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1 The Quantum Fourier Transform (QFT)

The Quantum Fourier Transform (QFT) is the quantum analogue of the classical Discrete Fourier Transform (DFT). It is a linear transformation on quantum bits and is a key component in many quantum algorithms, including Shor's algorithm and the phase estimation algorithm.

1.1 Definition

The QFT on an orthonormal basis $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ is defined as a unitary operator that acts on a basis state $|x\rangle$ as follows:

$$\mathrm{QFT}_N|x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i \frac{xy}{N}} |y\rangle$$

For quantum computing, we typically work with $N = 2^n$ for an n-qubit register. The definition becomes:

$$\mathrm{QFT}_{2^n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i \frac{xy}{2^n}} |y\rangle$$

The inverse QFT is defined by negating the exponent in the complex exponential:

$$\mathrm{QFT}_{2^n}^{-1}|y\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{-2\pi i \frac{xy}{2^n}} |x\rangle$$

2 The Quantum Phase Estimation Problem

The goal of the phase estimation algorithm is to estimate the phase ω of an eigenvalue of a unitary operator U. Specifically, if U has an eigenvector $|\psi\rangle$ with a corresponding eigenvalue $e^{2\pi i\omega}$, where $\omega \in [0, 1)$, we want to find an accurate estimate of ω.

2.1 Initial State

The algorithm begins with a specific input state. For this analysis, we assume the first part of the algorithm has already been performed, and we are given the following n-qubit state:

$$|\Psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{2\pi i\omega y}|y\rangle$$

Our goal is to apply the inverse QFT to this state and measure it to find an n-bit approximation of ω.

3 Analysis of the Measurement Probability

We apply the inverse QFT to the state $|\Psi\rangle$:

$$\begin{aligned}
\mathrm{QFT}_{2^n}^{-1}|\Psi\rangle &= \mathrm{QFT}_{2^n}^{-1}\left(\frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1} e^{2\pi i\omega y}|y\rangle\right) \\
&= \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1} e^{2\pi i\omega y}\left(\mathrm{QFT}_{2^n}^{-1}|y\rangle\right) \\
&= \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1} e^{2\pi i\omega y}\left(\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{-2\pi i\frac{yk}{2^n}}|k\rangle\right) \\
&= \frac{1}{2^n}\sum_{k=0}^{2^n-1}\sum_{y=0}^{2^n-1} e^{2\pi i\omega y-2\pi i\frac{yk}{2^n}}|k\rangle \\
&= \frac{1}{2^n}\sum_{k=0}^{2^n-1}\left(\sum_{y=0}^{2^n-1} e^{2\pi i y\left(\omega-\frac{k}{2^n}\right)}\right)|k\rangle
\end{aligned}$$

The expression in the parenthesis is the amplitude α_k for measuring the basis state $|k\rangle$.

Let's write the phase ω as an n-bit binary fraction, with a potential error term. Let a be the integer that represents the best n-bit approximation of $2^n\omega$. We can write:

$$2^n\omega = a + \delta$$

where $a = \lfloor 2^n\omega \rfloor$ is an integer and $-1/2 < \delta \leq 1/2$ is the error term. Substituting $\omega = \frac{a+\delta}{2^n}$ into the amplitude calculation:

$$\alpha_k = \frac{1}{2^n}\sum_{y=0}^{2^n-1} e^{2\pi i y\left(\frac{a+\delta}{2^n}-\frac{k}{2^n}\right)} = \frac{1}{2^n}\sum_{y=0}^{2^n-1} e^{\frac{2\pi i y}{2^n}(a-k+\delta)}$$

The probability of measuring the state $|k\rangle$ is $P(k) = |\alpha_k|^2$.

3.1 Case 1: The Phase is Exactly Representable ($\delta = 0$)

If ω can be perfectly represented by n bits, then $2^n\omega = a$ for some integer a, and the error $\delta = 0$. In this case, we measure the state $|a\rangle$. The amplitude for measuring $|k = a\rangle$ is:

$$\alpha_a = \frac{1}{2^n}\sum_{y=0}^{2^n-1} e^{\frac{2\pi i y}{2^n}(a-a+0)} = \frac{1}{2^n}\sum_{y=0}^{2^n-1} 1 = \frac{2^n}{2^n} = 1$$

The probability of measuring $|a\rangle$ is $P(a) = |\alpha_a|^2 = 1$. The algorithm succeeds with certainty.

3.2 Case 2: The Phase is Not Exactly Representable ($\delta \neq 0$)

If ω cannot be perfectly represented by n bits, we will measure the best approximation, $a = \mathrm{round}(2^n\omega)$, with high probability. The amplitude for measuring $|a\rangle$ is:

$$\alpha_a = \frac{1}{2^n}\sum_{y=0}^{2^n-1} e^{\frac{2\pi i y\delta}{2^n}} = \frac{1}{2^n}\sum_{y=0}^{2^n-1}\left(e^{\frac{2\pi i\delta}{2^n}}\right)^y$$

This is a geometric series with ratio $r = e^{2\pi i\delta/2^n}$. The sum is:

$$\alpha_a = \frac{1}{2^n}\frac{1-r^{2^n}}{1-r} = \frac{1}{2^n}\frac{1-e^{2\pi i\delta}}{1-e^{2\pi i\delta/2^n}}$$

The probability of measuring $|a\rangle$ is $P(a) = |\alpha_a|^2$. We use the identity $|1-e^{i\theta}|^2 = 4\sin^2(\theta/2)$:

$$P(a) = \frac{1}{2^{2n}}\frac{|1-e^{2\pi i\delta}|^2}{|1-e^{2\pi i\delta/2^n}|^2} = \frac{1}{2^{2n}}\frac{4\sin^2(\pi\delta)}{4\sin^2(\pi\delta/2^n)} = \frac{\sin^2(\pi\delta)}{2^{2n}\sin^2(\pi\delta/2^n)}$$

To find a lower bound, we use the inequality $|\sin(x)| \leq |x|$ in the denominator and $|\sin(\pi\delta)| \geq 2|\delta|$ for $|\delta| \leq 1/2$ in the numerator.

$$P(a) \geq \frac{(2\delta)^2}{2^{2n}(\pi\delta/2^n)^2} = \frac{4\delta^2}{2^{2n}\frac{\pi^2\delta^2}{2^{2n}}} = \frac{4}{\pi^2} \approx 0.405$$

This shows that even when the phase is not exact, we have at least a 40.5% chance of measuring the best n-bit approximation.

4 Improving the Probability of Success

A success probability of $\sim 40\%$ may not be sufficient. As noted in Nielsen & Chuang, we can improve this probability by adding a number of extra qubits, p, to our register. If we want to estimate ω to n bits of precision with a success probability of at least $1-\epsilon$, we should use a total of $n+p$ qubits, where:

$$p = \left\lceil \log_2\left(2 + \frac{1}{2\epsilon}\right)\right\rceil$$