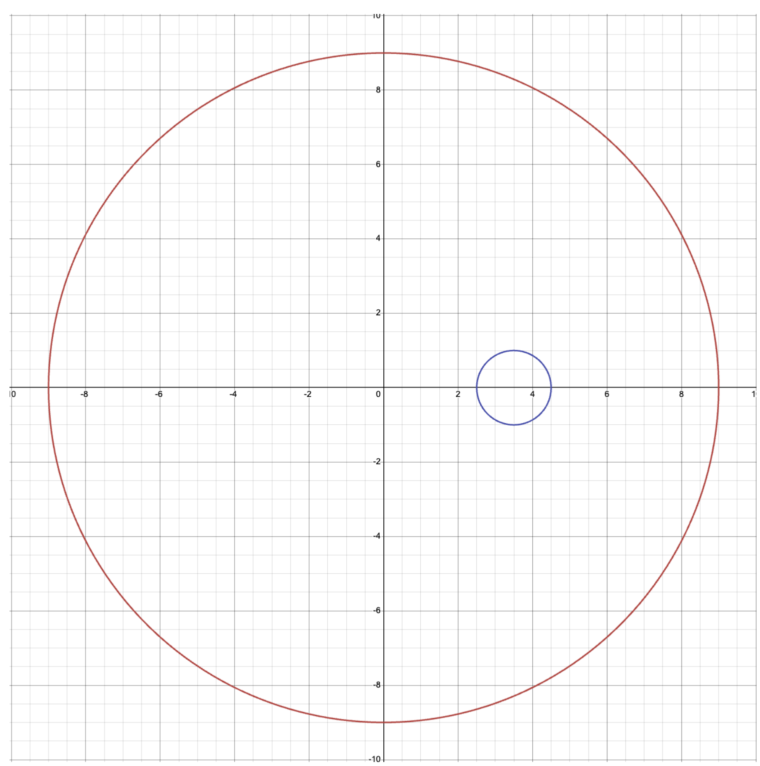
Putnam POTD November 9, 2025

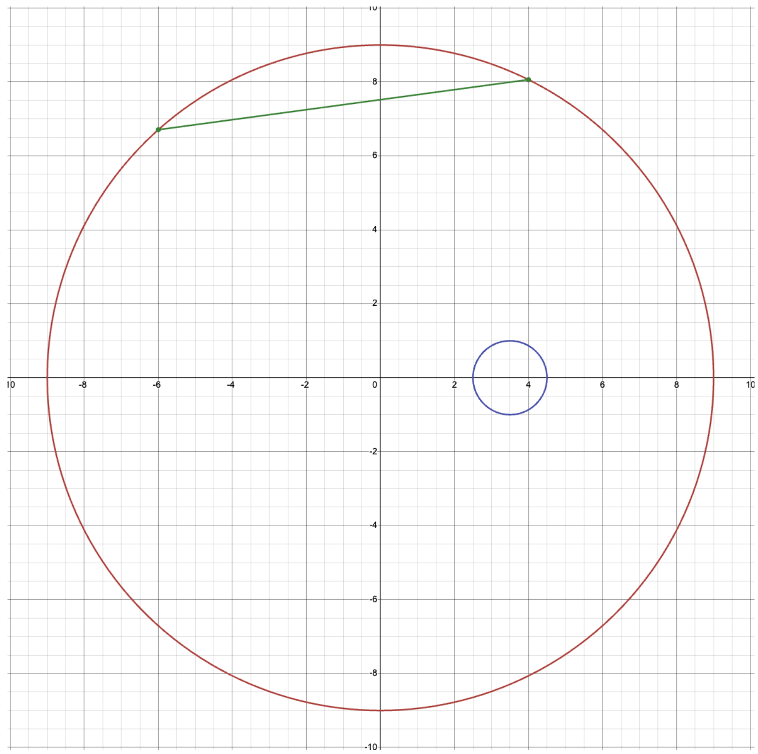
I. PROBLEM

(2024 A5) Consider a circle Ω with radius 9 and center at the origin $(0,0)$, and a disc Δ with radius 1 and center at $(r,0)$, where $0 \leq r \leq 8$. Two points P and Q are chosen independently and uniformly at random on Ω. Which value(s) of r minimize the probability that the chord $\overline{PQ}$ intersects Δ?

