Solutions of Assignment #1 part 1, Total points: 55
(Course: CS 401)

Problem 1 (20 points): Prof. Smart thinks he is smarter than all the students in this CS 401 class. He has made the following claim to show how smart he is.

> **Claim made by Prof. Smart**: Consider the stable matching problem as taught in class. Suppose that we have only two men, say m_1 and m_2, and two women, say w_1 and w_2, with their corresponding preference lists. Suppose also that the matching $m_1 - w_1$ and $m_2 - w_2$ is a stable matching. Prof. Smart claims that in that case the matching $m_1 - w_2$ and $m_2 - w_1$ can **never** be a stable matching.

Your task is to decide if Prof. Smart is indeed so smart. For this purpose, do the following.

- Either prove the claim made by Prof. Smart is indeed correct. Such a proof should work **no matter** what the preferences of the men and women are, as long as $m_1 - w_1$ and $m_2 - w_2$ is a stable matching.
- Or, prove the claim made by Prof. Smart is wrong by giving a counter-example. The counter-example should provide the preferences lists of every man and woman, and show that for these preference lists **both** $m_1 - w_1$, $m_2 - w_2$ and $m_1 - w_2$, $m_2 - w_1$ are indeed stable matchings.

Solution: The following preference lists constitute a counter-example since both both $m_1 - w_1$, $m_2 - w_2$ and $m_1 - w_2$, $m_2 - w_1$ are stable matchings.

	first	second
m_1	w_1	w_2
m_2	w_2	w_1
w_1	m_2	m_1
w_2	m_1	m_2

Problem 2 (35 points): Let $G = (V, E)$ be a **directed** graph and let s and t be two nodes of G. Let n and m be the number of nodes and edges of G, respectively. In the class we say how to decide if there is a path **from** s **to** t, namely, we start a (directed) BFS starting from s and check if t appears among the list of nodes that are visited during BFS. The purpose of the assignment is to decide if such a path exists under some **additional constraints**. Let u and v be two other nodes of G that are **not** s or t.

***(i)* [15 points]** Decide if G has a path from s and t that **avoids** using **both** the nodes u and v.

***(ii)* [20 points]** Decide if G has a path from s and t that **uses both** the nodes u and v.

Both of your algorithms should run in $O(m+n)$ time. You may assume that the graph is given in its adjacency list representation. If you are using BFS, there is no need to give codes for it; simply saying "do a BFS starting at such-and-such node" will suffice.

Solution:

(*i*) We delete all the edges in-coming to nodes u and v, *i. e.*, we delete all the edges of the form (x,u) or (x,v) where $x \in V - \{u,v\}$. Then we simply run a BFS starting at s and check if t can be reached.

To delete all the edges as mentioned above, we go through the adjacency list of every node except u and v. For each such list, we traverse the list keeping two pointers, pointer p_1 one at the current entry and pointer p_2 at the entry before the current entry. If the current entry is u or v, we change the link of entry to p_2 to skip u or v. Pictorially, it looks like as shown below for the adjacency list of node a:

$$
\begin{array}{c}
a \cdots\cdots \to \overset{\overset{p_2}{\downarrow}}{b} \to \overset{\overset{p_1}{\downarrow}}{c} \to u \to d \cdots\cdots \\
\Downarrow \\
a \cdots\cdots \to b \to \overset{\overset{p_2}{\downarrow}}{c} \to \overset{\overset{p_1}{\downarrow}}{u} \to d \cdots\cdots \\
\Downarrow \\
a \cdots\cdots \to b \to c \quad u \quad d \cdots\cdots
\end{array}
$$

(*ii*) We will use the notation $x \rightsquigarrow y$ to indicated a directed path from node x to node y. A path from s to t that uses both nodes u and v must be one of the following two types depending on whether node u is before or after node v:

(A) $s \rightsquigarrow u \rightsquigarrow v \rightsquigarrow t$

(B) $s \rightsquigarrow v \rightsquigarrow u \rightsquigarrow t$

For (A), we can use a BFS starting at s to find a path $s \rightsquigarrow u$, a BFS starting at u to find a path $u \rightsquigarrow v$, and a BFS starting at v to find a path $v \rightsquigarrow t$. (B) can be handled in a similar manner.