More Applications

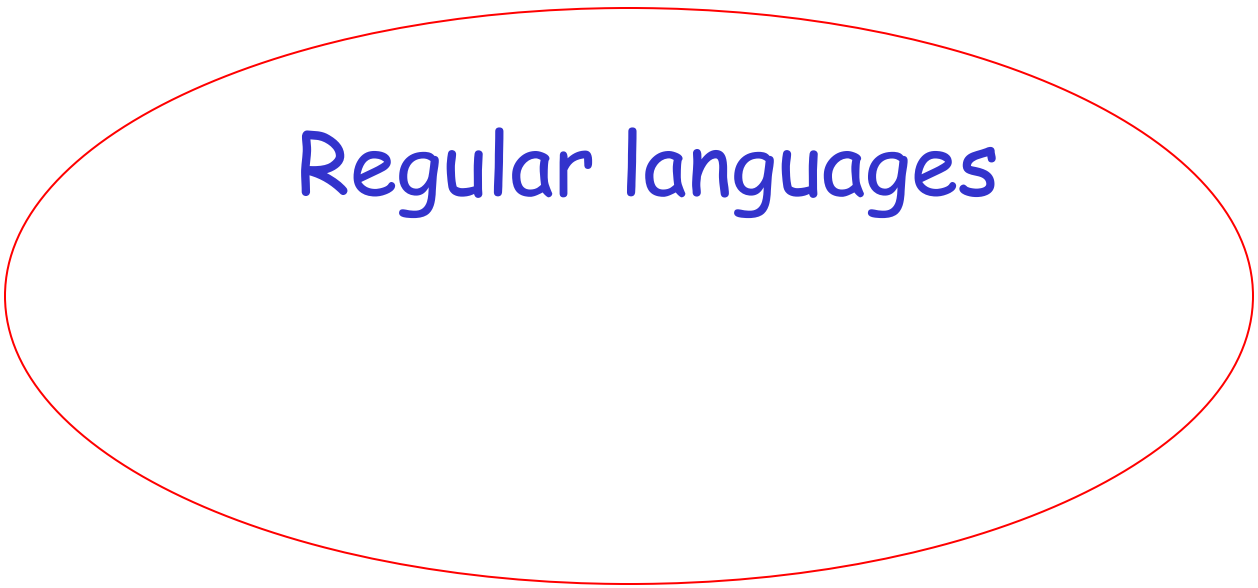
of

the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages $L = \{vv^R : v \in \Sigma^*\}$



Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

$$\boxed{\text{We pick } \; w = a^m b^m b^m a^m}$$

From the Pumping Lemma:

we can write: $w = a^m b^m b^m a^m = x\ y\ z$

with lengths: $|x\,y| \leq m, \quad |y| \geq 1$

$$w = xyz = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{...a}_{}}^{m}\underbrace{\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z}$$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$x\ y\ z = a^m b^m b^m a^m$ $y = a^k, \quad 1 \le k \le m$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$i = 0, 1, 2, \ldots$

Thus: $x\ y^2\ z \ \in\ L$

$$x\,y\,z = a^m b^m b^m a^m \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $\quad x\,y^2\,z \;\in\; L$

$$xy^2z = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a}_{y}\ldots a}^{m+k}\;\overbrace{b\ldots b}^{m}\,\overbrace{b\ldots b}^{m}\,\overbrace{a\ldots a}^{m} \;\square\; L$$

$$z = \ldots ab\ldots bb\ldots ba\ldots a$$

Thus: $\quad a^{m+k} b^m b^m a^m \;\in L$

$$a^{m+k} b^m b^m a^m \quad \in L \qquad\qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k} b^m b^m a^m \quad \notin L$$

CONTRADICTION!!!

