find that the transformed posterior function can be approximated by

$$n(\phi|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(\phi - \mu)^2\right]. \tag{19}$$

This is a normal density function, $n(|\mu, \sigma^2) = n(\phi|\lambda)$, for the statistical realization of transformed Poisson parameters ϕ, with a mean that is a transformation of the event-count datum $\mu = 2\sqrt{k}$ and variance $\sigma^2 = 1$; Figure 2 (red). With this, and after making mathematical manipulations similar to those in Section 3, we obtain the most likely estimate for the Poisson parameter; it is, as with (7), what we might have expected it to be,

$$\lambda_{ML} = k. \tag{20}$$

The Bayesian "credibility" intervals are

$$C_z(\lambda|k) = \left[\left(\sqrt{k} - \tfrac{1}{2}z\right)^2, \left(\sqrt{k} + \tfrac{1}{2}z\right)^2\right]. \tag{21}$$

In terms of mathematical equations, this is the same as the approximation we obtained for frequentist confidence intervals (10). From the Bayesian perspective, the credibility interval (21) is a measure of the probabilistic dispersion of the Poisson model parameter λ given a single datum k. Specifically, the probability is 68.3% (95.4%) that the model parameter is within a credible interval defined by a width parameter of $z = 1$ ($z = 2$). This interpretation is consistent with what many scientists seem to be thinking when they give "error bars", but it is very different from the frequentist interpretation assigned to confidence intervals, Section 3.

[14] In contrast to the argument Jeffreys makes for priors derived using equation (13), which, in our case, leads to the $1/\sqrt{\lambda}$ prior for Poisson statistics, earlier on in his book, *Jeffreys* [1961, chap. 3.3] argues that the need for a sort of scale invariance leads to a $1/\lambda$ prior. *Jaynes* [2003, section 6.15] and many geophysicists call the latter the "Jeffreys prior"; in other scientific communities, it is the former that usually bears that moniker [e.g., *Prosper et al.*, 2008]. While Jeffreys, himself, seems undecided as to which prior is "best", we choose to interpret Bayesian analysis with a bit of flexibility and accept the fact that different priors lead to different statistical inferences, each of which is consistent, in its own way, with the data. For the Poisson statistics considered here, equation (13) not only ensures invariance under changes of variables, but it also results in the nearly symmetric duality between the frequentist and Bayesian distributions shown in Figures 1 and 2. We find this to be appealing. And, indeed, some statisticians advocate a general approach to Bayesian analysis that seeks to maximize mathematical symmetry with frequentism [e.g., *Kass and Wasserman*, 1996, section 3.7].

5. Simple Estimates of Relative Accuracy

[15] How many Poisson events must be observed before we can make an accurate estimate of their occurrence rate? A rough answer to this question is obtained by dividing the width of the credibility interval (21) by the most likely parameter (20), giving $2z/\sqrt{k}$ as a measure of relative accuracy. An illustrative example: if we have observed only one, presumably very rare, geophysical event in a given duration of time τ, then the best estimate we can make for the occurrence rate is $1/\tau$, but the 68.3% credibility interval corresponds to a relative error of $2/\tau$. While each researcher will have his or her own requirements for accuracy, the width of Poisson credibility intervals corresponding to just a few events is sobering. To reduce the interval to width $1/\tau$, we would have to observe four events, and this would require, on average, making observations over a time duration that is four times longer than the original duration. Of course, a high level of accuracy can be obtained if we can count lots of events, but this is not usually possible for extreme events.

6. Forecasting Future Probabilities

[16] What is the probability of the occurrence of a rare, great geophysical event in, say, the next ten years? And, correspondingly, how certain is such a forecast? We can answer such questions by assuming stationarity, and using our estimated Poisson parameters, and their corresponding confidence-credibility intervals. With equation (1) and the most likely occurrence rate ρ_{ML}, the most likely probability that there will be 1 or more event in time T is

$$P_{ML}^{\geq 1} = \sum_{k=1}^{\infty} \frac{(\rho_{ML}T)^k}{k!} \exp(-\rho_{ML}T) \tag{22}$$

$$= 1 - \exp(-\rho_{ML}T). \tag{23}$$

Table 1. Poisson Event Occurrence Rates, 10-yr Occurrence Probabilities, and Corresponding Confidence-Credibility Intervals

