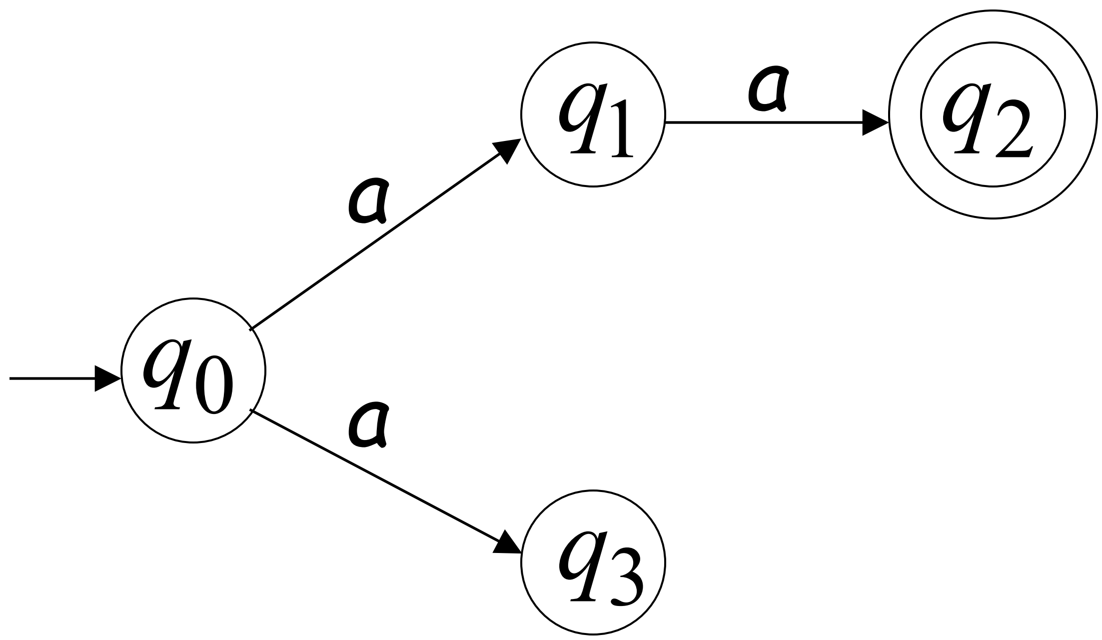
Non-Deterministic Finite Automata

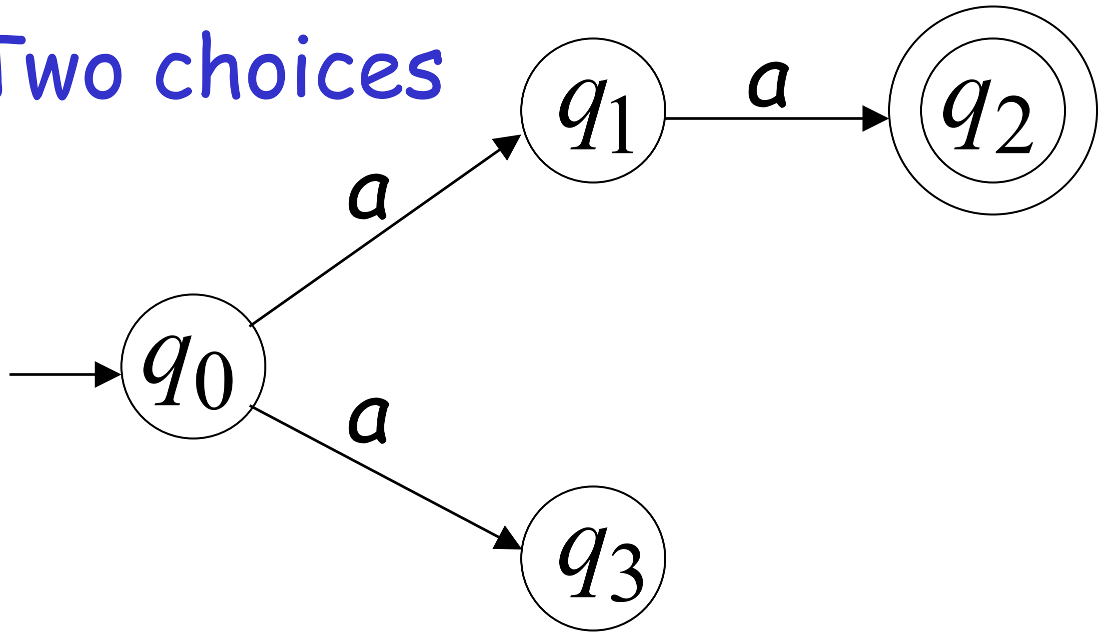
Nondeterministic Finite Automaton (NFA)

Alphabet = $\{a\}$

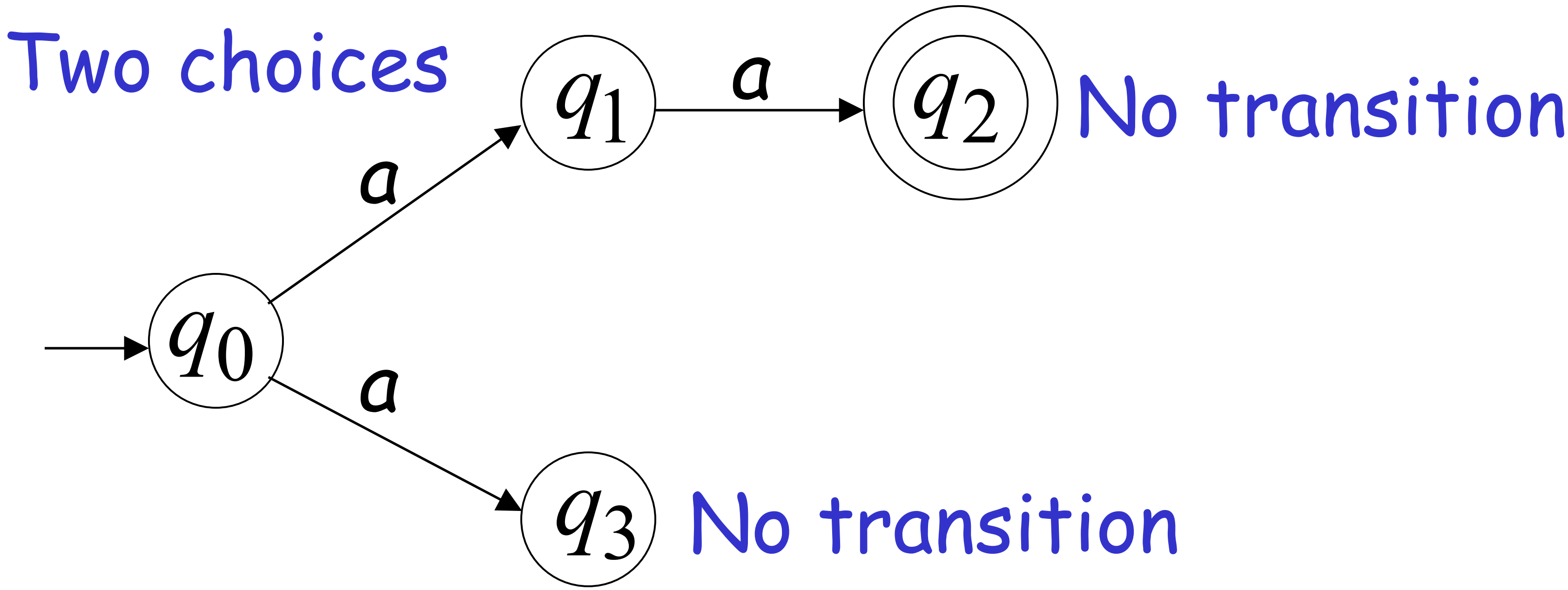


Alphabet = $\{a\}$

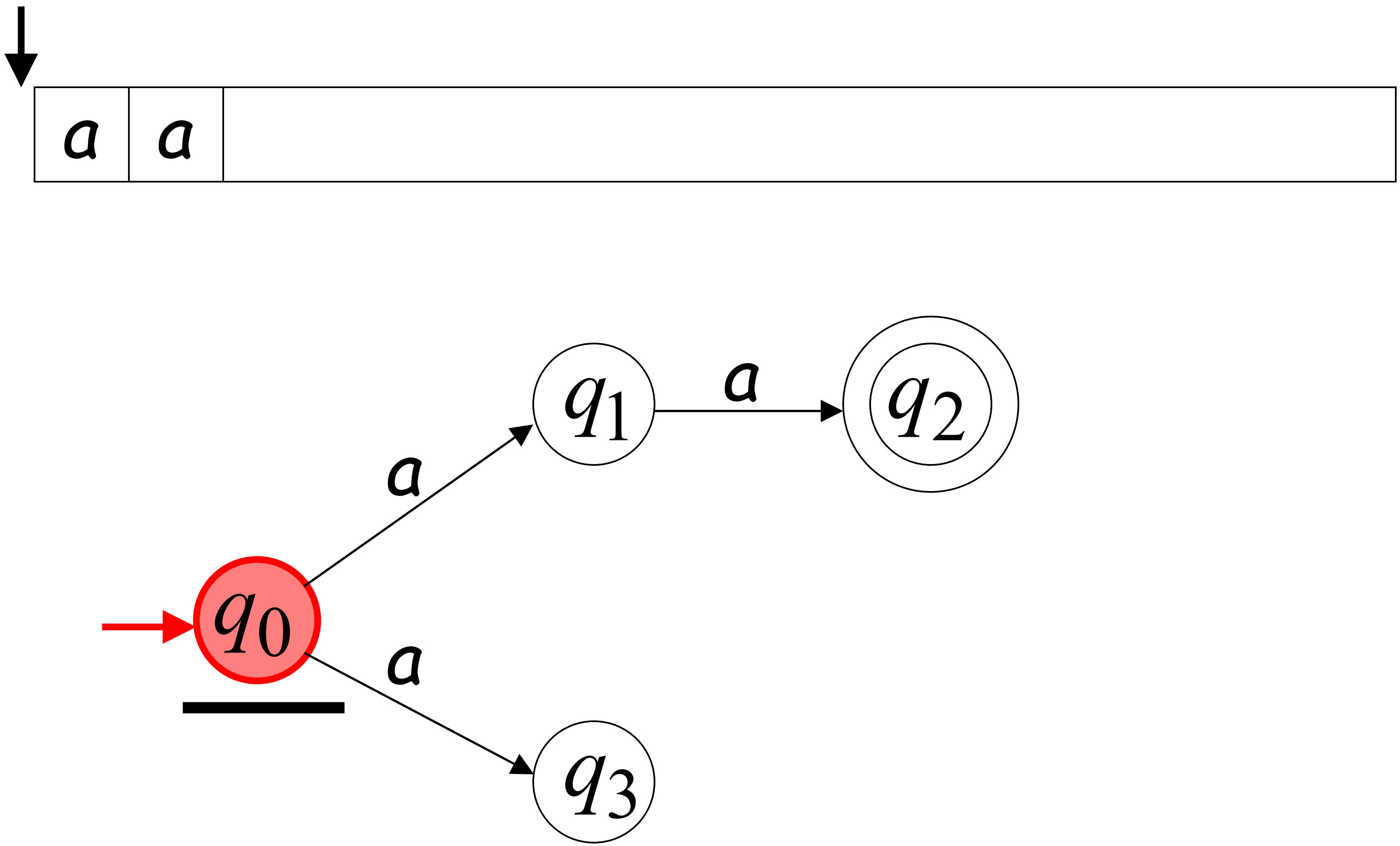
Two choices



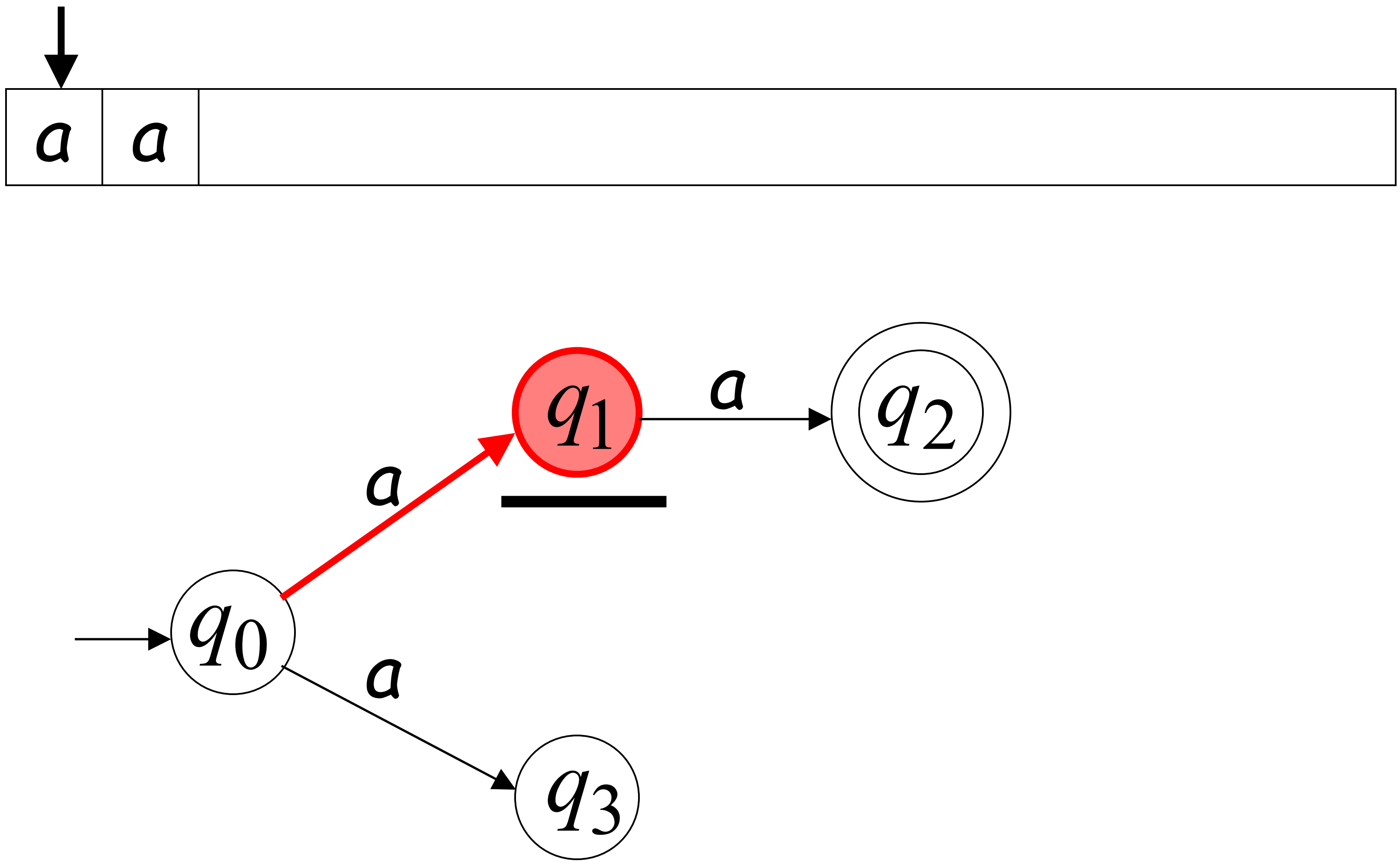
Alphabet = $\{a\}$



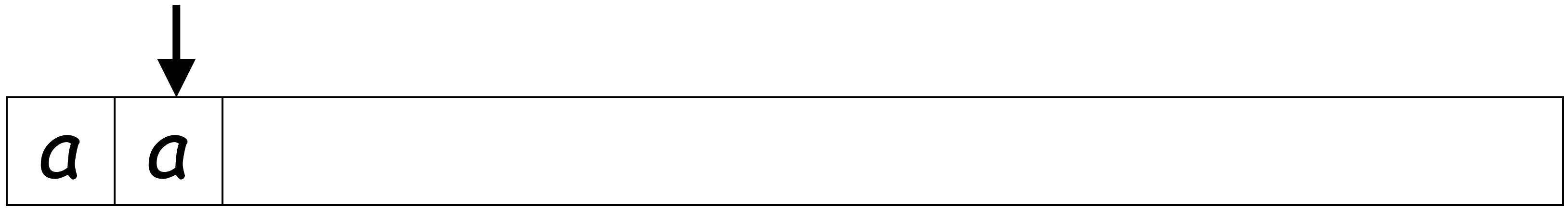
First Choice



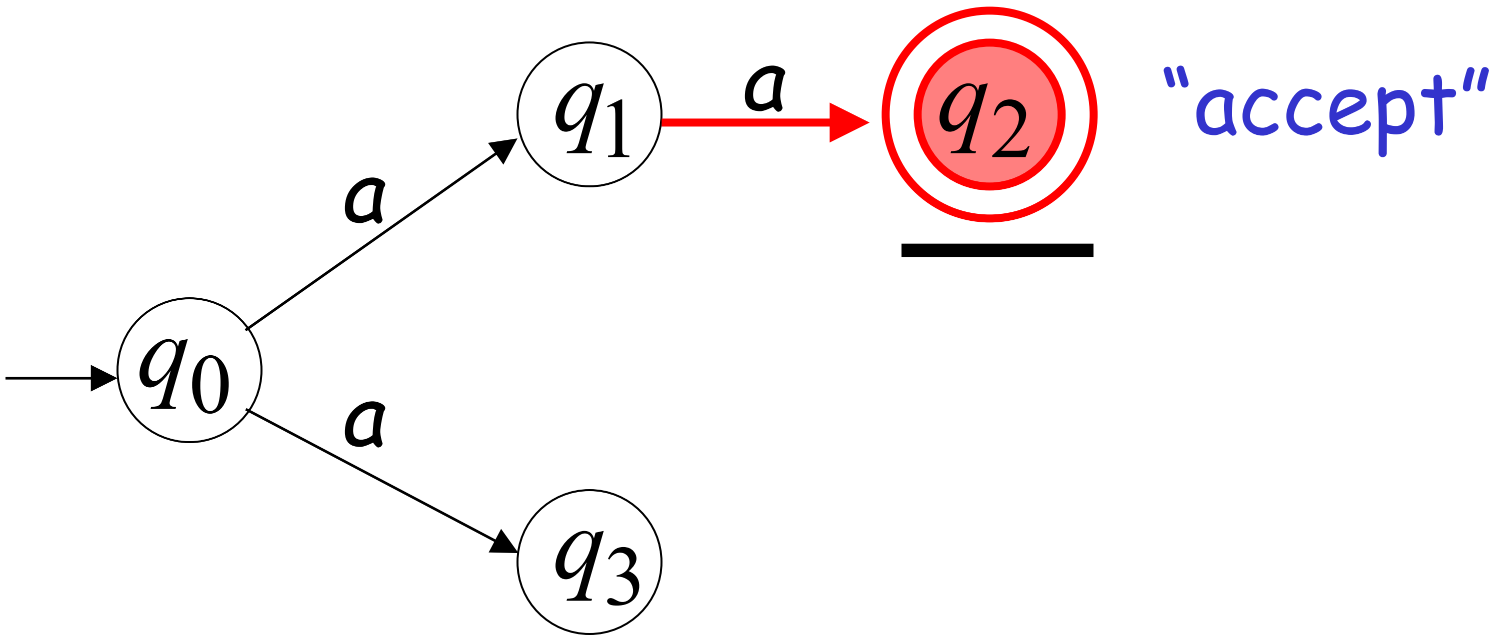
First Choice



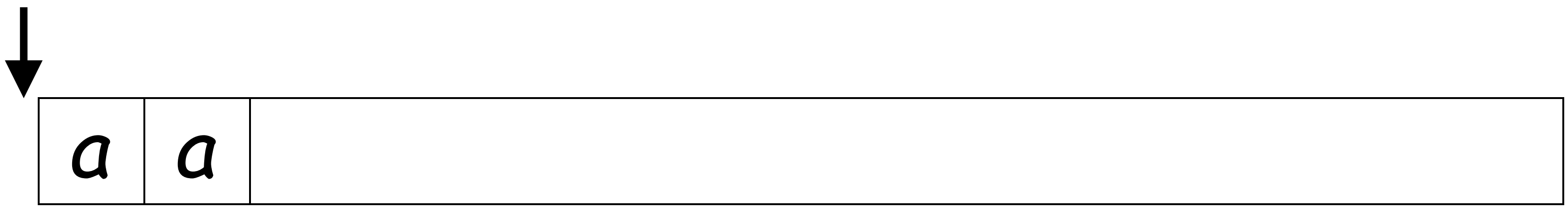
First Choice

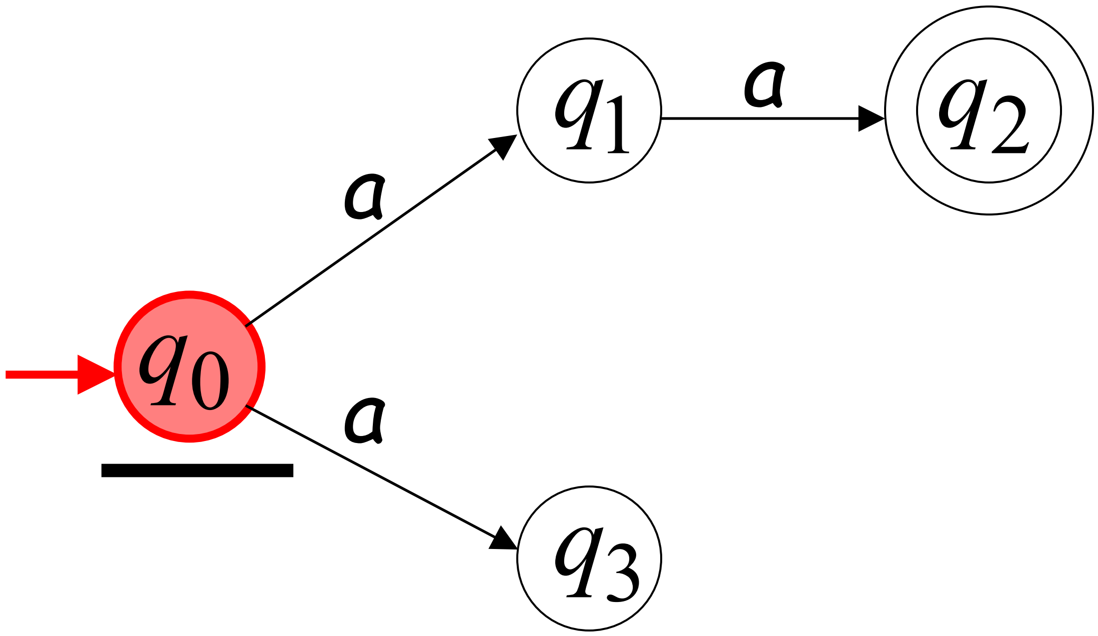


All input is consumed

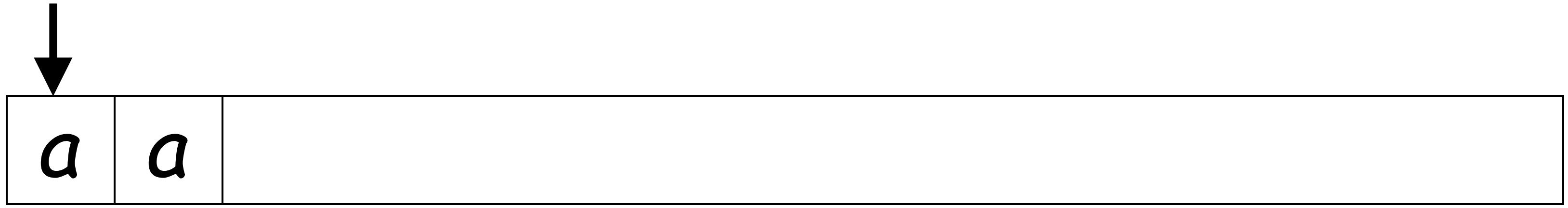


Second Choice

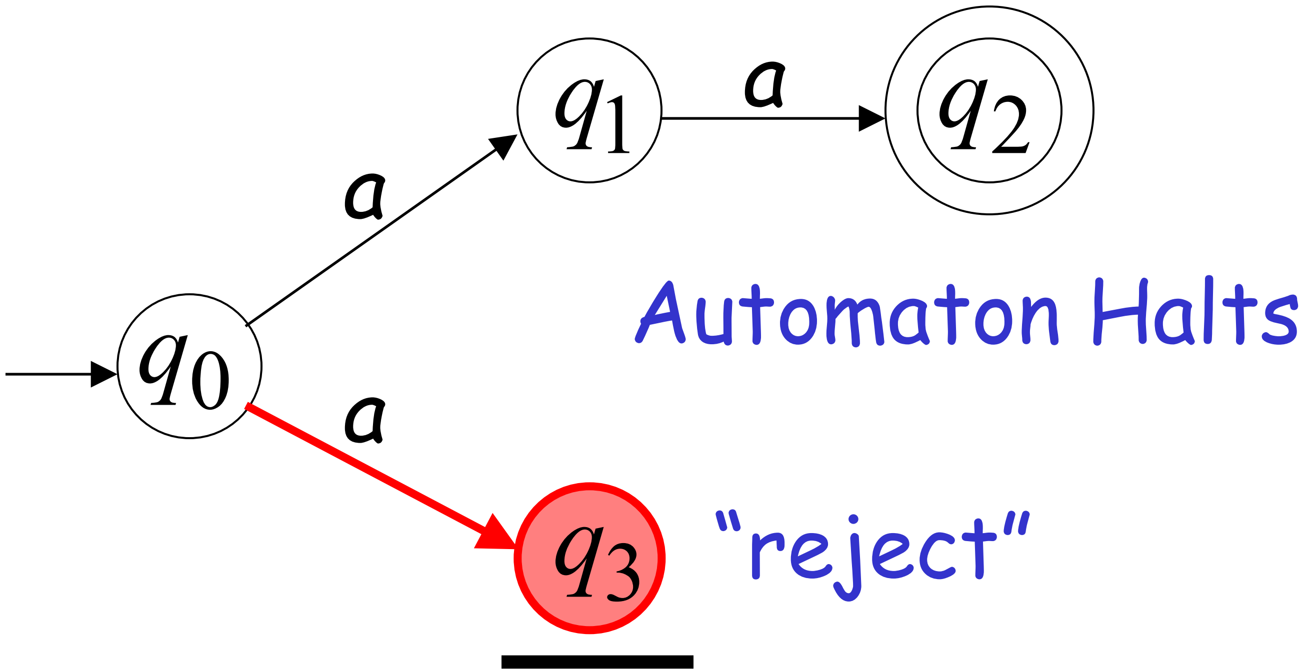




Second Choice



Input cannot be consumed



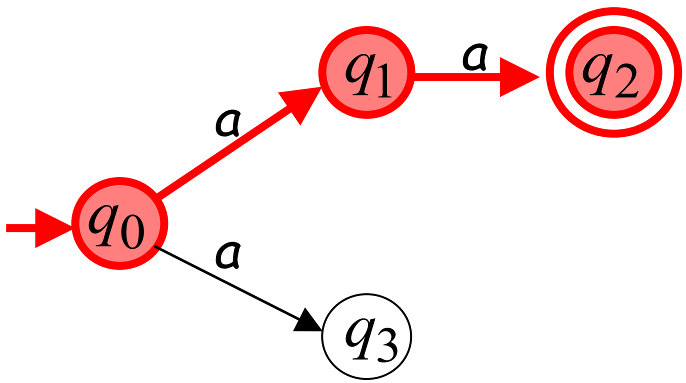
An NFA accepts a string:

if there is a computation of the NFA that accepts the string

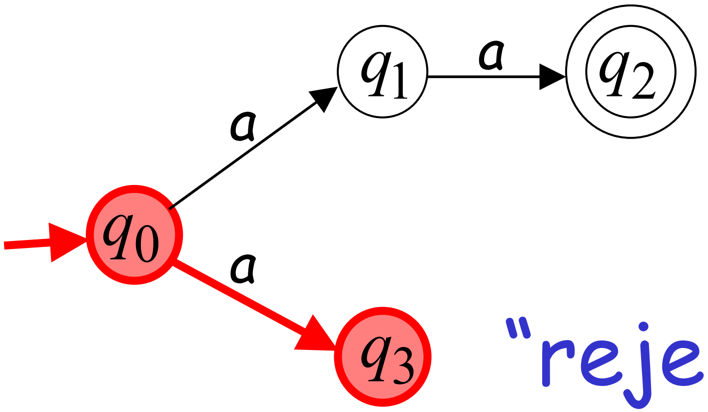
i.e., all the input string is processed and the automaton is in an accepting state

aa is accepted by the NFA:

"accept"



