CS 506 Lecture Notes: Linear Algebra, Operators in Dirac Notation

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1 Preliminaries:

1.1 Complex Conjugate of a Complex Number

Given a complex number $z = a + ib$, its complex conjugate is denoted by z^* and is defined as

$$z^* = a - ib.$$

Example. If $z = 2 + 3i$, then

$$z^* = 2 - 3i.$$

1.2 Inner Product:

In Dirac notation, the inner product between two vectors $|\psi\rangle$ and $\langle\phi|$ is written as $\langle\phi|\psi\rangle$. For complex vectors, the conjugate transpose (Hermitian transpose) is used.

Example. Consider the vectors

$$\langle\phi| = (1 \quad -i \ \ 2), \qquad |\psi\rangle = \begin{pmatrix} 2 \\ 3 \\ -i \end{pmatrix}.$$

Then the inner product is

$$\langle\phi|\psi\rangle = (1 \quad -i \ \ 2) \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 1 \cdot 2 + (-i) \cdot 3 + 2 \cdot (-i) = 2 - 5i$$

1.3 Outer Product

The outer product between a column vector $|\psi\rangle$ and a row vector $\langle\phi|$ produces a matrix. In Dirac notation, this is written as $|\phi\rangle\langle\psi|$.

Example. Let

$$|\phi\rangle = \begin{pmatrix} 1 \\ -2i \\ 2 \end{pmatrix}, \qquad \langle\psi| = (2 \;\; 3 \;\; -1).$$

Then the outer product is

$$|\phi\rangle\langle\psi| = \begin{pmatrix} 1 \\ -2i \\ 2 \end{pmatrix} (2 \;\; 3 \;\; -1) = \begin{pmatrix} 2 & 3 & -1 \\ -4i & -6i & 2i \\ 4 & 6 & -2 \end{pmatrix}.$$

1.4 General Form of Outer Product

In general, if

$$|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \langle b| = (b_1 \;\; b_2 \;\; b_3),$$

then the outer product $|a\rangle\langle b|$ is given by

$$|a\rangle\langle b| = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}.$$

Dimensions. If $|a\rangle$ is an $m \times 1$ column vector and $\langle b|$ is a $1 \times n$ row vector, then $|a\rangle\langle b|$ is an $m \times n$ matrix.

2 Linear Operators

2.1 Examples

Example 1. Recall the quantum NOT gate, is defined by the matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle.$$

where $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, the NOT gate swaps the two standard basis vectors.

Example 2. Consider the operator

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its action on the standard basis vectors is

$$A|0\rangle = -|1\rangle, \qquad A|1\rangle = |0\rangle.$$

Example 3. Recall the Hadamard gate is defined as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Its action on the standard basis vectors is

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

3 Writing Linear Operators in Terms of Basis Vectors

3.1 Identity Operator

The identity operator is given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Its action on the standard basis vectors is

$$I|0\rangle = |0\rangle, \qquad I|1\rangle = |1\rangle.$$

Example. Let us verify $I|0\rangle = |0\rangle$. Observe

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

Hence

$$\begin{aligned} I|0\rangle &= (|0\rangle\langle 0| + |1\rangle\langle 1|)|0\rangle \\ &= |0\rangle\langle 0|0\rangle + |1\rangle\langle 1|0\rangle \\ &= |0\rangle \cdot 1 + |1\rangle \cdot 0 = |0\rangle \end{aligned}$$

3.2 Linear Operators in Dirac Notation

Let $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ be an orthonormal basis for an n–dimensional vector space. Let T be an $n \times n$ linear operator, then any linear operator T can be written in terms of outer products of basis vectors as

$$T = \sum_{i,j} T_{ij} |b_i\rangle\langle b_j|,$$

where T_{ij} are the matrix elements of T. Let us demonstrate this with an example.

Example: NOT Operator. The NOT operator X is given by

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can decompose it into outer products:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In Dirac notation this becomes

$$X = |0\rangle \langle 1| + |1\rangle \langle 0| .$$

Equivalently, writing all components explicitly,

$$X = 0\,|0\rangle \langle 0| + 1\,|0\rangle \langle 1| + 1\,|1\rangle \langle 0| + 0\,|1\rangle \langle 1| .$$

General 2×2 **Operator Form** For a general operator in a two–dimensional basis $\{|b_1\rangle, |b_2\rangle\}$:

$$T = T_{11}\,|b_1\rangle \langle b_1| + T_{12}\,|b_1\rangle \langle b_2| + T_{21}\,|b_2\rangle \langle b_1| + T_{22}\,|b_2\rangle \langle b_2| .$$

Observing the equation, raises the following question. What is T_{ij} in dirac notation?

T_{ij} **in Dirac Notation:** The matrix element T_{kl} of operator T is obtained by

$$T_{k,l} = \langle b_k| T |b_l\rangle .$$

To see this, substitute the expansion:

$$\langle b_k| T |b_l\rangle = \sum_{i,j} T_{ij} \langle b_k| b_i\rangle\langle b_j |b_l\rangle .$$

Using orthonormality, we obtain

$$\langle b_k| T |b_l\rangle = T_{k,l}.$$

Thus, $\langle b_k| T |b_l\rangle$ picks out exactly the (k,l) entry of the operator matrix.