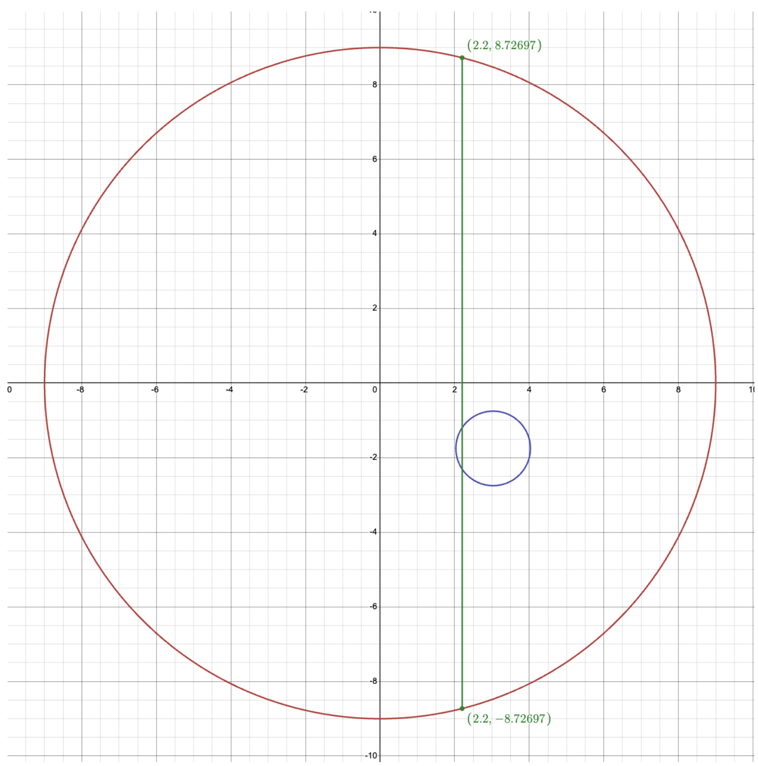
II. SOLUTION



We consider a configuration like the above. Here P and Q are uniformly distributed on the circle. We can first draw a sample PQ.



Let θ be the angle made between PQ and the y axis. Then we can make the rotation of the coordinate system by $-\theta$ degrees in the (counterclockwise) direction so that the new coordinates for the circle Δ is $(r\cos\theta, -r\sin\theta)$.



As shown, here is another configuration. Now we can determine if the segment PQ intersects Δ

by comparing the x coordinates. P and Q have coordinates $9(\cos\phi, \pm\sin\phi)$, and the critirea is therfore

$$|9\cos\phi - r\cos\theta| \leq 1$$
$$\phi \in [\cos^{-1}(\frac{r\cos\theta+1}{9}), \cos^{-1}(\frac{r\cos\theta-1}{9})].$$

Now all segments PQ parallel to the current PQ are valid, so we can represent this by considering all angles $\phi \in [0,\pi]$, so the probability that they interset is $\frac{1}{\pi}\left[\cos^{-1}(\frac{r\cos\theta-1}{9}) - \cos^{-1}(\frac{r\cos\theta+1}{9})\right]$, here we define $f(r,\theta) = \cos^{-1}(\frac{r\cos\theta-1}{9}) - \cos^{-1}(\frac{r\cos\theta+1}{9})$.

Now we can finally integrate over all possible $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We define θ to be negative if rotating results in the circle Δ to be in the first quadrant and negative otherwise (omitting the axes), then the total probability that the chord intersects Δ is

$$I(r) = \frac{1}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(r,\theta)d\theta$$

to find the minimum, we consider the derivative with respect to I to get

$$\frac{dI}{dr} = \frac{1}{\pi^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial r} f(r,\theta)d\theta.$$

$$\begin{aligned}
\frac{\partial}{\partial r}f(r,\theta) &= \frac{\partial}{\partial r}\left[\cos^{-1}(\frac{r\cos\theta-1}{9}) - \cos^{-1}(\frac{r\cos\theta+1}{9})\right] \\
&= \frac{\cos\theta}{9}\left[\frac{1}{\sqrt{1-\left(\frac{r\cos\theta+1}{9}\right)^2}} - \frac{1}{\sqrt{1-\left(\frac{r\cos\theta-1}{9}\right)^2}}\right] \\
&= \cos\theta\left[\frac{1}{\sqrt{80-r^2\cos^2\theta-2r\cos\theta}} - \frac{1}{\sqrt{80-r^2\cos^2\theta+2r\cos\theta}}\right]
\end{aligned}$$

Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $r \geq 0$, we find that

$$\begin{aligned}
80-r^2\cos^2\theta-2r\cos\theta &\leq 80-r^2\cos^2\theta+2r\cos\theta \\
\sqrt{80-r^2\cos^2\theta-2r\cos\theta} &\leq \sqrt{80-r^2\cos^2\theta+2r\cos\theta} \\
\frac{1}{\sqrt{80-r^2\cos^2\theta-2r\cos\theta}} &\geq \frac{1}{\sqrt{80-r^2\cos^2\theta+2r\cos\theta}} \\
\frac{1}{\sqrt{80-r^2\cos^2\theta-2r\cos\theta}} - \frac{1}{\sqrt{80-r^2\cos^2\theta+2r\cos\theta}} &\geq 0
\end{aligned}$$

Hence the integrand for $\frac{dI}{dr}$ is nonnegative, with zero if $r = 0$. Thus the answer is $\boxed{r = 0}$.