Properties
of
Context-Free languages

Union

Context-free languages

are closed under: **Union**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cup L_2$ is context-free

Example

Language	**Grammar**

$$L_1 = \{a^n b^n\}$$

$$S_1 \rightarrow aS_1b \mid \varepsilon$$

$$L_2 = \{ww^R\}$$

$$S_2 \rightarrow aS_2a \mid bS_2b \mid \varepsilon$$

Union

$$L = \{a^n b^n\} \cup \{ww^R\}$$

$$S \rightarrow S_1 \mid S_2$$

In general:

For context-free languages L_1, L_2
with context-free grammars G_1, G_2
and start variables S_1, S_2

The grammar of the **union** $L_1 \cup L_2$
has new start variable S
and additional production $S \to S_1 \mid S_2$

Concatenation

Context-free languages are closed under: **Concatenation**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 L_2$ is context-free

Example

Language | **Grammar**

$$L_1 = \{a^n b^n\}$$

$$S_1 \rightarrow aS_1b \mid \varepsilon$$

$$L_2 = \{ww^R\}$$

$$S_2 \rightarrow aS_2a \mid bS_2b \mid \varepsilon$$

Concatenation

$$L = \{a^n b^n\}\{ww^R\}$$

$$S \rightarrow S_1 S_2$$

In general:

For context-free languages $L_1, \; L_2$

with context-free grammars $G_1, \; G_2$

and start variables $S_1, \; S_2$

The grammar of the **concatenation** $L_1 L_2$

has new start variable S

and additional production $S \rightarrow S_1 S_2$

Star Operation

Context-free languages
are closed under: **Star-operation**

L is context free $\Rightarrow$ L^* is context-free

Example

Language

Grammar

$$L = \{a^n b^n\}$$

$$S \to aSb \mid \varepsilon$$

Star Operation

$$L = \{a^n b^n\}^*$$

$$S_1 \to SS_1 \mid \varepsilon$$

In general:

For context-free language L
with context-free grammar G
and start variable S

The grammar of the **star operation** L^*
has new start variable S_1
and additional production $S_1 \rightarrow SS_1 \mid \varepsilon$

Negative Properties of Context-Free Languages

Intersection

Context-free languages are **not** closed under: **intersection**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cap L_2$ **not** necessarily context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \to AC$$

$$A \to aAb \mid \varepsilon$$

$$C \to cC \mid \varepsilon$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \to AB$$

$$A \to aA \mid \varepsilon$$

$$B \to bBc \mid \varepsilon$$

Intersection

$L_1 \cap L_2 = \{a^n b^n c^n\}$ **NOT** context-free

Complement

Context-free languages are **not** closed under: **complement**

L is context free $\Longrightarrow$ $\overline{L}$ **not** necessarily context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \rightarrow AC$$

$$A \rightarrow aAb \mid \varepsilon$$

$$C \rightarrow cC \mid \varepsilon$$

Context-free:

$$S \rightarrow AB$$

$$A \rightarrow aA \mid \varepsilon$$

$$B \rightarrow bBc \mid \varepsilon$$

Complement

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

NOT context-free

Intersection
of
Context-free languages
and
Regular Languages

The intersection of

a context-free language and

a regular language

is a context-free language

$$\left.\begin{array}{ll} L_1 & \text{context free} \\ & \\ L_2 & \text{regular} \end{array}\right\} \Longrightarrow \begin{array}{c} L_1 \cap L_2 \\ \text{context-free} \end{array}$$

