CS 506 Lecture Notes

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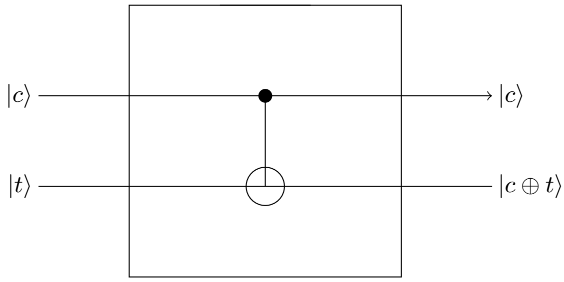
1 Phase Kickback

Consider a C-NOT gate:



When $|c\rangle = |0\rangle$ and $|t\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$, we have

$$\begin{aligned}
& \text{C-NOT}\left(|0\rangle\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
= \; & \text{C-NOT}\left(\frac{|0\rangle|0\rangle}{\sqrt{2}}\right) - \text{C-NOT}\left(\frac{|0\rangle|1\rangle}{\sqrt{2}}\right) \\
= \; & \frac{|0\rangle|0\rangle}{\sqrt{2}} - \frac{|0\rangle|1\rangle}{\sqrt{2}} \\
= \; & |0\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}}
\end{aligned}$$

where the C-NOT gate acts like I.

When $|c\rangle = |1\rangle$ and $|t\rangle = \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$, we have

$$\begin{aligned}
& \text{C-NOT}\left(|1\rangle\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
= \; & \text{C-NOT}\left(\frac{|1\rangle|0\rangle}{\sqrt{2}}\right) - \text{C-NOT}\left(\frac{|1\rangle|1\rangle}{\sqrt{2}}\right) \\
= \; & \frac{|1\rangle|1\rangle}{\sqrt{2}} - \frac{|1\rangle|0\rangle}{\sqrt{2}} \\
= \; & -|1\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}}
\end{aligned}$$

In general, for $b \in \{0, 1\}$,

$$\begin{aligned}
& \text{C-NOT}\left(|b\rangle\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
= \; & (-1)^b|b\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad \text{where} (-1)^b \text{ is the phase.} \\
& \text{C-NOT}\left((\alpha_0|0\rangle + \alpha_1|1\rangle)\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
= \; & (\alpha_0|0\rangle - \alpha_1|1\rangle)\frac{|0\rangle - |1\rangle}{\sqrt{2}}
\end{aligned}$$

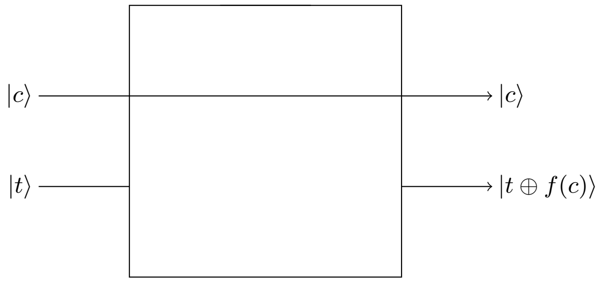
So that the eigenvalues of C-NOT are $(-1)^b = \pm 1$ with respect to the same eigenvector

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

and the eigenvalues are *kicked back* from the target register to the control register.

Scribe's Note I felt very confused about how the control qubit $|c\rangle$ can be regarded as $-|c\rangle$ as well. After reviewing the Measurement Postulate, I learned that we cannot physically distinguish between them, as they only differ by a phase.

Next, consider a general Control-f gate (u_f-gate):



where $f : \{0,1\} \to \{0,1\}$, and may be inreversible. Then

$$u_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

What are the eigenvalues and eigenvectors of u_f?

$$\begin{aligned}
& u_f|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
= \; & \frac{1}{\sqrt{2}}u_f(|0\rangle|x\rangle) - \frac{1}{\sqrt{2}}u_f(|1\rangle|x\rangle) \\
= \; & \frac{1}{\sqrt{2}}|x\rangle|0 \oplus f(x)\rangle - \frac{1}{\sqrt{2}}|x\rangle|1 \oplus f(x)\rangle \\
= \; & |x\rangle\left(\frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}}\right)
\end{aligned}$$

Case 1: $f(x) = 0$

$$u_f|x\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}} = |x\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Case 1: $f(x) = 1$

$$u_f|x\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}} = -|x\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

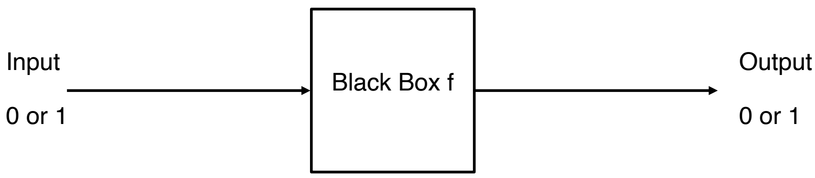
Hence there is a eigenvalues of u_f, $(-1)^{f(x)}$, with eigenvector $|x\rangle\frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