	Since	Threshold	k	τ (yrs)	Occurrence Rate ρ_{ML} (/100 yrs)	Occurrence Rate C_1 68.3% (/100 yrs)	Occurrence Rate C_2 95.4% (/100 yrs)	10-yr Occurrence Probability $P_{ML}^{\geq 1}$	10-yr Occurrence Probability C_1 68.3%	10-yr Occurrence Probability C_2 95.4%
Earthquakes	1868	$M \geq 9.0$	6	144	4.167	[2.639, 6.041]	[1.459, 8.263]	0.341	[0.232, 0.453]	[0.136, 0.562]
Earthquakes	1900	$M \geq 9.0$	5	112	4.464	[2.691, 6.684]	[1.364, 9.350]	0.360	[0.236, 0.487]	[0.128, 0.607]
Eruptions	969	VEI ≥ 7	2	1043	0.192	[0.080, 0.351]	[0.016, 0.559]	0.019	[0.008, 0.035]	[0.002, 0.054]
Eruptions	1815	VEI ≥ 6	5	197	2.538	[1.530, 3.800]	[0.776, 5.316]	0.224	[0.142, 0.316]	[0.075, 0.412]
Magnetic storms	1859	$-Dst \geq 589$ nT	3	153	1.961	[0.992, 3.256]	[0.350, 4.878]	0.178	[0.094, 0.278]	[0.034, 0.386]
Earthquakes	1868	$M \geq 9.5$	1	144	0.694	[0.174, 1.562]	[0.000, 2.778]	0.067	[0.017, 0.145]	[0.000, 0.243]
Earthquakes	1900	$M \geq 9.5$	1	112	0.893	[0.223, 2.009]	[0.000, 3.571]	0.085	[0.022, 0.182]	[0.000, 0.300]
Eruptions	1815	VEI ≥ 7	1	197	0.507	[0.127, 1.142]	[0.000, 2.030]	0.049	[0.013, 0.108]	[0.000, 0.184]
Magnetic storms	1859	$-Dst \geq 1760$ nT	1	153	0.654	[0.163, 1.471]	[0.000, 2.614]	0.063	[0.016, 0.137]	[0.000, 0.230]

Similar formulas apply for calculating probabilities for confidence-credibility intervals, equations (10) and (21).

7. Some Specific Results and Comments

[17] In Table 1, we give, for the geophysical events discussed in Section 2, most likely occurrence rates, most easily viewed in Bayesian terms, and 10-yr probabilities, most easily viewed in frequentist terms. We also give corresponding confidence-credibility intervals. We discuss some illustrative examples, starting with the results for multiple megaquakes. The most likely occurrence rate ρ_{ML} for $M \geq 9.0$ earthquakes since 1868 is 4.167 per century. This is higher than the 1–3 per century rate that *McCaffrey* [2008] estimates from plate tectonic parameters is more typical. His estimates, however, are mostly contained by our C_2 95.4% interval of [1.459, 8.263] per century. The most likely Poisson probability for the occurrence of at least one $M \geq 9.0$ earthquake in the next 10 yr is 0.341. The 95.4% interval for such earthquakes is [0.136, 0.562], the relative width of which reflects the small number of events used in the estimate. Similar results pertain for $M \geq 9.0$ earthquakes since 1900; we note that there is substantial overlap between their confidence-credibility intervals with those for $M \geq 9.0$ earthquakes since 1868.

[18] With respect to volcanic eruptions, the 5 VEI ≥ 6 events since 1815 correspond to an occurrence rate of 2.538 per century, and a 95.4% interval of [0.776, 5.316] per century. For the 2 VEI ≥ 7 events since 969, assuming that the Baitoushan eruption qualifies for this size, the corresponding occurrence rate of 0.192 per century, while for the 1 VEI ≥ 7 since 1815 it is 0.507 per century. This might seem, at first, to be a considerable difference, but we note that the C_1 68.3% intervals have considerable overlap; we can say that the two rate estimates are statistically indistinguishable. This is another example, of course, of the limited accuracy of statistical estimates for Poisson occurrence rates that are based on the observation of only a few events.

[19] For magnetic storms, if another event like the $-Dst \geq$ 1760 nT Carrington event of 1859 were to occur today, it could lead to an economic loss for the United States of \$1–2 trillion [*Baker et al.*, 2008]. Therefore, it is important to forecast the future occurrence probability of such a superstorm. It is also important to appreciate the limited accuracy of such a forecast. The most likely Poisson occurrence probability for another Carrington event in the next 10 yr is 0.063, or about half the 0.120 probability that *Riley* [2012] estimates by extrapolating from smaller events. His estimate, however, is contained by our 68.3% interval of [0.016, 0.137]. This serves as yet another example of the limited accuracy of statistical estimates; completely different methods give rates that are statistically indistinguishable. The 10-yr recurrence probability for a Carrington event is somewhere between vanishingly unlikely and surprisingly likely.

