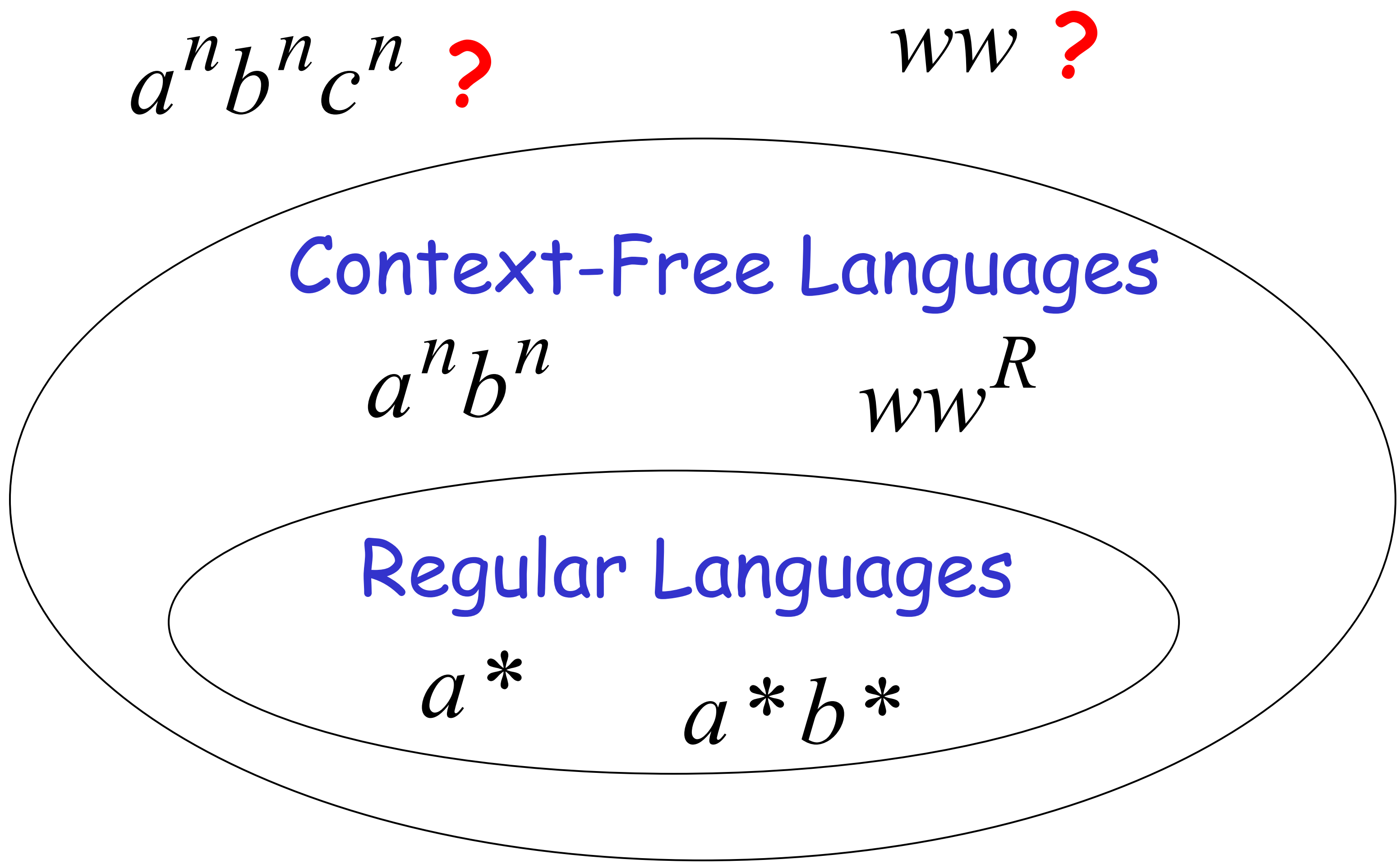
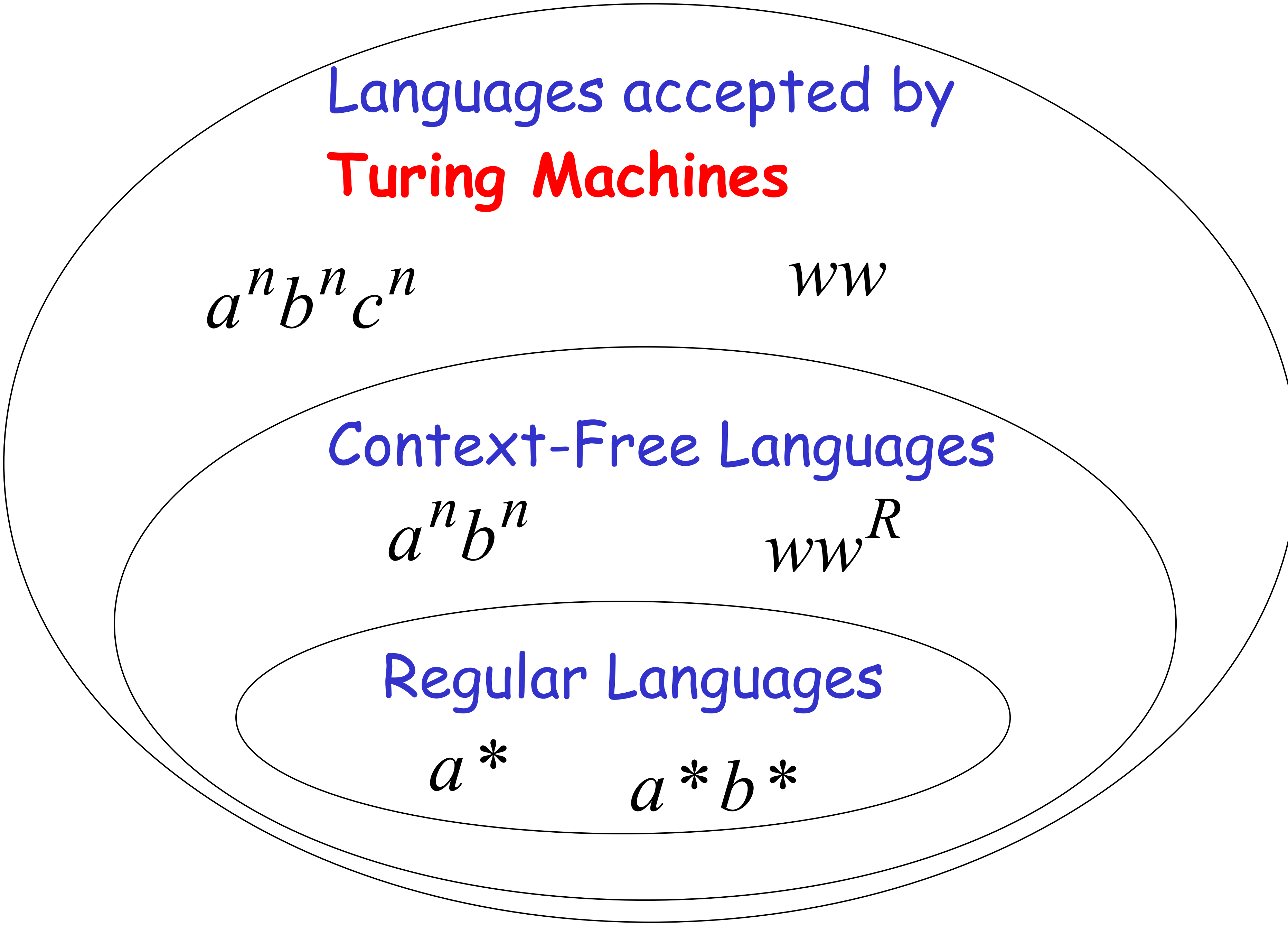
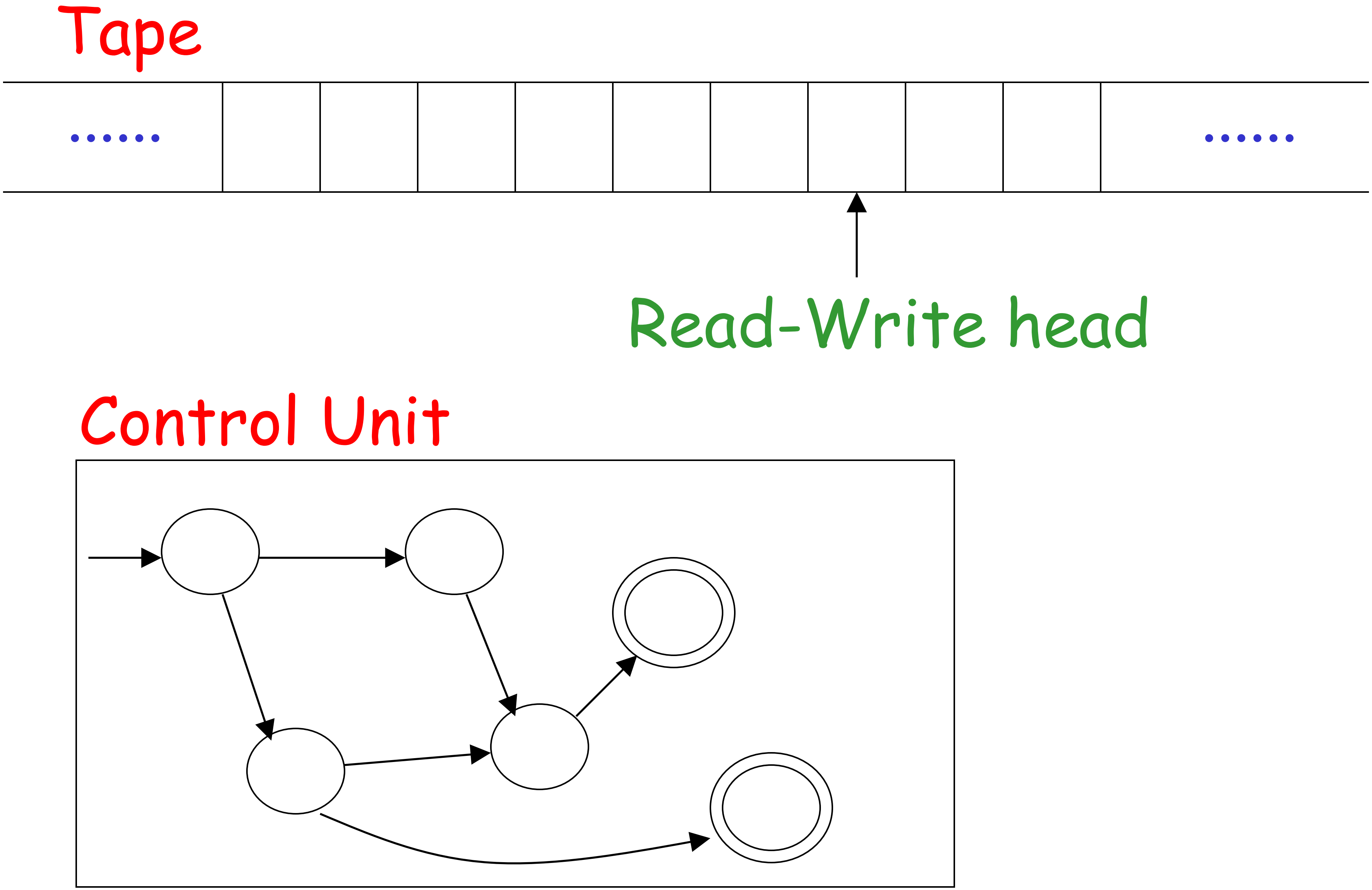
Turing Machines

The Language Hierarchy



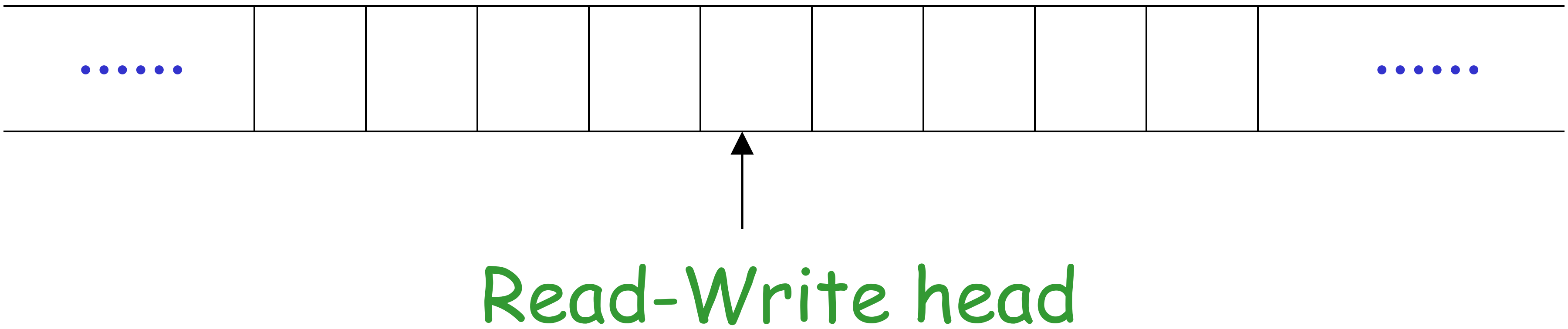

Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
Regular Languages
$a*$
$a*b*$

A Turing Machine

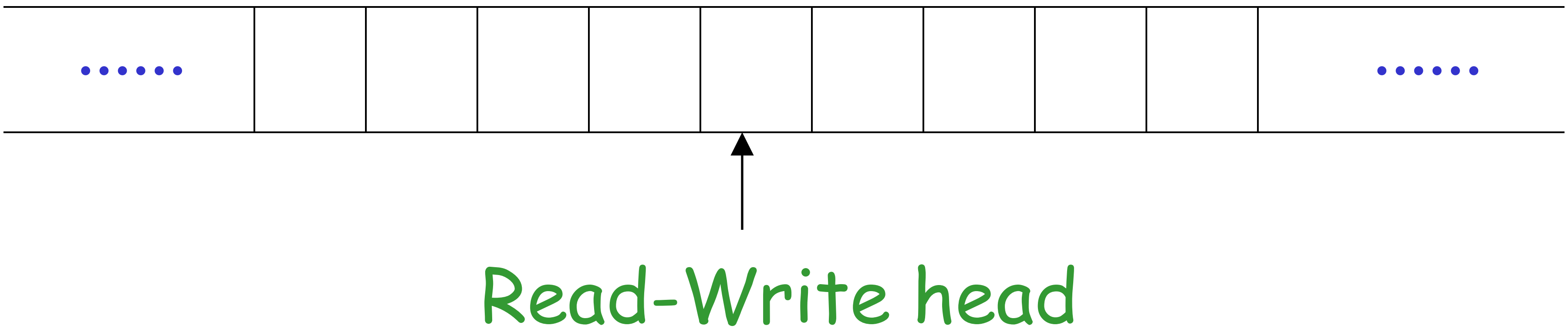


The Tape

No boundaries -- infinite length



The head moves Left or Right

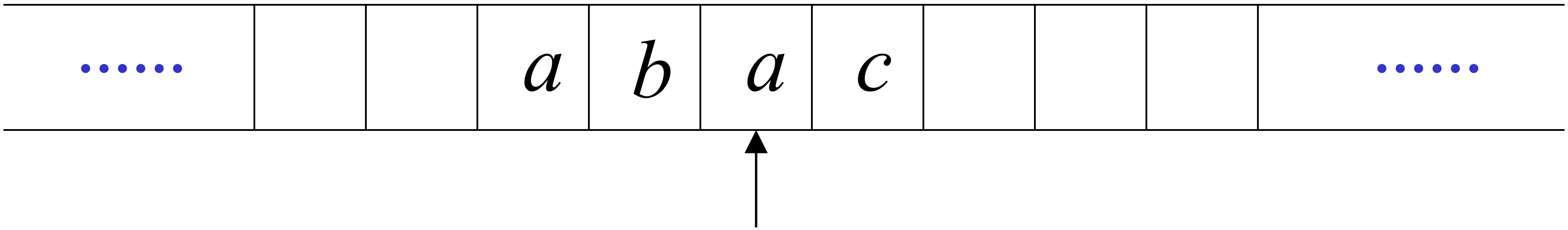


The head at each transition (time step):

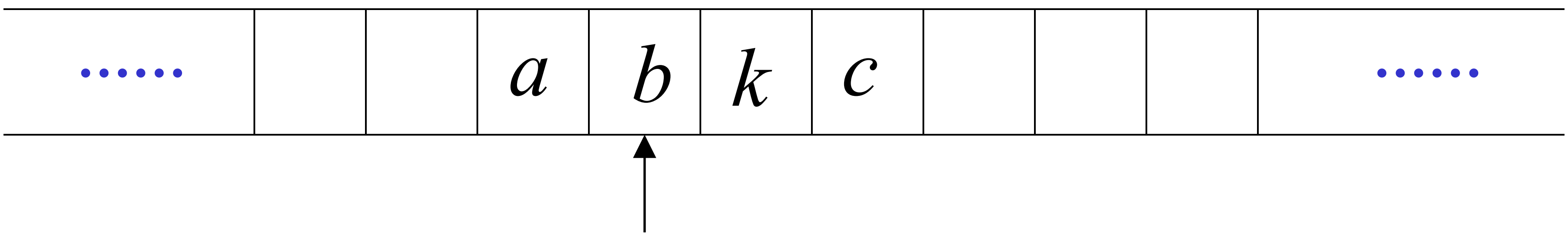
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

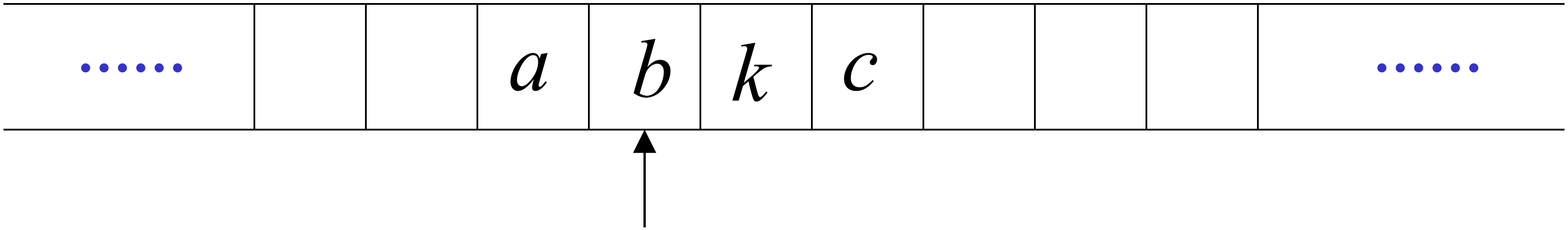


Time 1

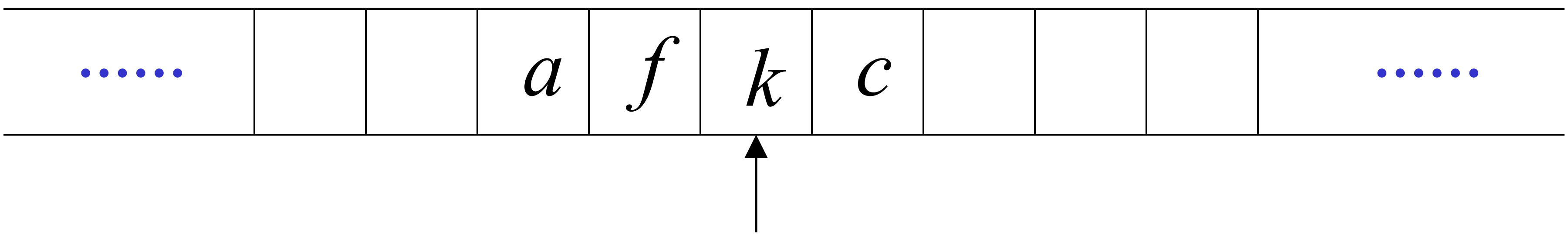


1. Reads a
2. Writes k
3. Moves Left

Time 1

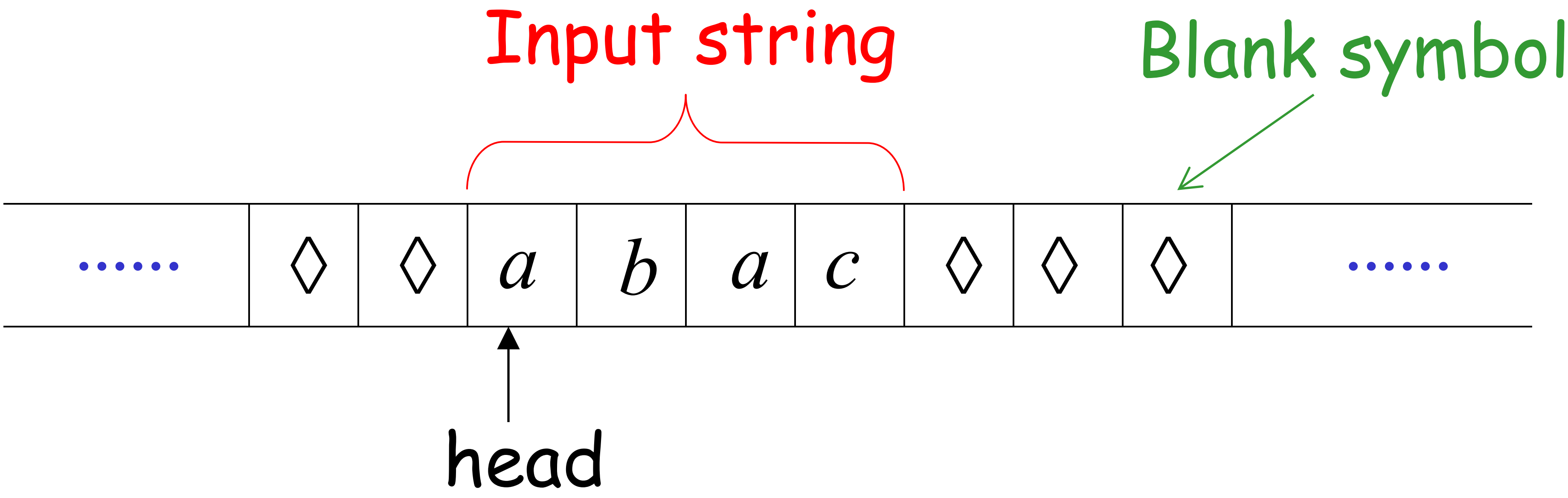


Time 2



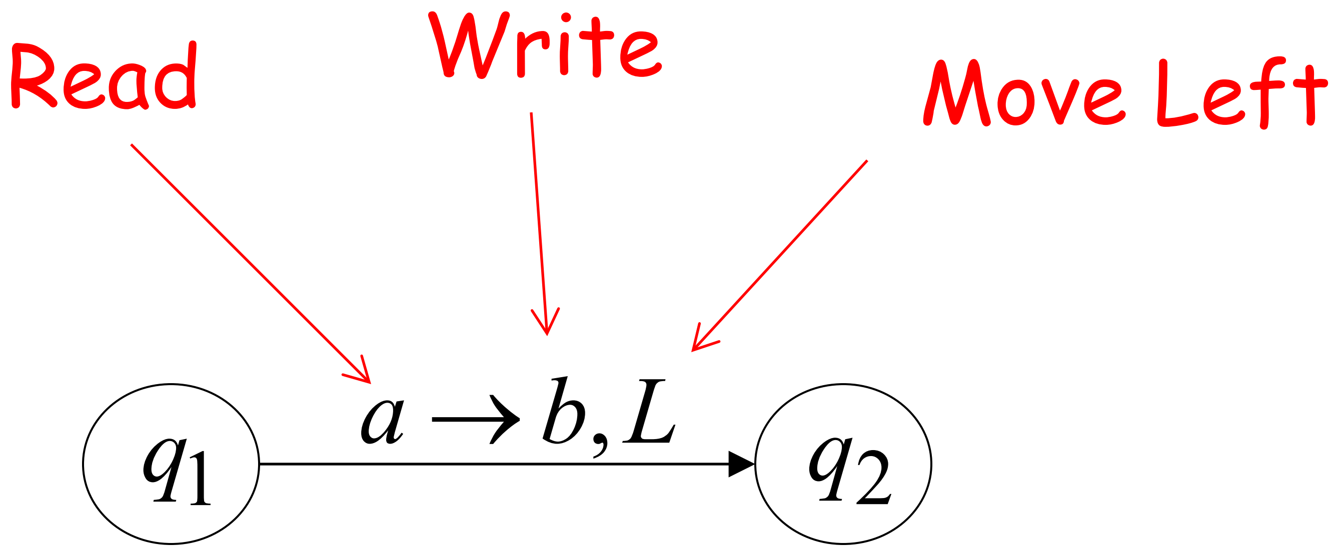
1. Reads b
2. Writes f
3. Moves Right

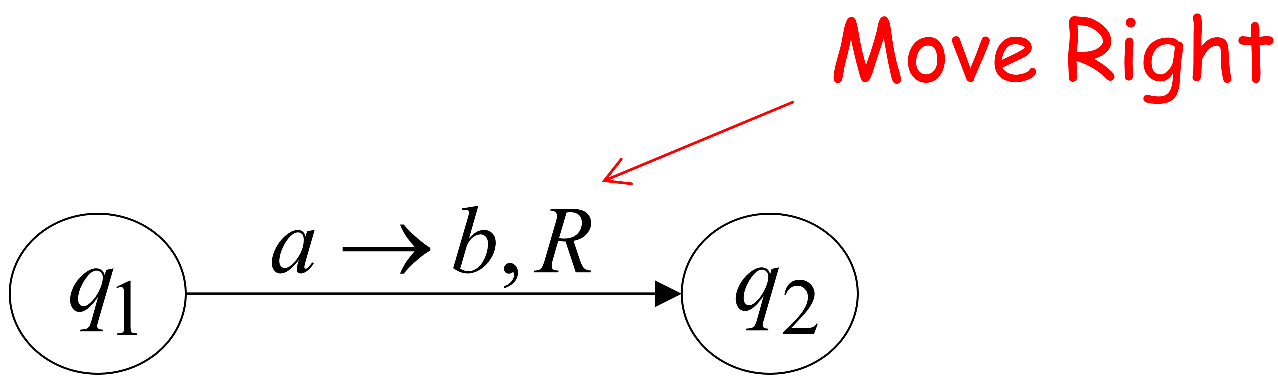
The Input String



Head starts at the leftmost position
of the input string

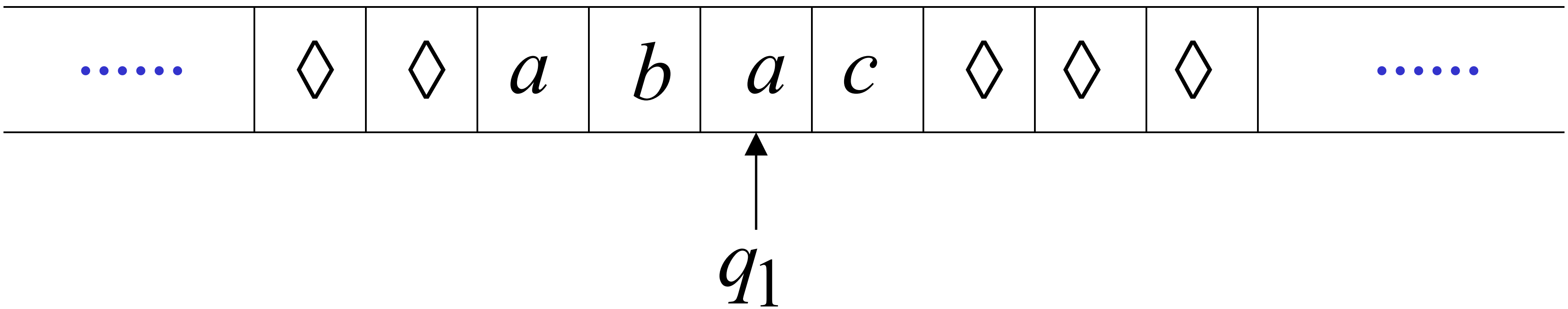
States & Transitions



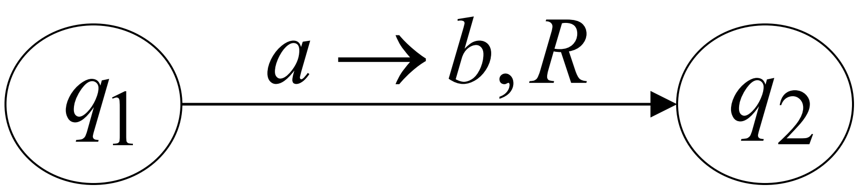


Example:

