# Pushdown Automata
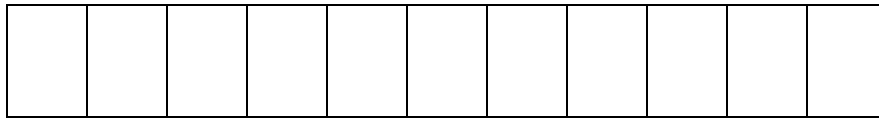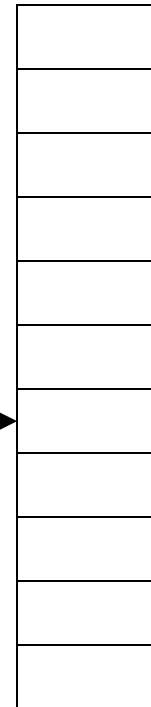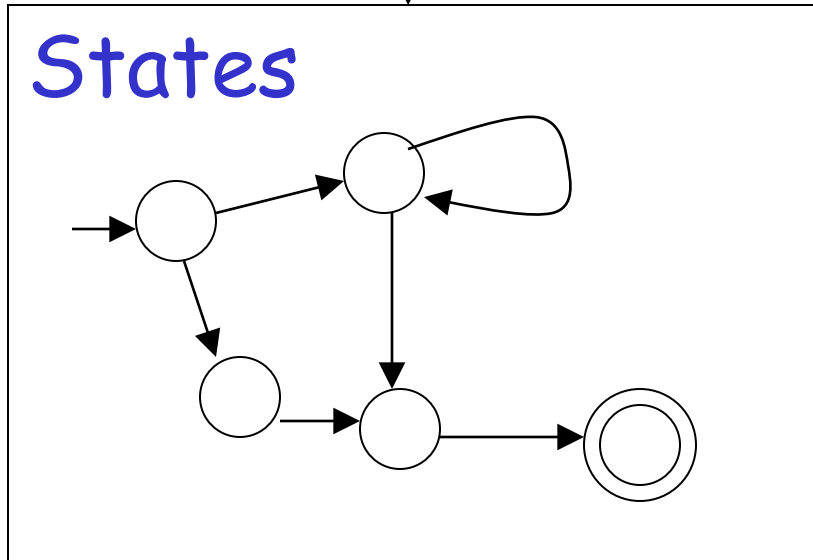# PDAs

# Pushdown Automaton -- PDA

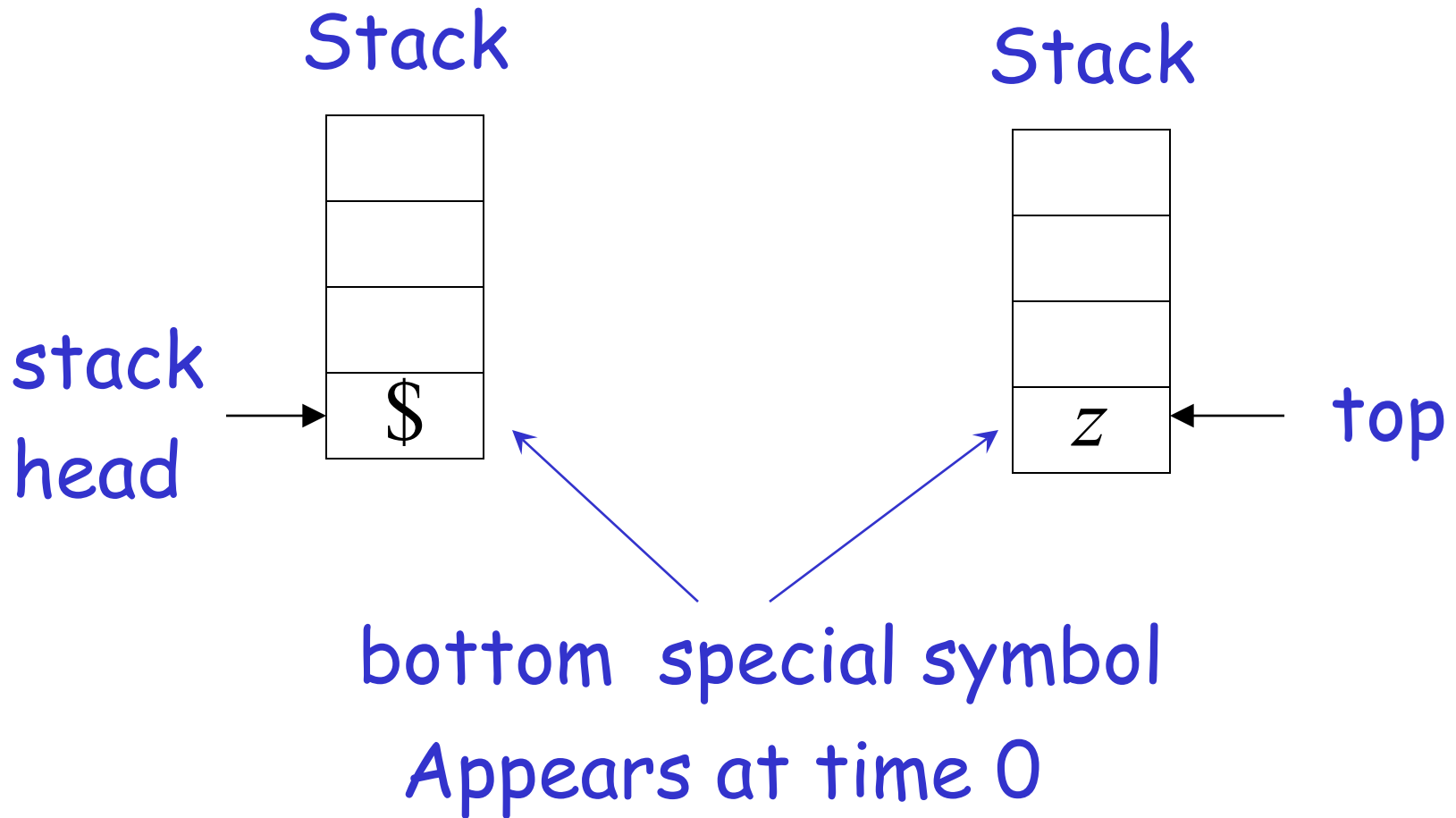Input String

Stack
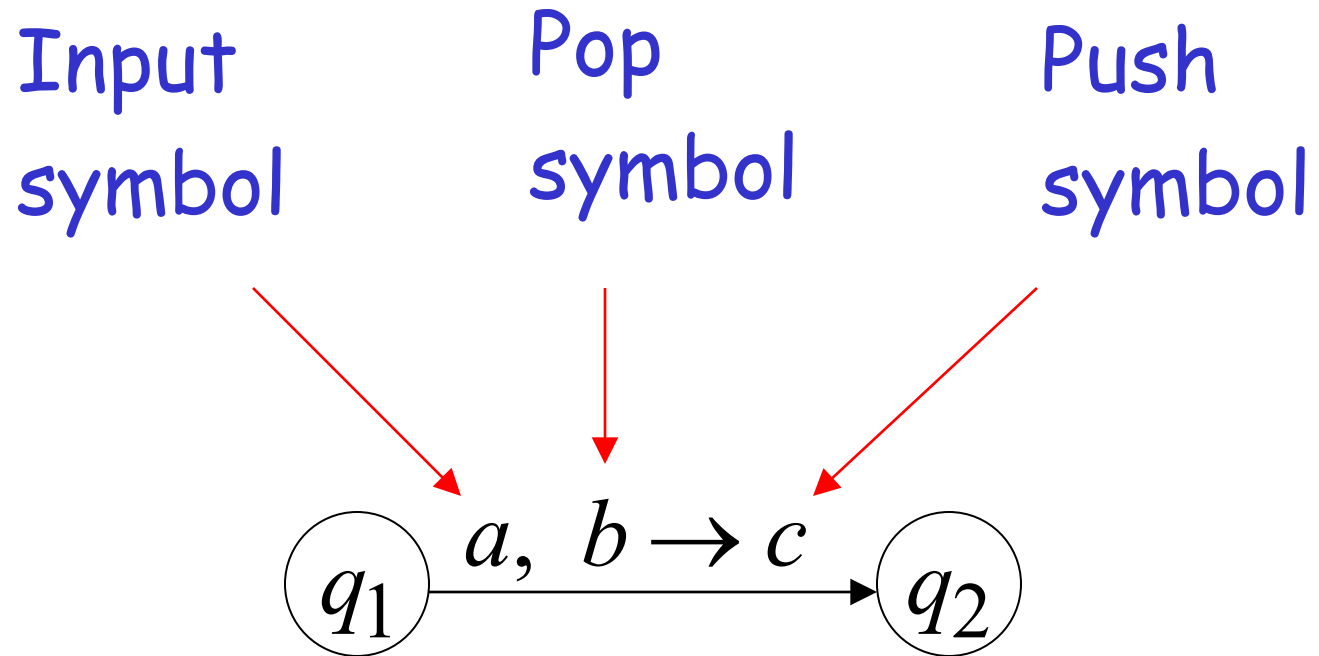
States

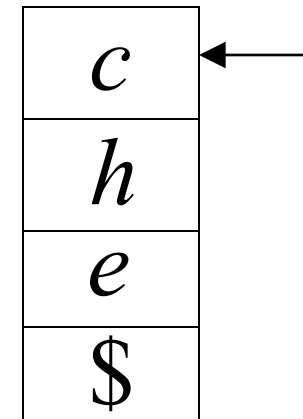# Initial Stack Symbol

Stack

Stack

stack
head → $\$$

$z$ ← top

bottom special symbol
Appears at time 0

# The States

Input symbol

Pop symbol

Push symbol

$$q_1 \quad a, \ b \rightarrow c \quad q_2$$

$q_1 \quad a, \ b \to c \quad q_2$

input

... | $a$ | ...

... | $a$ | ...

stack

$b$ ← top
$h$
$e$
$\$$

Replace

$c$ ←
$h$
$e$
$\$$

$q_1$  $a, \varepsilon \rightarrow c$  $q_2$

input

... $a$ ...

... $a$ ...

stack

$b$ ← top

$h$

$e$

$\$$

Push

$c$ ←

$b$

$h$

$e$

$\$$

$q_1$ $\quad a, \ b \to \varepsilon \quad$ $q_2$

input

| ... | $a$ | ... |

| ... | $a$ | ... |

stack

| $b$ | ← top
| $h$ |
| $e$ |
| $\$$ |

Pop

| $h$ | ←
| $e$ |
| $\$$ |

$q_1 \quad a,\ \varepsilon \to \varepsilon \quad q_2$

input

$\cdots \quad a \quad \cdots$

$\cdots \quad a \quad \cdots$

stack

| $b$ | ← top |
| $h$ | |
| $e$ | |
| $\$$ | |

No Change
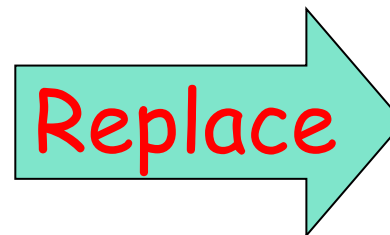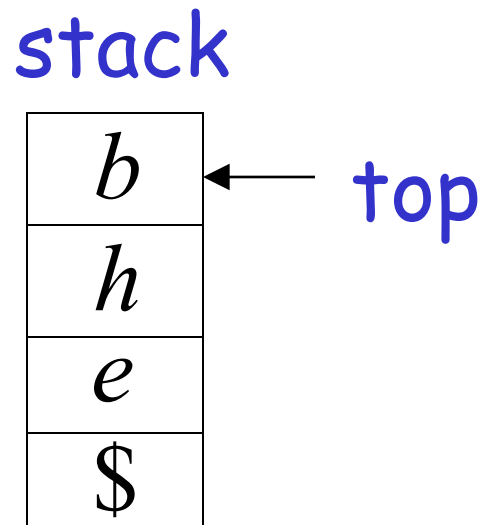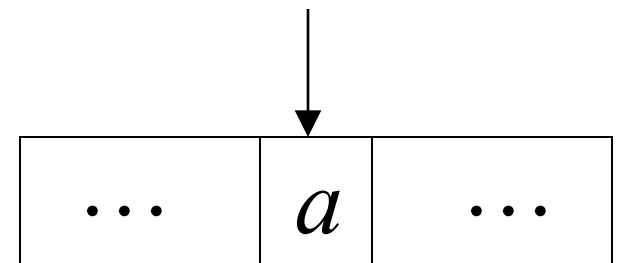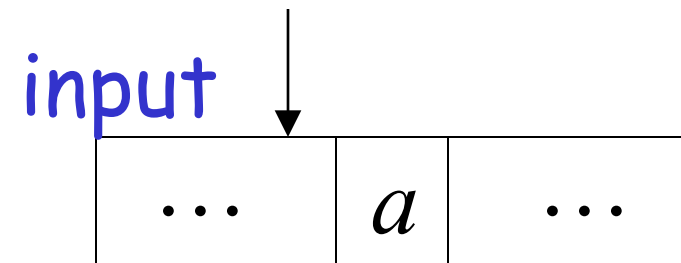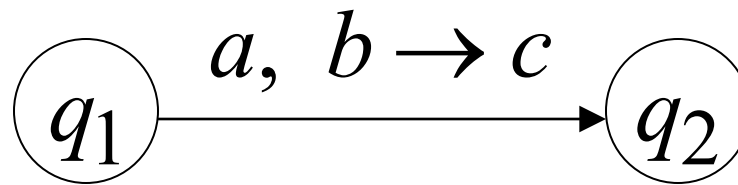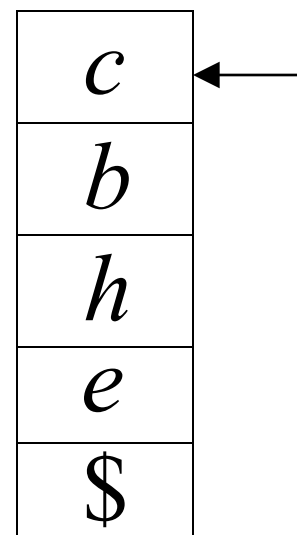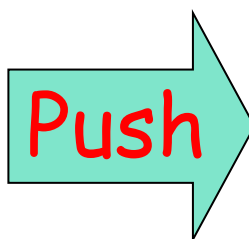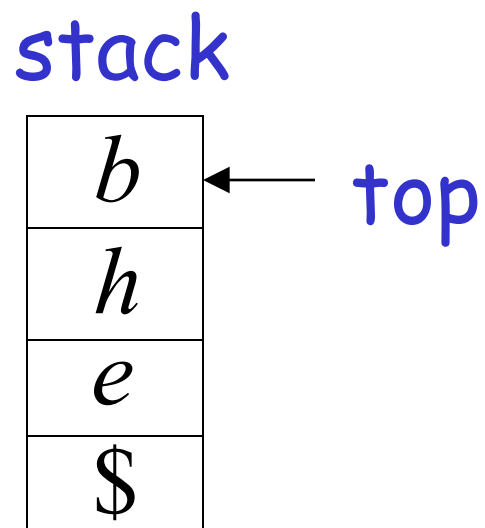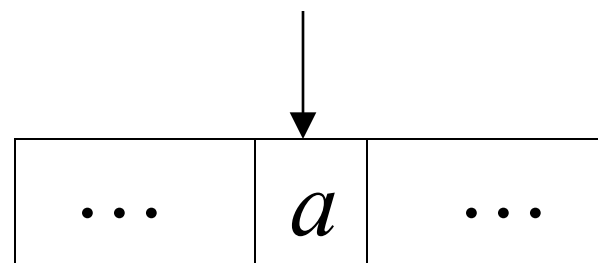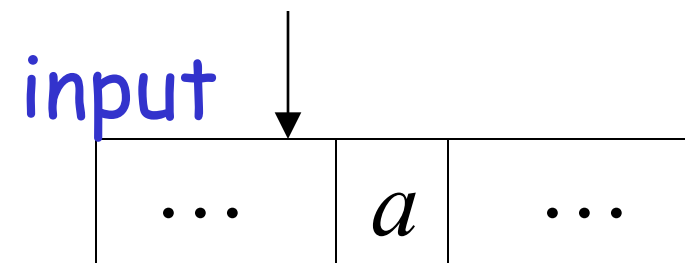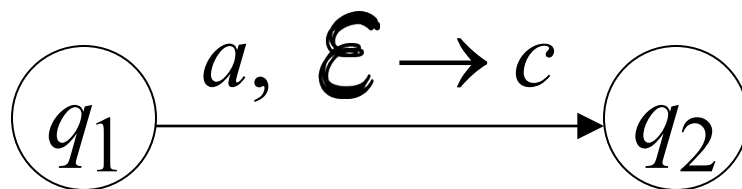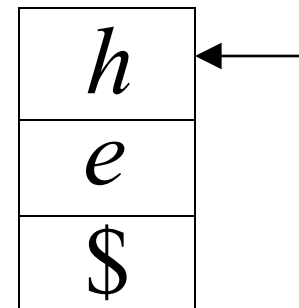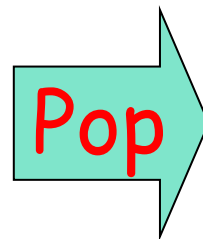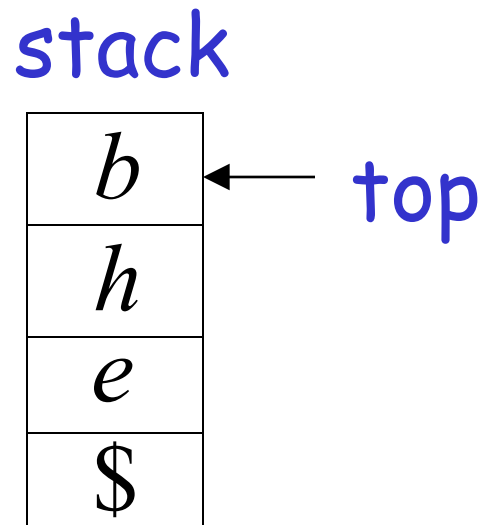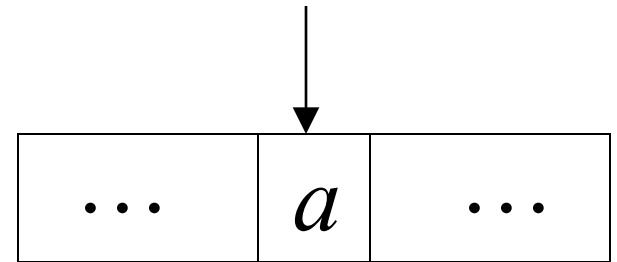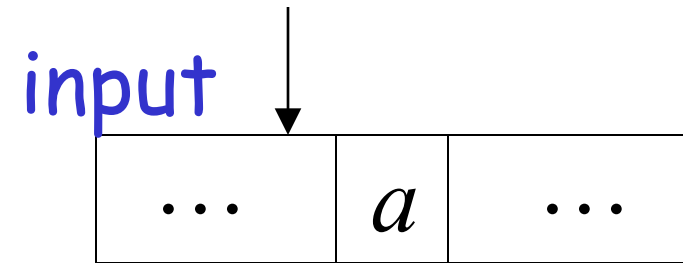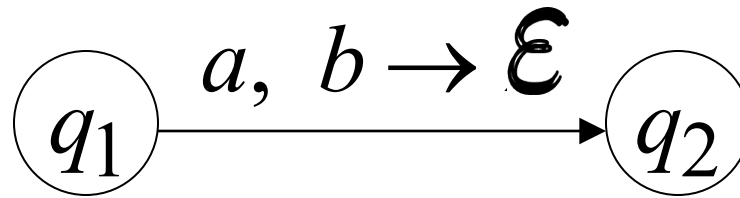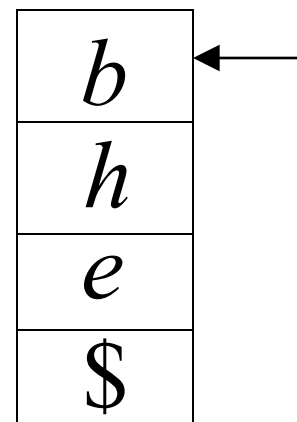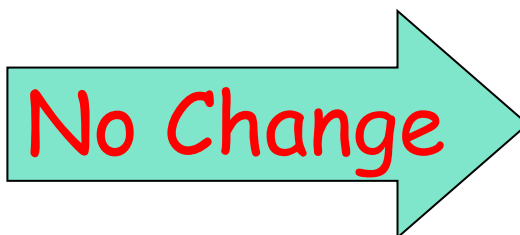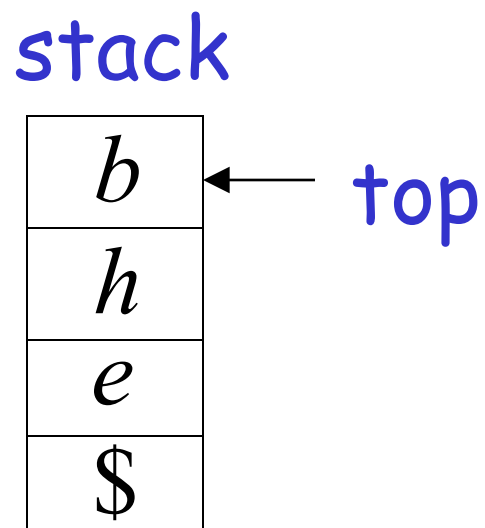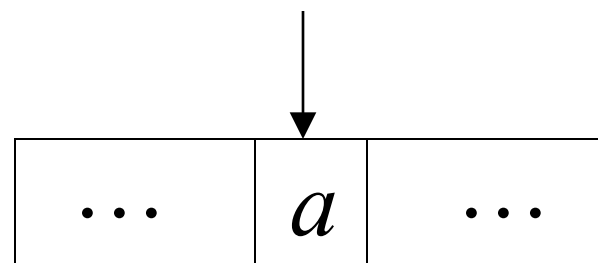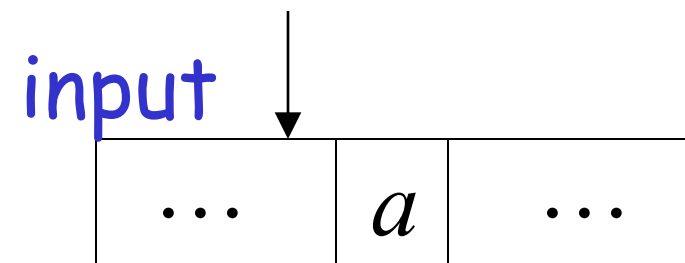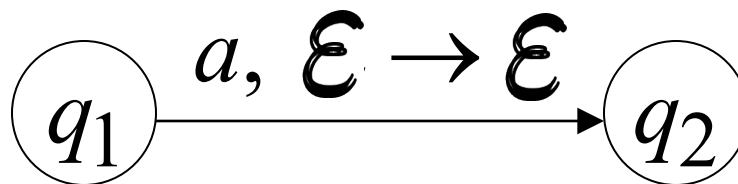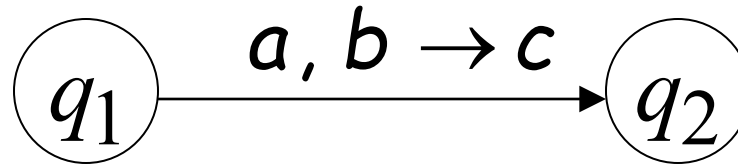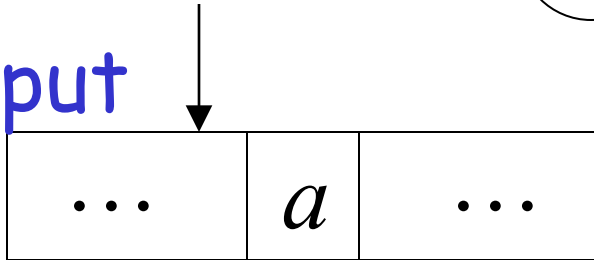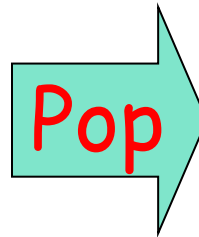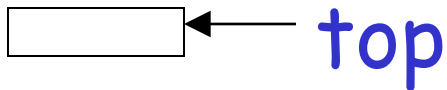
| $b$ | ← |
| $h$ | |
| $e$ | |
| $\$$ | |