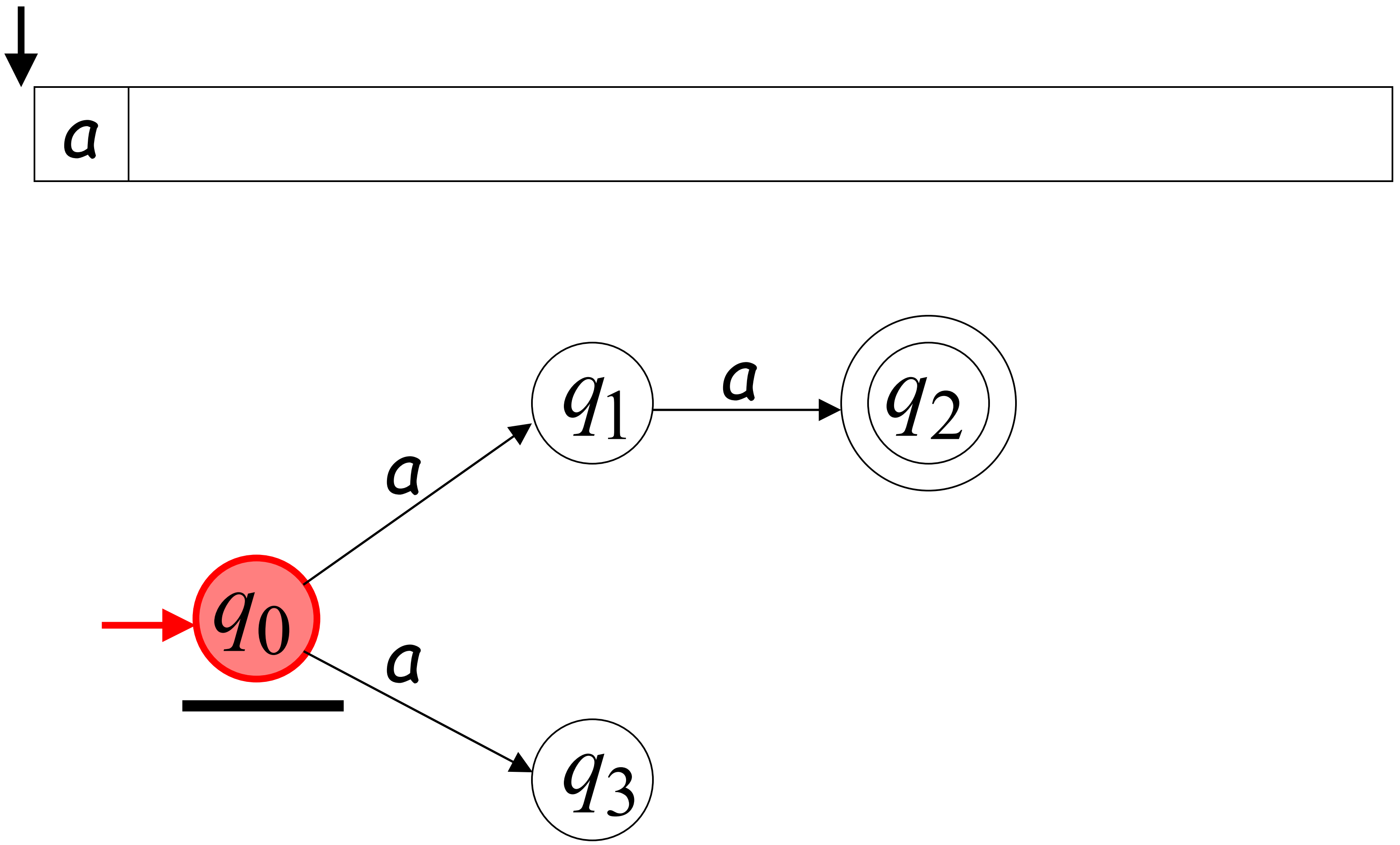
because this computation accepts aa



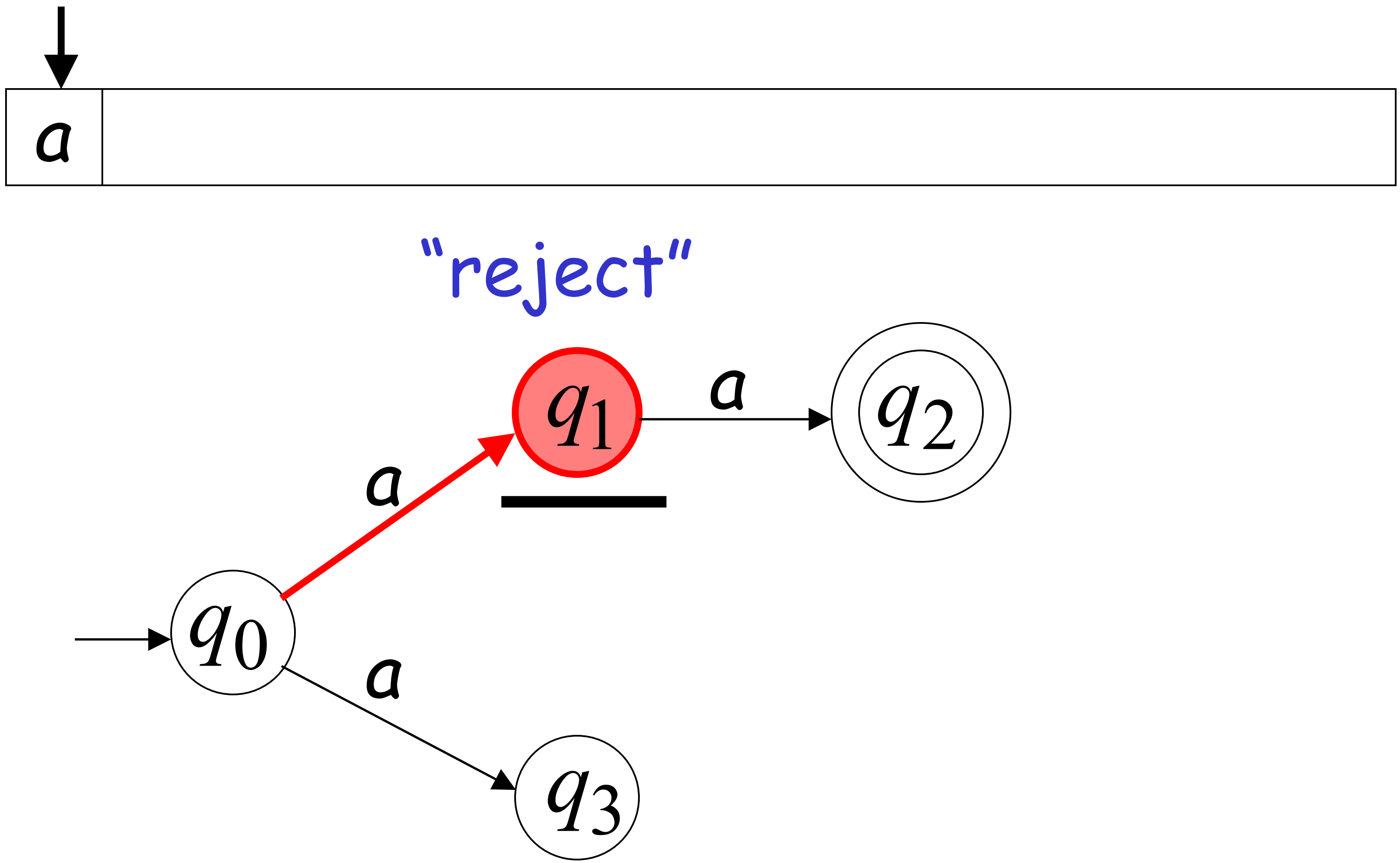
"reject"

this computation is ignored

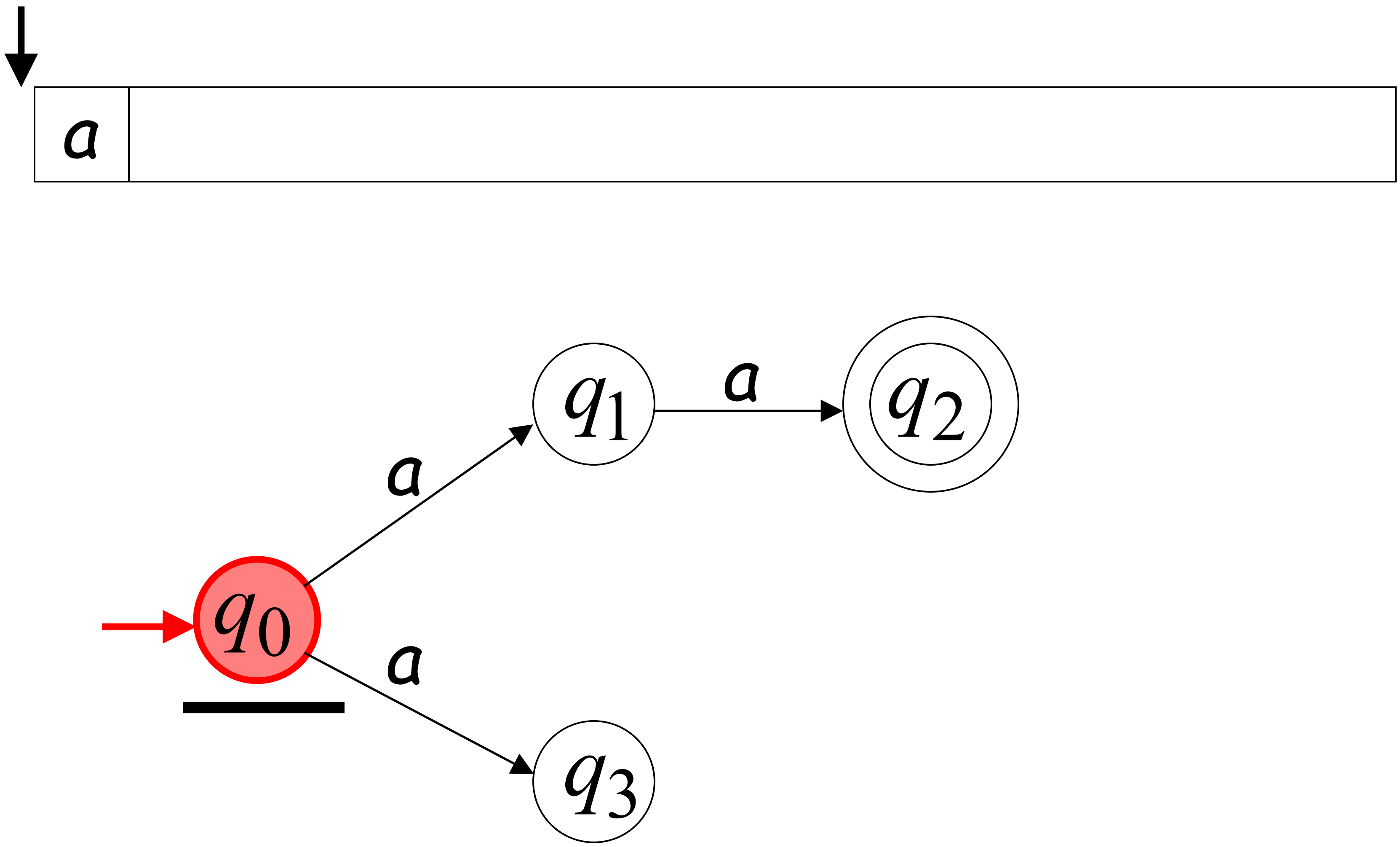
Rejection example



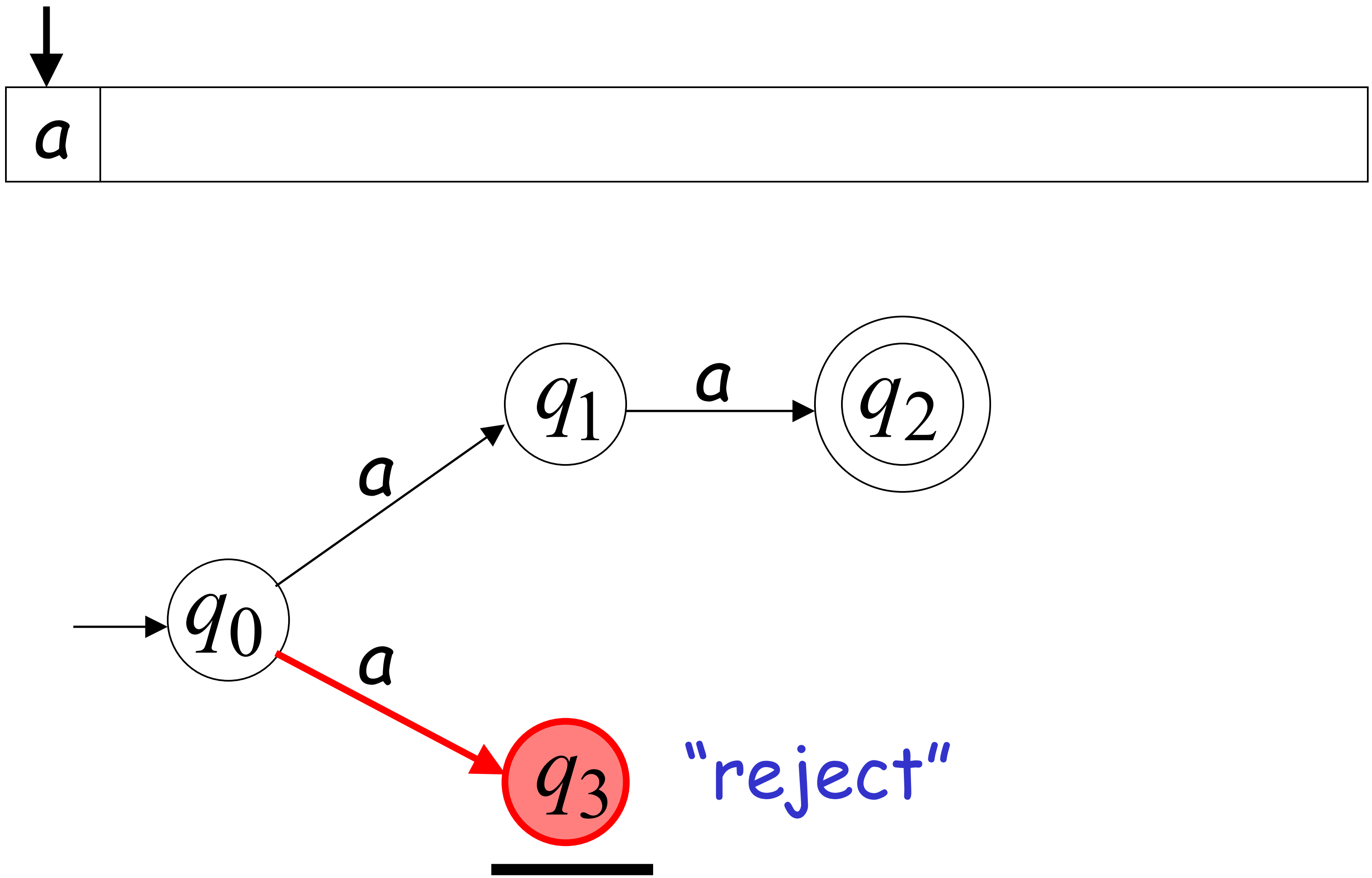
First Choice



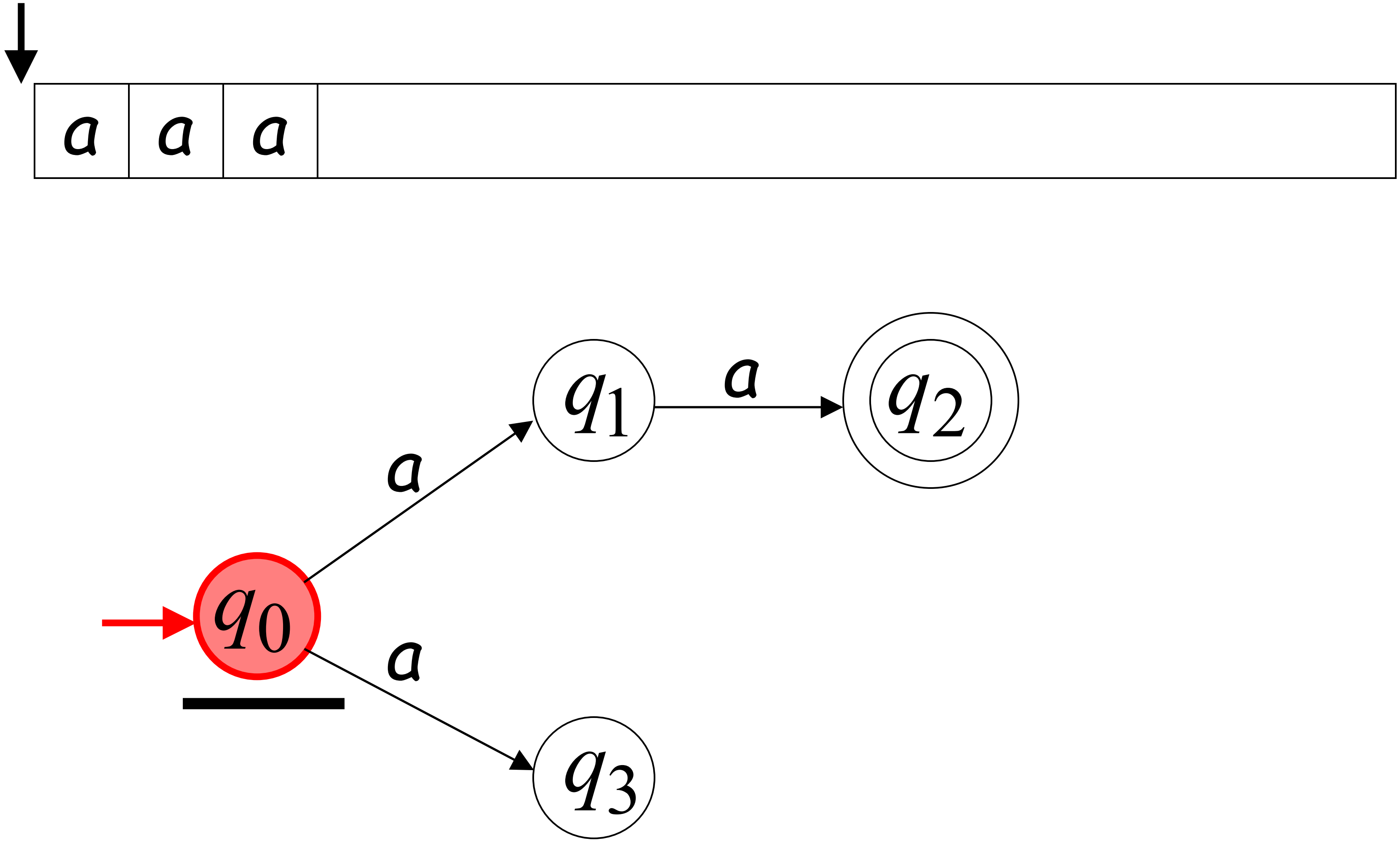
Second Choice



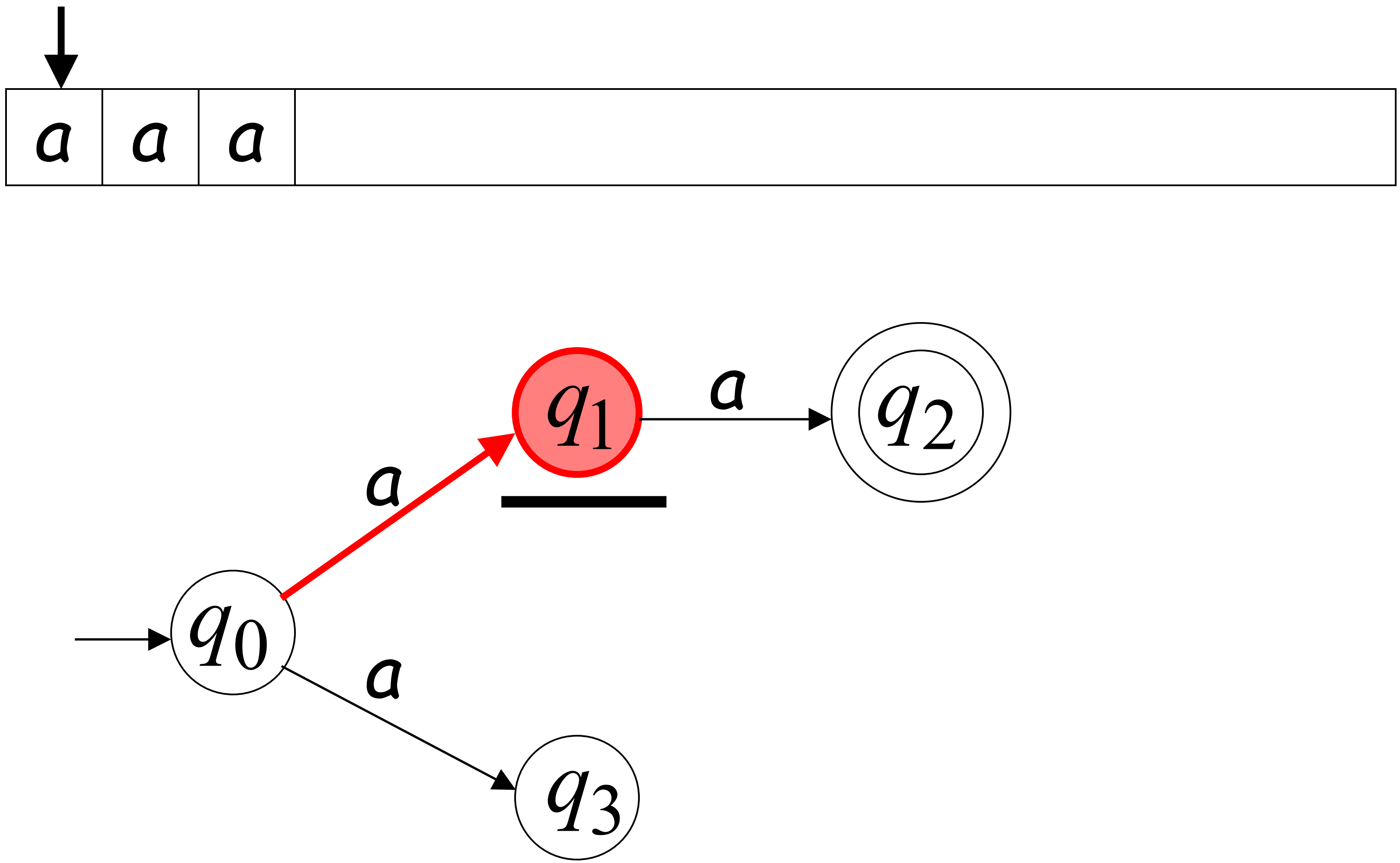
Second Choice



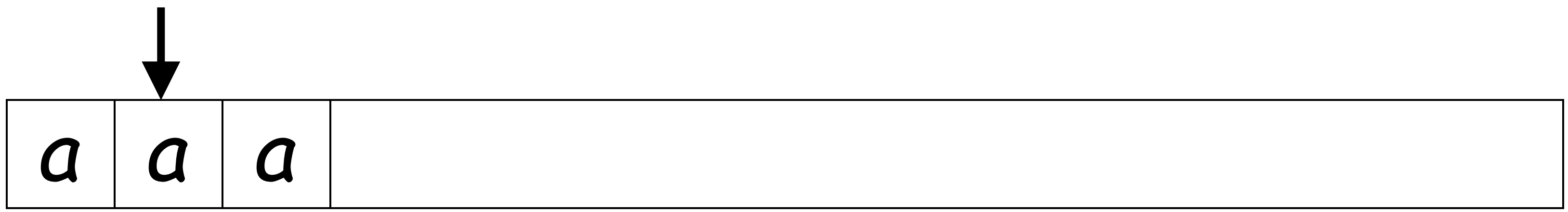
Another Rejection example



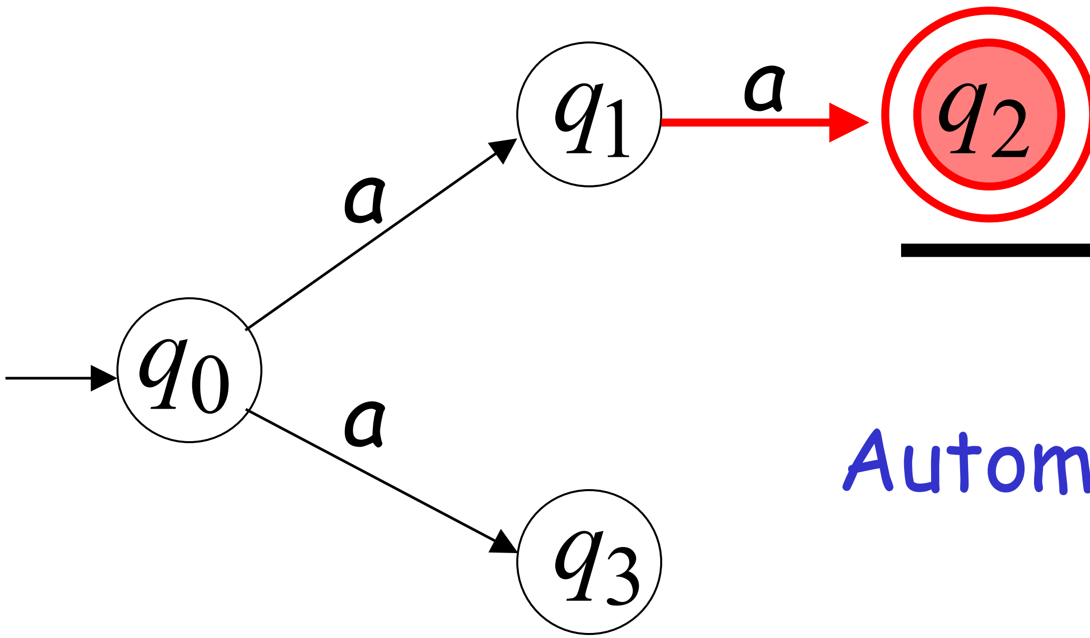
First Choice



First Choice



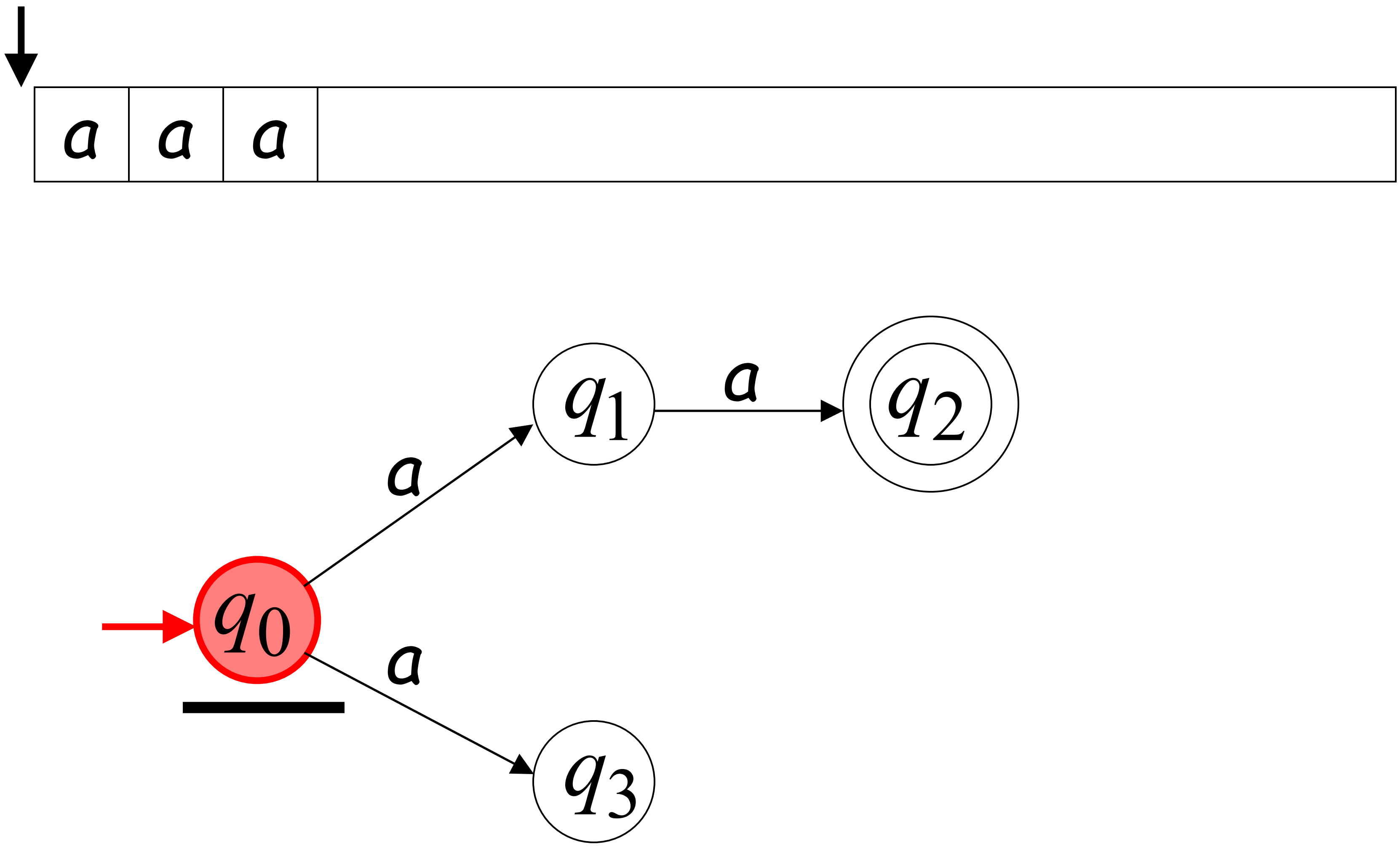
Input cannot be consumed



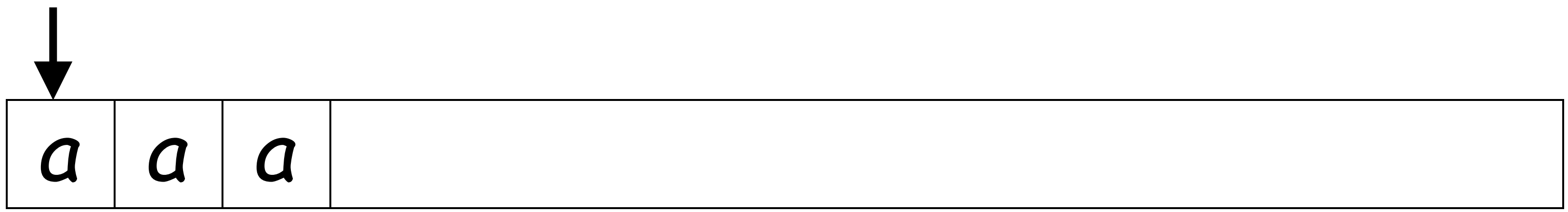
"reject"

Automaton halts

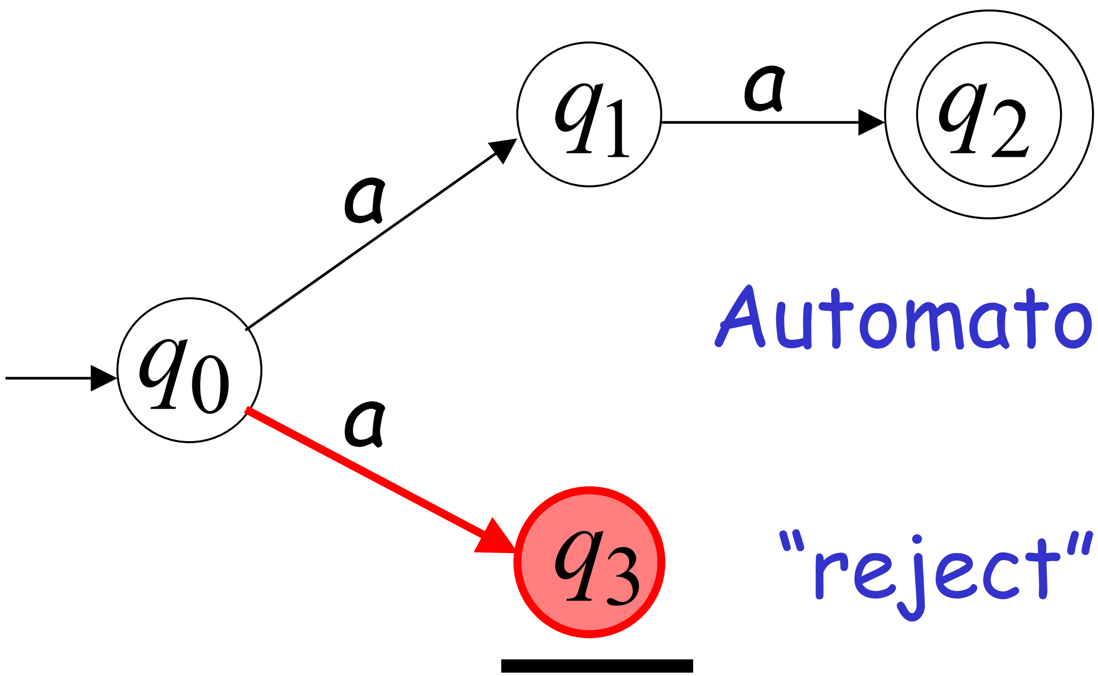
Second Choice



Second Choice



Input cannot be consumed



Automaton halts

"reject"

An NFA rejects a string:

if there is no computation of the NFA that accepts the string.