[20] To improve long-term estimates of extreme-event occurrence rates, future work will continue along several tracks, including development of (1) back-in-time historical catalogs that add to the data used to analyze extreme geophysical events, (2) statistical models that more completely exploit the information content of existing historical catalogs, (3) statistical models that make simultaneous exploitation of multiple types of data, and (4) physical models that are substantially more complete and realistic than those in present use. With respect to short-term forecasts, we understand that those for volcanic eruptions, based on local seismic activity and mountain deformation, are not amenable to pure statistical methods. On the other hand, short-term forecast methods, along the frequentist-Bayesian lines given here, might be developed for earthquake aftershock probabilities, conditional on the occurrence of a mainshock, or for magnetic storm probabilities, conditional on the existence and size of a sunspot. We predict, however, that it will be a long time before we can substantially reduce the uncertainty of long-term forecast probabilities of extreme geophysical events.

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Auxiliary Material for Paper 2012GL051431

Credible occurrence probabilities for extreme geophysical events:
Earthquakes, volcanic eruptions, magnetic storms

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Introduction

The auxiliary material provides elaboration on mathematical derivations outlined in the paper.

1. 2012gl051431-txts01.pdf
Text S1. Details of mathematical derivations outlined in the paper.

Derivation of equation (4) from (1)

The Poisson density function is

$$p(k|\lambda) = \frac{\lambda^k}{k!}\exp(-\lambda). \tag{1}$$

To make the transformation $k \to \nu$ given by equation (3) and, all the while, conserve unit quantities of probability, we must make a change of variables according to the equation

$$p(\nu|\lambda) = p(k(\nu)|\lambda) \cdot \left|\frac{\partial k}{\partial \nu}\right|$$

[e.g. *Marsden and Tromba*, 1981, Chapter 5.8]. Since

$$\frac{\partial k}{\partial \nu} = \frac{\nu}{2},$$

and $k! = \Gamma(k+1)$, we obtain

$$p(\nu|\lambda) = \frac{\nu}{2}\frac{\lambda^{\frac{1}{4}\nu^2}}{\Gamma\left(\frac{1}{4}\nu^2 + 1\right)}\exp(-\lambda), \tag{4}$$

which, as we have noted, is not a proper density function; it is not properly normalized, but this technical problem is diminished for $\lambda > 1$.

Derivation of equation (5) from (4)

The gamma function is

$$p(\nu|\lambda) = \frac{\nu}{2}\frac{\lambda^{\frac{1}{4}\nu^2}}{\Gamma\left(\frac{1}{4}\nu^2 + 1\right)}\exp(-\lambda). \tag{4}$$

From Sterling's approximation of the gamma function we have

$$\Gamma\left(\tfrac{1}{4}\nu^2 + 1\right) \simeq \sqrt{2\pi}\,\frac{\nu}{2}\left(\tfrac{1}{4}\nu^2\right)^{\frac{1}{4}\nu^2}\exp\left(-\tfrac{1}{4}\nu^2\right)$$

[e.g. *Spanier and Oldham*, 1987, Equations 43:5:3 and 43:6:6; *Korn and Korn*, 2000, Equations 21.4-7 and 21.4-10]. Next, we make a Taylor series expansion about 0 for small $\epsilon = \nu - 2\sqrt{\lambda}$,

$$\left(\tfrac{1}{4}\nu^2\right)^{\frac{1}{4}\nu^2} \simeq \lambda^{\frac{1}{4}\nu^2}\left(1 + \sqrt{\lambda}\epsilon + \tfrac{3}{4}\epsilon^2 + \cdots\right) \simeq \lambda^{\frac{1}{4}\nu^2}\exp(\sqrt{\lambda}\epsilon)\exp(\tfrac{3}{4}\epsilon^2),$$

with which the approximation of the gamma function becomes

$$\Gamma\left(\tfrac{1}{4}\nu^2+1\right) \simeq \sqrt{2\pi}\,\frac{\nu}{2}\lambda^{\frac{1}{4}\nu^2}\exp(\tfrac{1}{2}\epsilon^2)\exp(-\lambda).$$