# Pop from Empty Stack

$$q_1 \xrightarrow{\quad a\,,\,b \rightarrow c \quad} q_2$$

input

$$\boxed{\cdots \mid a \mid \cdots}$$

stack

**Pop** ⟹ Automaton halts!

$$\boxed{\phantom{xx}} \longleftarrow \text{top}$$

If the automaton attempts to pop from empty stack then it halts and rejects input

# Non-Determinism

## PDAs are non-deterministic

### Allowed non-deterministic transitions

$a, \ b \rightarrow c$ $\quad q_2$

$q_1$

$a, \ b \rightarrow c$ $\quad q_3$

$q_1 \quad \varepsilon, \ b \rightarrow c \quad q_2$

$\varepsilon -$ transition

# Example PDA

PDA $M$: $\qquad L(M) = \{a^n b^n : n \geq 0\}$

$$a, \varepsilon \rightarrow a \qquad\qquad b, a \rightarrow \varepsilon$$



$q_0 \xrightarrow{\varepsilon, \varepsilon \rightarrow \varepsilon} q_1 \xrightarrow{b, a \rightarrow \varepsilon} q_2 \xrightarrow{\varepsilon, \$ \rightarrow \$} q_3$

$$L(M) = \{a^n b^n : n \geq 0\}$$

Basic Idea:

1. Push the a's on the stack

2. Match the b's on input with a's on stack

3. Match found

$$a, \varepsilon \rightarrow a \qquad b, a \rightarrow \varepsilon$$

$q_0$ $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$ $q_1$ $\quad b, a \rightarrow \varepsilon \quad$ $q_2$ $\quad \varepsilon, \$ \rightarrow \$ \quad$ $q_3$

# Execution Example: Time 0

**Input**

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|---|---|

$$\boxed{\$}$$ ←

**Stack**

**current state**

$$a, \varepsilon \to a \qquad\qquad b, a \to \varepsilon$$

$$\to \underset{q_0}{\bigcirc} \quad \varepsilon, \varepsilon \to \varepsilon \quad \underset{q_1}{\bigcirc} \quad b, a \to \varepsilon \quad \underset{q_2}{\bigcirc} \quad \varepsilon, \$ \to \$ \quad \underset{q_3}{\bigcirc}$$

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Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

$\$$
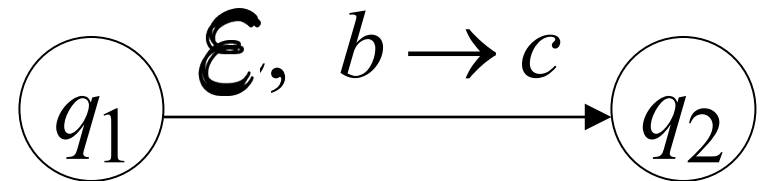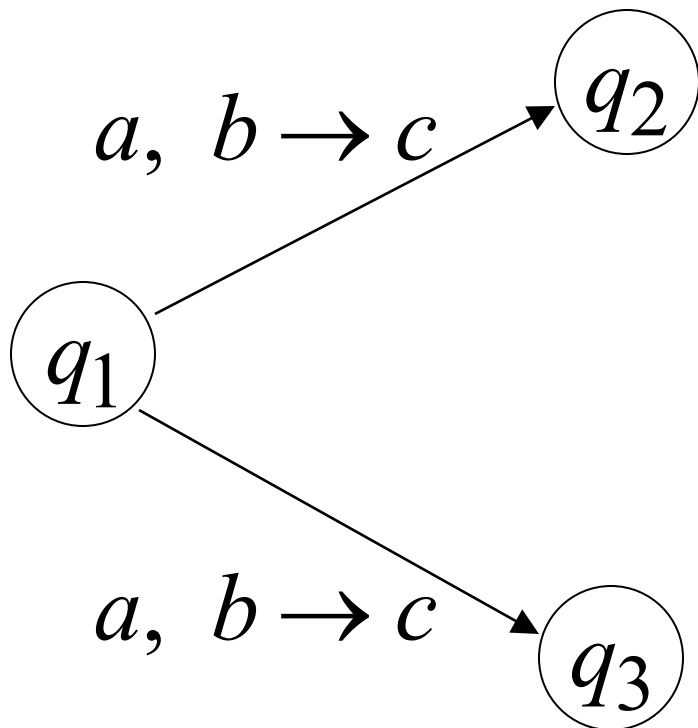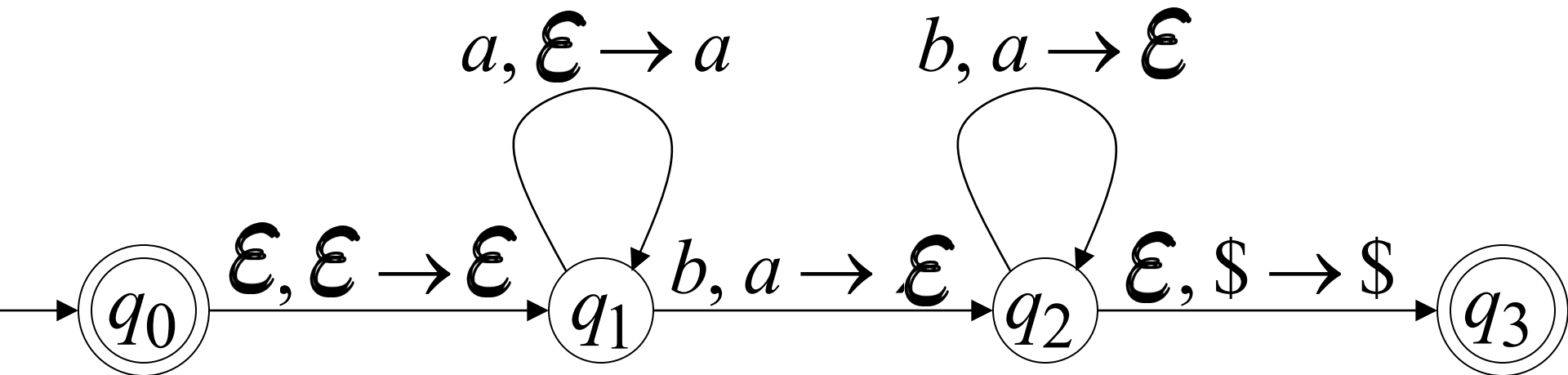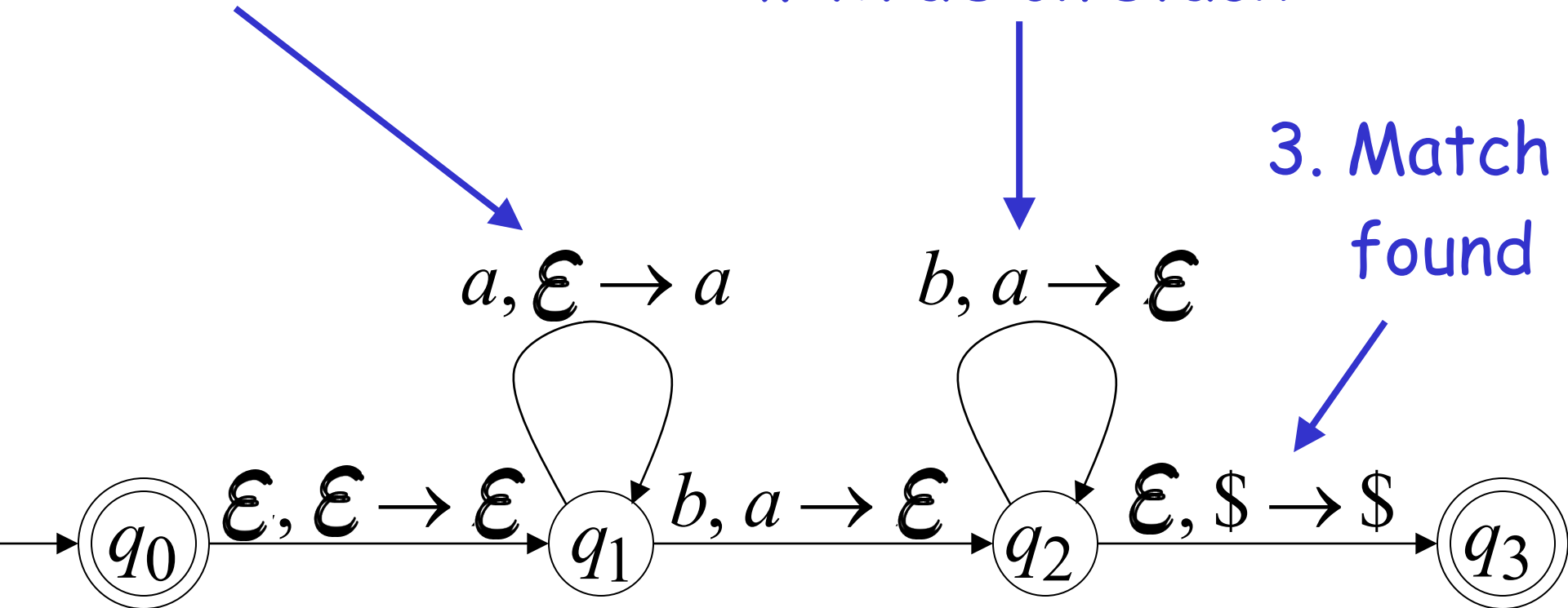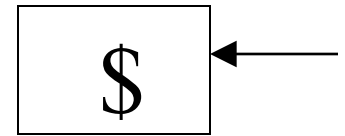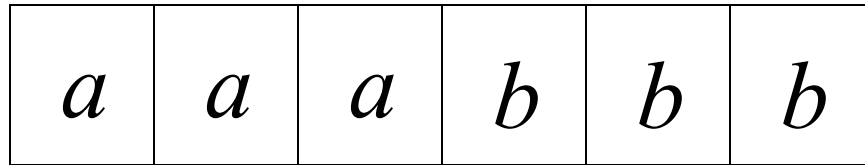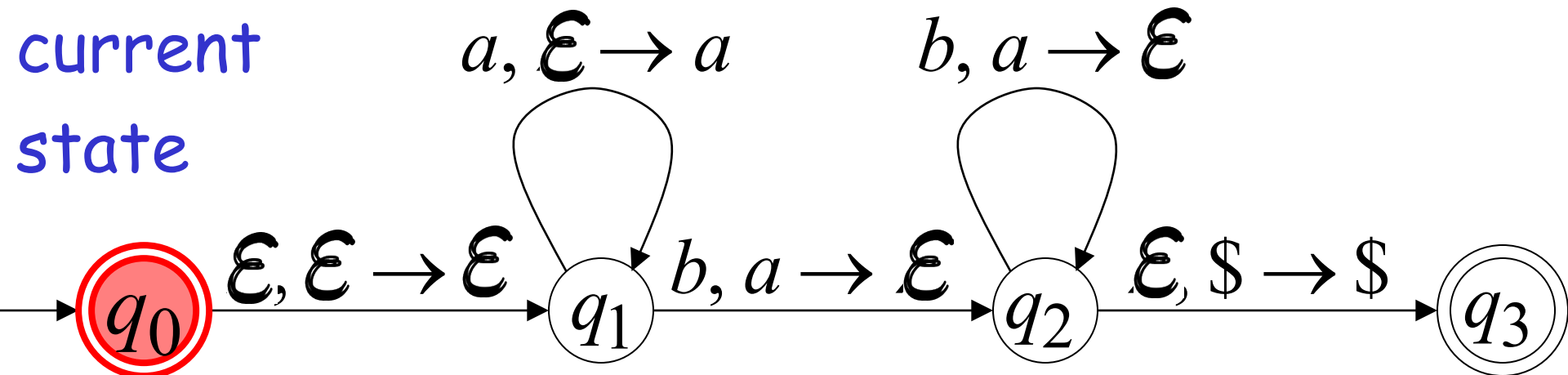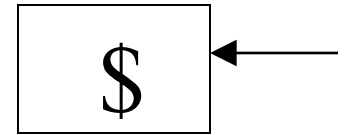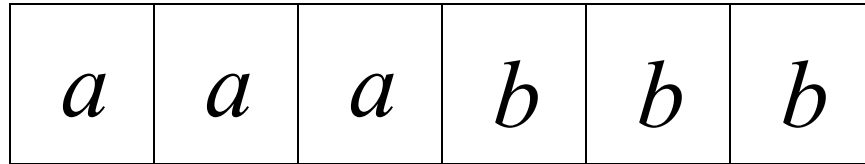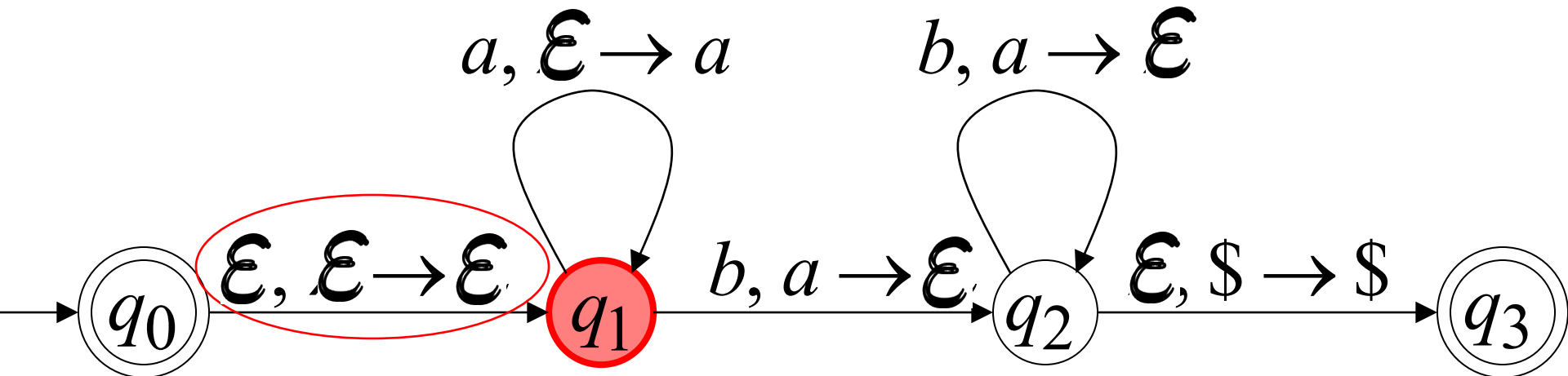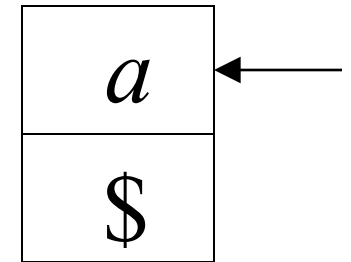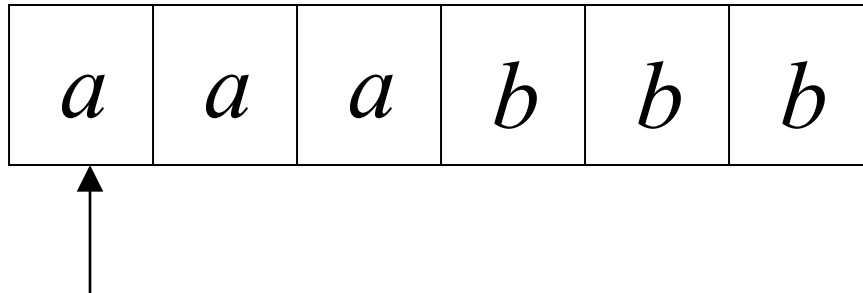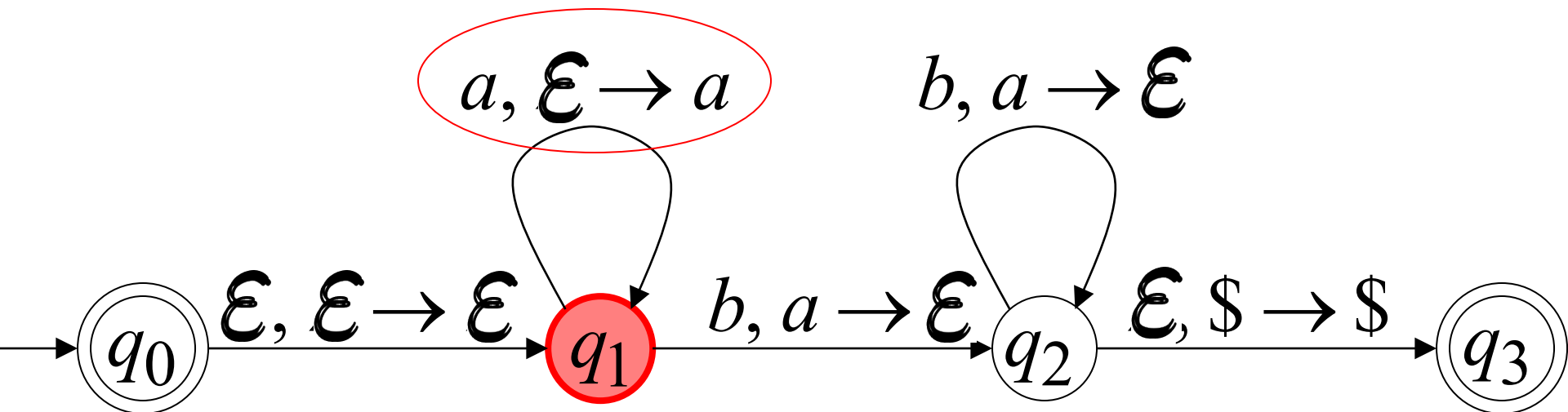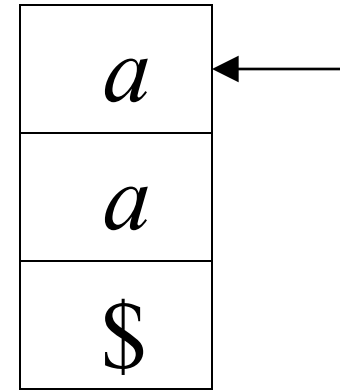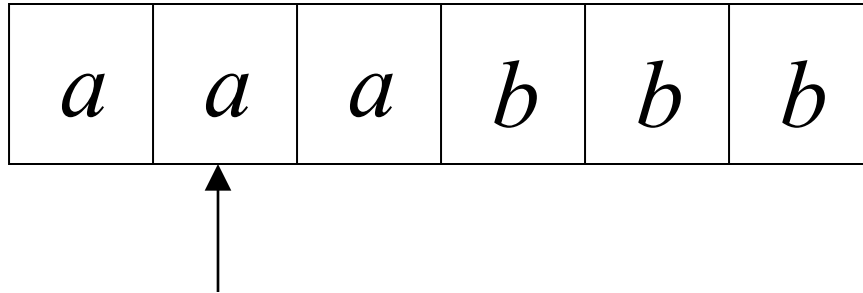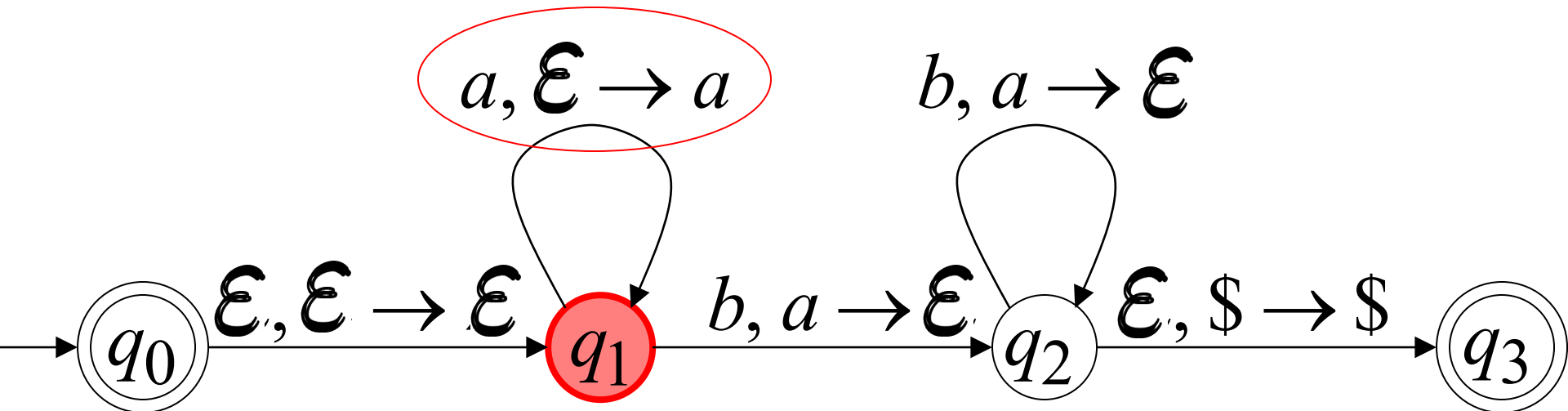
Stack

