For convenience, we will sometimes describe concepts somewhat informally if more formalism is not necessary for the purpose of this talk

***E.g.*, we may simply say**

Hilbert space

even though most of our discussions will pertain to

finite dimensional complex vector space

Welcome to the world of quantum mechanics

where intuitions can be misleading

and mysterious events may happen

Often we hear remarks like:

Explain the results **intuitively**

What is the **intuition** behind this claim?

This result makes no sense **intuitively**

⋮

In quantum world, “classical intuition” should be considered with a grain of salt

Albert Einstein (1905):

- light is composed of discrete quanta called **photons**
- received Nobel Prize (1921) for using this concept to explain photoelectric effects on metals

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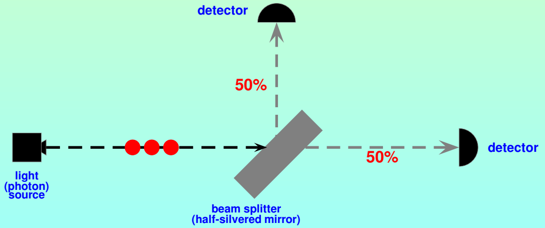
Experimental setup with one beam splitter



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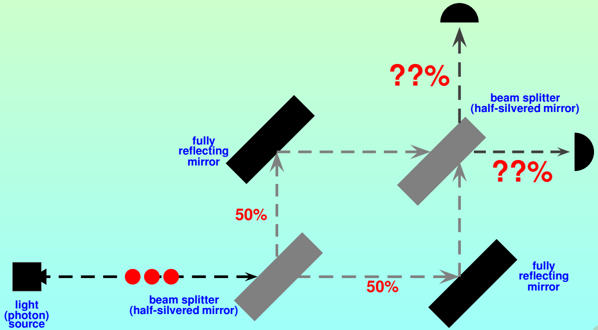

You may say: this is **intuitively** straightforward probability calculation !!

Elementary Quantum Mechanics

Why classical intuition may be confusing: an example from physics of light

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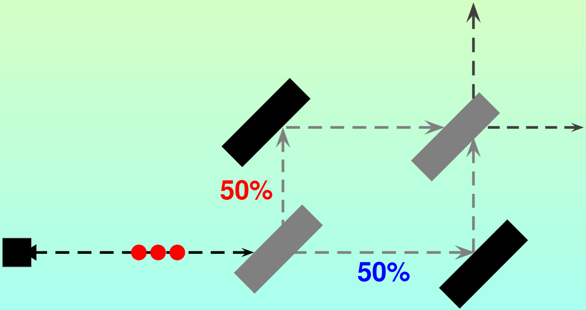

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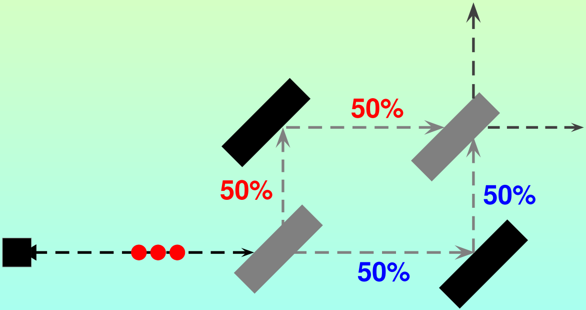

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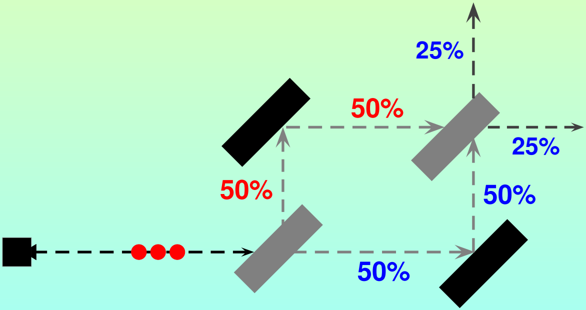