Machine M_1

PDA for L_1

context-free

Machine M_2

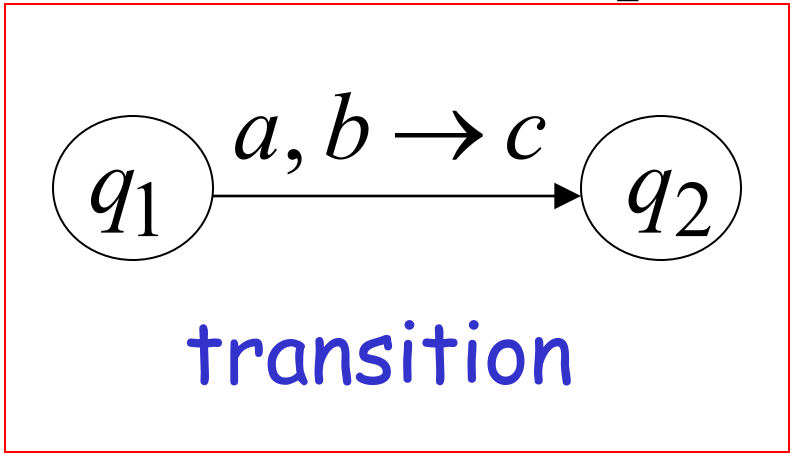
DFA for L_2

regular

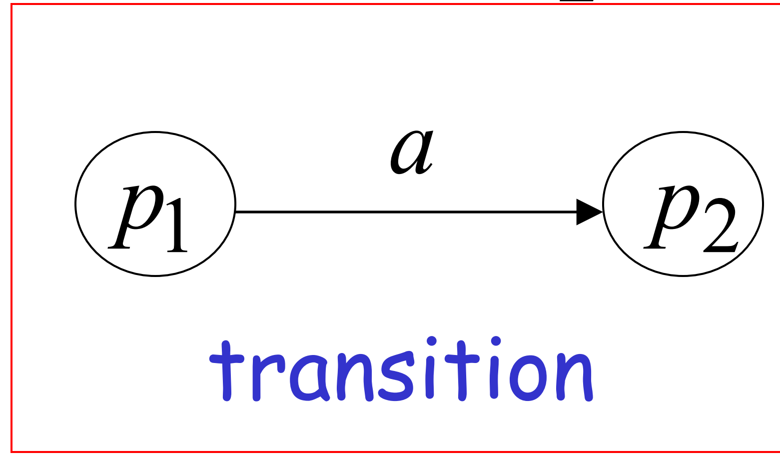
Construct a new PDA machine M that accepts $L_1 \cap L_2$