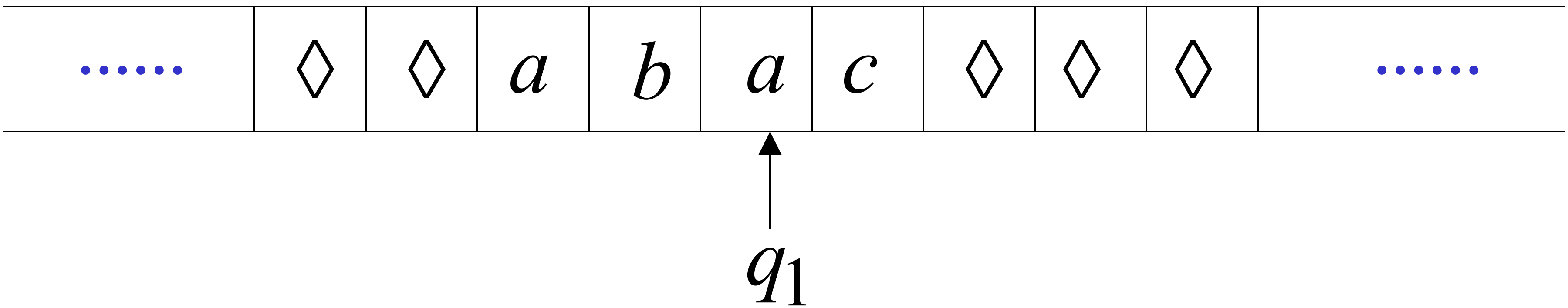
Time 1



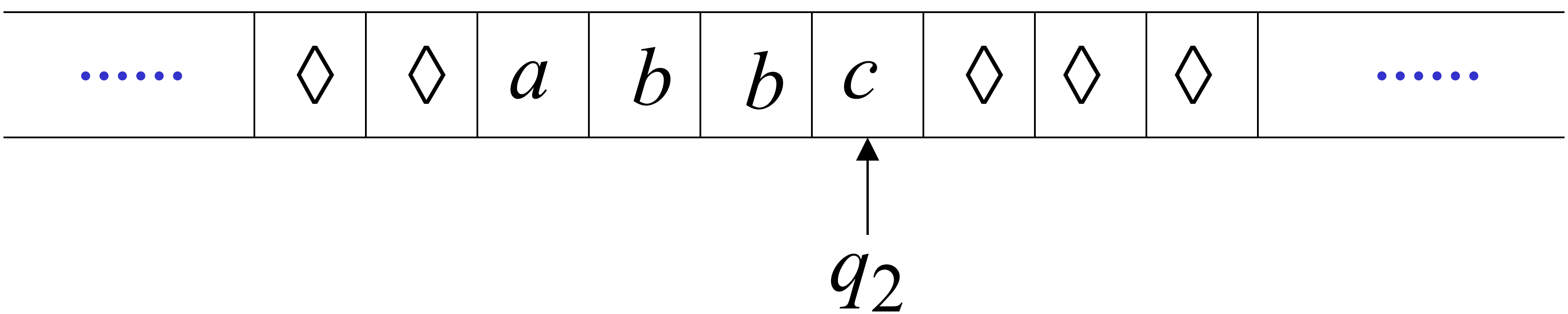
current state

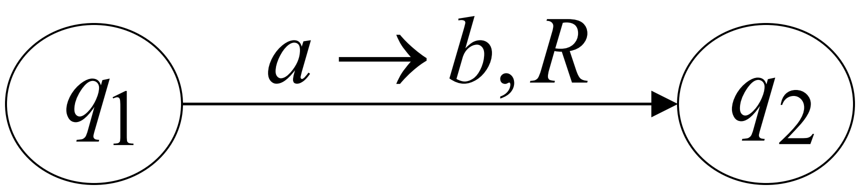


Time 1



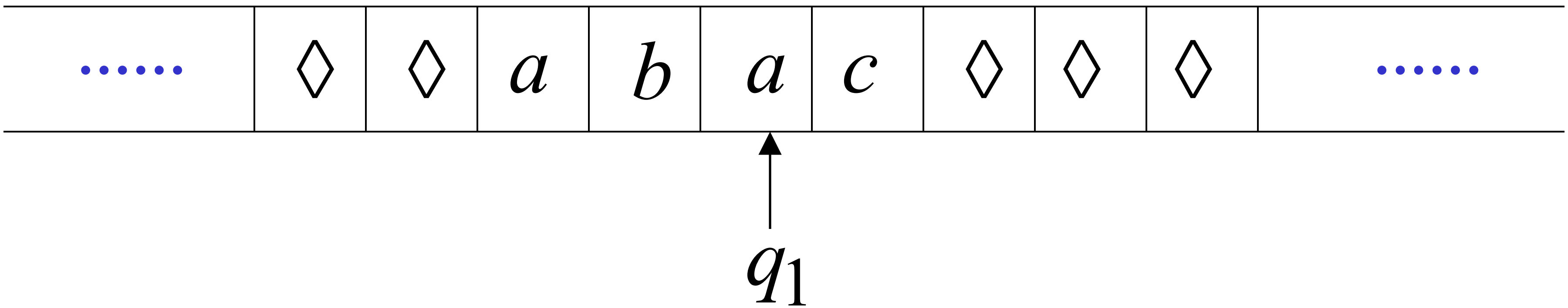
Time 2



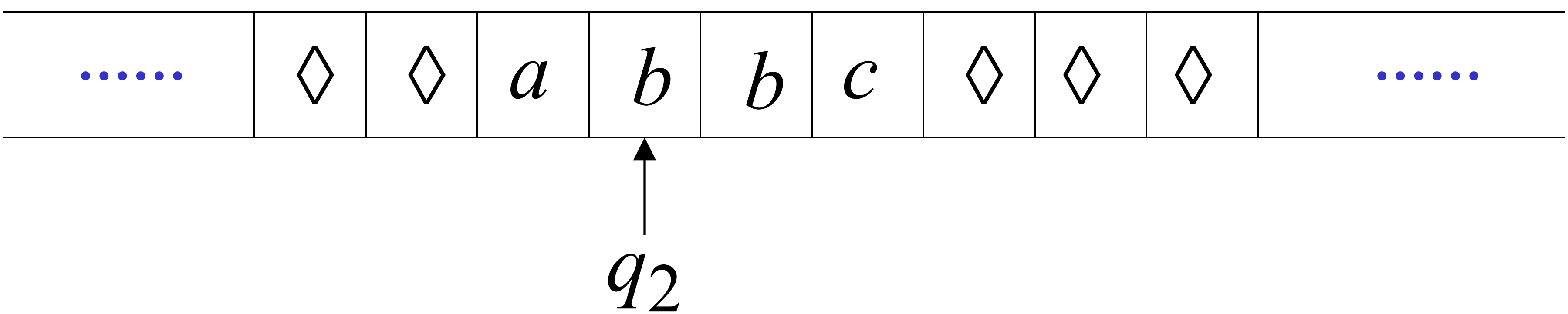


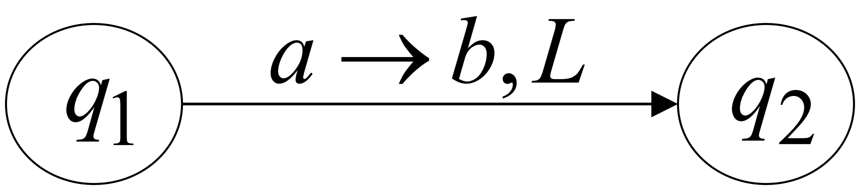
Example:

Time 1



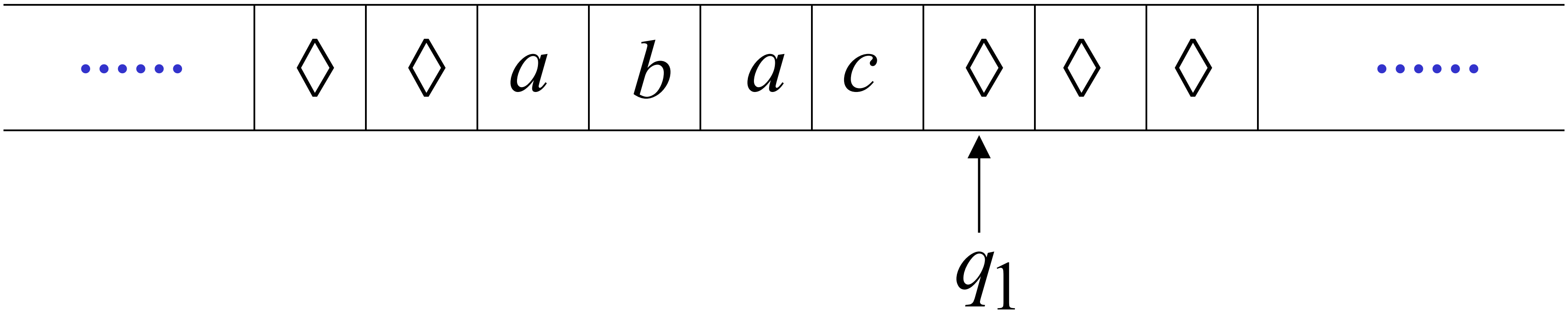
Time 2



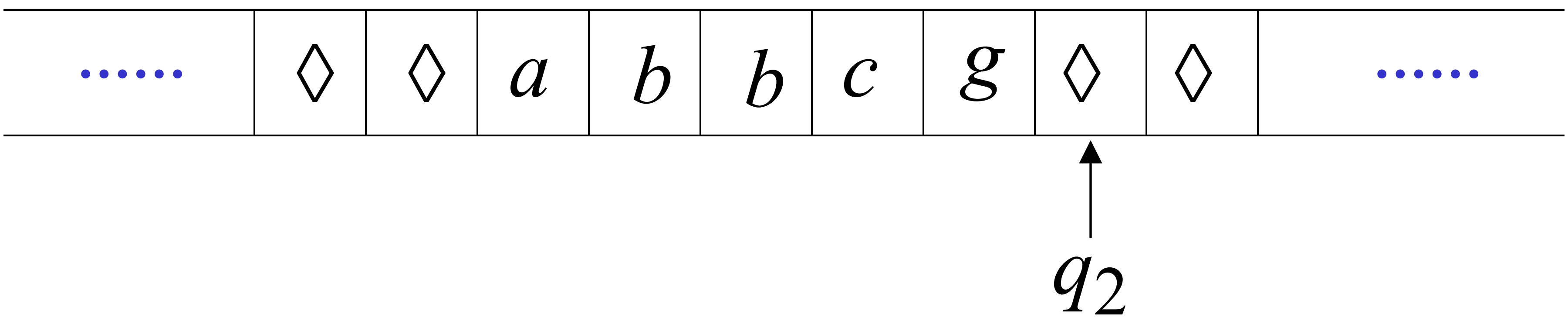


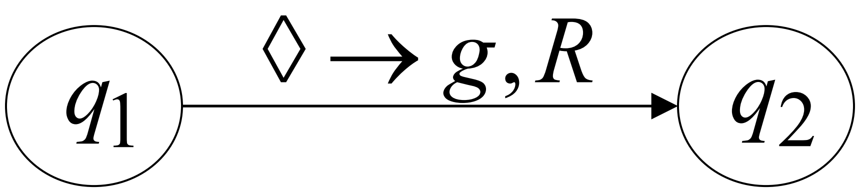
Example:

Time 1



Time 2

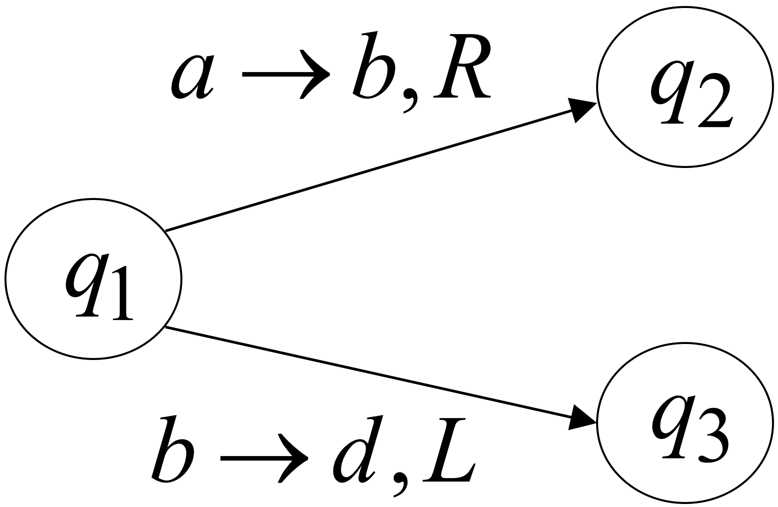




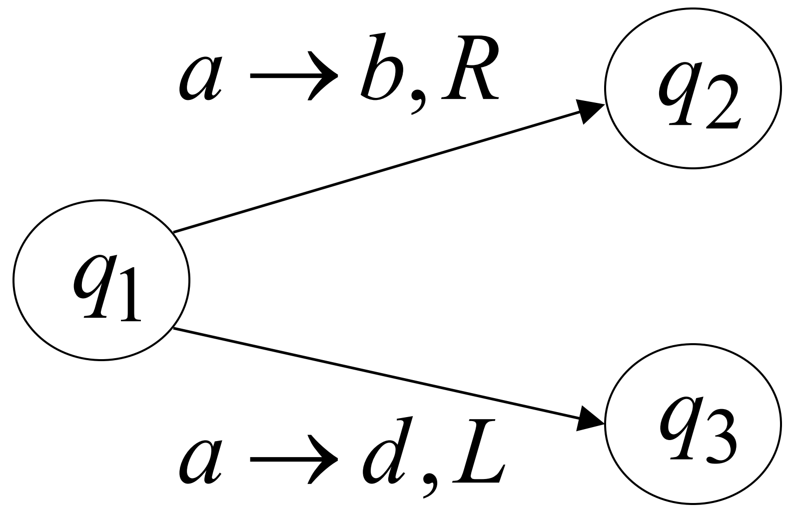
Determinism

Turing Machines are deterministic

Allowed



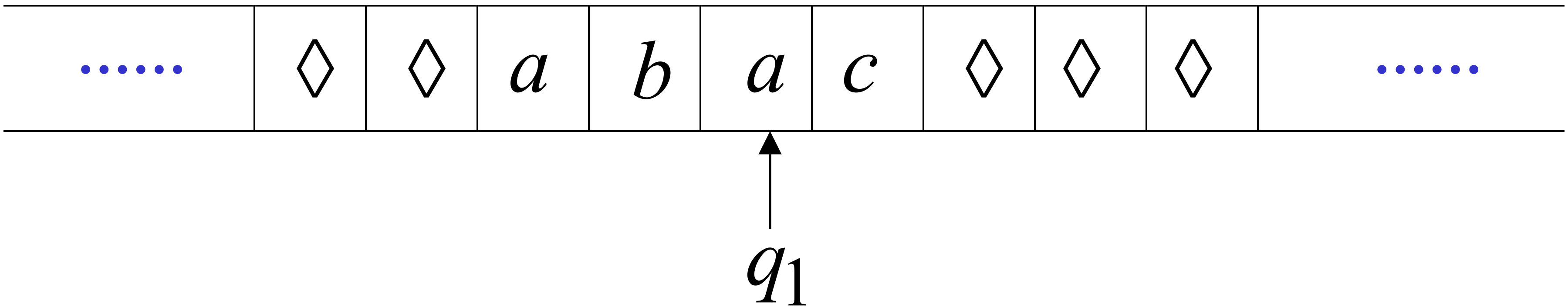
Not Allowed

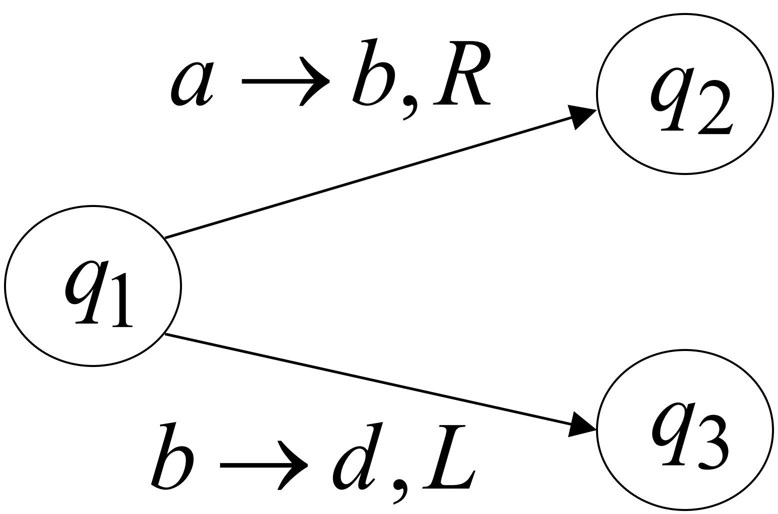


No epsilon transitions allowed

Partial Transition Function

Example:





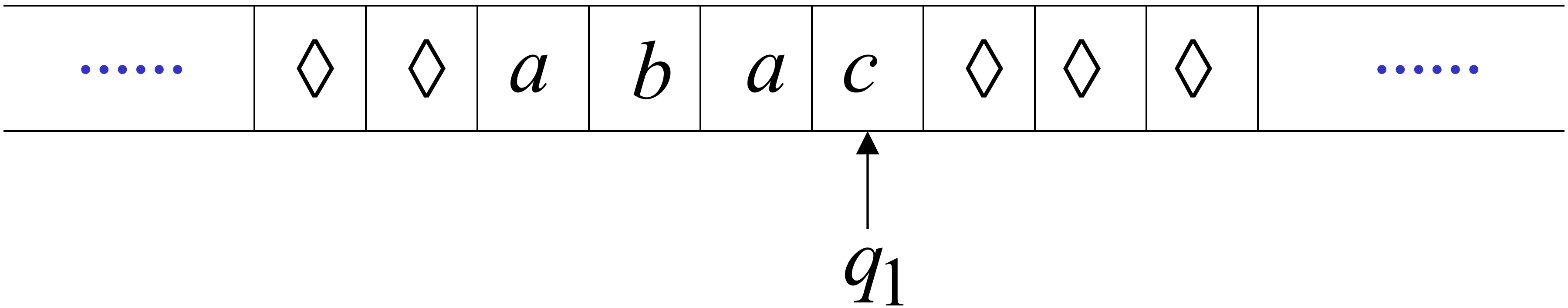
Allowed:

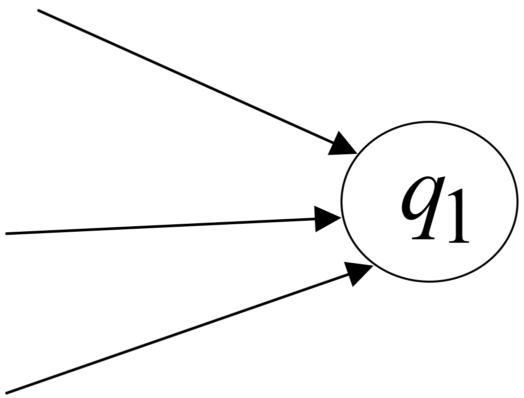
No transition for input symbol c

Halting

The machine ***halts*** in a state if there is no transition to follow

Halting Example 1:

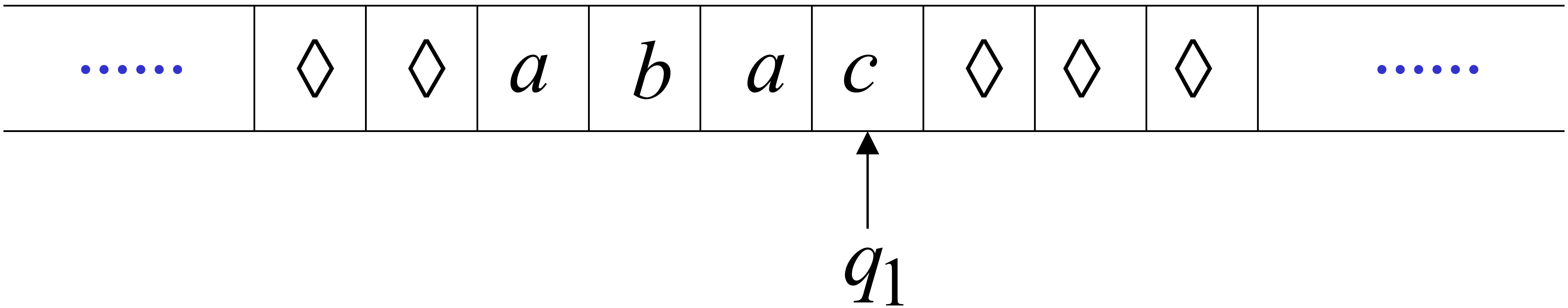


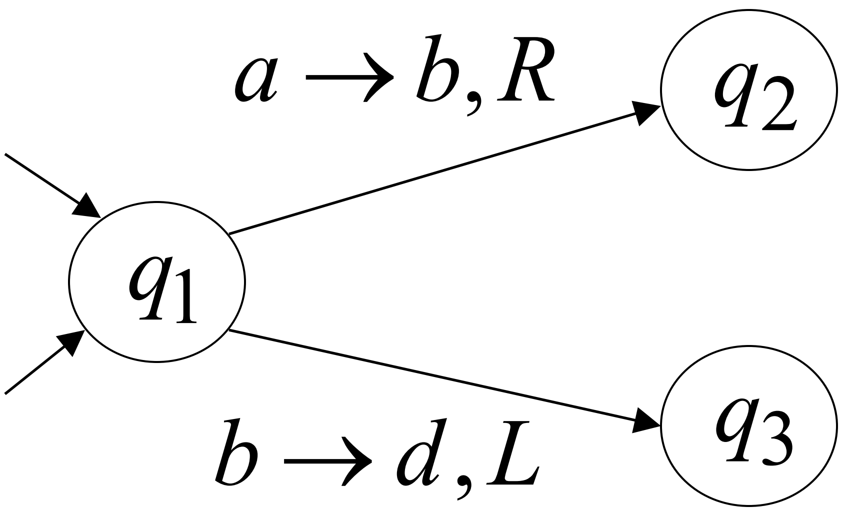


No transition from q_1

HALT!!!

Halting Example 2:

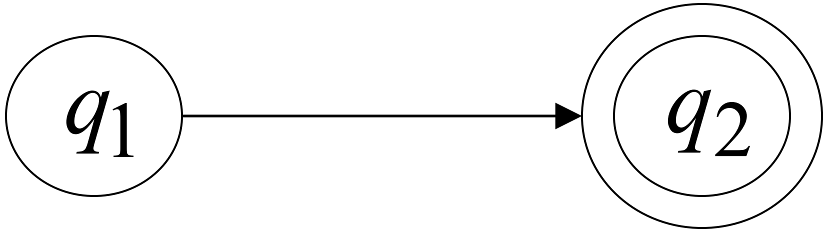




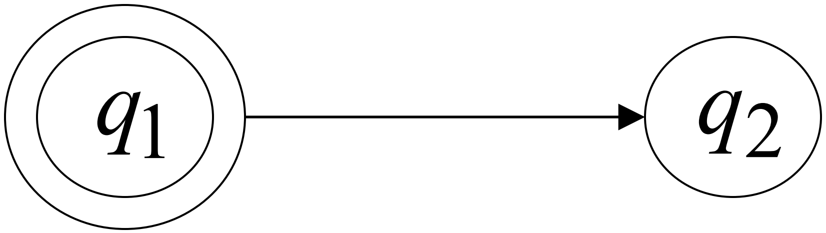
No possible transition from q_1 and symbol c

HALT!!!

Accepting States



Allowed



Not Allowed

- Accepting states have no outgoing transitions
- The machine halts and accepts

Acceptance

Accept Input string → If machine halts in an accept state

Reject Input string → If machine halts in a non-accept state

or

If machine enters an *infinite loop*

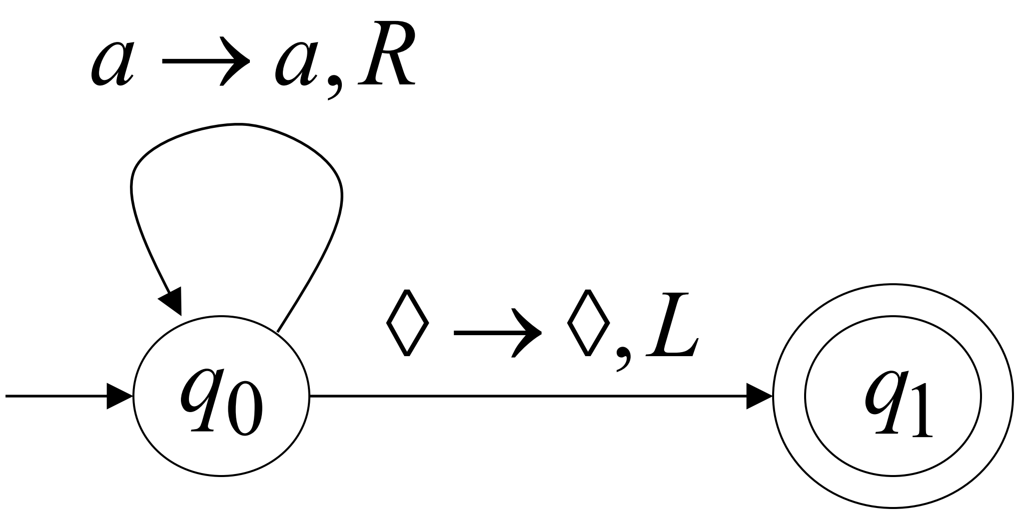
Observation:

In order to accept an input string, it is not necessary to scan all the symbols in the string