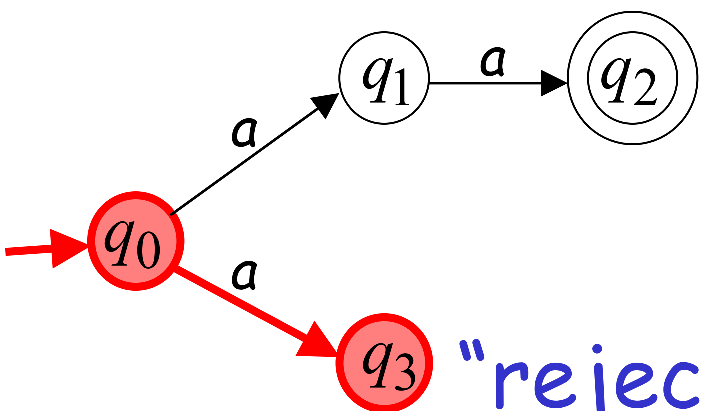
For each computation:

- All the input is consumed and the automaton is in a non final state

OR

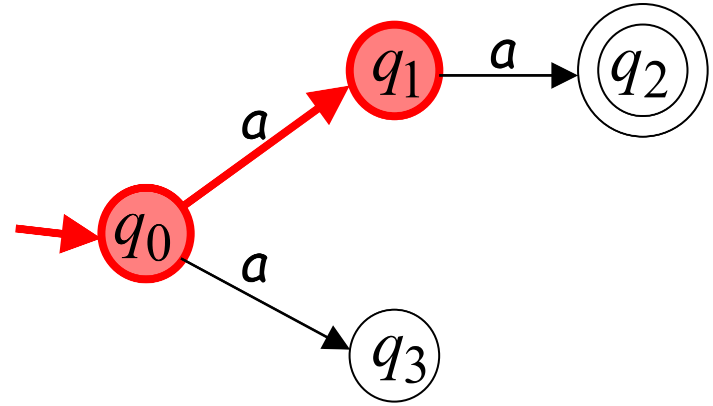
- The input cannot be consumed

a is rejected by the NFA:

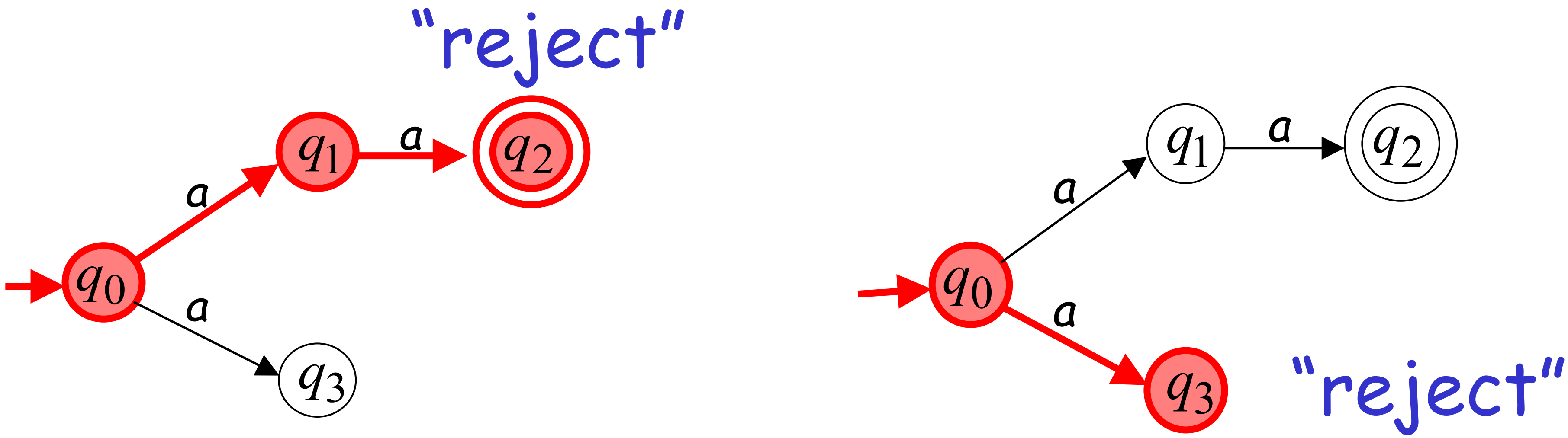


"reject"

"reject"

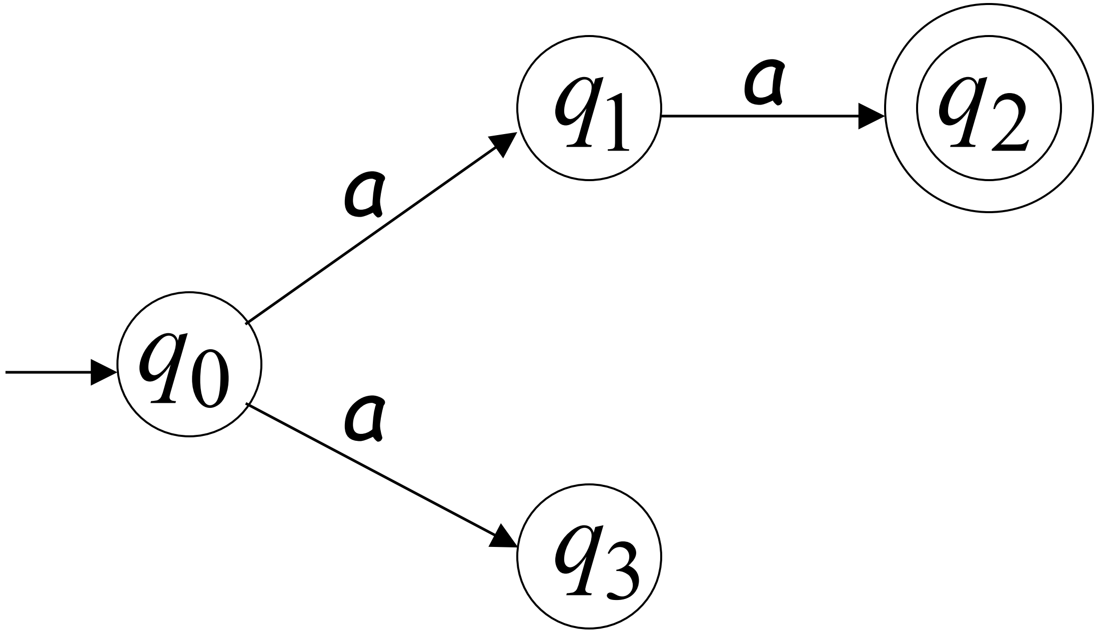


All possible computations lead to rejection aaa is rejected by the NFA:

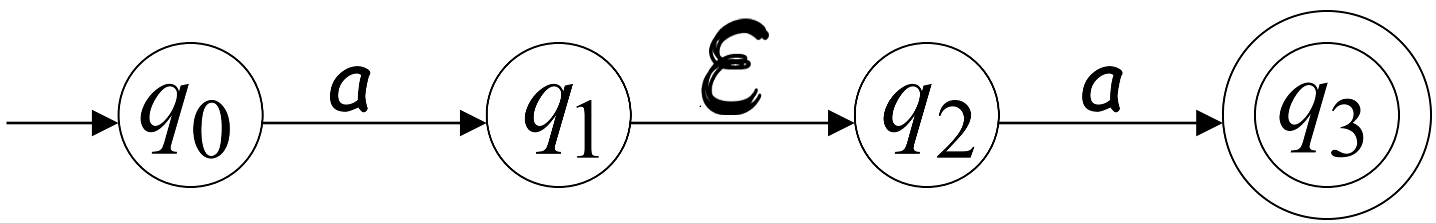


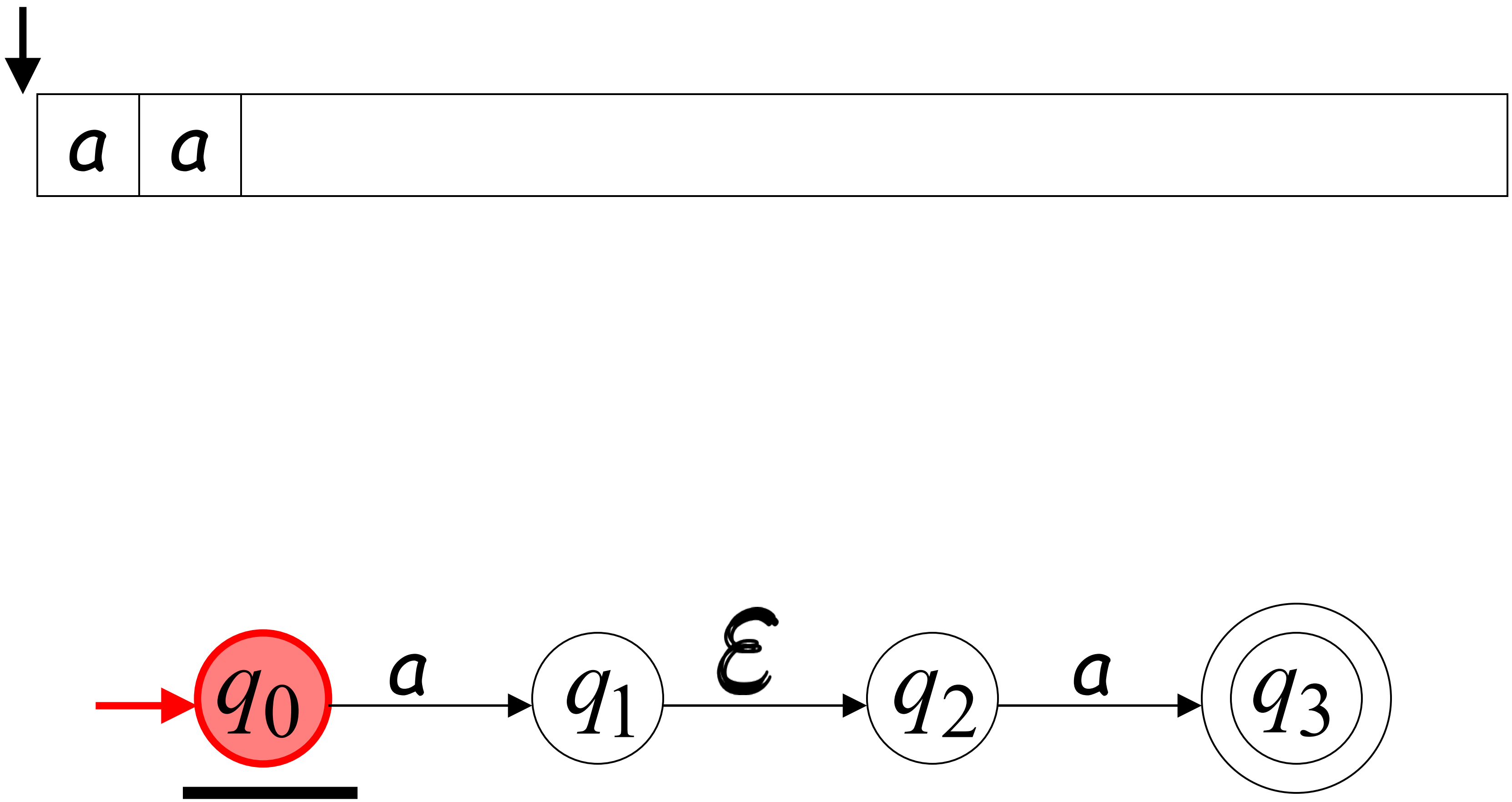
All possible computations lead to rejection

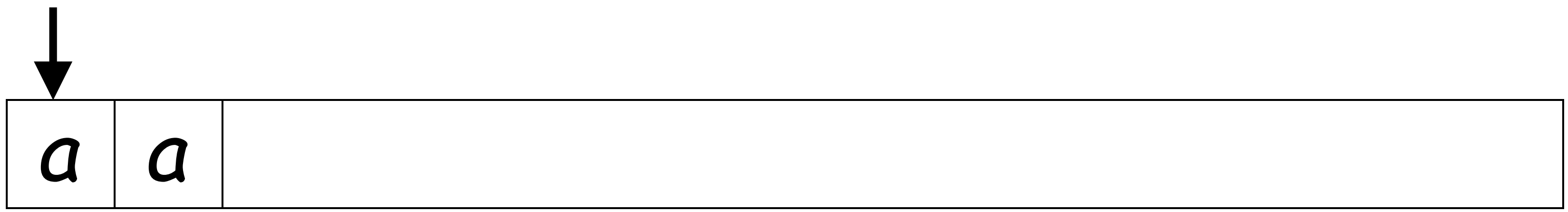
Language accepted: $L = \{aa\}$

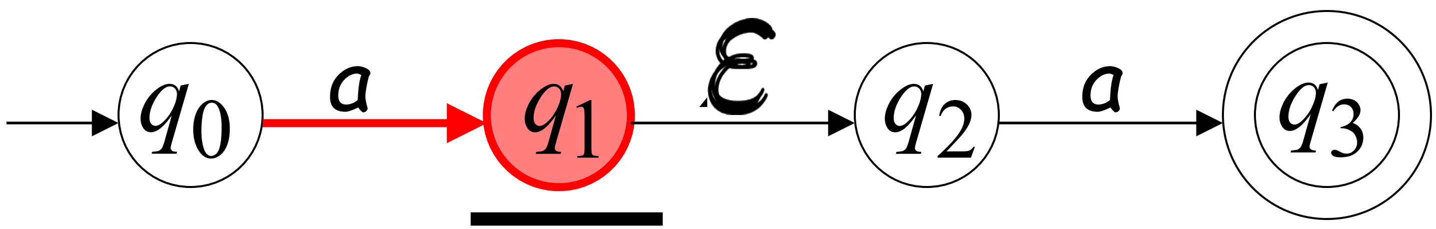


Lambda Transitions

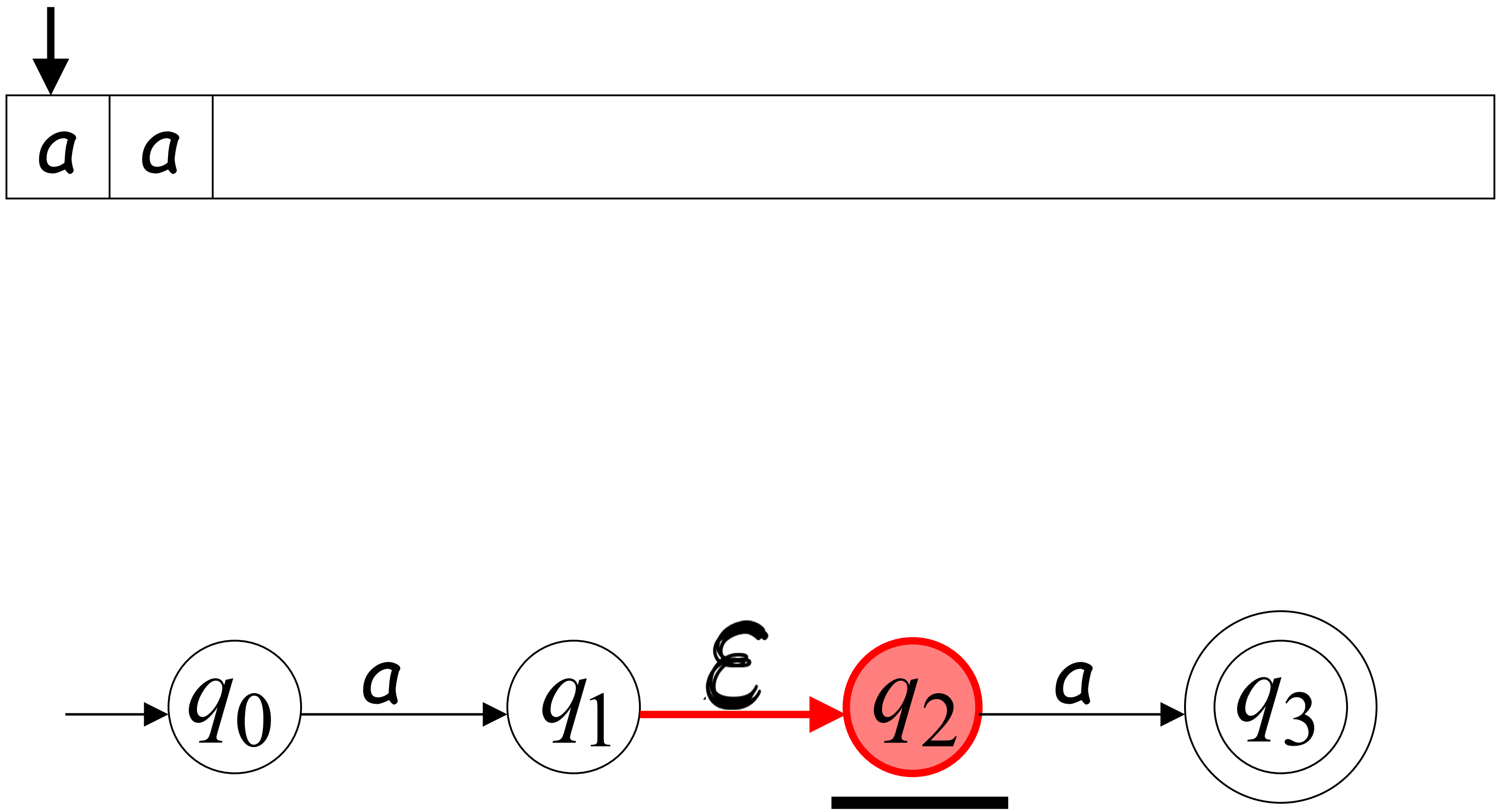



a
a
a
ε
a
q_0
q_1
q_2
q_3

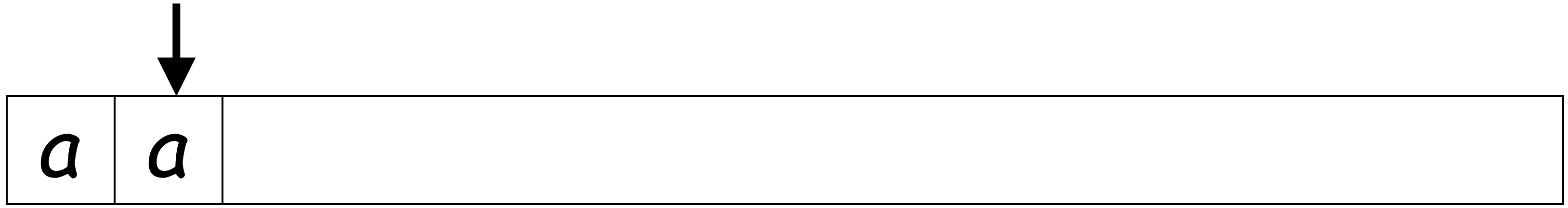

a
a

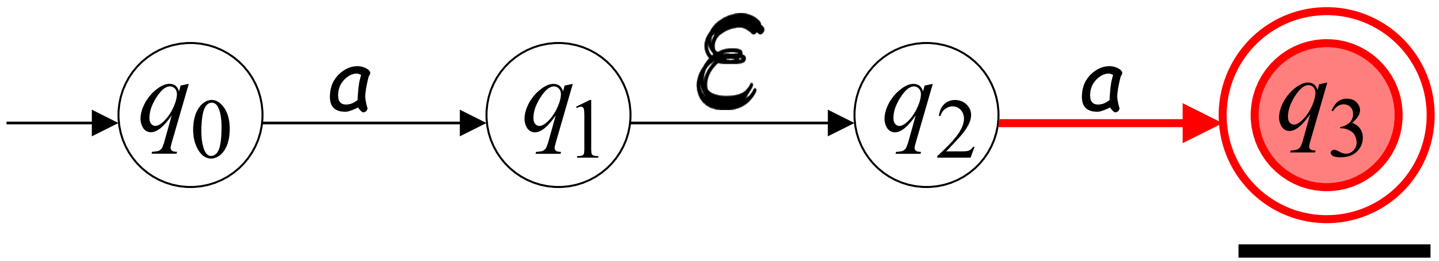

q_0
a
q_1
ε
q_2
a
q_3

input tape head does not move



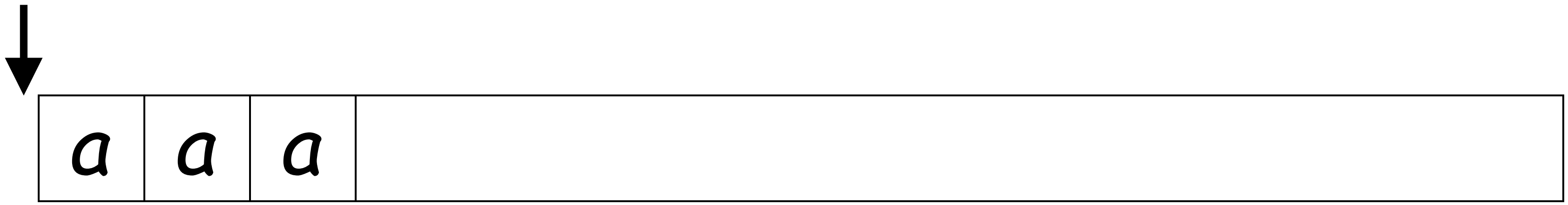
all input is consumed

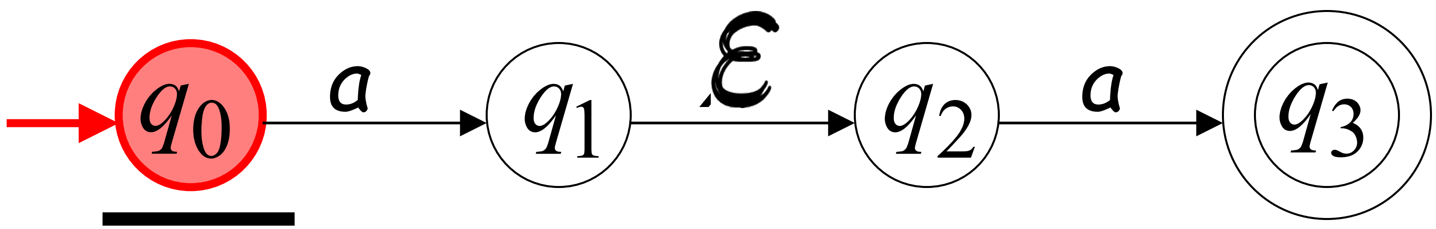


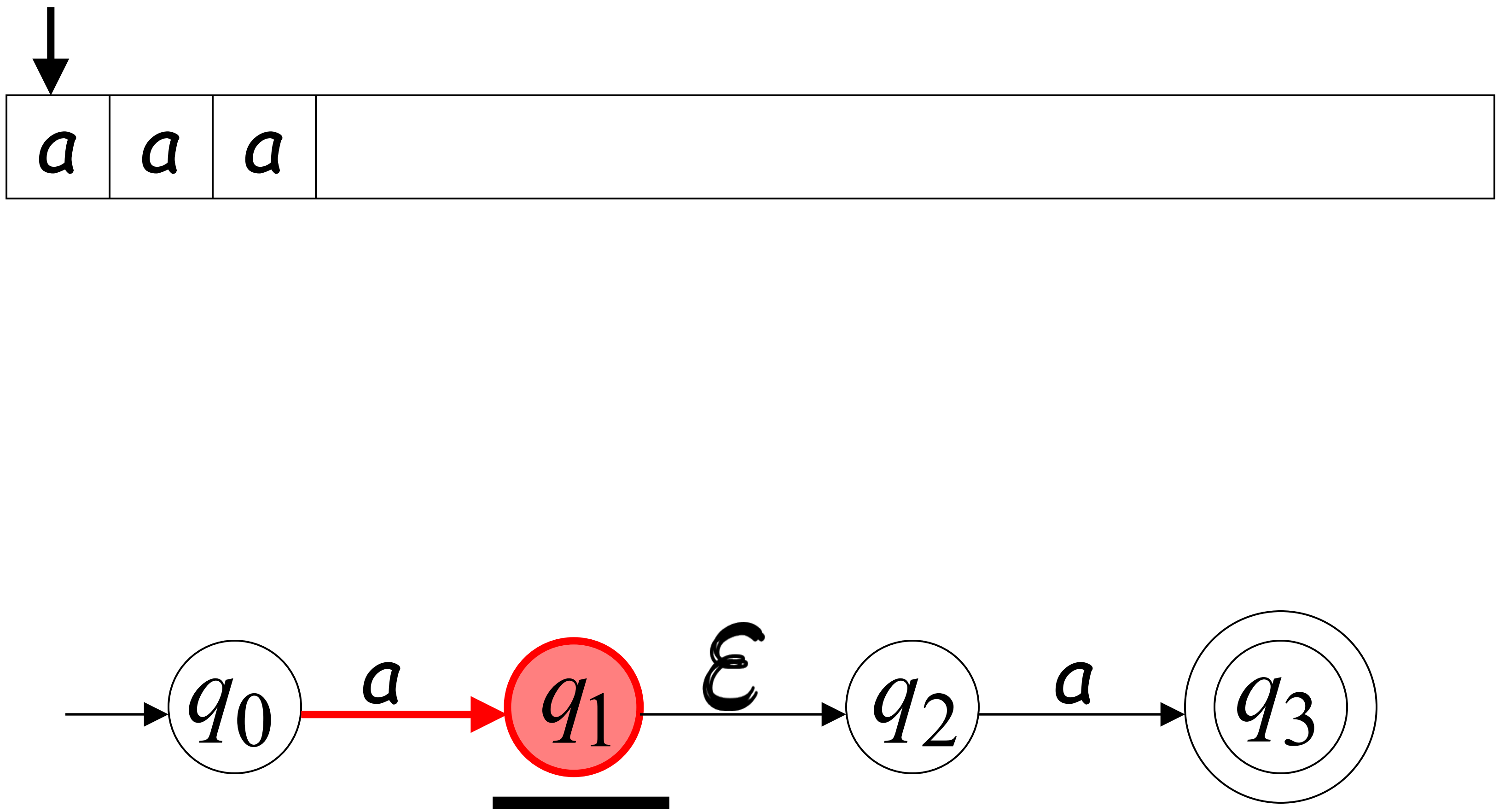


String aa is accepted

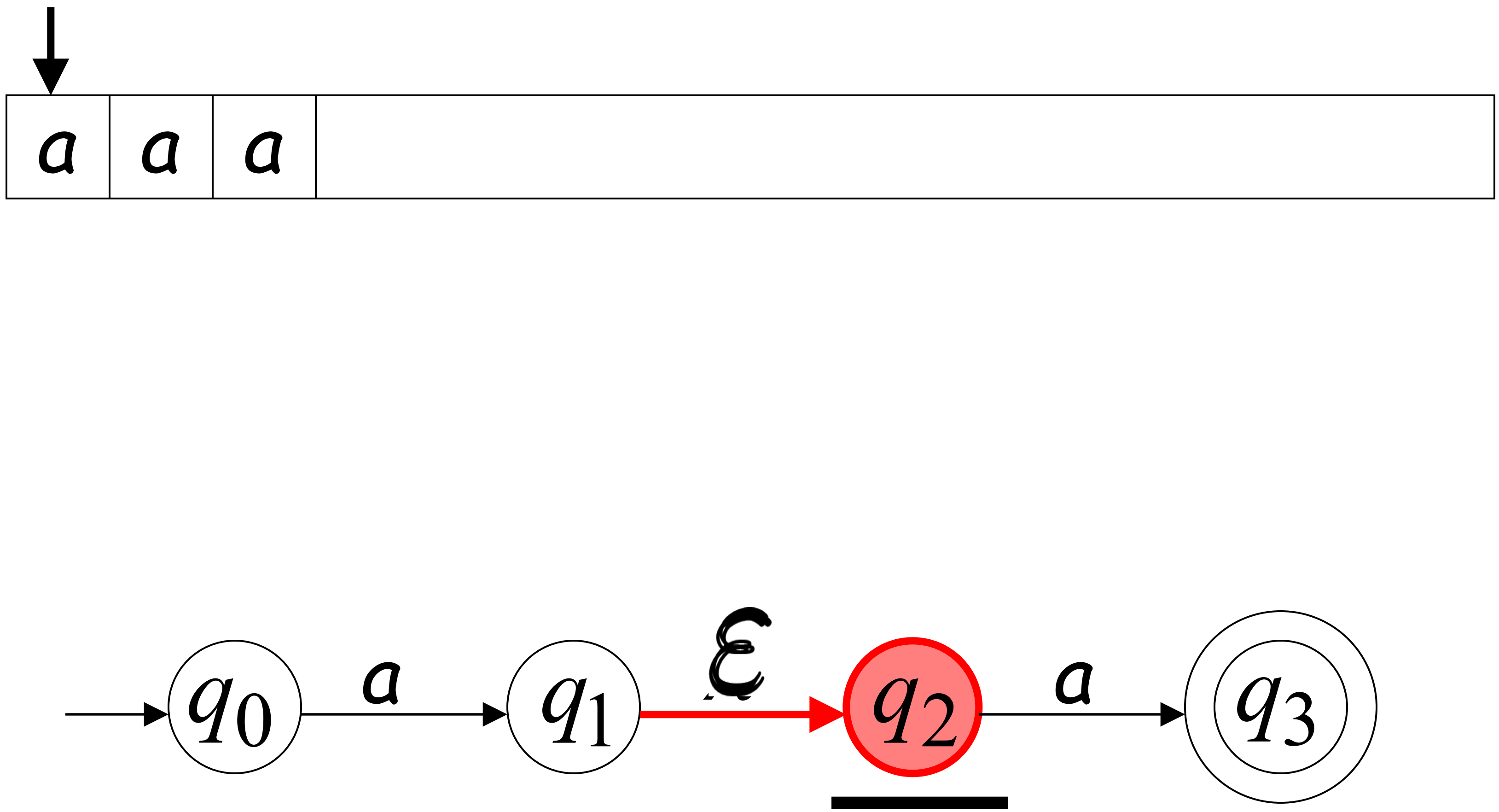
Rejection Example



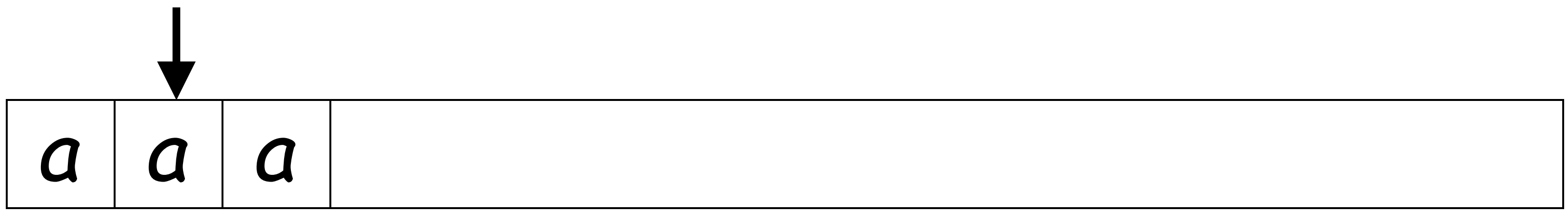



a
a
a
a
ε
a
q_0
q_1
q_2
q_3

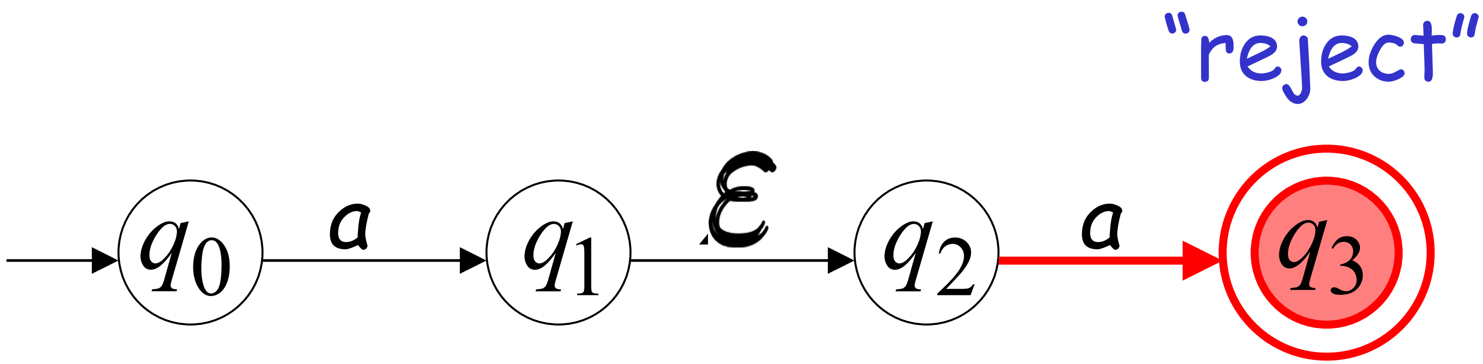
(read head doesn't move)