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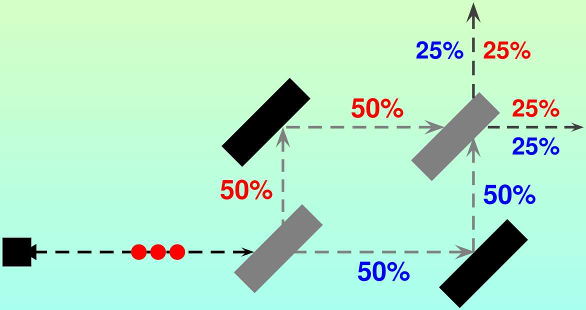

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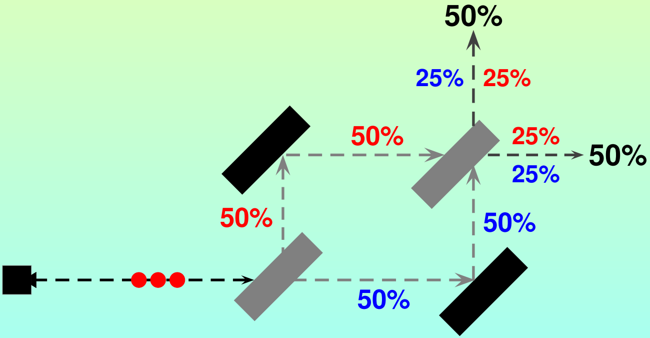

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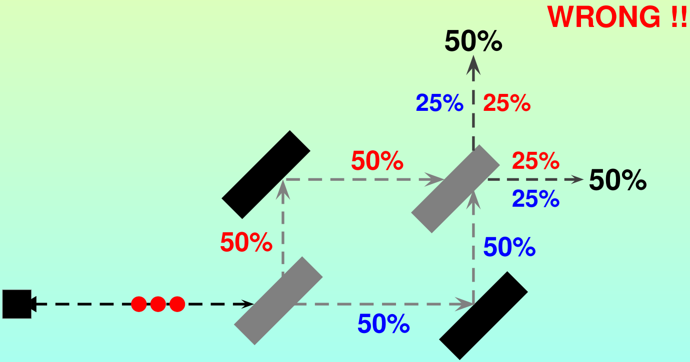

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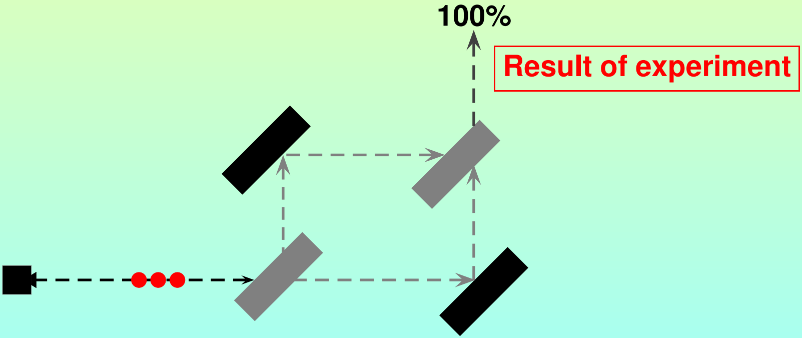
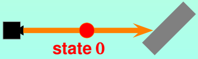


Elementary Quantum Mechanics

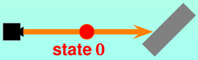
Why classical intuition may be confusing: an example from physics of light

Classical intuition



Classical intuition

When a photon hits the beam splitter



Elementary Quantum Mechanics

Why classical intuition may be confusing: an example from physics of light

Classical intuition

When a photon hits the beam splitter

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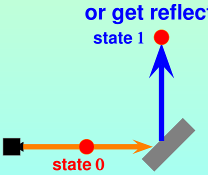

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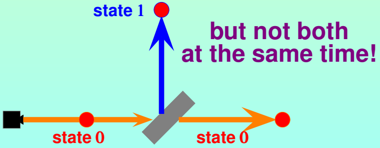

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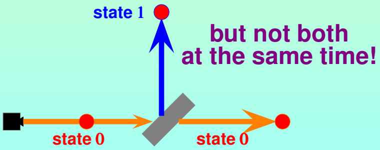


Quantum intuition (informally!)