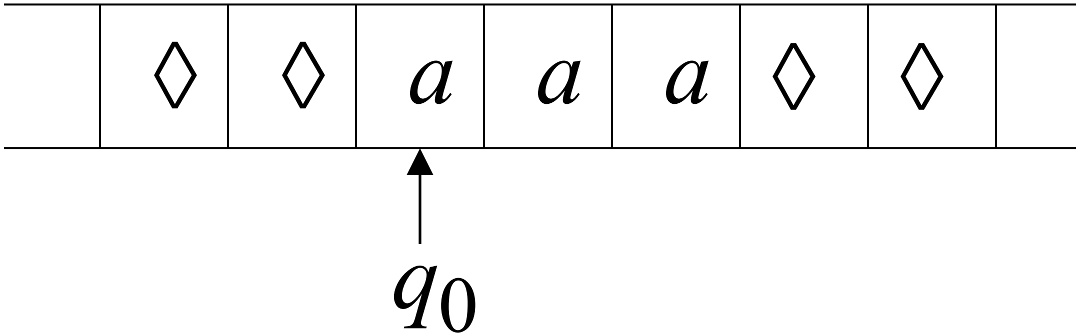
Turing Machine Example

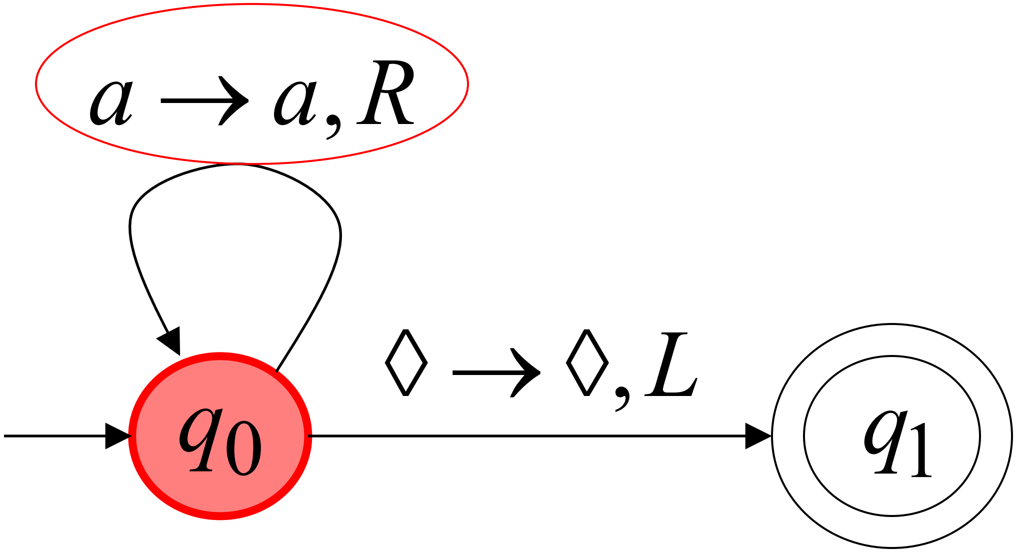
Input alphabet $\Sigma = \{a, b\}$

Accepts the language: a^*

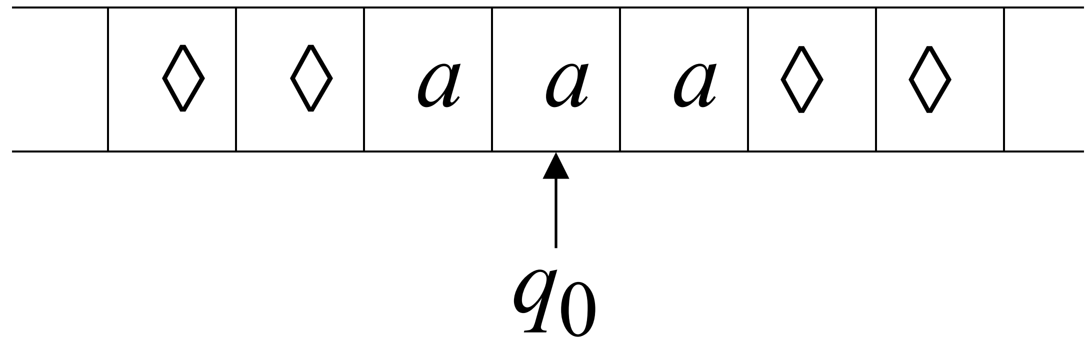


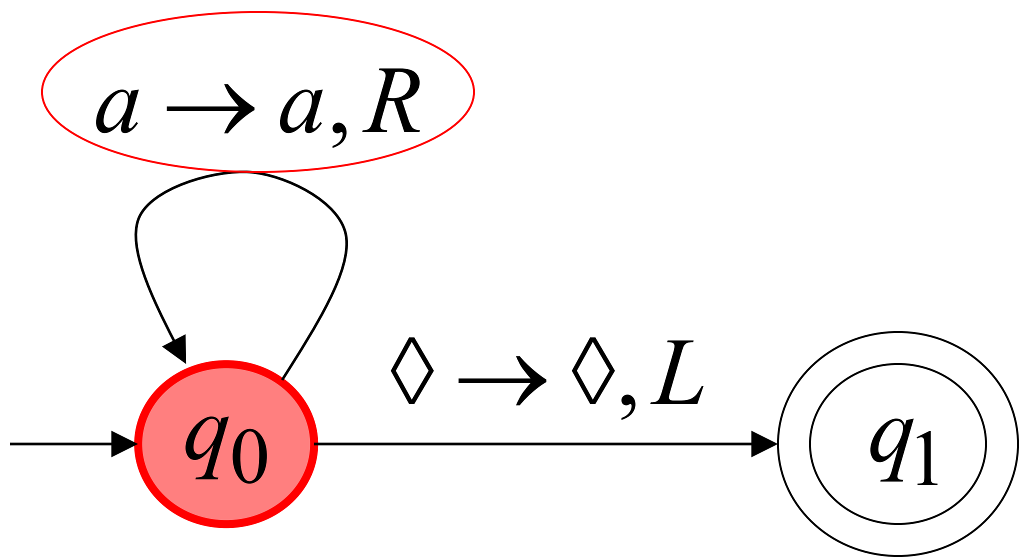
Time 0



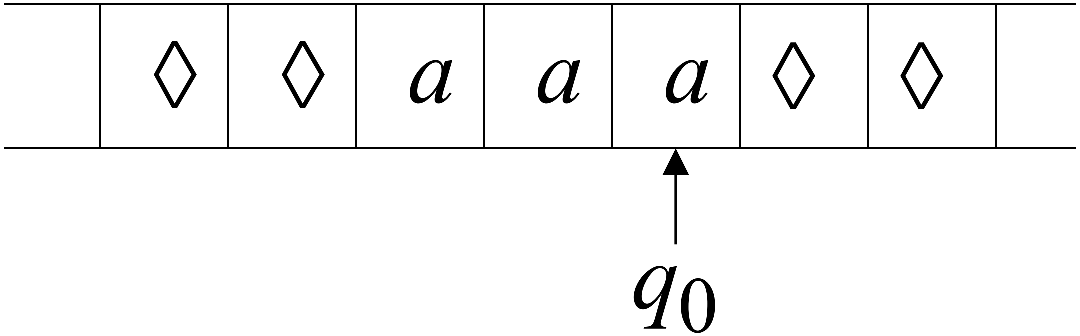


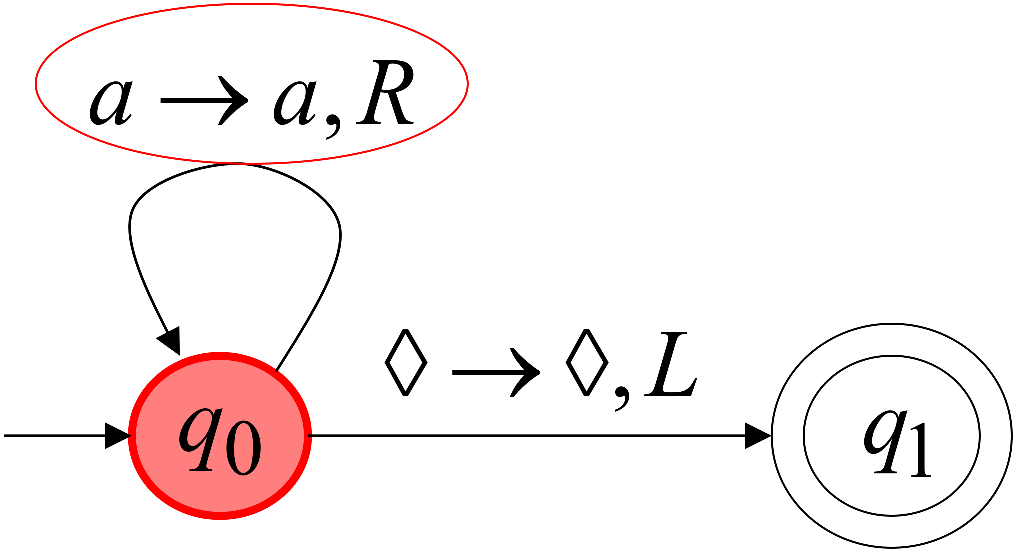
Time 1



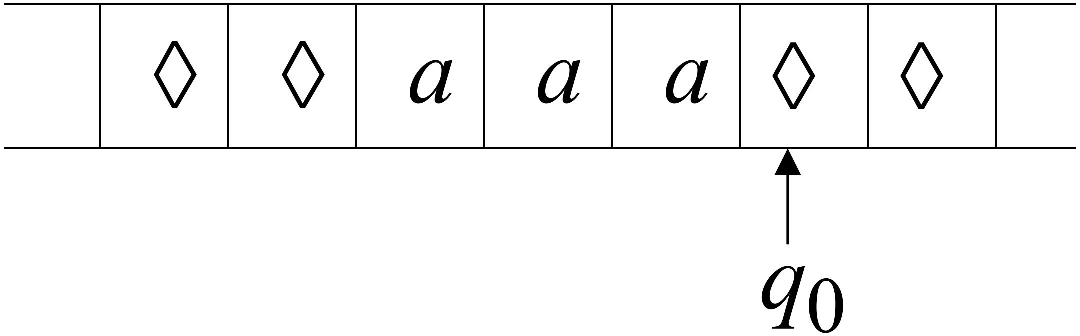


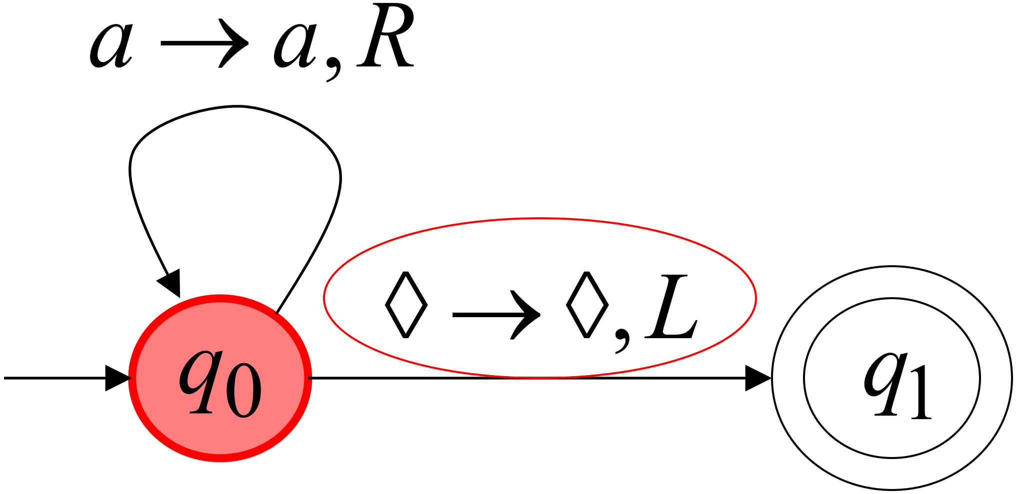
Time 2



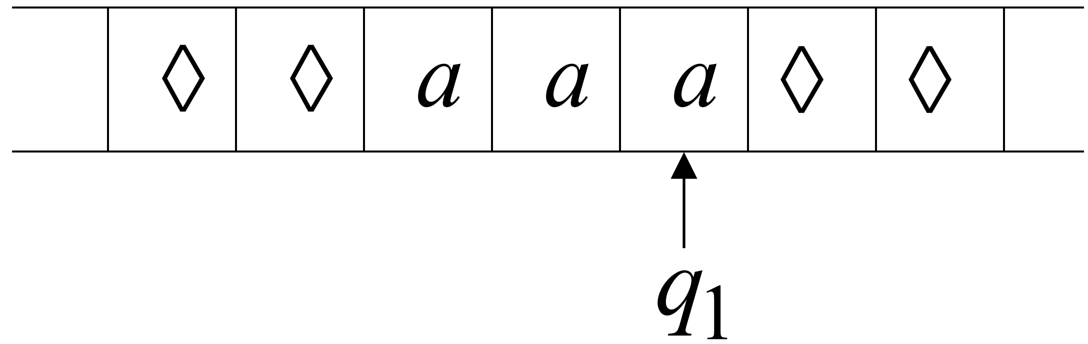


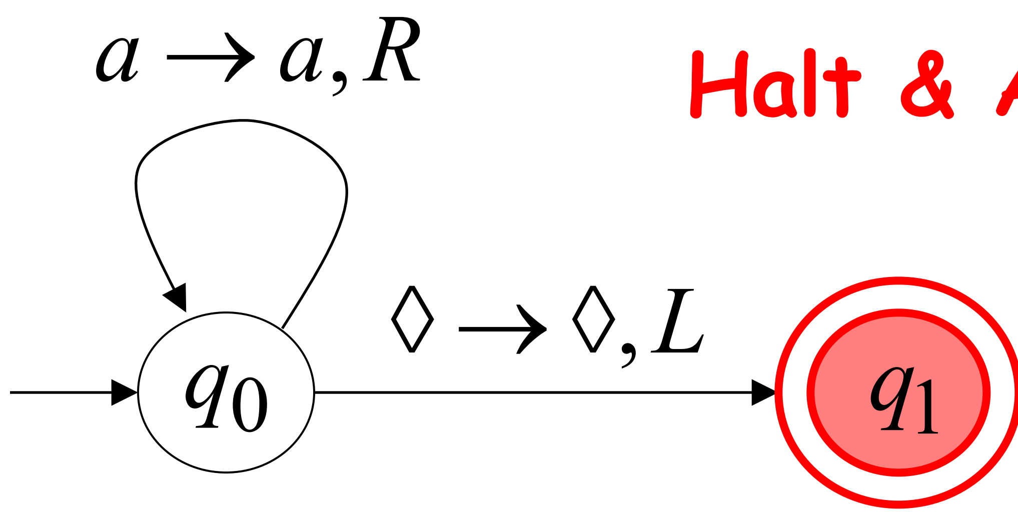
Time 3





Time 4

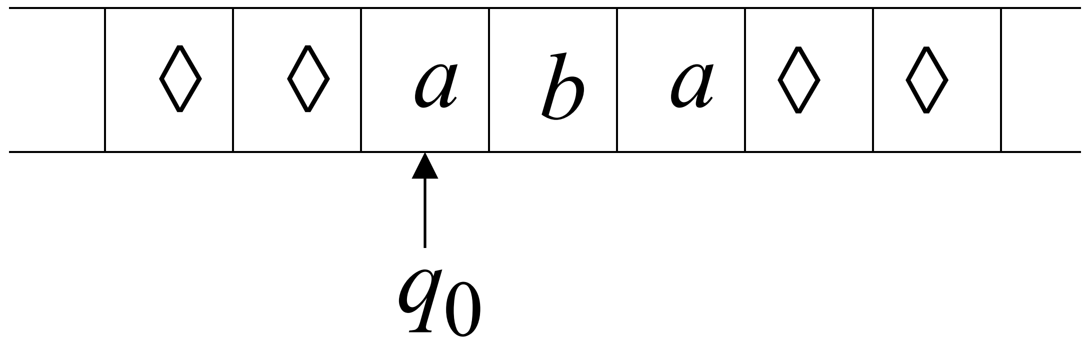


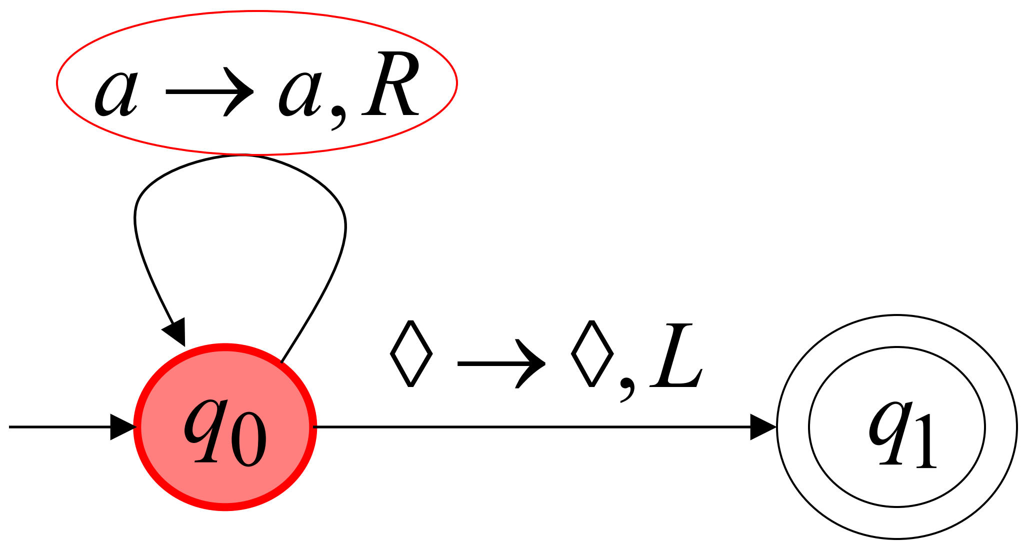


Halt & Accept

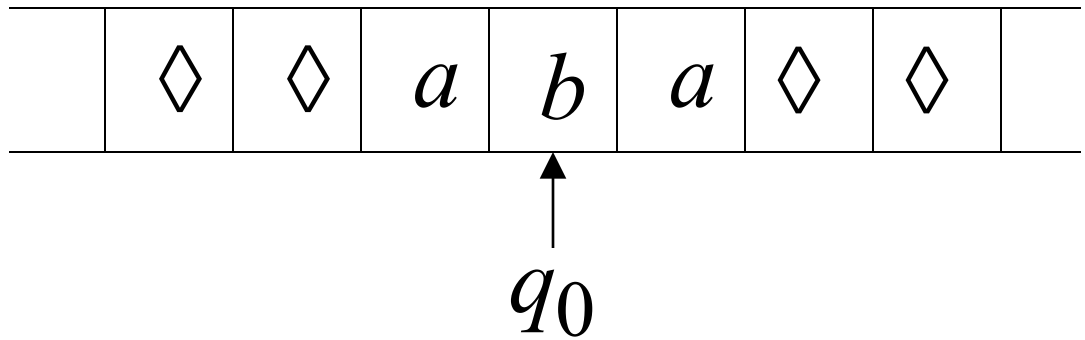
Rejection Example

Time 0



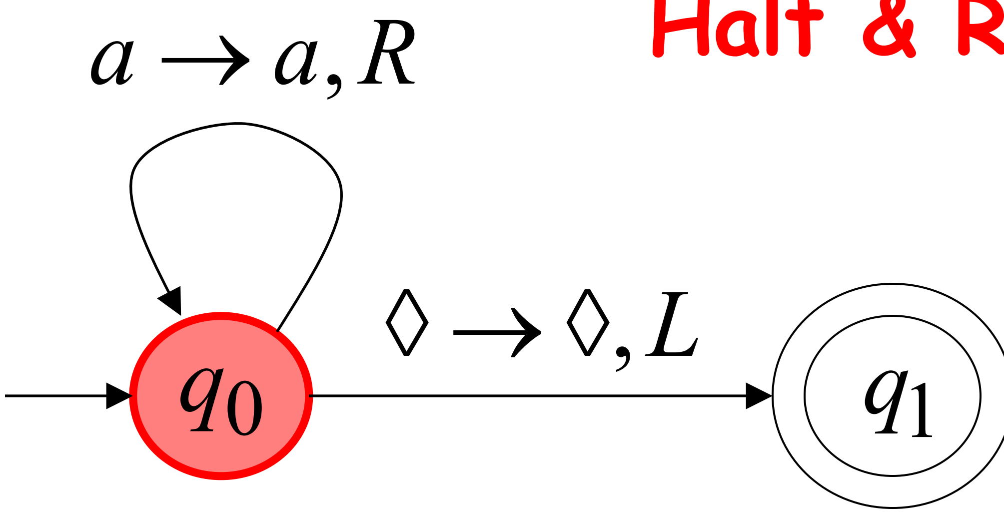


Time 1



No possible Transition

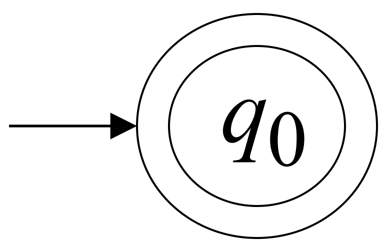
Halt & Reject



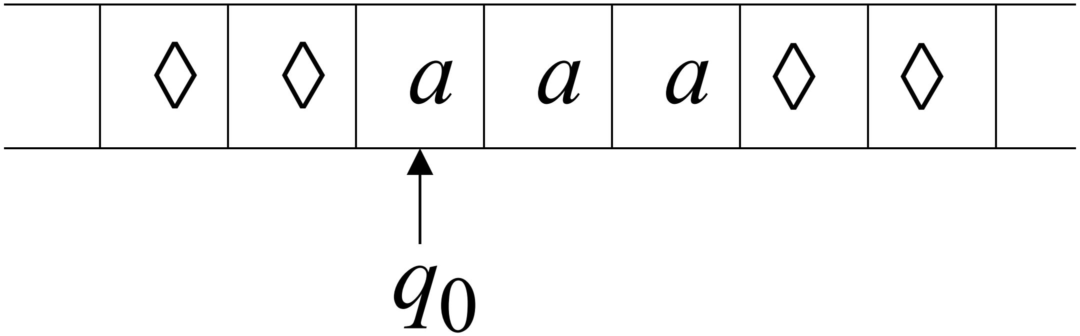
A simpler machine for same language

but for input alphabet $\Sigma = \{a\}$

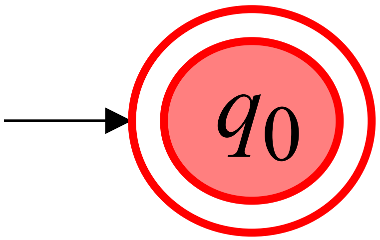
Accepts the language: a^*



Time 0



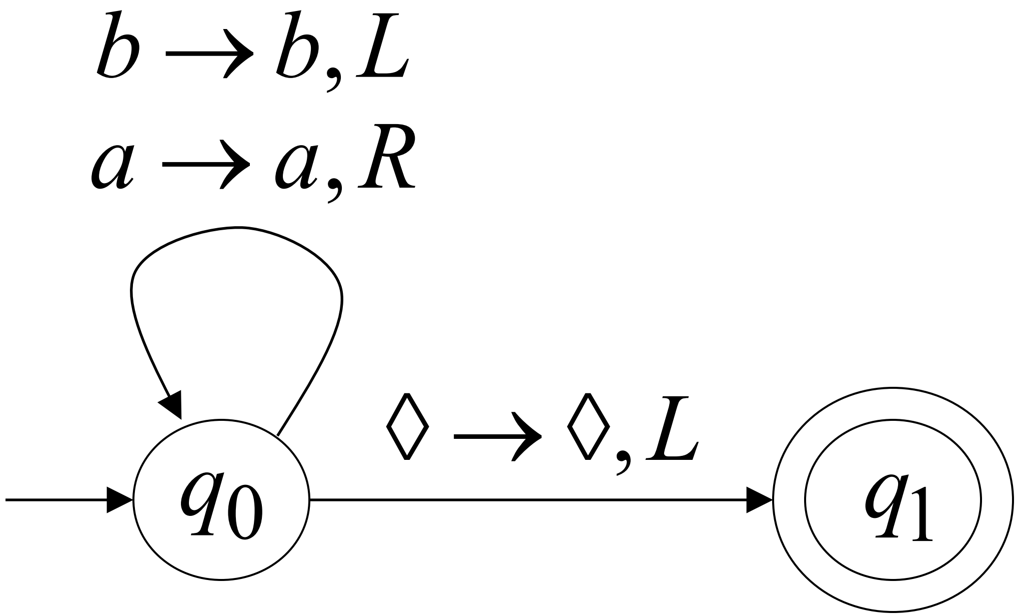
Halt & Accept



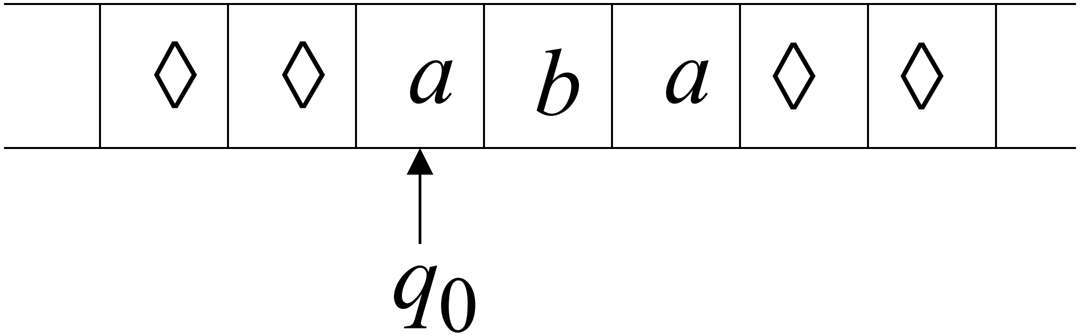
Not necessary to scan input

Infinite Loop Example

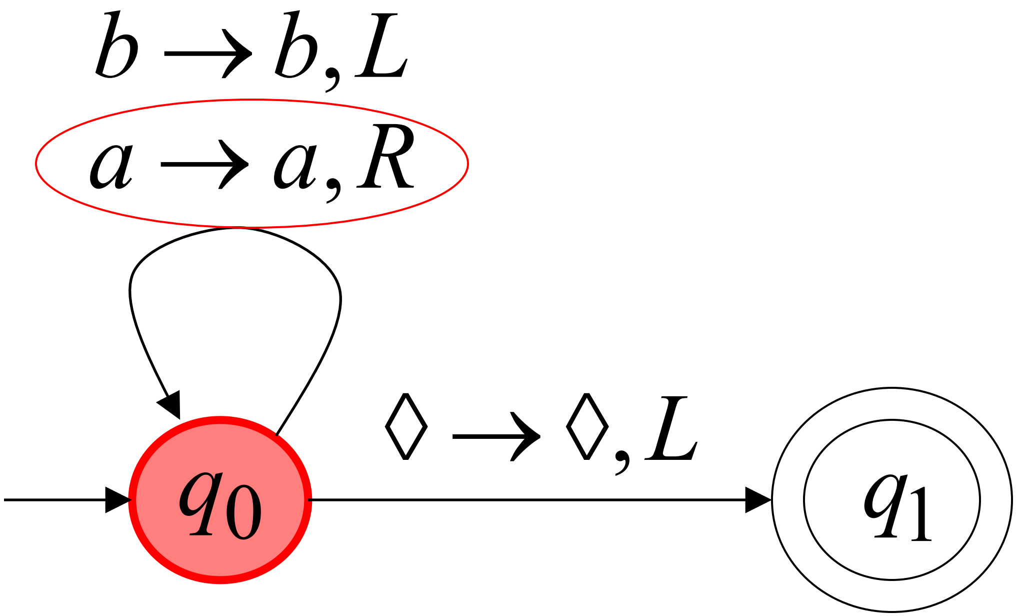
A Turing machine for language $a^* \cup b(a \cup b)^*$



Time 0

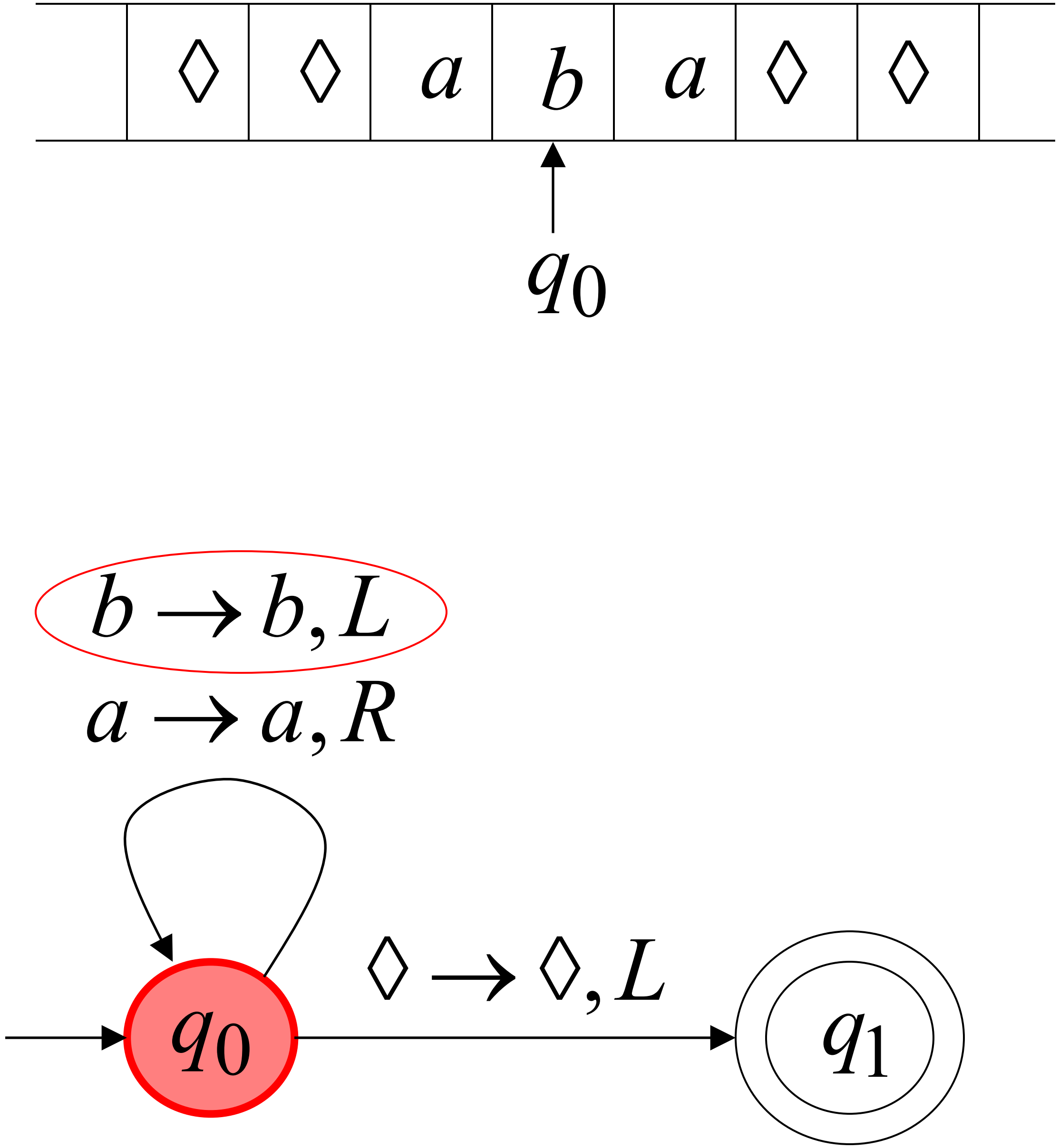


◊
◊
a
b
a
◊
◊
q_0

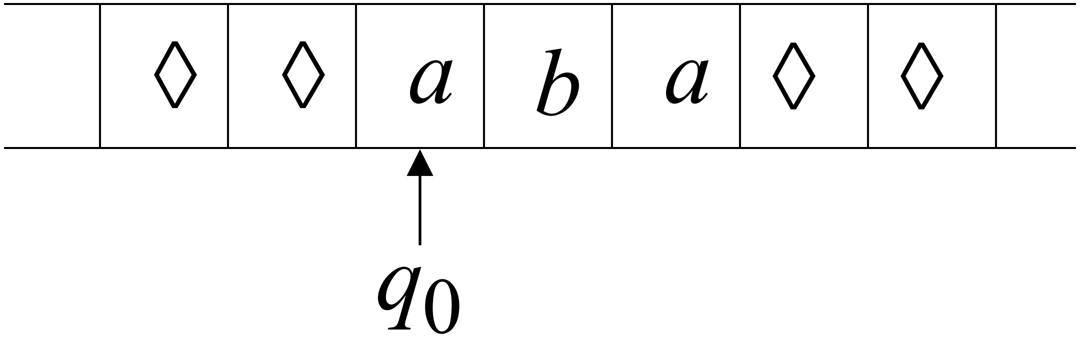


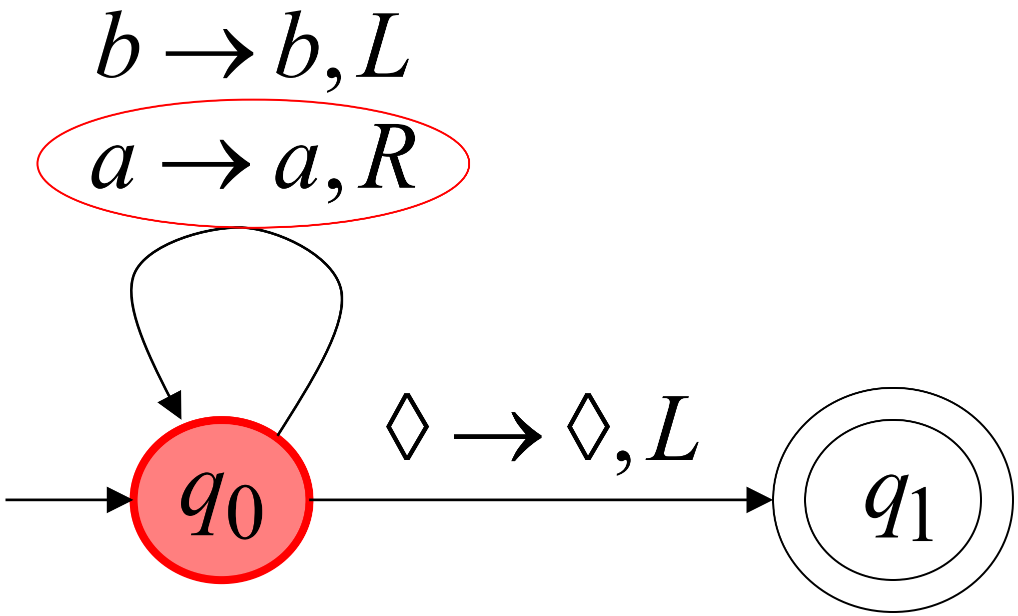
$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 1

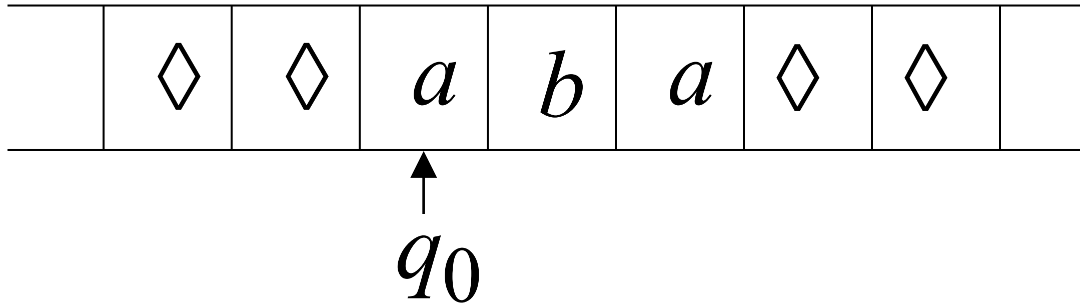

◊
◊
a
b
a
◊
◊
q_0
$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

Time 2

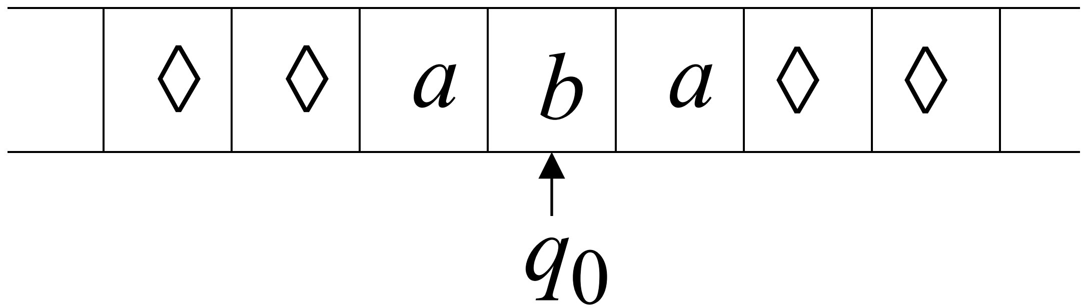

◊
◊
a
b
a
◊
◊
q_0


$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

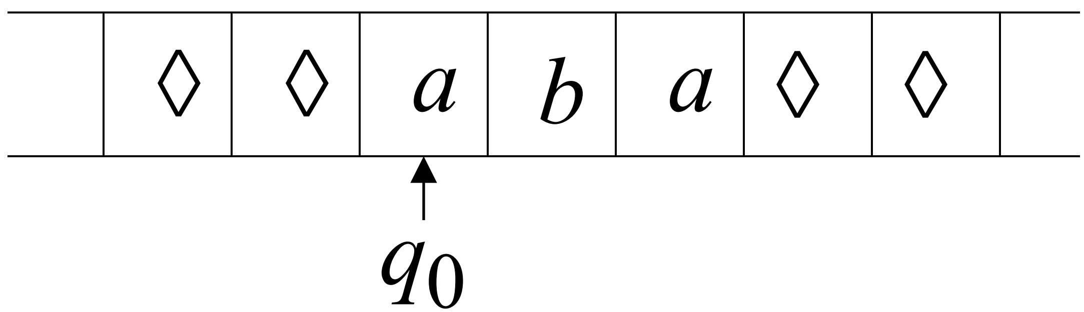
Time 2



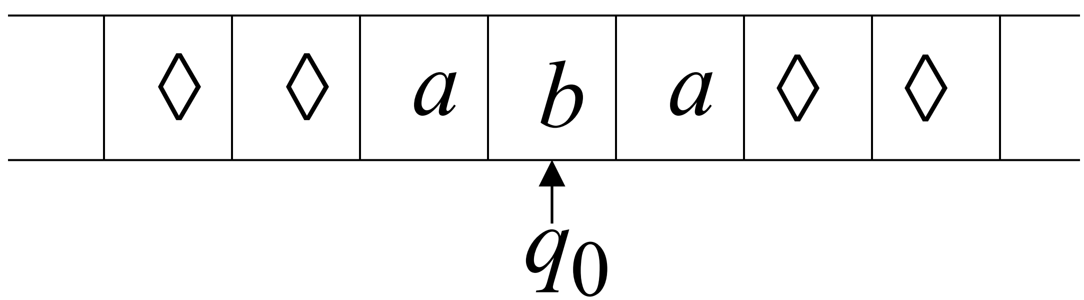
Time 3



Time 4



Time 5



Infinite loop

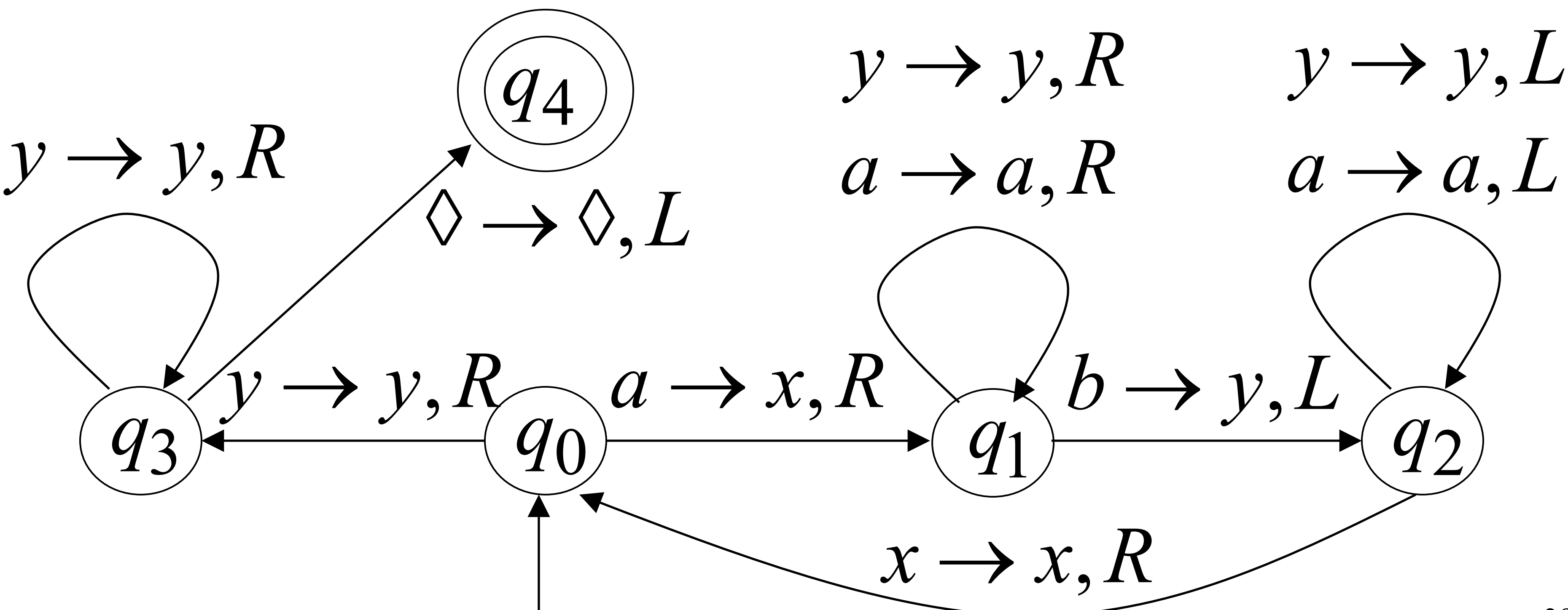
Because of the infinite loop:

- The accepting state cannot be reached
- The machine never halts
- The input string is rejected

Another Turing Machine Example

Turing machine for the language $\{a^n b^n\}$

$n \geq 1$



Basic Idea:

Match a's with b's:

Repeat:

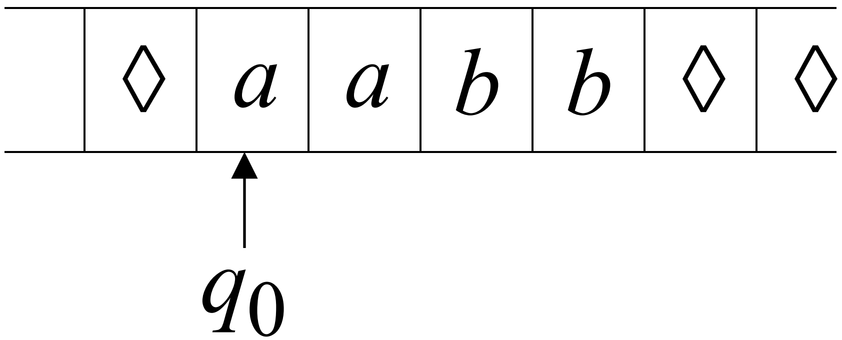
replace leftmost a with x

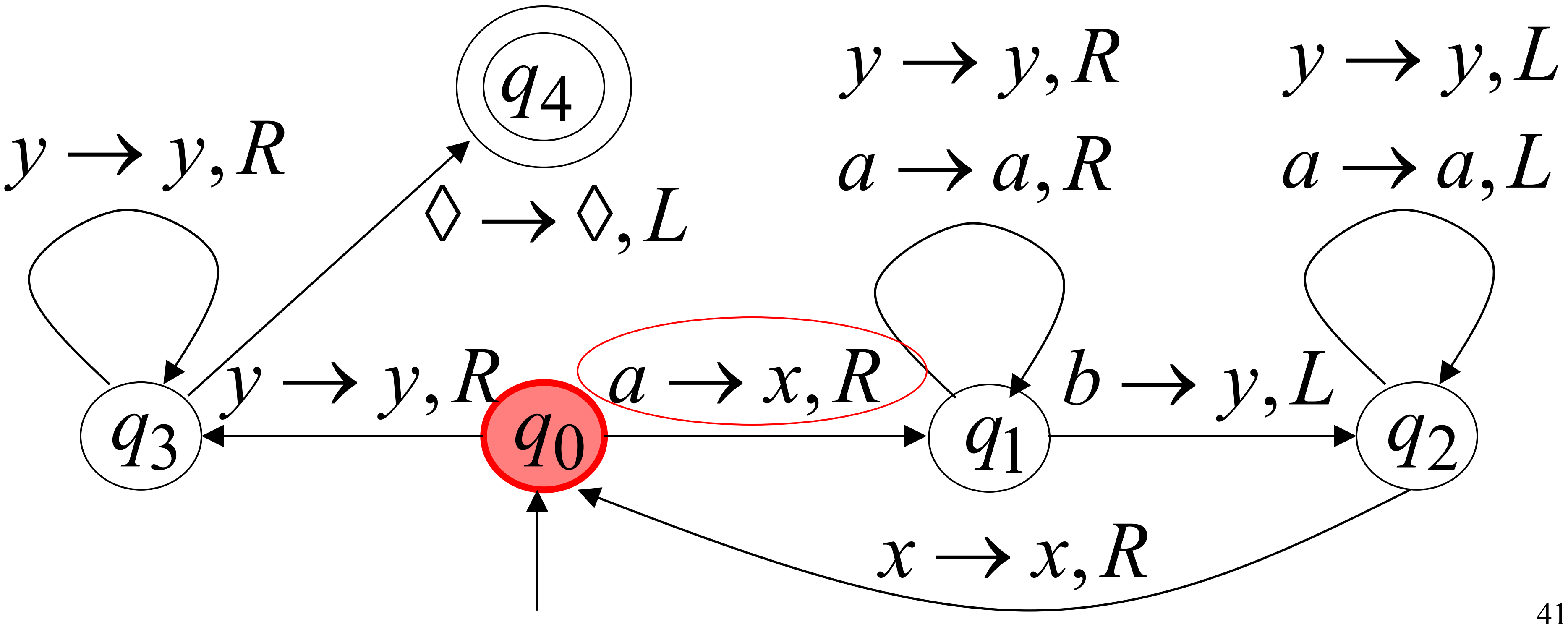
find leftmost b and replace it with y

Until there are no more a's or b's

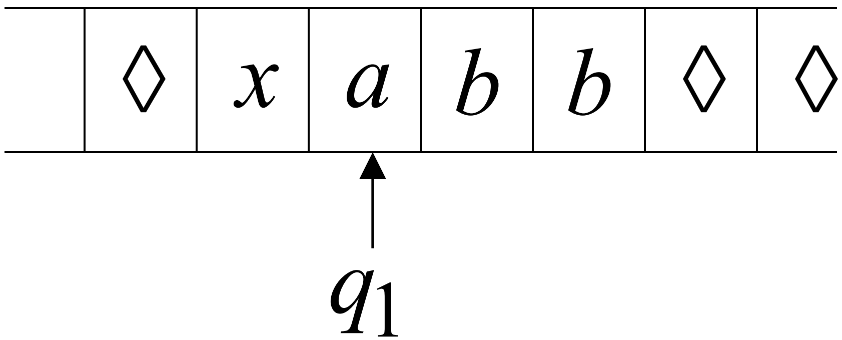
If there is a remaining a or b reject

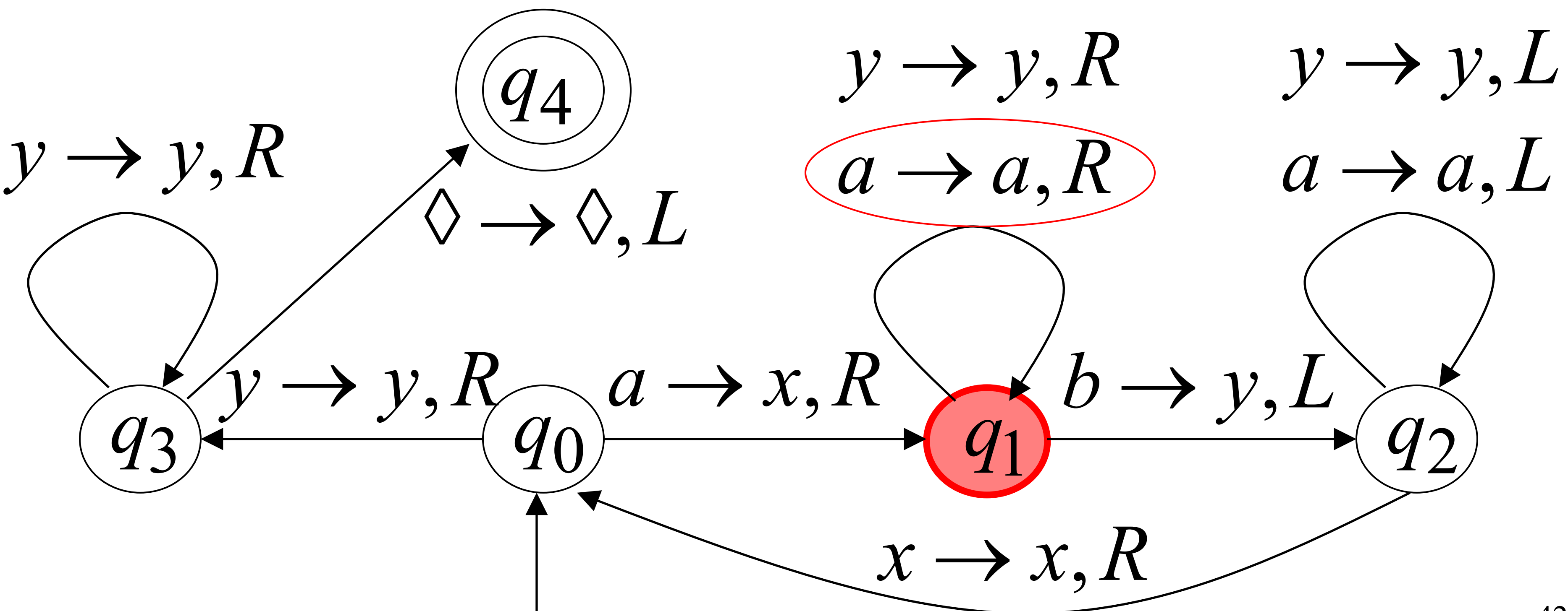
Time 0


◊
a
a
b
b
◊
◊
q_0

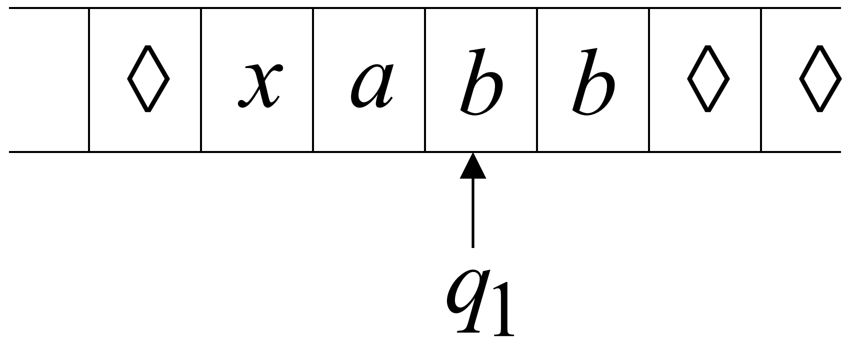

q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 1

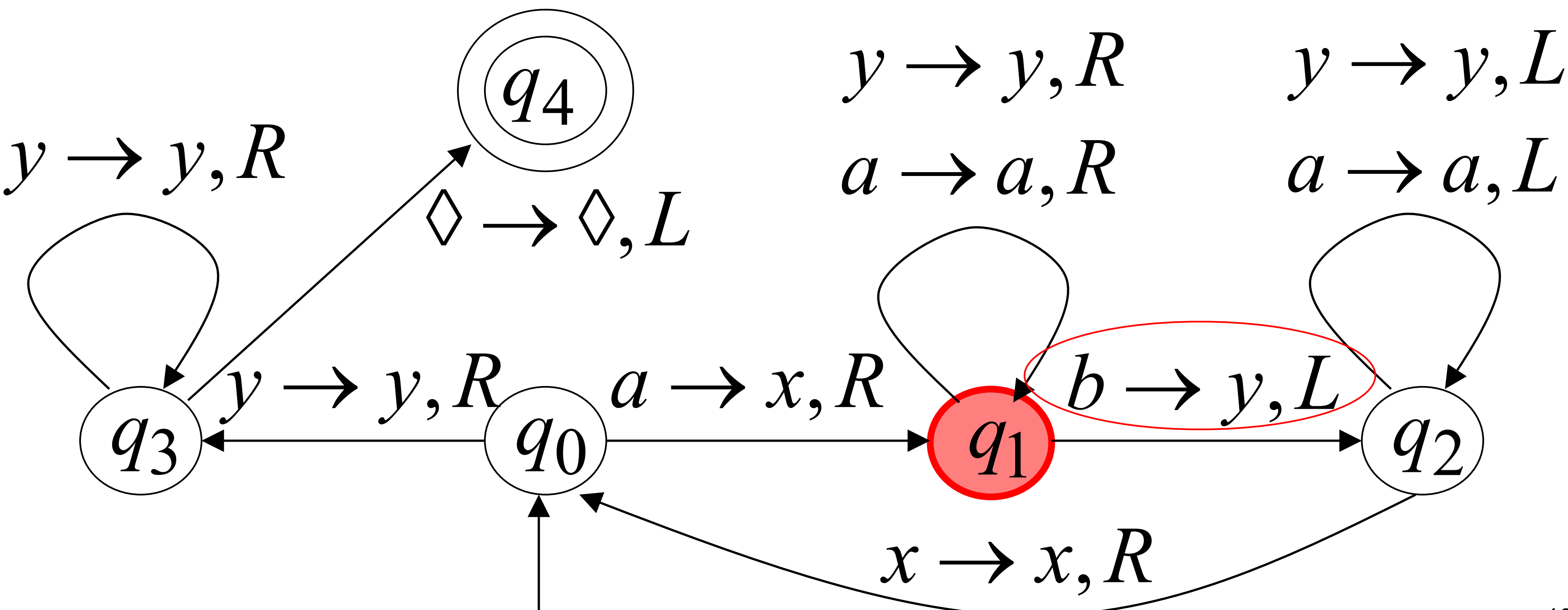

◊
x
a
b
b
◊
◊
q_1


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 2

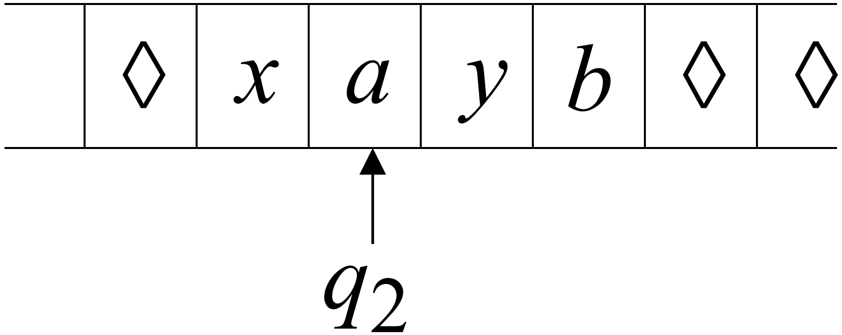


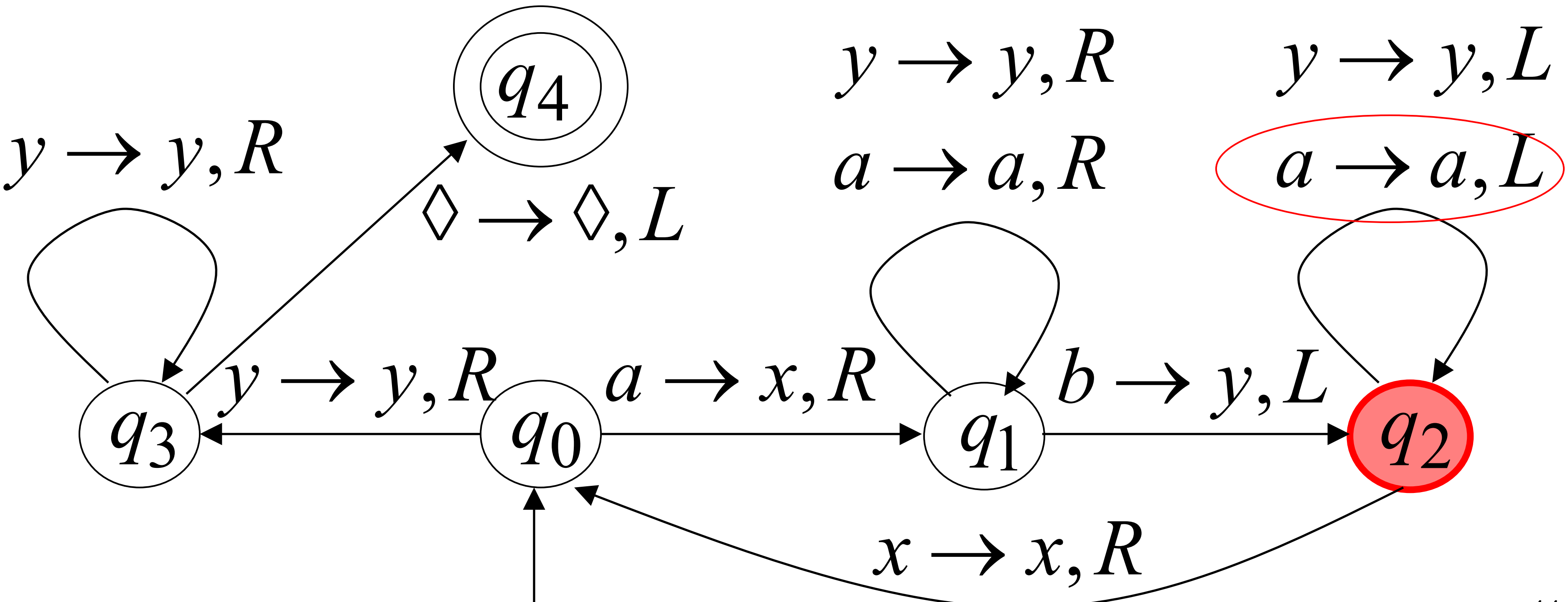
◊
x
a
b
b
◊
◊
q_1



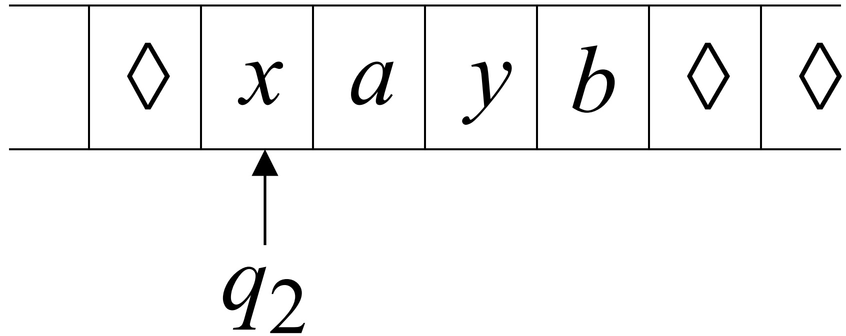
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

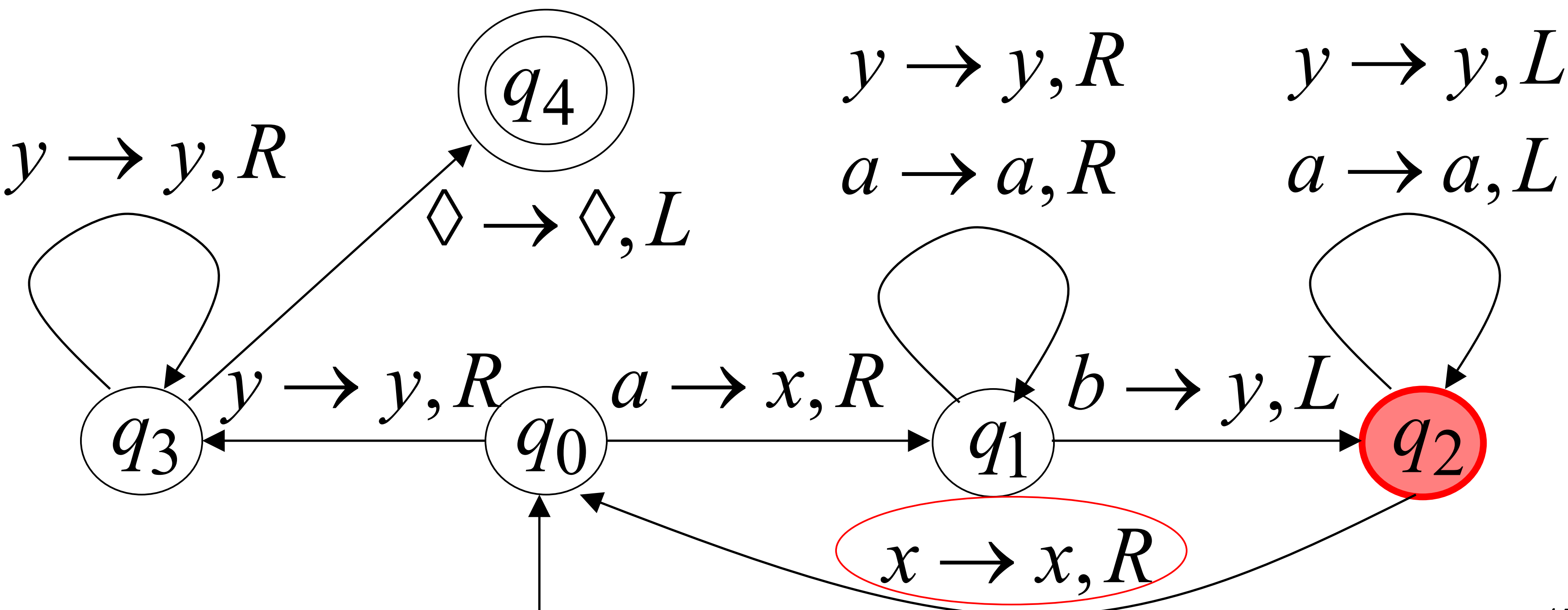
Time 3


$\Diamond$
x
a
y
b
$\Diamond$
$\Diamond$
q_2

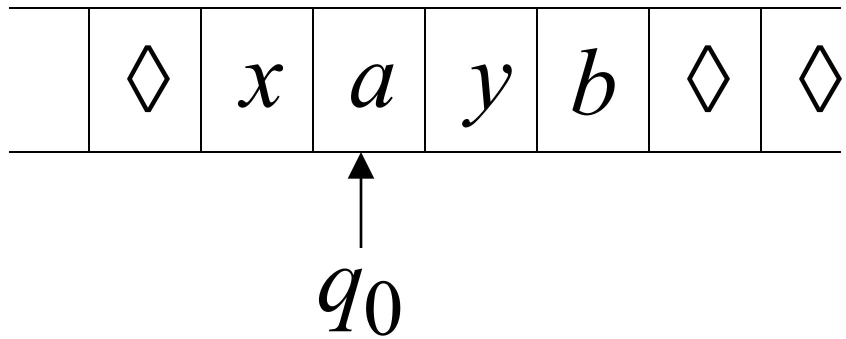

q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

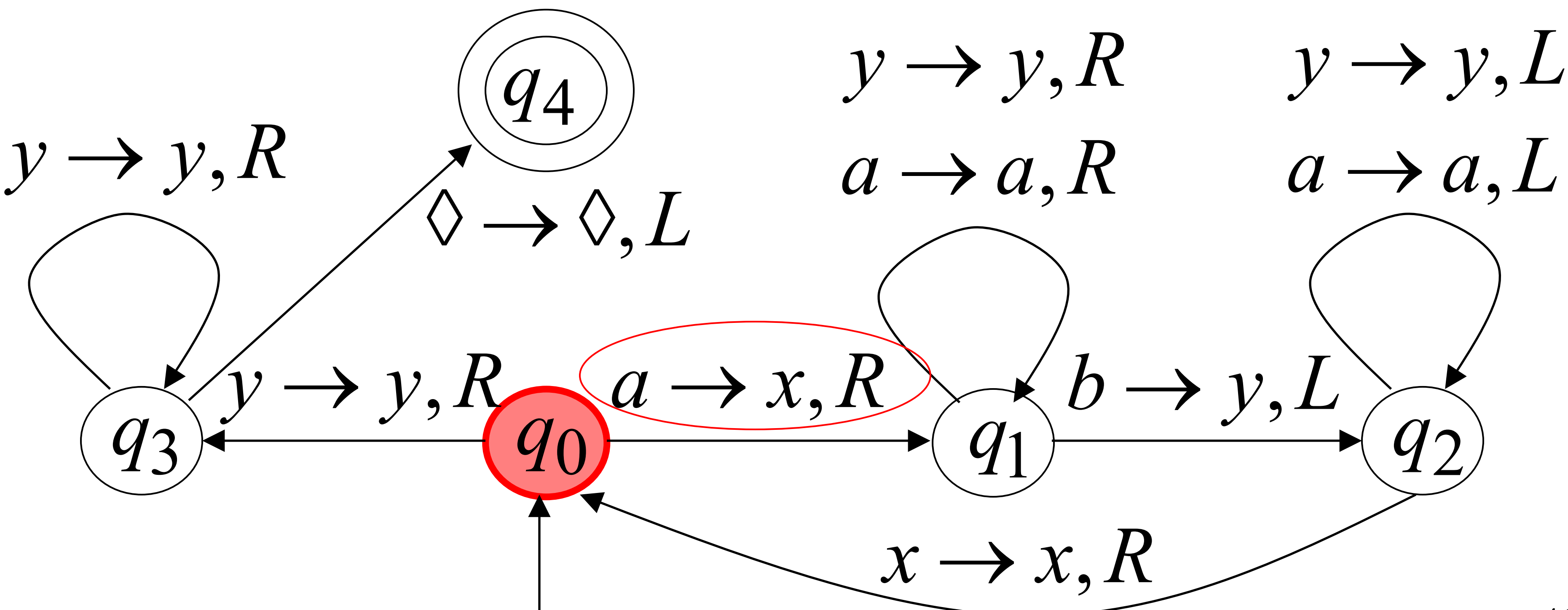
Time 4


◊
x
a
y
b
◊
◊
q_2

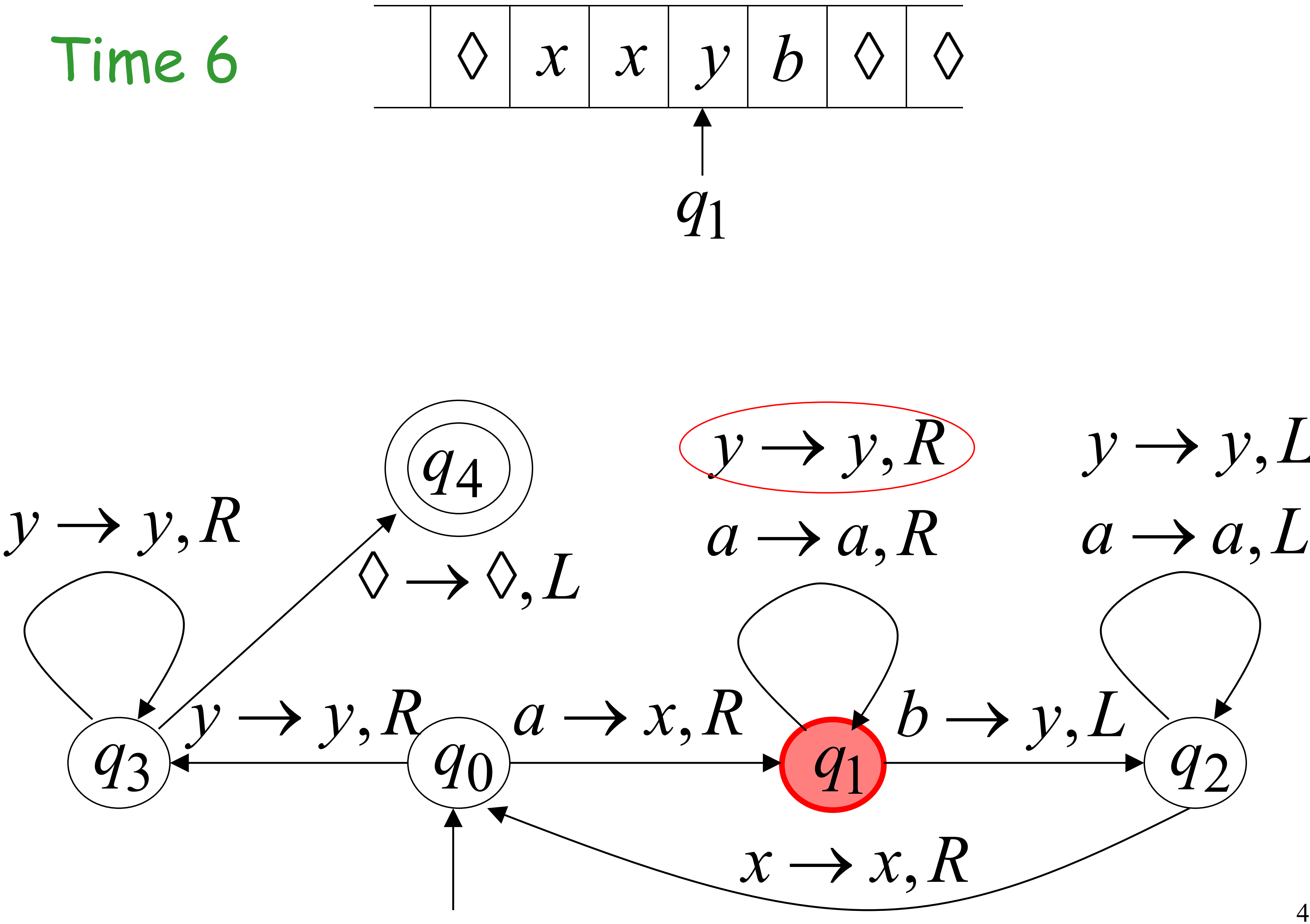

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 5


◊ x a y b ◊ ◊
q_0

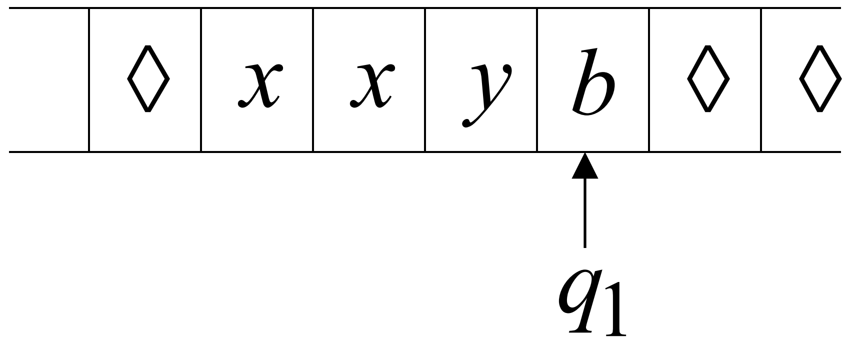

q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 6

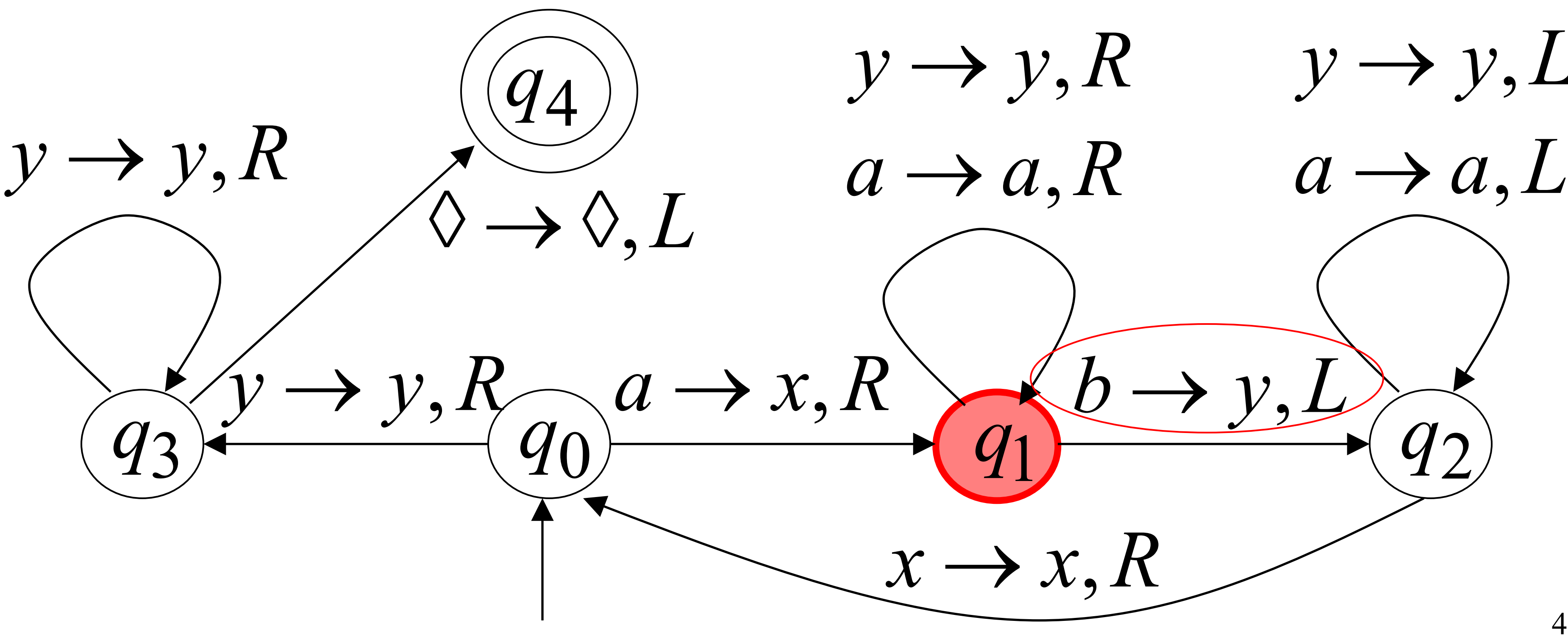


◊ x x y b ◊ ◊
q_1
q_4
$y \rightarrow y, R$
$y \rightarrow y, L$
$a \rightarrow a, R$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 7