And upon substitution into equation (4) we have

$$p(\nu|\lambda) = \frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\epsilon^2\right) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\left(\nu-2\sqrt{\lambda}\right)^2\right],$$

which can be recognized as a normal distribution with mean $\mu = 2\sqrt{\lambda}$ and unit variance,

$$n(\nu|\mu) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\left(\nu-\mu\right)^2\right]. \tag{5}$$

Derivation of equation (15) from (13)

Following the procedure given by *Jeffreys* [1961, Chapter 3.10], the prior for λ should ensure conservation of posterior probability under arbitrary changes of variables. In our case, this is accomplished by the prior

$$\pi(\lambda) \propto \sqrt{|I(\lambda)|}. \tag{13}$$

The Fisher information is given by

$$I(\lambda) = -E\left[\left.\frac{\partial^2}{\partial\lambda^2}\log\left\{\mathcal{L}(k|\lambda)\right\}\right|\lambda\right],$$

where $\mathcal{L}(k|\lambda)$ is the data likelihood, and where the expectation E is calculated relative to all possible event numbers k. In our case, for one realization of k Poisson events in a single duration of time, the likelihood is equal to the Poisson density function (1); that is, $\mathcal{L}(k|\lambda) = p(k|\lambda)$ and we have equation (14). The log likelihood is, then,

$$\log\left\{\mathcal{L}(k|\lambda)\right\} = \log\left\{p(k|\lambda)\right\} = k\log\lambda - \lambda - \log k!$$

Next, we differentiate,

$$\frac{\partial^2}{\partial\lambda^2}\log\left\{p(k|\lambda)\right\} = -k\lambda^{-2}.$$

The expected valued is obtained

$$-E\left[-k\lambda^{-2}\right] = \lambda^{-2}E[k] = \lambda^{-2}\sum_{k=0}^{\infty} k\, p(k|\lambda) = \frac{1}{\lambda}.$$

And, upon taking the square root, as per equation (13), we have

$$\pi(\lambda) \propto \frac{1}{\sqrt{\lambda}}. \tag{15}$$

Derivation of equation (18) from (16)

The posterior gamma density function is

$$g(\lambda|k) = \frac{\lambda^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2})} \exp(-\lambda). \tag{16}$$

To make the transformation $\lambda \to \phi$ given by equation (17) and, all the while, conserve unit quantities of probability, we must make a change of variables according to the equation

$$p(\phi|k) = p(\lambda(\phi)|k) \cdot \left|\frac{\partial\lambda}{\partial\phi}\right|.$$

Since

$$\frac{\partial\lambda}{\partial\phi} = \frac{\phi}{2},$$

we obtain

$$g(\phi|k) = \frac{1}{2^{2k}} \frac{\phi^{2k}}{\Gamma(k+\frac{1}{2})} \exp\left(-\tfrac{1}{4}\phi^2\right). \tag{18}$$

Derivation of equation (19) from (18)

The posterior gamma density function is

$$g(\phi|k) = \frac{1}{2^{2k}} \frac{\phi^{2k}}{\Gamma(k+\frac{1}{2})} \exp\left(-\tfrac{1}{4}\phi^2\right). \tag{18}$$

From Sterling's approximation of the gamma function we have

$$\Gamma\left(k+\tfrac{1}{2}\right) \simeq \sqrt{2\pi}\left(k+\tfrac{1}{2}\right)^{k} \exp\left(-k-\tfrac{1}{2}\right),$$

and for $k > 1$,

$$\Gamma\left(k+\tfrac{1}{2}\right) \simeq \sqrt{2\pi} k^{k} \exp\left(-k\right).$$

Next, we make a Taylor series expansion about 0 for small $\epsilon = \phi - 2\sqrt{k}$,

$$\phi^{2k} \simeq 2^{2k} k^{k}\left(1+\sqrt{k}\epsilon+\tfrac{1}{2}k\epsilon^{2}-\tfrac{1}{4}\epsilon^{2}+\cdots\right) \simeq \exp\left(\sqrt{k}\right)\exp\left(-\tfrac{1}{4}\epsilon^{2}\right).$$

And upon substitution into equation (18) we have

$$g(\phi|k) = \frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{1}{2}\epsilon^{2}\right) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\left(\phi-2\sqrt{k}\right)^{2}\right],$$

which can be recognized as a normal distribution with mean $\mu = 2\sqrt{k}$ and unit variance,

$$n\left(\phi|\mu\right) = \frac{1}{\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}(\phi-\mu)^{2}\right]. \tag{19}$$

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