The photon is in both states at the same time!
(even if counter-intuitive)

state 0

$$|0\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix},$$

(Dirac notation)

state 1

$$|1\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$$

(Dirac notation)

photon is allowed to be in a **complex super-imposed** state	what if we want to observe the photon ?
$a\,\vert 0\rangle + b\,\vert 1\rangle = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$	$\Pr\left[\text{photon is observed at state } \vert 0\rangle\right] = \vert a\vert^2$
$\vert a\vert^2 + \vert b\vert^2 = 1$	$\Pr\left[\text{photon is observed at state } \vert 1\rangle\right] = \vert b\vert^2$

Please note that

state of photon is $\begin{pmatrix} a \\ b \end{pmatrix}$ **(the real world)**

is not the same as saying

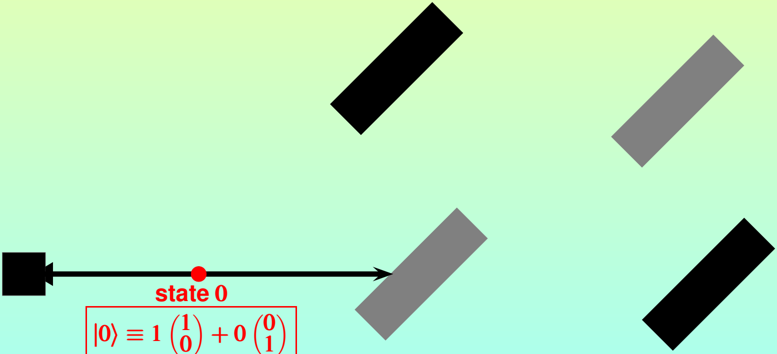
$$\Pr\left[\text{photon is at state } |0\rangle\right] = |a|^2$$
$$\Pr\left[\text{photon is at state } |1\rangle\right] = |b|^2$$

limitations of "any" measurement method

Elementary Quantum Mechanics

Quantum explanation of the beam splitter experiment

$i = \sqrt{-1}$



$$\boxed{|0\rangle \equiv 1\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix}}$$

Elementary Quantum Mechanics

Quantum explanation of the beam splitter experiment

$i = \sqrt{-1}$

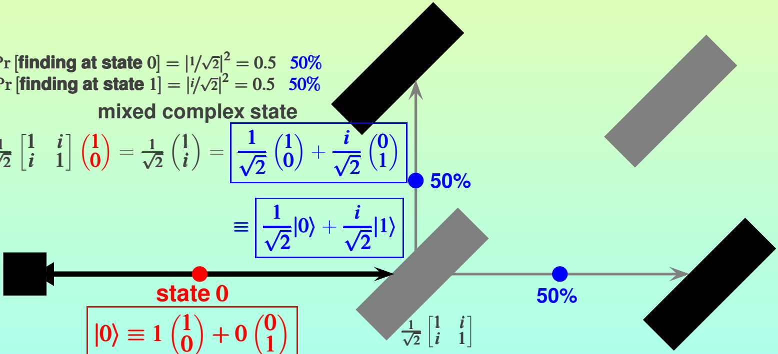
$\Pr[\textbf{finding at state } 0] = |1/\sqrt{2}|^2 = 0.5$ 50%

$\Pr[\textbf{finding at state } 1] = |i/\sqrt{2}|^2 = 0.5$ 50%

mixed complex state

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}}$$

$$\equiv \boxed{\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle}$$



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Elementary Quantum Mechanics

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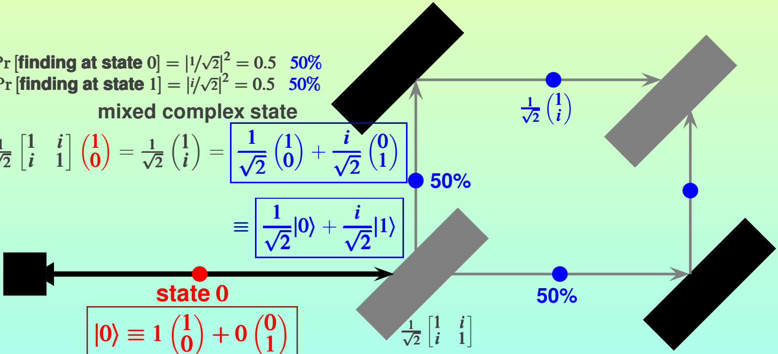

