CS 506: Introduction to Quantum Computing

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1 Recap

1.1 Complex Number, z

$$z = a + ib \qquad (a, b \in \mathbb{R},\ i = \sqrt{-1}) \tag{1}$$

$$\text{norm of } z: \quad |z| = \sqrt{a^2 + b^2} \tag{2}$$

1.2 Euler's identity

$$e^{ix} = \cos x + i \sin x \quad (\forall x \in \mathbb{R},\ i = \sqrt{-1}) \tag{3}$$

where, e is the base of the natural logarithm and cos and sin are the trigonometric functions cosine and sine respectively.

2 Operators

In quantum mechanics, we talk about operators. In math, we can call it a matrix with complex numbers, i.e., a complex matrix. For our case, it is always a square matrix (number of rows = number of columns).
An operator, T is a $n \times n$ complex matrix that acts on basis states $|b_1\rangle, |b_2\rangle, ..., |b_n\rangle$.
$\{|b_i\rangle\}_{i=1}^{n}$ is any set of orthonormal basis vectors.

2.1 T in Dirac notation

We can write T as

$$T = \sum_{i,j} \alpha_{ij}\ |b_i\rangle \langle b_j| \tag{4}$$

where, α_{ij} is a constant, $|b_i\rangle \langle b_j|$ is the outer product
So the operator is expressed as a linear combination of outer products weighted by the coefficients α_{ij}.

What is α_{ij} (in Dirac notation)?

Let us multiply T by $|b_j\rangle$

$$T|b_j\rangle = \left(\sum_{i,j} \alpha_{ij}\; |b_i\rangle\langle b_j|\right)|b_j\rangle \tag{5}$$

$$= \alpha_{ij}\,|b_i\rangle \tag{6}$$

in equation (5), we have

$$\langle b_i|b_j\rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$$

which is the standard orthonormality condition for the basis states.
Now,

$$\langle b_i|\,T\,|b_j\rangle = \alpha_{ij}\,\underbrace{\langle b_i|b_i\rangle}_{=1} = \alpha_{ij} \tag{7}$$

equation (4) can be written as,

$$T = \sum_{i,j} \langle b_i|\,T\,|b_j\rangle\; |b_i\rangle\langle b_j| \tag{8}$$

3 Identity Operator

It is a special kind of operator. Identity operator is an identity matrix. Multiplying any matrix A by the identity operator, we get the matrix A itself. I is an identity operator if,

$$IA = A \quad (\forall \text{ matrix } A) \tag{9}$$

$$I|0\rangle = |0\rangle\,, \quad I|1\rangle = |1\rangle$$

2×2 Identity matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

If I is a $n \times n$ identity matrix,

$$I = \sum_{i=0}^{n-1} |b_i\rangle\langle b_i| \tag{10}$$

I can be written as any orthonormal basis, not just standard basis.
2×2 Identity matrix can also be written in Hadamard basis,

$$I = (|+\rangle\langle +|) + (|-\rangle\langle -|)$$

4 Adjoint of an operator

For real matrix its called Transpose of the matrix, for complex matrix it's called Adjoint. Also called complex conjugate transpose and Hermitian conjugate.
For an operator T, it's corresponding adjoint is denoted by $T^\dagger$.

The adjoint $T^\dagger$ of a matrix/operator T is obtained by:

1. Taking the transpose of the matrix.
2. Taking the complex conjugate of each entry, i.e., change the sign of the imaginary part. Example, $(15-4i)^\dagger = 15+4i$

Example:

$$T = \begin{pmatrix} 5+3i & 7 \\ 91i & 0 \end{pmatrix}$$

$$T^\dagger = \begin{pmatrix} 5-3i & -91i \\ 7 & 0 \end{pmatrix}$$

For future reference

A quadratic form is an expression of the type,

$$Q(x) = x^T A x$$

$$\underbrace{\begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}}_{|x\rangle^\dagger = \langle x|} \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{|y\rangle}$$

$$\langle x| A |y\rangle = \sum_{ij} a_{ij} x_i y_j \tag{11}$$

What is $\langle x| A^\dagger |y\rangle$?

$$\langle x| A^\dagger |y\rangle = \sum_{ij} a_{ji} x_i y_j \tag{12}$$

Also,

$$\langle x| A |y\rangle^\dagger = \langle y| A^\dagger |x\rangle \tag{13}$$

5 Unitary Operator

Unitary operator is denoted by U. U is an unitary operator given that,

$$U^\dagger = U^{-1} \tag{14}$$

$$UU^\dagger = UU^{-1} = I \tag{15}$$

Consider a vector $|\psi\rangle$. Unitary matrix acts as 1. Therefore, norm is preserved.

$$\text{let, } |\phi\rangle = U|\psi\rangle$$

$$\| \, |\psi\rangle \, \| = \| \, |\phi\rangle \, \| \tag{16}$$

Let, $|\phi'\rangle = U|\phi\rangle$ and $|\psi'\rangle = U|\psi\rangle$. What is the relation between $\langle\phi|\psi\rangle$ and $\langle\phi'|\psi'\rangle$?
Now,

$$\langle\phi'| = (U|\phi\rangle)^\dagger \text{ and } \langle\psi'| = (U|\psi\rangle)^\dagger$$

$$\langle\phi'|\psi'\rangle = \langle\phi|U^\dagger|U|\psi\rangle$$

using equation (15)

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle \tag{17}$$

6 Hermitian Operator

In physics, Hermitian operators play a role to encode the total energy of a system, we'll see that briefly when we discuss Schrödinger's wave equation. In quantum computing, Hermitian operators represent observables (like energy, spin, etc.).
Hermitian operator H is one where

$$H^\dagger = H \tag{18}$$

In real matrices it is equivalent to a symmetric matrix ($A^T = A$).

