Putnam POTD November 13, 2025

I. PROBLEM

(1972 B3) Let G be a group and let $g, h \in G$ satisfy

$$ghg = hg^2h, \quad g^3 = 1,$$

and suppose $h^m = 1$ for some odd integer m. Prove that $h = 1$.

II. (WRONG) SOLUTION

Rewrite the given equation as

$$ghg = hg^{-1}h$$

Please let me know how to show this and when can I make these substitutions! We assume without proof that

$$g^{-1}hg^{-1} = hgh$$

then we can claim:

$$\begin{aligned} hghg &= h^2g^{-1}h \\ g^{-1}h &= h^2g^{-1}h \\ g^{-1} &= h^2g^{-1} \\ 1 &= h^2 \end{aligned}$$

and since $h^m = 1$ for some odd m, $h^{2n+1} = 1$ and thus $h = 1$.

III. SOLUTION

Let's work backwards: If we could show $gh^2 = h^2g$, then

$$\begin{aligned} h^2 &= g^{-1}h^2g \\ h^{2k} &= g^{-1}h^{2k}g \quad \forall k \in \mathbb{Z}^+ \end{aligned}$$

If $h^{2n+1} = 1$, let $k = n+1$, then

$$\begin{aligned} h^{2n+2} &= g^{-1}h^{2n+2}g \\ h &= g^{-1}hg \\ gh &= hg \\ ghg &= hg^{-1} \\ g^{-1}h &= hg^{-1}h \\ h &= 1. \end{aligned}$$

Now we show $h^2g = gh^2$, to see this:

$$gh^2 = ghgg^{-1}h = hg^{-1}hg^{-1}h = hg^{-1}ghg = h^2g,$$

and we are done.

IV. REMARK

To be frank, I am currently not great with solving these problems, and I did not come up with the idea to this solution.

The solution wrote: With hindsight, the correct approach is probably to work systematically through a set of increasingly complex expressions, trying to simplify them more than one way in order to get a new relation.