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Elementary Quantum Mechanics

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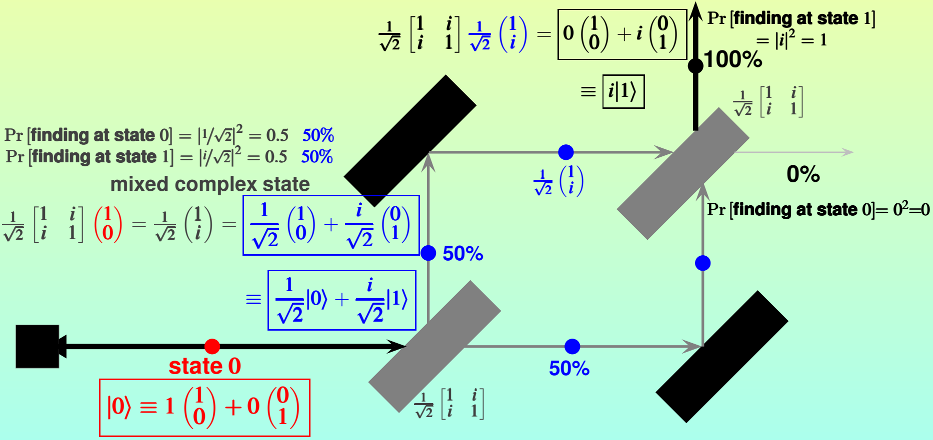


Summary of quantum explanation of the beam splitter experiment

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow[\text{linear operator}]{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\times} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \xrightarrow[\text{linear operator}]{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\times} i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$|0\rangle$ — **pure state**

$\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ — **mixed state**

$i\,|1\rangle$ — **pure state**

Consider a counter with $n = 2$ bits

Traditional Computer Science notation

states of counter	00	01	10	11
	0	1	2	3

Dirac notation

states of counter

$$\left.\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}2^n \equiv \underbrace{|00\rangle}_{n=2} \qquad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \equiv \underbrace{|01\rangle}_{n=2} \qquad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \equiv \underbrace{|10\rangle}_{n=2} \qquad \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \equiv \underbrace{|11\rangle}_{n=2}$$

Examples

$$\sqrt{\frac{2}{3}}\,|00\rangle + \frac{i}{\sqrt{3}}\,|11\rangle = \sqrt{\frac{2}{3}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{i}{\sqrt{3}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{2}{3}}\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\\frac{i}{\sqrt{3}}\end{pmatrix} = \begin{pmatrix}\sqrt{\frac{2}{3}}\\0\\0\\\frac{i}{\sqrt{3}}\end{pmatrix}$$

Complex conjugate, dot products etc.

Traditional math notations	Dirac notation
$\psi = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ \frac{i}{\sqrt{3}} \end{pmatrix}, \ \phi = \begin{pmatrix} \frac{i}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}$	$\lvert\psi\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ \frac{i}{\sqrt{3}} \end{pmatrix}, \ \lvert\phi\rangle = \begin{pmatrix} \frac{i}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}$
$\psi^* = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & \frac{-i}{\sqrt{3}} \end{pmatrix}$	$\langle\psi\rvert = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & \frac{-i}{\sqrt{3}} \end{pmatrix}$
$\psi^*\psi = \lvert\lvert\psi\rvert\rvert^2$	$\langle\psi\rvert\,\lvert\psi\rangle = \langle\psi\lvert\psi\rangle = \lvert\lvert\psi\rvert\rvert^2$
$\psi^* \cdot \phi = \begin{pmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & \frac{-i}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \frac{i}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} = 0$	$\langle\psi\rvert\,\lvert\phi\rangle = \langle\psi\lvert\phi\rangle = 0$

