CS506 Lecture Notes
Quantum Circuits and Linear Algebraic Formulation

Scribe: Victor Escudero

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1 Classical Circuits and Boolean Logic

Lets recall classical circuits, which are based on Boolean logic. Classical circuits operate on bits that take values in the set $\{0, 1\}$ and compute Boolean functions of the form

$$f : \{0, 1\}^n \to \{0, 1\}^m.$$

1.1 Elementary Boolean Gates

An *elementary gate* is a simple logic gate that performs a basic Boolean operation. The three fundamental Boolean gates are:

- **NOT gate**: Takes a single input x and outputs its negation $\neg x$.
- **AND gate**: Takes two inputs x and y and outputs $x \wedge y$.
- **OR gate**: Takes two inputs x and y and outputs $x \vee y$.

In addition to these gates, the **NAND gate** (NOT-AND) plays an important role in classical computation. The NAND gate outputs

$$\mathrm{NAND}(x, y) = \neg(x \wedge y).$$

A key theoretical result is that the NAND gate is *universal*, meaning that any Boolean circuit can be constructed using only NAND gates.

2 Quantum States and Linear Algebra

Quantum circuits differ fundamentally from classical circuits because they operate on *qubits* rather than classical bits.

2.1 Qubits as Vectors

A single qubit is represented as a vector in a two dimensional complex vector space. The computational basis states are the following two states of a *classical bit*, embedded into the quantum state space:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A qubit state is a linear combination of these basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

where $\alpha, \beta \in \mathbb{C}$ satisfy the normalization condition

$$|\alpha|^2 + |\beta|^2 = 1.$$

3 Quantum Gates as Operators

3.1 Unitary Operators

Quantum gates are represented mathematically by *unitary matrices*. A matrix U is unitary if

$$U^\dagger U = I,$$

$U^\dagger$ denotes the conjugate transpose of U. Unitarity guarantees that quantum evolution preserves probability.

3.2 The Quantum NOT Gate

The quantum analogue of the classical NOT gate is the **Pauli-X gate**, given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its action on the basis states is

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle.$$

Because quantum gates are linear operators, they act linearly on superpositions.

Example. Consider the qubit

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle.$$

Applying the NOT gate yields

$$X|\psi\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0\rangle.$$

The amplitudes are preserved but reassigned among the basis states.

4 Multi-Qubit Systems and Tensor Products

4.1 Tensor Product Representation

For multi qubit systems, joint states are described using the tensor product. The two qubit computational basis consists of

$$|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle.$$

If two qubits are in states $|\psi\rangle$ and $|\phi\rangle$, their combined state is

$$|\psi\rangle \otimes |\phi\rangle.$$

4.2 Product States and Entanglement

If a two qubit state can be written as a tensor product of two single qubit states, the qubits are not entangled. Otherwise, the state is entangled.

Example. The state

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

can be expanded and factored, and therefore represents a product state.

5 Controlled Operations and Entanglement

In quantum circuits, conditional logic is implemented using *controlled gates*. The most important example is the **Controlled NOT (CNOT)** gate.

The CNOT gate flips the target qubit if and only if the control qubit is in state $|1\rangle$. Its action on the computational basis states is

$$\begin{aligned} \mathrm{CNOT}|00\rangle &= |00\rangle, \\ \mathrm{CNOT}|01\rangle &= |01\rangle, \\ \mathrm{CNOT}|10\rangle &= |11\rangle, \\ \mathrm{CNOT}|11\rangle &= |10\rangle. \end{aligned}$$

Entanglement Example. Applying CNOT to the state

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes |0\rangle$$

produces

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

which cannot be factored and is therefore an entangled state.

6 Conclusion

Classical circuits compute Boolean functions using logic gates such as NOT, AND, OR, and NAND. NAND gates are universal. Quantum circuits represent states as vectors and gates as unitary operators acting linearly on those states. Multi qubit systems are described using tensor products, and entanglement arises when states cannot be factored. Controlled gates such as the CNOT gate enable conditional operations and play a central role in creating entanglement.