$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$

$q_0$  $\varepsilon, \varepsilon \rightarrow \varepsilon$  $q_1$  $b, a \rightarrow \varepsilon$  $q_2$  $\varepsilon, \$ \rightarrow \$$  $q_3$

Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

| $a$ |
|-----|
| $\$$ |

Stack

$$a, \varepsilon \rightarrow a$$

$$b, a \rightarrow \varepsilon$$

$$q_0 \quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad q_1 \quad b, a \rightarrow \varepsilon \quad q_2 \quad \varepsilon, \$ \rightarrow \$ \quad q_3$$

Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

| $a$ |
|-----|
| $a$ |
| $\$$ |

Stack

$$a, \varepsilon \rightarrow a$$

$$b, a \rightarrow \varepsilon$$

$q_0 \quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad q_1 \quad b, a \rightarrow \varepsilon \quad q_2 \quad \varepsilon, \$ \rightarrow \$ \quad q_3$$

Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

| $a$ |
|-----|
| $a$ |
| $a$ |
| $\$$ |

Stack

$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$   $q_0$   $b, a \rightarrow \varepsilon$   $q_1$   $\varepsilon, \$ \rightarrow \$$   $q_2$   $q_3$

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Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

Stack

$$a, \varepsilon \to a \qquad b, a \to \varepsilon$$

$$\varepsilon, \varepsilon \to \varepsilon \qquad b, a \to \varepsilon \qquad \varepsilon, \$ \to \$$$

$q_0 \qquad q_1 \qquad q_2 \qquad q_3$

**Input**

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|



**Stack**

$a$, $\varepsilon \rightarrow a$

$b$, $a \rightarrow \varepsilon$

$\varepsilon$, $\varepsilon \rightarrow \varepsilon$  $b$, $a \rightarrow \varepsilon$  $\varepsilon$, $\$ \rightarrow \$$

$q_0$  $q_1$  $q_2$  $q_3$

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Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

Stack

$a$

$\$$

$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$

$q_0$ $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$ $q_1$ $\quad b, a \rightarrow \varepsilon \quad$ $q_2$ $\quad \varepsilon, \$ \rightarrow \$ \quad$ $q_3$

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Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|
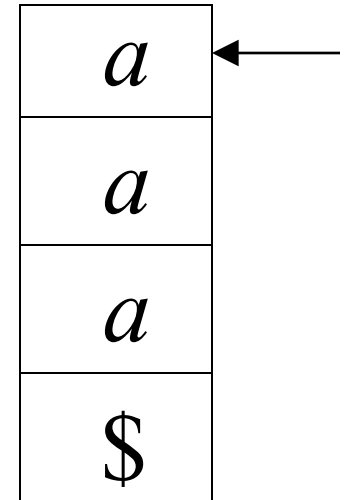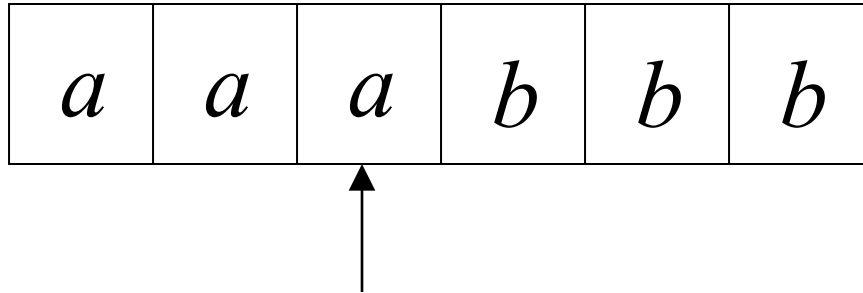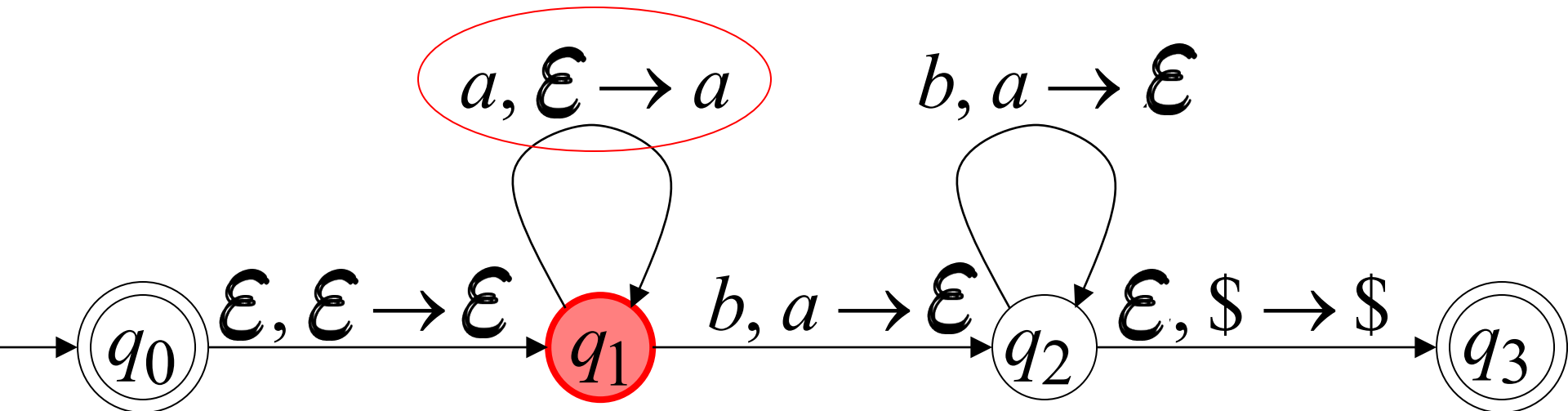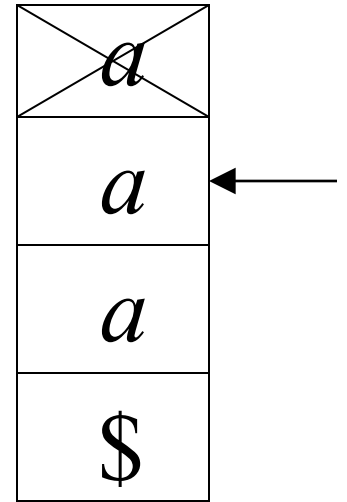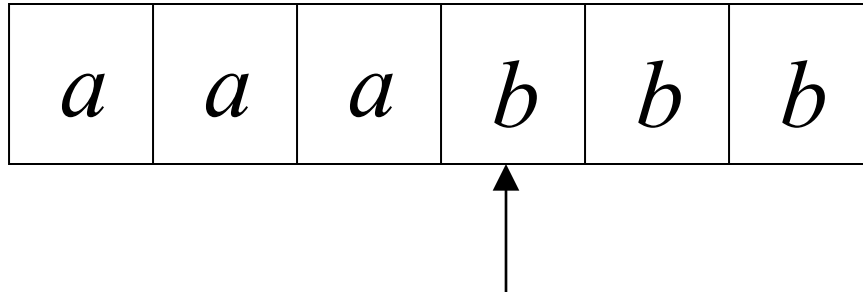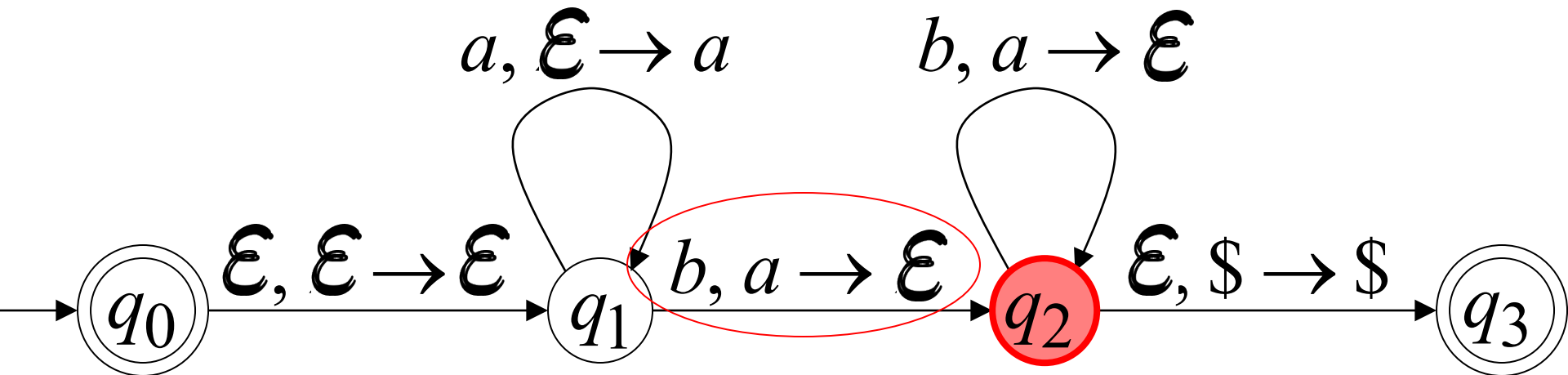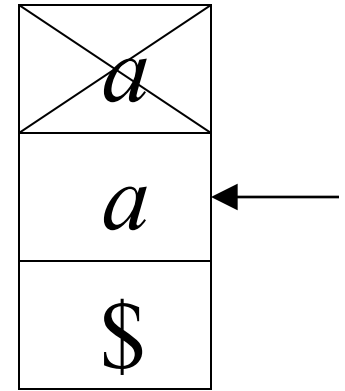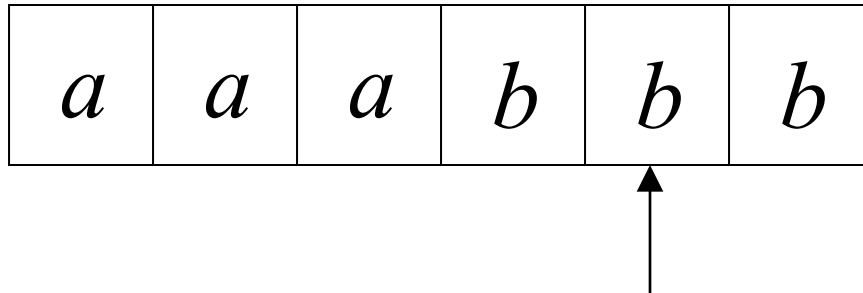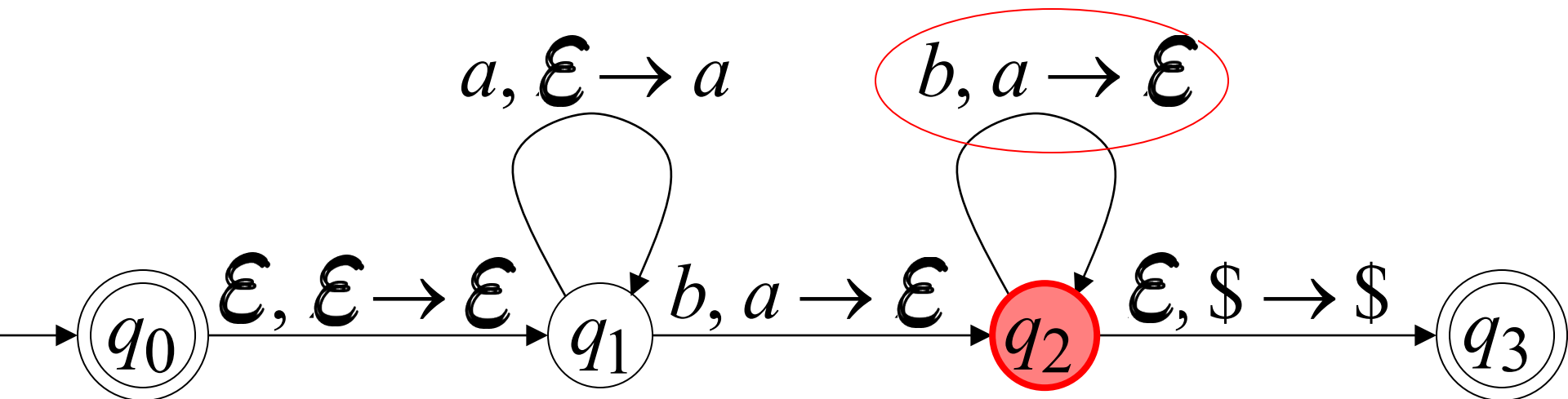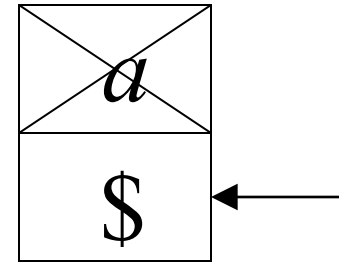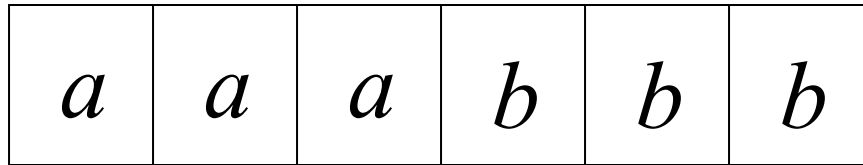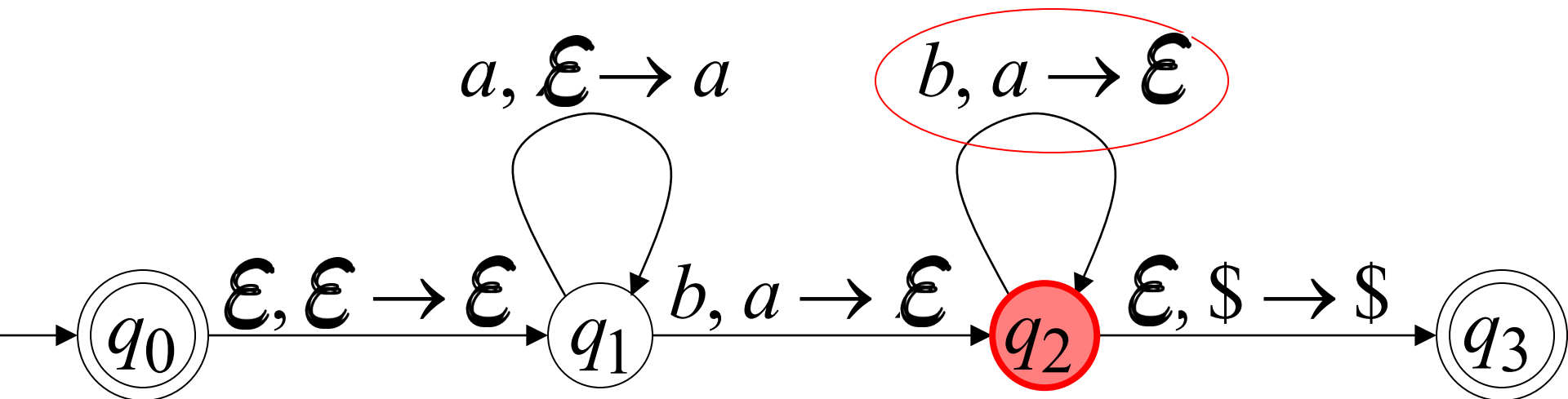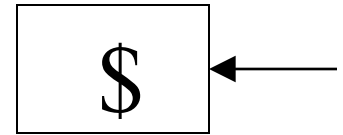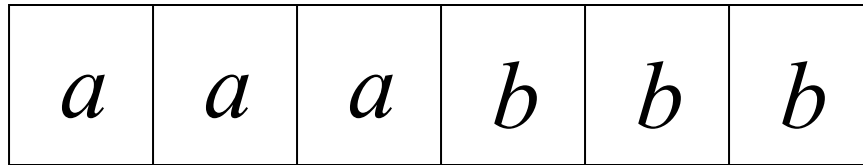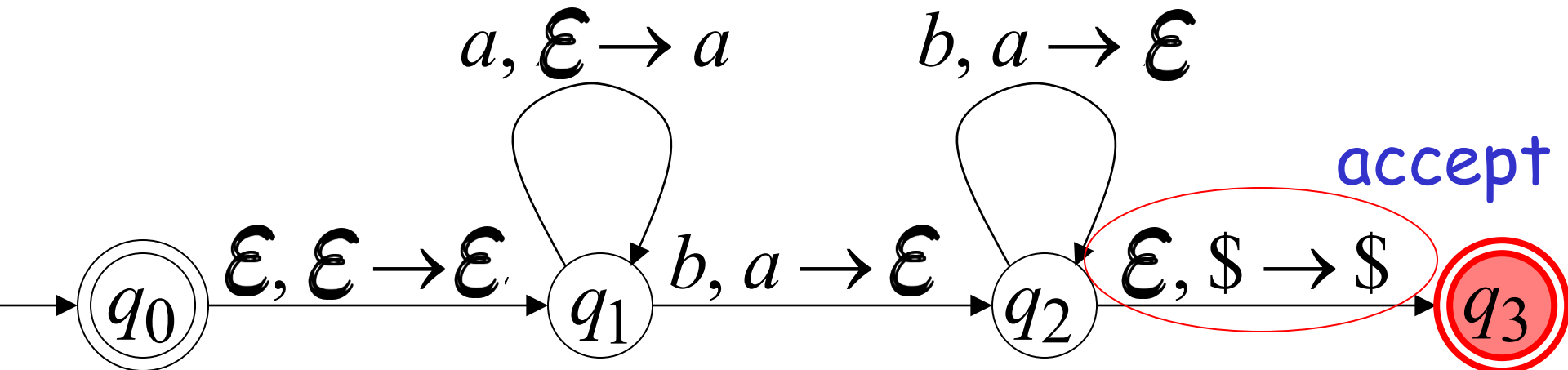
$\$$

Stack

$$a, \varepsilon \to a \qquad b, a \to \varepsilon$$

accept

$$q_0 \quad \varepsilon, \varepsilon \to \varepsilon \quad q_1 \quad b, a \to \varepsilon \quad q_2 \quad \varepsilon, \$ \to \$ \quad q_3$$

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A string is accepted if there is
a computation such that:

All the input is consumed
**AND**
The last state is an accepting state

we do not care about the stack contents
at the end of the accepting computation

# Rejection Example:   Time 0

## Input

| $a$ | $a$ | $b$ |
|-----|-----|-----|

$\$$

## Stack

current state

$$a, \varepsilon \rightarrow a \qquad b, a \rightarrow \varepsilon$$

$$q_0 \xrightarrow{\varepsilon, \varepsilon \rightarrow \varepsilon} q_1 \xrightarrow{b, a \rightarrow \varepsilon} q_2 \xrightarrow{\varepsilon, \$ \rightarrow \$} q_3$$

