CS 506: Introduction to Quantum Computing
Tensor Products, State Postulate, and the Bloch Sphere

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1 Review: Tensor Products

The tensor product $\otimes$ is used to combine two quantum objects into a composite object. For quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, the combined state can be written as:

$$|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle |\psi_2\rangle = |\psi_1\psi_2\rangle$$

1.1 Example: Three-Qubit Tensor Product

Consider the tensor product $|00\rangle \otimes |1\rangle$:

$$|00\rangle \otimes |1\rangle = |00\rangle |1\rangle = |001\rangle$$

In vector form:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2 Tensor Product of Operators

For quantum operators (gates) A and B, the tensor product $A \otimes B$ acts on composite systems:

$$(A \otimes B)(|\psi_1\rangle \otimes |\psi_2\rangle) = (A|\psi_1\rangle) \otimes (B|\psi_2\rangle) = |\phi_1\rangle \otimes |\phi_2\rangle$$

where $A|\psi_1\rangle = |\phi_1\rangle$ and $B|\psi_2\rangle = |\phi_2\rangle$.

2.1 Example: NOT $\otimes$ NOT Gate

The NOT $\otimes$ NOT gate acts on a 2-qubit system. The matrix representation is:

$$\text{NOT} \otimes \text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Computing the tensor product:

$$= \begin{pmatrix} 0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This is a $2^2 \times 2^2 = 4 \times 4$ matrix (2-qubit gate).

3 Properties of the Tensor Product

For a complex scalar $c \in \mathbb{C}$ and quantum states $|\psi_1\rangle, |\psi_2\rangle$:

3.1 Scalar Multiplication

$$c(|\psi_1\rangle \otimes |\psi_2\rangle) = (c\,|\psi_1\rangle) \otimes |\psi_2\rangle = |\psi_1\rangle \otimes (c\,|\psi_2\rangle)$$

3.2 Non-Commutativity

The tensor product is **not** commutative:

$$|\psi_1\rangle \otimes |\psi_2\rangle \neq |\psi_2\rangle \otimes |\psi_1\rangle$$

3.3 Distributivity

The tensor product distributes over addition:

$$|\psi_1\rangle \otimes (|\phi_1\rangle + |\phi_2\rangle) = |\psi_1\rangle \otimes |\phi_1\rangle + |\psi_1\rangle \otimes |\phi_2\rangle$$

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi_2\rangle = |\phi_1\rangle \otimes |\psi_2\rangle + |\phi_2\rangle \otimes |\psi_2\rangle$$

4 Two-Qubit State Expansion

Consider the tensor product of two general single-qubit states:

$$(\alpha_0\,|0\rangle + \alpha_1\,|1\rangle) \otimes (\beta_0\,|0\rangle + \beta_1\,|1\rangle)$$

Expanding using distributivity:

$$\begin{aligned} &= \alpha_0\beta_0\,|0\rangle\,|0\rangle + \alpha_0\beta_1\,|0\rangle\,|1\rangle + \alpha_1\beta_0\,|1\rangle\,|0\rangle + \alpha_1\beta_1\,|1\rangle\,|1\rangle \\ &= \alpha_0\beta_0\,|00\rangle + \alpha_0\beta_1\,|01\rangle + \alpha_1\beta_0\,|10\rangle + \alpha_1\beta_1\,|11\rangle \end{aligned}$$

In vector form:

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

5 The State Postulate

State Postulate: A quantum state is a complex vector of dimension n for some n that is a power of 2, whose norm is 1.

For a state $|\psi\rangle$:

$$\| \, |\psi\rangle \, \|^2 = \langle\psi|\psi\rangle = 1$$

If $|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle$ form an orthonormal basis, then:

$$|\psi\rangle = \alpha_1 |b_1\rangle + \alpha_2 |b_2\rangle + \cdots + \alpha_n |b_n\rangle$$

The normalization condition becomes:

$$\langle\psi|\psi\rangle = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 = 1$$

5.1 Example: 1 Qubit

For the state:

$$|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{\sqrt{2}}{\sqrt{3}} |1\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

Verification: $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{2}{3} = 1$ ✓

5.2 Example: 2 Qubits

For a 2-qubit state in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\begin{pmatrix} \frac{1}{\sqrt{12}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{12}} \end{pmatrix} = \frac{1}{\sqrt{12}} |00\rangle + \frac{1}{\sqrt{12}} |11\rangle$$

6 Geometric Representation of Qubits

6.1 Real Amplitudes and Trigonometric Form

For a single qubit with real amplitudes:

$$|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{\sqrt{2}}{\sqrt{3}} |1\rangle$$

We can write the amplitudes in trigonometric form:

$$\frac{1}{\sqrt{3}} = \cos\theta, \quad \frac{\sqrt{2}}{\sqrt{3}} = \sin\theta$$

Thus:

$$|\psi\rangle = \cos\theta \, |0\rangle + \sin\theta \, |1\rangle$$

6.2 Complex Amplitudes and Global Phase

For a qubit with complex amplitudes:

$$|\psi\rangle = \underbrace{\left(\frac{1}{\sqrt{6}} + i\frac{1}{\sqrt{3}}\right)}_{\alpha_0}|0\rangle + \underbrace{\left(\frac{1}{2} + i\frac{1}{2}\right)}_{\alpha_1}|1\rangle$$

