Detailed Explanations and Derivations from the Lecture

Conceptual Explanations

1. Measurement (Observation) Postulate of Quantum Mechanics

In quantum mechanics, measurement is described by an orthonormal basis $|b_i\rangle$. Any state can be written as

$$|\psi\rangle = \sum_i \alpha_i |b_i\rangle.$$

When a measurement in this basis is performed, the result is one of the basis states $|b_i\rangle$ with probability $p_i = |\alpha_i|^2$. After the measurement, the state collapses to the eigenstate corresponding to the observed outcome. The probabilities depend only on the squared magnitude of the amplitudes, not on any global phase factor.

2. Observation in the Hadamard Basis

Instead of the computational basis $|0\rangle, |1\rangle$, we can measure in the Hadamard basis $|+\rangle, |-\rangle$, where

$$|+\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

To calculate measurement probabilities in this basis, the state is re-expressed in terms of $|+\rangle$ and $|-\rangle$. The measurement outcome probabilities are then given by the squared modulus of the new coefficients.

3. Two-Qubit States and Partial Measurement

A two-qubit system uses the basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. We can measure both qubits together or measure only one qubit: * Measuring both gives the probability of each two-bit outcome directly. * Measuring only the first qubit involves summing the probabilities of all states sharing the same first-qubit value. After the first qubit is observed, the second qubit collapses to the conditional state.

4. Schrödinger's Wave Equation

Time evolution of a closed quantum system is governed by Schrödinger's equation:

$$i\frac{\hbar}{2\pi}\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

where $H(t)$ is the Hamiltonian operator (Hermitian). If the Hamiltonian is time-independent,

$$|\psi(t_2)\rangle = e^{-iH\frac{\hbar}{2\pi}(t_2 - t_1)}|\psi(t_1)\rangle.$$

This exponential of a Hermitian operator is unitary, which ensures that total probability is preserved.

5. Boolean Circuits and Quantum Circuits

Classical Boolean circuits compute functions of bits using logic gates such as AND, OR, NOT. Quantum circuits generalize this idea but all operations must be unitary (and therefore reversible). Classical logic gates are depicted to motivate the idea of reversible quantum gates.

6. No-Cloning Theorem

Quantum information cannot be copied perfectly. There is no unitary operator U such that

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$$

for every state $|\psi\rangle$. For two states $|\psi\rangle$ and $|\phi\rangle$, cloning would require

$$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2,$$

which holds only if $\langle\phi|\psi\rangle = 0$ or 1, i.e. for orthogonal or identical states. Thus universal cloning is impossible.

Detailed Derivations from the Lecture

Measurement in Computational Basis

Example state:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle.$$

* Probability of outcome $|0\rangle$: $(1/\sqrt{3})^2 = 1/3$. * Probability of outcome $|1\rangle$: $(\sqrt{2}/\sqrt{3})^2 = 2/3$. Adding a global phase factor $e^{i\theta}$ does not change these probabilities.

Measurement in Hadamard Basis

- Rewrite computational basis states:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle),\ |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

- For $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$, substitute and collect coefficients:

$$p_+ = \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right)^2, \quad p_- = \left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\right)^2.$$

Complex Amplitudes Example

For $|\psi\rangle = (\frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{6}})|0\rangle + (\frac{1}{2} + i\frac{1}{2})|1\rangle$: * $|0\rangle$ probability: $(1/\sqrt{3})^2 + (1/\sqrt{6})^2 = 1/2$. * $|1\rangle$ probability: $(1/2)^2 + (1/2)^2 = 1/2$.

Two-Qubit State and Partial Measurement

Example:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}|10\rangle.$$

* Measuring both qubits: * $|00\rangle$: 1/3, * $|10\rangle$: 2/3. * Measuring only the first qubit: * First qubit 0: 1/3, * First qubit 1: 2/3. * Conditional second qubit state when first qubit = 1:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

giving second-qubit outcomes each with probability 1/2.

More complex example:

$$|\psi\rangle = \frac{1}{\sqrt{11}}|00\rangle + \frac{\sqrt{5}}{\sqrt{11}}|01\rangle + \frac{\sqrt{2}}{\sqrt{11}}|10\rangle + \frac{\sqrt{3}}{\sqrt{11}}|11\rangle.$$

* Probabilities: * $|00\rangle$: 1/11, * $|01\rangle$: 5/11, * $|10\rangle$: 2/11, * $|11\rangle$: 3/11.

Schrödinger Equation

Starting from

$$i\frac{\hbar}{2\pi}\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

if H is time independent, integrate to find

$$|\psi(t_2)\rangle = e^{-iH\frac{\hbar}{2\pi}(t_2-t_1)}|\psi(t_1)\rangle.$$

This unitary evolution guarantees conservation of probability.

Boolean Circuit Diagram

Classical circuit drawn with inputs x1–x4 and gates AND, OR, NOT shows how classical logic is built from gates. This motivates the design of quantum circuits, which require reversible gates and unitary operations.

No-Cloning Theorem Proof

Assume there is a unitary operator U that clones states:

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle,\ U|\phi\rangle|0\rangle = |\phi\rangle|\phi\rangle.$$

Taking inner products gives

$$\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2.$$

This equation holds only when $\langle\phi|\psi\rangle = 0$ or 1. Therefore cloning is possible only when the states are orthogonal or identical, proving universal cloning is impossible.