Putnam POTD November 10, 2025

I. PROBLEM

(2021 B2) Determine the maximum value of the sum

$$S=\sum_{n=1}^{\infty}\frac{n}{2^n}(a_1a_2\cdots a_n)^{1/n}$$

over all sequences $a_1,a_2,a_3,\cdots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty}a_k=1.$$

II. SOLUTION

We may not directly get a good result by simply looking at the product $(a_1\ldots a_n)^{\frac{1}{n}}$, but this form is inviting to use AM-GM.

The step, then, is to see if we can fit it with something else. Note that

$$2^{n+1}(a_1\ldots a_n)^{\frac{1}{n}}=(2^{n(n+1)}a_1\ldots a_n)^{\frac{1}{n}}=(4a_1\ldots 4^na_n)^{\frac{1}{n}}\leq\frac{4a_1+\cdots+4^na_n}{n}$$

Going back to the original sum then,

$$\begin{aligned}
S=\sum_{n=1}^{\infty}\frac{n}{2^n}(a_1a_2\cdots a_n)^{1/n}&\leq\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{1}{2^{n+1}}\sum_{i=1}^{n}4^ia_i\\
&=\sum_{n=1}^{\infty}\frac{1}{2^{2n+1}}\sum_{i=1}^{n}4^ia_i\\
&=\sum_{i=1}^{\infty}\sum_{n=i}^{\infty}\frac{1}{2^{2n+1}}4^ia_i\\
&=\frac{1}{2}\sum_{i=1}^{\infty}4^ia_i\sum_{n=i}^{\infty}4^{-n}\\
&=\frac{1}{2}\sum_{i=1}^{\infty}4^ia_i4^{-i}\frac{4}{3}\\
&=\frac{2}{3}\sum_{i=1}^{\infty}a_i=\boxed{\frac{2}{3}}.
\end{aligned}$$

III. MOTIVATION

How did I (or many other solutions) come up with the right coefficients? The main thing is that we want to evaluate the series after making the boudning due to AM-GM. If we tried bounding naively, $(a_1 \dots a_n)^{\frac{1}{n}} \leq \sum_i^n \frac{a_i}{n}$, then

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{n}{2^n}(a_1 a_2 \cdots a_n)^{1/n} &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{n} a_i \\ &= \sum_{i=1}^{\infty} \sum_{n=i}^{\infty} \frac{a_i}{2^n} \end{aligned}$$

which does not simplify nicely into $\sum_{i=1}^{n} a_i$.

Refined Goal: We want to use some form of AM-GM to get this cancellation:

$$S \leq \sum_{n=1}^{\infty} \frac{n}{2^n} \left[\frac{\text{Some Arithmetic Mean}}{n} \right] = \sum_{n=1}^{\infty} \frac{1}{2^n} [\text{Some Arithmetic Mean}]$$

The naive AM-GM on a_k didn't work, so we need to use it on a modified or weighted sequence. Since the problem involves an exponential term 2^n, it's natural to guess an exponential weight, say c^k.Let's apply AM-GM to the sequence $x_k = c^k a_k$:

$$((c^1 a_1)(c^2 a_2) \cdots (c^n a_n))^{1/n} \leq \frac{\sum_{k=1}^{n} c^k a_k}{n}$$

Now, let's simplify that Left-Hand Side (LHS):

$$LHS = (c^{1+2+\cdots+n} \cdot a_1 \cdots a_n)^{1/n}$$

$$LHS = (c^{n(n+1)/2})^{1/n} \cdot (a_1 \cdots a_n)^{1/n}$$

$$LHS = c^{(n+1)/2} \cdot (GM_n)$$

So, our inequality is $c^{(n+1)/2} \cdot (GM_n) \leq \frac{\sum c^k a_k}{n}$.Isolating the GM_n term (which is what's in our sum):

$$(GM_n) \leq \frac{1}{c^{(n+1)/2}} \cdot \frac{\sum c^k a_k}{n}$$

The "Aha!" Moment: Matching the ExponentsNow, let's substitute this back into our "Refined Goal" expression from Step 2:

$$S \leq \sum_{n=1}^{\infty} \frac{n}{2^n} \left[\frac{1}{c^{(n+1)/2}} \cdot \frac{\sum_{k=1}^{n} c^k a_k}{n} \right]$$

The n terms cancel beautifully, just as we hoped!

$$S \leq \sum_{n=1}^{\infty} \left[\frac{1}{2^n \cdot c^{(n+1)/2}} \right] \sum_{k=1}^{n} c^k a_k$$

This still looks complicated. We need to make the denominator $D = 2^n \cdot c^{(n+1)/2}$ as simple as possible.

$$D = 2^n \cdot c^{n/2} \cdot c^{1/2} = \sqrt{c} \cdot (2 \cdot c^{1/2})^n$$

We want this to simplify into something that will create a clean geometric series after we swap the summations. The cleanest possible simplification would be if $2 \cdot c^{1/2}$ was equal to c. Or, more simply, what if $2^n \cdot c^{n/2}$ simplifies to c^n?We need $2^n \cdot c^{n/2} = c^n$.Divide by $c^{n/2}$: $2^n = c^{n/2}$.Take the n-th root: $2 = c^{1/2} = \sqrt{c}$.$c = 4$This is the "magic" number. By choosing $c = 4$, we are "matching" the 2^n from the problem with the $c^{(n+1)/2}$ that arises from the AM-GM.Let's see what happens when $c = 4$:The denominator D becomes:

$$D = 2^n \cdot 4^{(n+1)/2} = 2^n \cdot (2^2)^{(n+1)/2} = 2^n \cdot 2^{n+1} = 2^{2n+1} = 2 \cdot 4^n$$

Plugging this in:

$$S \leq \sum_{n=1}^{\infty} \left[\frac{1}{2 \cdot 4^n} \right] \sum_{k=1}^{n} 4^k a_k$$

$$2S \leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{4^k a_k}{4^n}$$

This is exactly the expression in the solution, and it's guaranteed to work because we can swap the summation and get a clean geometric series.