M simulates in parallel M_1 and M_2

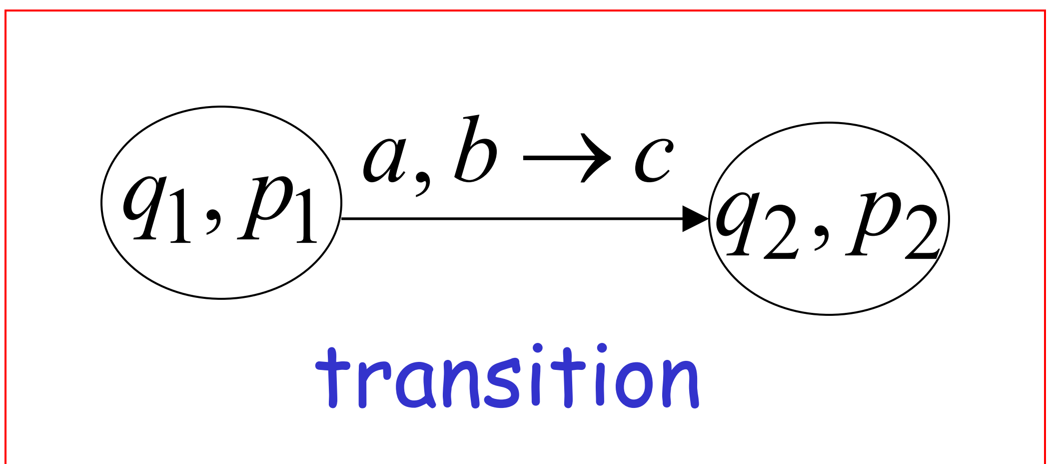
PDA M_1

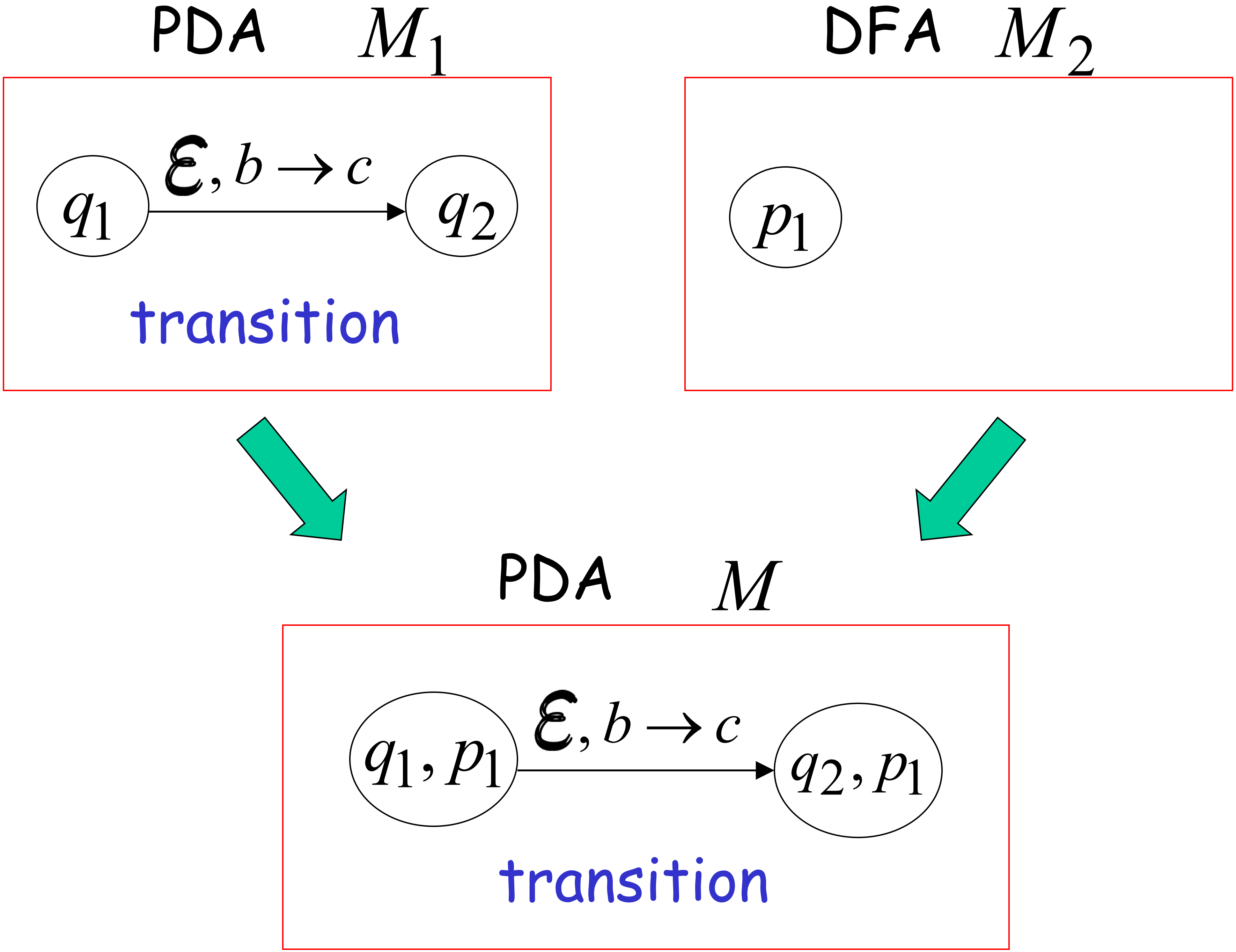


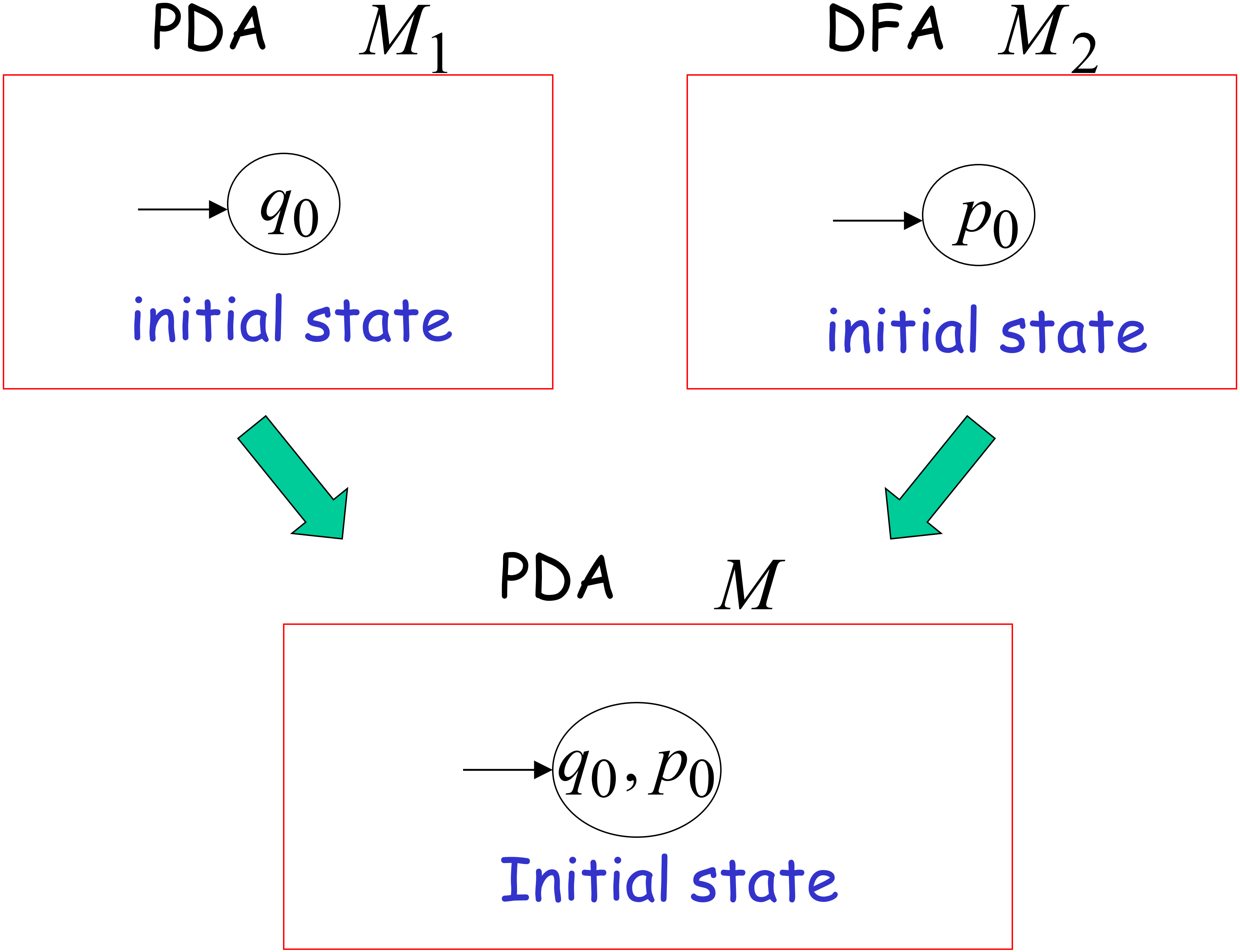
DFA M_2