Computing the squared magnitudes:

$$|\alpha_0|^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$$|\alpha_1|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

We can factor out a **global phase** $\left(\frac{1}{\sqrt{6}} + i\frac{1}{\sqrt{3}}\right)$:

$$|\psi\rangle = \left(\frac{1}{\sqrt{6}} + i\frac{1}{\sqrt{3}}\right)\left(|0\rangle + \frac{\frac{1}{2} + i\frac{1}{2}}{\frac{1}{\sqrt{6}} + i\frac{1}{\sqrt{3}}}|1\rangle\right)$$

The global phase can be written using Euler's formula:

$$\frac{1}{\sqrt{2}}e^{ix} = \frac{1}{\sqrt{2}}(\cos x + i\sin x) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{6}} + i\frac{\sqrt{2}}{\sqrt{3}}\right)$$

This simplifies to $\frac{1}{\sqrt{2}}e^{ix} = \cos x + i\sin x$.

Important: Global phases can be ignored (mostly), as they do not affect measurement outcomes.

6.3 General Qubit Form with Local Phase

The general form of a single qubit state is:

$$|\psi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle$$

The term $e^{i\phi}$ is called the **local phase** (or relative phase), which **cannot be ignored** as it affects interference and measurement outcomes.

Verification of normalization:

$$|e^{i\phi}|^2 = |\cos\phi + i\sin\phi|^2 = \cos^2\phi + \sin^2\phi = 1$$

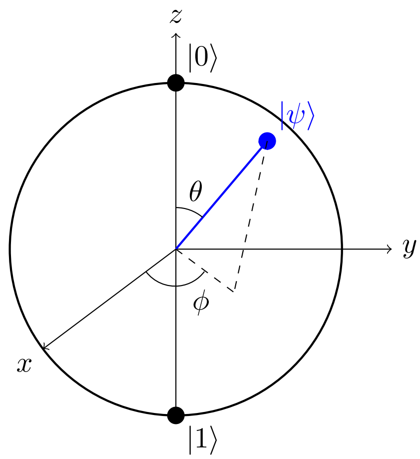
$$\|\,|\psi\rangle\,\|^2 = \cos^2\theta + |e^{i\phi}|^2\sin^2\theta = \cos^2\theta + \sin^2\theta = 1\ \checkmark$$

7 The Bloch Sphere

A general qubit state can be written as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

This maps to the **Bloch sphere**, a 3-dimensional sphere of radius 1 (unity).



The Cartesian coordinates on the Bloch sphere are:

$$\psi_x = \sin\theta\cos\phi$$
$$\psi_y = \sin\theta\sin\phi$$
$$\psi_z = \cos\theta$$

The state $|\psi\rangle$ corresponds to a point (ψ_x, ψ_y, ψ_z) on the unit sphere.

8 Time Evolution of Quantum States

8.1 Unitary Evolution

The time evolution of a quantum state is governed by a **unitary operator** U:

$$|\psi_1\rangle \longrightarrow |\psi_2\rangle = U|\psi_1\rangle$$

Unitary operators preserve the norm:

$$\| \, |\psi_2\rangle \, \|^2 = \| \, |\psi_1\rangle \, \|^2 = 1$$

Since U is unitary, its inverse $U^{-1} = U^\dagger$ exists, and:

$$U^{-1}|\psi_2\rangle = |\psi_1\rangle$$

8.2 Linearity of Quantum Mechanics

Quantum mechanics is linear. For a unitary operator U:

$$U(|\psi_1\rangle + |\psi_2\rangle) = U|\psi_1\rangle + U|\psi_2\rangle$$

Example: The NOT gate acting on a superposition:

$$\text{NOT}\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right) = \text{NOT}\left(\frac{1}{\sqrt{3}}|0\rangle\right) + \text{NOT}\left(\frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{3}}\text{NOT}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}\text{NOT}|1\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0\rangle$$

9 Schrödinger's Wave Mechanics

The non-relativistic Schrödinger equation describes continuous time evolution:

$$i\frac{\hbar}{2\pi}\frac{d\,|\psi(t)\rangle}{dt} = H(t)\,|\psi(t)\rangle$$

where:

- $\hbar$ is Planck's constant (used interchangeably with $\frac{h}{2\pi}$)
- $H(t)$ is the **Hamiltonian**, a **Hermitian matrix**: $H^\dagger = H$ (may vary with time)

Rearranging:

$$\frac{d\,|\psi(t)\rangle}{dt} = -\frac{iH(t)}{\hbar/(2\pi)}\,|\psi(t)\rangle$$

9.1 Connection to Differential Equations

This is analogous to the ordinary differential equation:

$$\frac{dy}{dx} = -\lambda y \quad \Rightarrow \quad y(x) = e^{-\lambda x}$$

For times t_1 and t_2 where $|t_1 - t_2|$ is small enough that $H(t_1) \approx H(t_2)$, the evolution can be approximated by:

$$|\psi(t_2)\rangle \approx e^{-iH(t_1-t_2)/\hbar}\,|\psi(t_1)\rangle$$