Putnam POTD November 15, 2025

I. PROBLEM

(2020 A2) Let k be a nonnegative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}.$$

II. SOLUTION

We first try out a few terms, let

$$f(k) = \sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}$$

- $f(1) = 4$
- $f(2) = 16$
- $f(3) = 64$
- $f(4) = 256$

so we suspect that $f(k) = 4^k = 2^{2k}$, then it suffices to show that

$$g(k) = \sum_{j=0}^{k} 2^{-j}\binom{k+j}{j} = 2^k$$

The few previous terms will be our base cases. Now for the inductive case, assume that $g(k) = 2^k$, consider $g(k+1)$:

$$\begin{aligned}
g(k+1) &= \sum_{j=0}^{k+1} 2^{-j}\binom{k+j+1}{j} \\
&= \sum_{j=0}^{k+1} 2^{-j}\binom{k+j+1}{k+1} \\
&= \sum_{j=0}^{k+1} 2^{-j}\left[\binom{k+j}{k} + \binom{k+j}{k+1}\right]
\end{aligned}$$

here we use $\binom{a+1}{j+1} = \binom{a}{j} + \binom{a}{j+1}$ (known as the committee argument), then

$$\begin{aligned}
g(k+1) &= \sum_{j=0}^{k+1} 2^{-j} \left[\binom{k+j}{k} + \binom{k+j}{k+1} \right] \\
&= \sum_{j=0}^{k+1} 2^{-j} \binom{k+j}{k} + \sum_{j=0}^{k+1} 2^{-j} \binom{k+j}{k+1} \\
&= \sum_{j=0}^{k} 2^{-j} \binom{k+j}{k} + 2^{-(k+1)} \binom{2k+1}{k+1} + \sum_{j=0}^{k+1} 2^{-j} \binom{k+j}{k+1} \\
&= g(k) + 2^{-(k+1)} \binom{2k+1}{k+1} + \sum_{j=0}^{k+1} 2^{-j} \binom{k+j}{k+1}
\end{aligned}$$

Now we consider

$$\begin{aligned}
L &= \sum_{j=0}^{k+1} 2^{-j} \binom{k+j}{k+1} \\
&= \sum_{j=1}^{k+1} 2^{-j} \binom{k+j}{k+1} \\
&= \sum_{i=0}^{k} 2^{-(i+1)} \binom{k+i+1}{k+1} \\
&= \sum_{i=0}^{k} 2^{-(i+1)} \binom{k+i+1}{i}
\end{aligned}$$

Note that this expression for L looks a lot like $g(k+1)$, so we exploit it:

$$\begin{aligned}
2L &= \sum_{i=0}^{k} 2^{-i} \binom{k+i+1}{i} \\
&= g(k+1) - 2^{-(k+1)} \binom{2k+2}{k+1}
\end{aligned}$$

thus going back to the expression for $g(k)$, we have

$$\begin{aligned}
g(k+1) &= g(k) + 2^{-(k+1)}\binom{2k+1}{k+1} + L \\
&= g(k) + 2^{-(k+1)}\binom{2k+1}{k+1} + \frac{1}{2}\left(g(k+1) - 2^{-(k+1)}\binom{2k+2}{k+1}\right) \\
\frac{1}{2}g(k+1) &= g(k) + 2^{-(k+1)}\binom{2k+1}{k+1} + -2^{-(k+2)}\binom{2k+2}{k+1} \\
&= g(k) + 2^{-(k+2)}\left[2\binom{2k+1}{k+1} - \binom{2k+2}{k+1}\right] \\
&= g(k) + 2^{-(k+2)}\left[\binom{2k+1}{k+1} + \binom{2k+1}{k} - \left(\binom{2k+1}{k} + \binom{2k+1}{k+1}\right)\right] \\
&= g(k) + 2^{-(k+2)}\left[\binom{2k+1}{k+1} - \binom{2k+1}{k+1}\right] \\
&= g(k) + 0 = g(k)
\end{aligned}$$

thus $g(k+1) = 2g(k) = 2^{k+1}$, and we are done.