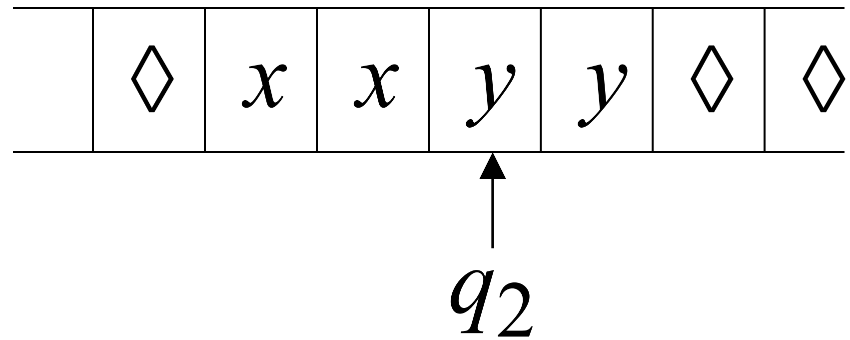


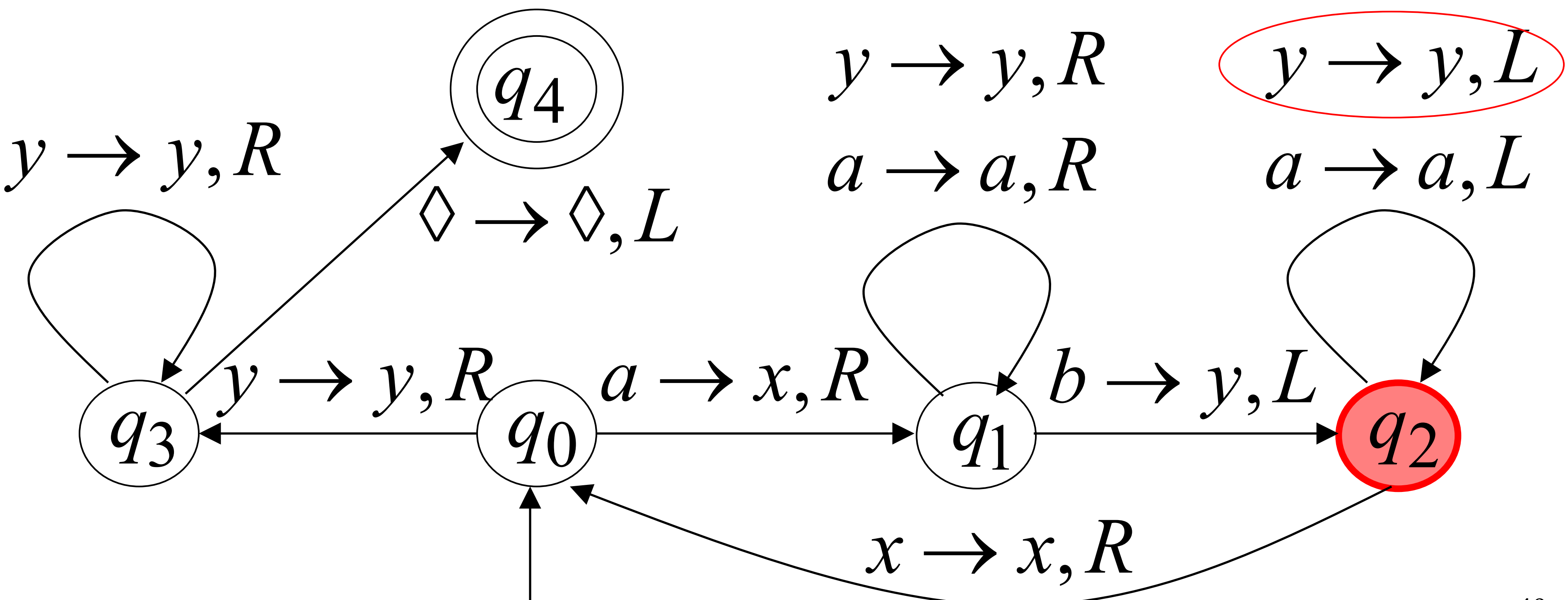
◊
x
x
y
b
◊
◊
q_1



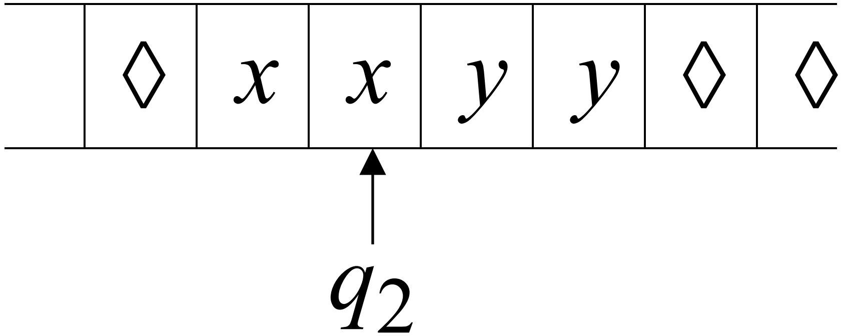
q_4
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

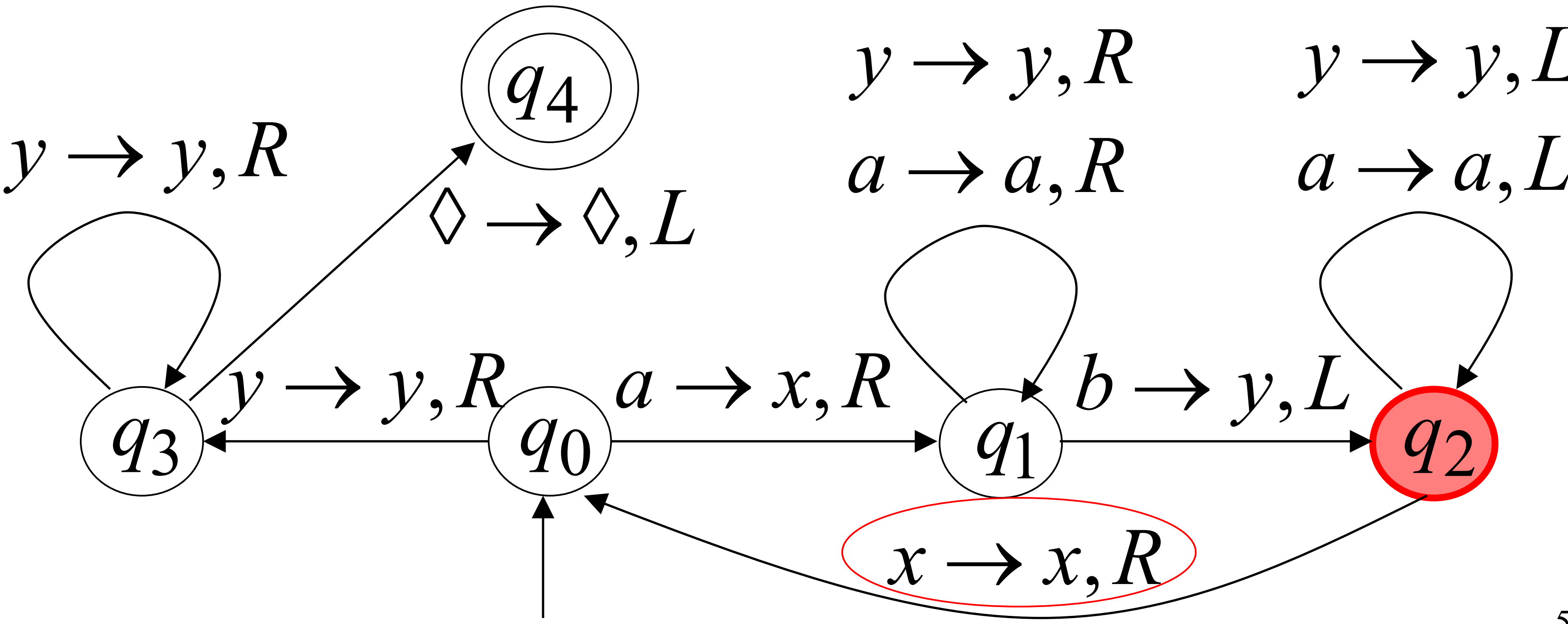
Time 8



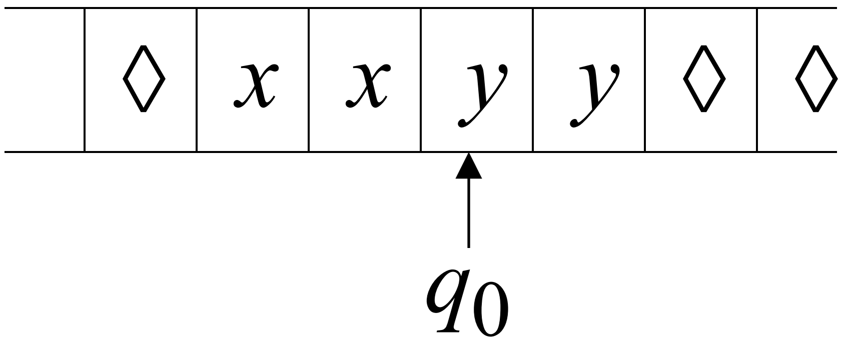


Time 9

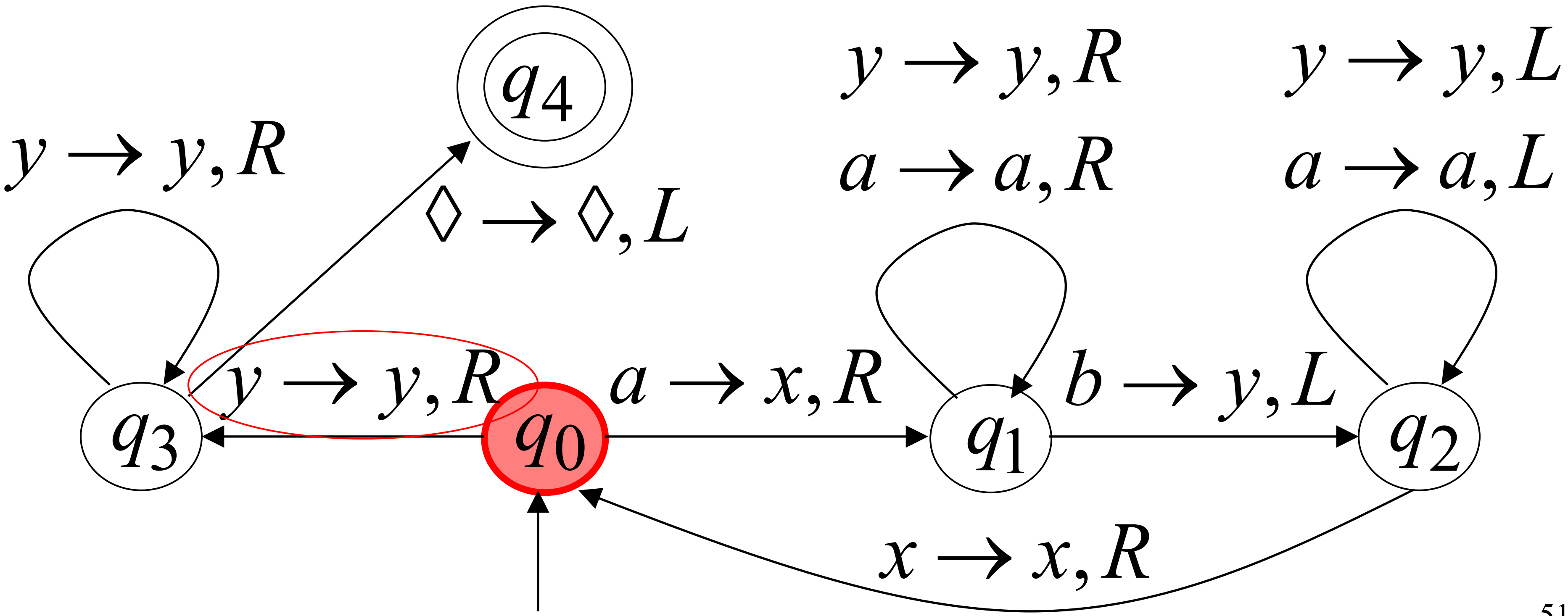

◊
x
x
y
y
◊
◊
q_2


$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_4
q_0
q_1
q_2
$x \to x, R$

Time 10

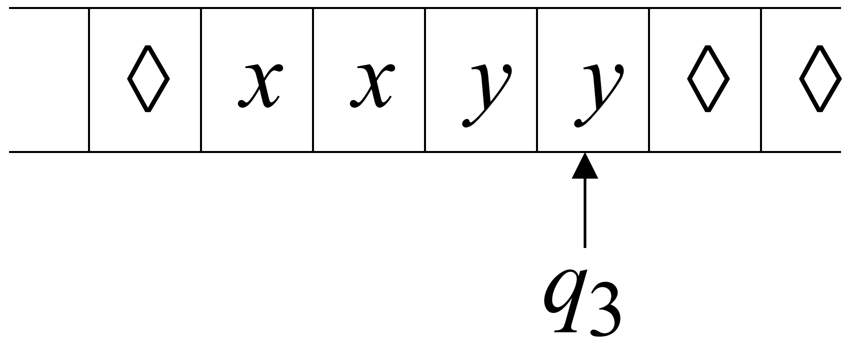


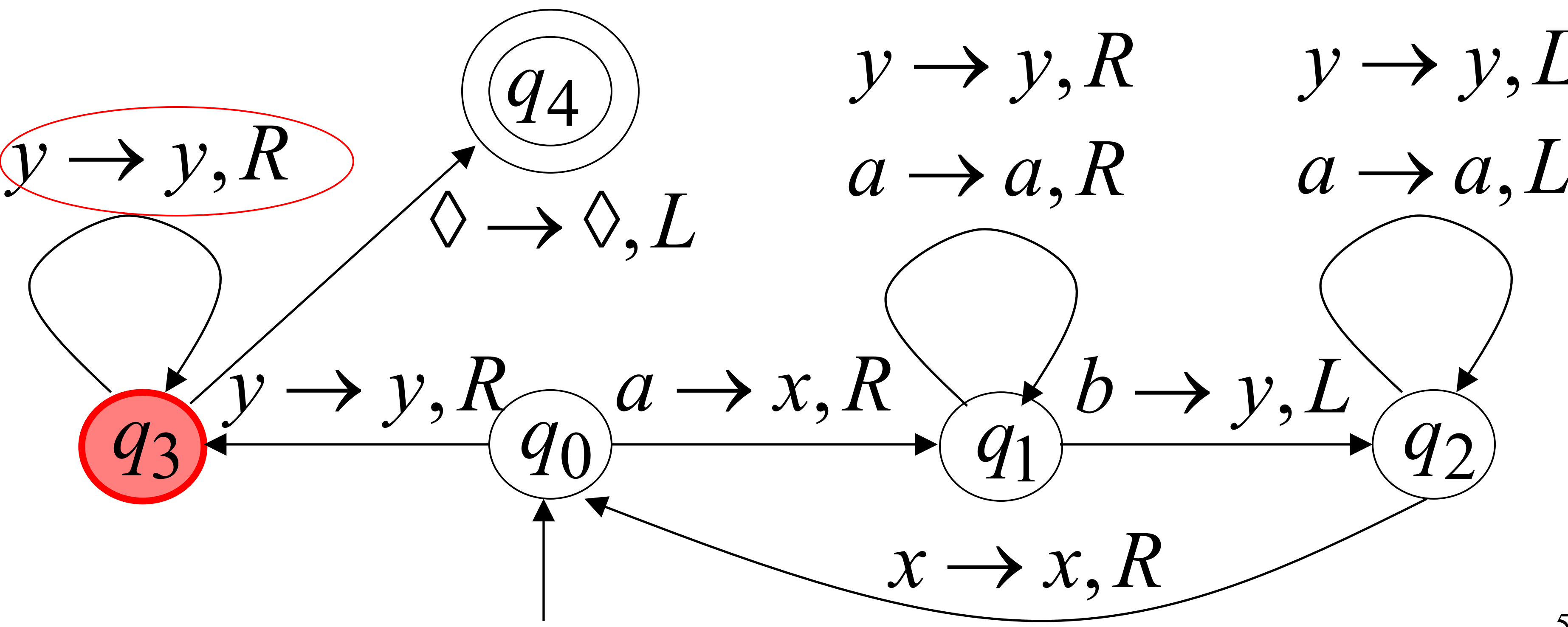
◊
x
x
y
y
◊
◊
q_0



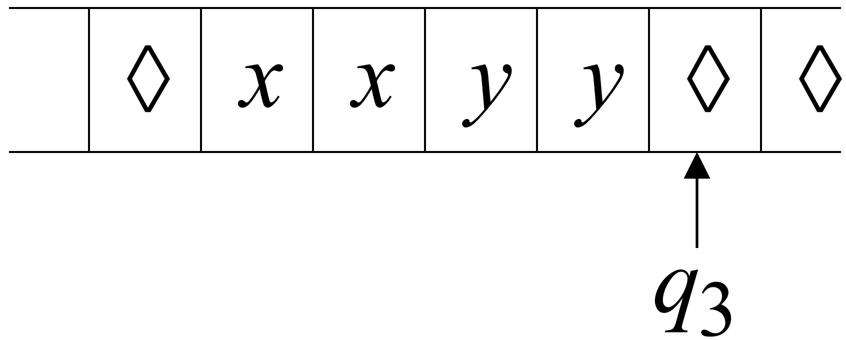
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

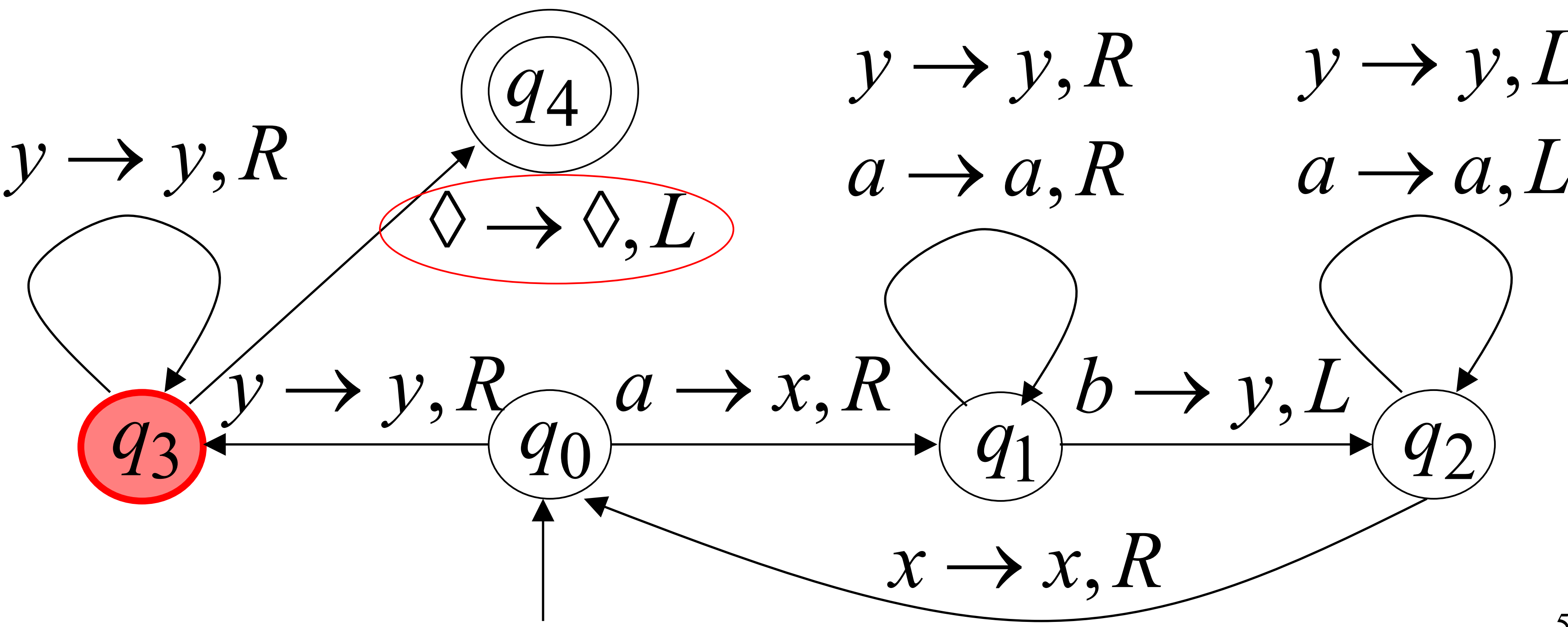
Time 11


◊
x
x
y
y
◊
◊
q_3

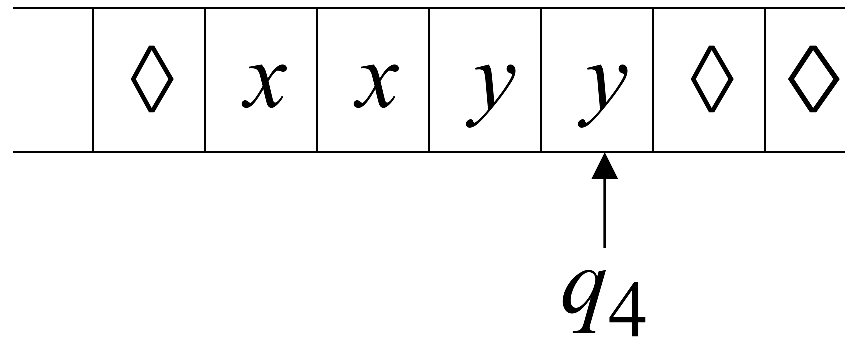

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 12

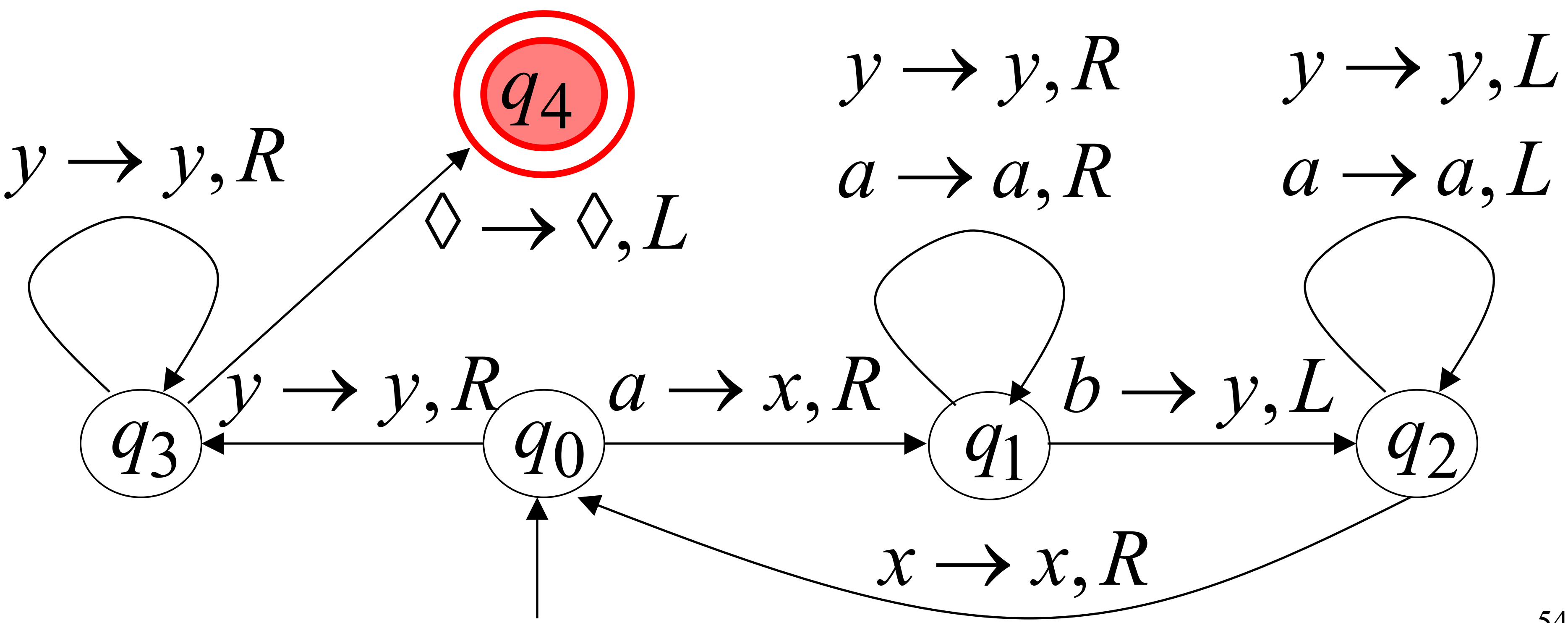

◊
x
x
y
y
◊
◊
q_3


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 13



Halt & Accept



Observation:

If we modify the
machine for the language $\{a^n b^n\}$

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

Formal Definitions for Turing Machines

Transition Function

$$q_1 \xrightarrow{a \rightarrow b, R} q_2$$

$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function

$$q_1 \xrightarrow{c \rightarrow d,L} q_2$$

$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

Input alphabet

Tape alphabet

States

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

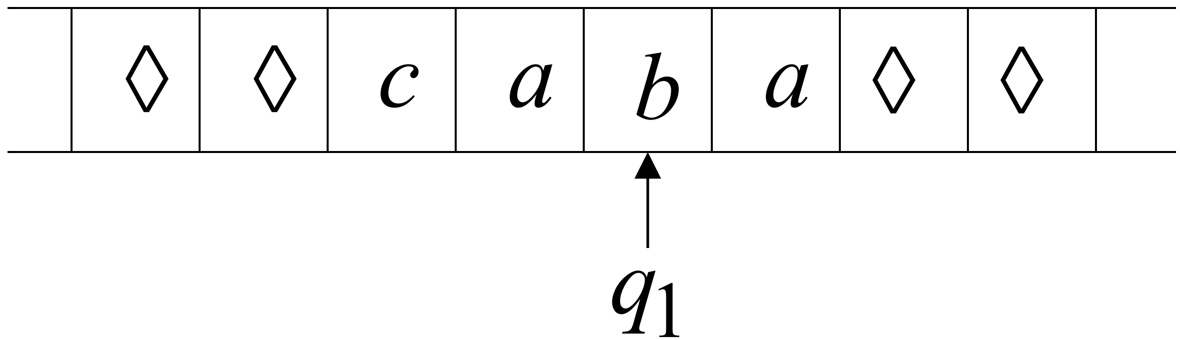
Transition function

Accept states

Initial state

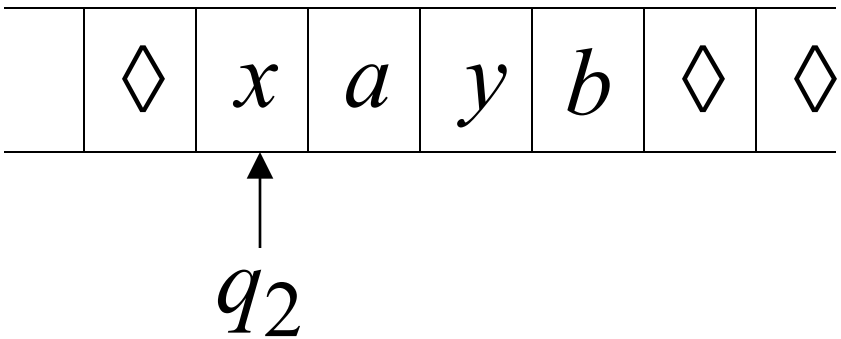
blank

Configuration

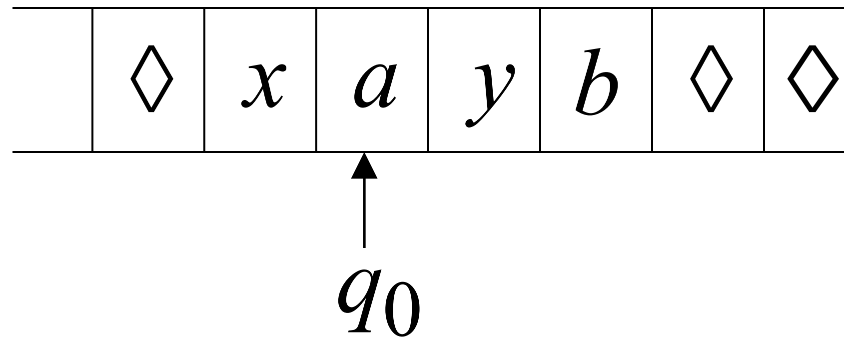


Instantaneous description: $ca\ q_1\ ba$

Time 4



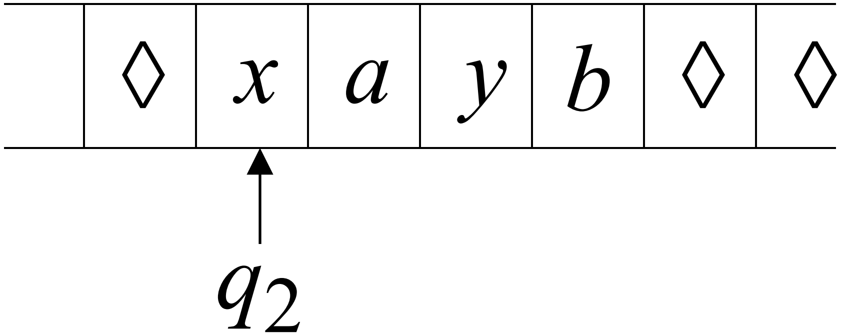
Time 5



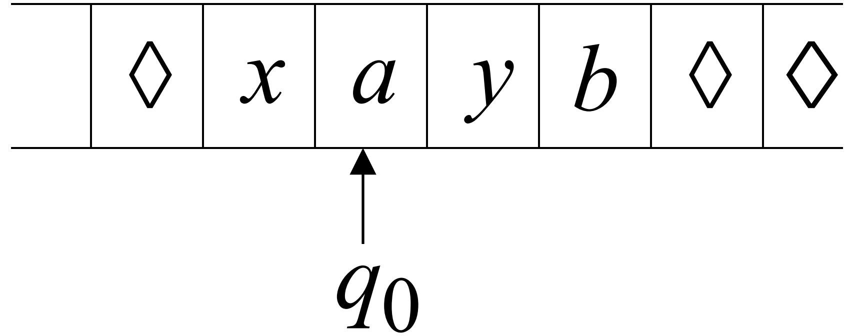
A Move: $q_2\ xayb \succ x\ q_0\ ayb$

(yields in one mode)