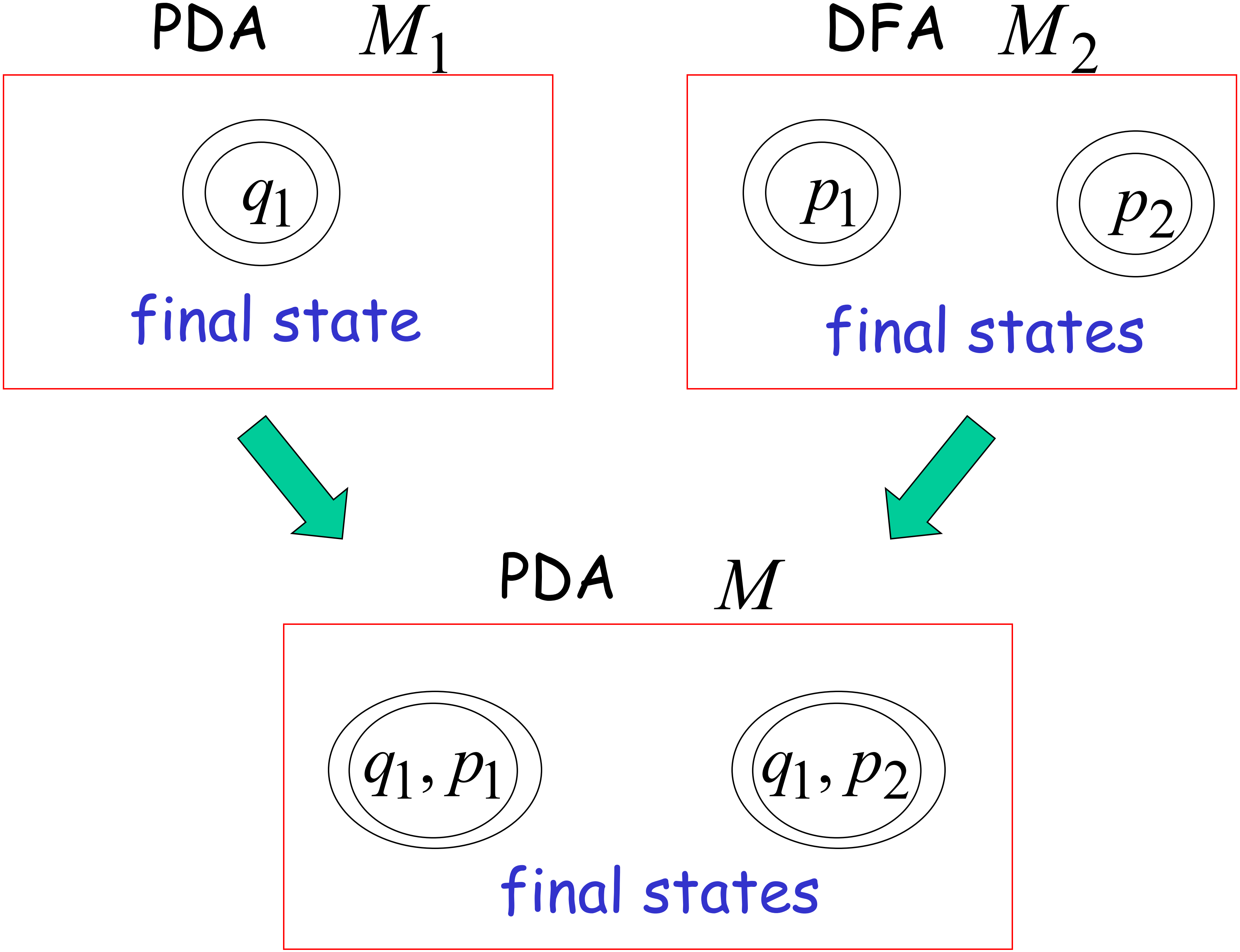


PDA M




PDA M_1
q_1
$\varepsilon, b \rightarrow c$
q_2
transition
DFA M_2
p_1
PDA M
q_1, p_1
$\varepsilon, b \rightarrow c$
q_2, p_1
transition


PDA M_1
q_0
initial state