Input cannot be consumed

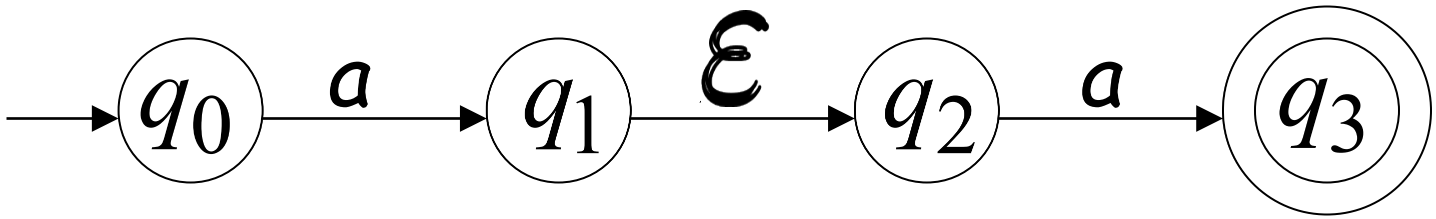


Automaton halts

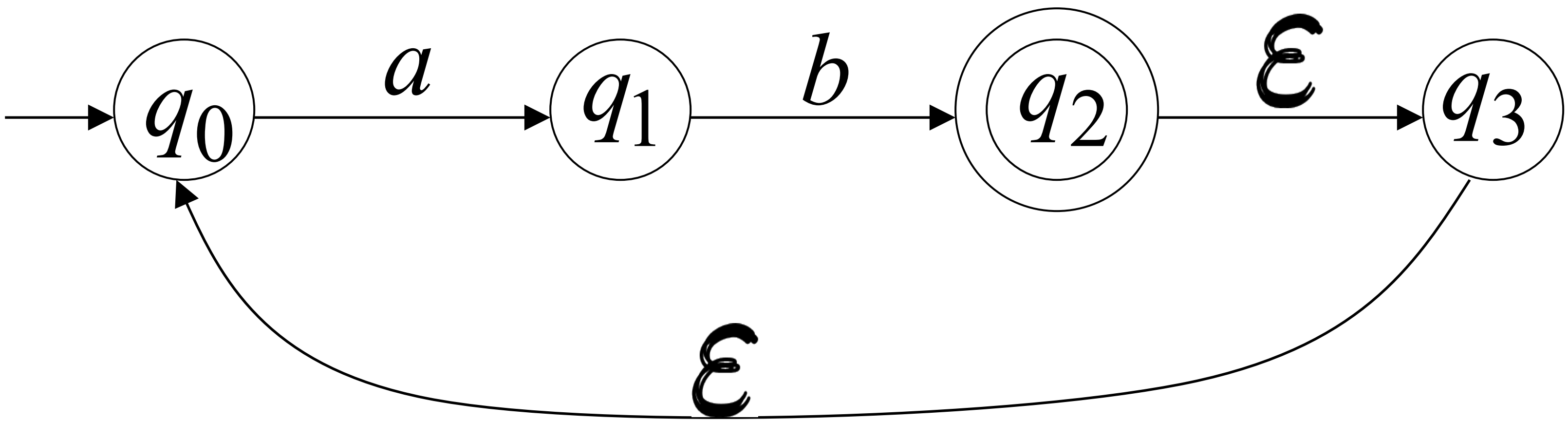


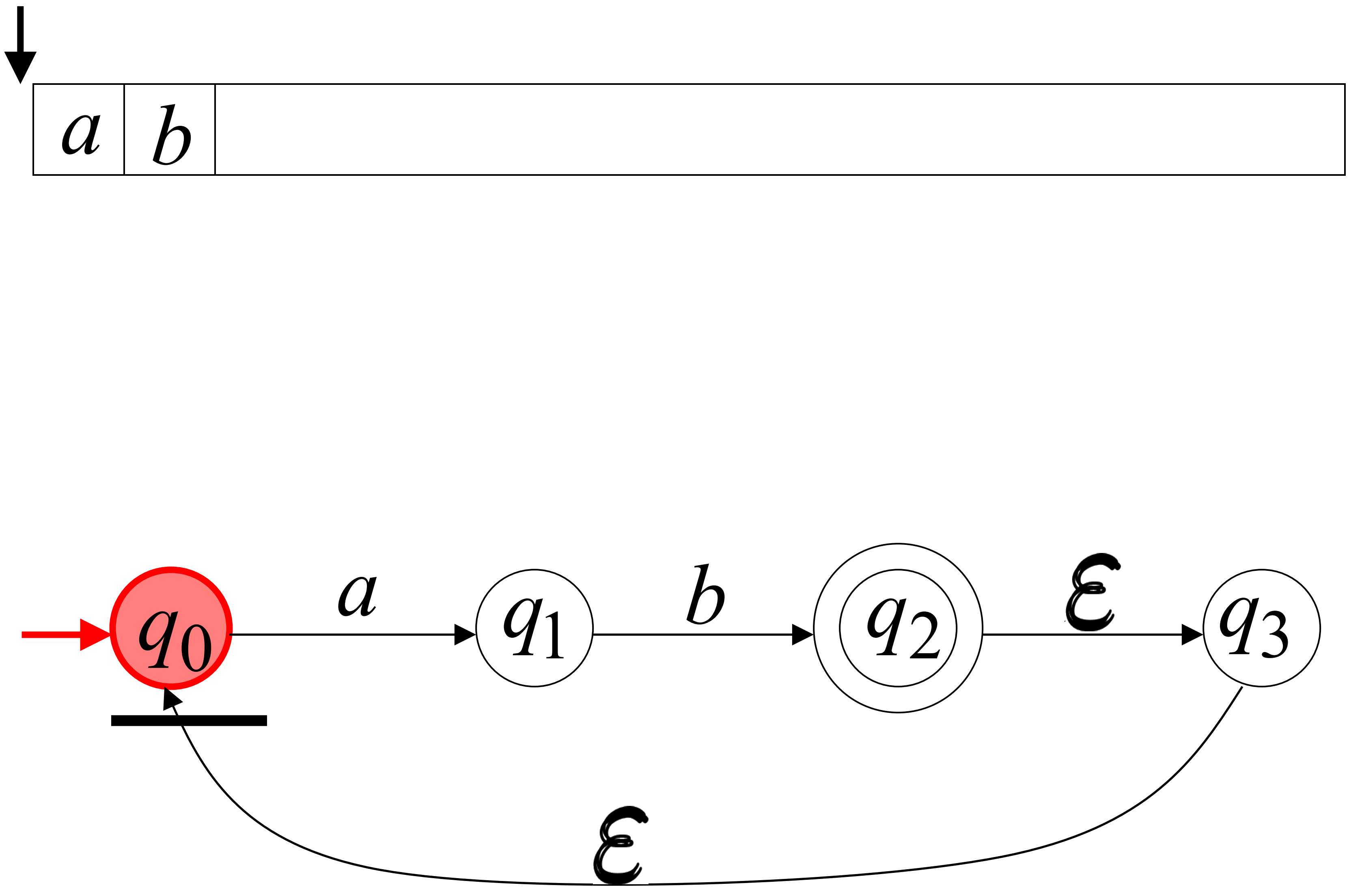
String aaa is rejected

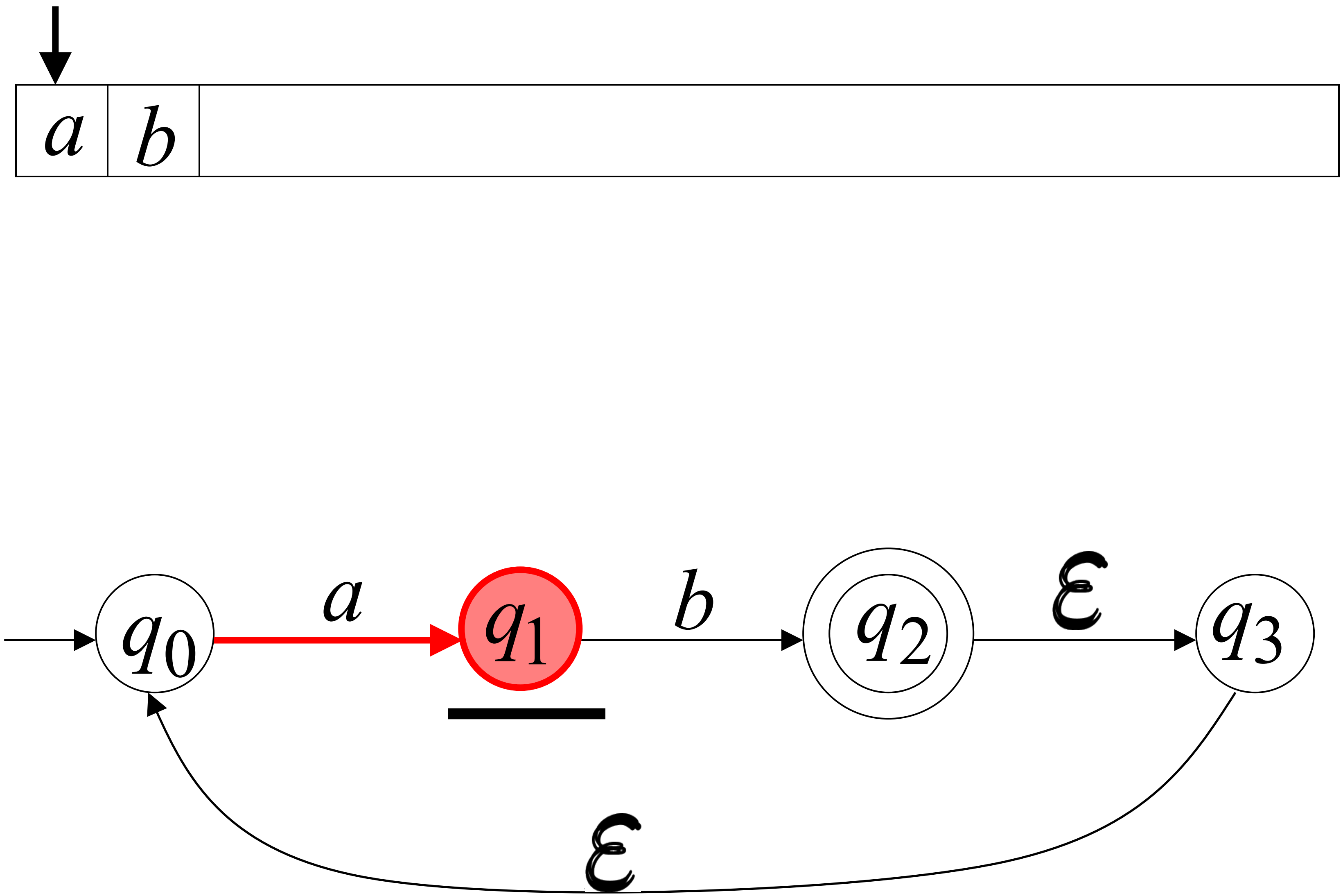
Language accepted: $L = \{aa\}$



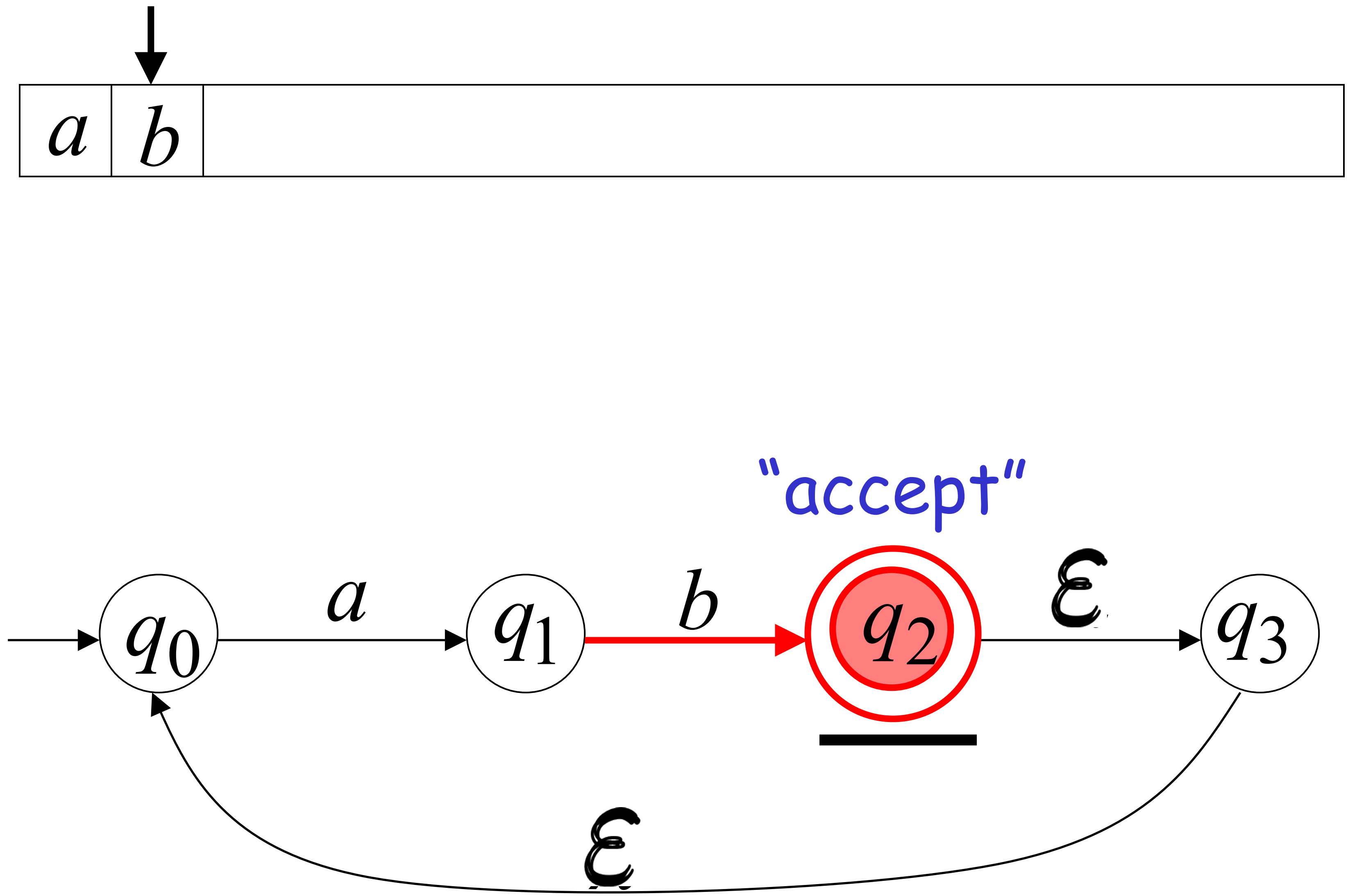
Another NFA Example




a
b
a
b
ε
q_0
q_1
q_2
q_3
ε

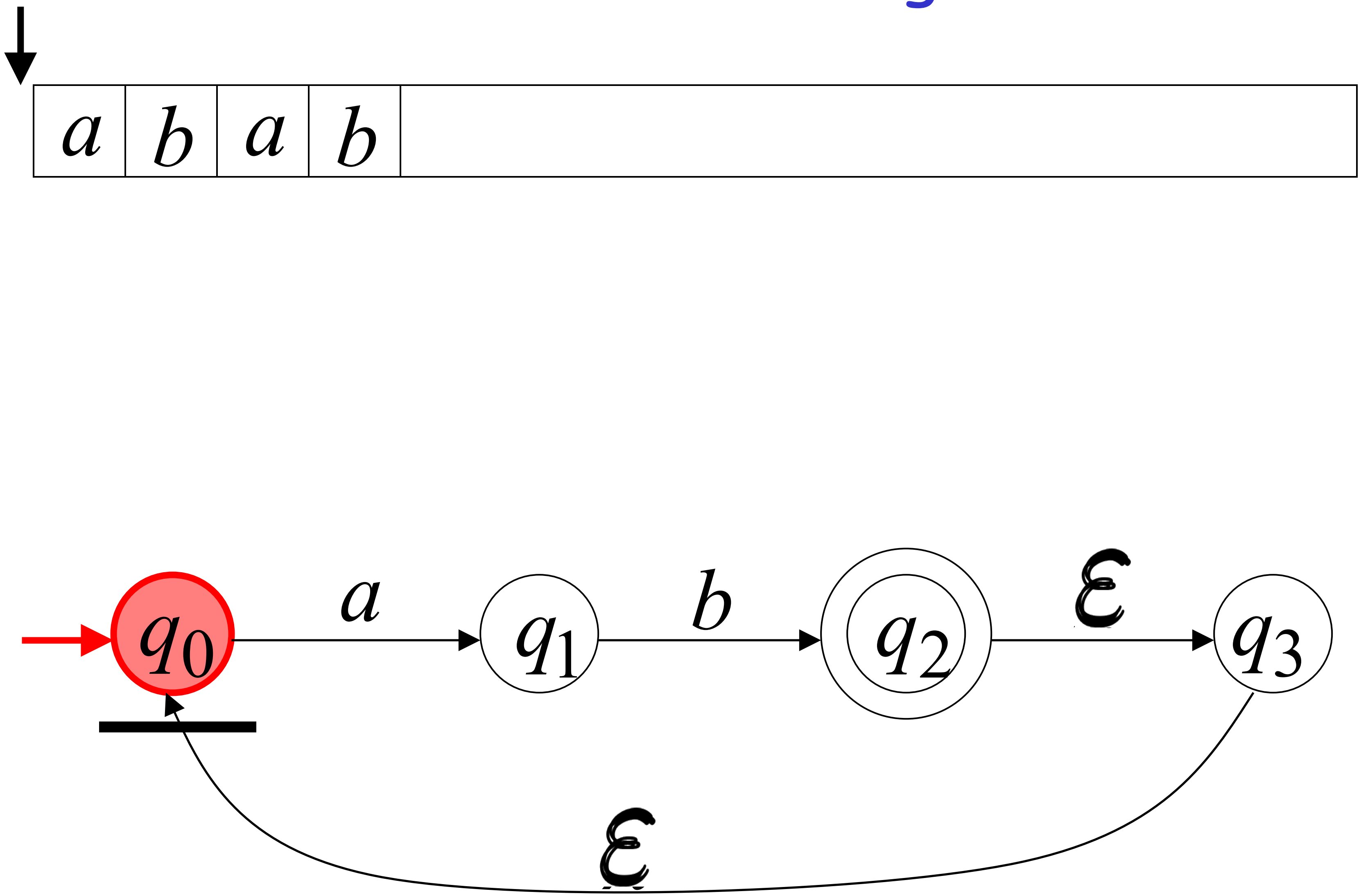


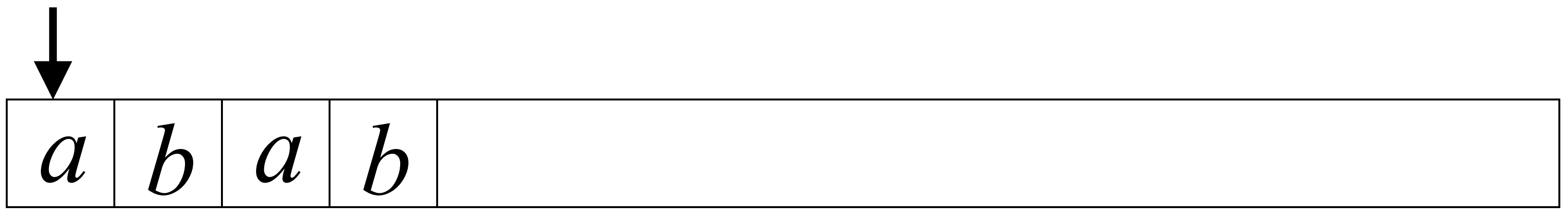
a
b
a
b
ε
ε
q_0
q_1
q_2
q_3

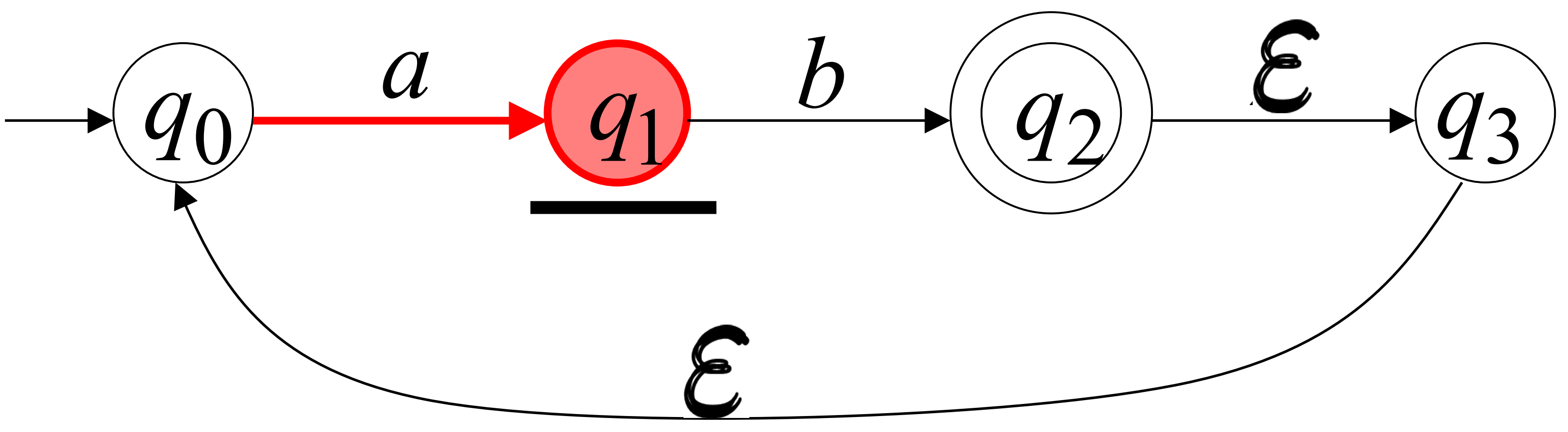


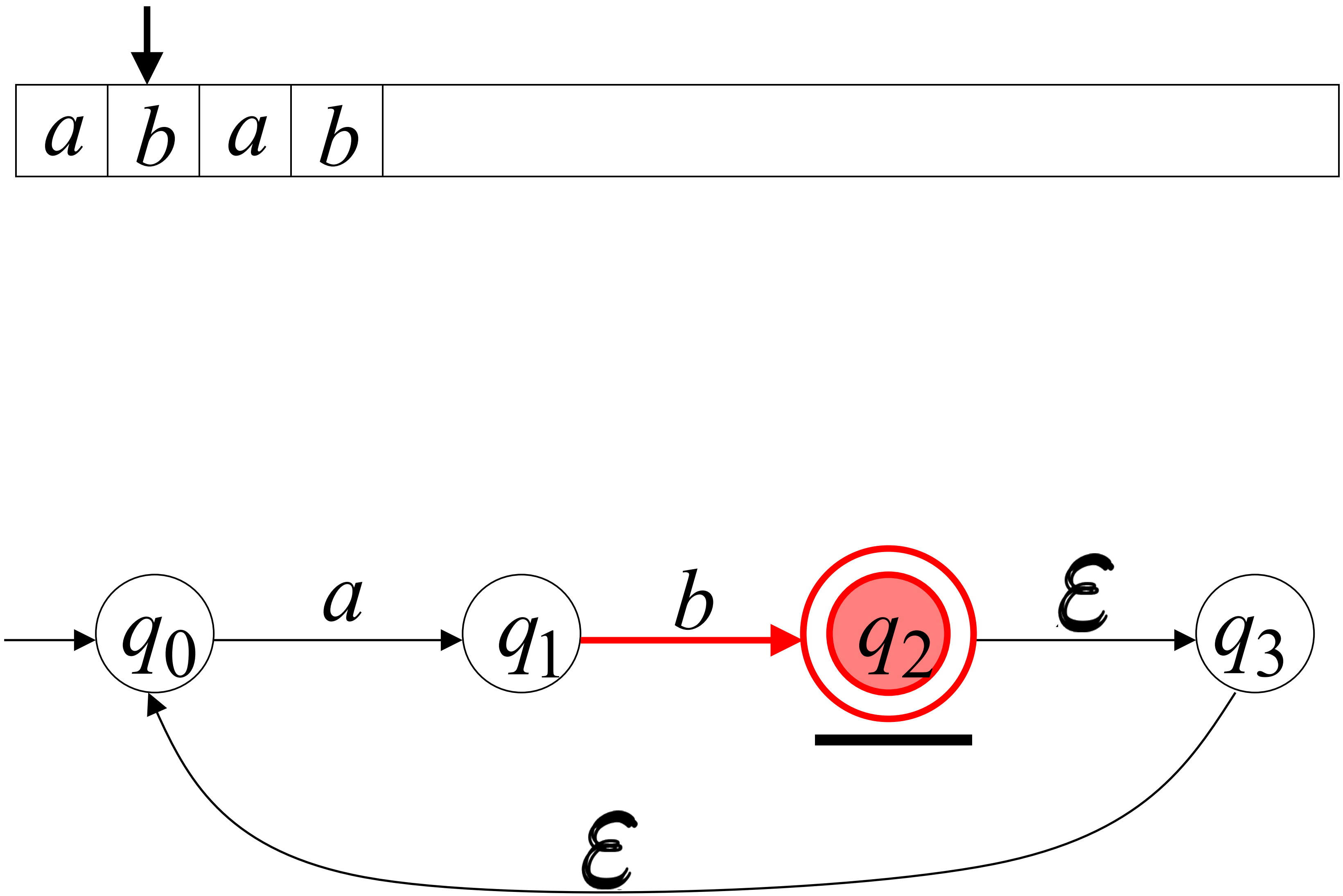
a b
"accept"
q_0 a q_1 b q_2 ε q_3
ε

Another String

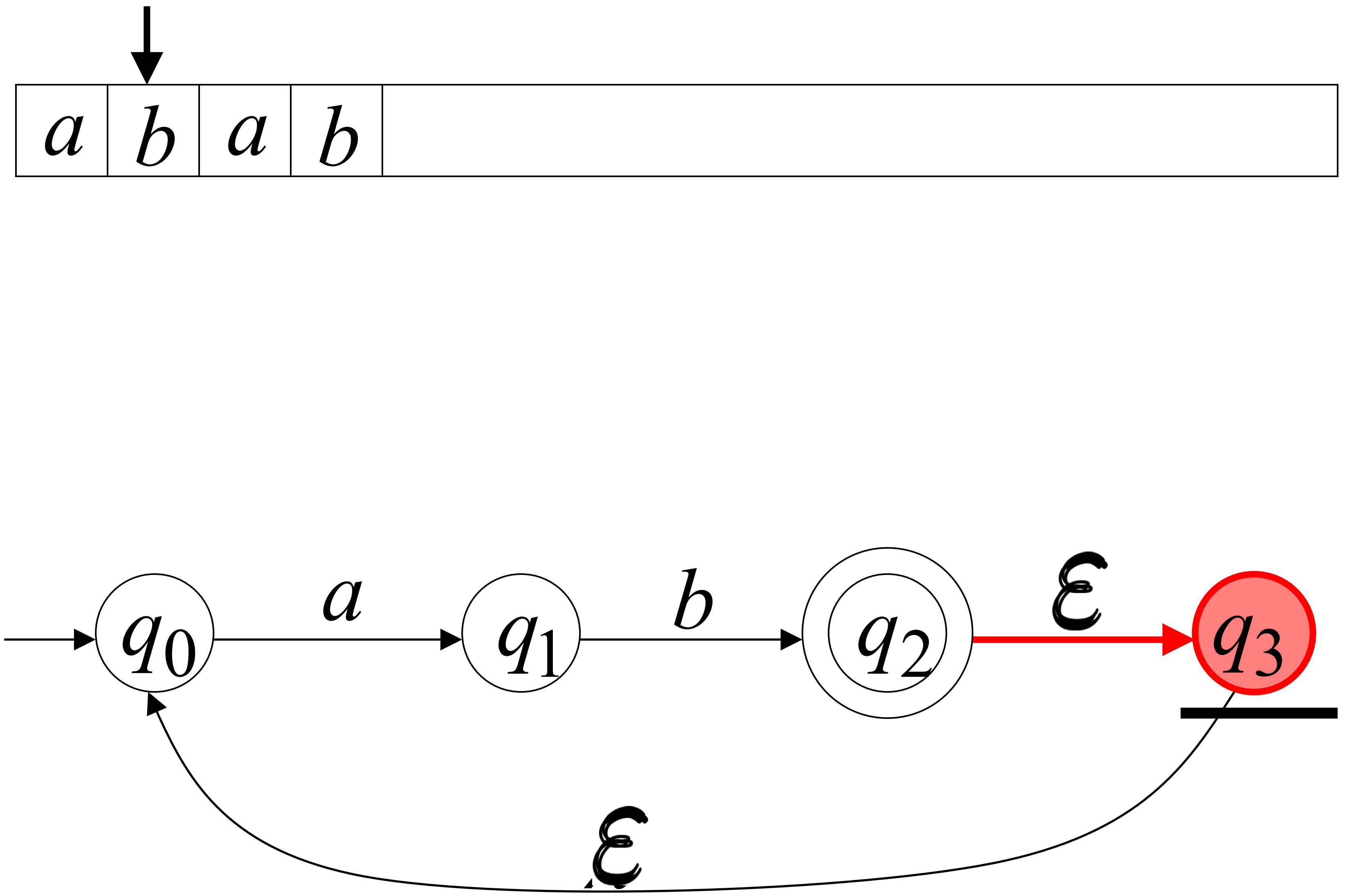



a
b
a
b

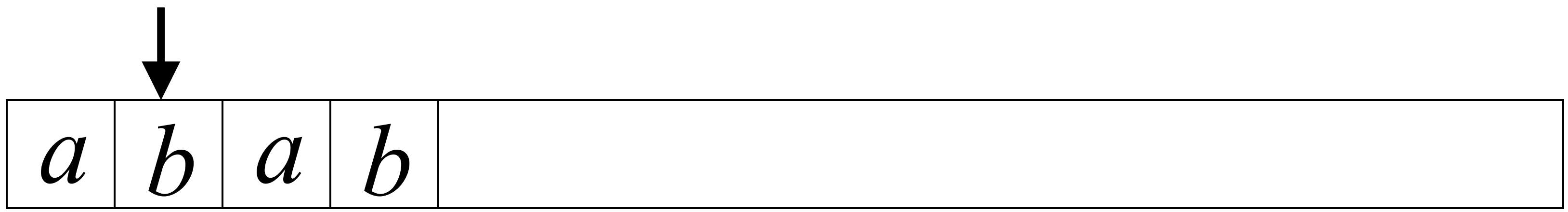

a
b
ε
ε
q_0
q_1
q_2
q_3

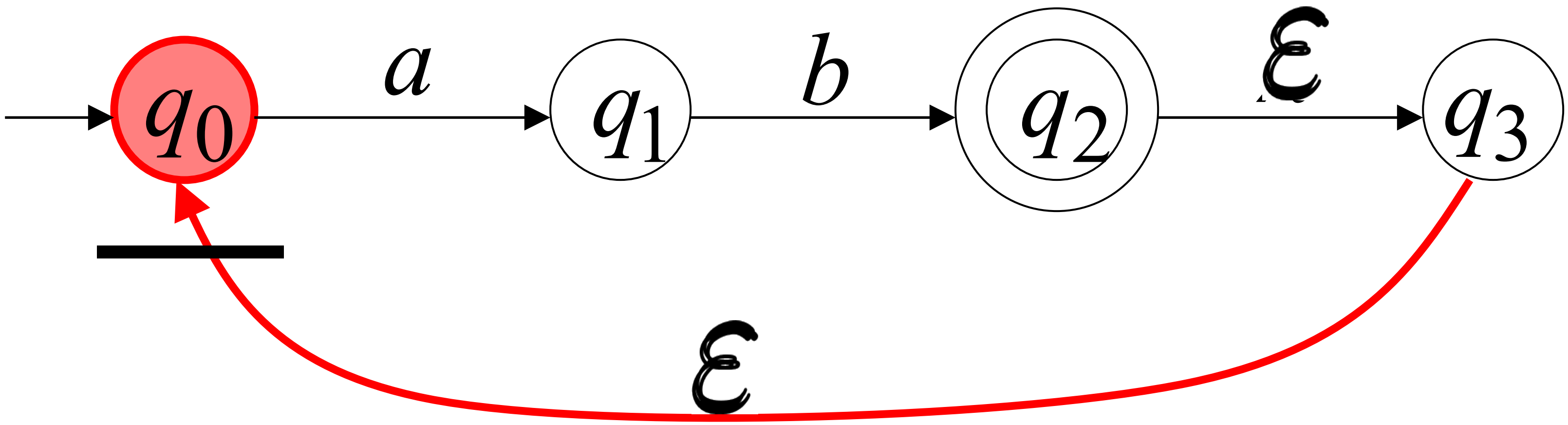


