CS 506 - Lecture Notes

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Topics: Super Dense Coding, Quantum Teleportation

2-Qubit System

Standard (Computational) Basis

For two qubits, the Hilbert space is $2 \times 2 = 4$-dimensional. The standard basis states are:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}.$$

Any 2-qubit state can be expressed as a linear combination

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \quad \sum |c_{ij}|^2 = 1.$$

Insight: These states are separable if they can be written as a tensor product of two single-qubit states. Otherwise, they are entangled.

Bell basis

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

These are unit norm vectors with cross product = 0. They cannot be expressed as tensor products of two vectors.

Properties:

- Orthonormal: $\langle B_{ij} | B_{kl} \rangle = \delta_{ik}\delta_{jl}$.
- Maximally entangled: cannot be factored into single-qubit states.
- Form a complete basis: any 2-qubit state can be expressed as a combination of them.

Insight: The Bell basis is crucial because measurement in this basis reveals correlations between qubits that are invisible in the computational basis.

Generalized Bell State

The generalized Bell state equation is

$$|\beta_{ij}\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle |j\rangle + (-1)^i |1\rangle |j \oplus 1\rangle \big), \quad i, j \in \{0, 1\}.$$

Insight: Each Bell state is maximally entangled and cannot be written as a tensor product of two single-qubit states. This marks the departure from separable quantum information.

Insight: If you run the entangling circuit (Hadamard + CNOT) *in reverse*, starting from $|B_{10}\rangle$, you recover the original input $|10\rangle$. This shows that Bell states are not only entangled outputs, but also reversible encodings of classical inputs.

Insight: The labels i, j act like "entanglement coordinates": j selects which pair of computational states are correlated, while i controls whether the correlation is positive or negative (phase). This becomes crucial in both superdense coding and teleportation because Bob's correction depends directly on (i, j).

Insight: Measuring in the Bell basis is powerful: it "projects" two qubits onto one of four maximally entangled patterns, extracting global correlation information that no single-qubit measurement could reveal.

Circuit Preparation of Bell States

To generate Bell states from separable inputs:

1. Start with $|10\rangle$ or $|00\rangle$.
2. Apply a Hadamard (H) gate to the first qubit.
3. Apply a CNOT with the first qubit as control and the second as target.

   Example:

$$|10\rangle \xrightarrow{H\otimes I} \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |B_{10}\rangle\,.$$

Insight: Running this circuit in reverse converts a Bell state back into the original computational basis input. This shows the transformation is unitary and reversible.

Quantum Superdense Coding

Alice wants to send 2 classical bits to Bob. Alice sends only one qubit (1 classical bit). Bob can figure out which 2 classical bits Alice meant.

Superdense Coding: A quantum communication protocol in which Alice sends *two classical bits of information* to Bob by transmitting only *one qubit*, provided that Alice and Bob already share an entangled pair. This shows how entanglement can be used to increase the capacity of classical communication.

Superdense Coding Protocol

1. Alice and Bob share an entangled pair

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

2. Alice wants to send 2 classical bits (00, 01, 10, or 11).

3. Depending on which 2 bits she wants to send, Alice applies a local Pauli:

Classical bits Alice wants to send	Alice applies
00	I
01	X
10	Z
11	XZ

4. This rotates the shared state into one of the four Bell states:

$$|B_{00}\rangle\,,\ |B_{01}\rangle\,,\ |B_{10}\rangle\,,\ |B_{11}\rangle\,.$$

5. Alice sends *her qubit* (only 1 qubit) to Bob.

6. Bob now has both qubits and measures in the Bell basis, distinguishing which $|B_{ij}\rangle$ was sent.

7. From that, Bob knows exactly which 2 classical bits Alice meant.

Key idea: 1 qubit transmission + 1 shared entangled pair $\Rightarrow$ 2 classical bits communicated. This protocol achieves 50% compression.

Quantum Teleportation

Quantum Teleportation: A quantum communication protocol in which Alice transmits an *unknown qubit state* to Bob by sending only *two classical bits*, provided that they already share an entangled pair. This shows how entanglement can be used to transfer quantum information without physically sending the qubit.

Quantum Teleportation Protocol

1. Alice wants to send an unknown qubit

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,.$$

2. Alice and Bob share an entangled pair

$$|B_{00}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

3. Alice combines her qubit $|\psi\rangle$ with her half of $|B_{00}\rangle$. Now the system has 3 qubits (2 with Alice, 1 with Bob).

4. Alice measures her 2 qubits in the *Bell basis*. This projects Bob's qubit into one of:
$$|\psi\rangle\,, \quad X\,|\psi\rangle\,, \quad Z\,|\psi\rangle\,, \quad XZ\,|\psi\rangle\,.$$

5. Alice sends 2 classical bits to Bob, telling him which Bell state she found.

6. Depending on those bits, Bob applies the right correction:

Alice's outcome	Bob applies
$\|B_{00}\rangle$	I
$\|B_{01}\rangle$	X
$\|B_{10}\rangle$	Z
$\|B_{11}\rangle$	XZ

7. After this correction, Bob's qubit is exactly
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,.$$

Key idea: Alice sends *two classical bits*, not a quantum state. The entanglement + classical communication allow Bob to reconstruct $|\psi\rangle$ perfectly.

