Putnam POTD November 10, 2025

I. PROBLEM

(2021 B2) Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

II. SOLUTION

We may not directly get a good result by simply looking at the product $(a_1 \ldots a_n)^{\frac{1}{n}}$, but this form is inviting to use AM-GM.

The step, then, is to see if we can fit it with something else. Note that

$$2^{n+1}(a_1 \ldots a_n)^{\frac{1}{n}} = (2^{n(n+1)} a_1 \ldots a_n)^{\frac{1}{n}} = (4a_1 \ldots 4^n a_n)^{\frac{1}{n}} \leq \frac{4a_1 + \cdots + 4^n a_n}{n}$$

Going back to the original sum then,

$$\begin{aligned}
S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n} &\leq \sum_{n=1}^{\infty} \frac{n}{2^n} \frac{1}{2^{n+1}} \sum_{i=1}^{n} 4^i a_i \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{2n+1}} \sum_{i=1}^{n} 4^i a_i \\
&= \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} \frac{1}{2^{2n+1}} 4^i a_i \\
&= \frac{1}{2} \sum_{i=1}^{\infty} 4^i a_i \sum_{n=i}^{\infty} 4^{-n} \\
&= \frac{1}{2} \sum_{i=1}^{\infty} 4^i a_i 4^{-i} \frac{4}{3} \\
&= \frac{2}{3} \sum_{i=1}^{\infty} a_i = \boxed{\frac{2}{3}}.
\end{aligned}$$