PUTNAM 2024 A

Math Competitions

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Problem 1. Let $n, k \in \mathbb{Z}^+$. The sequence in the ith row and jth column of an $n \times n$ grid contains the numbe $i + j - k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are $1, 2, \ldots, n$?

Problem 2. Two convex quadrilaterals are called partners if they have three verticies in common and they can be labelled $ABCD$ and $ABCE$ so that E is the reflection of D across the perpendicular bisector of the diagonal AC. Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent?

Problem 3. Let r_n be the smallet positive solution to $\tan x = x$, where the argument of tangent is in radians. Prove that

$$0 < r_{n+1} - r_n - \pi < \frac{1}{(n^2+n)\pi} \tag{1}$$

for $n \geq 1$.

Problem 4. Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1, \ldots, n\}$ and let

$$a_{n,k+1} = \begin{cases} a_{n,k} + 1 \text{if} m_{n,k} > a_{n,k}; \\ a_{n,k} \text{if} m_{n,k} = a_{n,k}; \\ a_{n,k} - 1 \text{if} m_{n,k} < a_{n,k}. \end{cases}$$

Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n\to\infty} E(n)/n$.

Problem 5. Let k and m be integers. For a positive integer n, let $f(n)$ be the number of integer sequences $x_1, \ldots, x_k, y_1, \ldots, y_m, z$ satisfying $1 \leq x_1 \leq \cdots \leq x_k \leq z \leq n$ and $1 \leq y_1 \leq \cdots \leq y_m \leq z \leq n$. Show that $f(n)$ can be expressed as a polynomial in n with nonnegative coefficients.

Problem 6. For a real number a, let $F_a(x) = \sum_{n\geq 1} n^a e^{2n} x^{n^2}$ for $0 \leq x < 1$. Find a real number c such that

$$\lim x \to 1^- F_a(x) e^{-1/(1-x)} = 0 \text{for all } a < c, \text{and} \tag{2}$$
$$\lim x \to 1^- F_a(x) e^{-1/(1-x)} = \infty \text{for all } a > c. \tag{3}$$