CS 506: Introduction to Quantum Computing

Lecture Notes — Deutsch–Jozsa Algorithm

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1 Problem Statement and Goal

We are given oracle access to a Boolean function

$$f : \{0,1\}^n \to \{0,1\}.$$

It is promised that f is either:

- **Constant:** $f(\mathbf{x}) = c$ for all $\mathbf{x} \in \{0,1\}^n$ and some $c \in \{0,1\}$.
- **Balanced:** $f(\mathbf{x}) = 0$ for exactly half the inputs and $f(\mathbf{x}) = 1$ for the rest.

Goal: Determine whether f is constant or balanced with as few oracle queries as possible.

Classical Baseline

A deterministic classical algorithm may require up to $1 + 2^{n-1}$ queries in the worst case. The Deutsch–Jozsa algorithm solves the same problem with **only one quantum query**.

2 Oracle Model and Notation

The oracle (unitary) for f acts as:

$$U_f \left|\mathbf{x}\right\rangle \left|y\right\rangle = \left|\mathbf{x}\right\rangle \left|y \oplus f(\mathbf{x})\right\rangle.$$

Here $\oplus$ denotes bitwise XOR. For $\mathbf{x}, \mathbf{z} \in \{0,1\}^n$,

$$\langle \mathbf{x} | \mathbf{z} \rangle = \sum_{i=1}^{n} x_i z_i \pmod 2$$

represents the mod-2 dot product.

Dirac Notation Recap

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}, \quad |-\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

3 Warm-up ($n = 1$): Deutsch's Problem

3.1 Objective

We first study the simplest case where $f : \{0,1\} \to \{0,1\}$. The function may be one of four possibilities:

$$f_1(x) = 0, \quad f_2(x) = 1, \quad f_3(x) = x, \quad f_4(x) = 1 - x.$$

Functions f_1, f_2 are constant; f_3, f_4 are balanced.

3.2 Circuit Overview

3.3 Step-by-Step Derivation

At the start:

$$|\psi_0\rangle = |0\rangle |1\rangle .$$

After Hadamards:

$$|\psi_1\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle).$$

After the oracle:

$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle .$$

Since the second qubit is $|-\rangle$, phase kickback gives:

$$U_f(|x\rangle |-\rangle) = (-1)^{f(x)} |x\rangle |-\rangle .$$

Therefore,

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left[(-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle\right] |-\rangle .$$

Finally, applying H on the first qubit:

$$H |\psi_2\rangle = \frac{1}{2} \left[(-1)^{f(0)} + (-1)^{f(1)}\right] |0\rangle + \frac{1}{2} \left[(-1)^{f(0)} - (-1)^{f(1)}\right] |1\rangle .$$

- If f is constant, amplitude of $|0\rangle = \pm 1$, so measurement yields 0.
- If f is balanced, amplitudes cancel and measurement yields 1.

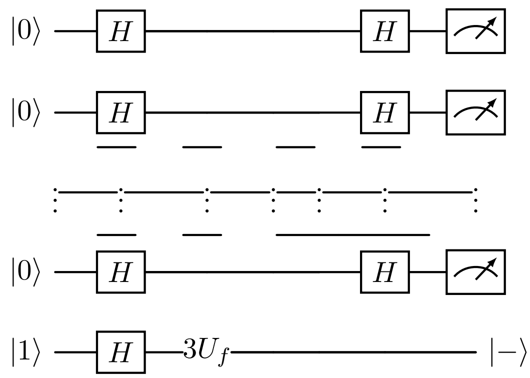
Thus, one query distinguishes constant from balanced perfectly.

4 The Deutsch–Jozsa Algorithm (n Qubits)

4.1 Timeline of the Algorithm

Stage	**Symbol**	**Description**
Initialization	t_0	Prepare $|0\rangle^{\otimes n}|1\rangle$
After Hadamards	t_1	Create uniform superposition and ancilla in $|-\rangle$
After Oracle	t_2	Apply U_f to imprint phase $(-1)^{f(x)}$
After Final Hadamards	t_3	Interfere amplitudes; measure first register

4.2 Circuit Diagram



4.3 Stage-by-Stage Evolution

At t_0: Initialization

$$|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle .$$

At t_1: After Hadamards

$$|\psi_1\rangle = \Big(\frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle\Big) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} |\mathbf{x}\rangle |-\rangle .$$

Each qubit is now in equal superposition of $|0\rangle$ and $|1\rangle$.