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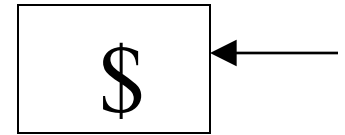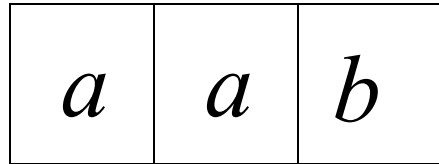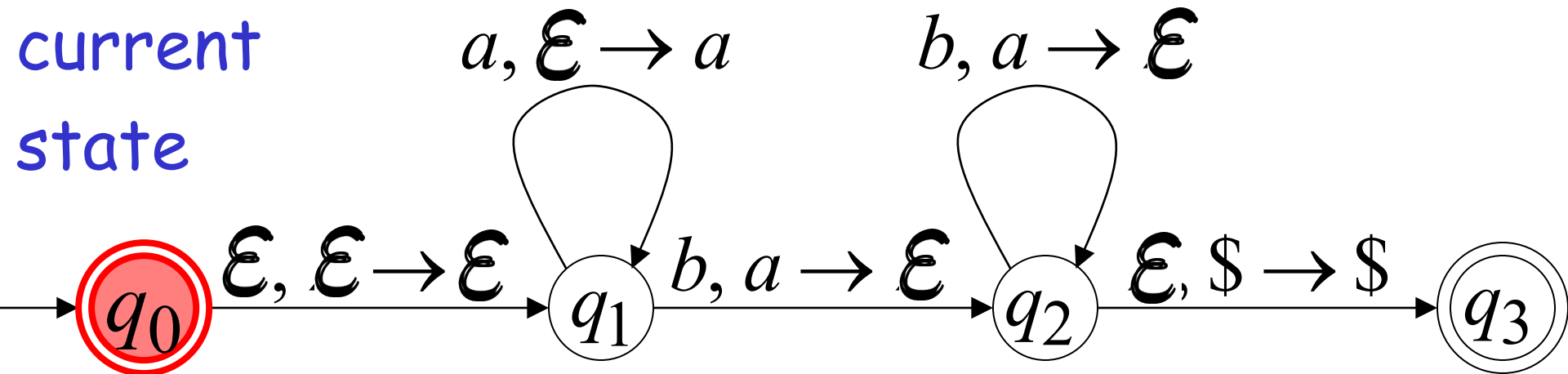
# Rejection Example: Time 1

Input

| $a$ | $a$ | $b$ |
|---|---|---|

$\uparrow$

$\boxed{\$}$ $\leftarrow$

Stack

current state

$$a, \varepsilon \rightarrow a \qquad b, a \rightarrow \varepsilon$$

$q_0 \quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad q_1 \quad b, a \rightarrow \varepsilon \quad q_2 \quad \varepsilon, \$ \rightarrow \$ \quad q_3$
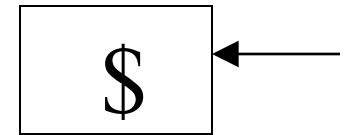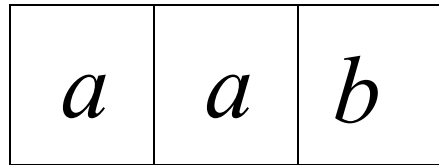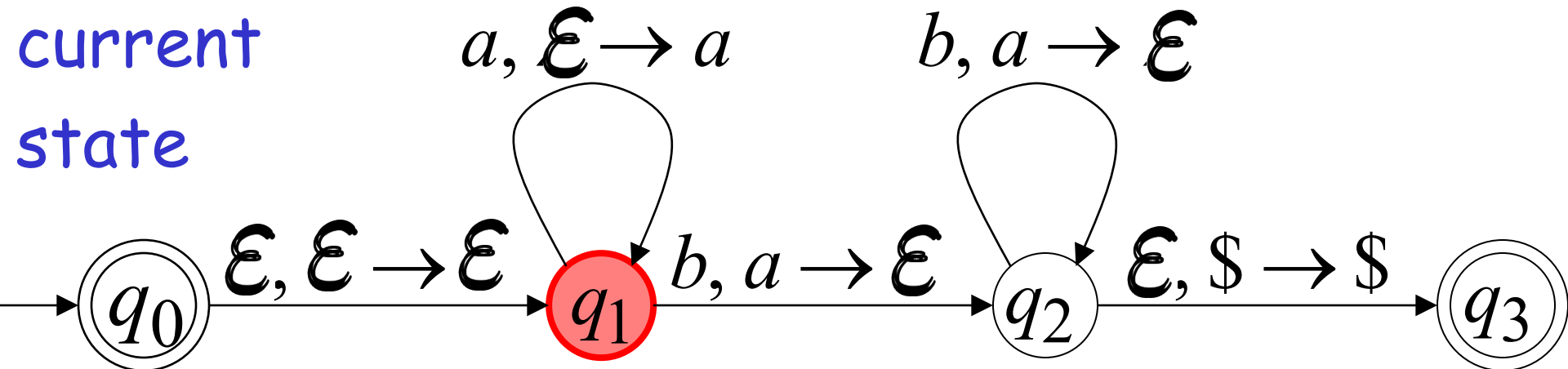
# Rejection Example: Time 2

## Input

| $a$ | $a$ | $b$ |
|-----|-----|-----|

↑

## Stack
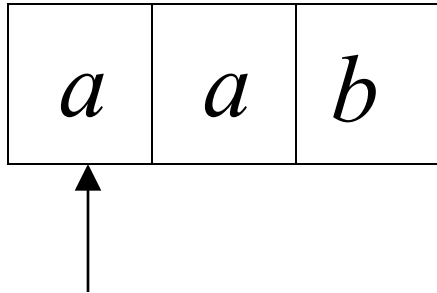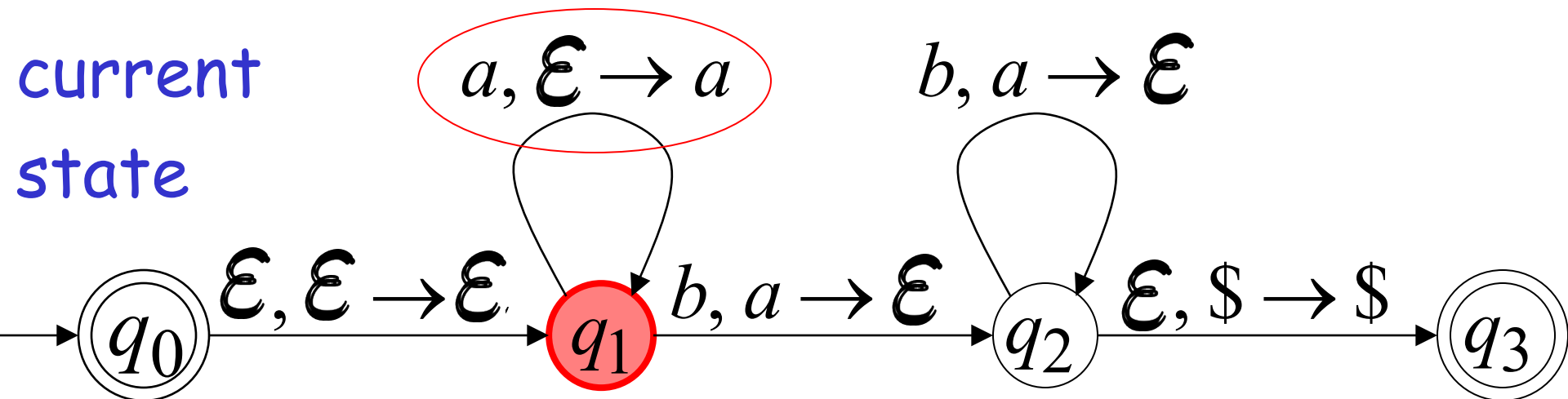
| $a$ |
|-----|
| $\$$ |

← 

current state

$$a, \varepsilon \rightarrow a$$

$$b, a \rightarrow \varepsilon$$

$$q_0 \quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad q_1 \quad b, a \rightarrow \varepsilon \quad q_2 \quad \varepsilon, \$ \rightarrow \$ \quad q_3$$

25

# Rejection Example: Time 3

**Input**

| $a$ | $a$ | $b$ |
|-----|-----|-----|

**Stack**

| $a$ |
|-----|
| $a$ |
| $\$$ |
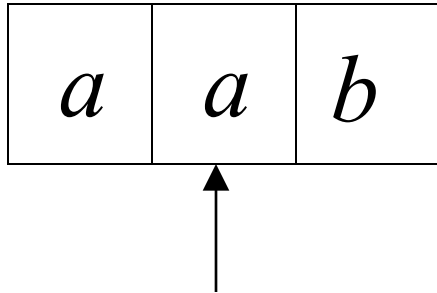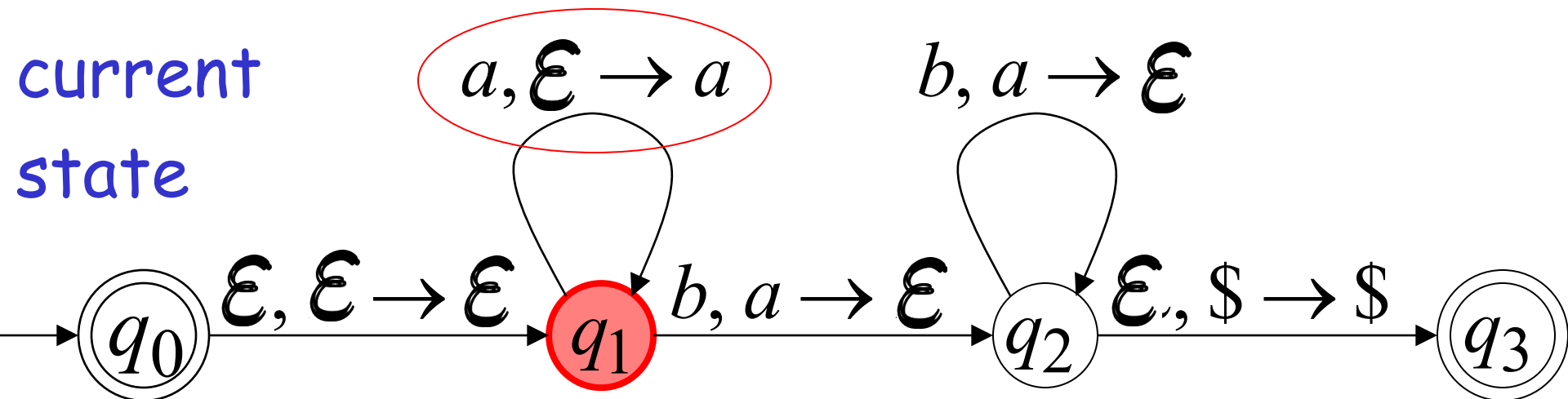
**current state**

$$a, \varepsilon \rightarrow a$$

$$b, a \rightarrow \varepsilon$$

$$q_0 \quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad q_1 \quad b, a \rightarrow \varepsilon \quad q_2 \quad \varepsilon, \$ \rightarrow \$ \quad q_3$$

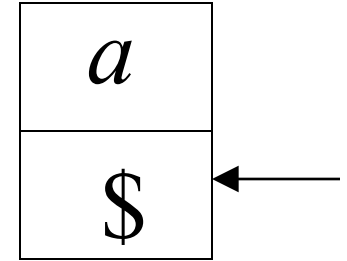# Rejection Example:  Time 4

## Input

| $a$ | $a$ | $b$ |
|-----|-----|-----|

Stack:
$$\boxed{\begin{array}{c} \text{\sout{$a$}} \\ a \\ \$ \end{array}}$$

**Stack**

current
state

$$q_0 \xrightarrow{\varepsilon, \varepsilon \to \varepsilon} q_1 \xrightarrow{b, a \to \varepsilon} q_2 \xrightarrow{\varepsilon, \$ \to \$} q_3$$

$q_1$ loop: $a, \varepsilon \to a$

$q_2$ loop: $b, a \to \varepsilon$

27
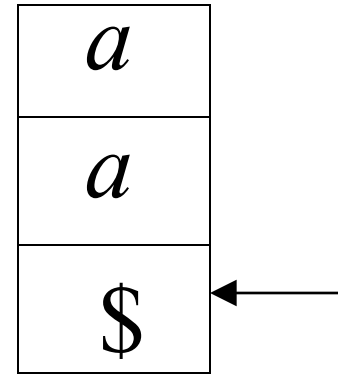
# Rejection Example: Time 4

## Input

| $a$ | $a$ | $b$ |
|---|---|---|
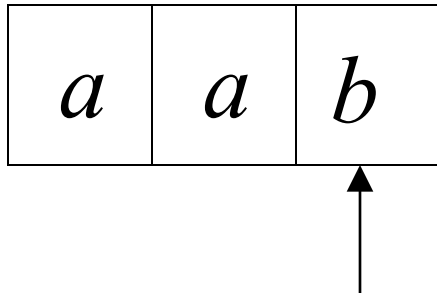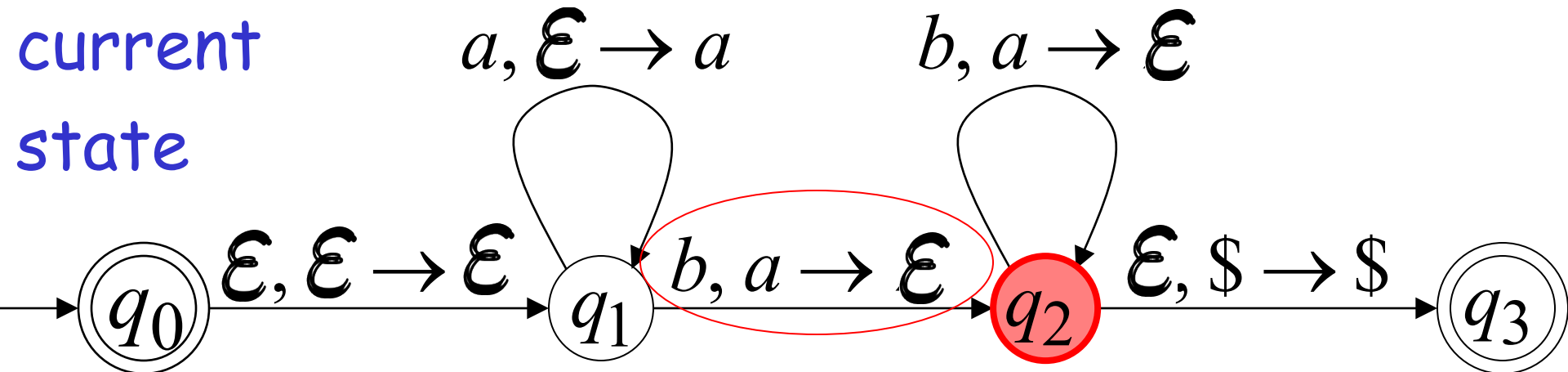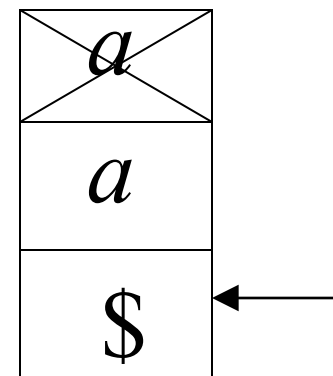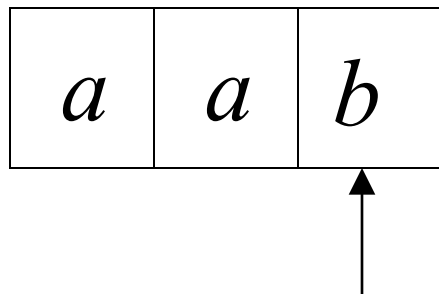
Stack
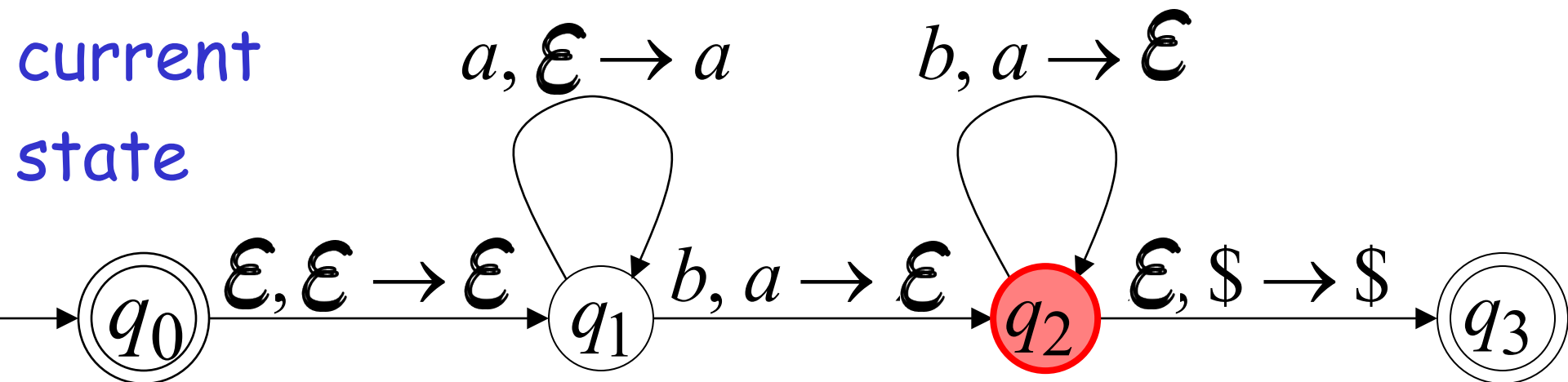
| $a$ |
|---|
| $a$ |
| $\$$ |

reject

current state



$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$

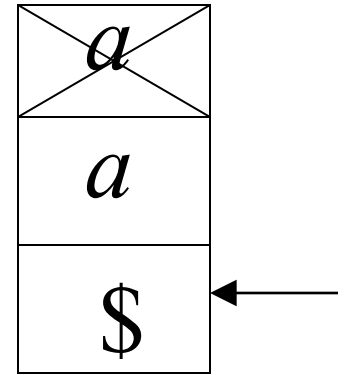$q_0$ $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$ $q_1$ $\quad b, a \rightarrow \varepsilon \quad$ $q_2$ $\quad \varepsilon, \$ \rightarrow \$ \quad$ $q_3$

28

There is no accepting computation for $aab$

The string $aab$ is rejected by the PDA



$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$b, a \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$   $q_1$   $q_2$   $q_3$
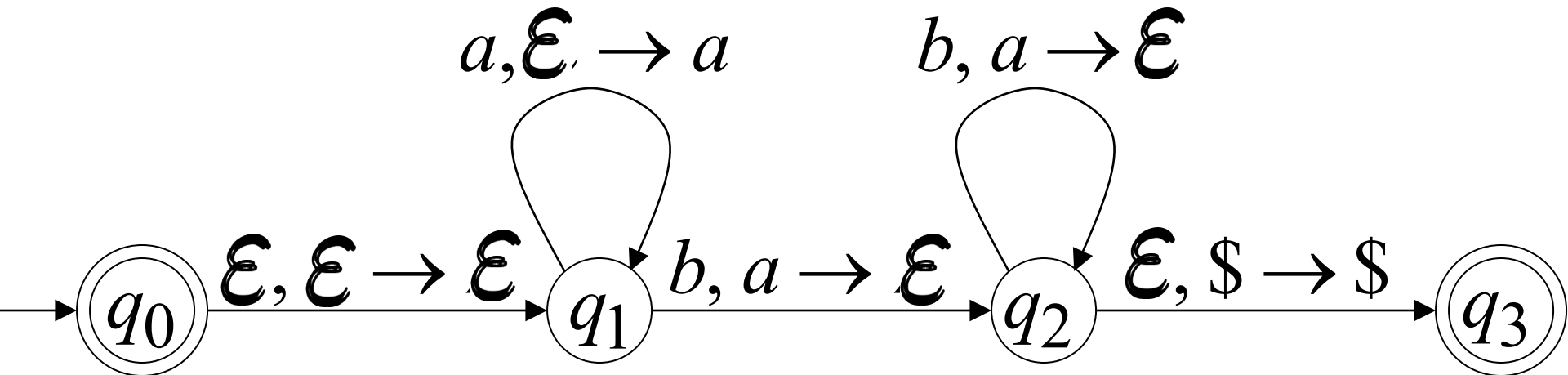
# Another PDA example

PDA $M:$ $L(M) = \{vv^R : v \in \{a,b\}^*\}$

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$



$\varepsilon, \varepsilon \to \varepsilon$ $\qquad$ $\varepsilon, \$ \to \$$

$q_0 \qquad q_1 \qquad q_2$

Basic Idea: $L(M) = \{vv^R : v \in \{a, b\}^*\}$

1. Push v on stack

2. Guess middle of input

3. Match $v^R$ on input with v on stack

4. Match found

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$q_0$ $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$ $q_1$ $\quad \varepsilon, \$ \rightarrow \$ \quad$ $q_2$

# Execution Example:

Input

| $a$ | $b$ | $b$ | $a$ |
|-----|-----|-----|-----|

$$\$$$

Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$$\varepsilon, \varepsilon \to \varepsilon \qquad \varepsilon, \$ \to \$$$