Hilbert space defines an inner product operation between two vectors (dot product shown before is an example of inner product)

Inner product operator $\langle\ ,\ \rangle$ satisfies the following three *axioms*:

linearity in second argument $\left\langle \mathbf{v}, \sum_i \lambda_i \mathbf{w}_i \right\rangle = \sum_i \lambda_i \langle \mathbf{v}, \mathbf{w}_i \rangle$, λ_i**'s are scalars**

conjugate symmetry $\forall\, \mathbf{v}, \mathbf{w}\colon\ \langle \mathbf{v}, \mathbf{w} \rangle$ **is complex conjugate of** $\langle \mathbf{w}, \mathbf{v} \rangle$

$$\forall\, \mathbf{v}, \mathbf{w}\colon\ \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle^*$$

positive definitiveness $\forall \mathbf{v}\colon\ \langle \mathbf{v}, \mathbf{v} \rangle \geq 0$

$\langle \mathbf{v}, \mathbf{v} \rangle = 0$ **only if** $\mathbf{v} = \mathbf{0}$

Elementary Quantum Mechanics

Dirac notations and other basic concepts (in complex Hilbert space)

Orthonormal basis of a Hilbert space $\mathcal{H}$

set of vectors $|b_1\rangle, |b_2\rangle, \ldots, |b_m\rangle$ such that

- $$\langle b_i \,|\, b_j\rangle = \underbrace{\delta_{i,j}}_{\text{Kronecker delta}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$ (mutually orthonormal)

- Any vector v can be written as a linear combination of basis:

$$|v\rangle = \lambda_1 |b_1\rangle + \lambda_2 |b_2\rangle + \cdots + \lambda_m |b_m\rangle$$

Example (dimension 2): standard basis

$|b_1\rangle = |0\rangle = \binom{1}{0}$ $|b_2\rangle = |1\rangle = \binom{0}{1}$

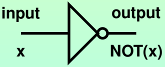
$$|v\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \sqrt{\frac{2}{3}}\,|0\rangle + \frac{1}{\sqrt{3}}\,|1\rangle$$

Example (dimension 2): Hadamard basis

$$|b_1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|b_1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Traditional CS view of circuit



truth table

x	NOT(x)
0	1
1	0

Linear algebraic view of circuit

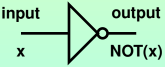
$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{bmatrix}? & ?\\? & ?\end{bmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{bmatrix}? & ?\\? & ?\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

Elementary Quantum Mechanics

Linear operators (in Hilbert space)

Traditional CS view of circuit



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Elementary Quantum Mechanics

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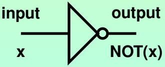

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More generally,

$$\underbrace{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}}_{\text{linear operator}}\begin{pmatrix}p_o\\p_1\end{pmatrix} = \begin{pmatrix}p_1\\p_0\end{pmatrix}$$

$$\text{NOT}\begin{pmatrix}p_o\\p_1\end{pmatrix}$$

more examples of linear operators and algebra in Dirac notation

$$\begin{array}{ccccc} & |0\rangle & \mapsto & |0\rangle \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}$$

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & = & \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ & |1\rangle & \mapsto & -|1\rangle \end{array}$$

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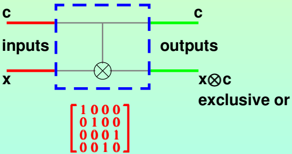
operator in Dirac notation: $|0\rangle\langle 0| - |1\rangle\langle 1|$

$$\begin{aligned} \Big(|0\rangle\langle 0| - |1\rangle\langle 1| \Big) |0\rangle &= |0\rangle\langle 0|0\rangle - |1\rangle\langle 1|0\rangle \\ &= |0\rangle \end{aligned}$$

$$\begin{aligned} \Big(|0\rangle\langle 0| - |1\rangle\langle 1| \Big) |1\rangle &= |0\rangle\langle 0|1\rangle - |1\rangle\langle 1|1\rangle \\ &= -|1\rangle \end{aligned}$$

another example: control-NOT operator



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underset{\substack{c=1 \\ x=0}}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} = \underset{\substack{c=1 \\ x=1}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

Unitary operators

operators in time evolution of quantum states in a closed system are always unitary

operator $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ **is unitary**

because

$$\underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{A^\dagger} \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

complex conjugate transpose

identity matrix

unitary operators are nice

- **preserves norms of vectors**
- **preserves inner products of vectors**

etc.