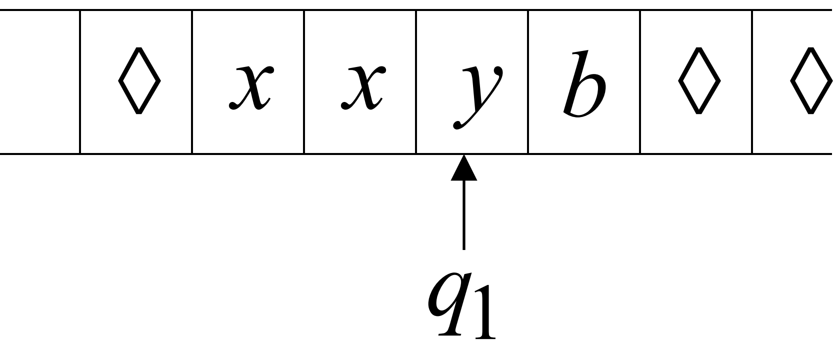
Time 4



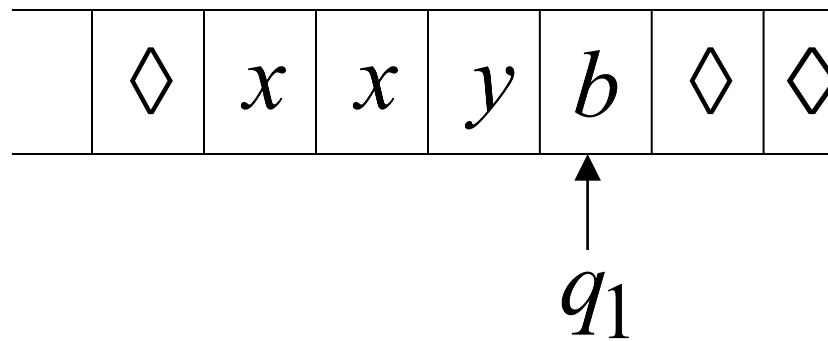
Time 5



Time 6



Time 7



A computation

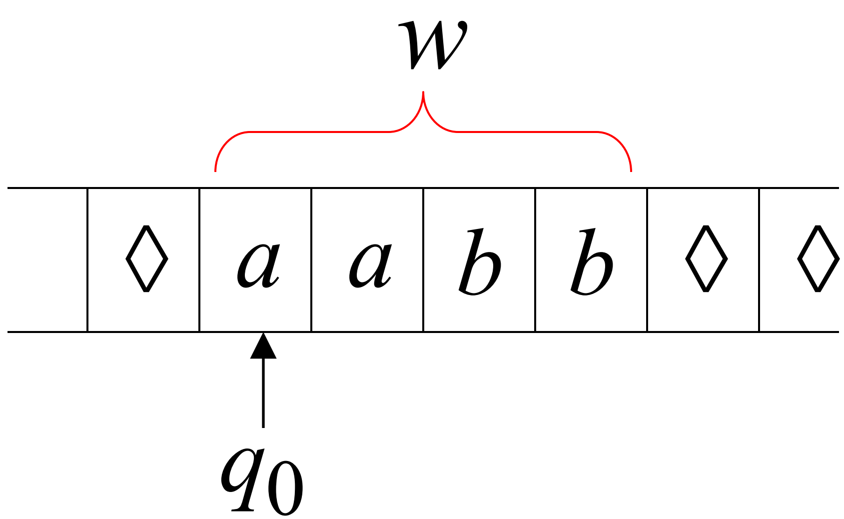
$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

$$q_2\, xayb \succ x\, q_0\, ayb \succ xx\, q_1\, yb \succ xxy\, q_1\, b$$

Equivalent notation: $$q_2\, xayb \overset{*}{\succ} xxy\, q_1\, b$$

Initial configuration: $q_0 \, w$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad q_0\, w \overset{*}{\succ} x_1\, q_f\, x_2\}$$

Initial state

Accept state

If a language L is accepted by a Turing machine M then we say that L is:

- Turing Recognizable

Other names used:

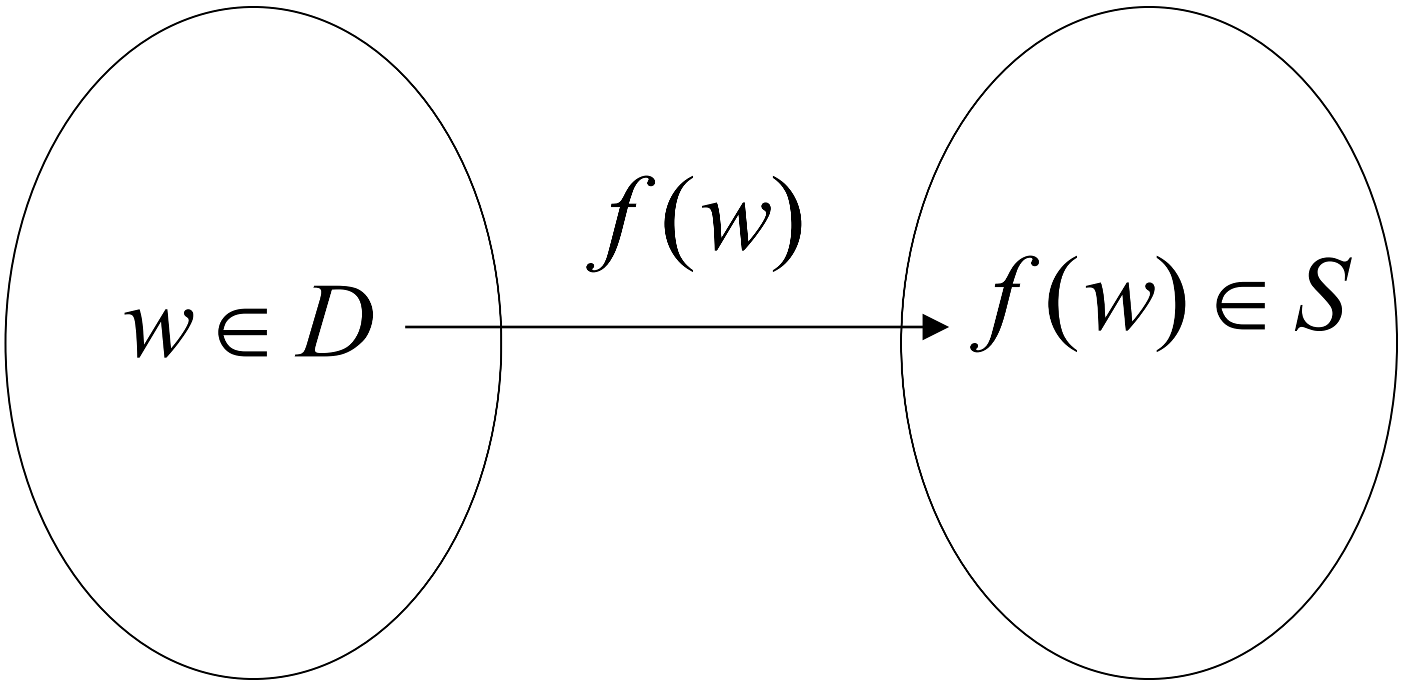
- Turing Acceptable
- Recursively Enumerable

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D

Result Region: S



A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

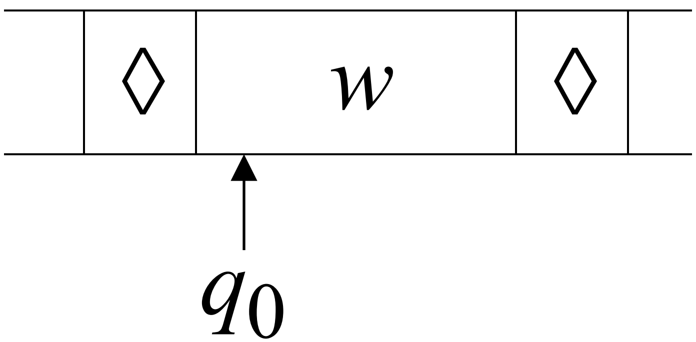
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

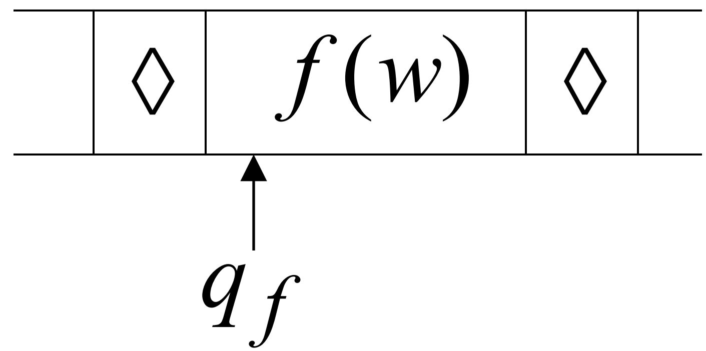
A function f is computable if
there is a Turing Machine M such that:

Initial configuration



initial state

Final configuration



accept state

For all $w \in D$ Domain

In other words:

A function f is computable if there is a Turing Machine M such that:

$$q_0 \, w \overset{*}{\succ} q_f \, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

Example

The function $f(x, y) = x + y$ is computable

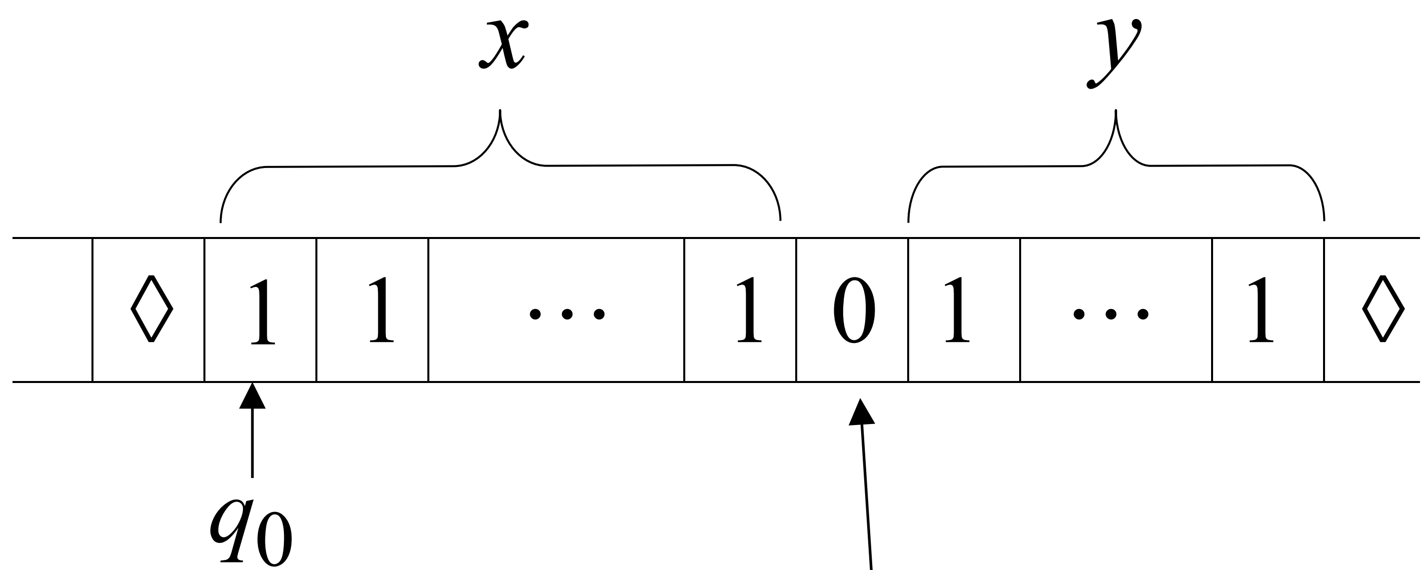
x, y are integers

Turing Machine:

Input string: $x0y$ unary

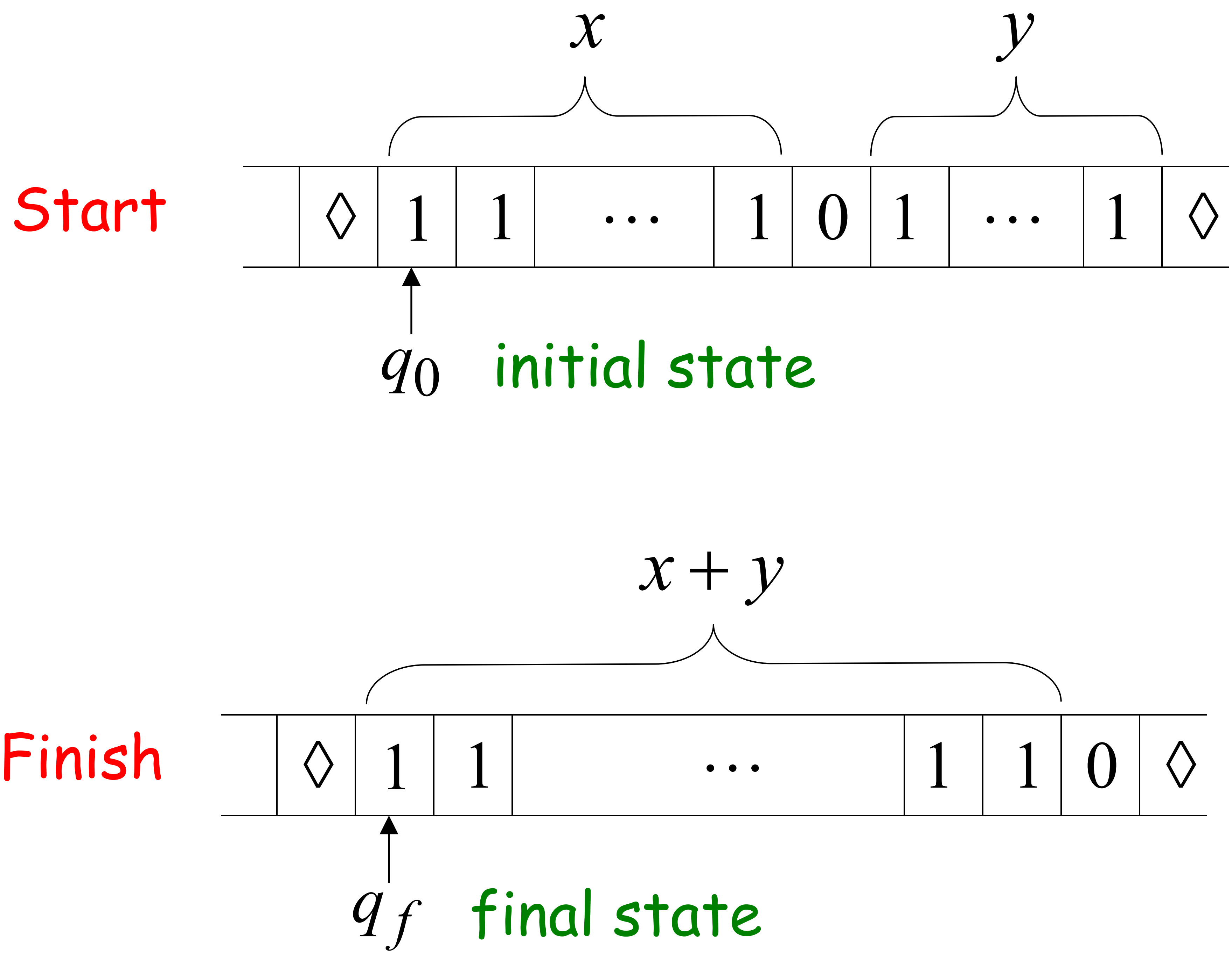
Output string: $xy0$ unary

Start

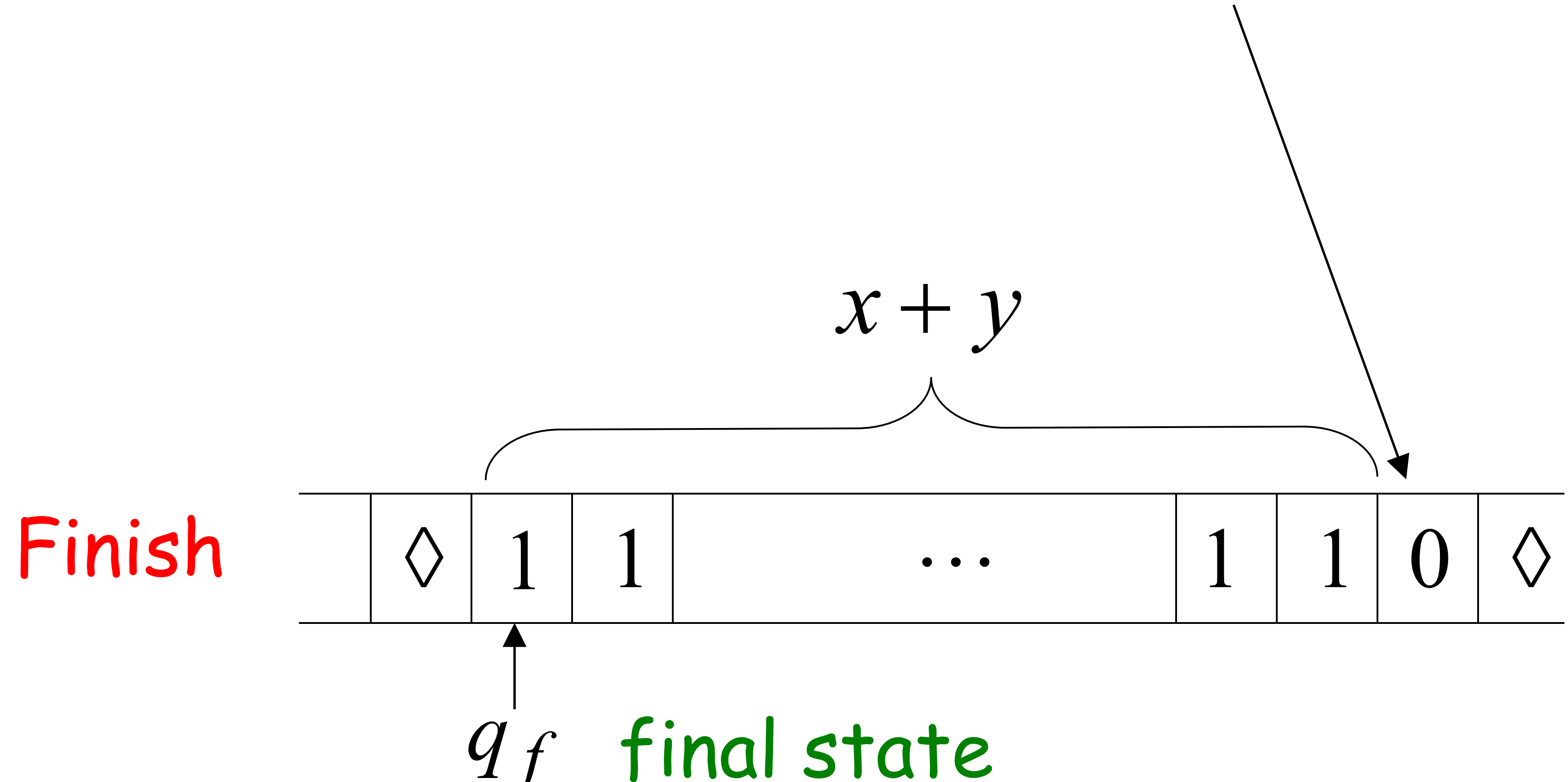


initial state

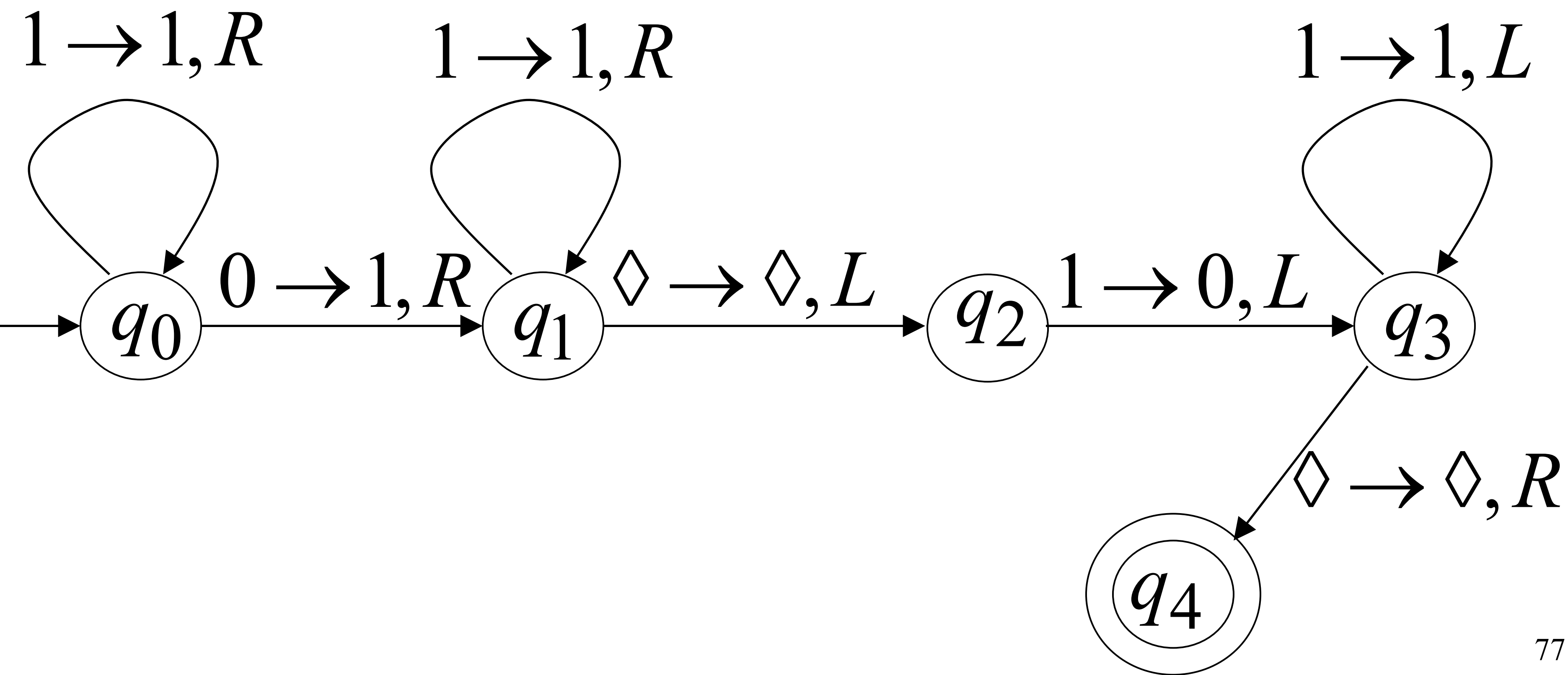
The 0 is the delimiter that separates the two numbers


Start
x
y
◊ 1 1 ... 1 0 1 ... 1 ◊
q0 initial state
Finish
x + y
◊ 1 1 ... 1 1 0 ◊
qf final state

The 0 here helps when we use the result for other operations



Turing machine for function $f(x, y) = x + y$



Execution Example:

$x = 11$ (=2)

$y = 11$ (=2)

