CS 506 – Lecture 3
Quantum States, Bases, and Reversible Gates

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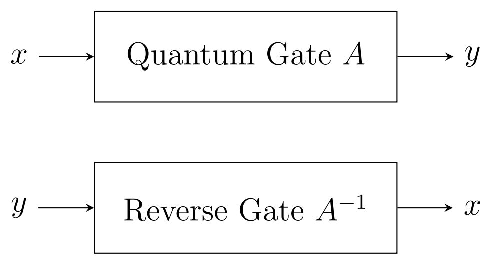
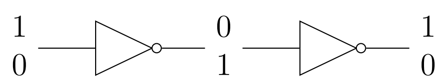
Reversibility of quantum gates (computing)

A reversible quantum gate A and its inverse A^{-1}:



NOT gate

The classical NOT gate is its own inverse: $\text{NOT}^{-1} = \text{NOT}$. Hence, it's a self-inverse gate.

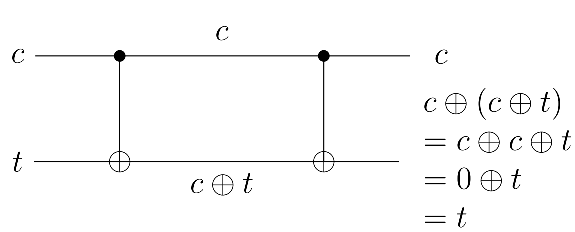


CNOT gate

The controlled-NOT (CNOT) gate on control c and target t acts as

$$\text{CNOT}(c, t) = (c,\, c \oplus t),$$

and applying it twice returns the original pair, so CNOT is also self-inverse.

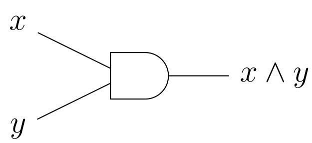


AND gate and reversible AND

The classical AND gate takes two bits and outputs a single bit

$$(x, y) \mapsto x \wedge y,$$

so different inputs (for example $(0, 0)$ and $(0, 1)$) can produce the same output 0. Therefore, the AND gate is not reversible by itself.



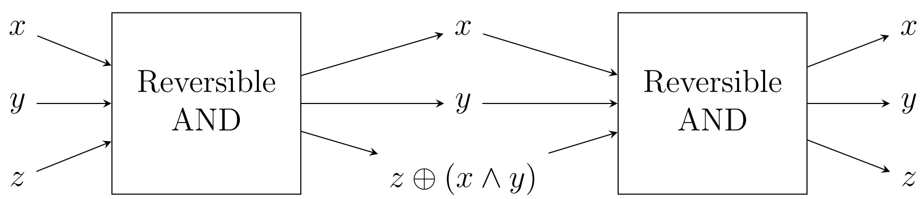
To make a reversible version, we keep the original inputs and add an additional input bit z. The reversible AND gate $R_\wedge$ acts as

$$R_\wedge(x, y, z) = \bigl(x,\ y,\ z \oplus (x \wedge y)\bigr).$$

All three outputs uniquely determine (x, y, z), so R is reversible. Moreover,

$$R_\wedge(R_\wedge(x, y, z)) = \bigl(x,\ y,\ z \oplus (x \wedge y) \oplus (x \wedge y)\bigr) = (x, y, z),$$

so the reversible AND gate is self-inverse.

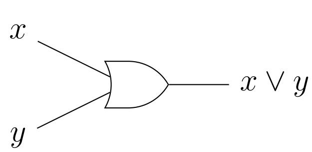


Reversible OR

The classical OR gate takes two bits and outputs a single bit

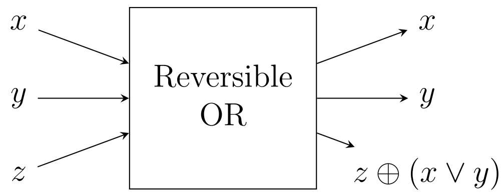
$$(x, y) \longmapsto x \vee y,$$

so it is not reversible by itself.



To obtain a reversible version, we keep the original inputs and add an extra input bit z. The reversible OR gate $R_\vee$ acts as

$$R_\vee(x, y, z) = \bigl(x,\ y,\ z \oplus (x \vee y)\bigr).$$

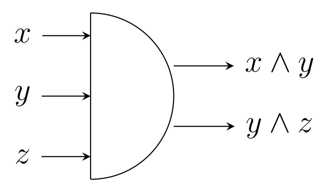


Multiple-output gates

General pattern. To make a gate with several output bits reversible, we add one extra input bit for every output bit, and XOR each output into its corresponding extra line.