Elementary Quantum Mechanics

Tensor products

Classical way of building composite systems

two individual probabilistic bits

$\Pr[\textbf{value = 0}] = 1/3 \quad \Pr[\textbf{value = 0}] = 1/2$

$\Pr[\textbf{value = 1}] = 2/3 \quad \Pr[\textbf{value = 1}] = 1/2$

$$\Downarrow$$

$00 \equiv 0$

$01 \equiv 1$

$10 \equiv 2$

$11 \equiv 3$

$\Pr[\textbf{value = 0}] = 1/6$

$\Pr[\textbf{value = 1}] = 1/6$

$\Pr[\textbf{value = 2}] = 1/3$

$\Pr[\textbf{value = 3}] = 1/3$

composite two bit system

Classical way of building composite systems

two individual probabilistic bits

$\Pr[\text{value} = 0] = 1/3$ $\Pr[\text{value} = 0] = 1/2$

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$00 \equiv 0$
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$\Pr[\text{value} = 0] = 1/6$
$\Pr[\text{value} = 1] = 1/6$
$\Pr[\text{value} = 2] = 1/3$
$\Pr[\text{value} = 3] = 1/3$

composite two bit system

Quantum way of building composite systems via tensor products

two individual quantum bits

$$\begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix} = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle \qquad \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$\Downarrow$

$00 \equiv 0$
$01 \equiv 1$
$10 \equiv 2$
$11 \equiv 3$

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{2}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

composite two bit quantum system

Tensor product in standard vector notation

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \otimes \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_0\, q_0 \\ p_0\, q_1 \\ p_1\, q_0 \\ p_1\, q_1 \end{pmatrix}$$

Tensor product in Dirac notation

$$\begin{aligned} & \left(p_0\,|0\rangle + p_1\,|1\rangle\right) \otimes \left(q_0\,|0\rangle + q_1\,|1\rangle\right) \\ = \;& p_0q_0\,|0\rangle \otimes |0\rangle \;+\; p_0q_1\,|0\rangle \otimes |1\rangle \;+\; p_1q_0\,|1\rangle \otimes |0\rangle \;+\; p_1q_1\,|1\rangle \otimes |1\rangle \\ = \;& p_0q_0\,|00\rangle \;+\; p_0q_1\,|01\rangle \;+\; p_1q_0\,|10\rangle \;+\; p_1q_1\,|11\rangle \end{aligned}$$

comment about notation

instead of $|x\rangle \otimes |y\rangle$, one often writes $|x\rangle\,|y\rangle$ or even simply $|xy\rangle$

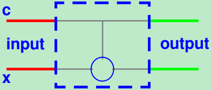
Entanglements

Linear operators can be used to combine two state vectors but sometimes they may act as a "super-glue"

$$\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$$

$$\boxed{\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \boxed{\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |0\rangle$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CNOT

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}}$$

Entanglements

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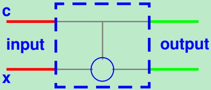
$\begin{pmatrix}1/2\\1/2\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix}$

$|0\rangle$



$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

CNOT

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{pmatrix}1/2\\0\\1/2\\0\end{pmatrix} = \boxed{\begin{pmatrix}1/2\\0\\0\\1/2\end{pmatrix}} \neq \begin{pmatrix}\alpha_0\\\alpha_1\end{pmatrix} \otimes \begin{pmatrix}\beta_0\\\beta_1\end{pmatrix}$$

for any $\alpha_0, \alpha_1, \beta_0, \beta_1$

the two bits are "entangled"