$q_0 \qquad q_1 \qquad q_2$

Input

| $a$ | $b$ | $b$ | $a$ |
|-----|-----|-----|-----|

| $a$ |
|-----|
| $\$$ |

Stack

$$a, \varepsilon \to a$$
$$b, \varepsilon \to b$$

$$a, a \to \varepsilon$$
$$b, b \to \varepsilon$$

$$\varepsilon, \varepsilon \to \varepsilon$$

$$\varepsilon, \$ \to \$$$

$q_0$ $\quad$ $q_1$ $\quad$ $q_2$

33

Input

| $a$ | $b$ | $b$ | $a$ |
|---|---|---|---|

Stack

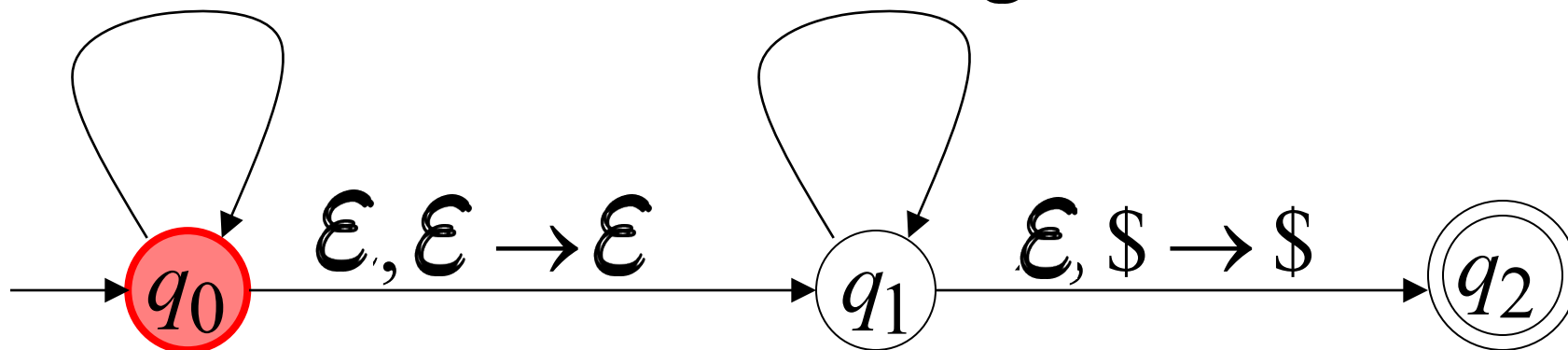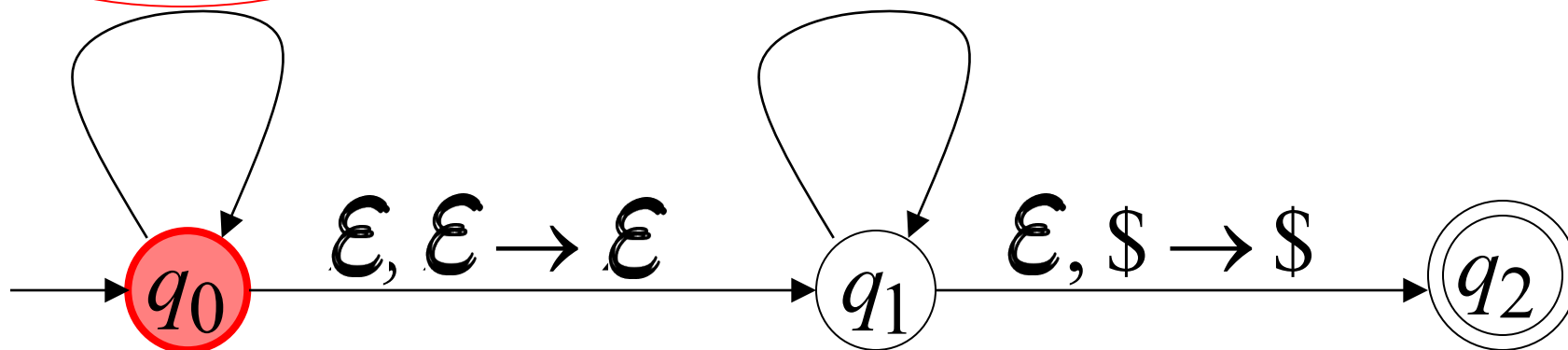| $b$ |
|---|
| $a$ |
| $\$$ |

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$q_0$  $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$  $q_1$  $\quad \varepsilon, \$ \rightarrow \$ \quad$  $q_2$

34

Input

| $a$ | $b$ | $b$ | $a$ |
|---|---|---|---|

Guess the middle
of string

Stack

| $b$ |
|---|
| $a$ |
| $\$$ |

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \$ \to \$$

$q_0$  $q_1$  $q_2$

35

Input

| $a$ | $b$ | $b$ | $a$ |
|-----|-----|-----|-----|

$b$

$a$

$\$$

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

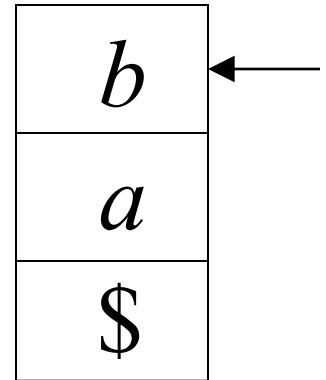$q_0$ $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$ $q_1$ $\quad \varepsilon, \$ \rightarrow \$ \quad$ $q_2$

36

Input

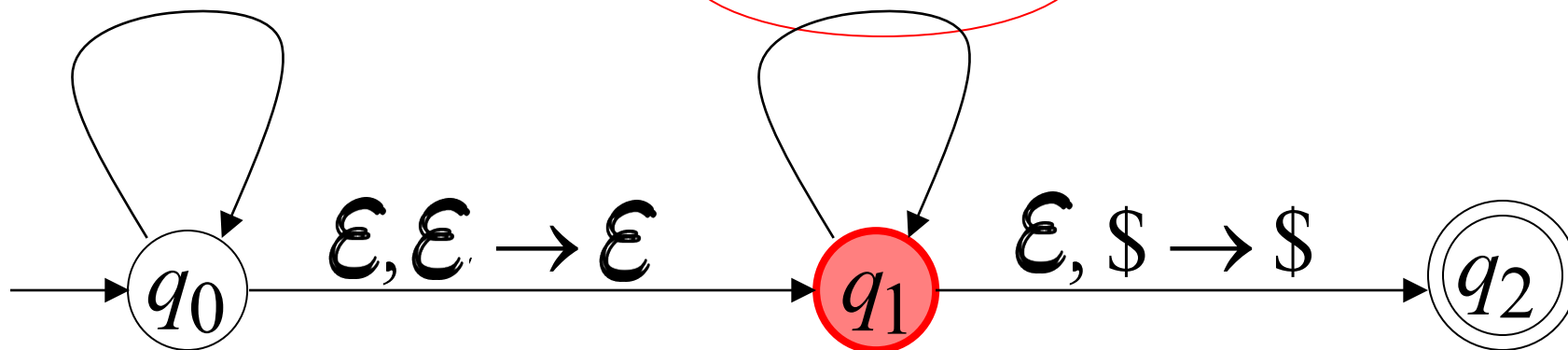| $a$ | $b$ | $b$ | $a$ |
|---|---|---|---|

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$ $q_1$ $q_2$

37

Input

| $a$ | $b$ | $b$ | $a$ |
|---|---|---|---|

$\$$

Stack
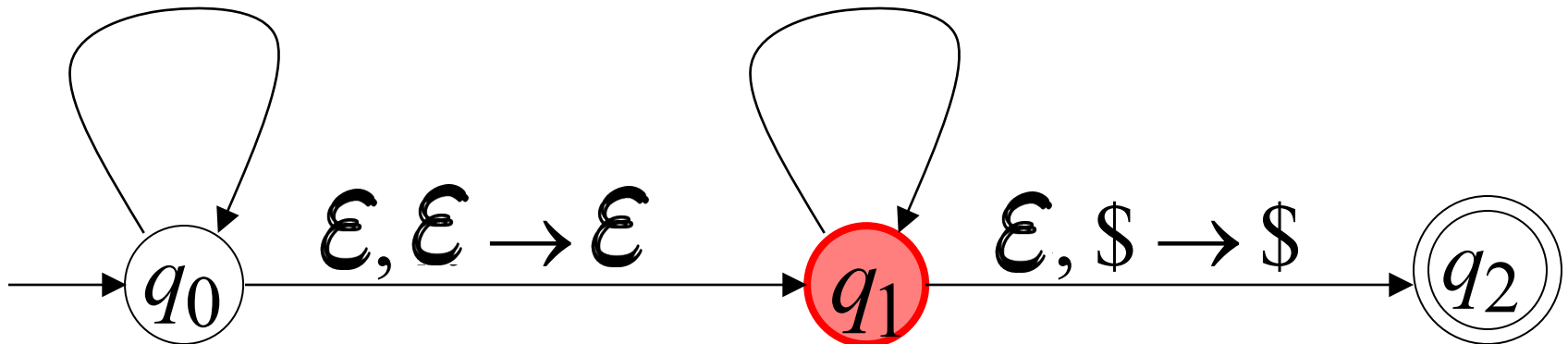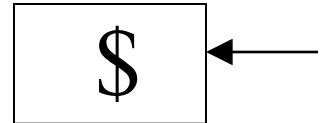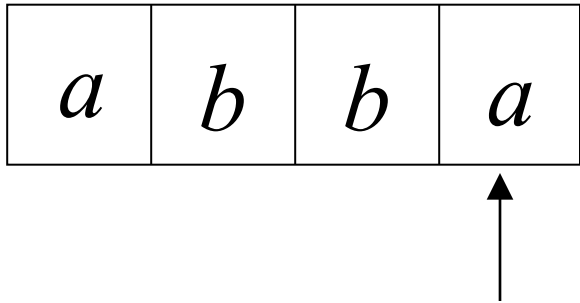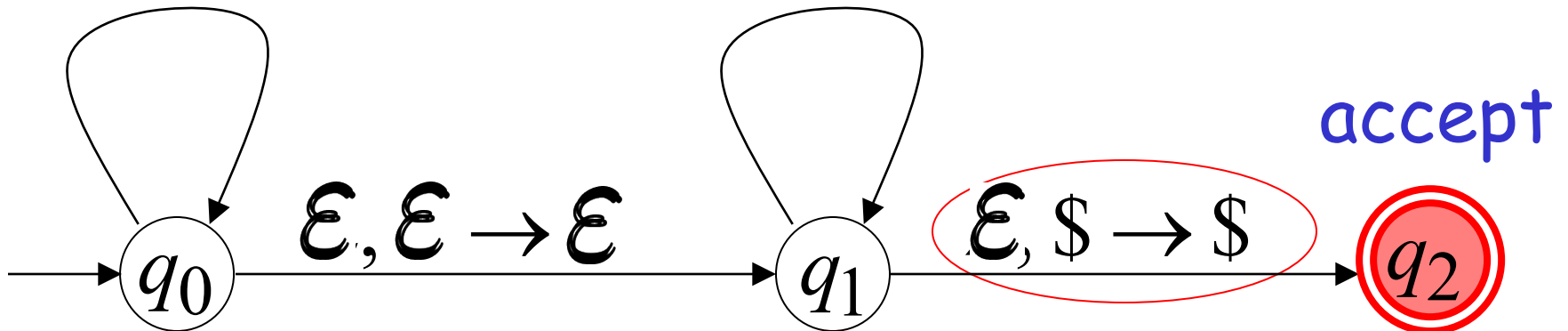
$a, \varepsilon \rightarrow a$
$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$
$b, b \rightarrow \varepsilon$

accept

$q_0$ $\qquad \varepsilon, \varepsilon \rightarrow \varepsilon \qquad$ $q_1$ $\qquad \varepsilon, \$ \rightarrow \$ \qquad$ $q_2$

# Rejection Example:

Input

| $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|

$\$$

Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \$ \to \$$

$q_0$     $q_1$     $q_2$

Input

| $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|

Stack

| $a$ |
|---|
| $\$$ |

$a, \mathcal{E} \rightarrow a$

$b, \mathcal{E} \rightarrow b$

$a, a \rightarrow \mathcal{E}$

$b, b \rightarrow \mathcal{E}$

$\mathcal{E}, \mathcal{E} \rightarrow \mathcal{E}$

$\mathcal{E}, \$ \rightarrow \$$

$q_0$    $q_1$    $q_2$

40

Input

| $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|

| $b$ |
|-----|
| $a$ |
| $\$$ |

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

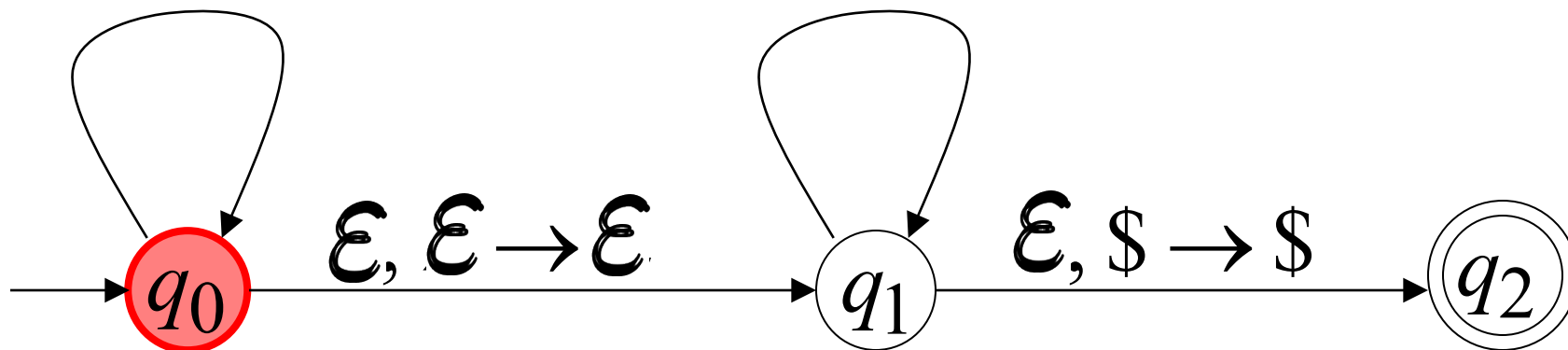$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$

$q_1$

$q_2$

41

## Input

| $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|

Guess the middle of string

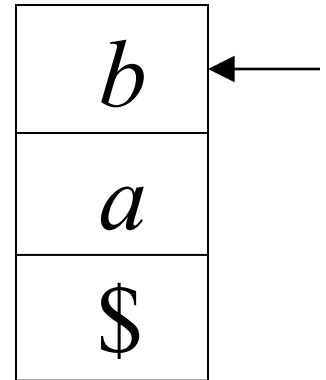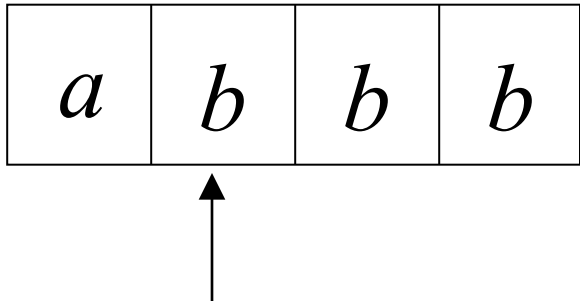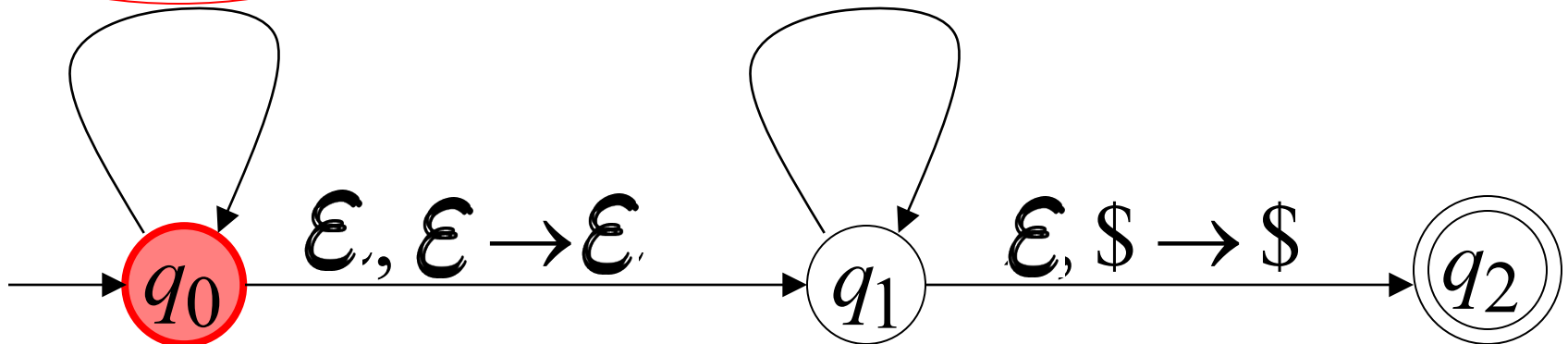| $b$ |
|-----|
| $a$ |
| $\$$ |

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$ $q_1$ $q_2$

Input

| $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|

Stack

$b$

$a$

$\$$

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$

$q_1$

$q_2$

# Input

There is no possible transition.

| $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|

↑
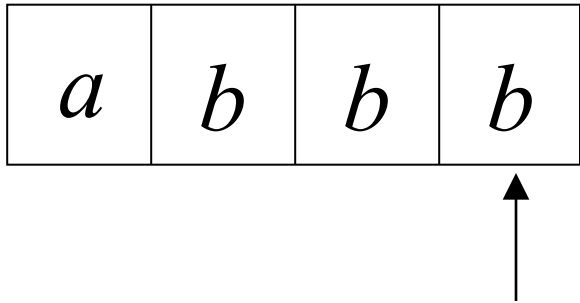
Input is not consumed

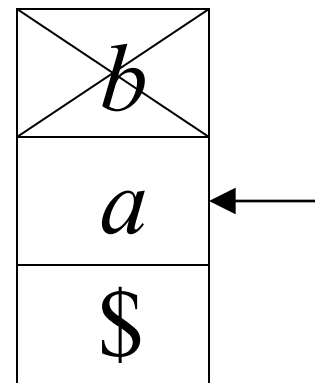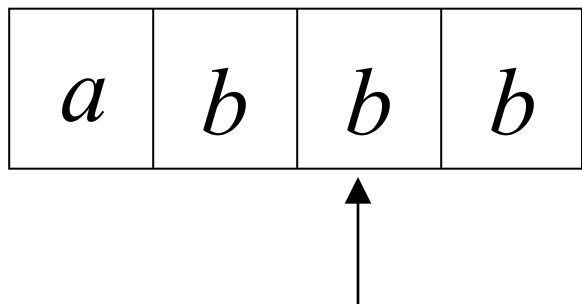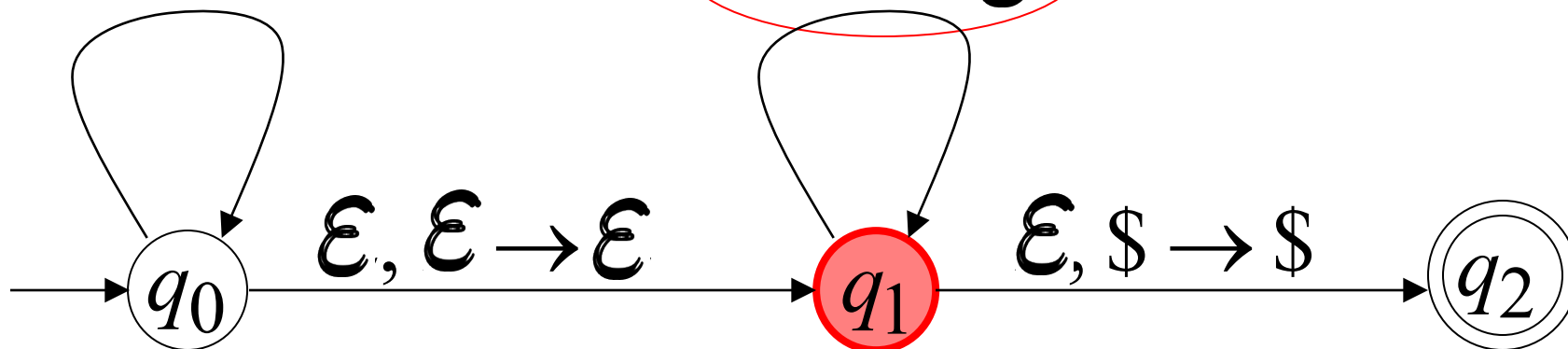| $a$ | ← |
|-----|---|
| $\$$ | |

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0 \qquad q_1 \qquad q_2$

44
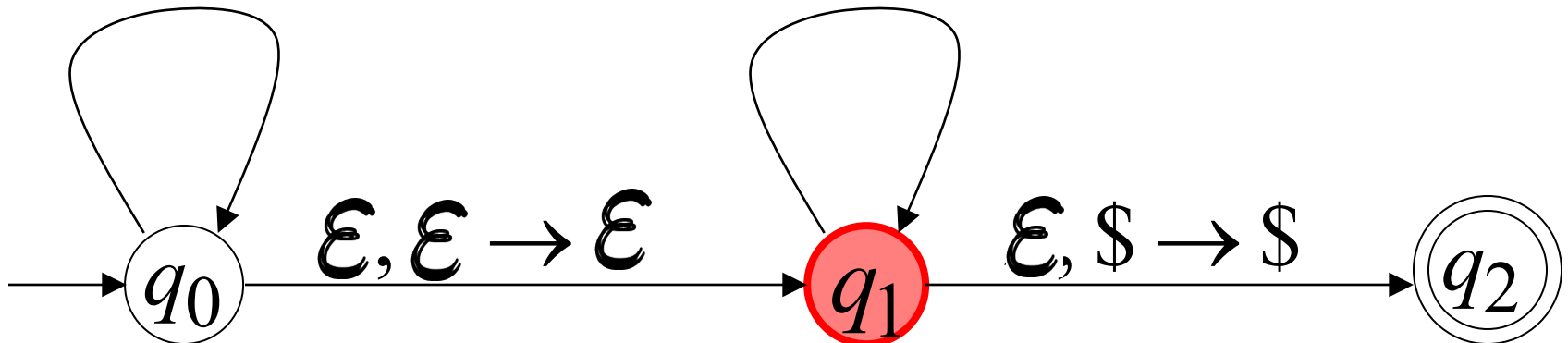
# Another computation on same string:

Input

Time 0
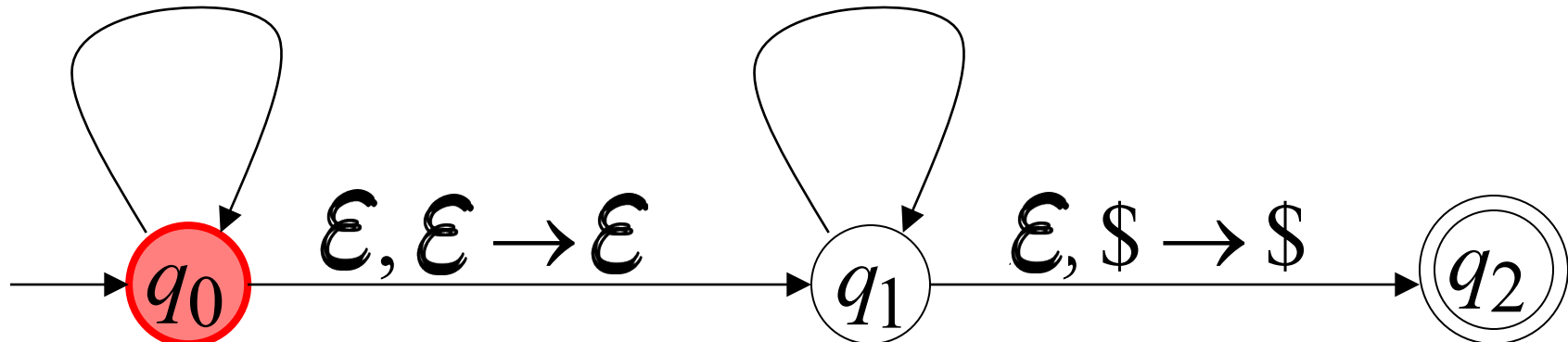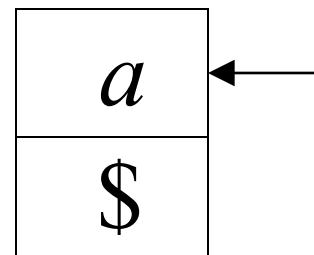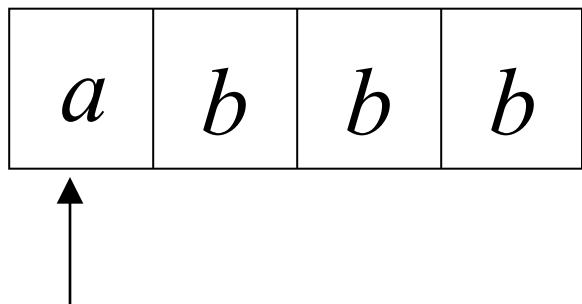
| $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|

$\$$

Stack

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$

$\varepsilon, \$ \to \$$

$q_0$     $q_1$     $q_2$

Input

| $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|

$a$

$\$$

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

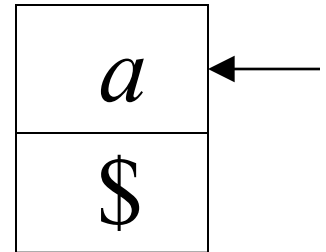$q_0$   $\varepsilon, \varepsilon \rightarrow \varepsilon$   $q_1$   $\varepsilon, \$ \rightarrow \$$   $q_2$

Input

| $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|

| $b$ |
|---|
| $a$ |
| $\$$ |

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$     $q_1$     $q_2$

47

Input

| $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|

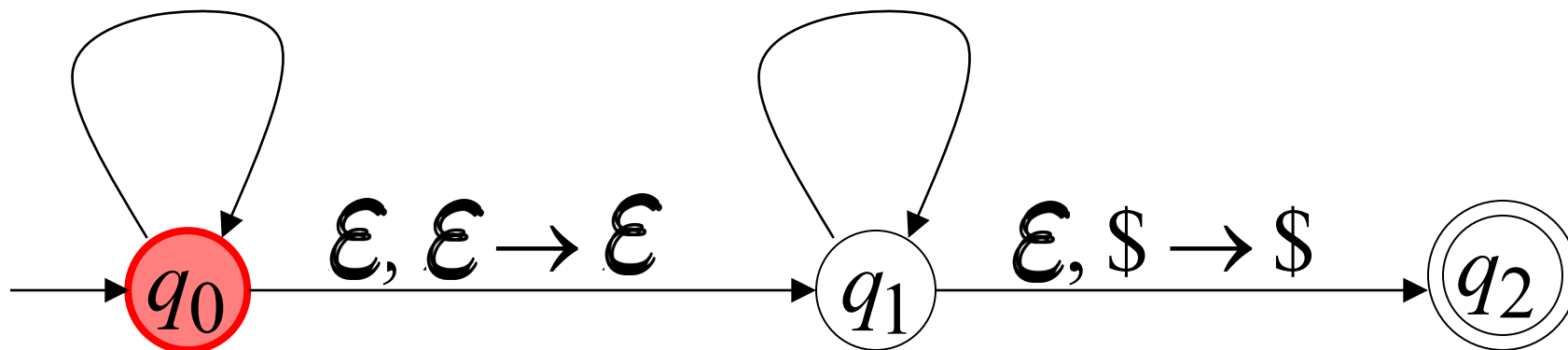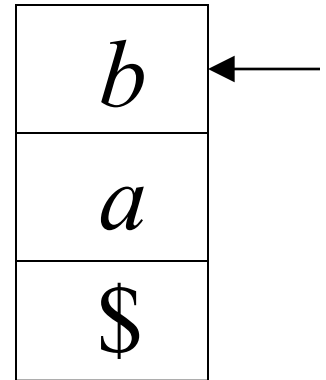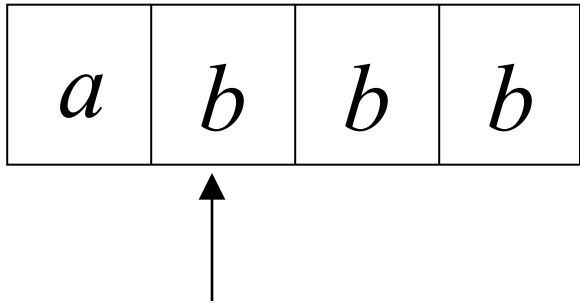| $b$ |
|---|
| $b$ |
| $a$ |
| $\$$ |

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$

$q_1$

$q_2$

48

Input

| $a$ | $b$ | $b$ | $b$ |
|---|---|---|---|

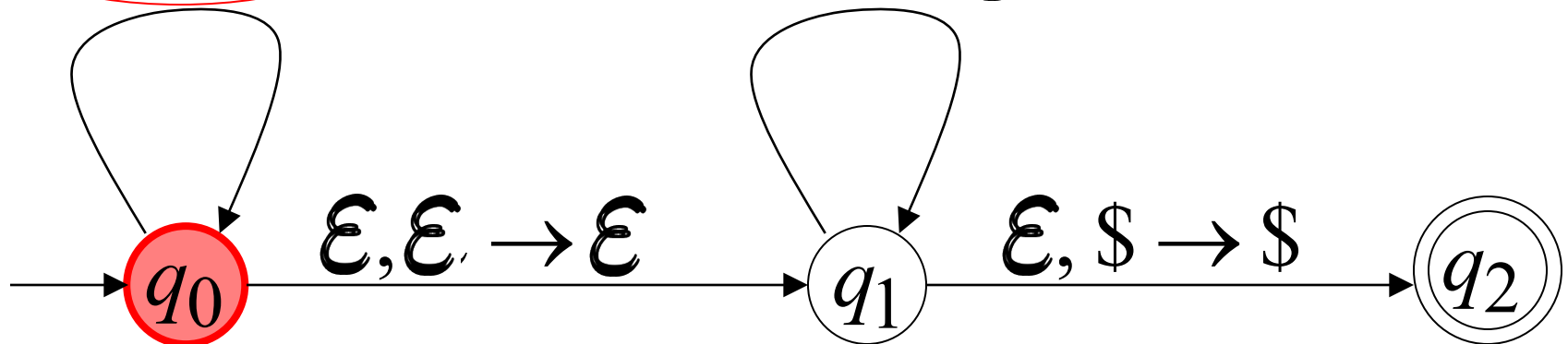| $b$ | ← |
|---|---|
| $b$ | |
| $b$ | |
| $a$ | |
| $\$$ | |

Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$

$q_1$

$q_2$

49
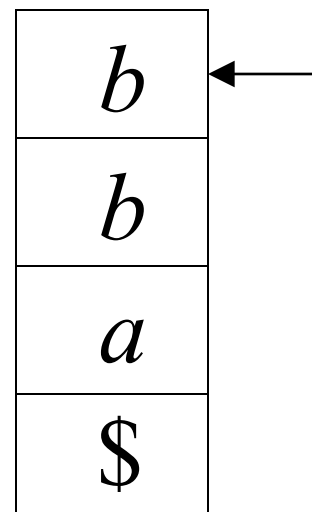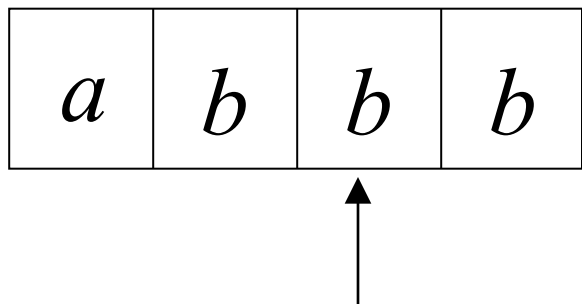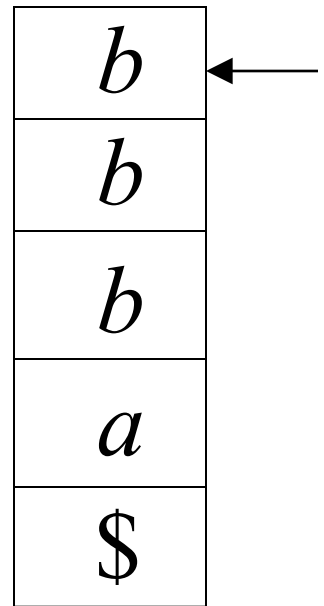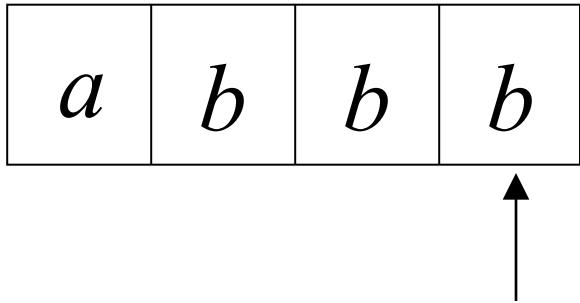
## Time 5

### Input

| $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|

No accept state is reached

### Stack

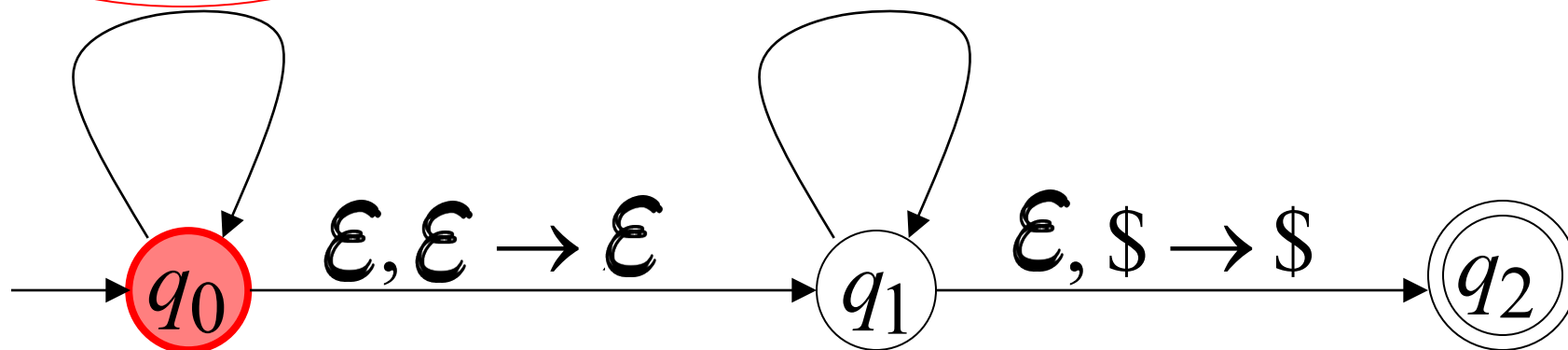| $b$ |
|-----|
| $b$ |
| $b$ |
| $a$ |
| $\$$ |

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

$\varepsilon, \$ \rightarrow \$$

$q_0$     $q_1$     $q_2$
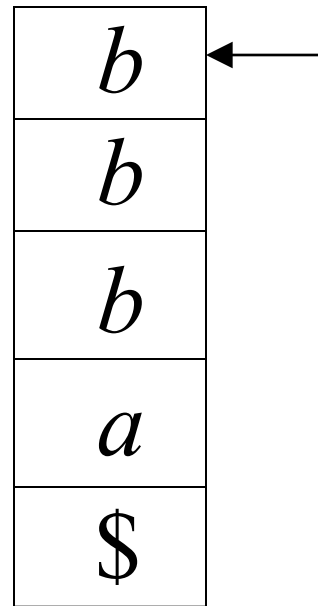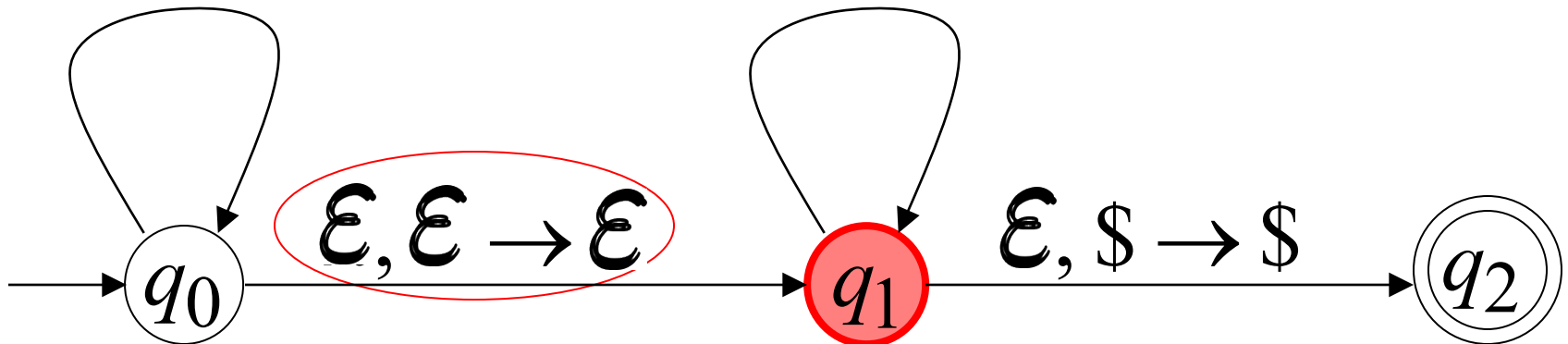
There is no computation
that accepts string $abbb$

$$abbb \notin L(M)$$

$a, \varepsilon \to a$

$b, \varepsilon \to b$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$\varepsilon, \varepsilon \to \varepsilon$
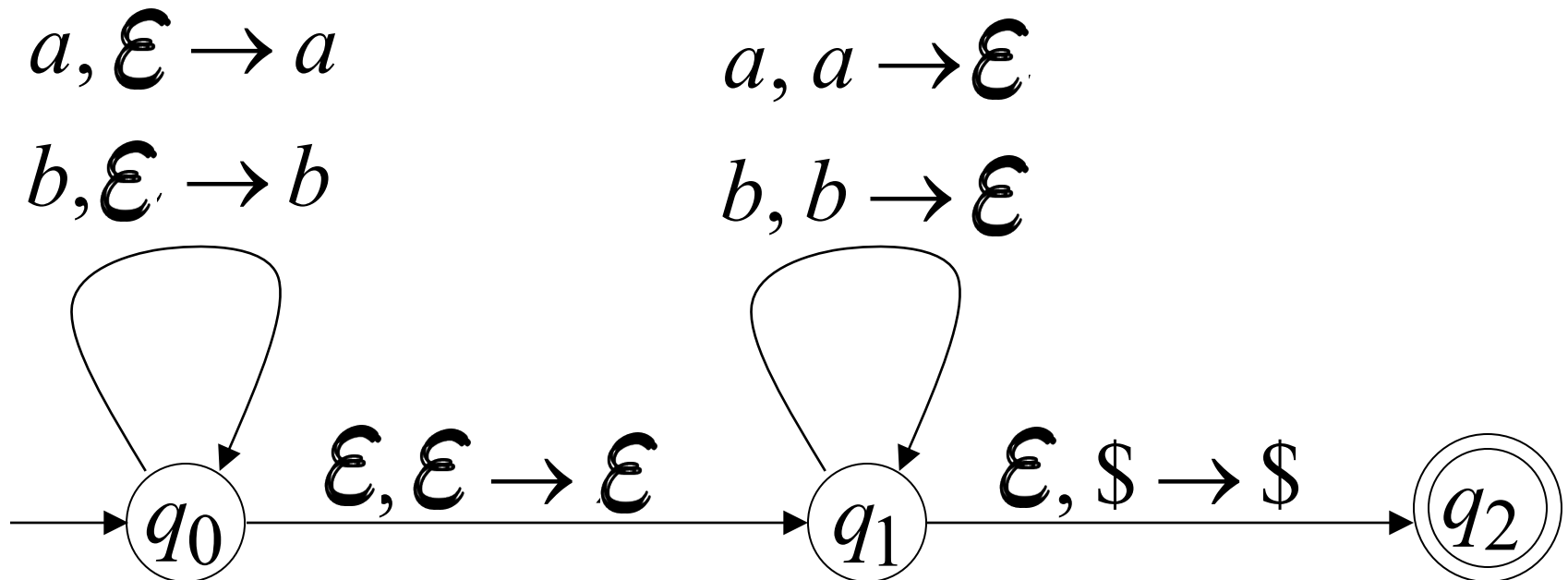
$\varepsilon, \$ \to \$$

$q_0$     $q_1$     $q_2$

# Pushing & Popping Strings



Input symbol

Pop string

Push string

$a, w_1 \rightarrow w_2$

$q_1$ → $q_2$

# Example:

$$q_1 \xrightarrow{a, eb \to cdf} q_2$$

input

$$\cdots \mid a \mid \cdots$$

$$\cdots \mid a \mid \cdots$$

stack

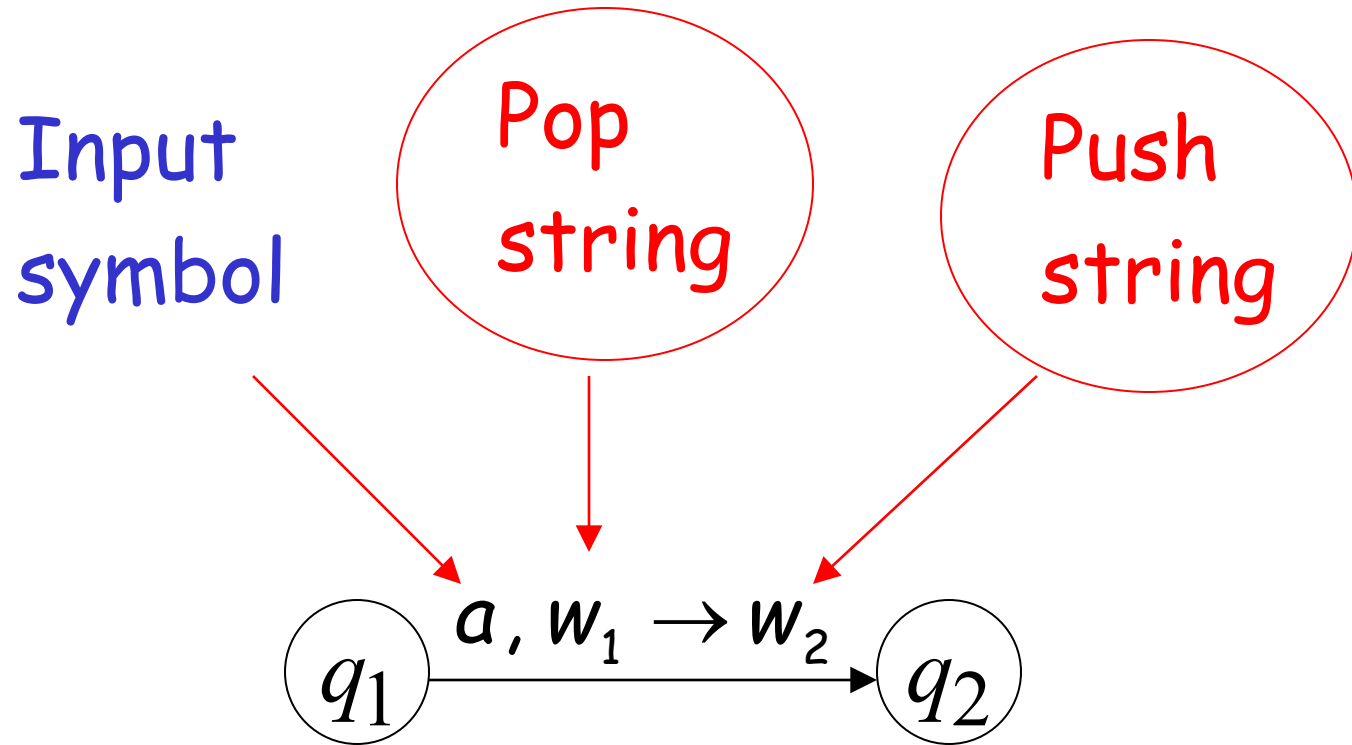pop string

| e | ← top |
| b | |
| h | |
| e | |
| $ | |

**Replace**

top → 

| c | push string |
| d | |
| f | |
| h | |
| e | |
| $ | |

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$q_1 \xrightarrow{a, eb \rightarrow cdf} q_2$

Equivalent transitions

pop

$q_1 \xrightarrow{a, e \rightarrow \varepsilon} \bigcirc \xrightarrow{a, b \rightarrow \varepsilon} \bigcirc$

