CS 506 Intro to Quantum Computing

Study Notes

1 Function of Operators

1.1 Series Expansion

For a function $f(x)$ of variable x, what is $f(T)$?

Series expansion: $f(T) = a_0 I + a_1 T + a_2 T^2 + \ldots$

General form: $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots$

1.2 Common Function Expansions

Exponential function:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{k!}x^k + \ldots \tag{1}$$

Sine function:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{2}$$

Cosine function: (holds for all x)

Euler's identity:

$$e^{ix} = \cos(x) + i\sin(x) \tag{3}$$

1.3 Matrix Exponential Example

For the NOT gate: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$e^X = I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \ldots \tag{4}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

Similarly:

$$e^X = I \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \ldots \tag{6}$$

2 Time Evolution of Closed Quantum Systems

$$|\psi_{\text{new}}\rangle = U\,|\psi_{\text{old}}\rangle \tag{7}$$

where U is the unitary operator.

3 Quantum States

3.1 Normalization

$|\psi\rangle$ is a column vector with normalization condition:

$$\langle\psi|\psi|\psi|\psi\rangle = 1 \tag{8}$$

3.2 Qubit States

2-qubit system:

$$\alpha_0 |00\rangle + \alpha_1 |01\rangle + \dots \tag{9}$$

with $|\alpha_0|^2 + |\alpha_1|^2 = 1$

1st qubit:

$$\frac{1}{\sqrt{3}} |00\rangle + \frac{1}{\sqrt{6}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \tag{10}$$

2nd qubit:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} \\ 1 \\ \sqrt{3} \\ 0 \end{pmatrix} \tag{11}$$

General version of 1 qubit:

$$\alpha_0 |0\rangle + \alpha_1 |1\rangle \tag{12}$$

$$\cos\frac{\theta}{2} |0\rangle + \sin\frac{\theta}{2} |1\rangle \tag{13}$$

$$= \cos\frac{\theta}{2} |0\rangle + e^{i\phi} \sin\frac{\theta}{2} |1\rangle \tag{14}$$

3.3 Bloch Sphere Representation

Block sphere coordinates with angles θ and ϕ.

4 NOT Gate Analysis

NOT gate: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Eigenvalues: $1, -1$ **Eigenvectors:** $|+\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$X = |+\rangle \langle+| - |-\rangle \langle-| \tag{15}$$
$$e^X = e^1 |+\rangle \langle+| + e^{-1} |-\rangle \langle-| \tag{16}$$

$$e^{\frac{i}{\sqrt{2}}X} = e^{\frac{i}{\sqrt{2}}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + e^{-\frac{i}{\sqrt{2}}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{17}$$

$$= \begin{pmatrix} \frac{e^{\frac{i}{\sqrt{2}}} + e^{-\frac{i}{\sqrt{2}}}}{2} & \frac{e^{\frac{i}{\sqrt{2}}} - e^{-\frac{i}{\sqrt{2}}}}{2} \\ \frac{e^{\frac{i}{\sqrt{2}}} - e^{-\frac{i}{\sqrt{2}}}}{2} & \frac{e^{\frac{i}{\sqrt{2}}} + e^{-\frac{i}{\sqrt{2}}}}{2} \end{pmatrix} \tag{18}$$

Using Euler's formula: $T_1 = I$, $T_2 = I$

$$\frac{I}{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad |T_1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |T_2\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{19}$$

$$\frac{I}{T} = e^{1} |T_1\rangle \langle T_1| + e^{-1} |T_2\rangle \langle T_2| \tag{20}$$

$$= e^{\frac{1}{\sqrt{2}}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + e^{-\frac{1}{\sqrt{2}}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{21}$$

$$= e^{\frac{1}{\sqrt{2}}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + e^{-\frac{1}{\sqrt{2}}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} \frac{e+1}{2e} & \frac{e-1}{2e} \\ \frac{e-1}{2e} & \frac{e+1}{2e} \end{pmatrix} = e^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{23}$$

5 Normal Operators and Spectral Decomposition

5.1 Properties of Normal Operators

T is a normal operator if $T^\dagger T = TT^\dagger$

T has eigenvalues $T_1, T_2, \ldots$

T has orthonormal eigenvectors $|T_1\rangle, |T_2\rangle, \ldots, |T_n\rangle$

$$T = \sqrt{\sum_i T_i} \tag{24}$$

$$(|T_i\rangle \langle T_i|)^2 = |T_i\rangle \langle T_i|T_i|T_i|T_i\rangle \langle T_i| \tag{25}$$

$$P^2 = |T_i\rangle \langle T_i| \quad (P \text{ is projector}) \tag{26}$$

$$(|T_i\rangle \langle T_i|)^3 = |T_i\rangle \langle T_i|T_i\rangle \tag{27}$$

$$(|T_i\rangle \langle T_i|)^k = |T_i\rangle \langle T_i| \quad \text{for all } k \tag{28}$$

If $i \neq j$: $|T_i\rangle \langle T_j| = 0$

Therefore: $|T_i\rangle \langle T_j| = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

6 Function of Operators - General Case

$$f(x) = a_0 + a_1 T + a_2 T^2 \tag{29}$$

$$f(T_m) = \sum_{m=0}^{\infty} a_m T_m^m \tag{30}$$

$$f(T) = \sum_{m=0}^{\infty} a_m \sum_i c_m \prod_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} T_k |T_k\rangle \langle T_k| \right)^m \tag{31}$$

For $m = 2, n = 3$:

$$(T_1 \left| T_1 \right\rangle \left\langle T_1 \right| + T_2 \left| T_2 \right\rangle \left\langle T_2 \right| + T_3 \left| T_3 \right\rangle \left\langle T_3 \right|)^2 \tag{32}$$

Using $(a + b + c)^2 = a^2 + b^2 + c^2$:

$$a^2 = (T_1 \left| T_1 \right\rangle \left\langle T_1 \right|)^2 = T_1^2 \left(\left| T_1 \right\rangle \left\langle T_1 \right|\right)^2 \tag{33}$$

$$ab = T_1 \left| T_1 \right\rangle \left\langle T_1 \right| \cdot T_2 \left| T_2 \right\rangle \left\langle T_2 \right| = T_1 T_2 \left\langle T_1 | T_2 | T_1 | T_2 \right\rangle = 0 \tag{34}$$

So: $(a + b + c)^2 = a^2 + b^2 + c^2$

$$f(T_m) = \sum_{m=0}^{\infty} a_m \sum_{i=0}^{\infty} T_i^m \left| T_i \right\rangle \left\langle T_i \right| \tag{35}$$

$$= \sum_{i=1}^{n} \left(\sum_{m=0}^{\infty} a_m T_i^m \right) \left| T_i \right\rangle \left\langle T_i \right| \tag{36}$$

$$= \sum_{i=1}^{n} f(T_i) \left| T_i \right\rangle \left\langle T_i \right| \tag{37}$$