In general

$$u_f(\alpha_0|0\rangle + \alpha_1|1\rangle)\frac{|0\rangle - |1\rangle}{\sqrt{2}} = \left((-1)^{f(0)}\alpha_0|0\rangle + (-1)^{f(1)}\alpha_1|1\rangle\right)\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

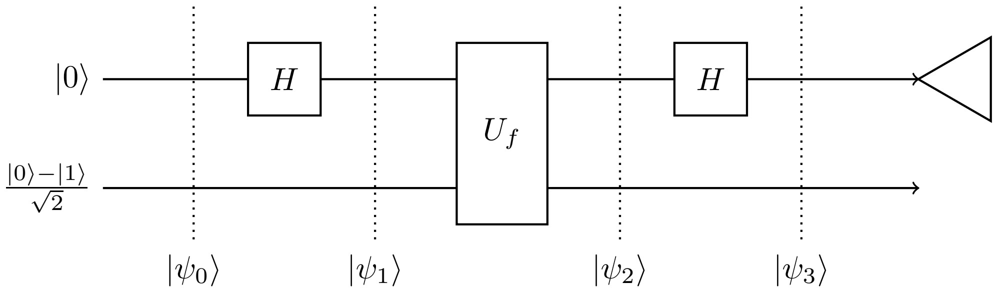
2 Deutsch Problem

Given a black box



where $f:\{0,1\}\to\{0,1\}$, how can we decide if f is constant or balanced?
The classical way needs to use 2 queries, $f(0)$ and $f(1)$:
If $f(0)=f(1)$ then f is constant;
If $f(0)\neq f(1)$ then f is balanced.
However, it only requries only 1 query in quantum computing.

Consider this circuit:



where we U_f is a Control-f (u_f-gate) defined previously.

$$u_f|x\rangle\,|y\rangle := |x\rangle\,|y\oplus f(x)\rangle\,.$$

Then we have:

$$\begin{aligned}
|\psi_0\rangle &= |0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
|\psi_1\rangle &= (H|0\rangle)\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \frac{|0\rangle+|1\rangle}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \frac{1}{\sqrt{2}}|0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}+\frac{1}{\sqrt{2}}|1\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
|\psi_2\rangle &= u_f\,|\psi_1\rangle\\
&= u_f\left(\frac{1}{\sqrt{2}}|0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}+\frac{1}{\sqrt{2}}|1\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\\
&= u_f\left(\frac{1}{\sqrt{2}}|0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)+u_f\left(\frac{1}{\sqrt{2}}|1\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\\
&= \frac{1}{\sqrt{2}}(-1)^{f(0)}|0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}+\frac{1}{\sqrt{2}}(-1)^{f(1)}|1\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= \frac{(-1)^{f(0)}|0\rangle+(-1)^{f(1)}|1\rangle}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= (-1)^{f(0)}\frac{|0\rangle+(-1)^{f(1)-f(0)}|1\rangle}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}}\\
&= (-1)^{f(0)}\frac{|0\rangle+(-1)^{f(0)\oplus f(1)}|1\rangle}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}}
\end{aligned}$$

- Case $f(0)\oplus f(1)=0$:

$$|\psi_2\rangle=(-1)^{f(0)}\frac{|0\rangle+|1\rangle}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}}.$$

Since $H\dfrac{|0\rangle+|1\rangle}{\sqrt{2}}=|0\rangle$

$$|\psi_3\rangle=(-1)^{f(0)}|0\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}.$$

Therefore observing the first qubit of $|\psi_3\rangle$ will always result 0.

- Case $f(0)\oplus f(1)=1$:

$$|\psi_2\rangle=(-1)^{f(0)}\frac{|0\rangle-|1\rangle}{\sqrt{2}}\frac{|0\rangle-|1\rangle}{\sqrt{2}}.$$

Since $H\dfrac{|0\rangle+|1\rangle}{\sqrt{2}}=|1\rangle$

$$|\psi_3\rangle=(-1)^{f(0)}|1\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}.$$

Therefore, observing the first qubit of $|\psi_3\rangle$ will always result in 1.

Hence, the circuit we built above can tell if f is constant or balanced by querying f only once.

Scribe's Note This quantum solution is not appealing to me at all, since we are querying one qubit to f instead of two classic bits. I cannot tell if it has any better performance. However, it may lead to a significant improvement in a more general problem in the next section.

3 Deutsch-Jozsa Problem

Consider a general version of the Deutsch Problem:

Given a function $f:\{0,1\}^n\to\{0,1\}$, we are granted that f is either constant or balanced, tell which case it actually is.

For classic computation, we need to query f at least $2^{n-1}+1$ times. But there is a way to query it only once using quantum computation, which will be introduced next class.