DFA M_2
p_0
initial state
PDA M
q_0, p_0
Initial state

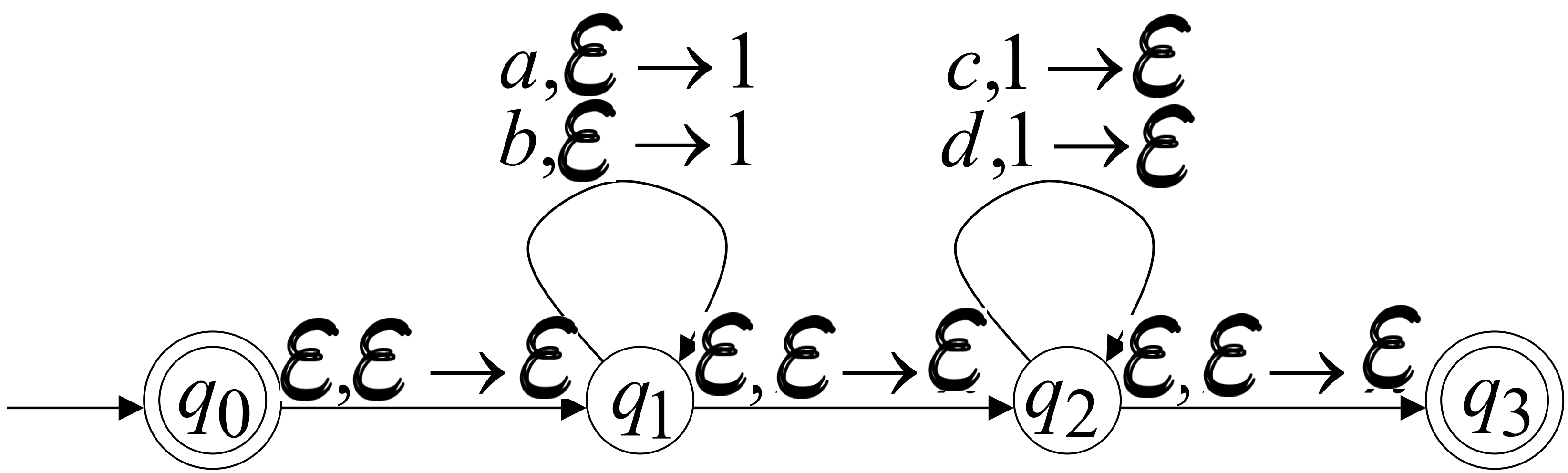

PDA M_1
q_1
final state
DFA M_2
p_1
p_2
final states
PDA M
q_1, p_1
q_1, p_2
final states

