CS 506: Introduction to Quantum Computing

Lecture 1 Notes

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1 Two Formulations of Quantum Mechanics

There are two equivalent formulations:

- **Schrödinger Picture**: State evolves with time, operators are fixed.
- **Heisenberg Picture**: Operators evolve with time, states are fixed.

Quantum computing mainly uses the Heisenberg picture due to its natural use of linear algebra and Hilbert space.

2 Heisenberg Matrix Mechanics

Quantum systems are represented using vectors and matrices inside a complex Hilbert space.

3 Types of Bits

3.1 Classical Bit

A classical bit can be either 0 or 1.

3.2 Probabilistic Bit

A probabilistic bit can be 0 or 1 with probability:

$$P(0) = P(1)$$

3.3 Qubit

A qubit is represented as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are called the computational basis states.

A general qubit:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and must satisfy:

$$|\alpha|^2 + |\beta|^2 = 1$$

4 Examples of Qubits

4.1 Real Amplitudes

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1\rangle$$

Vector form:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

4.2 Complex Amplitudes

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|1\rangle$$

Vector form:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ i\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

5 Two-Qubit System

Basis states:

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

Breakdown:

$$|00\rangle = |0\rangle|0\rangle, \quad |01\rangle = |0\rangle|1\rangle$$
$$|10\rangle = |1\rangle|0\rangle, \quad |11\rangle = |1\rangle|1\rangle$$

Vector forms:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

General 2-qubit state:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

with normalization:

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$$

6 Tensor Product

Tensor product combines qubits:

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Computation:

$$= \begin{pmatrix} 1(0,1) \\ 0(0,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dimensionality:

- 1 qubit $\rightarrow$ 2D
- 2 qubits $\rightarrow$ 4D
- n qubits $\rightarrow$ 2^n dimensions

6.1 Non-Commutativity

$$|0\rangle \otimes |1\rangle \neq |1\rangle \otimes |0\rangle$$

7 Tensor Product of Two States

Let:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + i\frac{\sqrt{2}}{\sqrt{3}}|1\rangle$$

Then:

$$|\psi_1\rangle \otimes |\psi_2\rangle = \frac{1}{\sqrt{6}}|00\rangle + \frac{i}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle + \frac{i}{\sqrt{3}}|11\rangle$$

Vector form:

$$\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} \end{pmatrix}$$

8 Bra–Ket Notation

A ket:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The corresponding bra is the conjugate transpose:

$$\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

9 Example with Complex Bra

Given:

$$|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ i\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

Then:

$$\langle\psi| = \begin{pmatrix} \frac{1}{\sqrt{3}} & -i\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

Inner product:

$$\langle\psi|\psi\rangle = \frac{1}{3} + \frac{2}{3} = 1$$

10 Next Topic

The next topic introduced was **Quantum Entanglement**, where multi-qubit states cannot be written as simple tensor products.