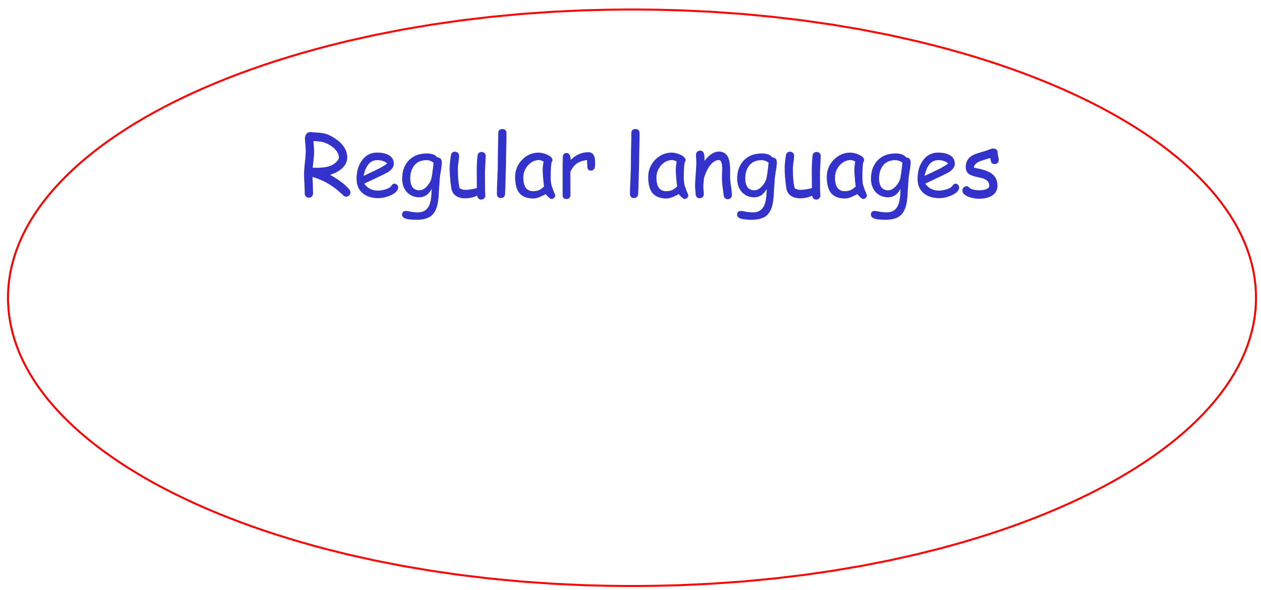
Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$



Theorem: The language

$$L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$$

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Proof: Use the Pumping Lemma

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Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n,l \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

$$\boxed{\text{We pick} \quad w = a^m b^m c^{2m}}$$

From the Pumping Lemma:

We can write $w = a^m b^m c^{2m} = x\ y\ z$

With lengths $|x\,y| \leq m, \quad |y| \geq 1$

$$w = xyz = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}a\ldots a}^{m}\overbrace{b\ldots b}^{m}\overbrace{c\ldots cc\ldots c}^{2m}$$

where z is the string from the remaining $a\ldots a$ through the final c.

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\ y^i\ z \ \in\ L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\ y^0\ z\ = xz\ \ \notin\ L$

$$x\ y\ z = a^m b^m c^{2m} \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $xz \in L$

$$xz = \underbrace{\overbrace{a...a}^{}}_{x}\underbrace{\overbrace{a...a}^{m-k}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}}_{z} \in L$$

Thus: $a^{m-k} b^m c^{2m} \in L$

$$a^{m-k}b^{m}c^{2m} \in L \qquad k \geq 1$$

BUT: $$L = \{a^n b^l c^{n+l} : n,l \geq 0\}$$

$$\Downarrow$$

$$a^{m-k}b^{m}c^{2m} \notin L$$

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END OF PROOF

Non-regular languages $L = \{a^{n!} : \ n \geq 0\}$

Regular languages

Theorem: The language $L = \{a^{n!} : n \geq 0\}$

is not regular

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : \ n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!} : \ n \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$

length $|w| \geq m$

$$\boxed{\text{We pick } w = a^{m!}}$$

From the Pumping Lemma:

We can write $w = a^{m!} = x\ y\ z$

With lengths $|x\ y| \leq m, \quad |y| \geq 1$

$$w = xyz = a^{m!} = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...aa...aa...a}_{z}$$

$$\overbrace{a...aa...aa...a}^{m}\overbrace{a...aa...a}^{m!-m}$$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x\,y\,z = a^{m!} \qquad y = a^k, \quad 1 \le k \le m$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\ y\ z = a^{m!} \qquad\qquad y = a^k,\quad 1 \le k \le m$$

From the Pumping Lemma: $x\ y^2\ z \in L$

$$xy^2z = \underbrace{\overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a}_{y}a\ldots a}^{m+k}\overbrace{a\ldots aa\ldots a}^{m!-m}}_{\text{}} \in L$$

$$x \quad y \quad y \quad z$$

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

Since: $L = \{a^{n!} : \; n \geq 0\}$

$\Downarrow$

There must exist p such that:

$$m!+k = p!$$

However: $m!+k \leq m!+m$ for $m > 1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m+1)$$

$$= (m+1)!$$

$$\Downarrow$$

$$m!+k < (m+1)!$$

$$\Downarrow$$

$m!+k \neq p!$ for any p

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!} : n \geq 0\}$

$$\Downarrow$$

$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

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