Introduction to Quantum Computing

CS 506
Lecture 1 Notes
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Quantum Mechanics

Two equivalent formulations:

$$\text{Heisenberg's matrix mechanics} \equiv \text{Schrödinger's wave mechanics.}$$

Two equivalent models of quantum computing:

$$\text{Circuit world of quantum} \equiv \text{Adiabatic model of quantum computing.}$$

Additional perspectives:

- Linear Algebraic View
- Dirac Notation

Classical Bits, Vectors, and Dirac Notation

Classical Bit	Vector View	Dirac Notation
0	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\lvert 0 \rangle$
1	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\lvert 1 \rangle$

Two-bit states

$$00 \quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |00\rangle$$

$$01 \quad \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |01\rangle$$

$$10 \quad \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |10\rangle$$

$$11 \quad \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad |11\rangle$$

Note: $i = \sqrt{-1}, \quad i^2 = -1.$

Orthonormal Basis

The computational basis vectors for a single qubit are

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} \in \mathbb{C}^2.$$

They *form an orthonormal basis* of $\mathbb{C}^2$. “Orthonormal” means:

- **Normalized:** $\langle 0|0\rangle = 1,\ \langle 1|1\rangle = 1.$
- **Orthogonal:** $\langle 0|1\rangle = 0,\ \langle 1|0\rangle = 0.$

Explicitly, bras are conjugate-transposes (Hermitian conjugates) of kets:

$$\langle 0| = |0\rangle^\dagger = \begin{bmatrix}1 & 0\end{bmatrix}, \qquad \langle 1| = |1\rangle^\dagger = \begin{bmatrix}0 & 1\end{bmatrix}.$$

So the inner products (row $\times$ column) evaluate as

$$\langle 0|\,|0\rangle = 1, \quad \langle 0|\,|1\rangle = 0, \quad \langle 1|\,|0\rangle = 0, \quad \langle 1|\,|1\rangle = 1.$$

Because they are orthonormal and linearly independent, they span $\mathbb{C}^2$. Thus any vector $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{C}^2$ can be written as

$$\begin{bmatrix}a\\b\end{bmatrix} = a\,|0\rangle + b\,|1\rangle\,.$$

Example Decomposition

$$\begin{bmatrix}2\\5\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix} \quad \Longleftrightarrow \quad 2\,|0\rangle + 5\,|1\rangle\,.$$

Four-Dimensional Vector Space

The standard orthonormal basis for a 4-dimensional vector space is

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

Example:

$$\begin{bmatrix}3\\2\\7\\4\end{bmatrix} = 3\,|00\rangle + 2\,|01\rangle + 7\,|10\rangle + 4\,|11\rangle\,.$$

Transpose and Bras

If

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix},$$

then

$$\langle 0| = |0\rangle^\dagger = \begin{bmatrix}1 & 0\end{bmatrix}, \quad \langle 1| = |1\rangle^\dagger = \begin{bmatrix}0 & 1\end{bmatrix}.$$

For example,

$$\langle 0|0\rangle = 1, \quad \langle 0|1\rangle = 0.$$

NOT Gate

The NOT (Pauli-X) gate flips the computational basis states:

$$X\,|0\rangle = |1\rangle\,, \quad X\,|1\rangle = |0\rangle\,.$$

It is represented by the unitary matrix

$$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

Important properties:

- X is unitary: $X^\dagger X = I$.
- X is self-inverse: $X^{-1} = X$.

Thus quantum operations are reversible:

$$X\,|0\rangle = |1\rangle\,, \quad X\,|1\rangle = |0\rangle\,, \quad X^{-1} = X.$$

Standard and Hadamard Basis

2-D Vector Space and Standard Basis

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hadamard Basis (Orthonormal)

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Expansion Between Bases (Step by Step)

1. Expressing $|+\rangle$ and $|-\rangle$ in terms of $|0\rangle$ and $|1\rangle$:

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle .$$

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle .$$

2. Expressing $|0\rangle$ and $|1\rangle$ in terms of $|+\rangle$ and $|-\rangle$: Start with the equations above:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Add the two equations:

$$|+\rangle + |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(2\,|0\rangle) = \sqrt{2}\,|0\rangle .$$

$$\Rightarrow |0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle).$$

Subtract the second equation from the first:

$$|+\rangle - |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(2\,|1\rangle) = \sqrt{2}\,|1\rangle .$$

$$\Rightarrow |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Summary:

$$\boxed{|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad |0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}}$$