they cannot be decoupled

Axioms of Tensor product operator $\otimes$

1. $c\left(|\psi_1\rangle \otimes |\psi_2\rangle\right) = \left(c\,|\psi_1\rangle\right) \otimes |\psi_2\rangle = |\psi_1\rangle \otimes \left(c\,|\psi_2\rangle\right)$
2. $\left(|\psi_1\rangle + |\psi_2\rangle\right) \otimes |\psi_3\rangle = |\psi_1\rangle \otimes |\psi_3\rangle + |\psi_2\rangle \otimes |\psi_3\rangle$
3. $|\psi_1\rangle \otimes \left(|\psi_2\rangle + |\psi_3\rangle\right) = |\psi_1\rangle \otimes |\psi_2\rangle + |\psi_1\rangle \otimes |\psi_3\rangle$

State space postulate

A quantum state is a unit vector in a complex Hilbert space

$$\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_k \end{pmatrix}, \quad \forall i \; p_i \in \mathbb{C}, \quad \sum_{i=1}^{k} |p_i|^2 = 1$$

For our purpose, $k = 2^n$ for some positive integer n

Time dynamics postulate

for a closed quantum system, the state evolves only via unitary operators

$$|\psi_{t+1}\rangle = U\,|\psi_t\rangle, \;\; U \text{ is unitary, } \textit{i. e.}, \; U^\dagger U = I$$

Composition of systems postulate

two quantum systems Q_1 and Q_2 can be combined to a composite system Q_3 using the tensor operator $\otimes$

if $|\psi_1\rangle$ is the state of Q_1 and $|\psi_2\rangle$ is the state of Q_2, then $|\psi_2\rangle \otimes |\psi_2\rangle$ is the state of the composite system

Measurement postulate (simplest version)

Suppose that a quantum system is in the following state

$$\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{k-1} \\ p_k \end{pmatrix} = p_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \cdots + p_k \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$p_1 \left|0\right\rangle + p_2 \left|1\right\rangle + \cdots + p_k \left|k\right\rangle$$

Suppose that we want to measure the state $\mathbf{P}$
Then, with probability $|p_i|^2$:

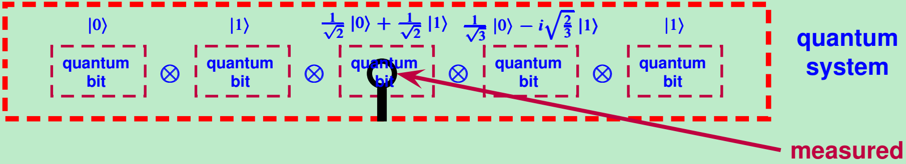
- **our measurement will show P to be in state $\left|i\right\rangle$**
- **P will change to the deterministic state $\left|i\right\rangle$**
  (measurement destroys quantum nature of the state P)

Measurement postulate is extremely "counter-intuitive" specially if we have a classical notion of "intuition"

$\vert 0\rangle$		$\vert 1\rangle$		$\frac{1}{\sqrt{2}}\vert 0\rangle + \frac{1}{\sqrt{2}}\vert 1\rangle$		$\frac{1}{\sqrt{3}}\vert 0\rangle - i\sqrt{\frac{2}{3}}\vert 1\rangle$		$\vert 1\rangle$
quantum bit	$\otimes$	quantum bit	$\otimes$	quantum bit	$\otimes$	quantum bit	$\otimes$	quantum bit

quantum system

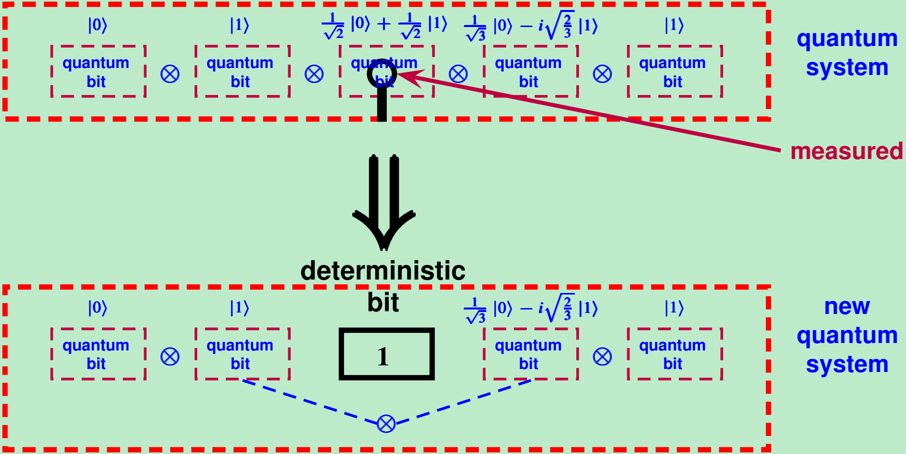
Measurement postulate is extremely "counter-intuitive" specially if we have a classical notion of "intuition"



