Simplifications of Context-Free Grammars

A Substitution Rule

$$S \rightarrow aB$$
$$A \rightarrow aaA$$
$$A \rightarrow abBc$$
$$B \rightarrow aA$$
$$B \rightarrow b$$

Substitute

$$B \rightarrow b$$

Equivalent grammar

$$S \rightarrow aB \mid ab$$
$$A \rightarrow aaA$$
$$A \rightarrow abBc \mid abbc$$
$$B \rightarrow aA$$

$$S \to aB \mid ab$$

$$A \to aaA$$

$$A \to abBc \mid abbc$$

$$B \to aA$$

Substitute

$$B \to aA$$

$$S \to \cancel{aB} \mid ab \mid aaA$$

$$A \to aaA$$

$$A \to \cancel{abBc} \mid abbc \mid abaAc$$

Equivalent grammar

In general: $A \to xBz$

$B \to y_1$

Substitute $B \to y_1$

$A \to xBz \mid xy_1z$

equivalent grammar

Nullable Variables

$\varepsilon-$production : $X \rightarrow \varepsilon$

Nullable Variable: $Y \Rightarrow \ldots \Rightarrow \varepsilon$

Example:

$$S \rightarrow aMb$$
$$M \rightarrow aMb$$
$$M \rightarrow \varepsilon$$

Nullable variable

$\varepsilon-$production

Removing $\varepsilon-$productions

$$S \to aMb$$
$$M \to aMb$$
$$\cancel{M \to \varepsilon}$$

Substitute
$$M \to \varepsilon$$

$$S \to aMb \mid ab$$
$$M \to aMb \mid ab$$

After we remove all the $\varepsilon-$productions
all the nullable variables disappear
(except for the start variable)

Unit-Productions

Unit Production: $X \to Y$

(a single variable in both sides)

Example:

$$S \to aA$$

$$A \to a$$

$$\boxed{A \to B}$$

$$\boxed{B \to A}$$

Unit Productions

$$B \to bb$$

Removal of unit productions:

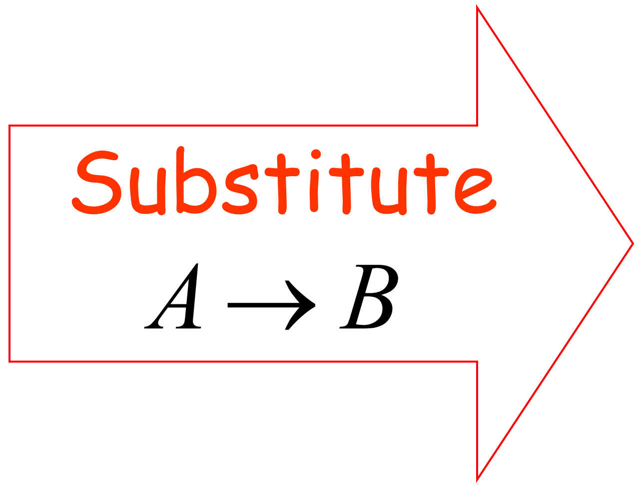
$$S \to aA$$

$$A \to a$$

$$\cancel{A \to B}$$

$$B \to A$$

$$B \to bb$$



$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A \mid B$$

$$B \to bb$$

Unit productions of form $X \to X$ can be removed immediately

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to A \mid \cancel{B}$$
$$B \to bb$$

Remove $B \to B$

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to A$$
$$B \to bb$$

$$S \rightarrow aA \mid aB$$

$$A \rightarrow a$$

$$\cancel{B \rightarrow A}$$

$$B \rightarrow bb$$

Substitute

$$B \rightarrow A$$

$$S \rightarrow aA \mid aB \mid aA$$

$$A \rightarrow a$$

$$B \rightarrow bb$$

Remove repeated productions

$$S \to aA \mid aB \mid \cancel{aA}$$
$$A \to a$$
$$B \to bb$$

Final grammar

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to bb$$

Useless Productions

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

$$S \rightarrow A$$

$$A \rightarrow aA$$ Useless Production

Some derivations never terminate...

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Another grammar:

$$S \to A$$

$$A \to aA$$

$$A \to \varepsilon$$

$$B \to bA$$

Useless Production

Not reachable from S

In general:

If there is a derivation

$$S \Rightarrow \ldots \Rightarrow xAy \Rightarrow \ldots \Rightarrow w \quad \in L(G)$$

consists of terminals

Then variable A is useful

Otherwise, variable A is useless

A production $A \rightarrow x$ is useless
if any of its variables is useless

Variables		Productions
	$S \rightarrow aSb$	
	$S \rightarrow \varepsilon$	
	$S \rightarrow A$	useless
useless	$A \rightarrow aA$	useless
useless	$B \rightarrow C$	useless
useless	$C \rightarrow D$	useless

Removing Useless Variables and Productions

Example Grammar:

$$S \rightarrow aS \mid A \mid C$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

$$C \rightarrow aCb$$

First: find all variables that can produce strings with only terminals or λ (possible useful variables)

$$S \rightarrow aS \mid A \mid C$$

$$A \rightarrow a$$

$$B \rightarrow aa$$

$$C \rightarrow aCb$$

Round 1: $\{A, B\}$

(the right hand side of production that has only terminals)