Time 0

	x			y	
◊	1	1	0	1	1	◊

$\uparrow$ q_0

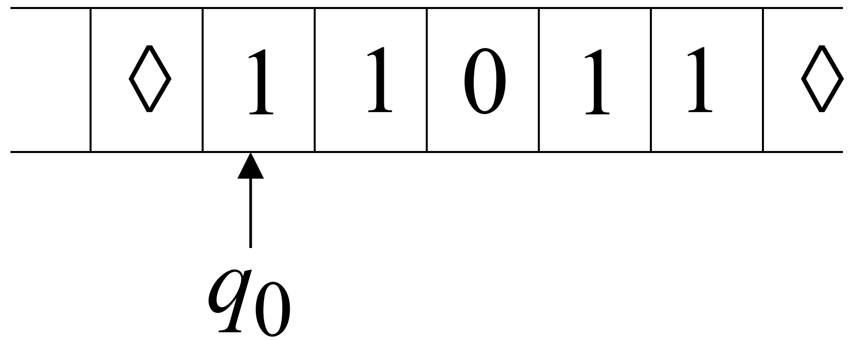
Final Result

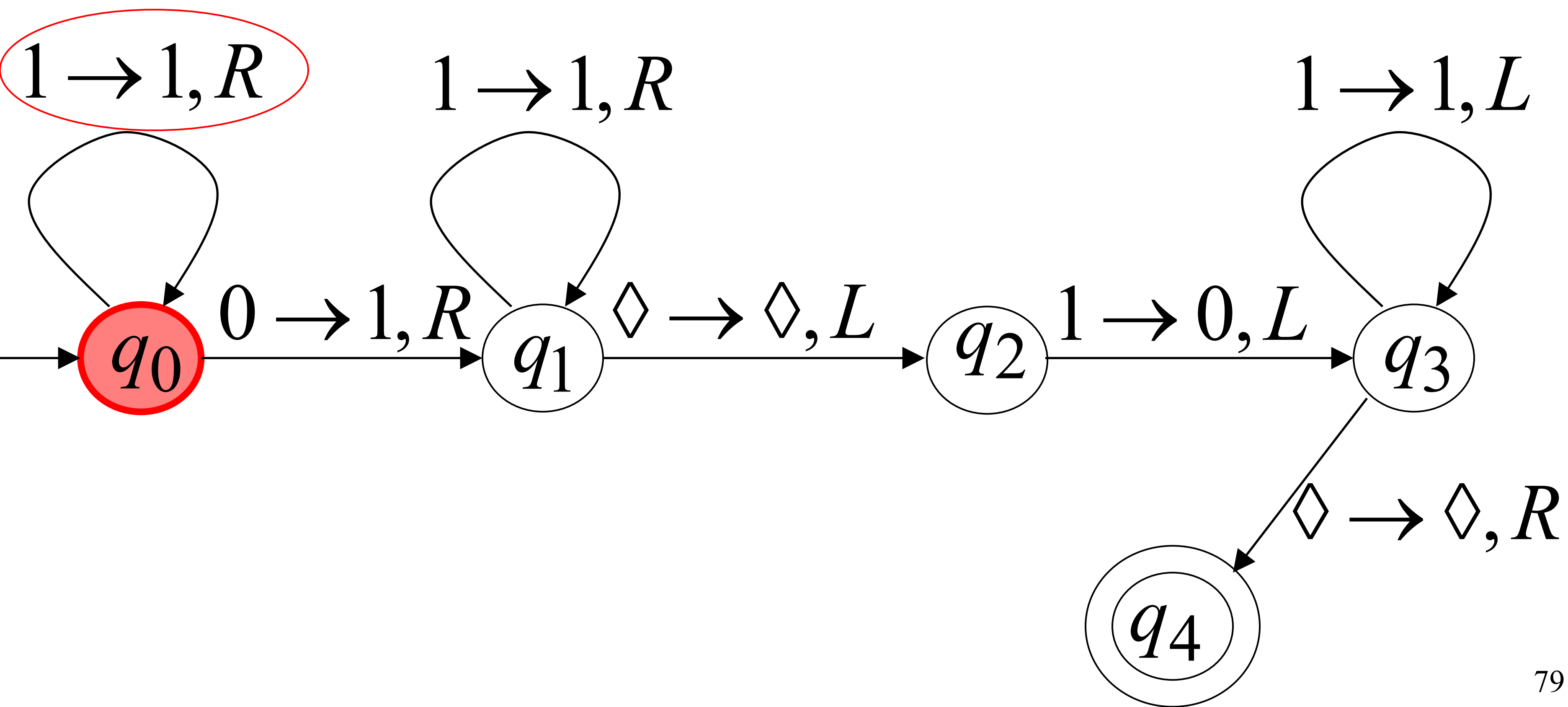
$x + y$

◊	1	1	1	1	0	◊

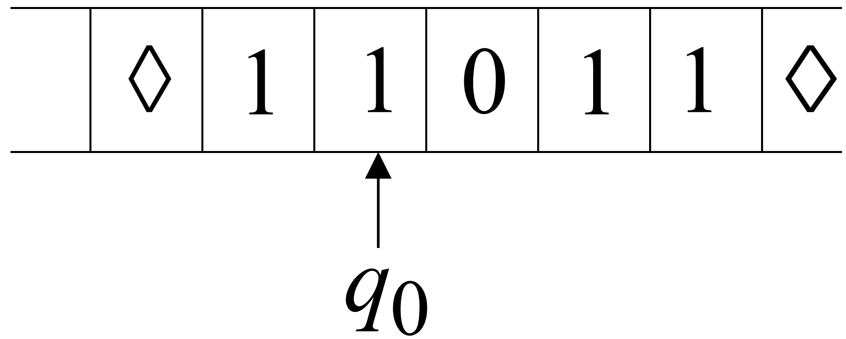
$\uparrow$ q_4

Time 0

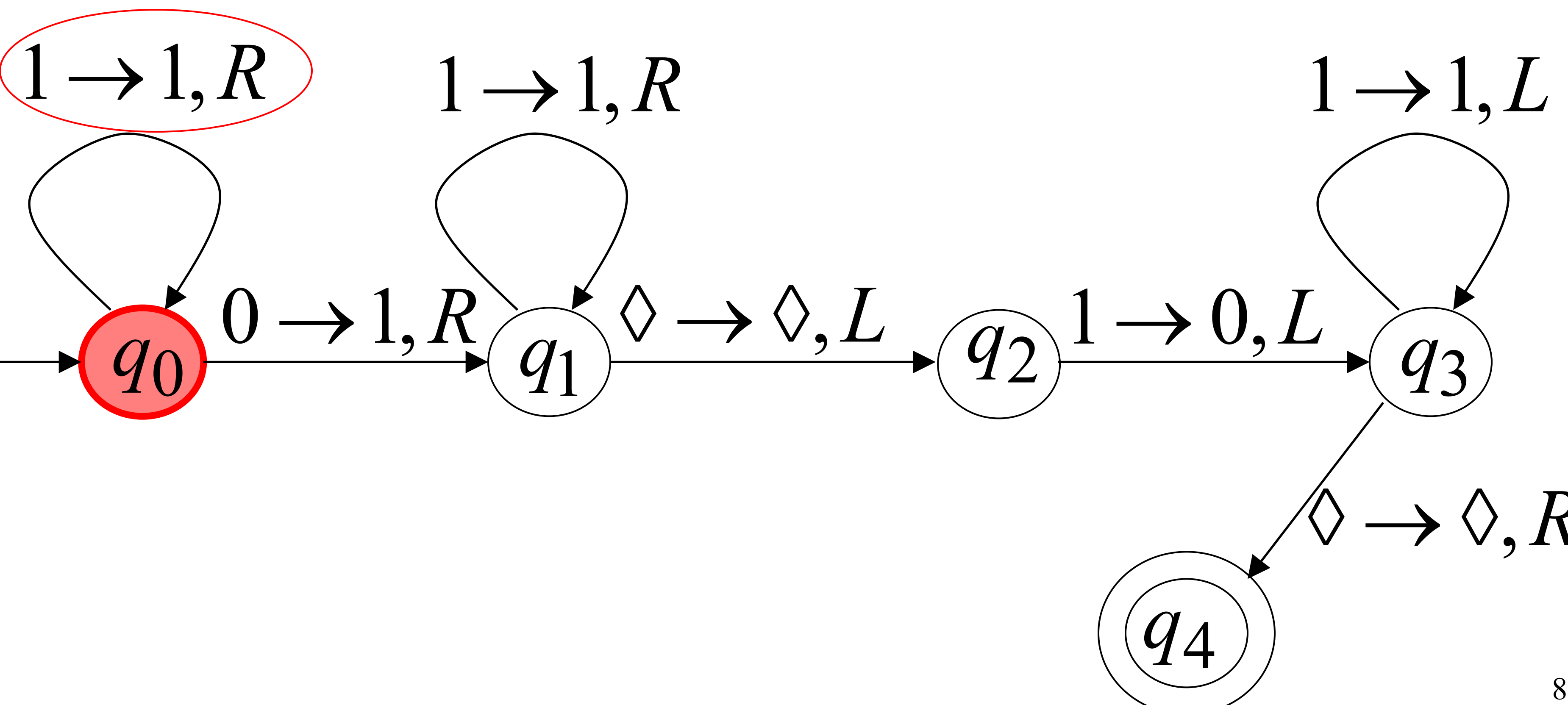

◊
1
1
0
1
1
◊
q_0


$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 1

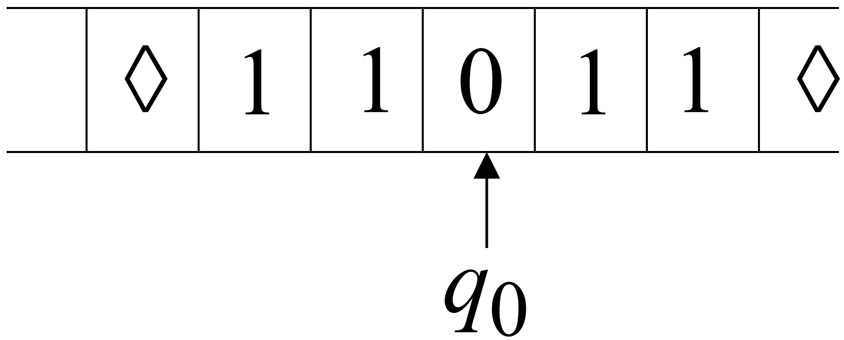


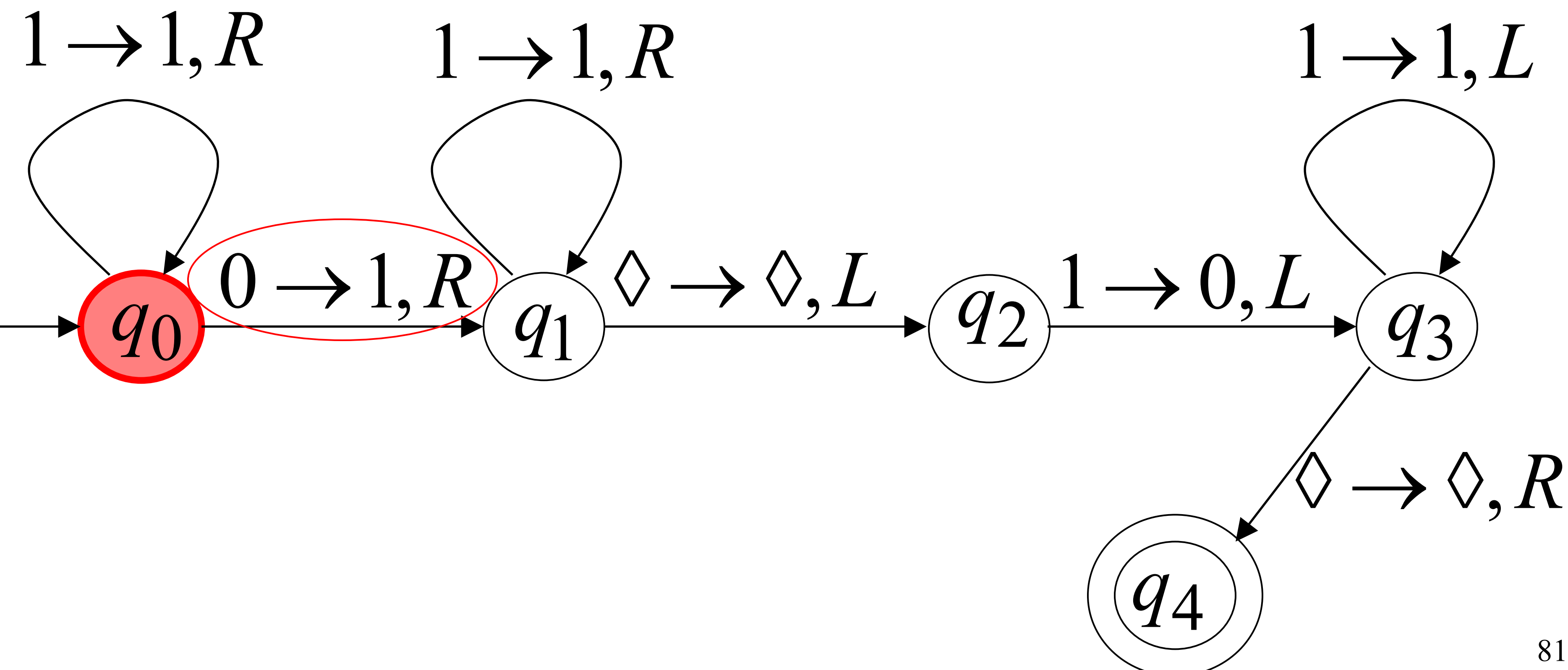
◊ 1 1 0 1 1 ◊
q_0



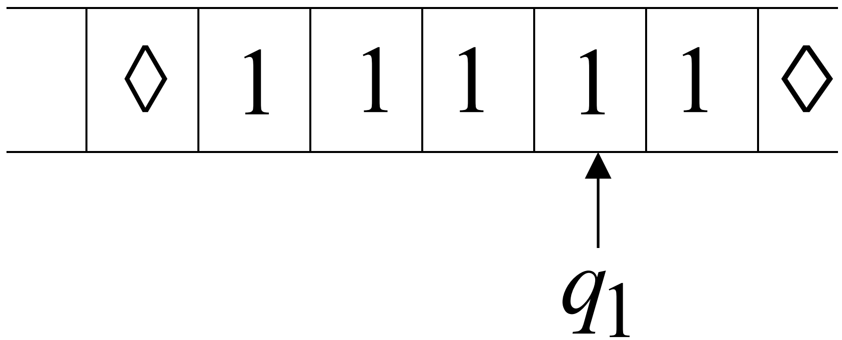
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

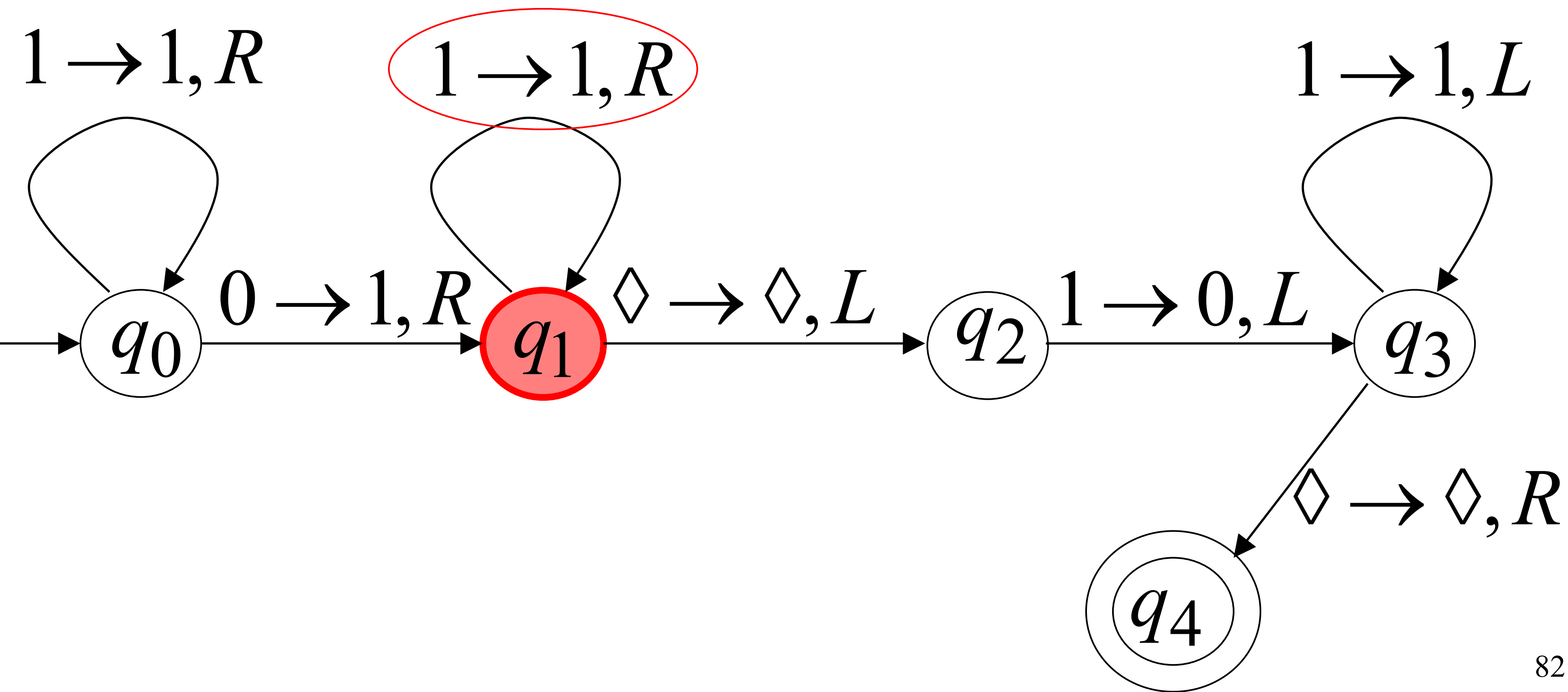
Time 2



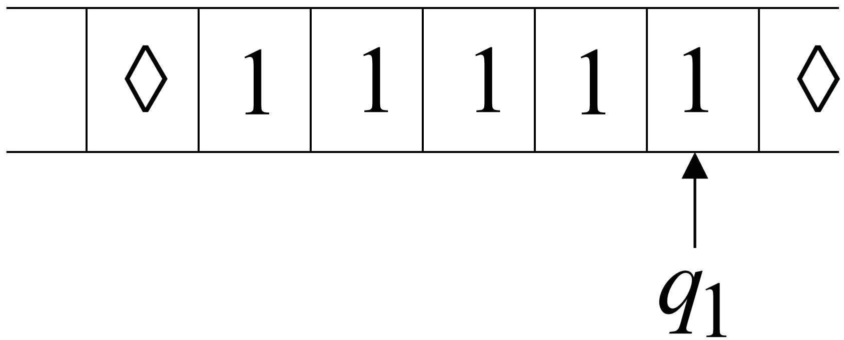


Time 3

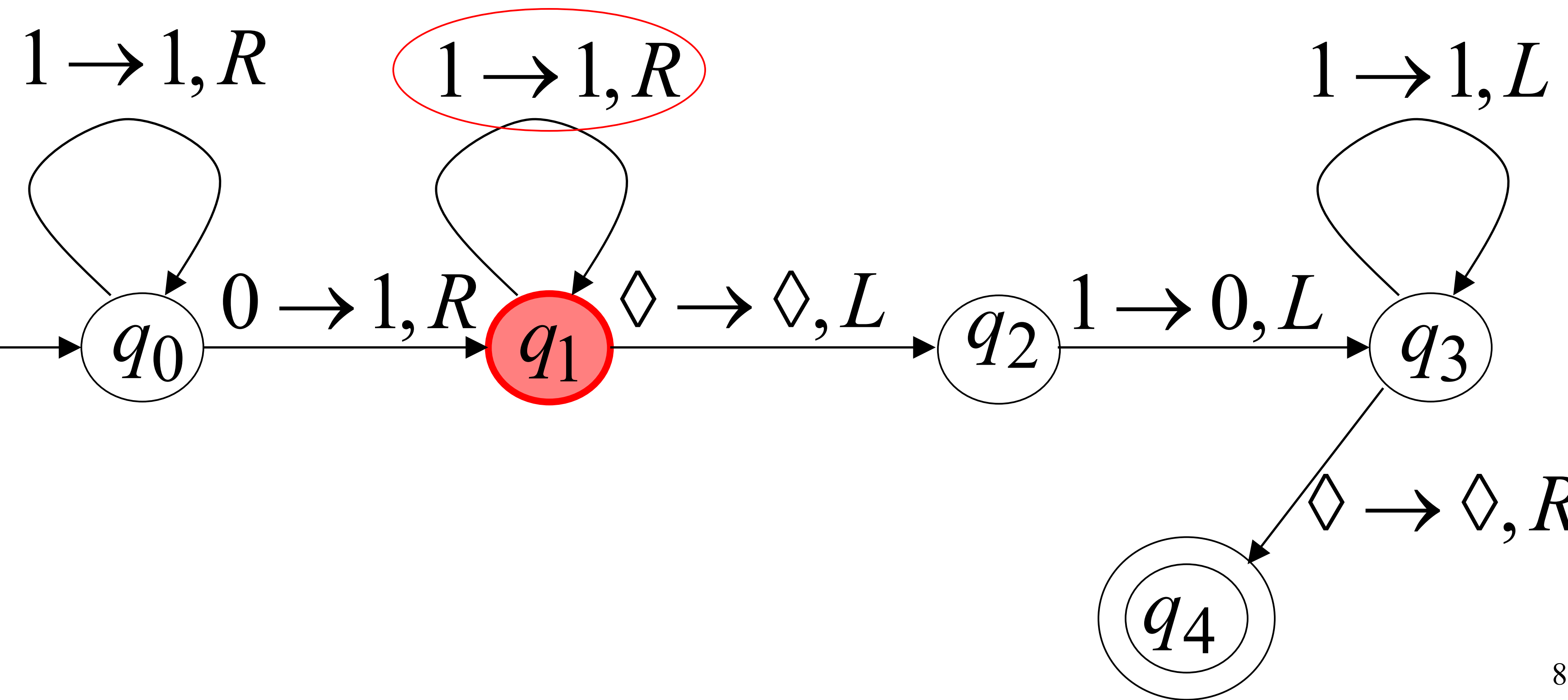

◊
1
1
1
1
1
◊
q_1


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

Time 4

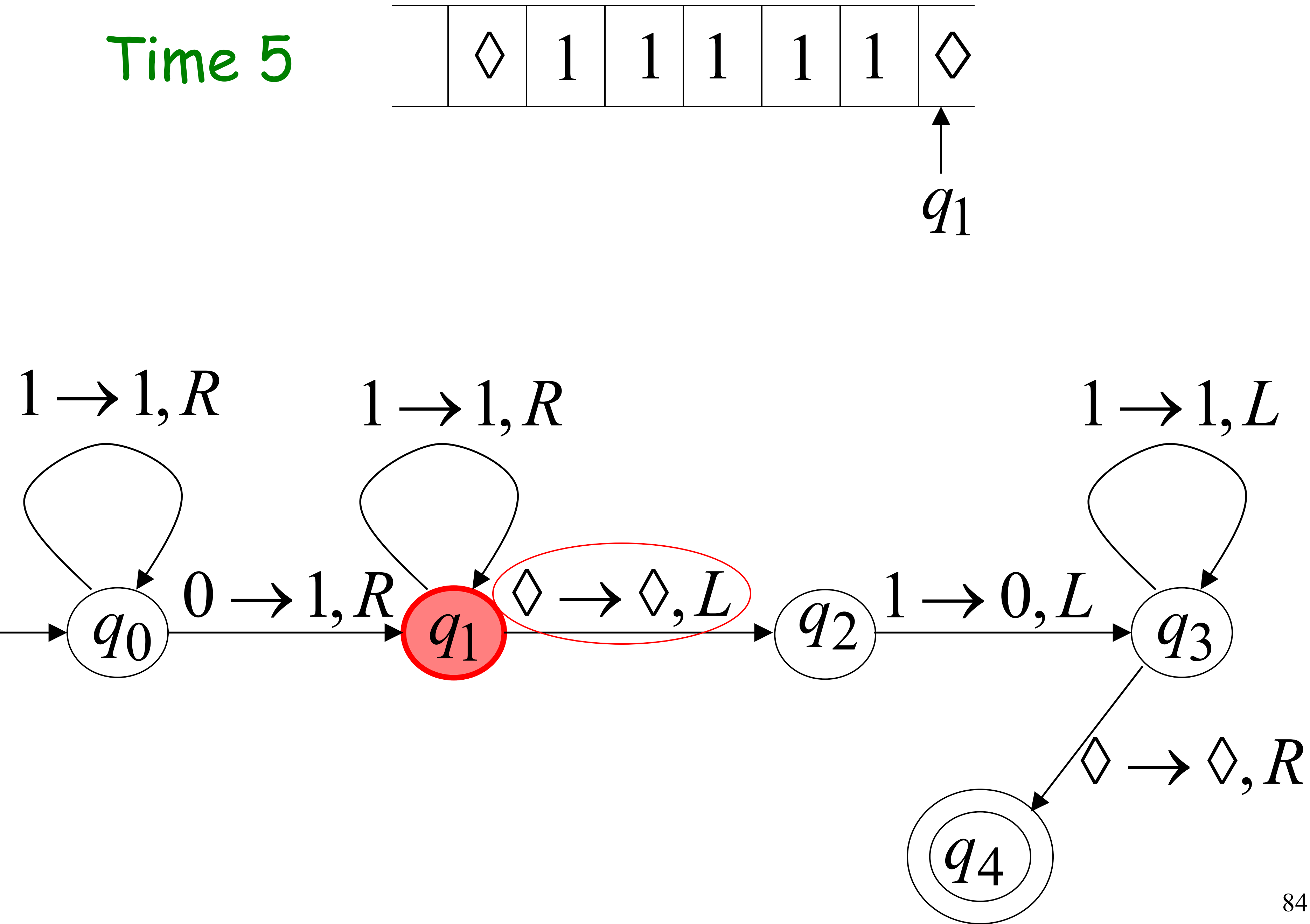


◊
1
1
1
1
1
◊
q_1



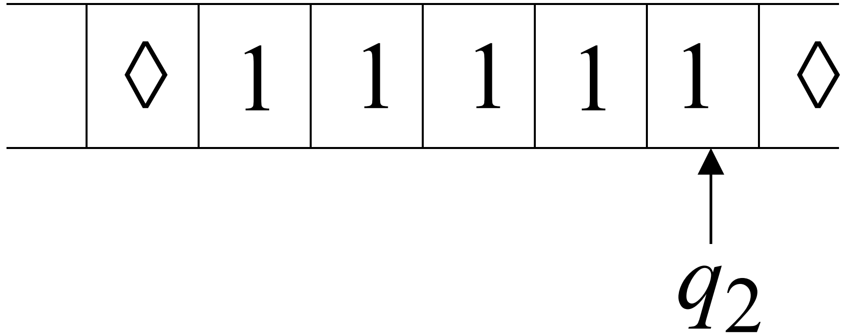
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

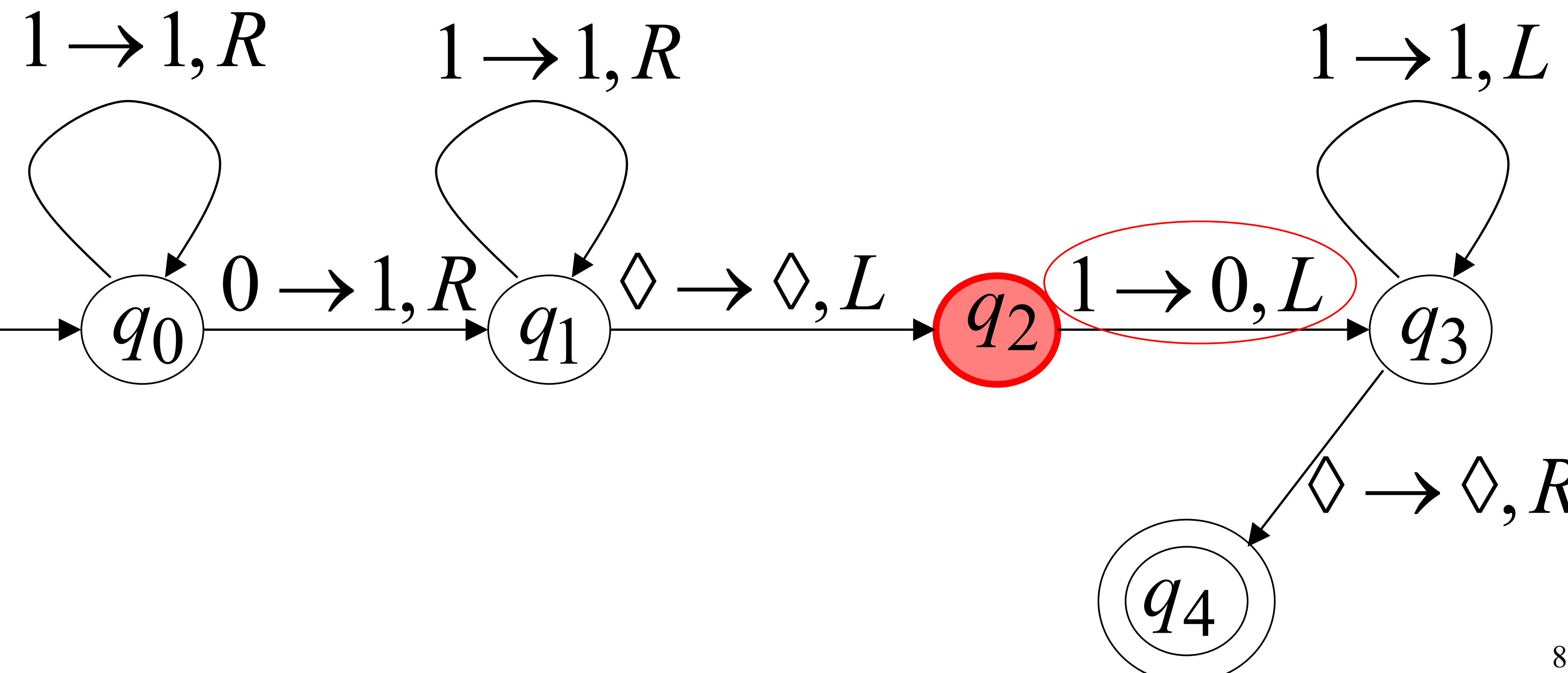
Time 5



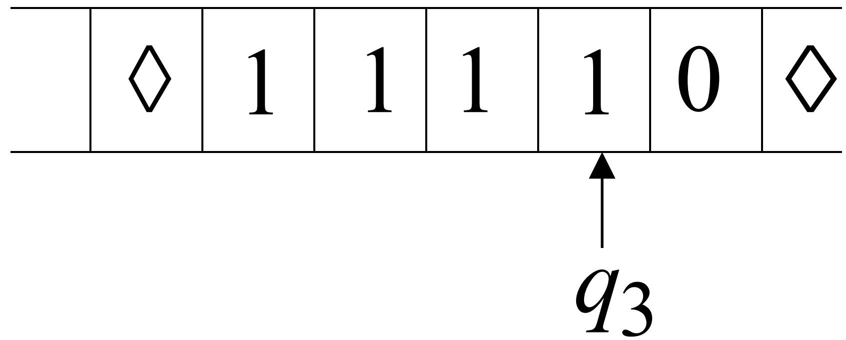
◊
1
1
1
1
1
◊
q_1
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