Regrouping into the Bell basis

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \qquad |B_{00}\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

$$|\psi\rangle \otimes |B_{00}\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle) \otimes \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

$$= \tfrac{1}{\sqrt{2}}\Big(\alpha\ |0\rangle\,|00\rangle + \alpha\ |0\rangle\,|11\rangle + \beta\ |1\rangle\,|00\rangle + \beta\ |1\rangle\,|11\rangle\Big).$$

$$= \tfrac{1}{\sqrt{2}}\Big(\alpha\ |00\rangle\,|0\rangle + \alpha\ |01\rangle\,|1\rangle + \beta\ |10\rangle\,|0\rangle + \beta\ |11\rangle\,|1\rangle\Big) \quad \text{(first two = Alice, last = Bob).}$$

$$|00\rangle = \tfrac{1}{\sqrt{2}}(|B_{00}\rangle + |B_{10}\rangle), \quad |11\rangle = \tfrac{1}{\sqrt{2}}(|B_{00}\rangle - |B_{10}\rangle),$$
$$|01\rangle = \tfrac{1}{\sqrt{2}}(|B_{01}\rangle + |B_{11}\rangle), \quad |10\rangle = \tfrac{1}{\sqrt{2}}(|B_{01}\rangle - |B_{11}\rangle).$$

$$\frac{1}{\sqrt{2}}\alpha\ |00\rangle\,|0\rangle = \frac{1}{\sqrt{2}}\alpha\,\frac{1}{\sqrt{2}}(|B_{00}\rangle + |B_{10}\rangle)\,|0\rangle = \frac{1}{2}\Big(\,|B_{00}\rangle\ \alpha\,|0\rangle + |B_{10}\rangle\ \alpha\,|0\rangle\,\Big),$$

$$\frac{1}{\sqrt{2}}\alpha\ |01\rangle\,|1\rangle = \frac{1}{\sqrt{2}}\alpha\,\frac{1}{\sqrt{2}}(|B_{01}\rangle + |B_{11}\rangle)\,|1\rangle = \frac{1}{2}\Big(\,|B_{01}\rangle\ \alpha\,|1\rangle + |B_{11}\rangle\ \alpha\,|1\rangle\,\Big),$$

$$\frac{1}{\sqrt{2}}\beta\ |10\rangle\,|0\rangle = \frac{1}{\sqrt{2}}\beta\,\frac{1}{\sqrt{2}}(|B_{01}\rangle - |B_{11}\rangle)\,|0\rangle = \frac{1}{2}\Big(\,|B_{01}\rangle\ \beta\,|0\rangle - |B_{11}\rangle\ \beta\,|0\rangle\,\Big),$$

$$\frac{1}{\sqrt{2}}\beta\ |11\rangle\,|1\rangle = \frac{1}{\sqrt{2}}\beta\,\frac{1}{\sqrt{2}}(|B_{00}\rangle - |B_{10}\rangle)\,|1\rangle = \frac{1}{2}\Big(\,|B_{00}\rangle\ \beta\,|1\rangle - |B_{10}\rangle\ \beta\,|1\rangle\,\Big).$$

$$|\psi\rangle \otimes |B_{00}\rangle = \frac{1}{2}\Big[\underbrace{|B_{00}\rangle\,\big(\alpha\,|0\rangle + \beta\,|1\rangle\,\big)}_{\text{Bob already has }|\psi\rangle} + \underbrace{|B_{01}\rangle\,\big(\alpha\,|1\rangle + \beta\,|0\rangle\,\big)}_{\text{Bob has }X|\psi\rangle}$$
$$+ \underbrace{|B_{10}\rangle\,\big(\alpha\,|0\rangle - \beta\,|1\rangle\,\big)}_{\text{Bob has }Z|\psi\rangle} + \underbrace{|B_{11}\rangle\,\big(\alpha\,|1\rangle - \beta\,|0\rangle\,\big)}_{\text{Bob has }XZ|\psi\rangle}\Big].$$

Alice measures	Send bits	Bob applies to recover $\|\psi\rangle$
$\|B_{00}\rangle$	00	I
$\|B_{01}\rangle$	01	X
$\|B_{10}\rangle$	10	Z
$\|B_{11}\rangle$	11	XZ

Insight: The "Bell basis is a goldmine." After the change of basis, each outcome leaves Bob with $|\psi\rangle$ up to a single Pauli (I, X, Z, or XZ). Alice's two classical bits just tell Bob which one to undo.

Insight: We see explicitly that Alice's measurement outcome (which Bell state she finds) determines the *form* of Bob's qubit. Each case is just $|\psi\rangle$ with a simple Pauli error (X, Z, or XZ).

Insight: Each possible measurement outcome gives Bob a state that is *one Pauli step away* from $|\psi\rangle$. Alice's 2 classical bits tell Bob exactly which correction to apply.

Insight: The Bell basis acts as a *bridge*: it allows us to rewrite the joint system in a way that reveals how Bob's qubit is correlated with Alice's measurement.

Insight: Teleportation avoids the "no-cloning" problem. Alice's original state is destroyed by measurement, so the qubit is not copied — it is faithfully transferred.

Insight: Teleportation separates the quantum and classical roles: entanglement carries the quantum "channel," while Alice's 2-bit classical message tells Bob which correction to apply. Both are essential.

Insight: Without entanglement, teleportation is impossible. Without the classical message, Bob's qubit remains scrambled. This is an example of quantum-classical hybrid communication.