Quantum Computation — Lecture Notes

Date: October 13, 2025

Topic: Phase Estimation and Quantum Fourier Transform (QFT)

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Overview (Lecture Summary)

Today's lecture linked **phase estimation** to the **Quantum Fourier Transform (QFT)**. We began by expressing a phase-encoded superposition as a product of single-qubit phase states, understood how the QFT and its inverse (QFT^{-1}) construct and remove these phases respectively, and concluded by analyzing circuit complexity ($\Theta(n^2)$) and approximation strategies.

Setup and Notation

We consider n qubits, with $N = 2^n$. The phase is represented as a binary fraction:

$$\omega = (0.\omega_1\omega_2 \dots \omega_n)_2 \in [0,1]$$

and an integer representation:

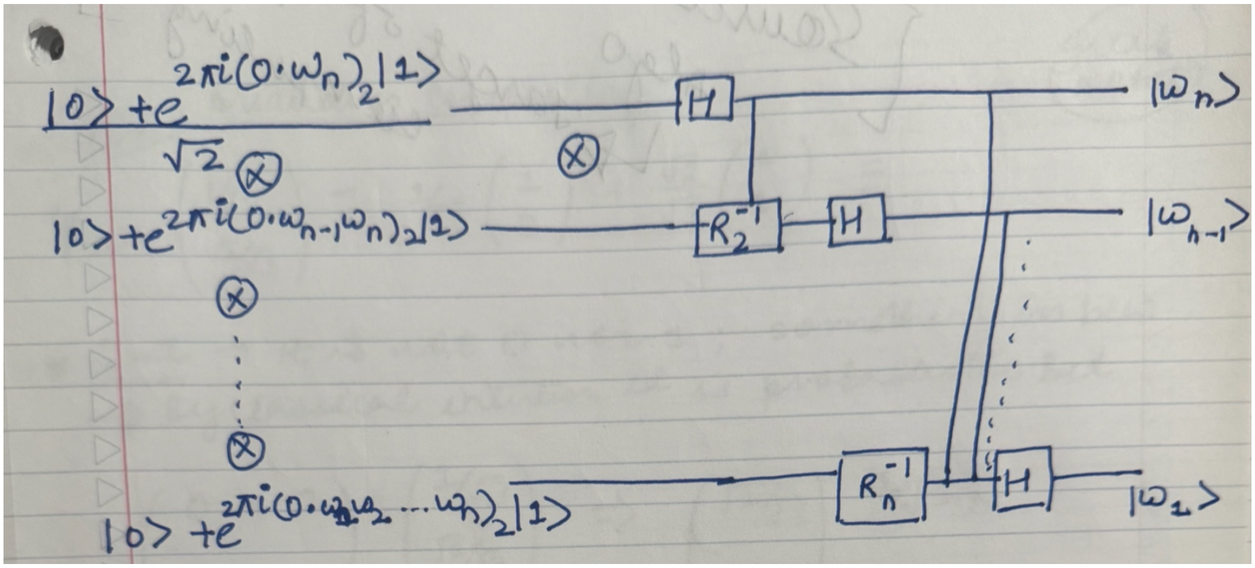
$$x = (x_1x_2 \dots x_n)_2 \in \{0,1,\dots,2^n-1\}, \quad \text{with } \omega = \frac{x}{2^n}.$$

We begin with the phase-encoded superposition:

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i \omega y} |y\rangle, \qquad y = \sum_{j=1}^{n} y_j 2^{n-j},\ y_j \in \{0,1\}. \tag{1}$$

Claim: Factorization into Single-Qubit Phases

$$\boxed{\frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i \omega y} |y\rangle = \bigotimes_{i=1}^{n} \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{2\pi i (0.\omega_i\omega_{i+1}\dots\omega_n)} |1\rangle \Big)} \tag{2}$$



Proof. Using $\omega = \frac{x}{2^n}$,

$$\omega y = \frac{x}{2^n} \sum_{j=1}^{n} y_j 2^{n-j} = \sum_{j=1}^{n} y_j x 2^{-j}, \tag{3}$$

$$e^{2\pi i \omega y} = \prod_{j=1}^{n} e^{2\pi i y_j x 2^{-j}}. \tag{4}$$

Then:

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{y_1,\ldots,y_n \in \{0,1\}} \left(\prod_{j=1}^{n} e^{2\pi i y_j x 2^{-j}} \right) |y_1 \ldots y_n\rangle = \bigotimes_{j=1}^{n} \frac{|0\rangle + e^{2\pi i x 2^{-j}} |1\rangle}{\sqrt{2}}.$$

Finally, $x2^{-j} = (0.\omega_j \omega_{j+1} \ldots \omega_n)_2$, giving the required factorization. □

Interpretation

Each qubit j carries the phase $e^{2\pi i (0.\omega_j \omega_{j+1} \ldots \omega_n)}$. This structure matches what the (inverse) QFT produces using Hadamard gates and controlled rotations R_k.

Inverse QFT

The circuit below reproduces the exact **board diagram** from the lecture, showing the uncomputation of phases using R_k^{-1} gates and Hadamards, moving from the bottom qubit upward.

Context Reminder

Starting from the tensor-product phase state

$$|\psi\rangle = \bigotimes_{i=1}^{n} \frac{|0\rangle + e^{2\pi i(0.\omega_i \ldots \omega_n)}\,|1\rangle}{\sqrt{2}},$$

the inverse QFT removes each phase sequentially:

- Apply controlled R_k^{-1} gates from lower qubits to the current qubit.
- Apply a Hadamard gate to the current qubit.
- Move upward and repeat until all phases are peeled off.

Special Case: Last Qubit

- If $\omega_n = 0$: $\frac{|0\rangle+|1\rangle}{\sqrt{2}} \xrightarrow{H} |0\rangle$.
- If $\omega_n = 1$: $\frac{|0\rangle-|1\rangle}{\sqrt{2}} \xrightarrow{H} |1\rangle$.

Inverse QFT Formula and Gate Counts

$$QFT_{2^n}^{-1}\,|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{-2\pi i x y/2^n}\,|y\rangle\,,$$

$$\#H = n, \quad \#(\text{controlled } R_k) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}, \quad \text{Total size: } \Theta(n^2).$$

Approximate QFT: For practical circuits, small-angle rotations R_k for large k may be neglected. Keeping only $O(\log T)$ rotations per line yields an overall error $< 1/T$, maintaining polynomial complexity.

Professor's Key Remarks

- The binary expansion of ω allows a clean tensor factorization of the phase state.
- QFT adds these phases; QFT^{-1} removes them top-to-bottom.
- The complete QFT uses $\Theta(n^2)$ gates; approximate QFT achieves similar results with fewer rotations.
- The final measurement order (bottom to top) reflects the inverse QFT's operation order.