Example:

context-free

$$L_1 = \{w_1 w_2 : |w_1| = |w_2|, w_1 \in \{a,b\}^*, w_2 \in \{c,d\}^*\}$$

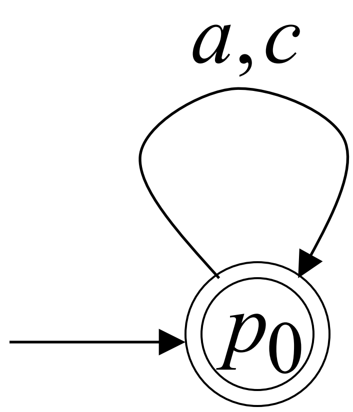
PDA M_1



regular

$$L_2 = \{a, c\}^*$$

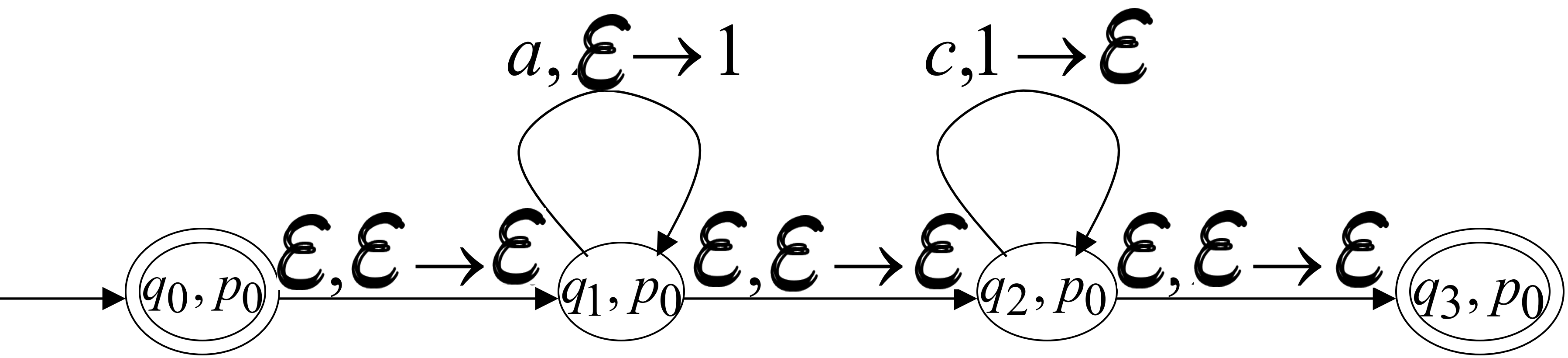
DFA M_2



context-free

Automaton for: $L_1 \cap L_2 = \{a^n c^n : n \geq 0\}$

PDA M



In General:

M simulates in parallel M_1 and M_2

M accepts string w if and only if

M_1 accepts string w and

M_2 accepts string w

$$L(M) = L(M_1) \cap L(M_2)$$

Therefore:

M is PDA

$\Downarrow$

$L(M_1) \cap L(M_2)$ is context-free

$\Downarrow$

$L_1 \cap L_2$ is context-free

Applications
of
Regular Closure

The intersection of

a context-free language and

a regular language

is a context-free language

Regular Closure

L_1 context free

L_2 regular

$\left.\begin{array}{l} \\ \\ \end{array}\right\} \Rightarrow$

$L_1 \cap L_2$

context-free

An Application of Regular Closure

Prove that: $$L = \{a^n b^n : \;\; n \neq 100, n \geq 0\}$$

is context-free

We know:

$$\{a^n b^n : n \geq 0\}$$ is context-free

We also know:

$$L_1 = \{a^{100}b^{100}\}$$ is regular

$$\Downarrow$$

$$\overline{L_1} = \{(a+b)^*\} - \{a^{100}b^{100}\}$$ is regular

$\{a^n b^n\}$

context-free

$\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$

regular

(regular closure)

$\{a^n b^n\} \cap \overline{L_1}$

context-free

$$\{a^n b^n\} \cap \overline{L_1} = \{a^n b^n : \ n \neq 100, n \geq 0\} = L$$

is context-free

Another Application of Regular Closure

Prove that: $L = \{w: \; n_a = n_b = n_c\}$

is **not** context-free

If $L=\{w: \quad n_a=n_b=n_c\}$ is context-free

(regular closure)

Then $L \cap \{a^*b^*c^*\} = \{a^n b^n c^n\}$

context-free regular context-free

Impossible!!!

Therefore, L is **not** context free