Lecture Notes: Phase Estimation Problem

Introduction to Quantum Computing, CS 506
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Quick warm-up

- We label computational basis states by *binary* or *decimal*:

Binary	Decimal
$(101)_2$	5
$(0.1011)_2$	$\frac{11}{16}$

- Binary fractions: $(0.b_1b_2b_3\cdots)_2 = \sum_{j\geq 1} b_j 2^{-j}$.
- For example: $(0.1011)_2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = \frac{11}{16}$.

One-qubit($n = 1$)

Consider the input is a 1-qubit state:

1. For $\omega = (0.1)_2$, we have $e^{2\pi\omega y} = e^{2\pi y}$ and,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{i\pi y} |y\rangle$$

$$|\psi\rangle = \frac{|0\rangle + \mathrm{e}^{\pi \mathrm{i}} |1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = H |1\rangle$$

2. Similarly for $\omega = (0.0)_2$,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{y=0}^{1} e^{i\pi y} |y\rangle$$

$$|\psi\rangle = \frac{|0\rangle + \mathrm{e}^{\pi \mathrm{i}} |1\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = H |1\rangle$$

So, if $\omega = (0.x_1)_2 \in \{0, \frac{1}{2}\}$,

$$|\psi_\omega\rangle = \frac{|0\rangle + \mathrm{e}^{2\pi \mathrm{i}(0.x_1)} |1\rangle}{\sqrt{2}} = \frac{|0\rangle + (-1)^{x_1} |1\rangle}{\sqrt{2}}$$

applying a Hadamard,

$$H\left(\frac{|0\rangle + (-1)^{x_1} |1\rangle}{\sqrt{2}}\right) = |x_1\rangle,$$

so a single H reveals the only fractional bit.

Special case: ω has exactly n fractional bits

Assume

$$\omega = (0.x_1x_2\cdots x_n)_2 \qquad \text{(no bits after } x_n). \tag{1}$$

Binary expansion of ω is,

$$\omega = (0.\omega_1\omega_2\omega_3\cdots)_2 = \sum_{j\geq 1}\omega_j 2^{-j}, \qquad \omega_j \in \{0,1\}. \tag{2}$$

Multiplying by powers of two shifts the binary point:

$$\begin{aligned} \omega &= 0\cdot\omega_1\omega_2\ldots\omega_k\ldots\omega_n \\ 2^1\omega &= \omega_1\cdot\omega_2\omega_3\ldots\omega_k\ldots\omega_n \\ 2^2\omega &= \omega_1\omega_2\cdot\omega_3\ldots\omega_k\ldots\omega_{k+1} \end{aligned}$$

So, we can write,

$$2^k\omega = \omega_1\omega_2\cdots\omega_k.\,\omega_{k+1}x_{k+2}\cdots = \text{(integer)} + (0.\omega_{k+1}\omega_{k+2}\cdots)_2. \tag{3}$$

Hence we have,

$$\begin{aligned} e^{2\pi i\omega} &= e^{2\pi i(0\cdot\omega_1\omega_2\cdots)_2} \\ e^{2\pi i(2^k\omega)} &= e^{2\pi i(\omega_1\cdots\omega_k\cdot\omega_{k+1}\cdots)_2} \\ &= e^{2\pi i(\omega_1\cdots\omega_k)_2}\cdot e^{2\pi i(0,\omega_{k+1}\omega_{k+2}\cdots)_2} \\ &= e^{2\pi i(0.\omega_{k+1}\omega_{k+2}\cdots)_2} \end{aligned}$$

$$(\text{Since } \mathrm{e}^{2\pi\mathrm{i}\cdot\text{integer}} = 1) \quad \longrightarrow \mathrm{e}^{2\pi\mathrm{i}\,(2^k\omega)} = \mathrm{e}^{2\pi\mathrm{i}\,(0.\omega_{k+1}\omega_{k+2}\cdots)_2}. \tag{4}$$

Claim.