$\varepsilon, \varepsilon \rightarrow \varepsilon$

push

$\bigcirc \xrightarrow{a, \varepsilon \rightarrow f} \bigcirc \xrightarrow{a, \varepsilon \rightarrow d} \bigcirc \xrightarrow{a, \varepsilon \rightarrow c} q_2$

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# Another PDA example

$$L(M) = \{w \in \{a,b\}^* : \quad n_a(w) = n_b(w)\}$$

PDA $M$

$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$
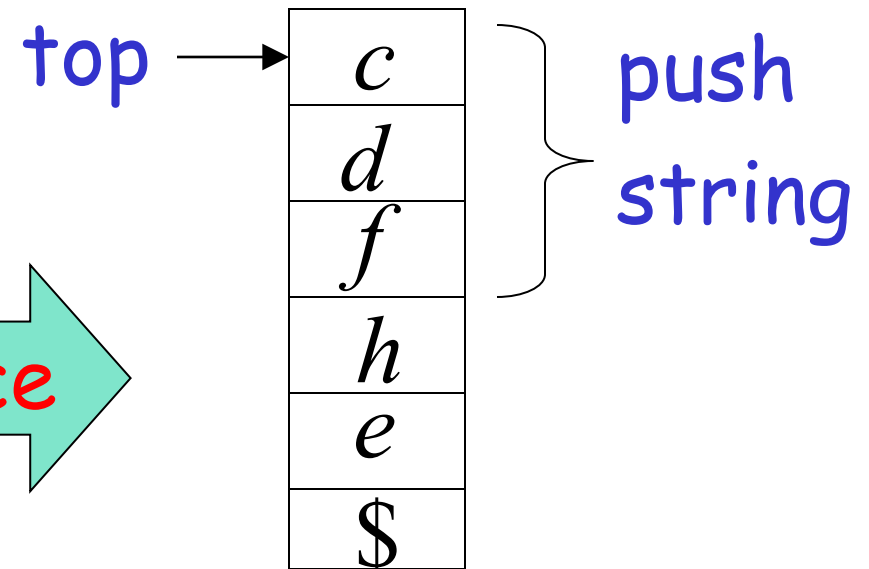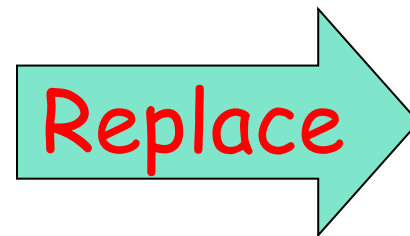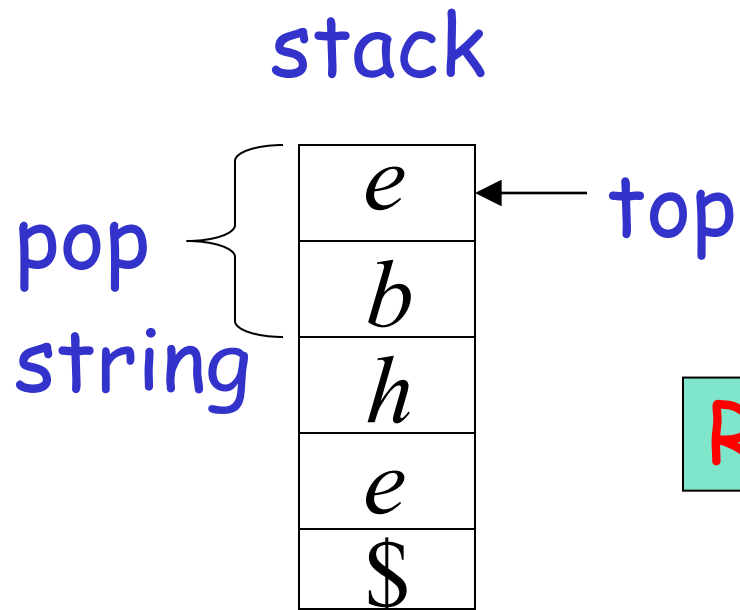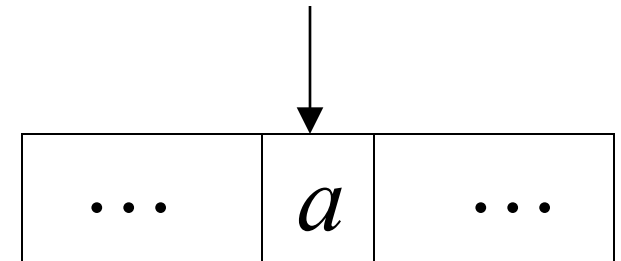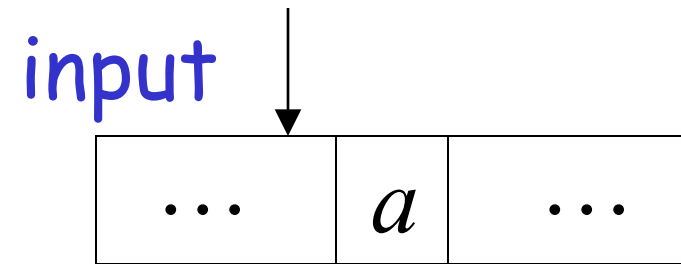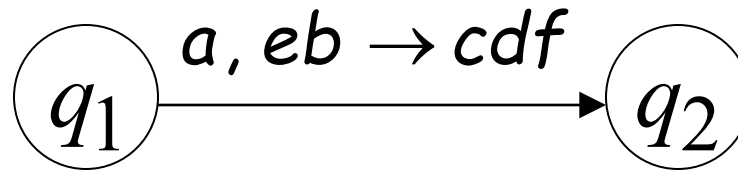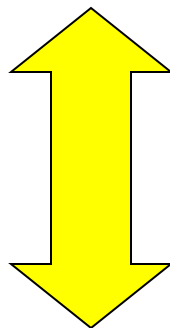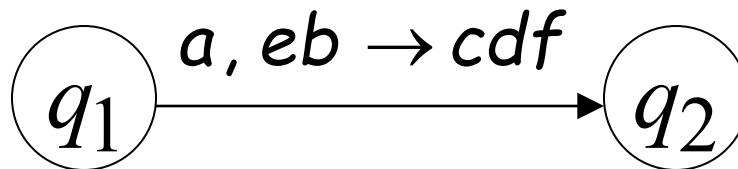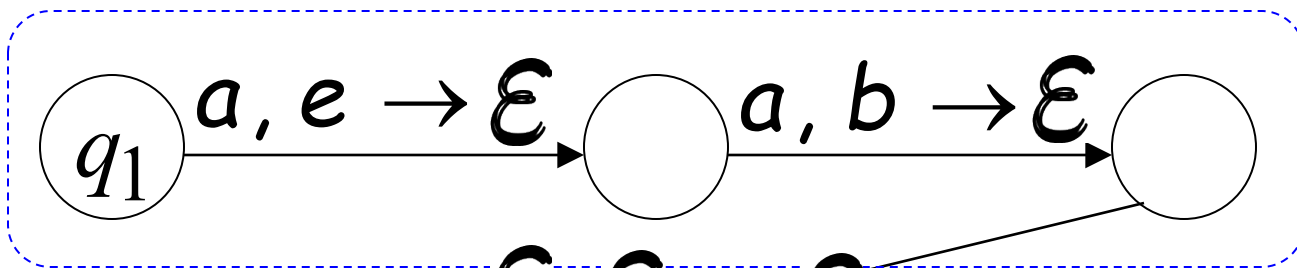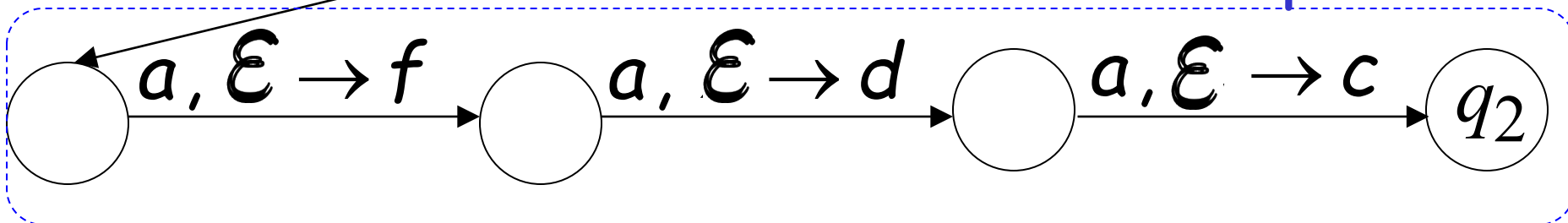
$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon \qquad b, 0 \rightarrow \varepsilon$
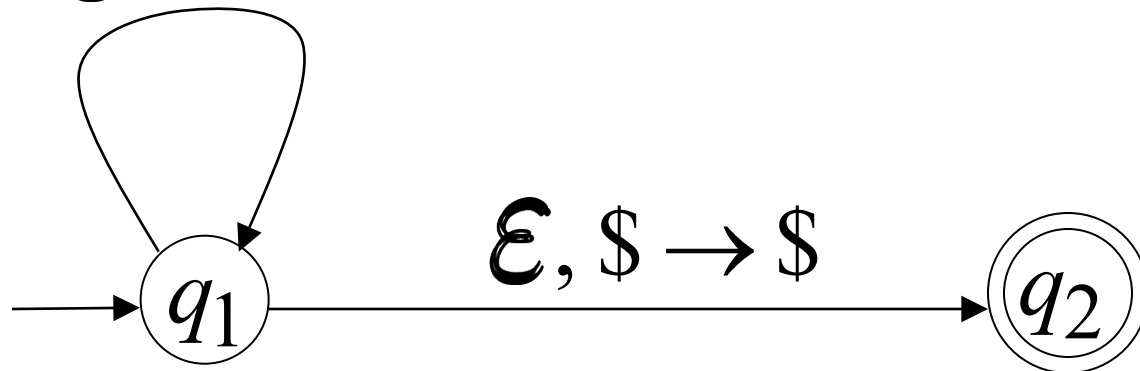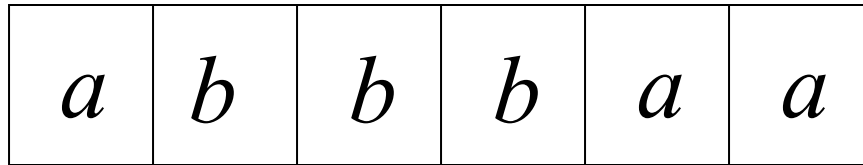
$\varepsilon, \$ \rightarrow \$$

$q_1 \qquad q_2$

# Execution Example:

Input

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|-----|-----|-----|-----|-----|-----|

$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$
$$a, 0 \to 00 \qquad b, 1 \to 11$$
$$a, 1 \to \varepsilon \qquad b, 0 \to \varepsilon$$

| $\$$ |
|------|

Stack

current
state

$$\varepsilon, \$ \to \$$$

$q_1 \qquad q_2$

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**Input**

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|-----|-----|-----|-----|-----|-----|

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$
$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$
$$a, 1 \rightarrow \varepsilon \qquad b, 0 \rightarrow \varepsilon$$

| 0 |
|---|
| $\$$ |

**Stack**

$$\varepsilon, \$ \rightarrow \$$$

$q_1 \qquad q_2$

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Input

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|-----|-----|-----|-----|-----|-----|

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \varepsilon \qquad \boxed{b, 0 \rightarrow \varepsilon}$$

$$\varepsilon, \$ \rightarrow \$$$

$q_1$ $\qquad$ $q_2$

Stack

| 0 |
|---|
| $\$$ |

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Input

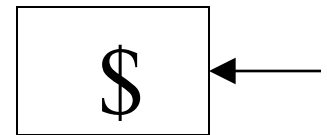| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|-----|-----|-----|-----|-----|-----|

$a, \$ \rightarrow 0\$$  $b, \$ \rightarrow 1\$$
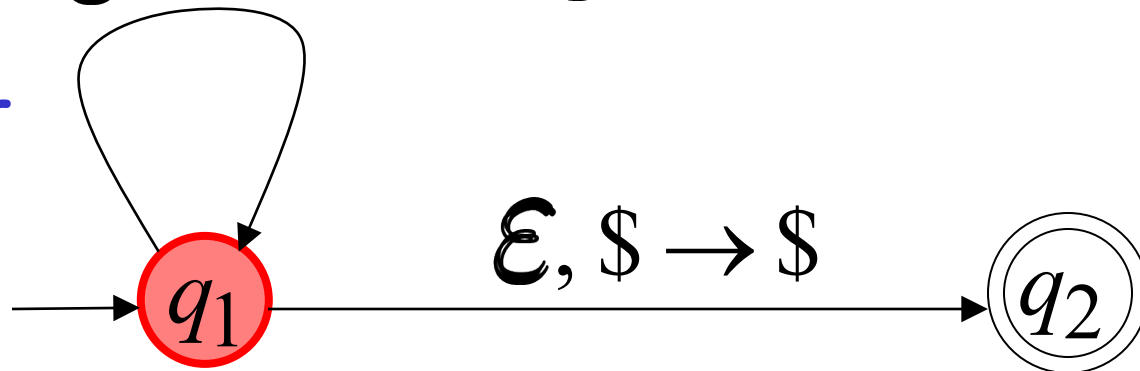
$a, 0 \rightarrow 00$  $b, 1 \rightarrow 11$

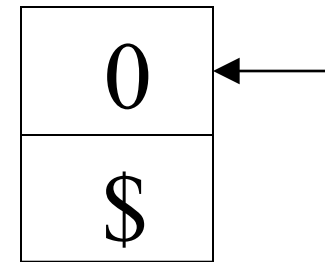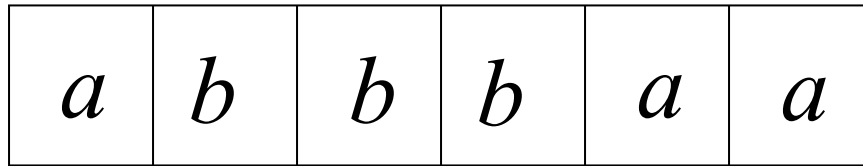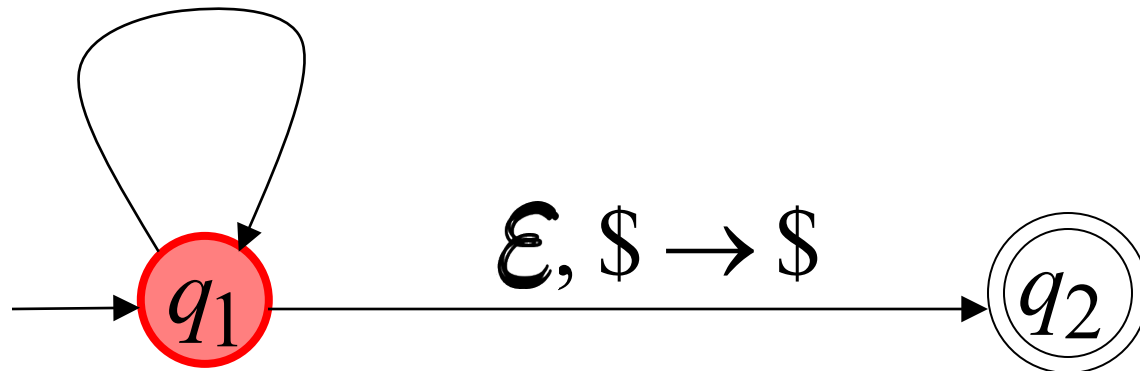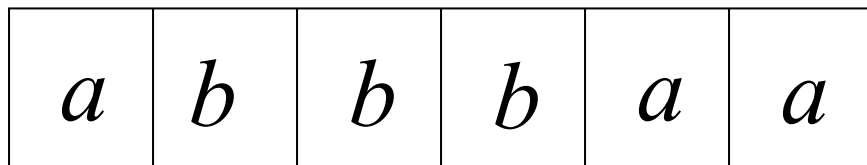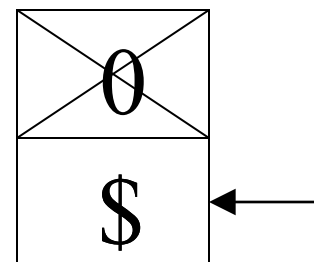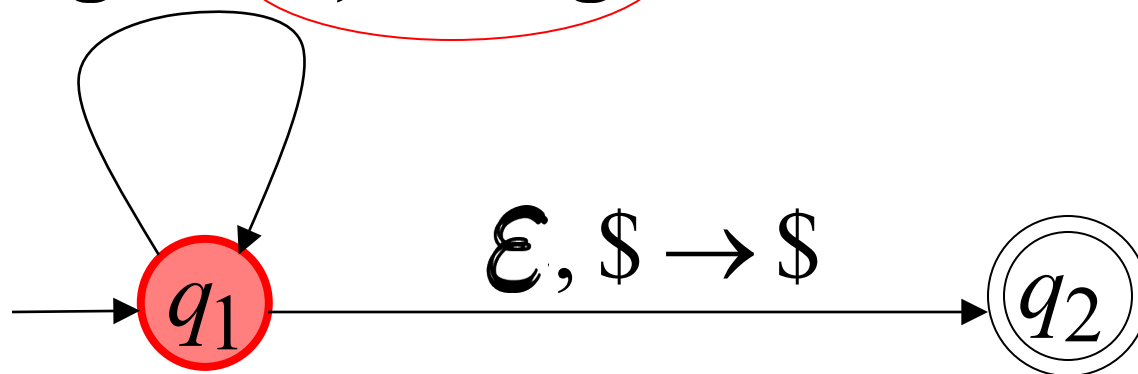$a, 1 \rightarrow \varepsilon$  $b, 0 \rightarrow \varepsilon$

| 1 |
|---|
| \$ |

Stack

$\varepsilon, \$ \rightarrow \$$

$q_1$  $q_2$

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Input

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|-----|-----|-----|-----|-----|-----|

| 1 |
|---|
| 1 |
| $ |

Stack

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \varepsilon \qquad b, 0 \rightarrow \varepsilon$$

$$\varepsilon, \$ \rightarrow \$$$

$q_1 \qquad q_2$

## Input

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|-----|-----|-----|-----|-----|-----|

Stack:

| 1 |
|---|
| 1 |
| $ |

**Stack**

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \varepsilon \qquad b, 0 \rightarrow \varepsilon$$

$$\varepsilon, \$ \rightarrow \$$$

$q_1 \qquad q_2$

Input

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|---|---|---|---|---|---|

$a, \$ \to 0\$$     $b, \$ \to 1\$$

$a, 0 \to 00$     $b, 1 \to 11$

$a, 1 \to \varepsilon$     $b, 0 \to \varepsilon$

Stack

| 1 |
|---|
| $\$$ |

$\varepsilon, \$ \to \$$

$q_1$      $q_2$

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Input

| $a$ | $b$ | $b$ | $b$ | $a$ | $a$ |
|---|---|---|---|---|---|

$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$
$$a, 0 \to 00 \qquad b, 1 \to 11$$
$$a, 1 \to \varepsilon \qquad b, 0 \to \varepsilon$$

$\boxed{\$}$ ←

Stack

accept

$\varepsilon, \$ \to \$$

$q_1$ ⟶ $q_2$

# Formalities for PDAs

Transition function:
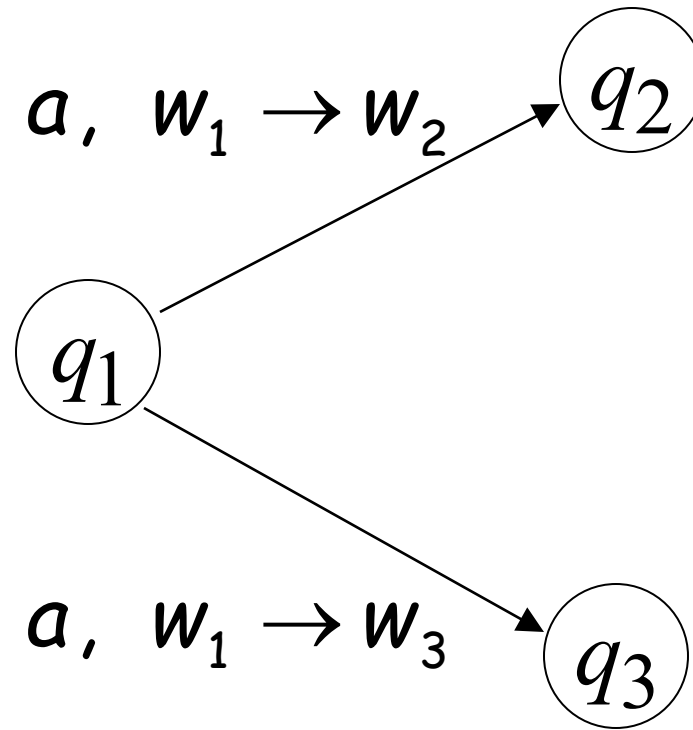
$$\delta(q_1, a, w_1) = \{(q_2, w_2)\}$$

Transition function:

$$\delta(q_1, a, w_1) = \{(q_2, w_2), \ (q_3, w_3)\}$$

# Formal Definition

Pushdown Automaton (PDA)

