Novel metrics and nearest-neighbor distance distributions in high dimensional bioinformatics data

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Abstract

Nearest-neighbor projected distance regression (NPDR) is a feature selection algorithm that is able to detect interactions in high dimensional data. The performance of NPDR and other nearest neighbor methods depends on the metric for computing neighborhoods and the expected moments of the distribution of pairwise distances for the given data type. We derive the expected number of neighbors for adaptive radius neighborhood algorithms. We derive general analytical expressions for distributional properties of pairwise distances for L_q metrics for Gaussian and uniform data with p attributes and m instances. These expressions are applicable to the analysis of gene expression data. We derive similar analytical expressions for a new metric for genome-wide association study data (categorical predictors) and a new metric for resting-state fMRI data (correlation-based predictors). In addition, we derive expected minimum and maximum distances using extreme value theory and we consider the effect of correlation in the data.

Author summary

Introduction

Feature selection that relies on nearest neighbor algorithms in order to determine relative feature importance require an understanding of distributional properties for a variety of different metrics. For continuous data, L^q metrics with $q = 1$ or $q = 2$ are those most commonly used in this context. For data from standard normal or standard uniform distributions, the asymptotic behavior of the L^q metrics is known. However, detailed derivations of these distance distribution asymptotics are not commonly found or mentioned in the literature on nearest-neighbor distance based feature selection. Furthermore, there is much work to be done to better understand new metrics in discrete data, such as, genome-wide association studies (GWAS) data or correlation data like resting-state fMRI (rs-fMRI). Optimal choices of neighborhood selection parameters, such as, fixed-radius or fixed-k depend on distance distributional properties with respect to the instance dimension. As neighborhood order increases, nearest neighbor distance based algorithms get better at detecting main effects. On the other hand, their ability to detect interaction effects decreases as neighborhood order increases. These different statistical effects impact distance distributions by introducing positive skewness and increased variance, which can lead to changes in neighborhood inclusion. In order to understand how statistical effects impact distance distributions in continuous and discrete data types, we first derive distance asymptotics for null data where instances are independently and identically distributed and there is no correlation between features. Using these derivations, we can then determine how statistical effects and correlation change distance distributional properties from the null case.

1 Derivations of distance asymptotics for common metrics used in continuous data

The distance between instances i and j in the data set $X^{m \times p}$ of m instances and p attributes is calculated in the space of all attributes ($a \in \mathcal{A}$, $|\mathcal{A}| = p$) using a metric such as

$$D_{ij}^{(q)} = \left(\sum_{a \in \mathcal{A}} |\mathrm{d}_{ij}(a)|^q \right)^{1/q}, \tag{1}$$

which is typically Manhattan ($q = 1$) but may also be Euclidean ($q = 2$). The quantity $\mathrm{d}_{ij}(a)$, known as a "diff" in Relief literature, is the projection of the distance between instances i and j onto the attribute a dimension. The function $\mathrm{d}_{ij}(a)$ supports any type of attributes (e.g., numeric and categorical). For example, the projected difference between two instances i and j for a continuous numeric ($\mathrm{d}^{\mathrm{num}}$) attribute a may be

$$\begin{aligned} \mathrm{d}_{ij}^{\mathrm{num}}(a) &= \mathrm{diff}(a, (i, j)) \\ &= |\hat{X}_{ia} - \hat{X}_{ja}|, \end{aligned} \tag{2}$$

where $\hat{X}$ represents the standardized data matrix X. We use a simplified $\mathrm{d}_{ij}(a)$ notation in place of the $\mathrm{diff}(a, (i, j))$ notation that is customary in Relief-based methods. We omit the division by $\max(a) - \min(a)$ used by Relief to constrain scores to the interval from -1 to 1. As we show in subsequent sections, NPDR scores are standardized regression coefficients with corresponding P values, so any scaling operation at this stage is unnecessary for comparing attribute scores. The numeric $\mathrm{d}_{ij}^{\mathrm{num}}(a)$ projection is simply the absolute difference between row elements i and j of the data matrix $X^{m \times p}$ for the attribute column a.

We define the NPDR neighborhood set $\mathcal{N}$ of ordered pair indices as follows. Instance i is a point in p dimensions, and we designate the topological neighborhood of i as N_i. This neighborhood is a set of other instances trained on the data $X^{m \times p}$ and depends on the type of Relief neighborhood method (e.g., fixed-k or adaptive radius) and the type of metric (e.g., Manhattan or Euclidean). If instance j is in the neighborhood of i ($j \in N_i$), then the ordered pair $(i, j) \in \mathcal{N}$ for the projected-distance regression analysis. The ordered pairs constituting the neighborhood can then be represented as nested sets:

$$\mathcal{N} = \{\{(i, j)\}_{i=1}^{m}\}_{\{j \neq i : j \in N_i\}}. \tag{3}$$

The cardinality of the set $\{j \neq i : j \in N_i\}$ is k_i, the number of nearest neighbors for subject i.