We claim the following identity which is so important.

$$\begin{aligned} \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}\mathrm{e}^{2\pi\mathrm{i}\omega y}\,|y\rangle \;=\; &\left(\frac{|0\rangle + e^{2\pi i\left(2^{n-1}\omega\right)}|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle + e^{2\pi i\left(2^{n-2}\omega\right)}|1\rangle}{\sqrt{2}}\right)\otimes\cdots \\ &\cdots\otimes\left(\frac{|0\rangle + e^{2\pi i\left(2^{n-(n-1)}\omega\right)}|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle + e^{2\pi i\left(2^{n-n}\omega\right)}|1\rangle}{\sqrt{2}}\right). \end{aligned}$$

$$\begin{aligned} = &\left(\frac{|0\rangle + e^{2\pi i(0\cdot\omega_n)_2}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + e^{2\pi i(0\cdot\omega_{n-1}\omega_2)_2}|1\rangle}{\sqrt{2}}\right) \\ &\left(\frac{|0\rangle + e^{2\pi i(0\cdot\omega_{n-2}\omega_{n-1}\omega_n)_2}|1\rangle}{\sqrt{2}}\right)\cdots\cdot\left(\frac{|0\rangle + e^{2\pi i(0\cdot\omega_2\cdots\omega_n)_2}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle + e^{2\pi i(0\cdot\omega_1\cdots\omega_n)_2}|1\rangle}{\sqrt{2}}\right) \end{aligned}$$

Worked examples: n=3

$$\begin{aligned} &\frac{1}{\sqrt{2^3}}e^{2\pi i\omega(000)_2}|000\rangle + \frac{1}{\sqrt{2^3}}e^{2\pi i\omega(001)_3}|000\rangle + \frac{1}{\sqrt{2^3}}e^{2\pi i\omega(010)_2}|010\rangle + \overbrace{\frac{1}{\sqrt{2^3}}e^{2\pi i\omega(011)2}|011\rangle}^{\frac{1}{\sqrt{2^3}}e^{2\pi i\omega\left(0\times 2^2+1\times 2^3+1\times 2^0\right)}} \\ &+\frac{1}{\sqrt{2^3}}e^{2\pi i\omega(100)2}|100\rangle + \frac{1}{\sqrt{2^3}}e^{2\pi i\omega(101)_2}\,|101\rangle + \frac{1}{\sqrt{2^3}}e^{2\pi i\omega(110)_2}\,|110\rangle + \frac{1}{\sqrt{2^3}}e^{2\pi i\omega(111)_2}\,|111\rangle \end{aligned}$$

$$\longrightarrow \left(\frac{|0\rangle}{\sqrt{2}}+\frac{e^{2\pi i\omega 2^2}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle}{\sqrt{2}}+\frac{e^{2\pi i\omega 2^1}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle}{\sqrt{2}}+\frac{e^{2\pi i\omega 2^0}|1\rangle}{\sqrt{2}}\right)$$

$$\begin{aligned}
\longrightarrow\ & \frac{e^{2\sin\left(0\times 2^2\right)}\cdot|0\rangle}{\sqrt{2}}\frac{e^{2\pi i\omega\left(1\times 2^1\right)}|1\rangle}{\sqrt{2}}\cdot\frac{e^{2\pi i\omega\left(1\times 2^0\right)}|1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2^3}}e^{2\pi i\omega\left(0\times 2^2+1\times 2^1+1\times 2^0\right)}\,|011\rangle \\
&= \frac{1}{\sqrt{2^3}}e^{2\pi i\omega(011)_2}|011\rangle
\end{aligned}$$

Now, consider, $\omega = (0\cdot\omega_1\omega_2\cdots\omega_n)_2$, we can write,

$$\begin{aligned}
&\frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}e^{2\pi i\omega y}|y\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2-1}e^{\,2\pi i\frac{x}{2^n}y|\,y)} \\
&= \frac{1}{\sqrt{2^n}}\sum_{y_1=0}^{1}\sum_{y_2=0}^{1}\cdots\sum_{y_n=0}^{1}e^{2\pi i\frac{x}{2^n}y}\,|y_1y_2\ldots y_n\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{y=0}^{1}\sum_{y=0}^{1}\ldots\sum_{y=0}^{1}e^{2\pi i}\left(\sum_{l=1}^{n}y_l2^l\right)
\end{aligned}$$

So, we have,

$$\begin{aligned}
|y\rangle &= |\,y_{n-1}\,y_{n-2}\,\cdots\,y_1\,y_0\,\rangle \qquad y_i\in\{0,1\} \\
y &= 2^{n-1}y_{n-1}+2^{n-2}y_{n-2}+\ldots+2^1y_1+2^0y_0=0 \\
&= \sum_{n-l=1}^{n}y_{n-l}2^{n-l}
\end{aligned}$$