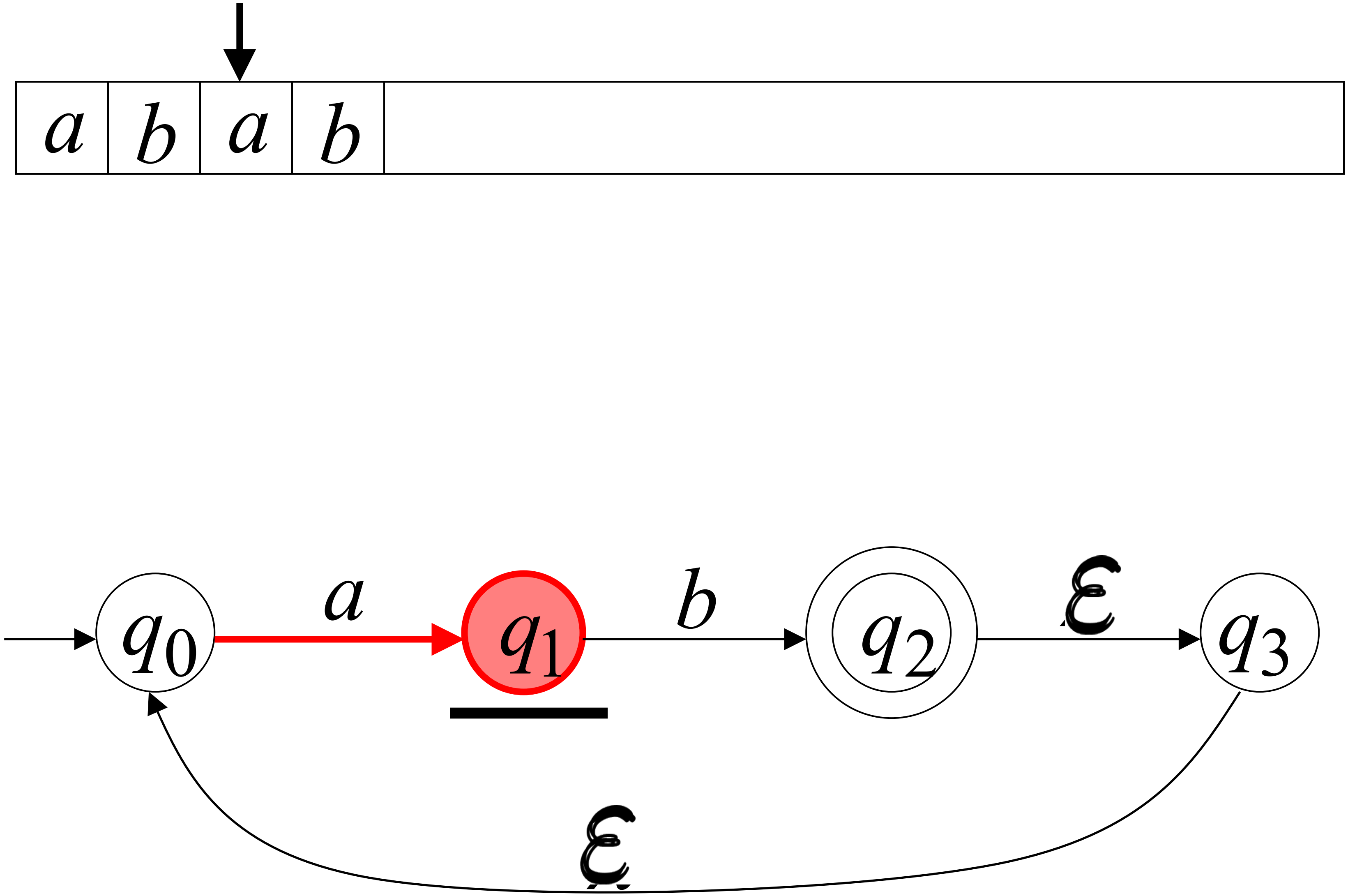
a
b
a
b
q_0
a
q_1
b
q_2
ε
q_3
ε

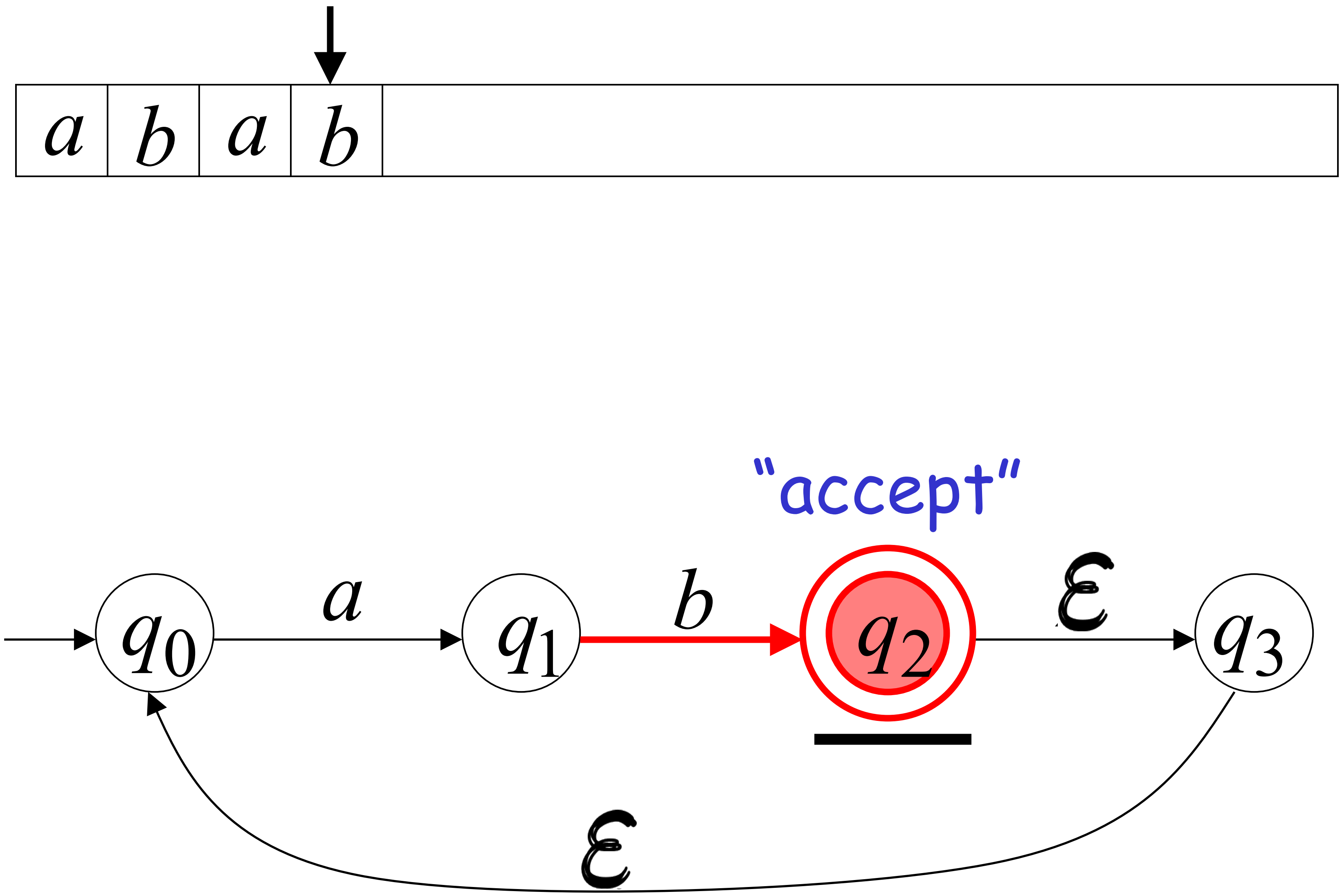


a
b
a
b
a
b
ε
ε
q_0
q_1
q_2
q_3


a
b
a
b


q_0
a
q_1
b
q_2
ε
q_3
ε

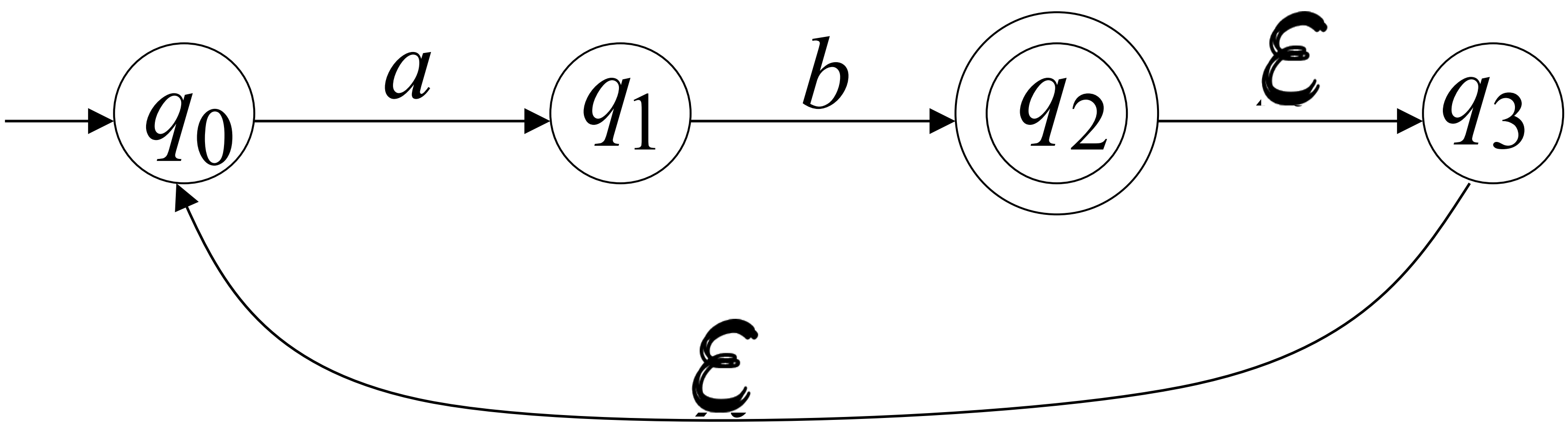

a
b
a
b
a
b
ε
ε
q_0
q_1
q_2
q_3


a
b
a
b
"accept"
a
b
ε
ε
q_0
q_1
q_2
q_3

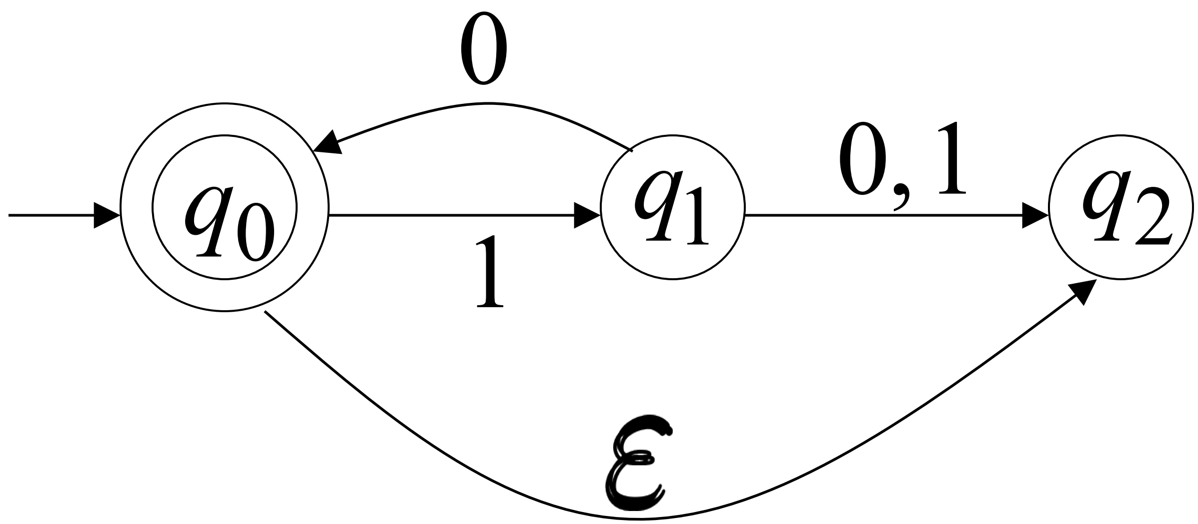
Language accepted

$$L = \{ab,\ abab,\ ababab,\ \ldots\}$$

$$= \{ab\}^+$$

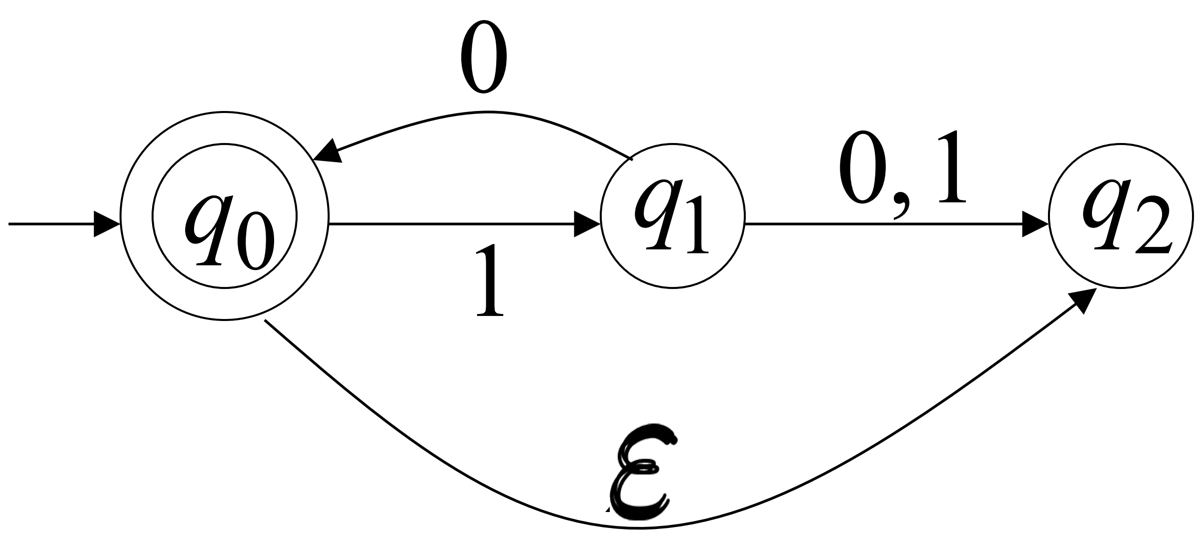


Another NFA Example



Language accepted

$$L(M) = \{\varepsilon,\ 10,\ 1010,\ 101010,\ \ldots\}$$
$$= \{10\}^*$$



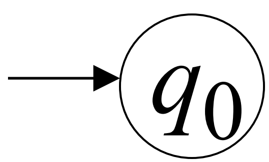
(redundant state)

Remarks:

- The ε symbol never appears on the input tape

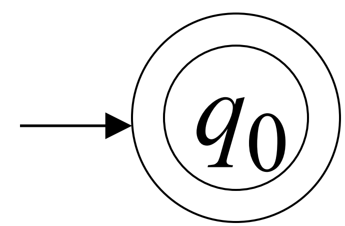
- Simple automata:

M_1



$L(M_1) = \{\}$

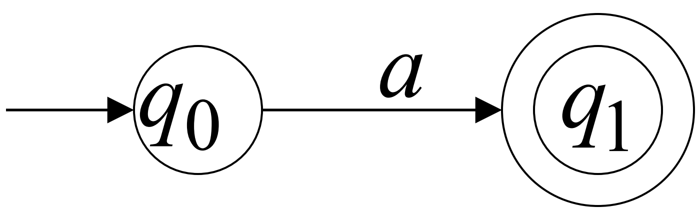
M_2



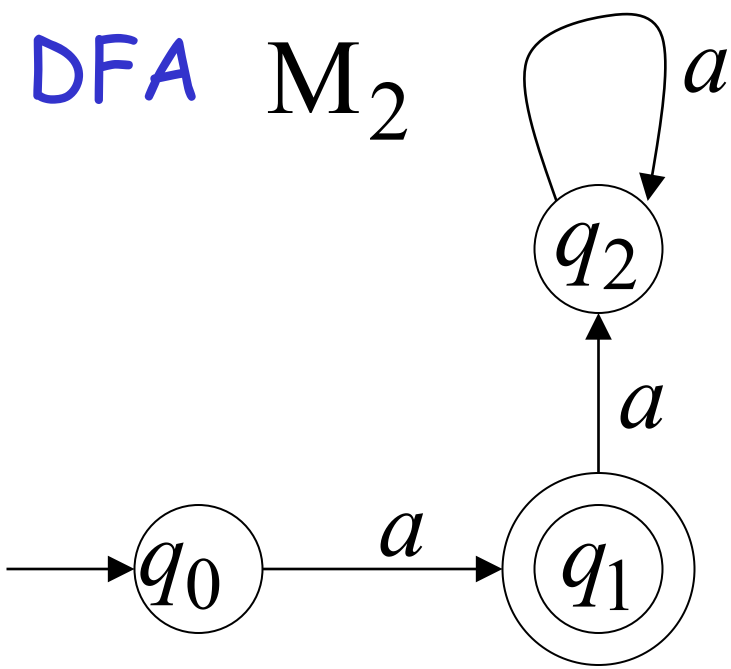
$L(M_2) = \{\varepsilon\}$

• NFAs are interesting because we can express languages easier than DFAs

NFA M_1



DFA M_2



$$L(M_1) = \{a\}$$

$$L(M_2) = \{a\}$$

Formal Definition of NFAs

$$M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$$

$Q:$ Set of states, i.e. $\{q_0, q_1, q_2\}$

$\Sigma:$ Input aplhabet, i.e. $\{a, b\}$ $\qquad \varepsilon \notin \Sigma$

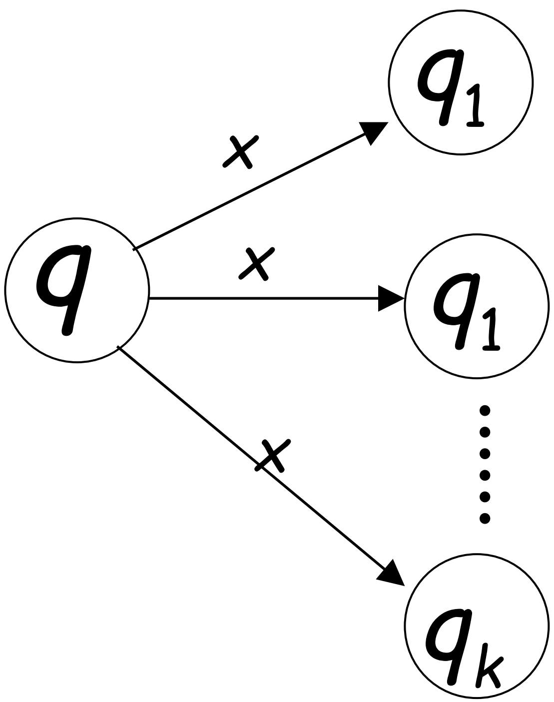
$\delta:$ Transition function

$q_0:$ Initial state

$F:$ Accepting states

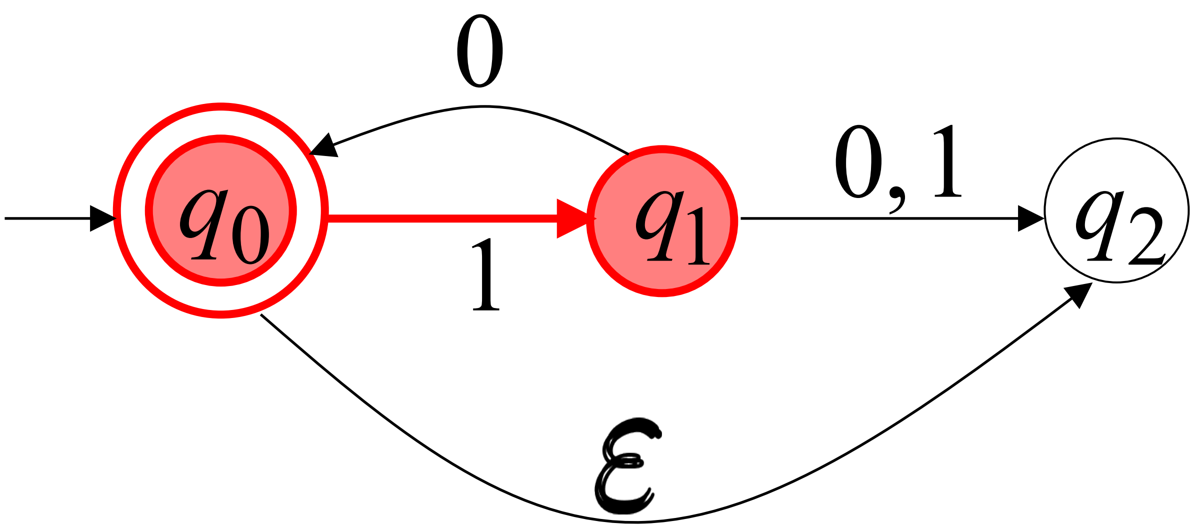
Transition Function δ

$$\delta(q, x) = \{q_1, q_2, \ldots, q_k\}$$

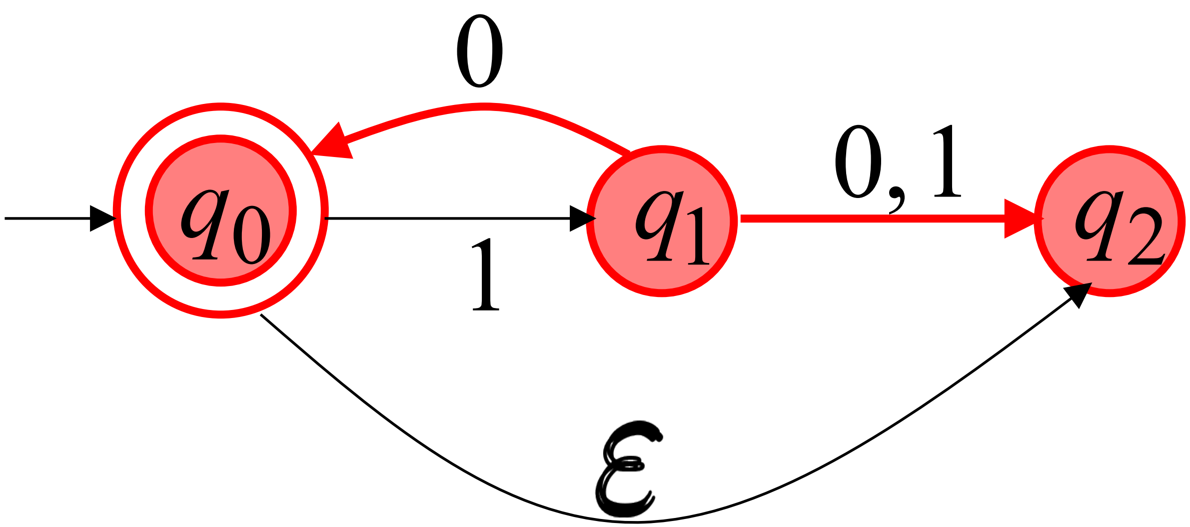


resulting states with following one transition with symbol x

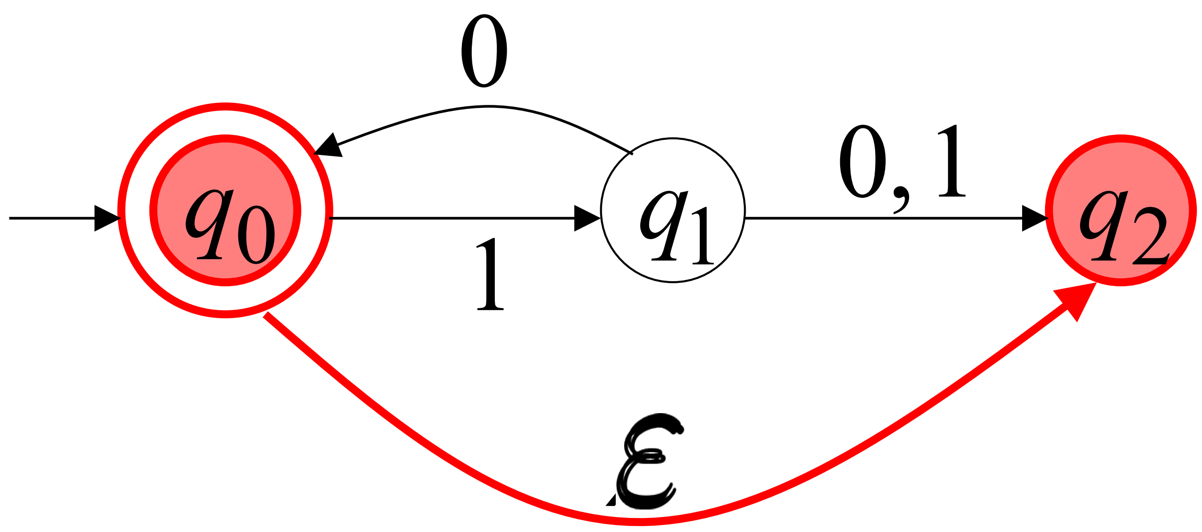
$$\delta(q_0, 1) = \{q_1\}$$



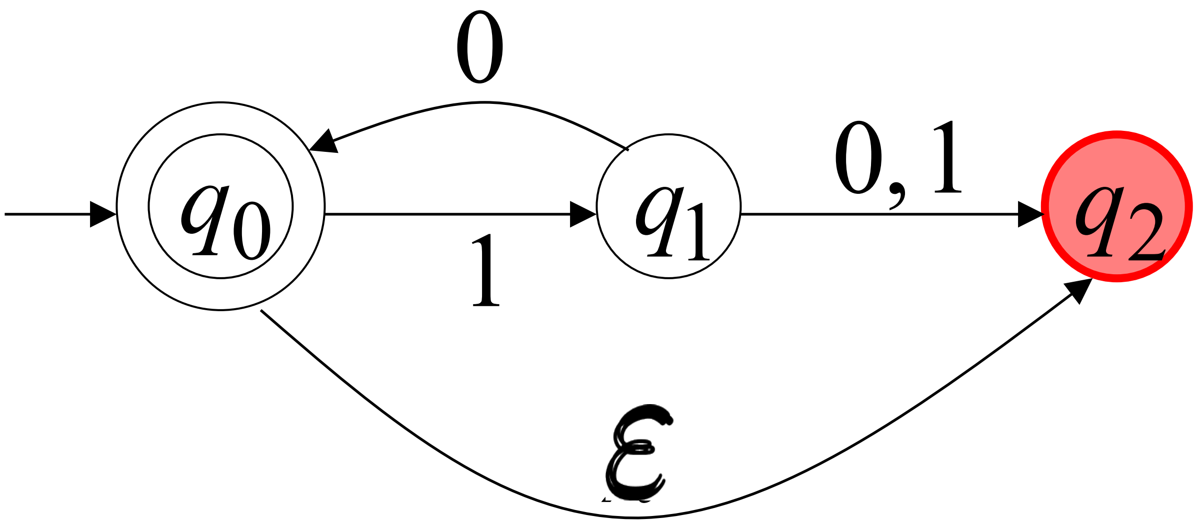
$$\delta(q_1, 0) = \{q_0, q_2\}$$



$$\delta(q_0, \varepsilon) = \{q_2\}$$



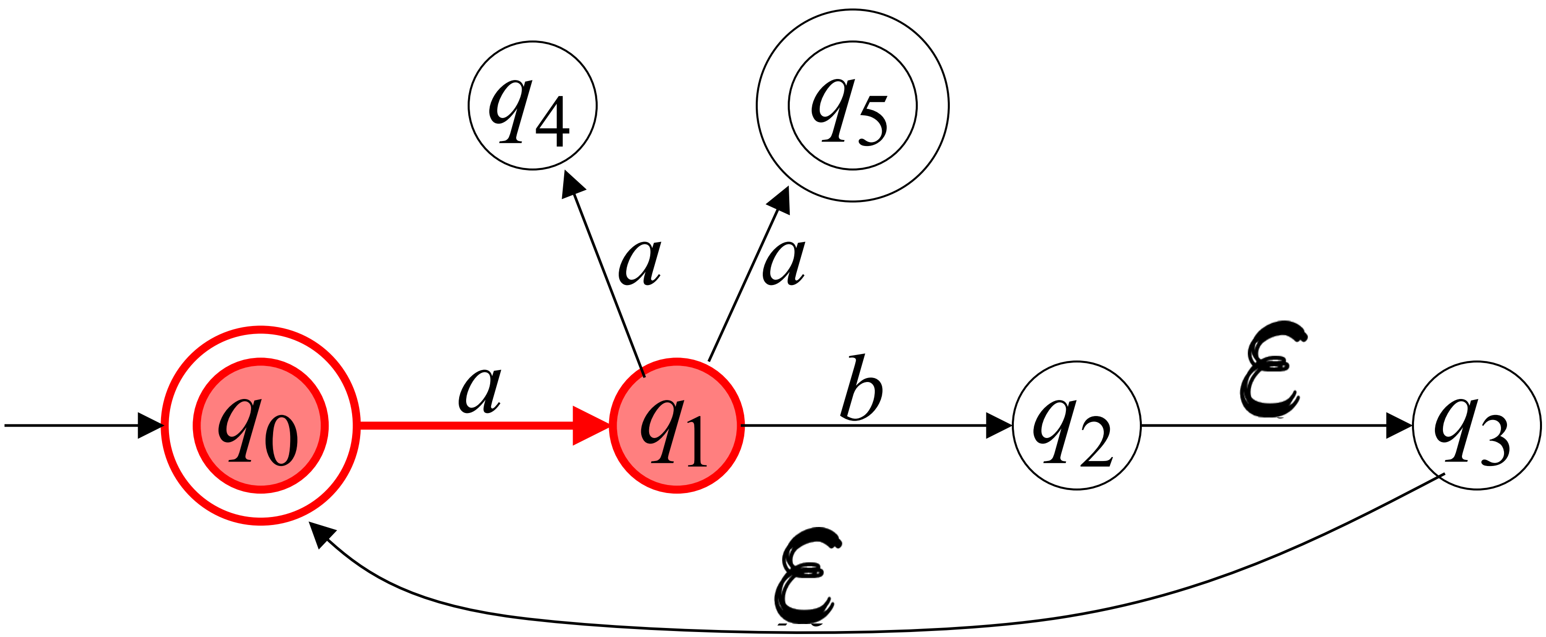
$$\delta(q_2, 1) = \varnothing$$



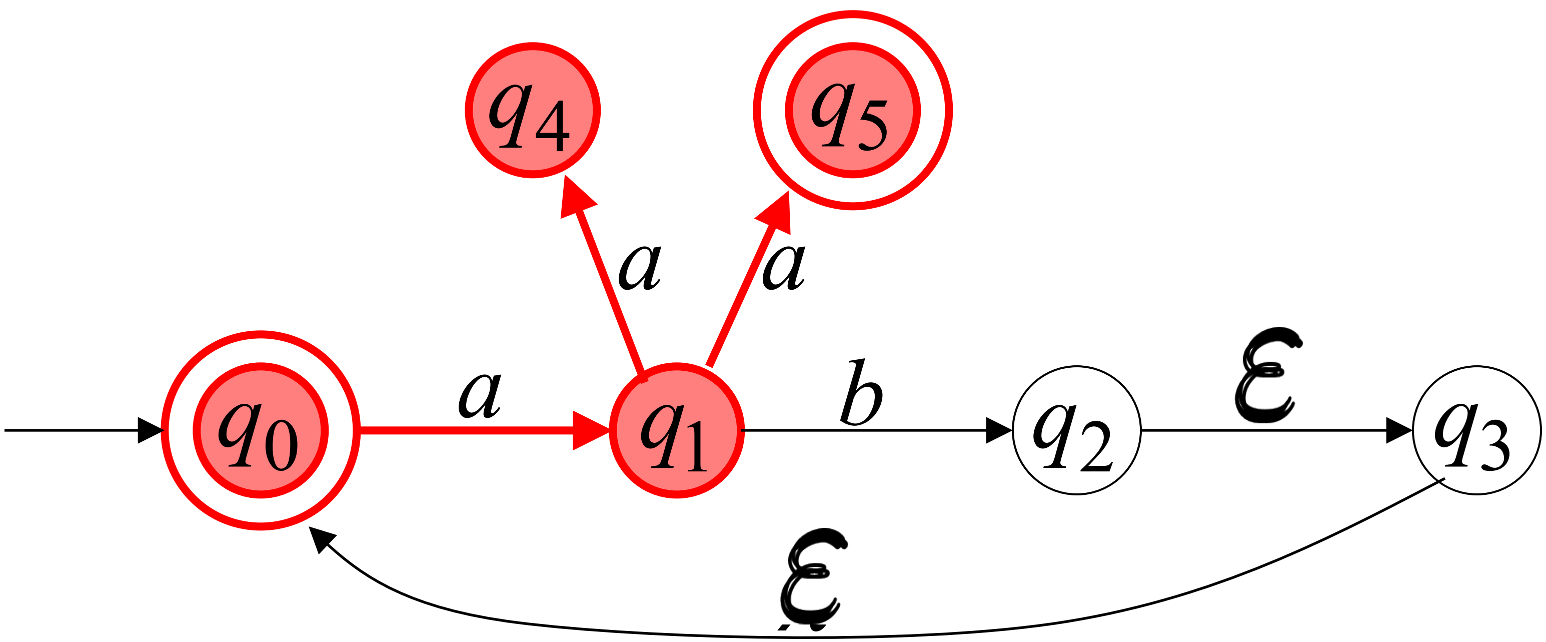
Extended Transition Function δ^*

Same with δ but applied on strings

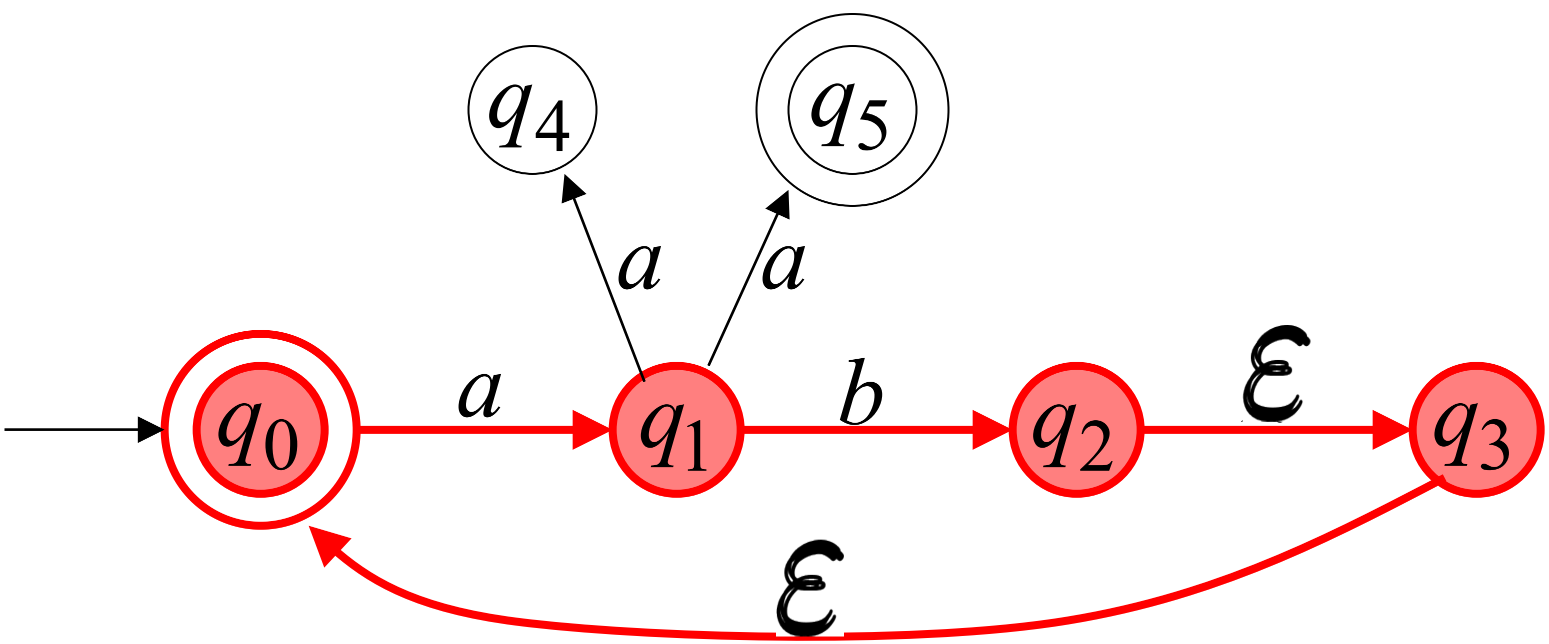
$$\delta^*(q_0, a) = \{q_1\}$$



$$\delta^*(q_0, aa) = \{q_4, q_5\}$$



$$\delta^*(q_0, ab) = \{q_2, q_3, q_0\}$$



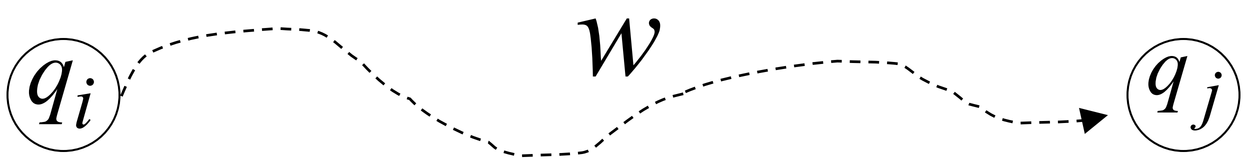
Special case:

for any state q

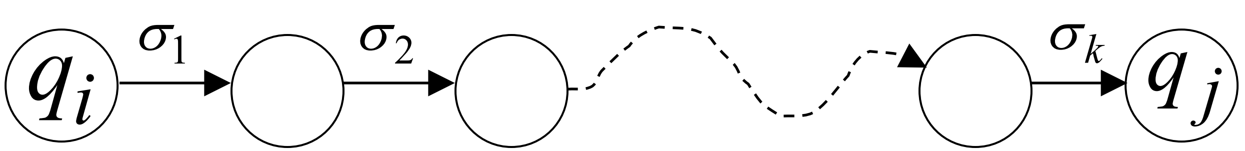
$$q \in \delta^*(q, \varepsilon)$$

In general

$q_j \in \delta^*(q_i, w)$: there is a walk from q_i to q_j with label w



$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



The Language of an NFA M

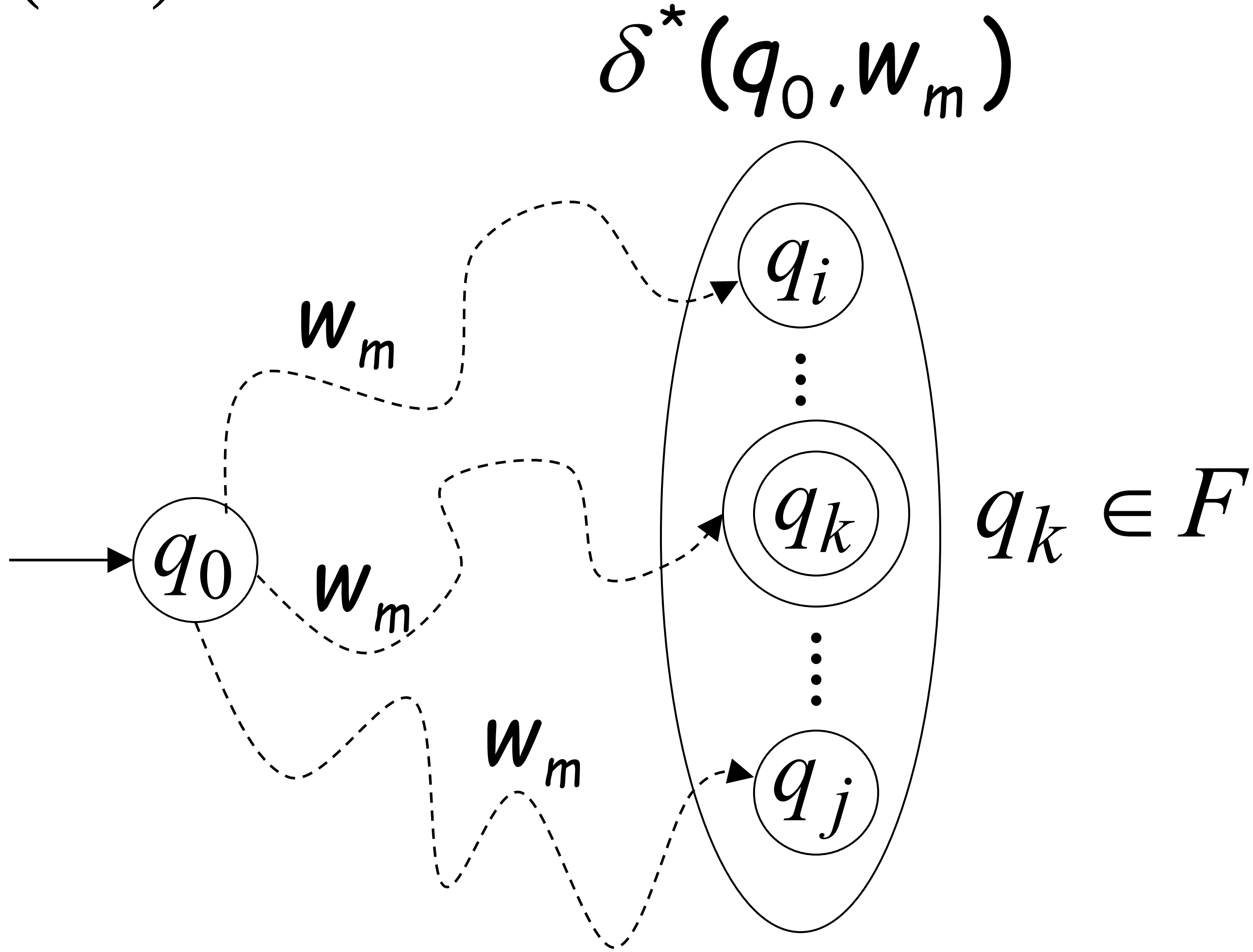
The language accepted by M is:

$$L(M) = \{w_1, w_2, \ldots w_n\}$$

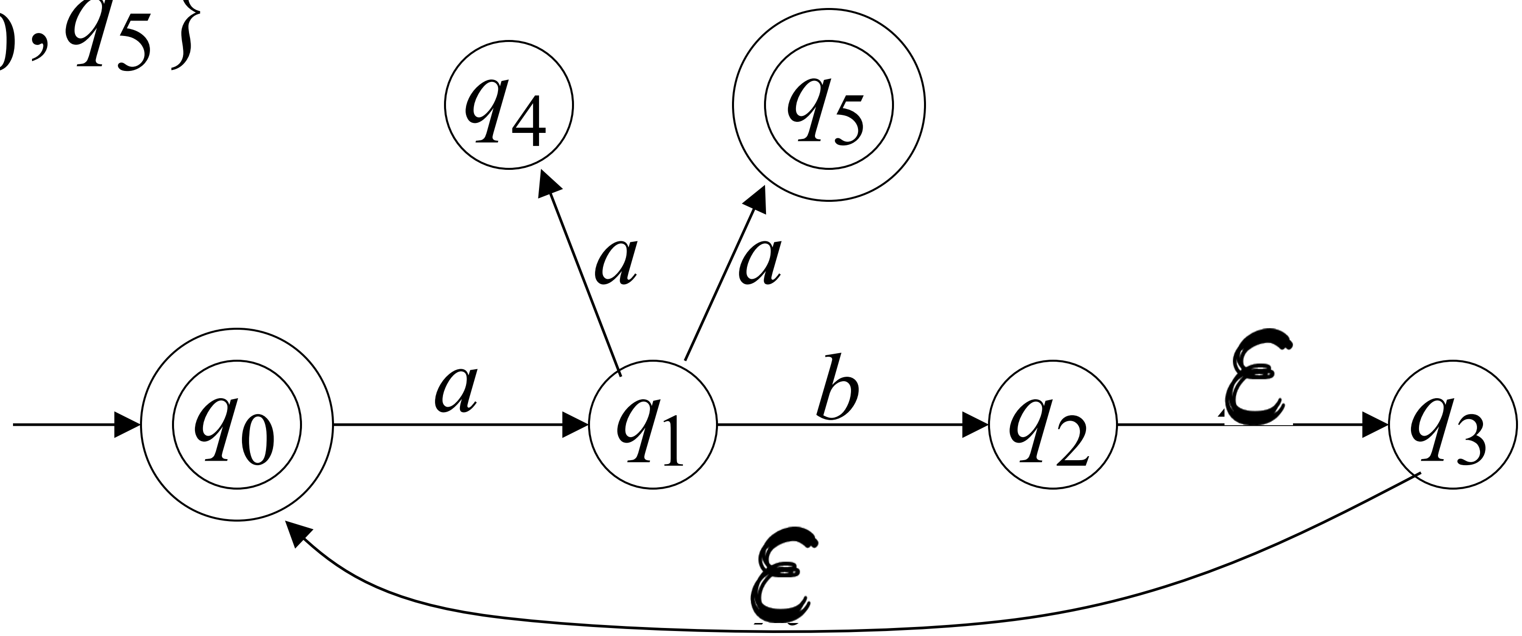
where $\delta^*(q_0, w_m) = \{q_i, \ldots, q_k, \ldots, q_j\}$

and there is some $q_k \in F$ (accepting state)

$w_m \in L(M)$



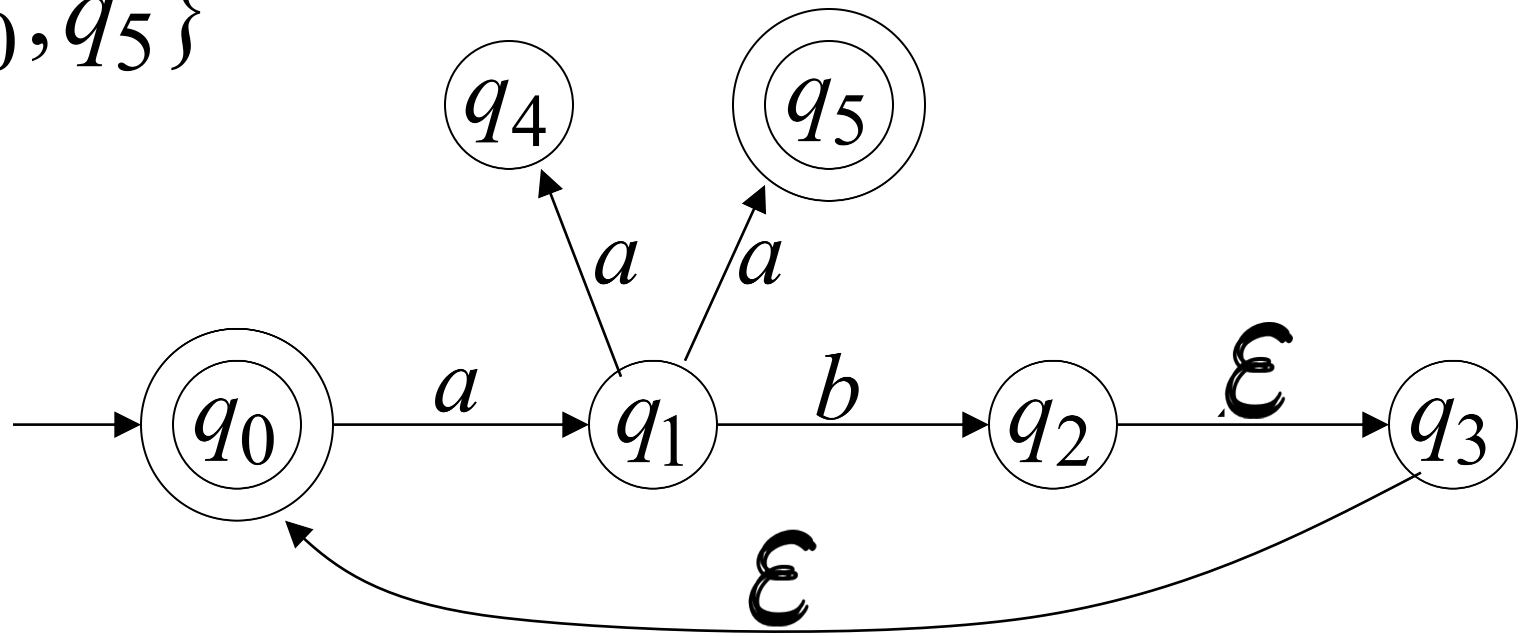
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aa) = \{q_4, \underline{q_5}\} \Longrightarrow aa \in L(M)$$

$$q_5 \in F$$

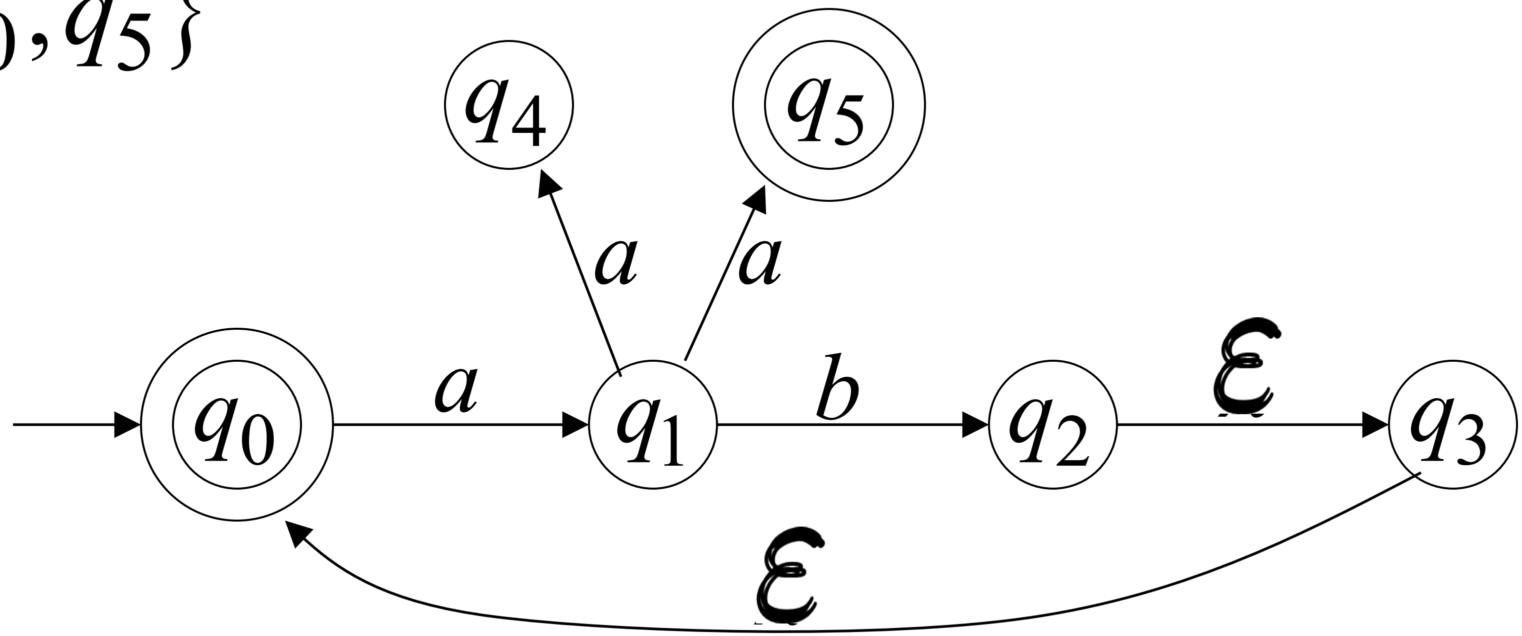
$F = \{q_0, q_5\}$



$$\delta^*(q_0, ab) = \{q_2, q_3, \underline{q_0}\} \Longrightarrow ab \in L(M)$$

$$\underline{q_0} \searrow \in F$$

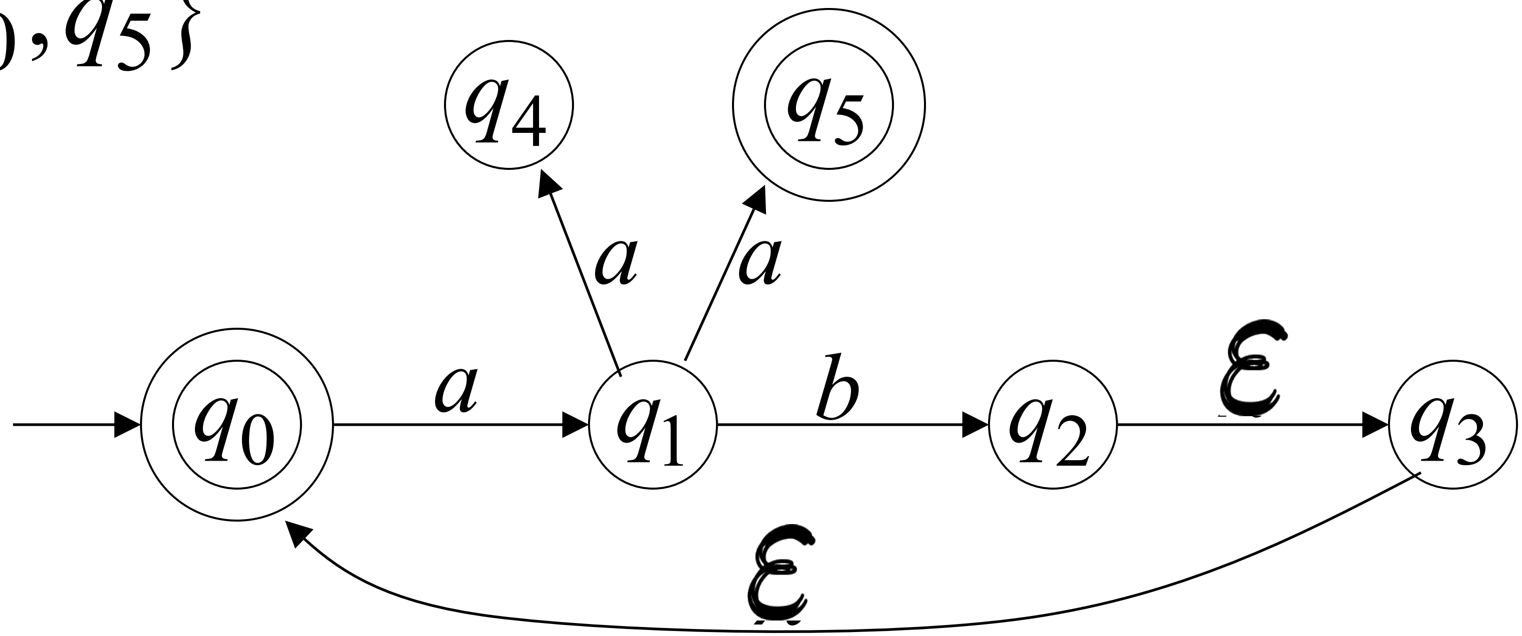
$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, abaa) = \{q_4, \underline{q_5}\} \Rightarrow aaba \in L(M)$$

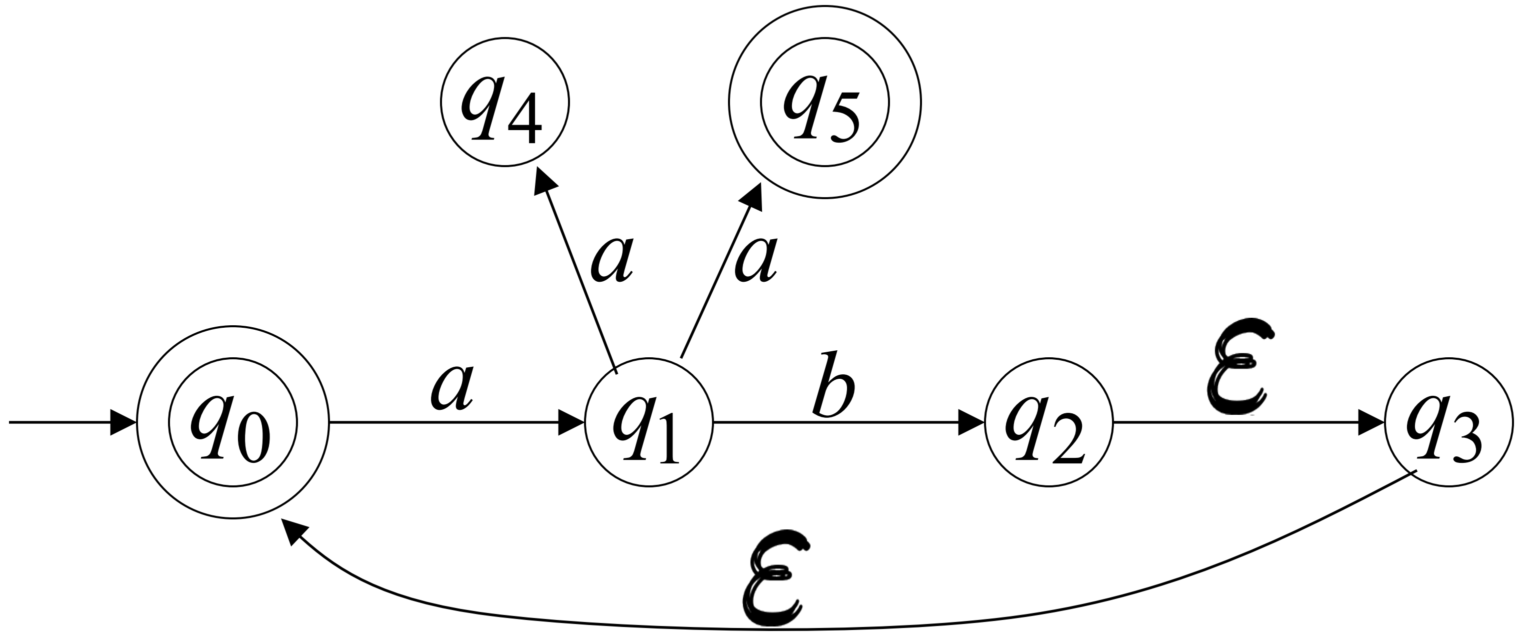
$$\in F$$

$$F = \{q_0, q_5\}$$



$$\delta^*(q_0, aba) = \{q_1\} \Rightarrow aba \notin L(M)$$

$$\{q_1\} \notin F$$



$$L(M) = \{ab\}^* \cup \{ab\}^* \{aa\}$$

NFAs accept the Regular Languages

Equivalence of Machines

Definition:

