

# More Applications of the Pumping Lemma

# The Pumping Lemma:

- Given a infinite regular language  $L$
- there exists an integer  $m$  (critical length)
- for any string  $w \in L$  with length  $|w| \geq m$
- we can write  $w = x y z$
- with  $|x y| \leq m$  and  $|y| \geq 1$
- such that:  $x y^i z \in L \quad i = 0, 1, 2, \dots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$



Regular languages

**Theorem:** The language

$$L = \{vv^R : v \in \Sigma^*\} \quad \Sigma = \{a, b\}$$

is not regular

**Proof:** Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction  
that  $L$  is a regular language

Since  $L$  is infinite  
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let  $m$  be the critical length for  $L$

Pick a string  $w$  such that:  $w \in L$

and length  $|w| \geq m$

We pick  $w = a^m b^m b^m a^m$

From the Pumping Lemma:

we can write:  $w = a^m b^m b^m a^m = x y z$

with lengths:  $|x y| \leq m, \quad |y| \geq 1$

$$w = xyz = \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{b \dots b}_{m} \underbrace{b \dots b}_{m} \underbrace{b \dots b}_{m} \underbrace{a \dots a}_{m}$$

$x \quad y \quad z$

Thus:  $y = a^k, \quad 1 \leq k \leq m$

$$x y z = a^m b^m b^m a^m \quad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma:  $x y^i z \in L$   
 $i = 0, 1, 2, \dots$

Thus:  $x y^2 z \in L$



$$x y z = a^m b^m b^m a^m \quad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma:  $x y^2 z \in L$

$$xy^2z = \overbrace{a \dots a}^{m+k} \overbrace{a \dots a}^m \overbrace{a \dots a}^m \overbrace{a \dots a}^m \square L$$

$\underbrace{\hspace{1.5cm}}_x \quad \underbrace{\hspace{1.5cm}}_y \quad \underbrace{\hspace{1.5cm}}_y \quad \underbrace{\hspace{4cm}}_z$

Thus:  $a^{m+k} b^m b^m a^m \in L$

$$a^{m+k}b^mb^ma^m \in L \quad k \geq 1$$

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**BUT:**  $L = \{vv^R : v \in \Sigma^*\}$



$$a^{m+k}b^mb^ma^m \notin L$$

**CONTRADICTION!!!**

Therefore: Our assumption that  $L$   
is a regular language is not true

**Conclusion:**  $L$  is not a regular language

END OF PROOF

# Non-regular languages

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$



Regular languages

**Theorem:** The language

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

is not regular

**Proof:** Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

Assume for contradiction  
that  $L$  is a regular language

Since  $L$  is infinite  
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

Let  $m$  be the critical length of  $L$

Pick a string  $w$  such that:  $w \in L$  and  
length  $|w| \geq m$

We pick  $w = a^m b^m c^{2m}$

From the Pumping Lemma:

We can write  $w = a^m b^m c^{2m} = x y z$

With lengths  $|x y| \leq m, |y| \geq 1$

$$w = xyz = \overbrace{a \dots a}^m \overbrace{a \dots a}^m \overbrace{a \dots a b \dots b c \dots c}^{2m}$$
$$\underbrace{\hspace{1.5cm}}_x \underbrace{\hspace{1.5cm}}_y \underbrace{\hspace{4.5cm}}_z$$

Thus:  $y = a^k, 1 \leq k \leq m$



$$x y z = a^m b^m c^{2m}$$

$$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma:  $x y^i z \in L$   
 $i = 0, 1, 2, \dots$

Thus:  $x y^0 z = xz \notin L$

$$x y z = a^m b^m c^{2m} \quad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma:  $xz \in L$

$$xz = \overbrace{a \dots a}^{m-k} \overbrace{a \dots a}^m \overbrace{b \dots b}^m \overbrace{c \dots c}^{2m} \in L$$

$$\underbrace{\hspace{1.5cm}}_x \underbrace{\hspace{4.5cm}}_z$$

Thus:  $a^{m-k} b^m c^{2m} \in L$

$$a^{m-k} b^m c^{2m} \in L \quad k \geq 1$$

---

**BUT:**  $L = \{a^n b^l c^{n+l} : n, l \geq 0\}$



$$a^{m-k} b^m c^{2m} \notin L$$

**CONTRADICTION!!!**

Therefore: Our assumption that  $L$   
is a regular language is not true

**Conclusion:**  $L$  is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^{n!} : n \geq 0\}$$



Regular languages

**Theorem:** The language  $L = \{a^{n!} : n \geq 0\}$   
is not regular

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

**Proof:** Use the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction  
that  $L$  is a regular language

Since  $L$  is infinite  
we can apply the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Let  $m$  be the critical length of  $L$

Pick a string  $w$  such that:  $w \in L$

length  $|w| \geq m$

We pick  $w = a^{m!}$



From the Pumping Lemma:

We can write  $w = a^{m!} = x y z$

With lengths  $|x y| \leq m, |y| \geq 1$

$$w = xyz = a^{m!} = \overbrace{a \dots a}^m \overbrace{a \dots a}^{m!-m}$$

The diagram illustrates the decomposition of the string  $w = a^{m!}$  into  $xyz$ . The string is represented as a sequence of  $m!$  'a' characters. A top brace labeled  $m$  spans the first  $m$  'a's, and another top brace labeled  $m!-m$  spans the remaining  $m!-m$  'a's. Below the string, three bottom braces identify the segments  $x$ ,  $y$ , and  $z$ . The  $x$  brace is under the first few 'a's, the  $y$  brace is under the next few 'a's, and the  $z$  brace is under the rest of the string. The combined length of  $x$  and  $y$  is indicated to be at most  $m$ , and the length of  $y$  is at least 1.

Thus:  $y = a^k, 1 \leq k \leq m$

$$x y z = a^{m!}$$

$$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma:  $x y^i z \in L$   
 $i = 0, 1, 2, \dots$

Thus:  $x y^2 z \in L$

$$x y z = a^{m!}$$

$$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma:  $x y^2 z \in L$

$$xy^2z = \overbrace{a \dots a a \dots a a \dots a a \dots a a \dots a a \dots a}^{m+k} \overbrace{a \dots a a \dots a a \dots a a \dots a a \dots a}^{m!-m} \in L$$

$x$

$y$

$y$

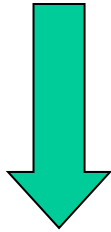
$z$

Thus:  $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

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Since:  $L = \{a^{n!} : n \geq 0\}$



There must exist  $p$  such that:

$$m!+k = p!$$

However:  $m!+k \leq m!+m$  for  $m > 1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m + 1)$$

$$= (m + 1)!$$



$$m!+k < (m + 1)!$$



$$m!+k \neq p! \quad \text{for any } p$$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

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**BUT:**  $L = \{a^{n!} : n \geq 0\}$



$$a^{m!+k} \notin L$$

**CONTRADICTION!!!**

Therefore: Our assumption that  $L$   
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