Spectral Theorem and Applications

January 28, 2026

1 Spectral Theorem

A **normal operator** A satisfies

$$A^\dagger A = AA^\dagger.$$

Unitary and Hermitian operators are normal.

Let T be a normal operator represented by an $n \times n$ matrix. Let

$$|T_1\rangle, |T_2\rangle, \ldots, |T_n\rangle$$

be the eigenvectors of T with corresponding eigenvalues

$$T_1, T_2, \ldots, T_n \in \mathbb{C}.$$

The eigenvectors form an orthonormal basis, and T is diagonal in its own eigenbasis:

$$T = \sum_{i=1}^{n} T_i |T_i\rangle \langle T_i|.$$

1.1 Real Matrix Case

For a real normal matrix A satisfying

$$AA^T = A^T A,$$

we can write

$$T = P\Lambda P^T,$$

where P is an orthogonal (unitary) matrix and Λ is diagonal with eigenvalues as diagonal elements.

2 Example: NOT Operator

The NOT operator is

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its action on basis states:

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle.$$

2.1 Eigenvalues and Eigenvectors of X

Solve

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

This gives

$$y_2 = \lambda y_1, \quad y_1 = \lambda y_2,$$

implying

$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1.$$

Eigenvalue $\lambda = 1$:

$$|x_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Eigenvalue $\lambda = -1$:

$$|x_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

2.2 Spectral Decomposition of X

$$X = 1 \left|x_1\right\rangle \left\langle x_1\right| - 1 \left|x_2\right\rangle \left\langle x_2\right|.$$

3 Projectors

The operators $|T_i\rangle \langle T_i|$ satisfy:

$$(|T_i\rangle \langle T_i|)^2 = |T_i\rangle \langle T_i|,$$

and for any $m \geq 1$,

$$(|T_i\rangle \langle T_i|)^m = |T_i\rangle \langle T_i|.$$

4 Functions of Operators

Let $f(x)$ be expanded as a power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Then

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \cdots.$$

Using the spectral decomposition of T:

$$f(T) = \sum_{i=1}^{n} f(T_i) \left|T_i\right\rangle \left\langle T_i\right|.$$

5 Taylor Series

5.1 Exponential Function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Valid for all x.

5.2 Euler's Identity

$$e^{ix} = \cos x + i \sin x.$$

From the series expansion:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

6 Matrix Exponential Example

Using the spectral decomposition of X:

$$e^X = e^1 \left|x_1\right\rangle \left\langle x_1\right| + e^{-1} \left|x_2\right\rangle \left\langle x_2\right|.$$

Explicitly,

$$e^X = \begin{pmatrix} \frac{e+e^{-1}}{2} & \frac{e-e^{-1}}{2} \\ \frac{e-e^{-1}}{2} & \frac{e+e^{-1}}{2} \end{pmatrix}.$$