Putnam POTD November 12, 2025

I. PROBLEM

(2017 A2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

II. SOLUTION

Given the form of the question, it is inviting to use induction. However, and I must point out that this could just be my algebraic skills, but the given form

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

does not have a directly good algebraic form to make proofs pertaining our claim that $Q_n(x) \in \mathbb{Z}[x]$. Instead, we may be motivated to find an alternative form of the expression: Let's observe the terms:

- $Q_0(x) = 1$
- $Q_1(x) = x$
- $Q_2(x) = x^2 - 1$
- $Q_3(x) = x^3 - 2$
- $Q_4(x) = x^4 - 3x^2 + 1$

Do you see

$$Q_n(x) = xQ_{n-1}(x) - Q_{n-2}?$$

Now we need to that this is indeed the case. Since we are given

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)},$$

the justification lies in this expression leading to the linear recurrence.

We first assume via induction that the linear expression is valid up to Q_{n-1}, then consider

$$\begin{aligned}
Q_n(x) &= \frac{(Q_{n-1}(x))^2-1}{Q_{n-2}(x)} \\
&= \frac{x^2Q_{n-2}^2 - 2xQ_{n-2}Q_{n-3} + Q_{n-3}^2 - 1}{Q_{n-2}(x)} \\
&= \frac{x^2Q_{n-2}^2 - 2xQ_{n-2}Q_{n-3} + Q_{n-2}Q_{n-4}}{Q_{n-2}(x)} \\
&= x^2Q_{n-2} - 2xQ_{n-3} + Q_{n-4} \\
&= x^2Q_{n-2} - xQ_{n-3} - (xQ_{n-3} - Q_{n-4}) \\
&= x^2Q_{n-2} - xQ_{n-3} - Q_{n-2} \\
&= x(xQ_{n-2} - Q_{n-3}) - Q_{n-2} \\
&= xQ_{n-1} - Q_{n-2},
\end{aligned}$$

and we have derived the linear recurrence!

Now assume again via induction that for all natural numbers $i < m$, $Q_i \in \mathbb{Z}[x]$, then

$$Q_m = xQ_{m-1} - Q_{m-2}$$

since $\mathbb{Z}[x]$ is closed under addition and multiplication, Q_m is an integer.

III. OBSERVATIONS

The expression $Q_n(x) = \frac{(Q_{n-1}(x))^2-1}{Q_{n-2}(x)}$, seems quite familar. I remember an AIME problem with a similar setting, but to find the 2021th (I made that up) term, given the inital expresion involving a ratio between several terms. The most common idea was to write down a few terms to gain some intution.

The Core Idea: Linear vs. Quadratic RecurrenceThe problem's "trick" is that a messy, non-linear (quadratic) recurrence relation is equivalent to a simple, linear one.The Given Recurrence (Quadratic):$Q_n(x) = \frac{(Q_{n-1}(x))^2-1}{Q_{n-2}(x)}$This is hard to work with.

If you rearrange it, you get

$$Q_n(x)Q_{n-2}(x) = (Q_{n-1}(x))^2 - 1$$

The "Hidden" Recurrence (Linear):

$$Q_n(x) = xQ_{n-1}(x) - Q_{n-2}(x)$$

This is very easy to work with. If you assume this recurrence is true, the proof is trivial:Base Cases: $Q_0 = 1$ and $Q_1 = x$ are polynomials with integer coefficients.Inductive Step: Assume Q_{n-1} and Q_{n-2} are polynomials with integer coefficients. Then $Q_n = x \cdot Q_{n-1} - Q_{n-2}$ is just the product, sum, and difference of such polynomials, which must also be a polynomial with integer coefficients.