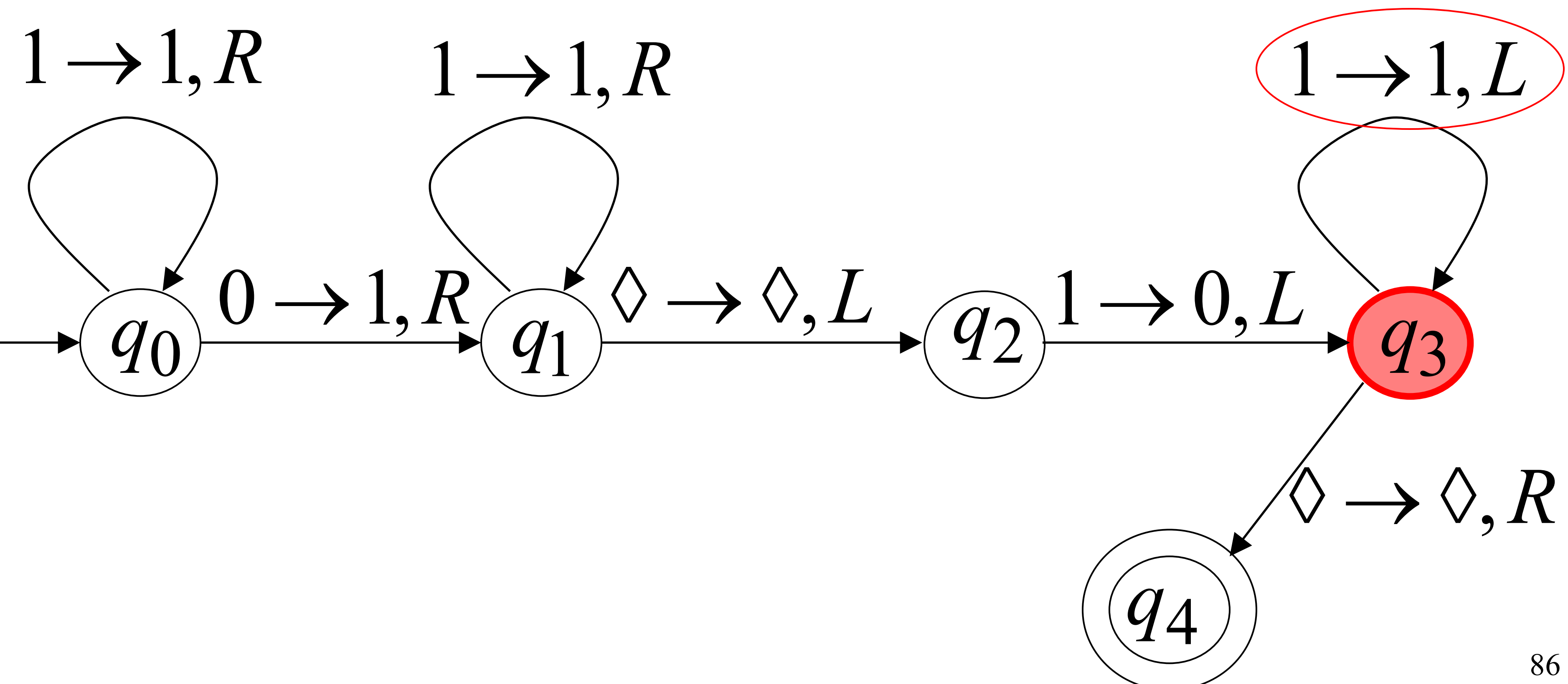
Time 6


◊ 1 1 1 1 1 ◊
q_2

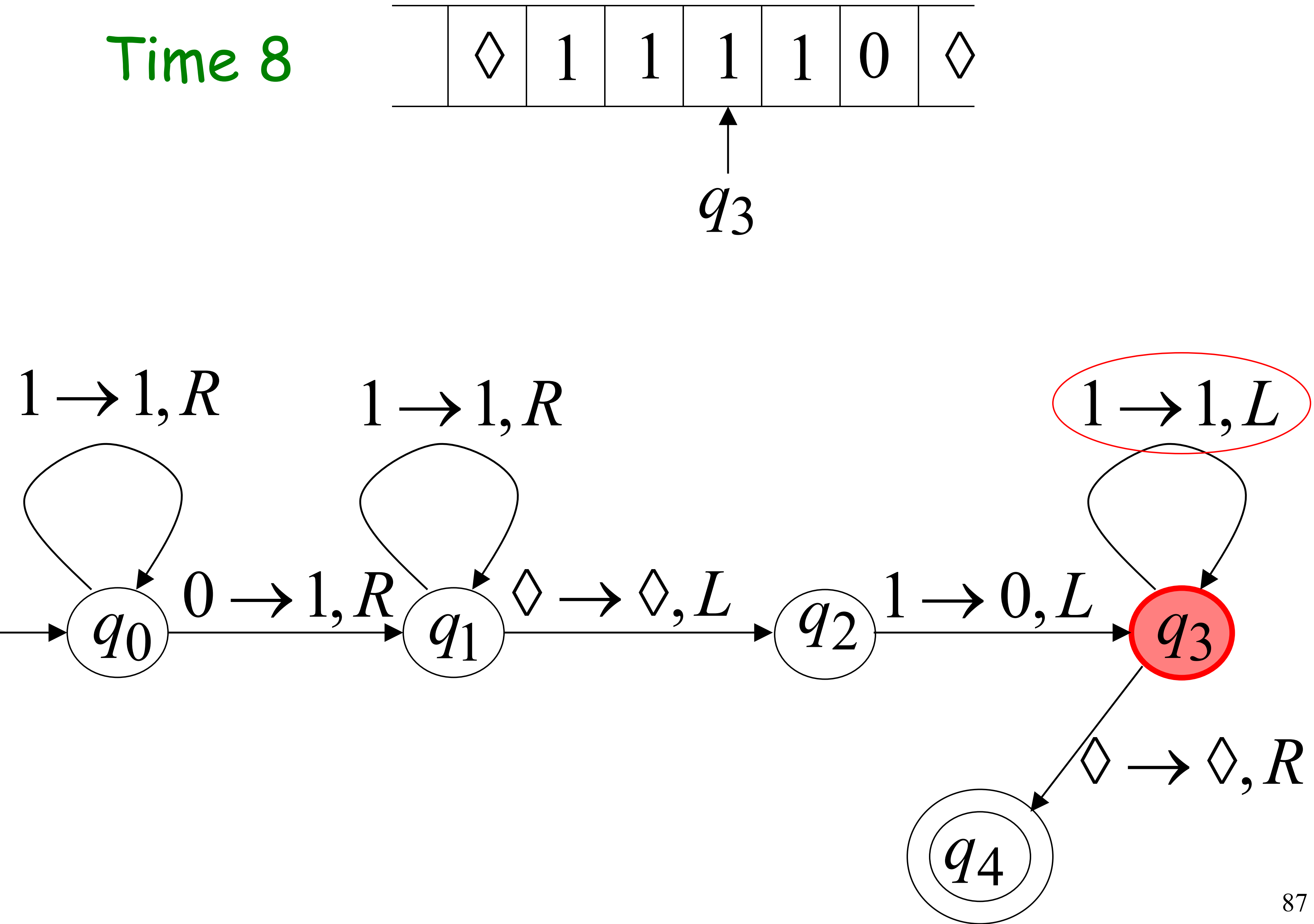

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 7

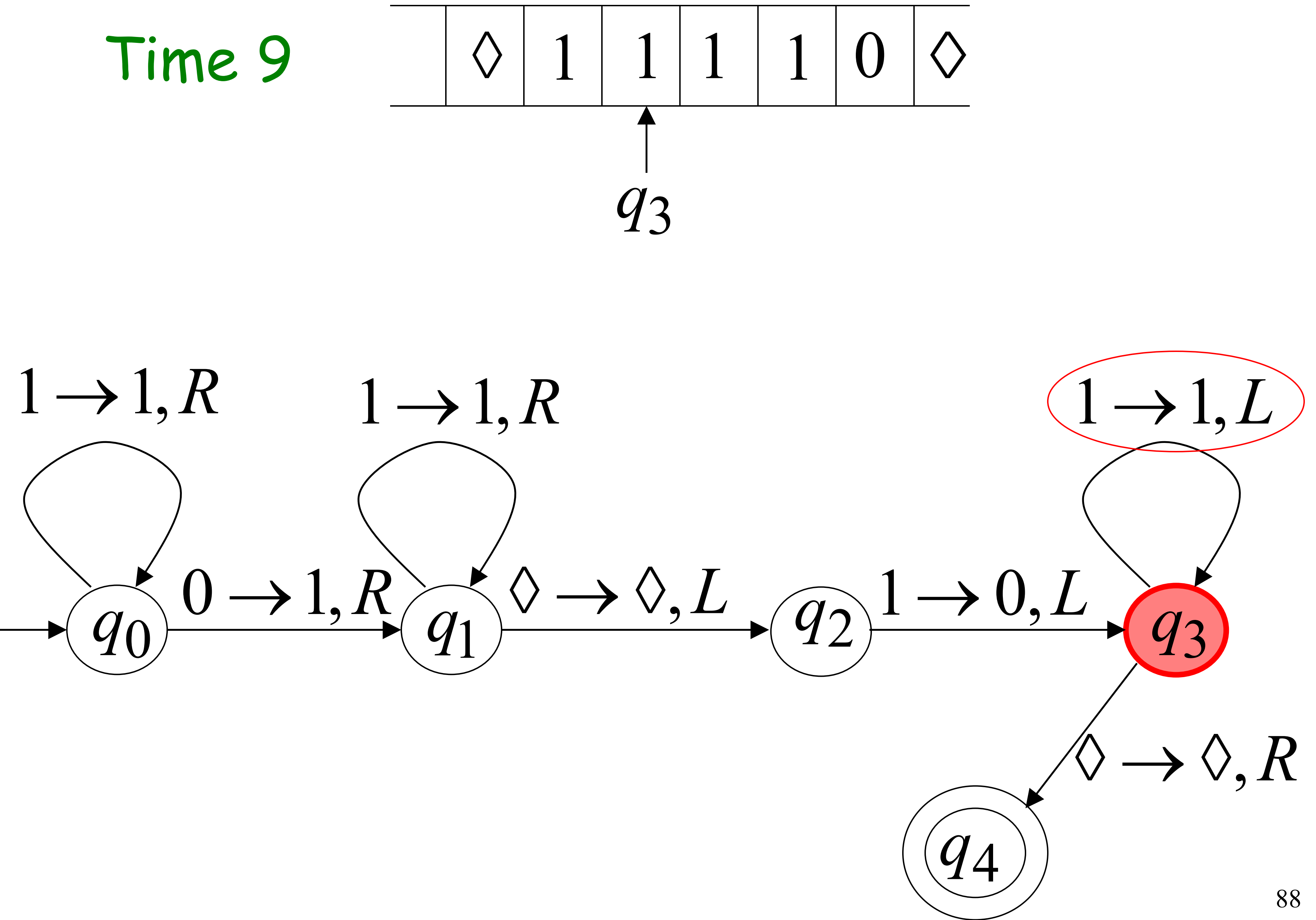

◊ 1 1 1 1 0 ◊
q_3


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 8

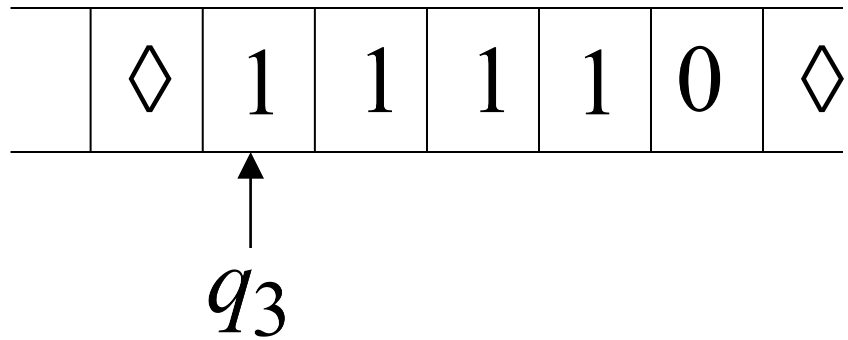

◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

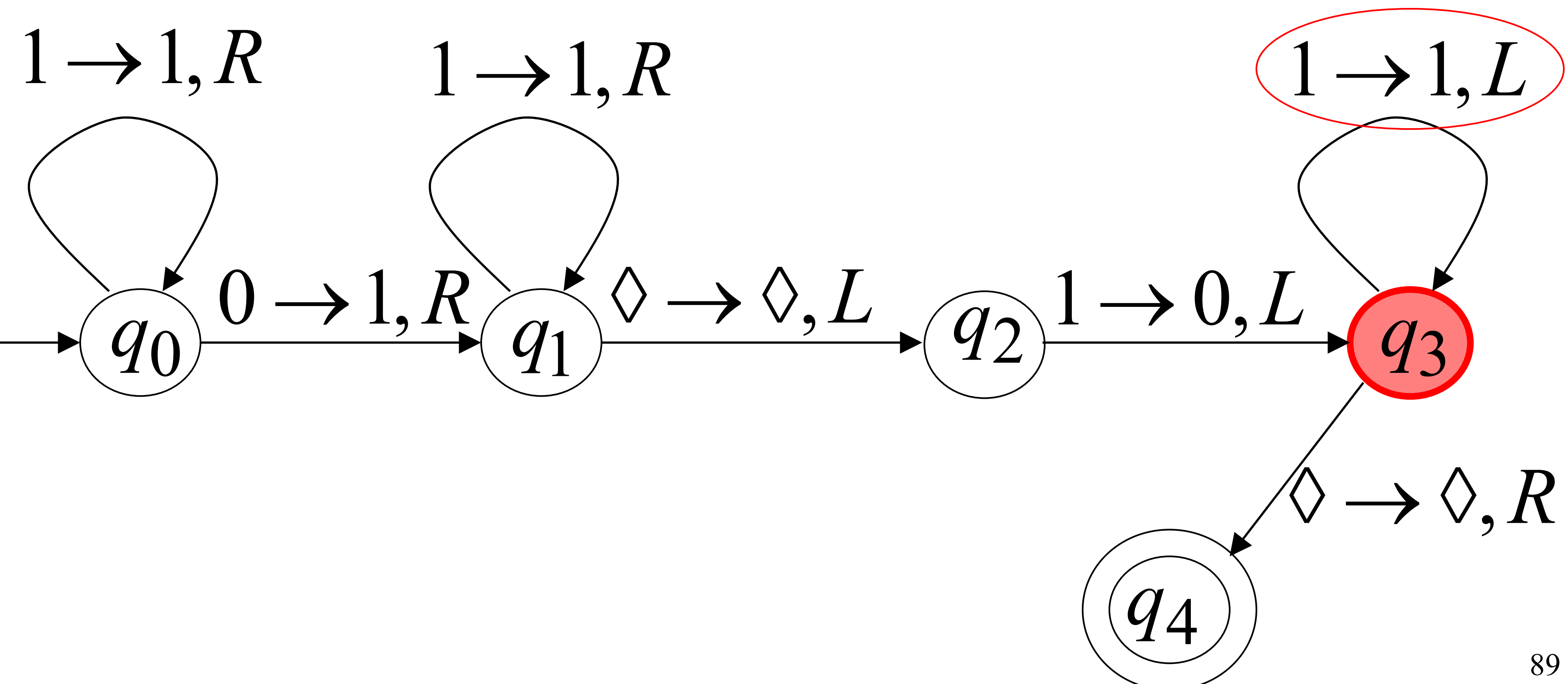
Time 9



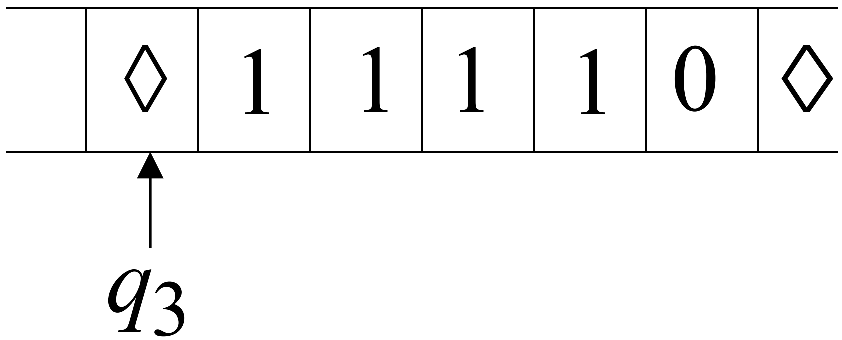
◊
1
1
1
1
0
◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$◊ \rightarrow ◊, L$
$1 \rightarrow 0, L$
$◊ \rightarrow ◊, R$
q_0
q_1
q_2
q_3
q_4

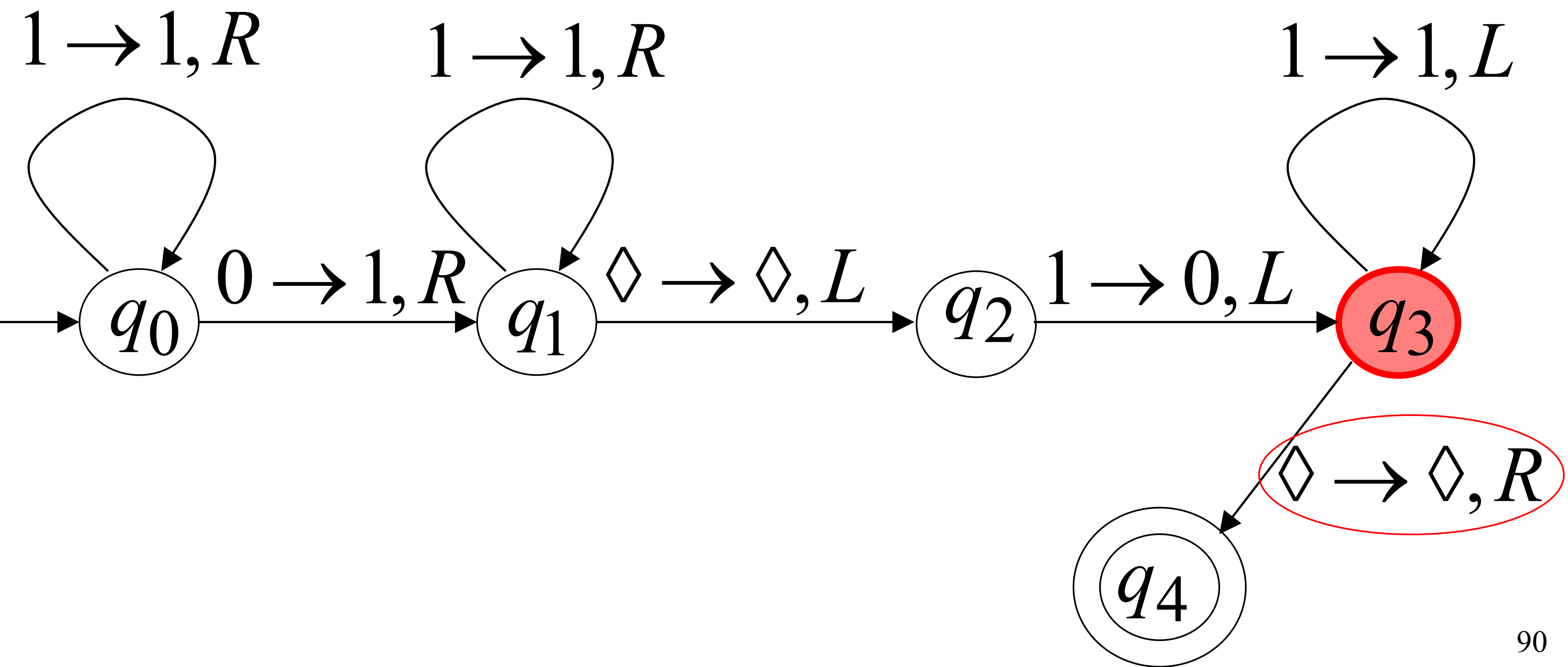
Time 10


◊ 1 1 1 1 0 ◊
q_3

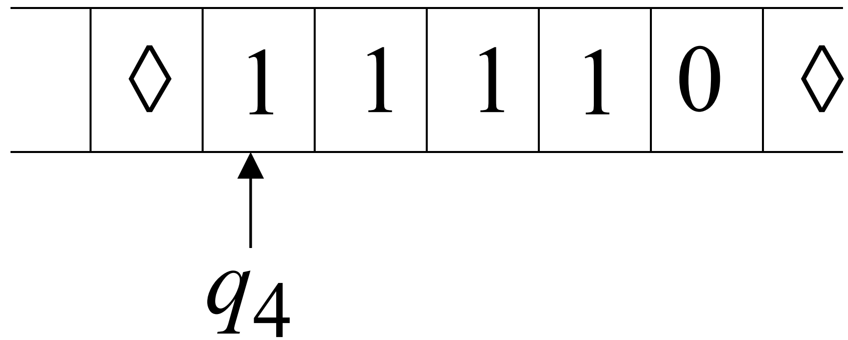

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

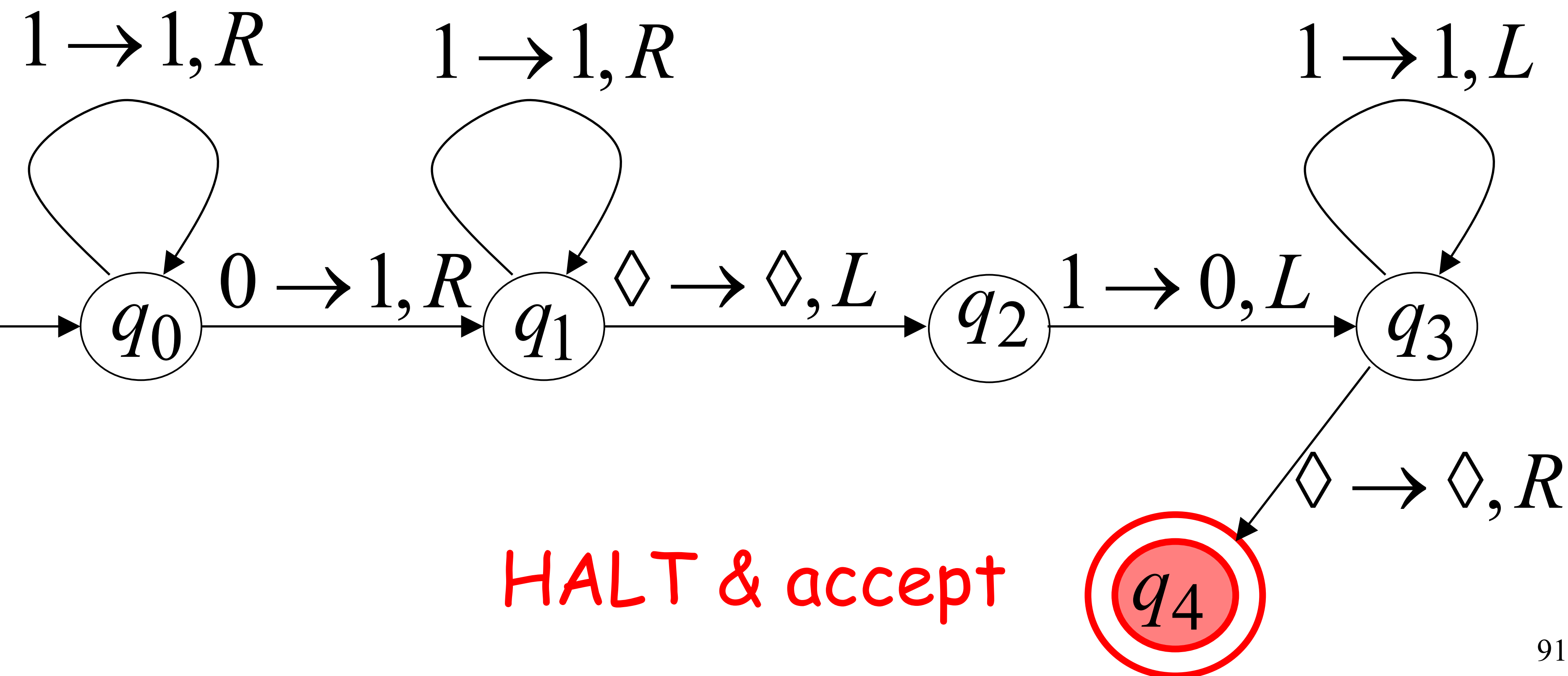
Time 11


◊
1
1
1
1
0
◊
q_3


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 12


◊
1
1
1
1
0
◊
q_4


$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\lozenge \to \lozenge, L$
$1 \to 0, L$
$\lozenge \to \lozenge, R$
q_0
q_1
q_2
q_3
q_4
HALT & accept

Another Example

The function $f(x) = 2x$ is computable

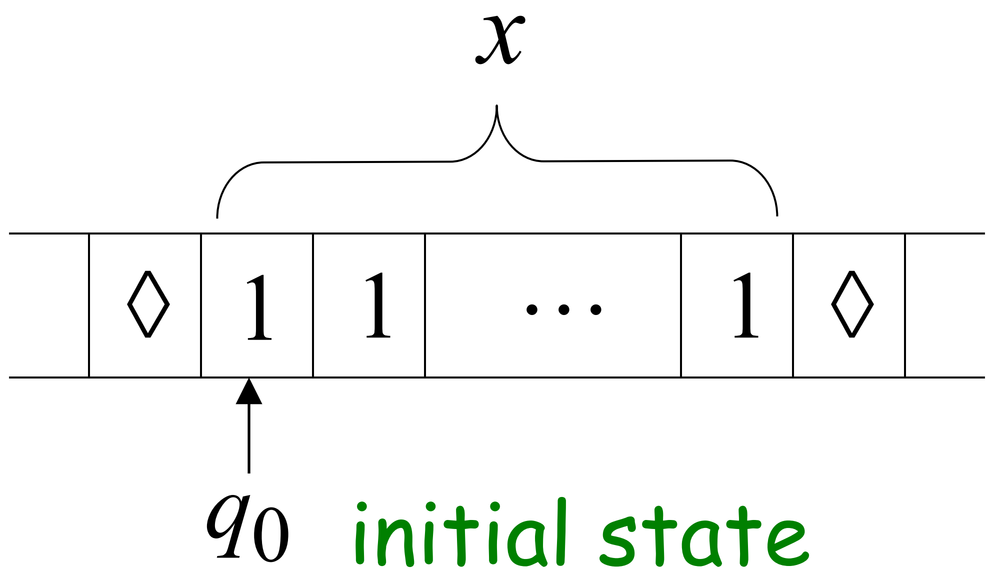
x is integer

Turing Machine:

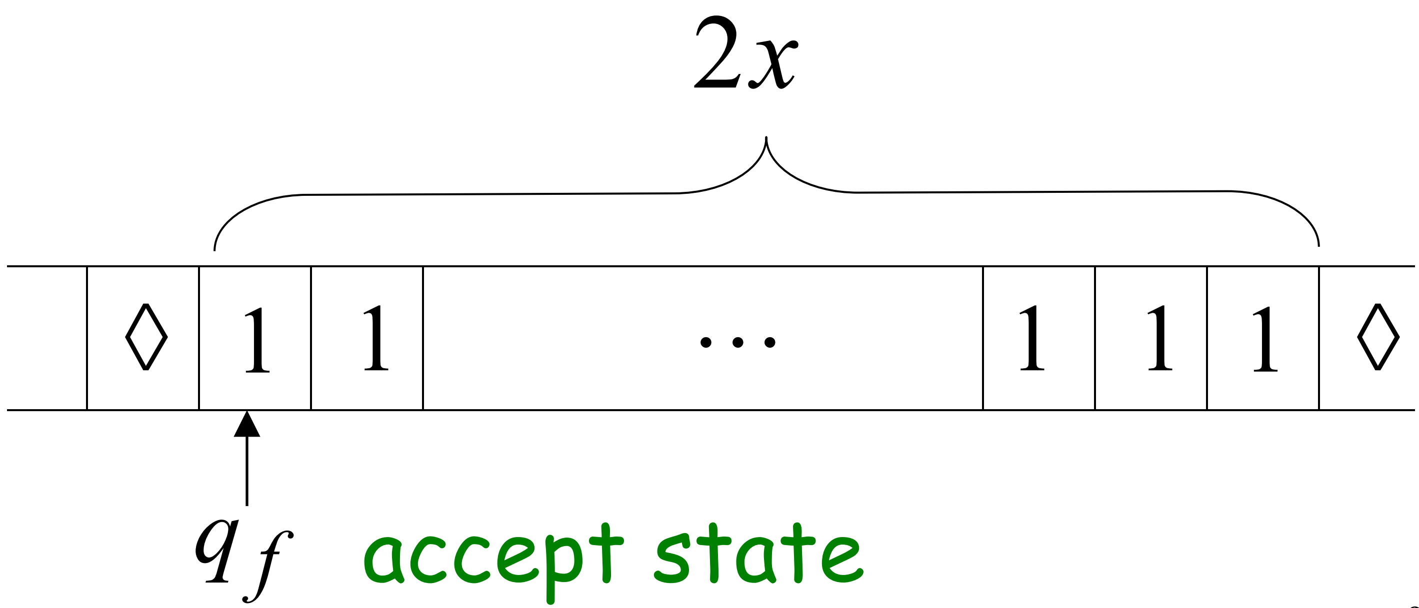
Input string: x unary

Output string: xx unary

Start



Finish

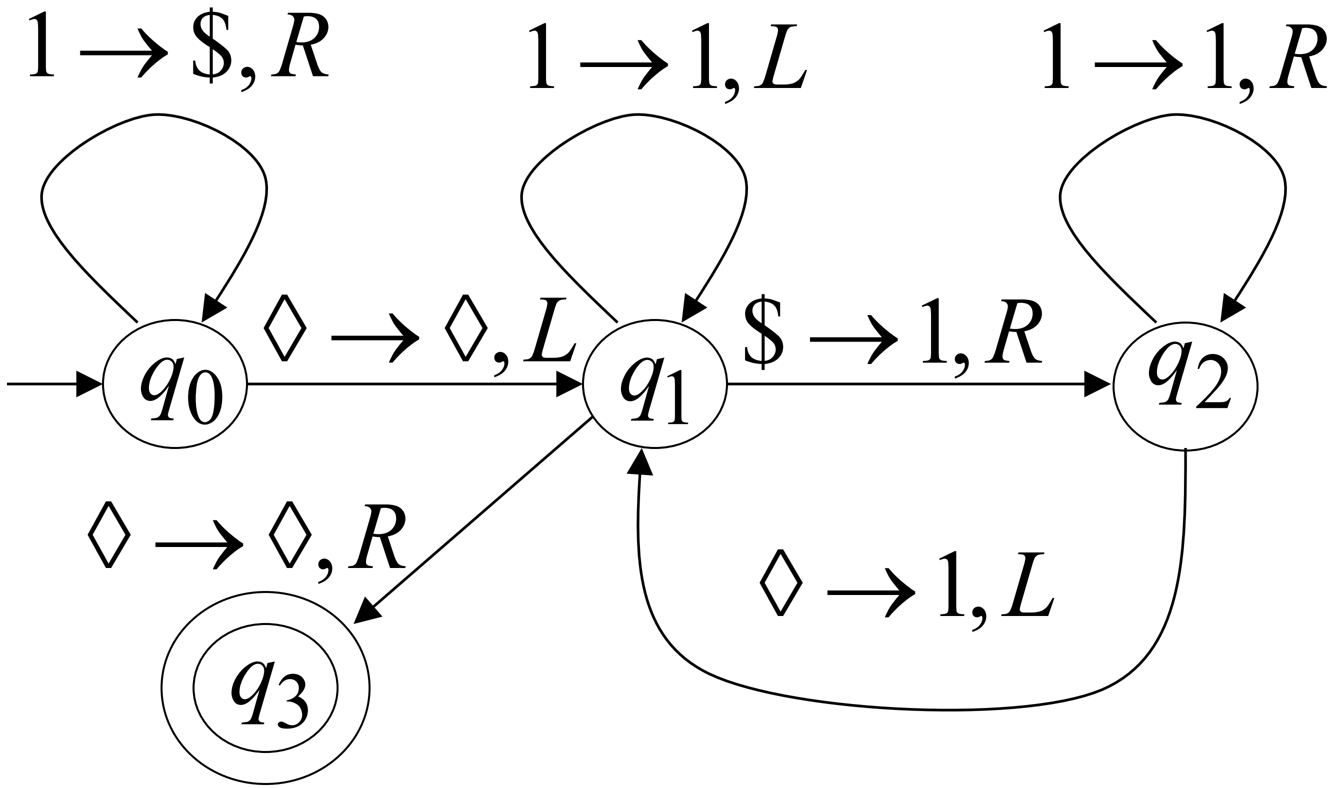


Turing Machine Pseudocode for $f(x) = 2x$

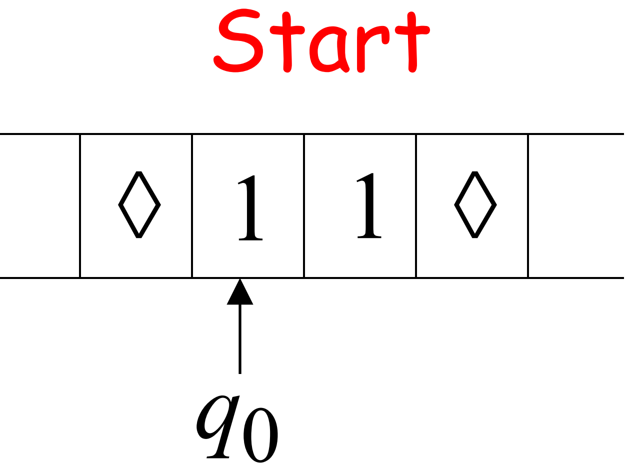
- Replace every 1 with \$
- Repeat:
  - Find rightmost \$, replace it with 1
  - Go to right end, insert 1

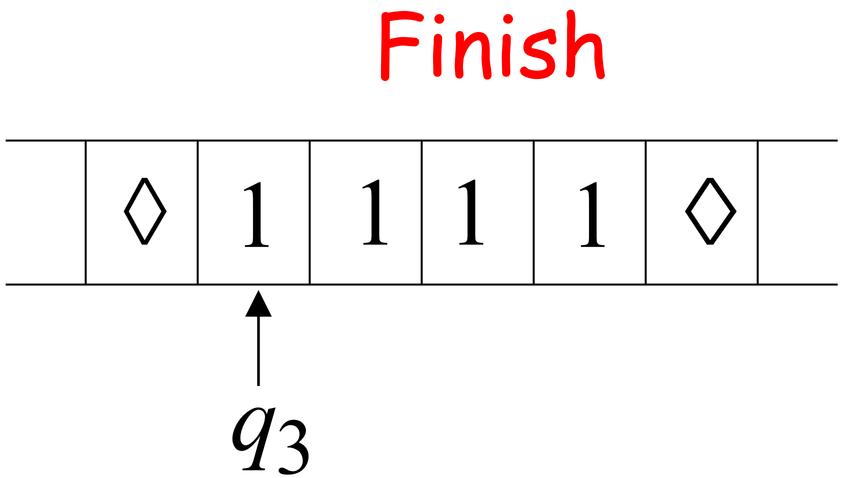
  Until no more \$ remain

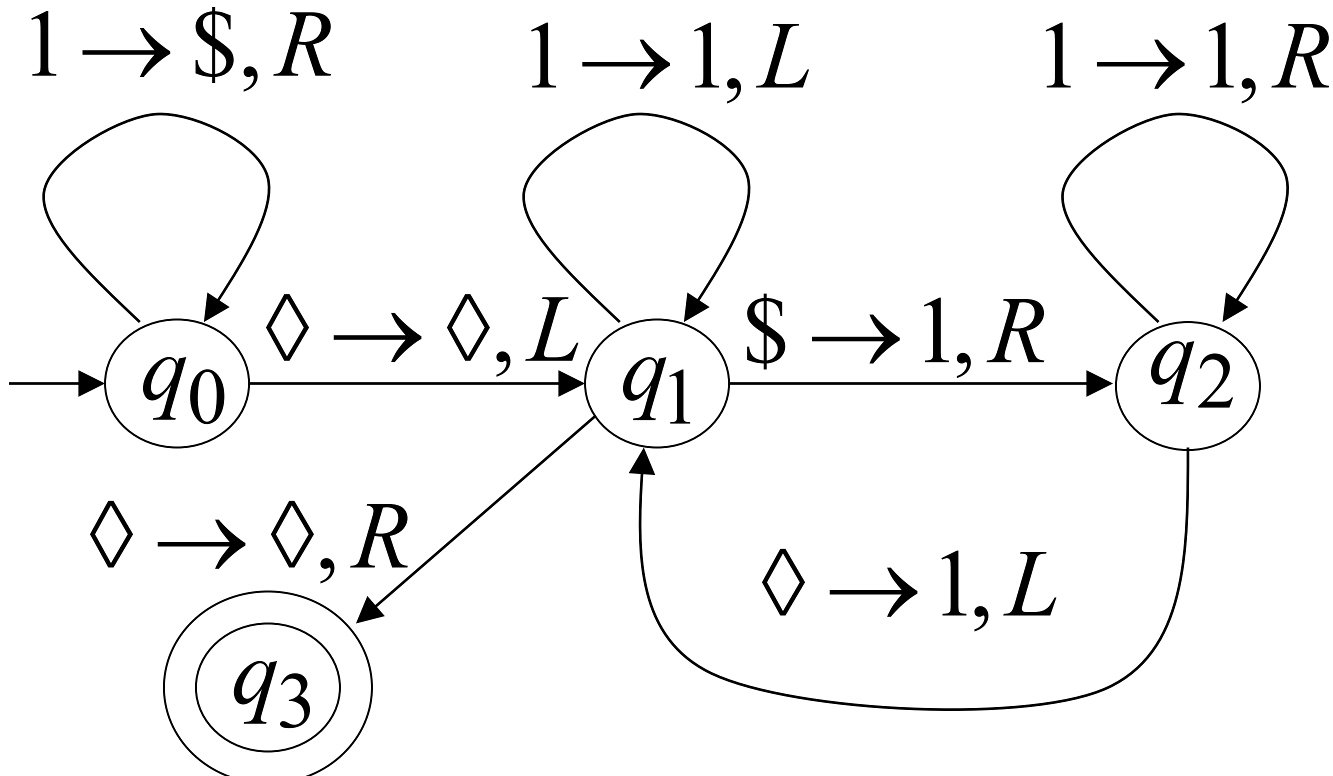
Turing Machine for $f(x) = 2x$



Example


Start
◊
1
1
◊
q_0


Finish
◊
1
1
1
1
◊
q_3


$1 \to \$, R$
$1 \to 1, L$
$1 \to 1, R$
$\Diamond \to \Diamond, L$
$\$ \to 1, R$
$\Diamond \to \Diamond, R$
$\Diamond \to 1, L$
q_0
q_1
q_2
q_3

Another Example

The function is computable

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

  Match a 1 from x with a 1 from y

  Until all of x or y is matched

- If a 1 from x is not matched

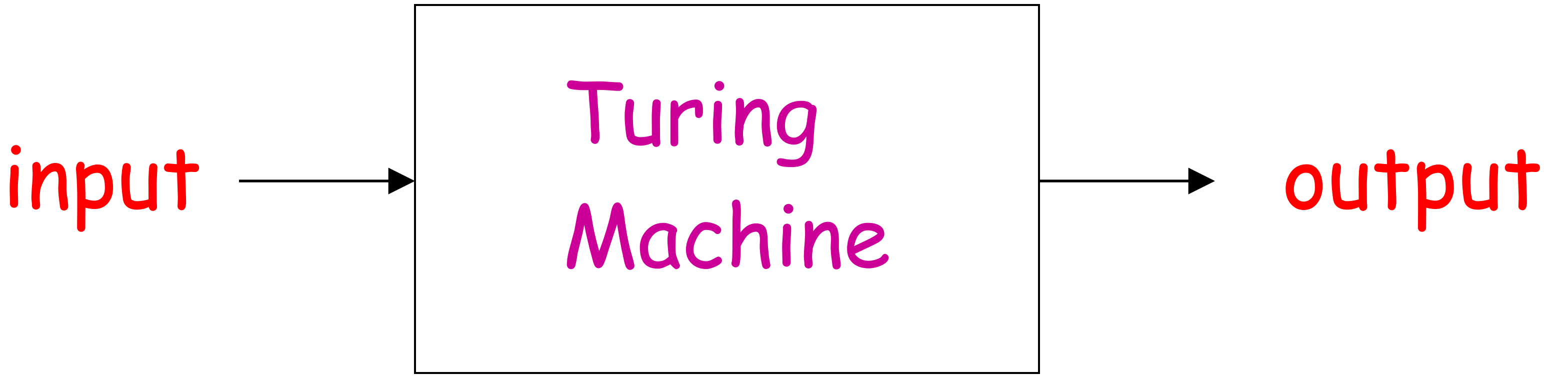
  erase tape, write 1 $(x > y)$

  else

  erase tape, write 0 $(x \leq y)$

Combining Turing Machines

Block Diagram



Example:

$$f(x,y) = \begin{cases} x+y & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases}$$