For future reference

Is Hermitian operator also an unitary operator?
No, a Hermitian operator is not necessarily a unitary operator, though it is possible for an operator to be both.
Also,

$$e^{-iH} \text{ is a unitary operator} \tag{19}$$

where, H is a Hermitian operator.

Proof:

$$\begin{aligned}
(e^{-iH})^{\dagger} e^{-iH} &= e^{(-iH)^{\dagger}} e^{-iH} \\
&= e^{iH^{\dagger}} e^{-iH} \\
&= e^{iH} e^{-iH} && \text{(from equation (18))} \\
&= e^{iH-iH} \\
&= e^{0} \\
&= I
\end{aligned}$$

7 Eigenvalue and Eigenvector

λ is an eigenvalue with corresponding eigenvector $|\phi\rangle$. T is an operator.

$$T|\phi\rangle = \lambda|\phi\rangle \tag{20}$$

If T is a Hermitian operator, then all eigenvalues of T are real ($\lambda \in \mathbb{R}$). Some of the eigenvalues can be repeated, but the corresponding eigenvectors can be chosen to be orthonormal. The eigenvectors corresponding to different eigenvalues are automatically orthogonal.

8 Trace of a matrix

Trace of a square matrix A is the sum of diagonals of A.

$$Tr(A) = \sum_i a_{ii} \tag{21}$$

How do we write $Tr(A)$ in Dirac notation?
Example:

$$\underbrace{\begin{pmatrix} 2 & 0 & 1 \\ 5 & 9 & 7 \\ -1 & 2 & 5+3i \end{pmatrix}}_{A}$$

standard basis for dimension 3

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{|b_1\rangle}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{|b_2\rangle}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{|b_3\rangle}$$

$$\begin{aligned}
Tr(A) &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
&= 2 + 9 + 5 + 3i
\end{aligned}$$

$Tr(A)$ in Dirac notation

$$Tr(A) = \sum_{i=1}^{n} \langle b_i | A | b_i \rangle \tag{22}$$

9 Normal Operator

An operator N is normal if

$$N^\dagger N = N N^\dagger \tag{23}$$

Is unitary operator normal?

$$\begin{aligned} UU^\dagger &= UU^{-1} \\ &= I \end{aligned} \qquad \begin{aligned} U^\dagger U &= U^{-1}U \\ &= I \end{aligned} \qquad \text{(from equation (14))}$$

Therefore, unitary operator is normal.
Is Hermitian operator normal?

$$\begin{aligned} TT^\dagger &= TT \\ &= T^2 \end{aligned} \qquad \begin{aligned} T^\dagger T &= TT \\ &= T^2 \end{aligned} \qquad \text{(from equation (18))}$$

Therefore, Hermitian operator is normal.

10 Projector Operator

A projector operator P is an operation that satisfies

$$P^2 = P \tag{24}$$

An operator P on a Hilbert space is an orthogonal projector if it satisfies

$$\begin{aligned} P^2 &= P \text{ , and} \\ P &= P^\dagger \end{aligned} \tag{25}$$

11 Spectral Theorem

Spectral theorem states the relationship of a matrix with it's eigenvalues and eigenvectors. What we are going to see works for Hermitian and unitary operators, in other words, works for more general class of matrices called normal operators. Spectral Theorem applies for normal operators.

Suppose T is a normal operator ($n \times n$ matrix).

T's eigenvalues:	T_1,	T_2,	... ,	T_n
T's corresponding eigenvectors:	$\|T_1\rangle$,	$\|T_2\rangle$,	... ,	$\|T_n\rangle$

Spectral Theorem states that

$$T = \sum_{i=1}^{n} T_i \left| T_i \right\rangle \left\langle T_i \right| \tag{26}$$

Example:

NOT gate, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{array}{lcc} & X_1 & X_2 \\ \text{eigenvalues of } X\text{:} & 1 & -1 \\ \text{corresponding eigenvectors of } X\text{:} & \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\ & \left|X_1\right\rangle & \left|X_2\right\rangle \end{array}$$

There are Hadamard basis $\left|+\right\rangle$ and $\left|-\right\rangle$.
Check if Spectral Theorem is satisfying:

$$\begin{aligned} X_1 \left|X_1\right\rangle \left\langle X_1\right| + X_2 \left|X_2\right\rangle \left\langle X_2\right| &= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix} = X \end{aligned}$$

12 Functions of an Operator

The correspondence between the Schrödinger picture and the Heisenberg picture in quantum mechanics boils down to the idea of taking functions of operators.
We know what a NOT gate X is. We also know what e is. $e^0 = 1, e^1 \approx 2.71828, ...$
What is e^X?
If the function $f(A)$ that is applied has a Taylor series expansion (which e^x has), we can calculate the value of $f(A)$.

For future reference

Note 1. We cannot duplicate in the quantum world (**No Cloning Theorem**).