Round 2: $\{A, B, S\}$

(the right hand side of a production has terminals and variables of previous round)

This process can be generalized

Then, remove productions that use variables other than $\{A, B, S\}$

$$S \rightarrow aS \mid A \mid \cancel{C}$$
$$A \rightarrow a$$
$$B \rightarrow aa$$
$$\cancel{C \rightarrow aCb}$$

$$\Rightarrow$$

$$S \rightarrow aS \mid A$$
$$A \rightarrow a$$
$$B \rightarrow aa$$

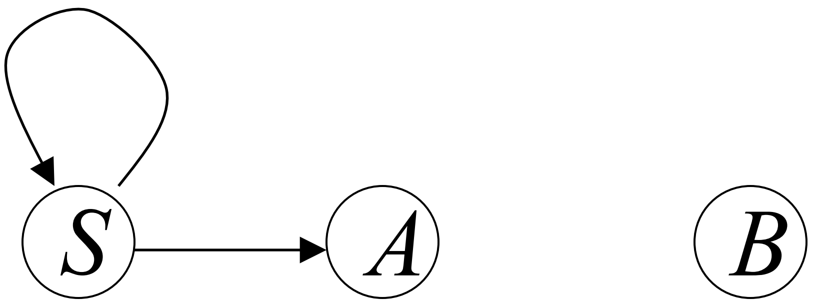
Second: Find all variables reachable from S

Use a Dependency Graph where nodes are variables

$$S \to aS \mid A$$

$$A \to a$$

$$B \to aa$$



unreachable

Keep only the variables reachable from S

$$S \to aS \mid A$$

$$A \to a$$

$$\cancel{B \to aa}$$

Final Grammar

$$S \to aS \mid A$$

$$A \to a$$

Contains only useful variables

Removing All

Step 1: Remove Nullable Variables

Step 2: Remove Unit-Productions

Step 3: Remove Useless Variables

This sequence guarantees that unwanted variables and productions are removed

Normal Forms
for
Context-free Grammars

Chomsky Normal Form

Each productions has form:

$A \to BC$ or $A \to a$

variable variable terminal

Examples:

$$S \to AS$$
$$S \to a$$
$$A \to SA$$
$$A \to b$$

Chomsky
Normal Form

$$S \to AS$$
$$S \to AAS$$
$$A \to SA$$
$$A \to aa$$

Not Chomsky
Normal Form

Conversion to Chomsky Normal Form

Example:

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

Not Chomsky Normal Form

We will convert it to Chomsky Normal Form

Introduce new variables for the terminals:

$$T_a, T_b, T_c$$

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to ABT_a$$

$$A \to T_a T_a T_b$$

$$B \to AT_c$$

$$\boxed{\begin{aligned} T_a &\to a \\ T_b &\to b \\ T_c &\to c \end{aligned}}$$

Introduce new intermediate variable V_1
to break first production:

$$\boxed{S \to ABT_a}$$
$$A \to T_a T_a T_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\boxed{\begin{array}{l} S \to AV_1 \\ V_1 \to BT_a \end{array}}$$
$$A \to T_a T_a T_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_2

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$\boxed{A \to T_aT_aT_b}$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$\Rightarrow$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$\boxed{\begin{array}{l} A \to T_aV_2 \\ V_2 \to T_aT_b \end{array}}$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Final grammar in Chomsky Normal Form:

Initial grammar

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to AV_1$$

$$V_1 \to BT_a$$

$$A \to T_aV_2$$

$$V_2 \to T_aT_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

In general:

From any context-free grammar
(which doesn't produce ε)
not in Chomsky Normal Form

we can obtain:
an equivalent grammar
in Chomsky Normal Form

The Procedure

First remove:

Nullable variables

Unit productions

(Useless variables optional)

Then, for every symbol a:

New variable: T_a

Add production $T_a \rightarrow a$

In productions with length at least 2 replace a with T_a

Productions of form $A \rightarrow a$ do not need to change!

Replace any production $A \to C_1 C_2 \cdots C_n$

with
$$A \to C_1 V_1$$
$$V_1 \to C_2 V_2$$
$$\cdots$$
$$V_{n-2} \to C_{n-1} C_n$$

New intermediate variables: $V_1, V_2, \ldots, V_{n-2}$

Observations

- Chomsky normal forms are good for parsing and proving theorems

- It is easy to find the Chomsky normal form for any context-free grammar

Greinbach Normal Form

All productions have form:

$$A \to a\,V_1 V_2 \cdots V_k \qquad k \geq 0$$

symbol

variables

Examples:

$$S \to cAB$$

$$A \to aA \mid bB \mid b$$

$$B \to b$$

Greinbach
Normal Form

$$S \to abSb$$

$$S \to aa$$

Not Greinbach
Normal Form

Conversion to Greinbach Normal Form:

$$S \to abSb$$

$$S \to aa$$

$$\Rightarrow$$

$$S \to aT_bST_b$$

$$S \to aT_a$$

$$T_a \to a$$

$$T_b \to b$$

Greinbach
Normal Form

Observations

- Greinbach normal forms are very good for parsing strings (better than Chomsky Normal Forms)

- However, it is difficult to find the Greinbach normal of a grammar