At t_2: Oracle Application and Phase Kickback

$$U_f(|\mathbf{x}\rangle |-\rangle) = (-1)^{f(\mathbf{x})} |\mathbf{x}\rangle |-\rangle .$$

Therefore,

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})} |\mathbf{x}\rangle |-\rangle .$$

At t_3: Apply Hadamard to Each Input Qubit Using the identity

$$H^{\otimes n} |\mathbf{x}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in\{0,1\}^n} (-1)^{\langle \mathbf{x}|\mathbf{z}\rangle} |\mathbf{z}\rangle ,$$

we have

$$|\psi_3\rangle = \frac{1}{2^n} \sum_{\mathbf{z}\in\{0,1\}^n} \Big(\sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})+\langle \mathbf{x}|\mathbf{z}\rangle} \Big) |\mathbf{z}\rangle |-\rangle .$$

The amplitude of each basis state $|\mathbf{z}\rangle$ is:

$$\alpha(\mathbf{z}) = \frac{1}{2^n} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})+\langle \mathbf{x}|\mathbf{z}\rangle}.$$

4.4 Measurement and Analysis

We measure the first n-qubit register.

For $\mathbf{z} = \mathbf{0}^n$,

$$\alpha(\mathbf{0}^n) = \frac{1}{2^n} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})} = \begin{cases} \pm 1, & \text{if } f \text{ is constant,} \\ 0, & \text{if } f \text{ is balanced.} \end{cases}$$

$$P(\mathbf{0}^n) = |\alpha(\mathbf{0}^n)|^2 = \begin{cases} 1, & f \text{ constant,} \\ 0, & f \text{ balanced.} \end{cases}$$

Interpretation of Interference

In the constant case, all phases are identical and constructively interfere at $|\mathbf{0}^n\rangle$. In the balanced case, half of the amplitudes are $+1$ and half are -1, cancelling perfectly.

5 Expanded Example ($n = 2$)

Let $f(x_1x_2) = x_1 \oplus x_2$ (balanced).

Step 1: Initial Superposition

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)\,|-\rangle\,.$$

Step 2: Apply Oracle (phase)

$$|\psi_2\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|-\rangle\,.$$

Step 3: Apply $H^{\otimes 2}$ Compute each term's contribution:

$$\begin{aligned} H^{\otimes 2}|00\rangle &= \tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \\ H^{\otimes 2}|01\rangle &= \tfrac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle), \\ H^{\otimes 2}|10\rangle &= \tfrac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle), \\ H^{\otimes 2}|11\rangle &= \tfrac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle). \end{aligned}$$

Combining them shows that the $|00\rangle$ term cancels completely:

$$\Rightarrow \text{no amplitude at } |00\rangle \text{ (balanced case).}$$

Result: Measurement never yields 00; probability $= 0$, confirming f is balanced.

6 Detailed Identities and Notes

Single-qubit Hadamards.

$$H|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Tensor Product Identity. For $\mathbf{x}, \mathbf{z} \in \{0,1\}^n$,

$$H^{\otimes n} |\mathbf{x}\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{z}\in\{0,1\}^n} (-1)^{\langle \mathbf{x}|\mathbf{z}\rangle} |\mathbf{z}\rangle .$$

Phase Kickback Reminder. Since $|-\rangle$ is an eigenstate of X with eigenvalue -1:

$$U_f(|\mathbf{x}\rangle |-\rangle) = (-1)^{f(\mathbf{x})} |\mathbf{x}\rangle |-\rangle .$$

7 Summary

- The algorithm encodes function values as relative phases.
- Hadamard gates create and later recombine superpositions.
- Constructive interference occurs only for the all-zero state in the constant case.
- A single query distinguishes the two promised categories with certainty.

Final Insight

The Deutsch–Jozsa algorithm was the first to demonstrate an exponential separation between classical and quantum query complexity, setting the stage for later algorithms such as Simon's and Shor's.

Notation: $\oplus$ denotes bitwise exclusive OR. Inner products are taken mod 2.