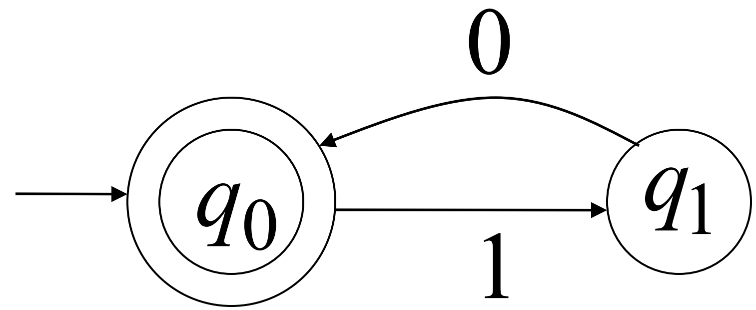
Machine M_1 is equivalent to machine M_2

if $L(M_1) = L(M_2)$

Example of equivalent machines

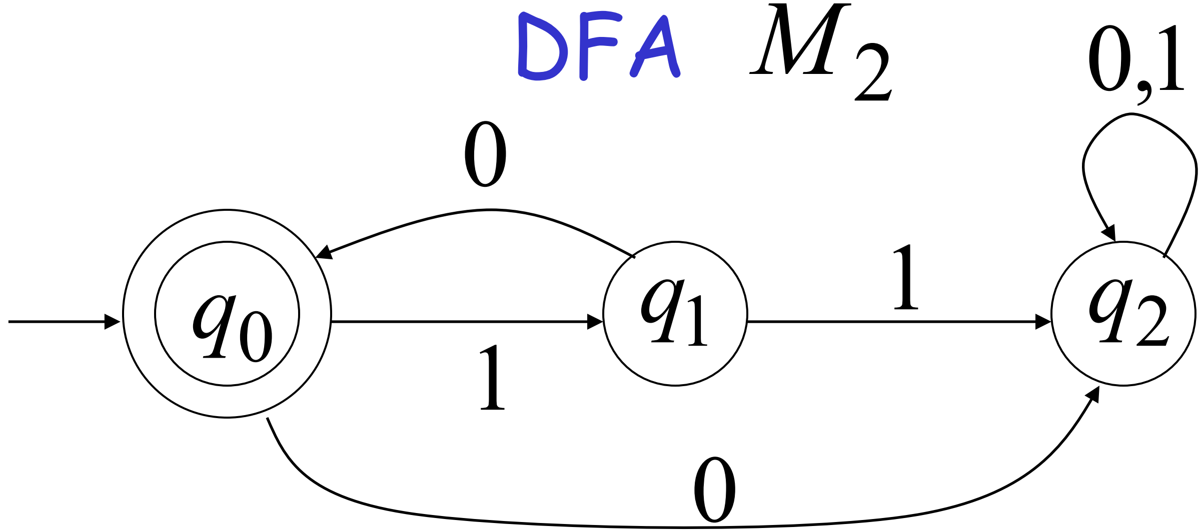
NFA M_1

$$L(M_1) = \{10\}^*$$



DFA M_2

$$L(M_2) = \{10\}^*$$



Theorem:

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Languages
accepted
by DFAs

NFAs and DFAs have the same computation power, accept the same set of languages

Proof: we only need to show

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

AND

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Proof-Step 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Every DFA is trivially an NFA

$\Downarrow$

Any language L accepted by a DFA is also accepted by an NFA

Proof-Step 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{accepted} \\ \text{by NFAs} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

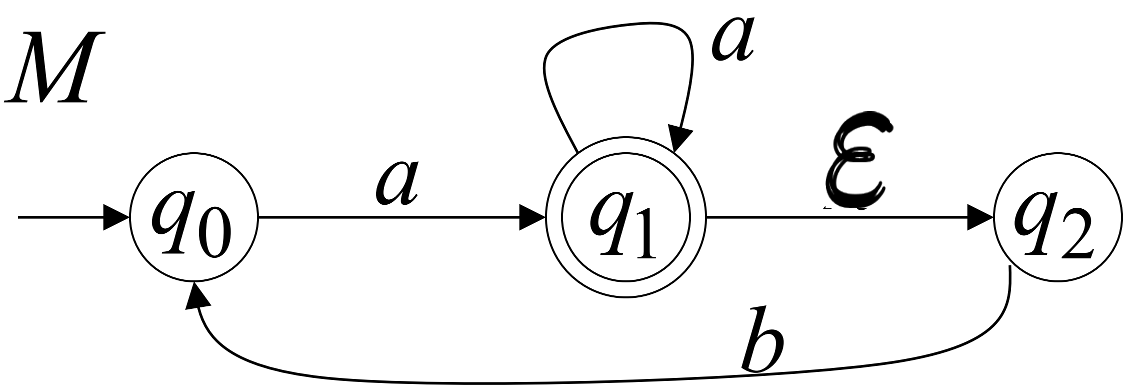
Any NFA can be converted to an equivalent DFA

$\Downarrow$

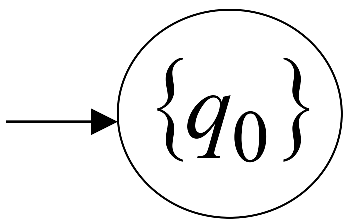
Any language L accepted by an NFA is also accepted by a DFA

Conversion NFA to DFA

NFA M

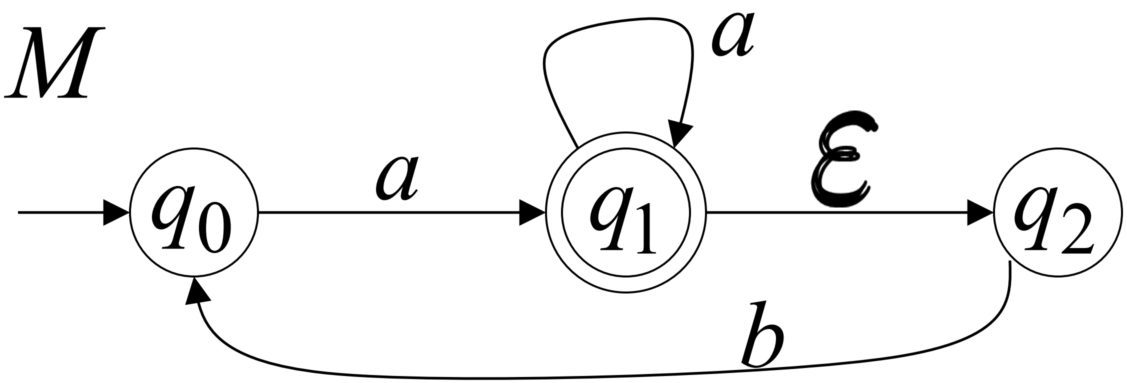


DFA M'

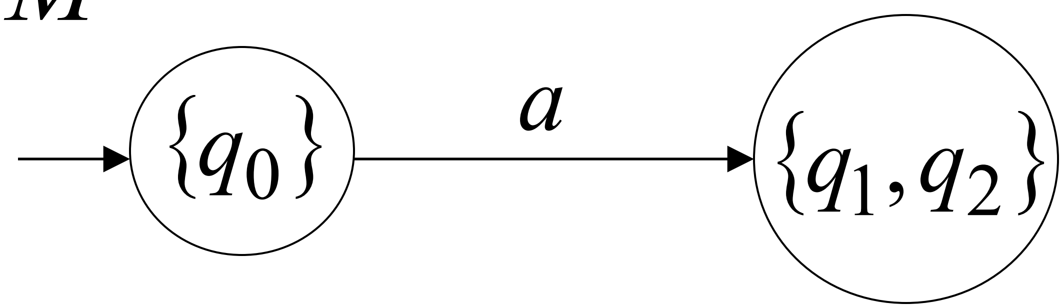


$$\delta^*(q_0, a) = \{q_1, q_2\}$$

NFA M

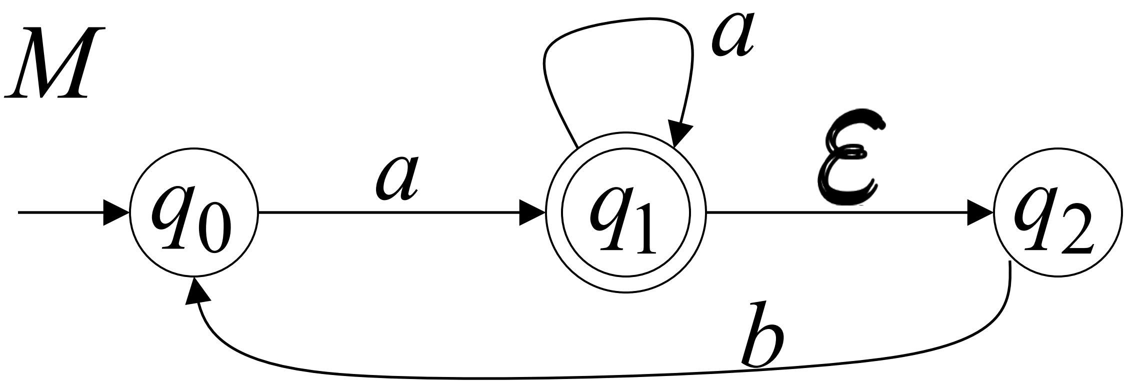


DFA M'

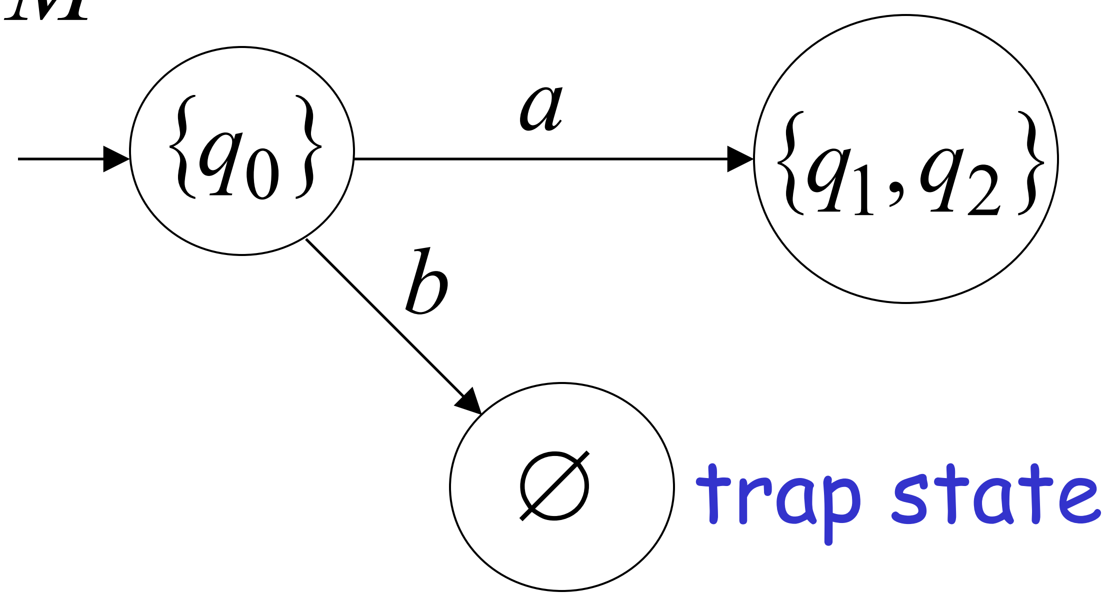


$$\delta^*(q_0, b) = \varnothing \quad \text{empty set}$$

NFA M



DFA M'

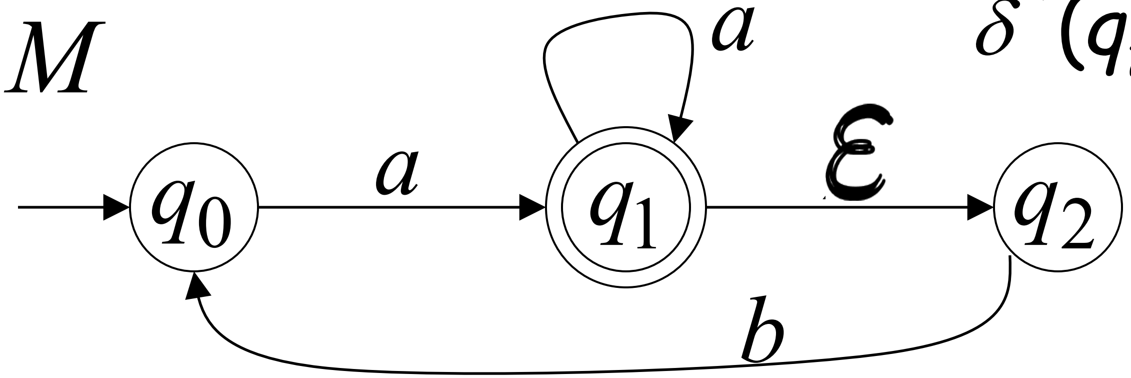


$$\delta^*(q_1, a) = \{q_1, q_2\}$$
$$\delta^*(q_2, a) = \varnothing$$

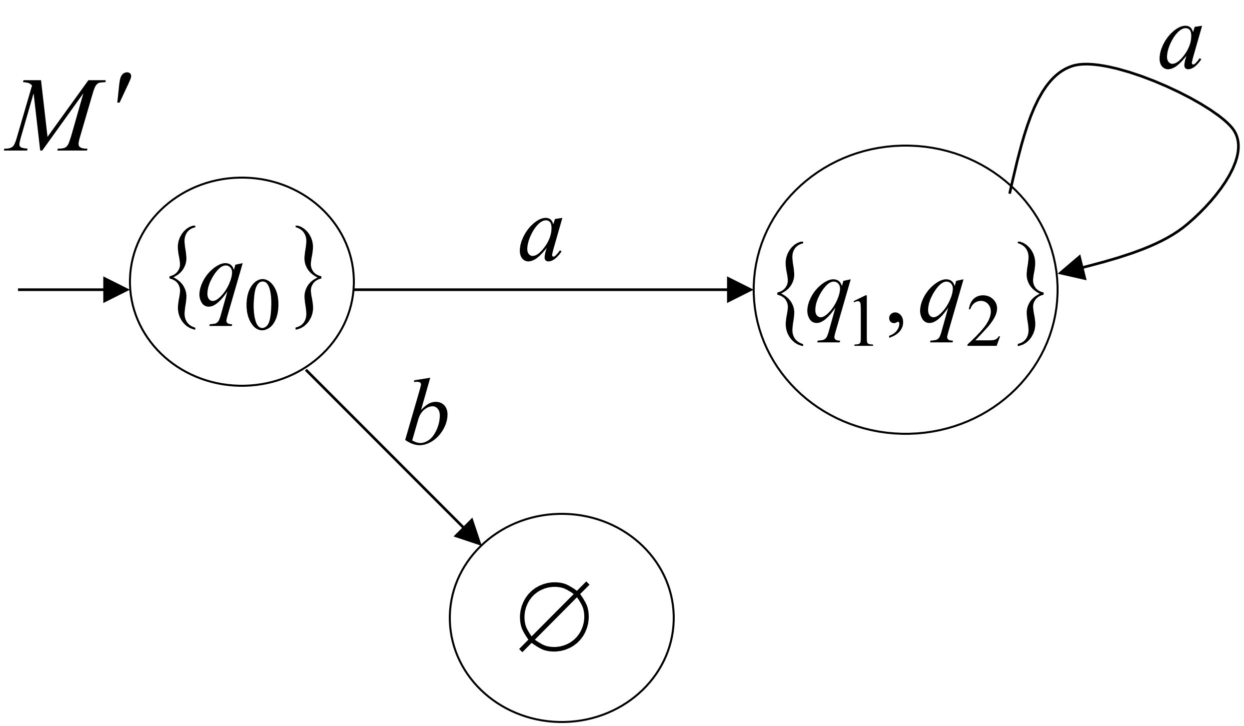
union

$$\{q_1, q_2\}$$

NFA M



DFA M'

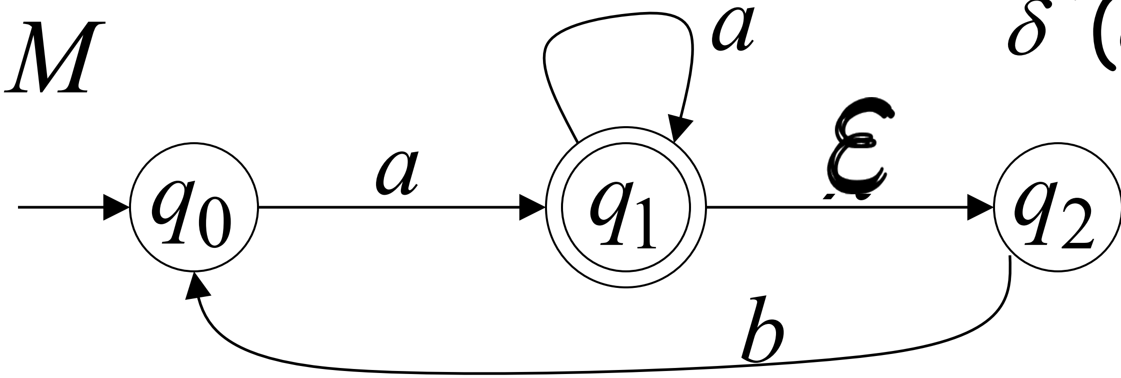


$$\delta^*(q_1, b) = \{q_0\}$$
$$\delta^*(q_2, b) = \{q_0\}$$

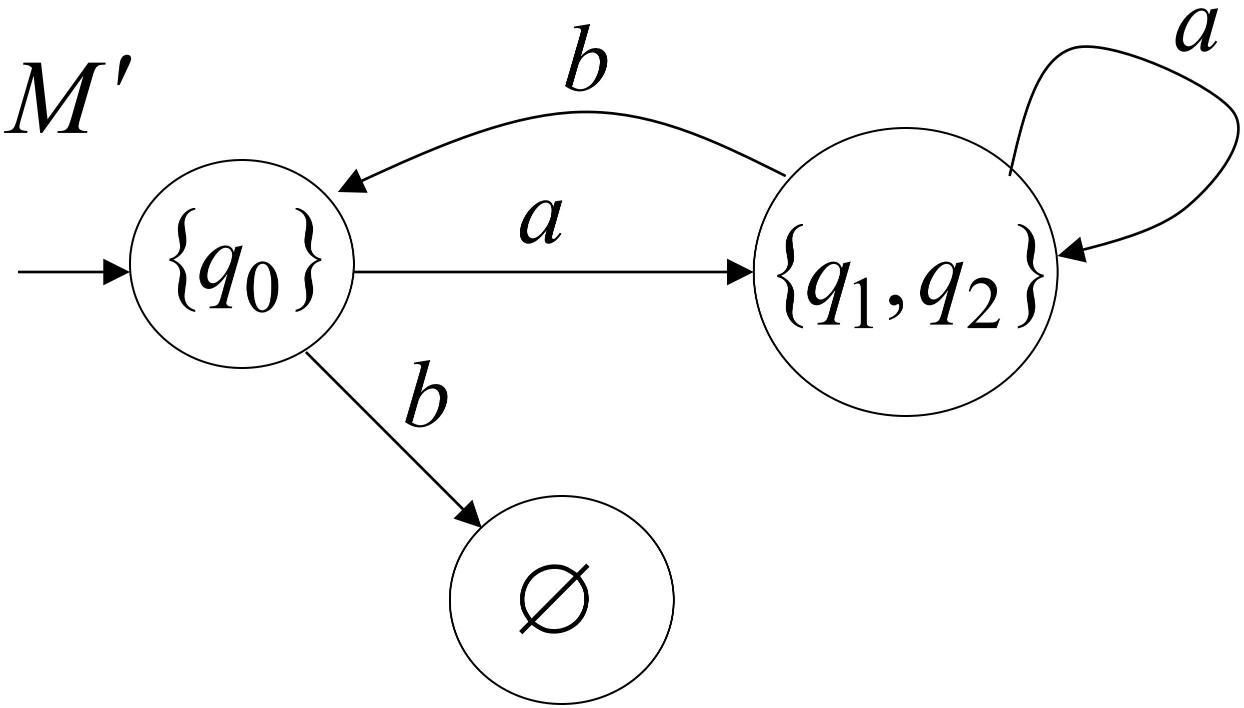
union

$\{q_0\}$

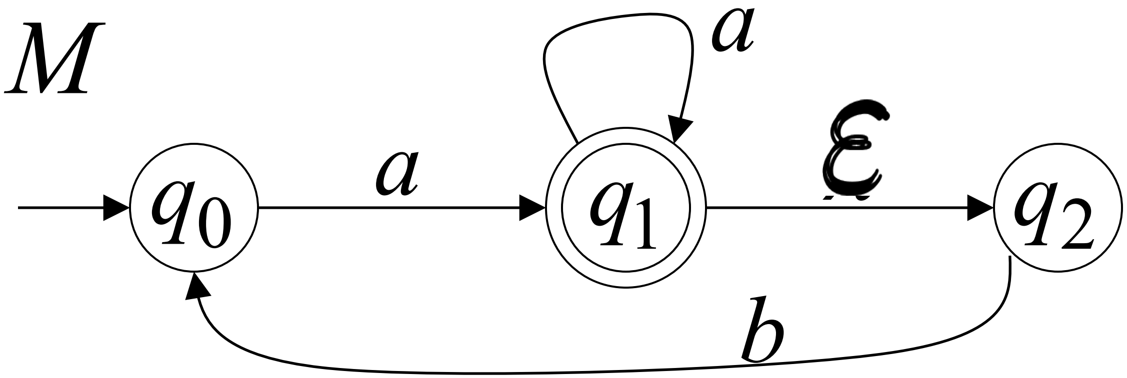
NFA M



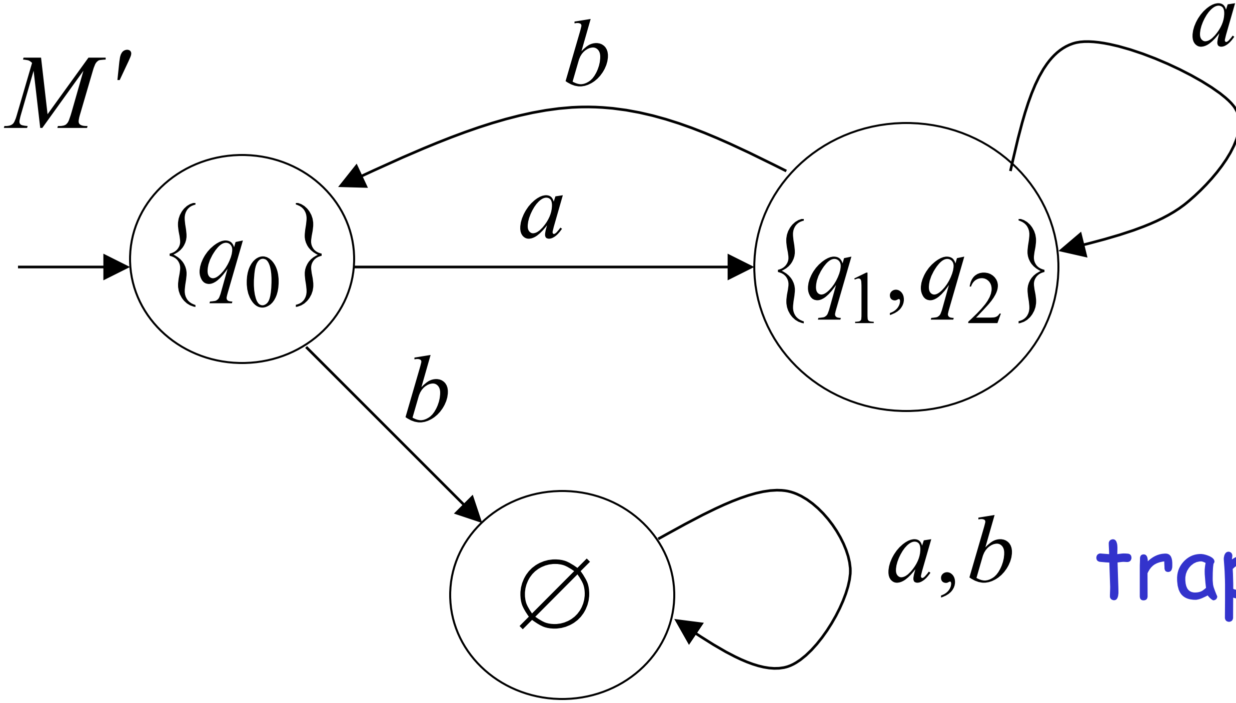
DFA M'



NFA M



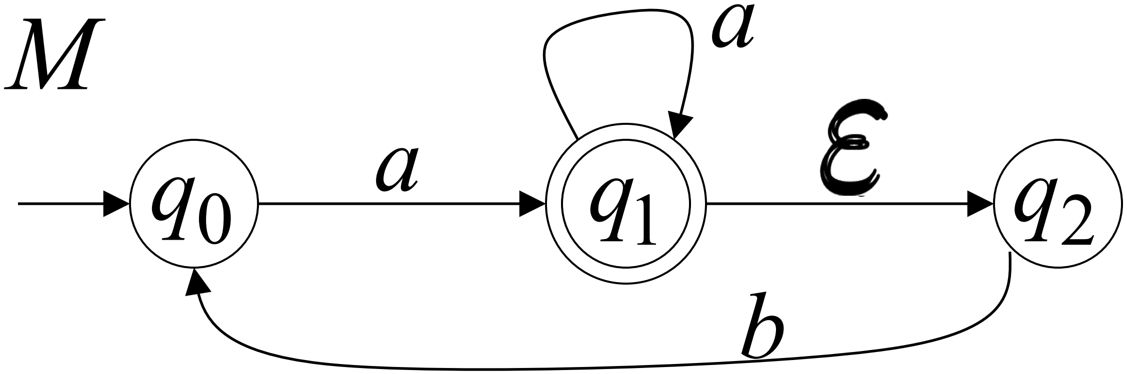
DFA M'



trap state

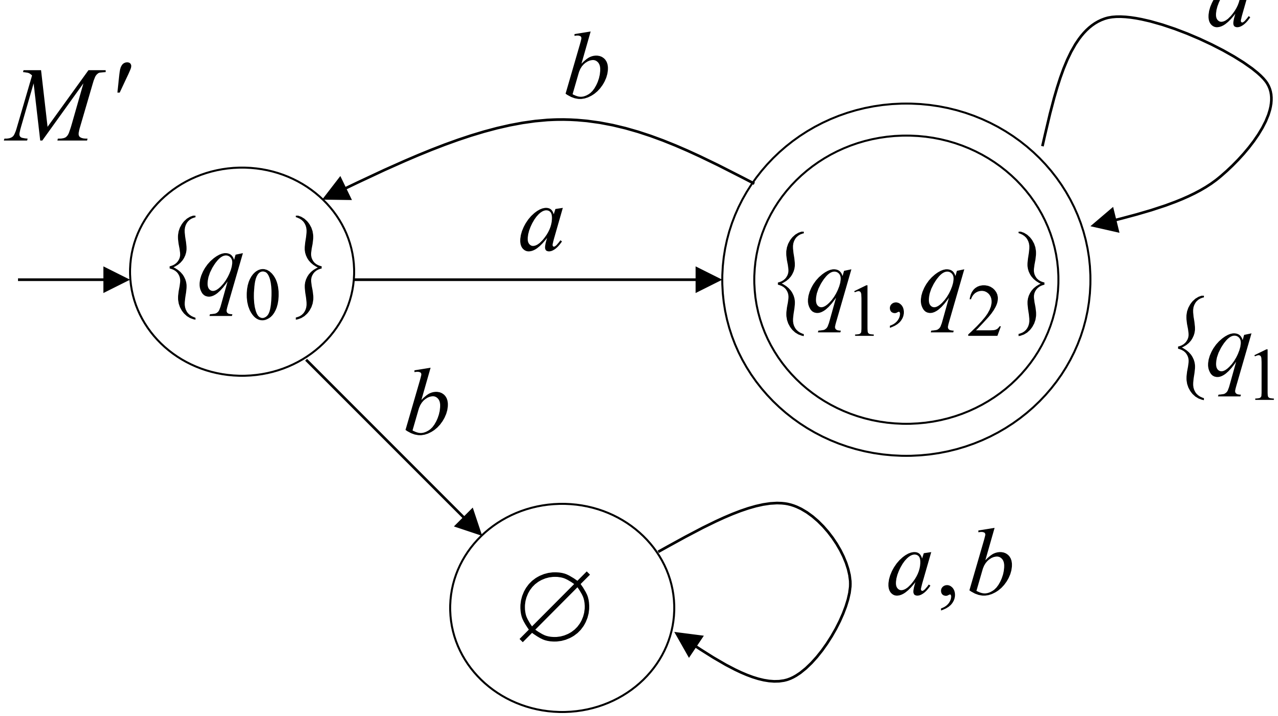
END OF CONSTRUCTION

NFA M



$q_1 \in F$

DFA M'



