Lecture Notes: Quantum Computing
(27th August 2025)

At the start of the lecture, we discussed Dirac notation and the idea of an orthonormal basis in quantum computing. We then looked at how a quantum state can be represented. The state vector

$$|\psi\rangle = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

can also be written in terms of the computational basis states as

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$$

Next, we studied the action of the NOT gate. The NOT gate is represented by the operator

$$\text{NOT} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which flips the computational standard states:

$$\text{NOT}|0\rangle = |1\rangle \qquad \text{NOT}|1\rangle = |0\rangle$$

We then applied the NOT gate to the state $|\psi\rangle$.

$$\text{NOT}|\psi\rangle = \text{NOT}\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

By the distributive property, the NOT operator can be applied to each term separately:

$$\text{NOT}|\psi\rangle = \text{NOT}\left(\frac{1}{\sqrt{3}}|0\rangle\right) + \text{NOT}\left(\frac{\sqrt{2}}{\sqrt{3}}|1\rangle\right)$$

Since scalar values are not affected by operators, they can be taken out of each term:

$$\text{NOT}|\psi\rangle = \frac{1}{\sqrt{3}}\left(\text{NOT}|0\rangle\right) + \frac{\sqrt{2}}{\sqrt{3}}\left(\text{NOT}|1\rangle\right)$$

Finally, applying the action of the NOT gate on the standard vectors, we obtain

$$\text{NOT}|\psi\rangle = \frac{1}{\sqrt{3}}|1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0\rangle$$

Thus, the NOT gate swaps the magnitudes of $|0\rangle$ and $|1\rangle$ while keeping the state properly normalized.

The Controlled-NOT (CNOT) Gate

After studying the NOT gate, we moved on to the Controlled-NOT (CNOT) gate, which is one of the most important two-qubit gates in quantum computing. The CNOT gate acts on two qubits: a *control qubit* and a *target qubit*. Its action is such that the target qubit is flipped (i..., a NOT operation is applied) if and only if the control qubit is in the state $|1\rangle$. If the control qubit is $|0\rangle$, the target qubit remains unchanged.

Matrix Representation

The matrix representation of the CNOT gate in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is given by

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We can understand the structure of this matrix as follows: - The **top-left** 2×2 **block** is the identity matrix, meaning the states $|00\rangle$ and $|01\rangle$ remain unchanged. - The **bottom-right** 2×2 **block** is the reverse of the identity matrix, which is the NOT gate, meaning the states $|10\rangle$ and $|11\rangle$ are swapped. - All other entries are zeros, ensuring the matrix is unitary.

XOR Interpretation

The action of the CNOT gate can also be described in terms of classical binary variables. If c denotes the control bit and t the target bit, then the output is given by

$$(c, t) \mapsto (c,\, c \oplus t)$$

where $\oplus$ denotes the XOR operation. This means that the control bit c remains unchanged, and the target bit t is flipped if $c = 1$.

The truth table for this transformation is shown below:

c	t	Output
0	0	$(0, 0)$
0	1	$(0, 1)$
1	0	$(1, 1)$
1	1	$(1, 0)$

Columns from Matrix Transformations

We can now clearly see how each standard vector is transformed by the CNOT gate. Each of these transformations corresponds directly to one column of the CNOT matrix:

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle \xrightarrow{\text{CNOT}} \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle$$

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01\rangle \xrightarrow{\text{CNOT}} \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01\rangle$$

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10\rangle \xrightarrow{\text{CNOT}} \begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11\rangle$$

$$\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = |11\rangle \xrightarrow{\text{CNOT}} \begin{bmatrix}0\\0\\1\\0\end{bmatrix} = |10\rangle$$

Together, these four transformations form the four columns of the CNOT matrix.

Tensor Product

In quantum computing, multi-qubit systems are described using the tensor product of single-qubit states. If we have a qubit in state $|0\rangle$ and another in state $|1\rangle$, then their joint state is written as

$$|0\rangle \otimes |1\rangle$$

In general, if M_1 is an $m \times m$ matrix and M_2 is an $n \times n$ matrix, then

$$M_1 \otimes M_2 = \begin{bmatrix} a_{11}M_2 & a_{12}M_2 & \cdots & a_{1m}M_2 \\ a_{21}M_2 & a_{22}M_2 & \cdots & a_{2m}M_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}M_2 & a_{m2}M_2 & \cdots & a_{mm}M_2 \end{bmatrix}$$

Two-Qubit Systems

In a two-qubit system, we assign one qubit as the first bit and the other as the second bit. For example:

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

Normalization

Whenever we use matrices to represent qubits and their superpositions, square-root terms usually appear. This happens because every state vector must be normalized: the sum of the squared magnitudes of its coefficients must equal 1. For example, the state

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

is normalized since

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

CNOT and Entanglement

Let us consider the state

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |\psi_2\rangle = |0\rangle$$

The joint state is

$$|\psi_1\rangle \otimes |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Applying the CNOT gate gives

$$\text{CNOT}\left(\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)\right) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

This state is entangled: once processed through CNOT, the two qubits cannot be written as a tensor product of individual qubits.

The CNOT example demonstrates that two qubits can become inseparably linked. This phenomenon is called **quantum entanglement**. The EPR Paradox was mentioned during the discussion of quantum entanglement.

An entangled state cannot be factored into a product of single-qubit states. For instance:

There **do not** exist states $|\psi_1\rangle$ and $|\psi_2\rangle$ such that

$$\textbf{the entangled state}\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi_1\rangle \otimes |\psi_2\rangle$$

The $\sqrt{\textbf{NOT}}$ Gate

We are representing NOT gate in form of fractional power. The $\sqrt{\text{NOT}}$ gate is defined such that

$$\left(\sqrt{\text{NOT}}\right)^2 = \text{NOT}$$

A representation of $\sqrt{\text{NOT}}$ is

$$\sqrt{\text{NOT}} = \frac{1}{\sqrt{2i}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

Verification

We now check that squaring this matrix gives the NOT gate:

$$\left(\sqrt{\text{NOT}}\right)^2 = \left(\frac{1}{\sqrt{2i}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\right)\left(\frac{1}{\sqrt{2i}}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\right)$$

Factor out the scalar:

$$= \frac{1}{2i}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

Multiply the matrices:

$$= \frac{1}{2i}\begin{bmatrix} i \cdot i + 1 \cdot 1 & i \cdot 1 + 1 \cdot i \\ 1 \cdot i + i \cdot 1 & 1 \cdot 1 + i \cdot i \end{bmatrix}$$

Simplify each entry:

$$= \frac{1}{2i}\begin{bmatrix} -1 + 1 & i + i \\ i + i & 1 - 1 \end{bmatrix} = \frac{1}{2i}\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \text{NOT}$$

Thus, Applying $\sqrt{\text{NOT}}$ twice is equivalent to applying NOT once.

Reversibility of Quantum Gates

Every quantum gate must be reversible, meaning its action can be undone by applying its inverse operation. In contrast, classical gates like AND and OR are not reversible, but quantum gates always are. We will explore reversibility in more detail in the next lecture.