Putnam POTD November 11, 2025

I. PROBLEM

(1996 A3) Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

II. SOLUTION

We can model this problem as a bipartite graph $K_{20,6}$ with 20 student nodes and 6 classes. Color an edge green if a student chooses a class and yellow if a student does not, then we are curious if a subgraph $K_{5,2}$ arises where all edges have the same color.

This problem strikes a strong resemblence to the classic Ramsey theory number $R_{3,3}$, which asks for the smallest n such that a complete graph K_n with two coloring that has a subgraph of K_3 with all edges of the same color. Please review this solution before proceeding.

Now we can consider the number of monochromatic edges. Say a student chooses k classes, then he contributes $\binom{k}{2} + \binom{6-k}{2}$ monochromatic edges. The minimum number of such edges he can contribute is

$$m_{min} = \min_{k \in [0,6]} \binom{k}{2} + \binom{6-k}{2} = 6.$$

Thus the min total number of monochromatic edges is

$$M = 20 * m_{min} = 120.$$

And there are $\binom{6}{2} = 15$ course pairs, so the min number of courses each course pair could have is $120/15 = 8$ from the pigonhole principle.

For a particular course pair in this case, let x be the number of students that chose this course and y be the numebr of students that did not chose this course, then $x + y \geq 8$. We want to show that this implies $x \geq 5$ or $y \geq 5$, however, this might not be true as $x = 4, y = 4$ provides a counterexample.