4 Matrix Definitions:

4.1 Transpose of a Real Matrix

For a real matrix A, the transpose of A is denoted by A^T. The (i,j)-th entry of A^T is the (j,i)-th entry of A, i.e.,

$$(A^T)_{ij} = A_{ji}.$$

Example. If

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ then } A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

4.2 Complex Matrices and the Adjoint

For a complex matrix A, the adjoint (also called the *Hermitian conjugate* or *Complex conjugate*) is denoted by $A^\dagger$ or A^+. The (i, j)-th entry of $A^\dagger$ is defined as

$$(A^\dagger)_{ij} = \overline{A_{ji}},$$

where the bar denotes complex conjugation.
Equivalently, the adjoint is obtained by taking the transpose of the matrix and then complex-conjugating each entry.

Example. If

$$A = \begin{pmatrix} 1 & i \\ 2 & -i \end{pmatrix} \text{ then, } A^\dagger = \begin{pmatrix} 1 & 2 \\ -i & i \end{pmatrix}.$$

4.3 Unitary Matrices

A matrix U is called *unitary* if

$$UU^\dagger = U^\dagger U = I,$$

where I is the identity matrix.

Example. Consider the NOT (Pauli-X) operator

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now,

$$XX^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Thus, $X^\dagger = X$ and since $XX^\dagger = I$, X is unitary.

Action of a Unitary Operator on Basis States Consider

$$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \text{ then, } U^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

The action of U on the standard basis vectors is

$$U|0\rangle = |0\rangle, \qquad U|1\rangle = i|1\rangle.$$

Finally, we verify unitarity:

$$UU^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

From this condition it follows that

$$U^\dagger = U^{-1},$$

that is, the adjoint of U is equal to its inverse. Hence, every unitary operator is invertible (reversible).

4.4 Hermitian (Self-Adjoint) Matrices

A matrix A is called *Hermitian* or *self-adjoint* if

$$A = A^\dagger.$$

Real Matrices. For real matrices, the adjoint reduces to the transpose. If $A = A^T$ implies that A is symmetric.

Complex Matrices. For complex matrices, Hermitian means $A^\dagger = A$, i.e., the matrix is equal to its conjugate transpose.

Hermitian vs. Unitary. A Hermitian matrix is not necessarily unitary. If a matrix A is unitary it satisfies,

$$AA^\dagger = I,$$

whereas Hermitian only requires $A = A^\dagger$. If A is Hermitian, then

$$AA^\dagger = A^2,$$

which equals I only in special cases. Thus, in general Hermitian $\not\Rightarrow$ unitary.

4.5 Projection Operators

A matrix P is called a *projector* if :

$$P^2 = P.$$

An *orthogonal projector* additionally satisfies

$$P = P^\dagger.$$

5 Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A nonzero vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq \mathbf{0}$$

is called an *eigenvector* of A if there exists a scalar λ such that

$$Ax = \lambda x.$$

The scalar λ is called the corresponding *eigenvalue*.

Eigenvalue Equation in Dirac Notation. Let T be an $n \times n$ linear operator. In Dirac notation, an eigenvector $|\psi\rangle$ satisfies

$$T |\psi\rangle = \lambda |\psi\rangle .$$

Here, $|\psi\rangle$ is represented as a column vector and λ is a scalar eigenvalue.

Eigenvalues of Real Symmetric Matrices. All eigenvalues of a real symmetric matrix are real. This result can be shown using induction or via the spectral theorem, covered in the next lecture.

5.1 Trace of a Matrix

The trace of a square matrix A is defined as the sum of its diagonal elements:

$$\mathrm{Tr}(A) = \sum_i A_{ii}.$$

Trace in Dirac Notation Let $\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\}$ be an orthonormal basis. Then any operator A can be written as

$$A = \sum_{i,j} A_{ij} |b_i\rangle \langle b_j| , \text{ where } A_{ij} = \langle b_i| A |b_j\rangle .$$

The trace of A can be expressed as

$$\mathrm{Tr}(A) = \sum_i \langle b_i| A |b_i\rangle .$$

5.2 Normal Operators

An operator A is called *normal* if

$$AA^\dagger = A^\dagger A.$$

Unitary Operators Are Normal. A unitary operator U satisfies $UU^\dagger = I$. Then

$$U^\dagger U = (UU^\dagger)^\dagger = I^\dagger = I, \text{ so } UU^\dagger = U^\dagger U.$$

Thus, every unitary operator is normal.

Hermitian Operators Are Normal If A is Hermitian, i.e., $A = A^\dagger$, then

$$AA^\dagger = A^2 = A^\dagger A.$$

Hence, every Hermitian operator is normal.

Remark. All eigenvalues of a Hermitian operator are real.

6 Spectral Theorem for Normal Operators

The spectral theorem for normal operators states that any normal matrix can be expressed as a combination of outer products and operators. A more detailed analysis and proof of the spectral theorem will be presented in the next lecture.