Elementary Quantum Mechanics

Measurement postulate

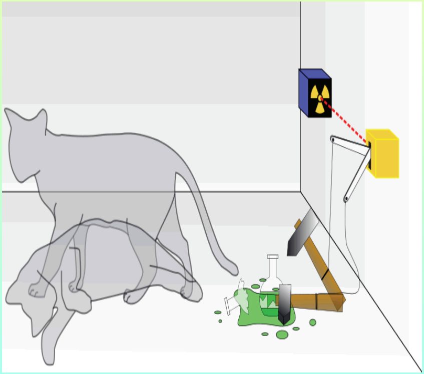
Measurement postulate is extremely "counter-intuitive" specially if we have a classical notion of "intuition"



note that the measurement postulate is not dependent on the actual method of measurement used

it does not matter if we measure by shining a beam of light, or by using radiation detector, or by some indirect method

Schrödinger's cat: a thought experiment (Erwin Schrödinger, 1935)



a transparent box containing

- a nice cat (secured by chain)
- flask containing poison
- Geiger counter with small amount of radioactive substance

  (state of radioactive atoms evolve according to quantum mechanics rules)

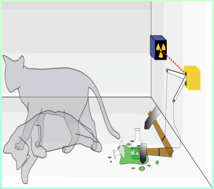
at any time t:

- one of the radioactive atoms may decay, but also perhaps none decays
- if an atom decays, a relay releases a hammer that shatters a small flask, releases poison, and kills cat

Measurement postulate (a seemingly paradoxical situation)

Schrödinger's cat: a thought experiment

at any time t, is the cat dead or alive ?



According to quantum mechanics, the state of the entire box is a superposition of various states

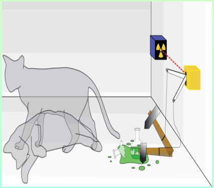
$$\underbrace{\alpha_1 |\psi_1\rangle}_{\text{cat is alive}} + \underbrace{\alpha_2 |\psi_2\rangle}_{\text{cat is dead}} + \underbrace{\alpha_3 |\psi_3\rangle}_{\text{cat is dead}} + \underbrace{\alpha_4 |\psi_4\rangle}_{\text{cat is alive}} + \cdots\cdots\cdots$$

thus, the cat is in some sense both dead and alive

Measurement postulate (a seemingly paradoxical situation)

Schrödinger's cat: a thought experiment

at any time t, is the cat dead or alive ?



According to quantum mechanics, the state of the entire box is a superposition of various states

$$\underset{\substack{\Uparrow\\ \text{cat}\\ \text{is}\\ \text{alive}}}{\alpha_1\,|\psi_1\rangle} \;+\; \underset{\substack{\Uparrow\\ \text{cat}\\ \text{is}\\ \text{dead}}}{\alpha_2\,|\psi_2\rangle} \;+\; \underset{\substack{\Uparrow\\ \text{cat}\\ \text{is}\\ \text{dead}}}{\alpha_3\,|\psi_3\rangle} \;+\; \underset{\substack{\Uparrow\\ \text{cat}\\ \text{is}\\ \text{alive}}}{\alpha_4\,|\psi_4\rangle} \;+\; \ldots\ldots\ldots$$

thus, the cat is in some sense both dead and alive

But, according to measurement postulate, if you look at the cat, then:
the cat will change to an alive or a dead state permanently