Solutions for Assignment #2
(Course: CS 301)

Problem 1**:** Let L be a regular language over some alphabet Σ. Prove that the following language is also regular:

$$L_{\text{strange}} = \left\{ b_2\, b_1\, b_4\, b_3 \ldots b_{2n}\, b_{2n-1} \,\middle|\, b_1\, b_2 \ldots b_{2n-1}\, b_{2n} \in L \right\}$$

For example, if the language L over $\Sigma = \{0, 1, 2\}$ contains $01\,21\,11$ then L_{strange} contains $10\,12\,11$.

Solution: Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting L. We construct a DFA M_{strange} that accepts L_{strange}. M_{strange} will process input symbols in pairs. On seeing the first symbol, say α, in a pair, M_{strange} stores a in its finite control. Then, on seeing the second symbol, say β, in that pair, M_{strange} behaves like M on the input $\beta\alpha$. More formally, Let Q' be the set of $|Q| \times |\Sigma|$ states

$$Q' = \left\{ q_a \,|\, q \in Q,\, a \in \Sigma \right\}$$

Then, $M_{\text{strange}} = \big(Q \cup Q', \Sigma, \delta_{\text{strange}}, q_0, F \big)$, where

- $\delta_{\text{strange}}(q, \alpha) = q_\alpha$ for every $\alpha \in \Sigma$ and every $q \in Q$, and
- $\delta_{\text{strange}}(q_\alpha, \beta) = \delta\big(\delta(q, \beta),\, \alpha \big)$ for every $q_\alpha \in Q'$ and every $\beta \in \Sigma$.

Problem 2**:** Let L be the following language over the alphabet $\Sigma = \{a, b\}$:

$$L = \left\{ a^m b^n \,|m \geq 1,\, n \geq 1,\, m \geq n^2 \right\}$$

For example, $0^{2^2} 1^2 = 0^4 1^2 = 000011$ and $0^{3^2+2} 1^3 = 0^{11} 1^3 = 00000000000111$ is in L but $0^3 1^2 = 00011$ is not in L.

Using pumping lemma prove that L is *not* regular.

Solution: Let $p \geq 1$ be the constant (the pumping length) of the pumping lemma. Consider the string $z = a^{p^2}\, b^p \in L$. We write z as uvw where $|uv| \leq p$ and $|v| \geq 1$. Thus, uv is entirely contained in the leftmost p a's of z, which in turn implies that $u = a^q$ and $v = a^r$ for some two integers q, r satisfying $r \geq 1$ and $q + r \leq p$. By pumping lemma, $uv^i w \in L$ for every $i = 0, 1, 2, \ldots$. Consider the string $uv^0 w = uw$ by setting $i = 0$. Then, $uw = a^{p^2 - r}\, b^p$ and, since $r \geq 1$, $a^{p^2-r}\, b^p \notin L$, providing the desired contradiction.

Problem 3**:** Write down a context-free grammar (CFG) for the following language L over alphabet $\Sigma = \{0, 1, \#\}$:

$$L = \left\{ 0^{n+3}\, \#\, 1^n \,\middle|\, n \geq 0 \right\}$$

Solution: The CFG is $\big(\{S, A\}, \{0, 1, \#\}, S, P \big)$ where the set of productions P is as given below:

$$\begin{array}{rcl} S & \rightarrow & 0S1 \,|\, A \\ A & \rightarrow & 000\# \end{array}$$