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

Accept states

States

Input alphabet

Stack alphabet

Transition function

Initial state

Stack start symbol

# Instantaneous Description

$$(q, \ u, \ s)$$

Current state

Remaining input

Current stack contents
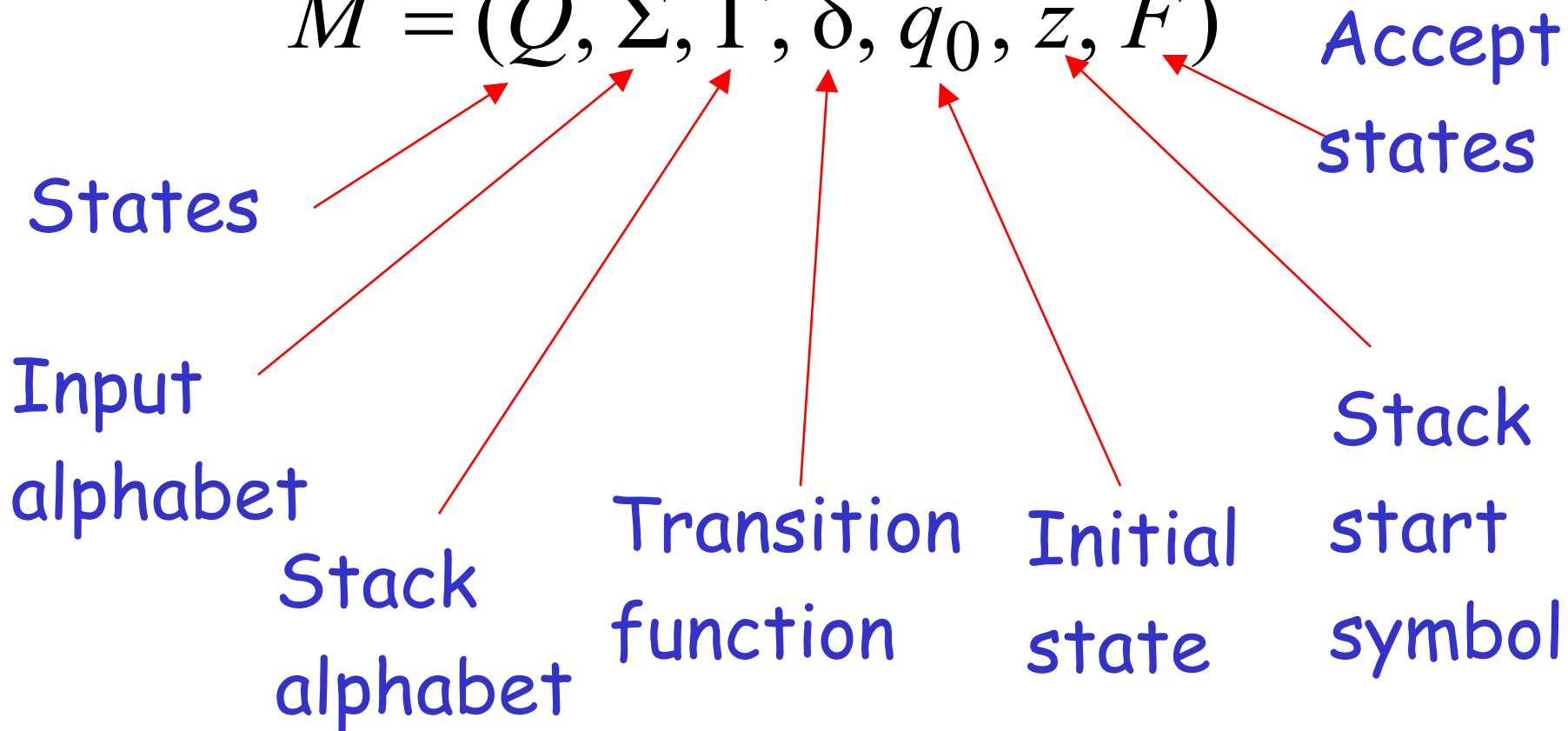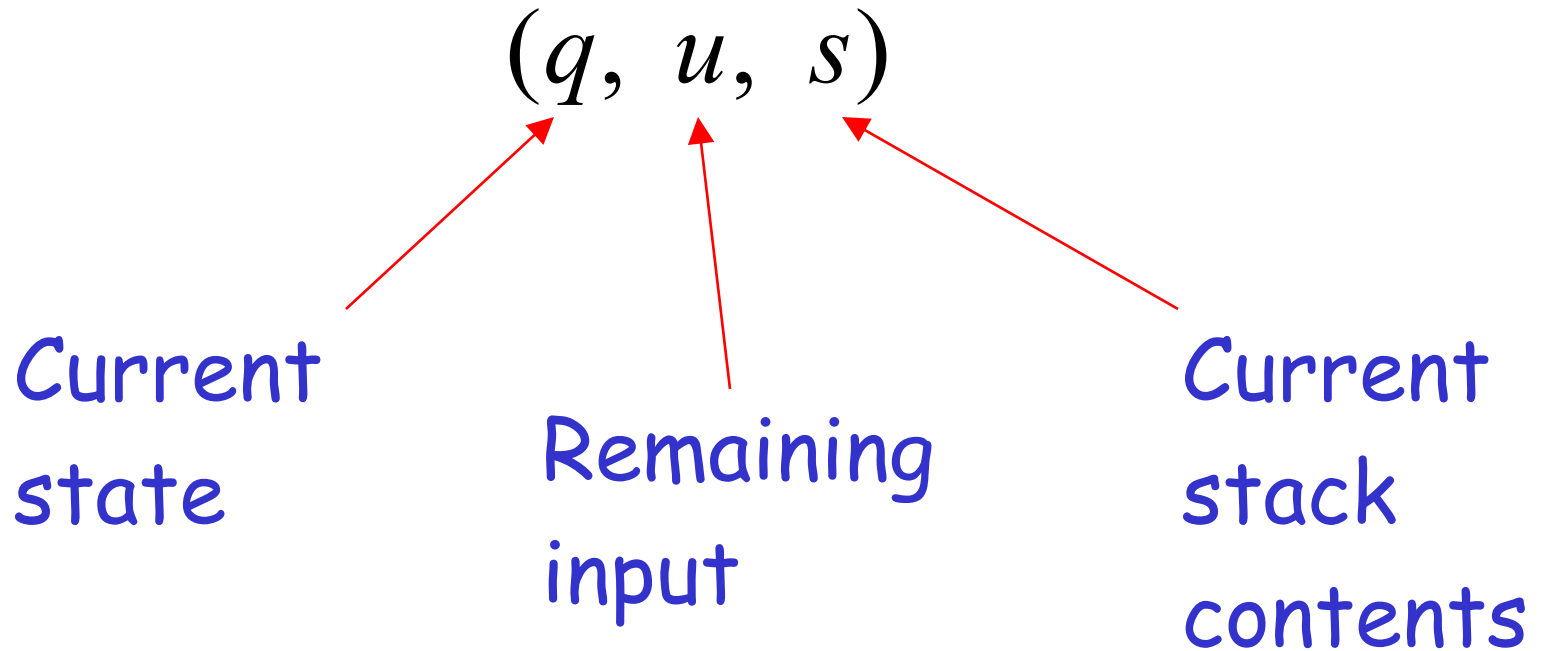
# Example:    Instantaneous Description

$$(q_1, bbb, aaa\$)$$

Time 4:    Input
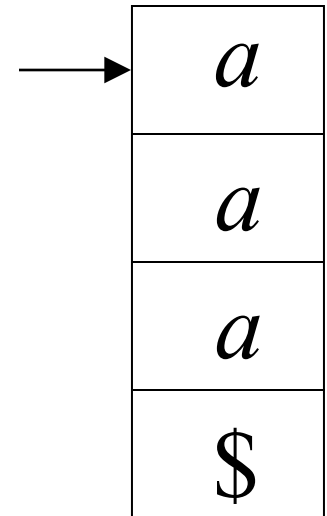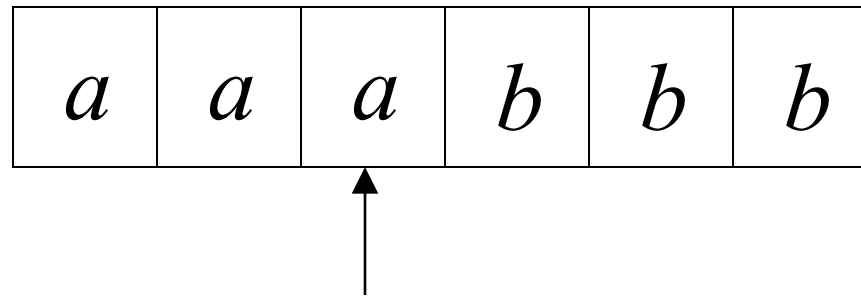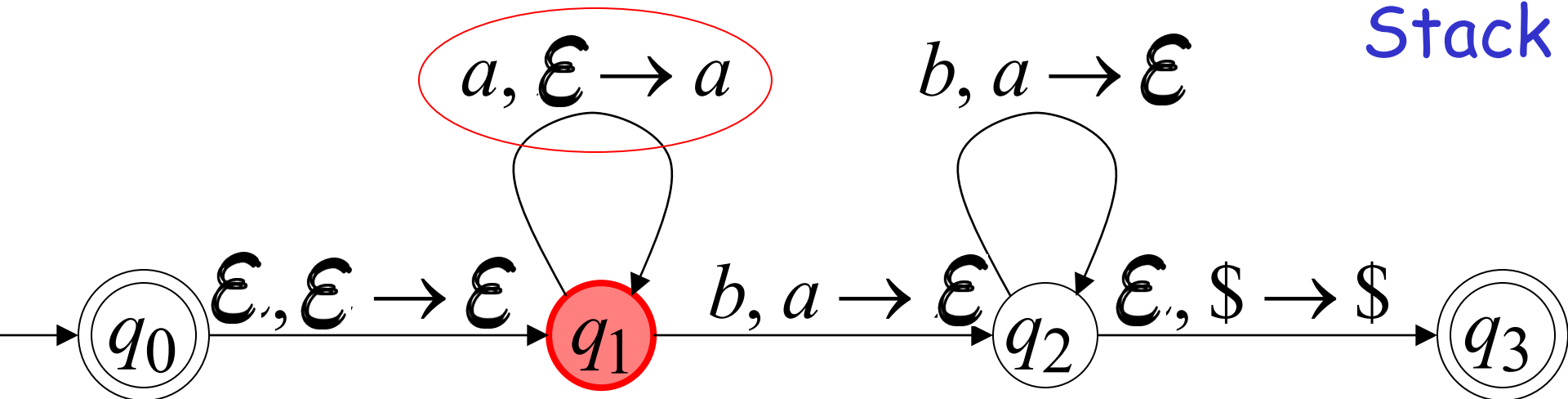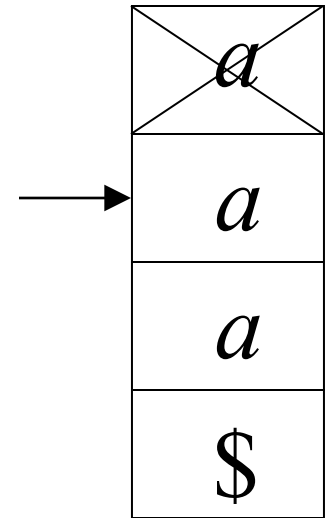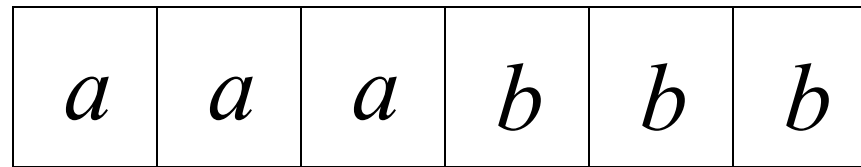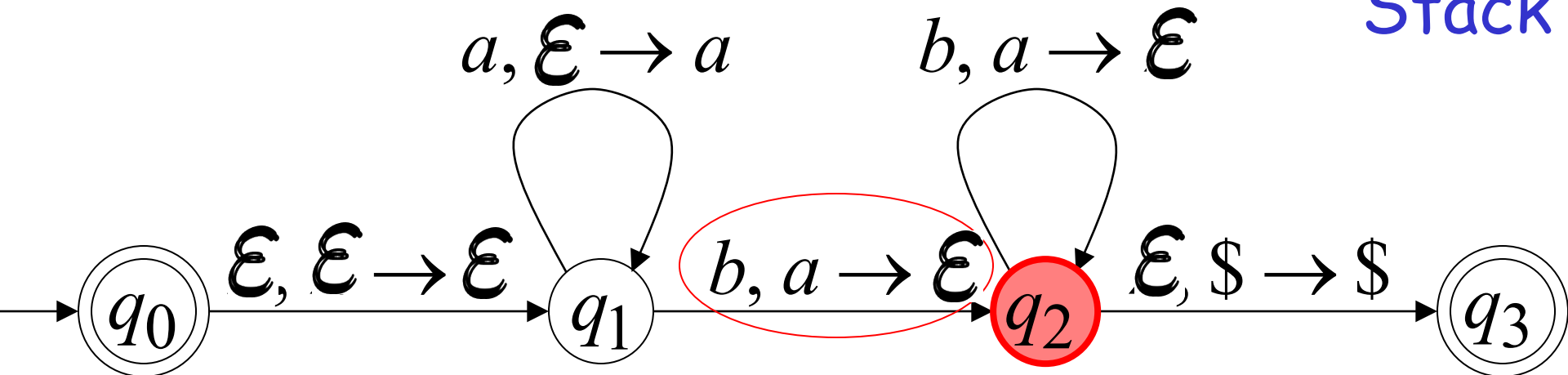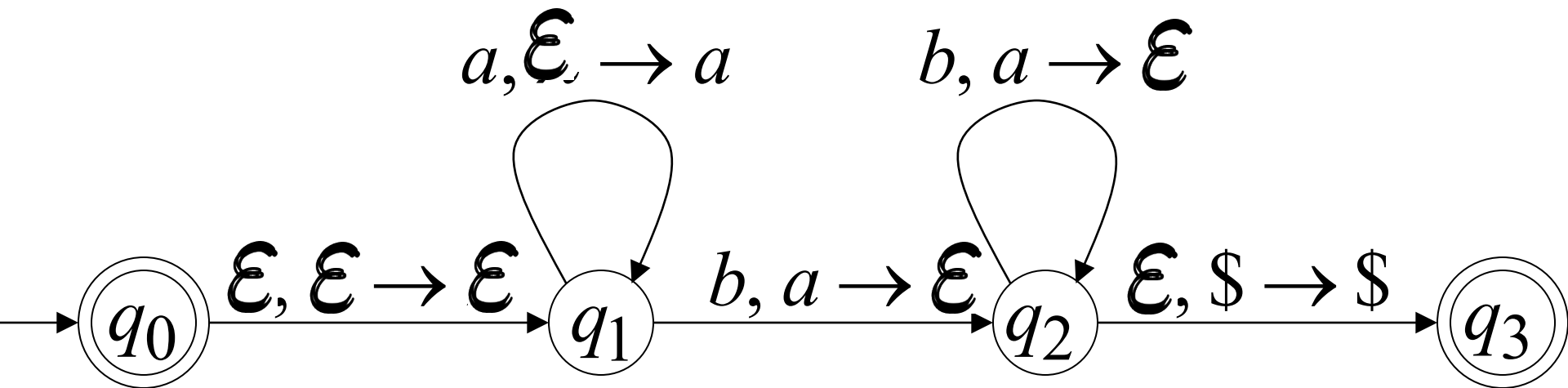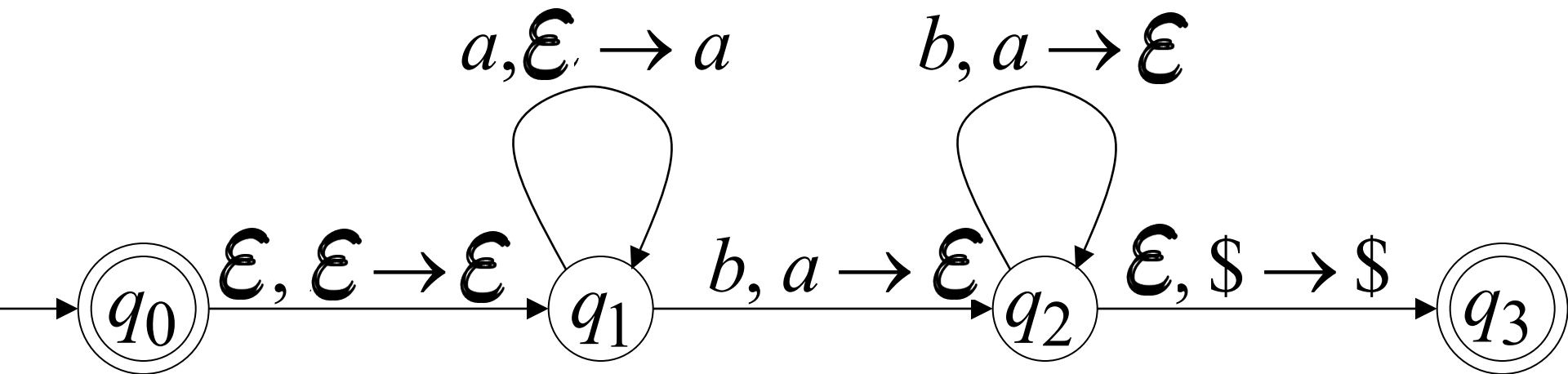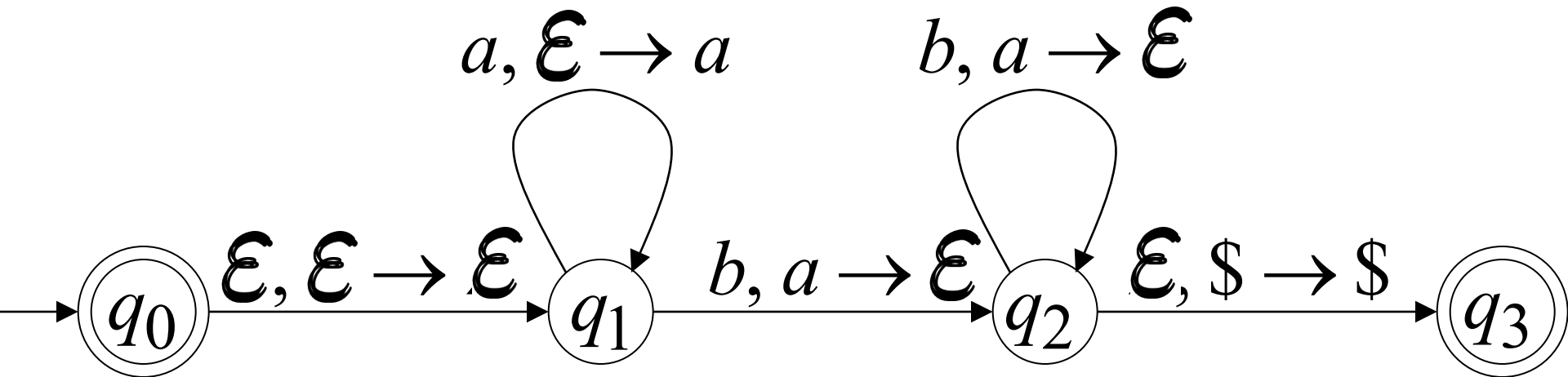
| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

| $a$ |
|-----|
| $a$ |
| $a$ |
| $\$$ |

Stack

$a, \varepsilon \rightarrow a$

$b, a \rightarrow \varepsilon$

$q_0$ $\;\; \varepsilon, \varepsilon \rightarrow \varepsilon \;\;$ $q_1$ $\;\; b, a \rightarrow \varepsilon \;\;$ $q_2$ $\;\; \varepsilon, \$ \rightarrow \$ \;\;$ $q_3$

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# Example: Instantaneous Description

$$(q_2, bb, aa\$)$$

**Time 5:**

Input

| $a$ | $a$ | $a$ | $b$ | $b$ | $b$ |
|-----|-----|-----|-----|-----|-----|

Stack

$a, \varepsilon \rightarrow a$ $\qquad$ $b, a \rightarrow \varepsilon$

$q_0$ $\quad \varepsilon, \varepsilon \rightarrow \varepsilon \quad$ $q_1$ $\quad b, a \rightarrow \varepsilon \quad$ $q_2$ $\quad \varepsilon, \$ \rightarrow \$ \quad$ $q_3$

We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4                    Time 5

# A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$$

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \varepsilon, \$) \succ (q_3, \varepsilon, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \stackrel{*}{\succ} (q_3, \varepsilon, \$)$$

# Language of PDA

Language $L(M)$ accepted by PDA $M$ :

$$L(M) = \{w : \quad (q_0, w, z) \overset{*}{\succ} (q_f, \varepsilon, s)\}$$

Initial state

Accept state

Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

$$aaabbb \in L(M)$$

PDA $M$:

$$a, \varepsilon \to a \qquad b, a \to \varepsilon$$

$$q_0 \quad \varepsilon, \varepsilon \to \varepsilon \quad q_1 \quad b, a \to \varepsilon \quad q_2 \quad \varepsilon, \$ \to \$ \quad q_3$$

$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \varepsilon, \$)$$

$$a^n b^n \in L(M)$$

PDA $M$ :

$$a, \varepsilon \rightarrow a \qquad b, a \rightarrow \varepsilon$$

$$\varepsilon, \varepsilon \rightarrow \varepsilon \qquad b, a \rightarrow \varepsilon \qquad \varepsilon, \$ \rightarrow \$$$

$q_0 \quad q_1 \quad q_2 \quad q_3$

Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

PDA $M$ :

$$a, \varepsilon \to a \qquad\qquad b, a \to \varepsilon$$

$$\begin{array}{ccccc} q_0 & \xrightarrow{\ \varepsilon, \varepsilon \to \varepsilon\ } & q_1 & \xrightarrow{\ b, a \to \varepsilon\ } & q_2 & \xrightarrow{\ \varepsilon, \$ \to \$\ } & q_3 \end{array}$$