A. Remark 1: A motivation for the problem

Remark 1: The "Real" Identity (Chebyshev Polynomials)This is the most important piece of background. This problem is a disguised version of a standard identity for Chebyshev Polynomials.Chebyshev Polynomials of the Second Kind ($U_n(y)$) are defined by the exact same linear recurrence, just with a slightly different variable:

$$U_0(y) = 1, U_1(y) = 2y, \ldots, U_n(y) = 2y \cdot U_{n-1}(y) - U_{n-2}(y)$$

The Connection: If you make the substitution $x = 2y$ (or $y = x/2$), our polynomials $Q_n(x)$ are identical to $U_n(y)$. Therefore, $Q_n(x) = U_n(x/2)$.The original problem's identity

$$Q_n(x)Q_{n-2}(x) = (Q_{n-1}(x))^2 - 1$$

is just a famous, standard identity for Chebyshev polynomials, which is

$$U_n(y)U_{n-2}(y) = (U_{n-1}(y))^2 - 1.$$

This identity is usually proven using the trigonometric definition $U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$, where it becomes a simple trig identity.

B. Remark 2: Matrix Proof & Cassini's Identity

This remark connects the problem to another famous example: the Fibonacci sequence.

Fibonacci Recurrence: $F_n = F_{n-1} + F_{n-2}$

Cassini's Identity: $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$

Notice the pattern: a second-order linear recurrence ($F_n = \ldots$) leads to a quadratic identity (Cassini's).

Our problem is the same:

Linear Recurrence: $Q_n = xQ_{n-1} - Q_{n-2}$

Quadratic Identity: $Q_nQ_{n-2} - Q_{n-1}^2 = -1$, or $(Q_{n-1})^2 - Q_nQ_{n-2} = 1$.

The matrix proof is the standard way to prove all of these identities.The matrix $T = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ is the "transition matrix" for the linear recurrence.

The identity $TM_n = M_{n+1}$ just says "applying the transition matrix to the state at step n gives the state at step $n+1$."

The core of the proof is that $\det(M_{n+1}) = \det(TM_n) = \det(T)\det(M_n)$.

Since $\det(T) = (x)(0) - (-1)(1) = 1$, the determinant never changes!

$$\det(M_n) = \det(M_{n-1}) = \cdots = \det(M_1). \quad \det(M_1) = \begin{vmatrix} P_0 & P_1 \\ P_{-1} & P_0 \end{vmatrix} = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore, $\det(M_n) = (P_{n-1})^2 - P_nP_{n-2} = 1$ for all n.

This is a beautiful and general technique. For the Fibonacci sequence, the transition matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which has a determinant of -1. This is precisely why the F_n identity has $(-1)^n$ in it, while the Q_n identity has $1^n = 1$.

As mentioned, in general, **a second-order linear recurrence ($F_n = \ldots$) leads to a quadratic identity (Cassini's).**

C. Remark 3: Combinatorial Interpretation

The third way to understand these polynomials, this time through combinatorics.

They define a new set of polynomials, $R_n(x)$, called Fibonacci Polynomials (though this name is used for a few different things).

Definition: $R_n(x)$ is the generating function for the number of ways to tile a $1 \times n$ board with 1×1 squares (which count as "x") and 1×2 dominoes (which count as "1").

Example $R_3(x)$:Tile (1,1,1): $x \cdot x \cdot x = x^3$Tile (1, 2): $x \cdot 1 = x$Tile (2, 1): $1 \cdot x = x$Total: $R_3(x) = x^3 + 2x$

Recurrence for $R_n(x)$: To tile a $1 \times n$ board, you can either:Start with a 1×1 square (cost: x) and then tile the remaining $1 \times (n-1)$ board. This gives $x \cdot R_{n-1}(x)$.Start with a 1×2 domino (cost: 1) and then tile the remaining $1 \times (n-2)$ board. This gives $1 \cdot R_{n-2}(x)$.

This gives the linear recurrence

$$R_n(x) = xR_{n-1}(x) + R_{n-2}(x).$$

This is almost our Q_n recurrence, but with a + sign instead of a -.
We can then connect them with a change of variables:

$$Q_n(x) = i^{-n} R_n(ix)$$

This algebraic substitution transforms one linear recurrence into the other. The complex combinatorial identity for R_n is just another version of Cassini's identity, which, when transformed, becomes the identity for Q_n.
See related:
(1993 A2) Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.