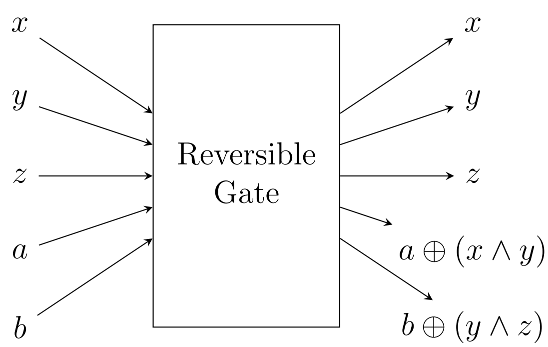
As an example, consider a classical gate with three inputs and two outputs

$$(x, y, z) \longmapsto \big(x \wedge y,\ y \wedge z\big).$$



To make this reversible, we add two extra inputs a and b and define

$$R(x, y, z, a, b) = \big(x,\ y,\ z,\ a \oplus (x \wedge y),\ b \oplus (y \wedge z)\big).$$



Quantum states as column vectors

A quantum state $|\psi\rangle$ of dimension 2^n (always a power of 2 for us) is represented as a column vector whose 2-norm is 1.

$$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Its norm is

$$\big\||01\rangle\big\| = \sqrt{\langle 01|01\rangle} = \sqrt{(0\ 1\ 0\ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} = \sqrt{1} = 1.$$

Another example is

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}.$$

Then

$$\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{\left(\tfrac{1}{\sqrt{2}} \ \ \tfrac{-i}{\sqrt{2}} \ 0 \ 0\right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = 1.$$

By definition, the bra $\langle\psi|$ is the complex-conjugate transpose of the ket $|\psi\rangle$.

Inner product of two states

Consider the two 2-qubit states

$$|\psi\rangle = \frac{1}{\sqrt{3}}|01\rangle + \frac{\sqrt{2}}{\sqrt{3}}|11\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix},$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{i}{\sqrt{2}}|11\rangle = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}.$$

Their inner product is

$$\langle\psi|\phi\rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{i}{\sqrt{3}}.$$

Orthonormal bases

An orthonormal basis of a Hilbert space of dimension m (must be a power of 2) is a collection of states

$$|b_1\rangle,\ |b_2\rangle,\ \ldots,\ |b_m\rangle$$

such that

$$\langle b_i|b_i\rangle = 1 \quad \text{for all } i, \qquad \langle b_i|b_j\rangle = 0 \quad \text{for all } i \neq j.$$

Fact. Any other state can be written as a linear combination of the basis states:

$$|\psi\rangle = \sum_{i=1}^{m} \alpha_i |b_i\rangle.$$

Example: $m = 2$

For a single qubit ($m = 2$), the standard (computational) basis is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle.$$

For instance,

$$\begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} \end{pmatrix} = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{i}{\sqrt{3}}|1\rangle.$$

Example: $m = 2^2 = 4$

For two qubits, the Hilbert space has dimension $m = 4$ and the standard (computational) basis consists of the four vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle.$$

Hadamard (plus/minus) basis

Another common single-qubit basis is the Hadamard (or plus/minus) basis, given by the states

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Adding and subtracting these states shows how to recover the computational basis:

$$|+\rangle + |-\rangle = \frac{2}{\sqrt{2}}|0\rangle, \qquad \frac{1}{\sqrt{2}}\big(|+\rangle + |-\rangle\big) = |0\rangle,$$

$$|+\rangle - |-\rangle = \frac{2}{\sqrt{2}}|1\rangle, \qquad \frac{1}{\sqrt{2}}\big(|+\rangle - |-\rangle\big) = |1\rangle.$$

Example: rewriting a state in the Hadamard basis

Consider the state

$$|\psi\rangle = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle.$$

Using $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$, we get

$$\begin{aligned} |\psi\rangle &= \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \\ &= \frac{1}{\sqrt{3}}(|+\rangle + |-\rangle) + \frac{1}{\sqrt{6}}(|+\rangle - |-\rangle) \\ &= \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|+\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|-\rangle. \end{aligned}$$

Bell basis

Another important two-qubit basis is the Bell basis. In the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ we define

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix},$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle - |11\rangle\big) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Inner product in Dirac notation

Using the same states as before,

$$|\psi\rangle = \frac{1}{\sqrt{3}}|01\rangle + \frac{\sqrt{2}}{\sqrt{3}}|11\rangle, \qquad |\phi\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{i}{\sqrt{2}}|11\rangle,$$

compute the inner product by expanding:

$$\begin{aligned}\langle\psi|\phi\rangle &= \left(\frac{1}{\sqrt{3}}\langle 01| + \frac{\sqrt{2}}{\sqrt{3}}\langle 11|\right)\left(\frac{1}{\sqrt{2}}|10\rangle + \frac{i}{\sqrt{2}}|11\rangle\right) \\ &= \frac{1}{\sqrt{6}}\langle 01|10\rangle + \frac{i}{\sqrt{6}}\langle 01|11\rangle + \frac{1}{\sqrt{3}}\langle 11|10\rangle + \frac{i}{\sqrt{3}}\langle 11|11\rangle.\end{aligned}$$

By orthonormality of the computational basis,

$$\langle 01|10\rangle = \langle 01|11\rangle = \langle 11|10\rangle = 0, \qquad \langle 11|11\rangle = 1,$$

so

$$\langle\psi|\phi\rangle = \frac{i}{\sqrt{3}}.$$