$\{q_1, q_2\} \in F'$

General Conversion Procedure

Input: an NFA M

Output: an equivalent DFA M'

with $L(M) = L(M')$

The NFA has states $q_0, q_1, q_2, \ldots$

The DFA has states from the power set

$$\emptyset, \{q_0\}, \{q_1\}, \{q_0, q_1\}, \{q_1, q_2, q_3\}, \ldots.$$

Conversion Procedure Steps

step

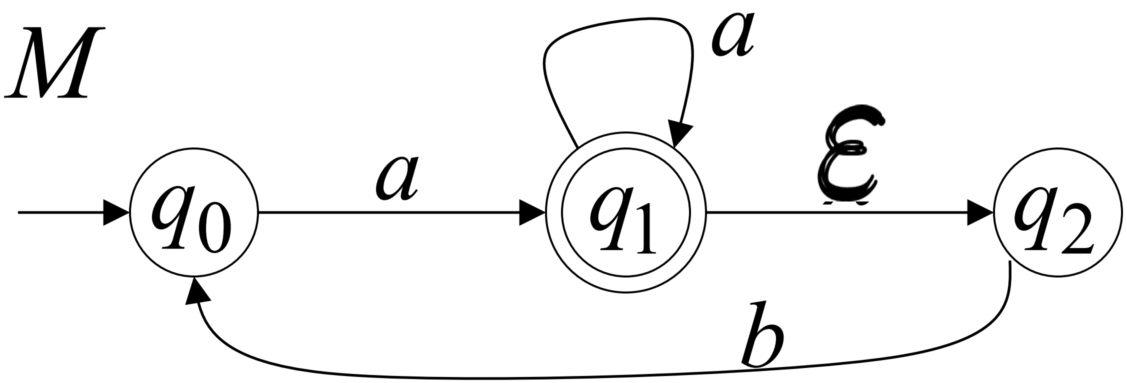
1. Initial state of NFA: q_0

↓

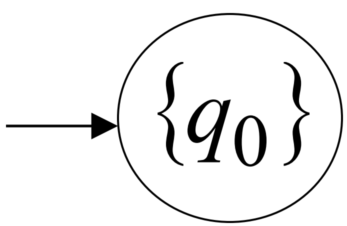
Initial state of DFA: $\{q_0\}$

Example

NFA M



DFA M'



step

2. For every DFA's state $\{q_i, q_j, \ldots, q_m\}$

compute in the NFA

$$\left.\begin{aligned} &\delta^*(q_i, a) \\ \cup\, &\delta^*(q_j, a) \\ &\ldots \\ \cup\, &\delta^*(q_m, a) \end{aligned}\right\} = \{q'_k, q'_l, \ldots, q'_n\}$$

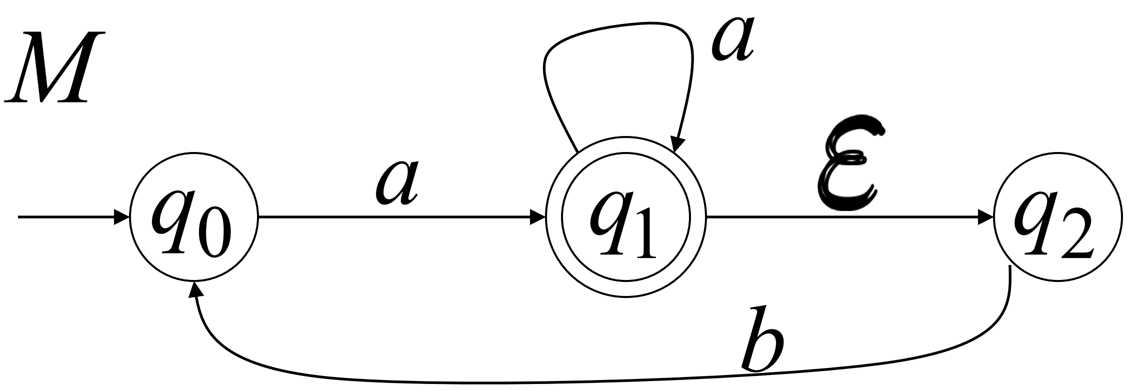
Union

add transition to DFA

$$\delta(\{q_i, q_j, \ldots, q_m\},\ a) = \{q'_k, q'_l, \ldots, q'_n\}$$

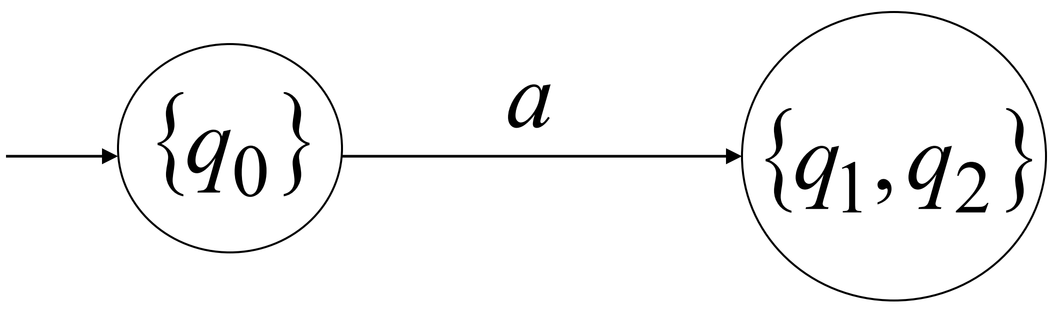
Example $\delta^*(q_0, a) = \{q_1, q_2\}$

NFA M



$$\delta(\{q_0\}, a) = \{q_1, q_2\}$$

DFA M'

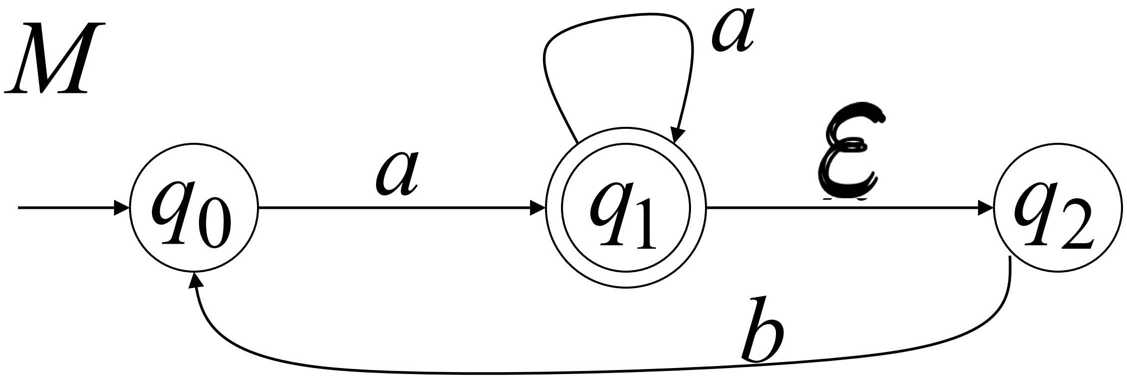


step

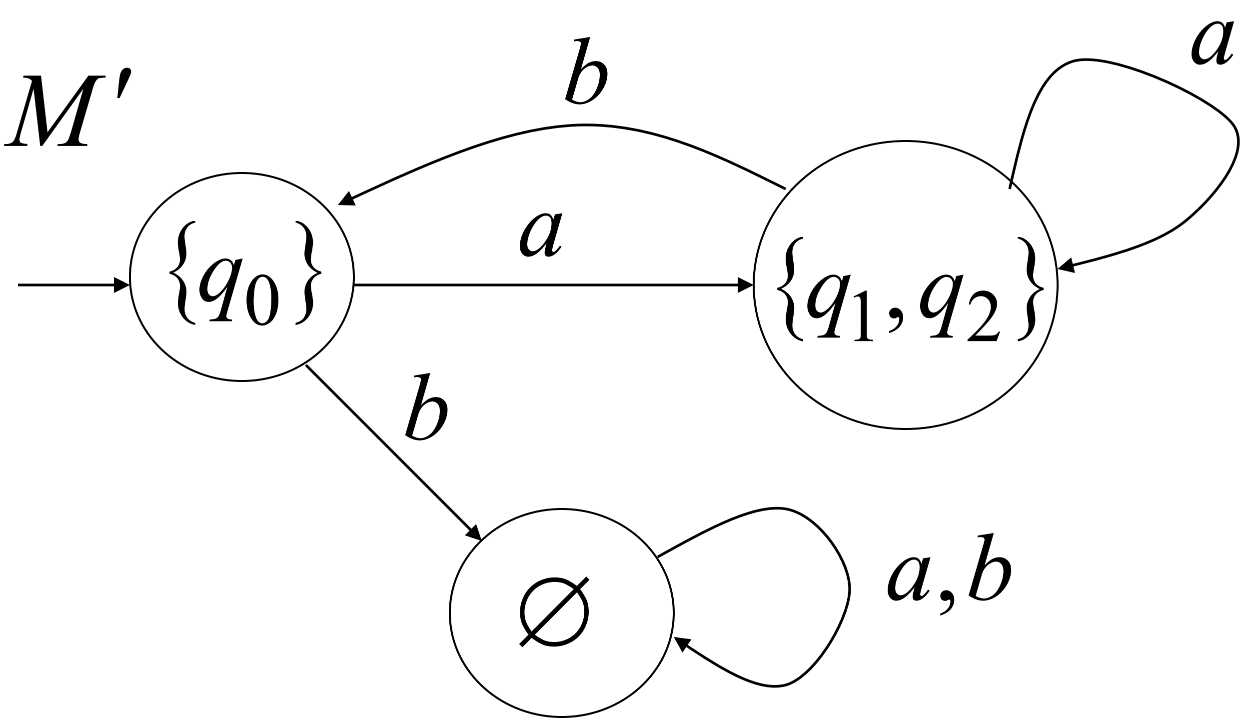
3. Repeat Step 2 for every state in DFA and symbols in alphabet until no more states can be added in the DFA

Example

NFA M



DFA M'



step

4. For any DFA state $\{q_i, q_j, \ldots, q_m\}$

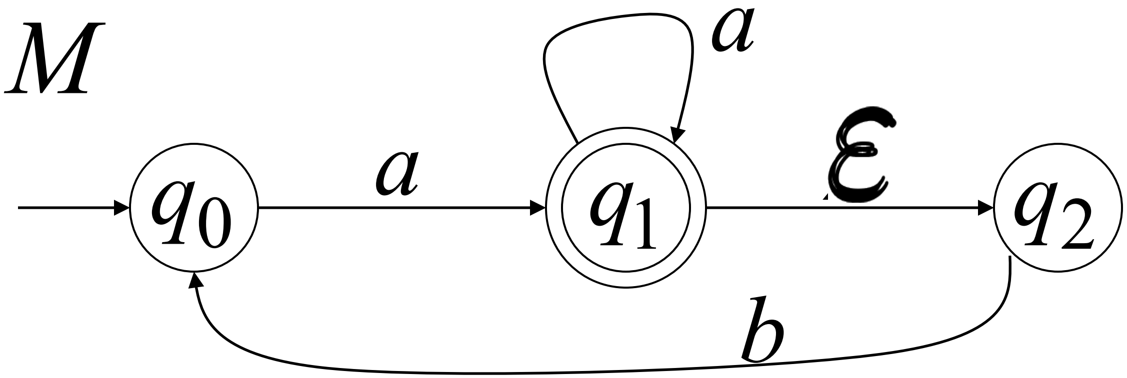
if some q_j is accepting state in NFA

Then, $\{q_i, q_j, \ldots, q_m\}$

is accepting state in DFA

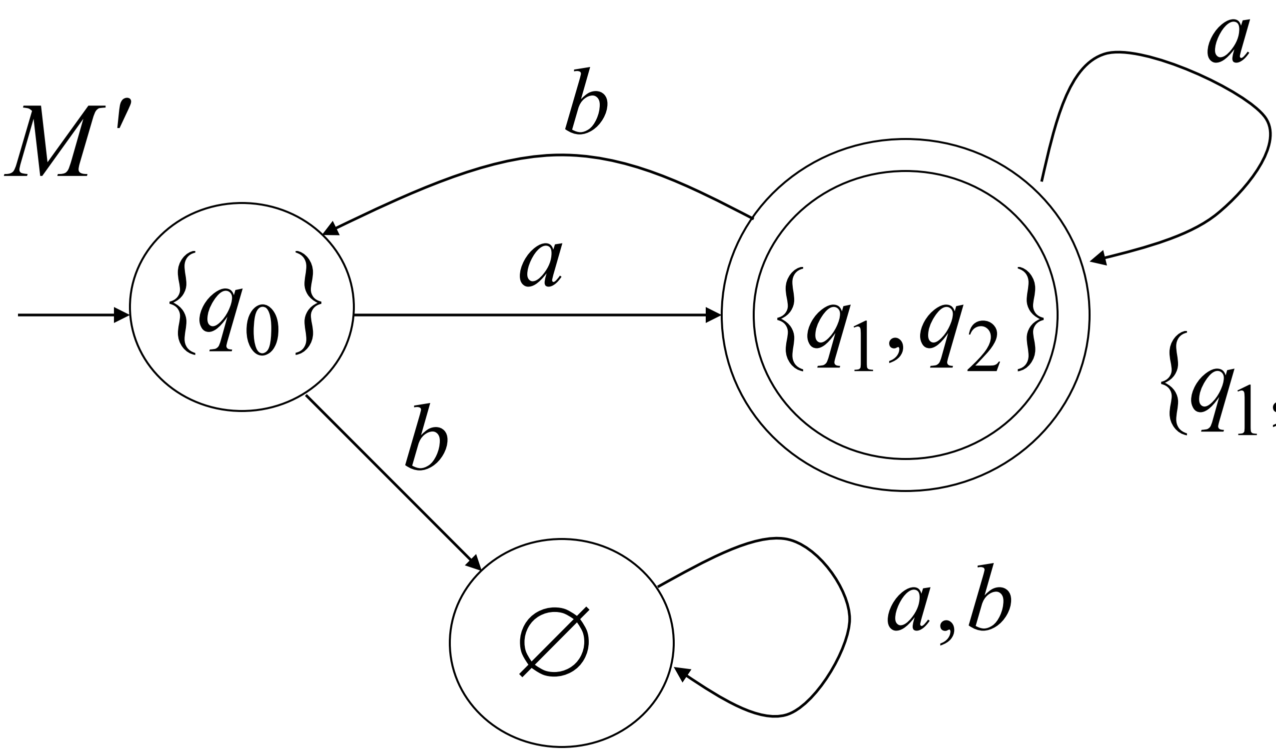
Example

NFA M



$q_1 \in F$

DFA M'



$\{q_1, q_2\} \in F'$

Lemma:

If we convert NFA M to DFA M'

then the two automata are equivalent:

$$L(M) = L(M')$$

Proof:

We only need to show: $L(M) \subseteq L(M')$

AND

$$L(M) \supseteq L(M')$$

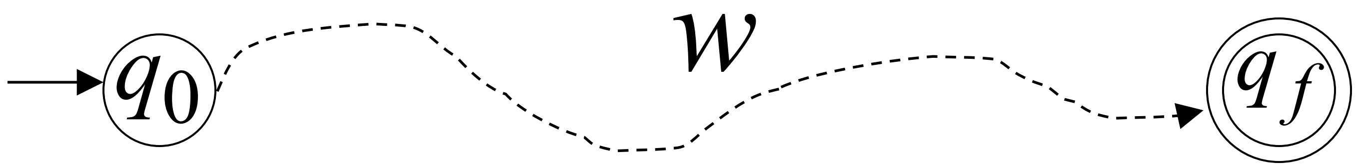
First we show: $L(M) \subseteq L(M')$

We only need to prove:

$$w \in L(M) \quad \Longrightarrow \quad w \in L(M')$$

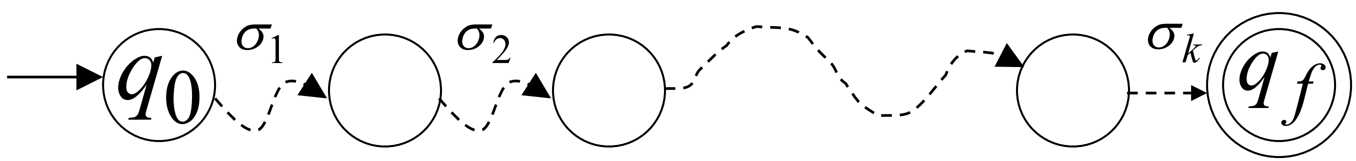
NFA

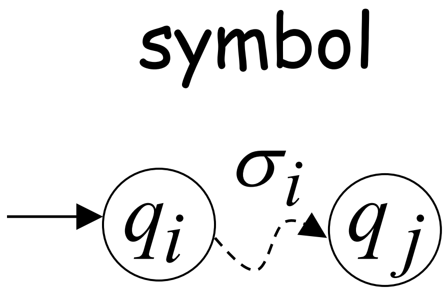
Consider $w \in L(M)$



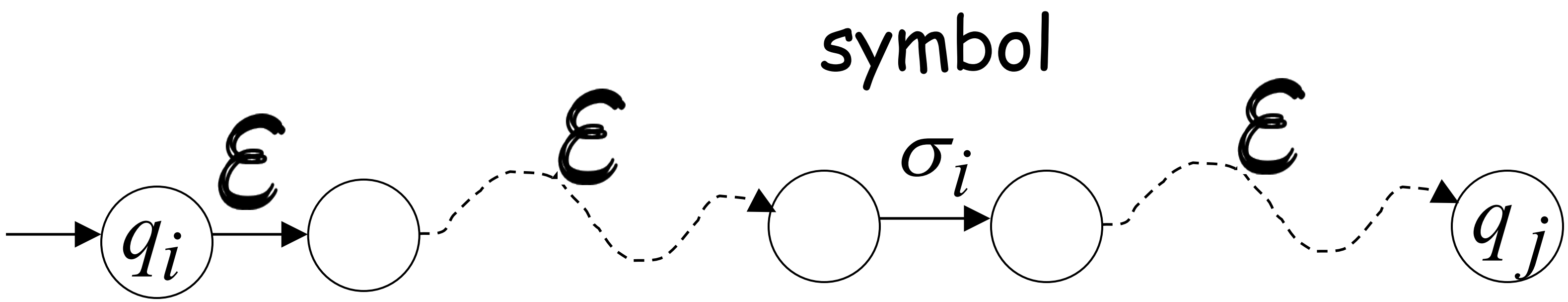
symbols

$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$



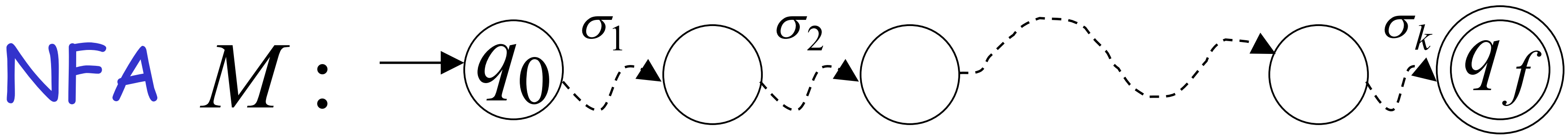


denotes a possible sub-path like

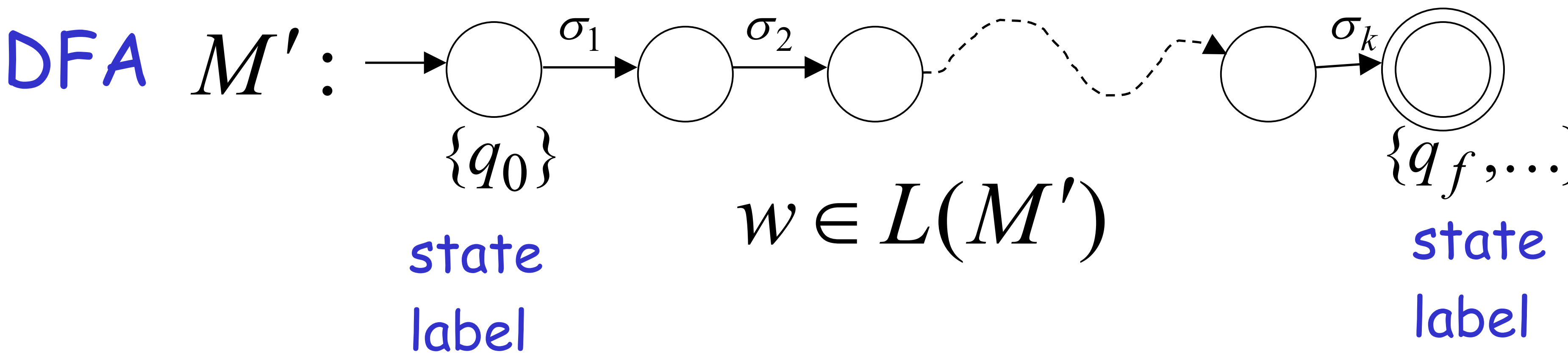


We will show that if $w \in L(M)$

$w = \sigma_1 \sigma_2 \cdots \sigma_k$



then

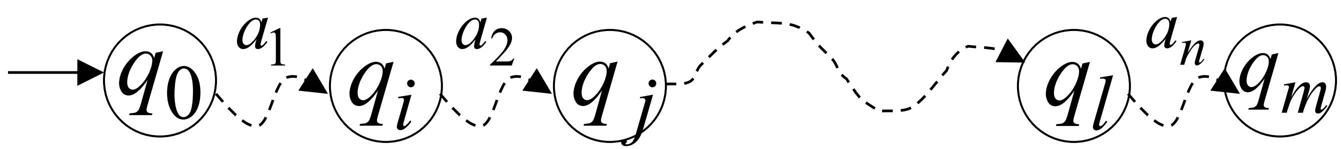


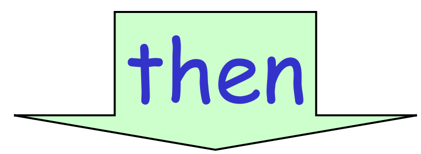
$w \in L(M')$

More generally, we will show that if in M:

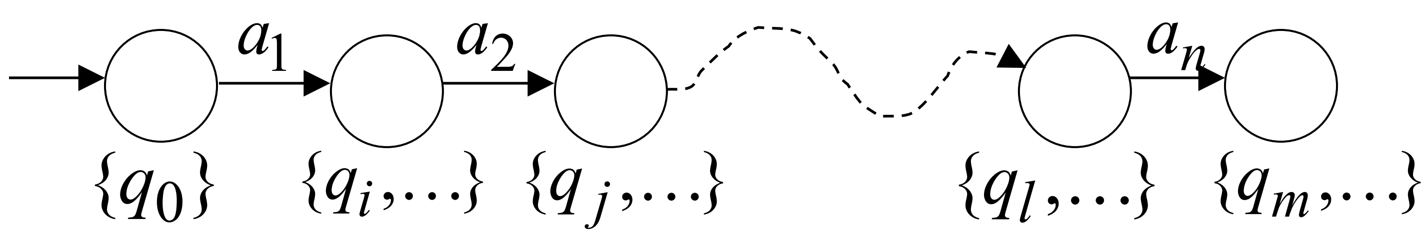
(arbitrary string) $v = a_1 a_2 \cdots a_n$

NFA $M:$





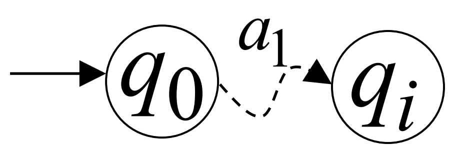
DFA $M':$



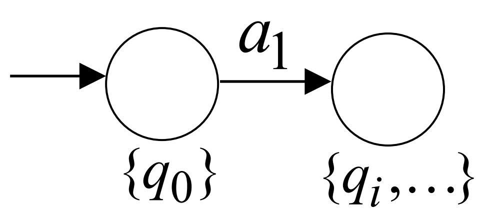
Proof by induction on $|v|$

Induction Basis: $|v| = 1$ $\quad v = a_1$

NFA $M:$



DFA $M':$



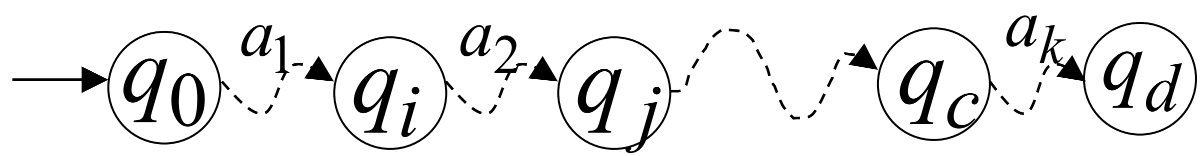
is true by construction of M'

Induction hypothesis: $1 \leq |v| \leq k$

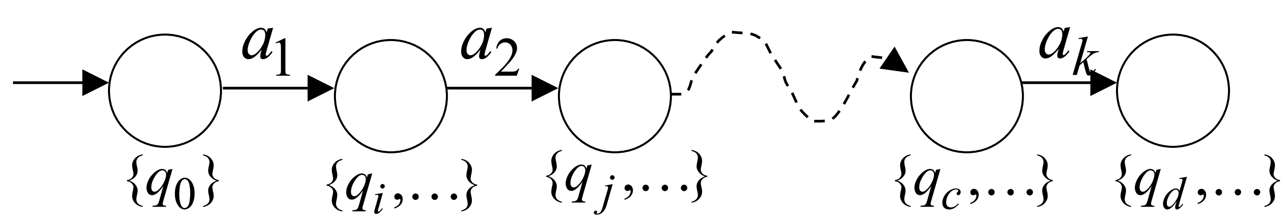
$$v = a_1 a_2 \cdots a_k$$

Suppose that the following hold

NFA $M:$



DFA $M':$

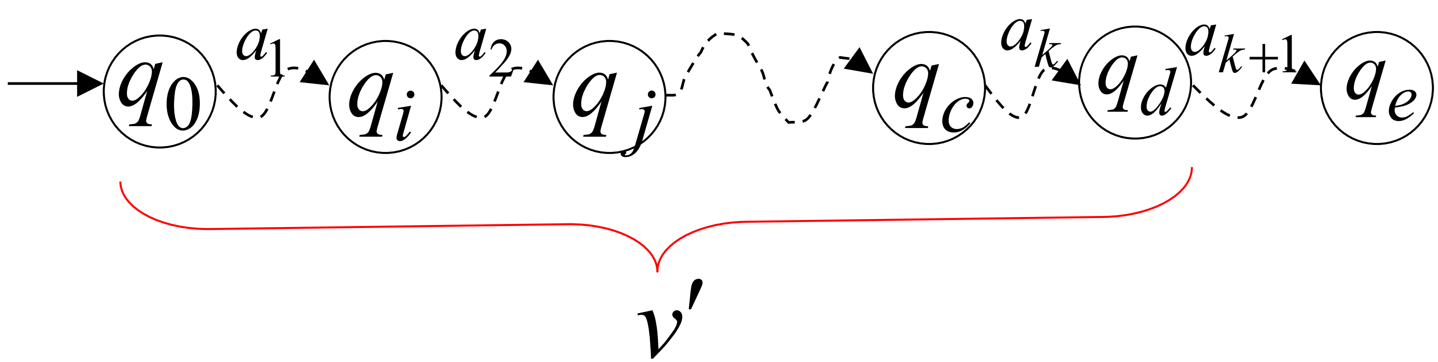


Induction Step: $|v| = k+1$

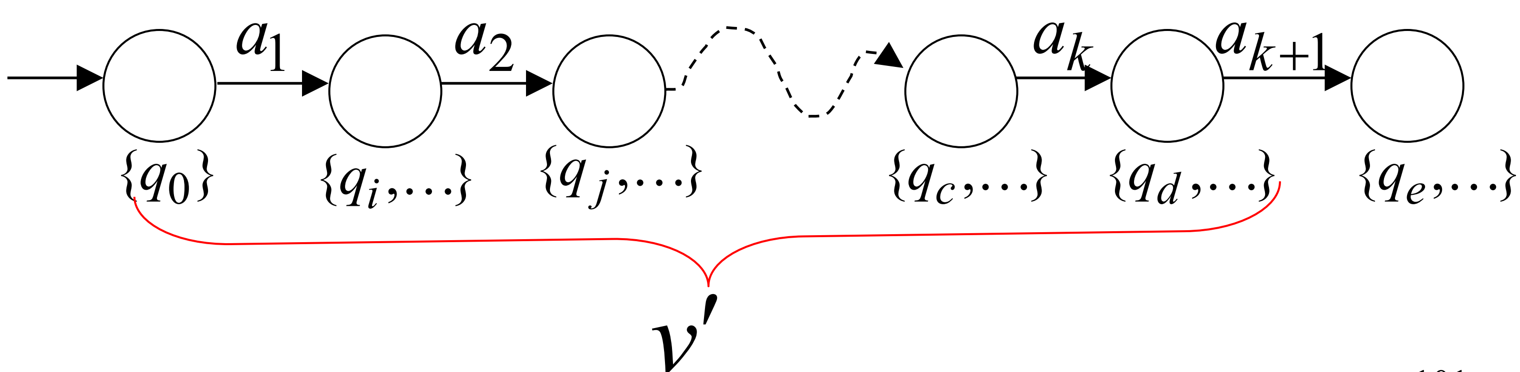
$$v = \underbrace{a_1 a_2 \cdots a_k}_{v'} a_{k+1} = v' a_{k+1}$$

Then this is true by construction of M'

NFA M:



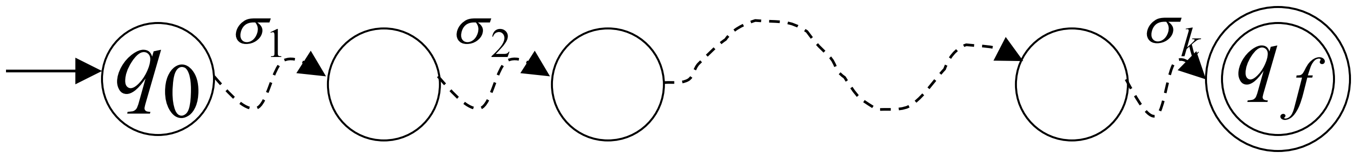
DFA M':



Therefore if $w \in L(M)$

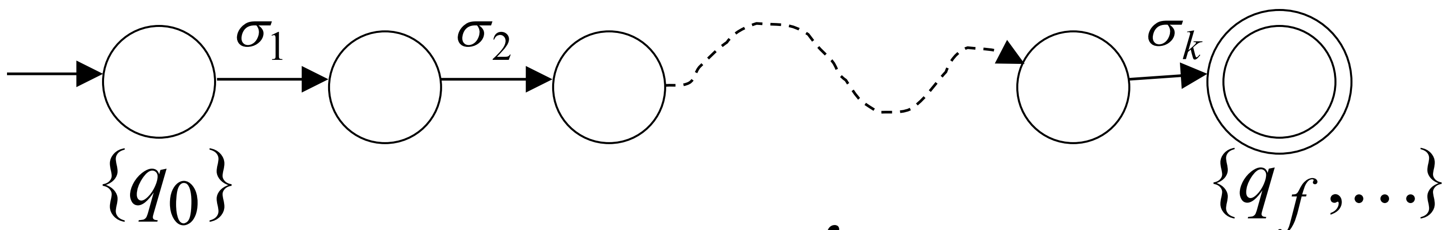
$$w = \sigma_1 \sigma_2 \cdots \sigma_k$$

NFA $M:$



then

DFA $M':$



$$w \in L(M')$$

We have shown: $L(M) \subseteq L(M')$

With a similar proof

we can show: $L(M) \supseteq L(M')$

Therefore: $L(M) = L(M')$

END OF LEMMA PROOF