1.1 Distribution of pairwise distances

Suppose that $X_{ia}, X_{ja} \overset{iid}{\sim} \mathcal{F}_X(\mu_X, \sigma_X^2)$ for two fixed and distinct instances $(i, j) \in \mathcal{N}$ and a fixed attribute $a \in \mathcal{A}$. $\mathcal{F}_X$ represents any data distribution with mean μ_X and variance σ_X^2.

It is clear that $|X_{ia} - X_{ja}|^q = |d_{ij}(a)|^q$ is another random variable. Let $Z_a^q \sim \mathcal{F}_{Z^q}(\mu_{z^q}, \sigma_{z^q}^2)$ be the random variable such that

$$Z_a^q = |d_{ij}(a)|^q = |X_{ia} - X_{ja}|^q, \quad a \in \mathcal{A}. \tag{4}$$

Furthermore, the collection $\{Z_a^q | a \in \mathcal{A}\}$ is a random sample of size p of mutually independent random variables. Hence, the sum of Z_a^q over all $a \in \mathcal{A}$ is asymptotically normal by the Classical Central Limit Theorem (CCLT). More explicitly, this implies that

$$\left(D_{ij}^{(q)}\right)^q = \sum_{a\in\mathcal{A}} |d_{ij}(a)|^q = \sum_{a\in\mathcal{A}} |X_{ia} - X_{ja}|^q = \sum_{a\in\mathcal{A}} Z_a^q \;\dot\sim\; \mathcal{N}\left(\mu_{z^q}p, \sigma^2_{z^q}p\right) \tag{5}$$

Consider the smooth function $g(z) = z^{1/q}$ that is continuously differentiable for $z > 0$. Assuming that $\mu_{z^q} > 0$, the Delta Method [2] can be applied to show that

$$\begin{aligned}
g\left(\left(D_{ij}^{(q)}\right)^q\right) &= g\left(\sum_{a\in\mathcal{A}}^{p} Z_a^q\right) \\
&= \left(\sum_{a\in\mathcal{A}} |X_{ia} - X_{ja}|^q\right)^{1/q} \\
&= D_{ij}^{(q)} \;\dot\sim\; \mathcal{N}\left(g\left(\mu_{z^q}p\right), \left[g\prime\left(\mu_{z^q}p\right)\right]^2 \sigma^2_{z^q}p\right) \\
\Rightarrow\quad D_{ij}^{(q)} &\;\dot\sim\; \mathcal{N}\left(\left(\mu_{z^q}p\right)^{1/q}, \frac{\sigma^2_{z^q}p}{q^2\left(\mu_{z^q}p\right)^{2\left(1-\frac{1}{q}\right)}}\right)
\end{aligned} \tag{6}$$

Therefore, the distance between two fixed, distinct instances i and j given by Eq. 1 is asymptotically normal. Specifically, when $q = 2$, the distribution of $D_{ij}^{(2)}$ asymptotically approaches $\mathcal{N}\left(\sqrt{\mu_{z^2}p}, \frac{\sigma^2_{z^2}}{4\mu_{z^2}}\right)$. When p is small, however, we observe empirically that a closer estimate of the sample mean is

$$\begin{aligned}
\mathrm{E}\left(D_{ij}^{(2)}\right) &= \sqrt{\mathrm{E}\left[\left(D_{ij}^{(2)}\right)^2\right] - \mathrm{Var}\left(D_{ij}^{(2)}\right)} \\
&= \sqrt{\mu_{z^2}p - \frac{\sigma^2_{z^2}}{4\mu_{z^2}}}.
\end{aligned} \tag{7}$$

One can readily verify the normality of distances between independent instances through sampling from any data distribution and plotting the histogram of pairwise distances. Histograms for the case of standard normal data and Manhattan ($q = 1$) and Euclidean ($q = 2$) metrics are shown in Figs. 1 and 2, respectively. For these simulated distances, we fixed the number of instances $m = 100$ and varied the number of attributes p from 10 to 10000. For even a rather small number of predictors, as in the case of $p = 10$, the sample distances are approximately normal. As p increases, the normality becomes stronger.

For distance based learning methods, all pairwise distances are used to determine relative importances for attributes. The collection of all distances above the diagonal in an $m \times m$ distance matrix does not satisfy the independence assumption used in the previous derivations. This is because of the redundancy that is inherent to the distance matrix calculation. However, this collection is still asymptotically normal with mean and variance approximately equal to those given in Eq. 6. Hence, all fixed-radius methods will use a fixed radius that is some fraction of the expected pairwise distance for a given metric and data type. This implies that the probability of a fixed instance j being within a fixed radius of a given instance i can be parameterized by the expected pairwise distance and the variance of the pairwise distance. This probability is obtained by evaluating the normal cumulative distribution function (CDF), with corresponding mean and variance, at the quantile given by some function of the fixed radius. Therefore, we can derive the expected number of neighbors in the neighborhood of a fixed instance i. In other words, for sufficiently large data sets, the sample mean of the number of neighbors in a given neighborhood is well approximated by the product between the total number of possible neighbors and the expected probability of an instance being

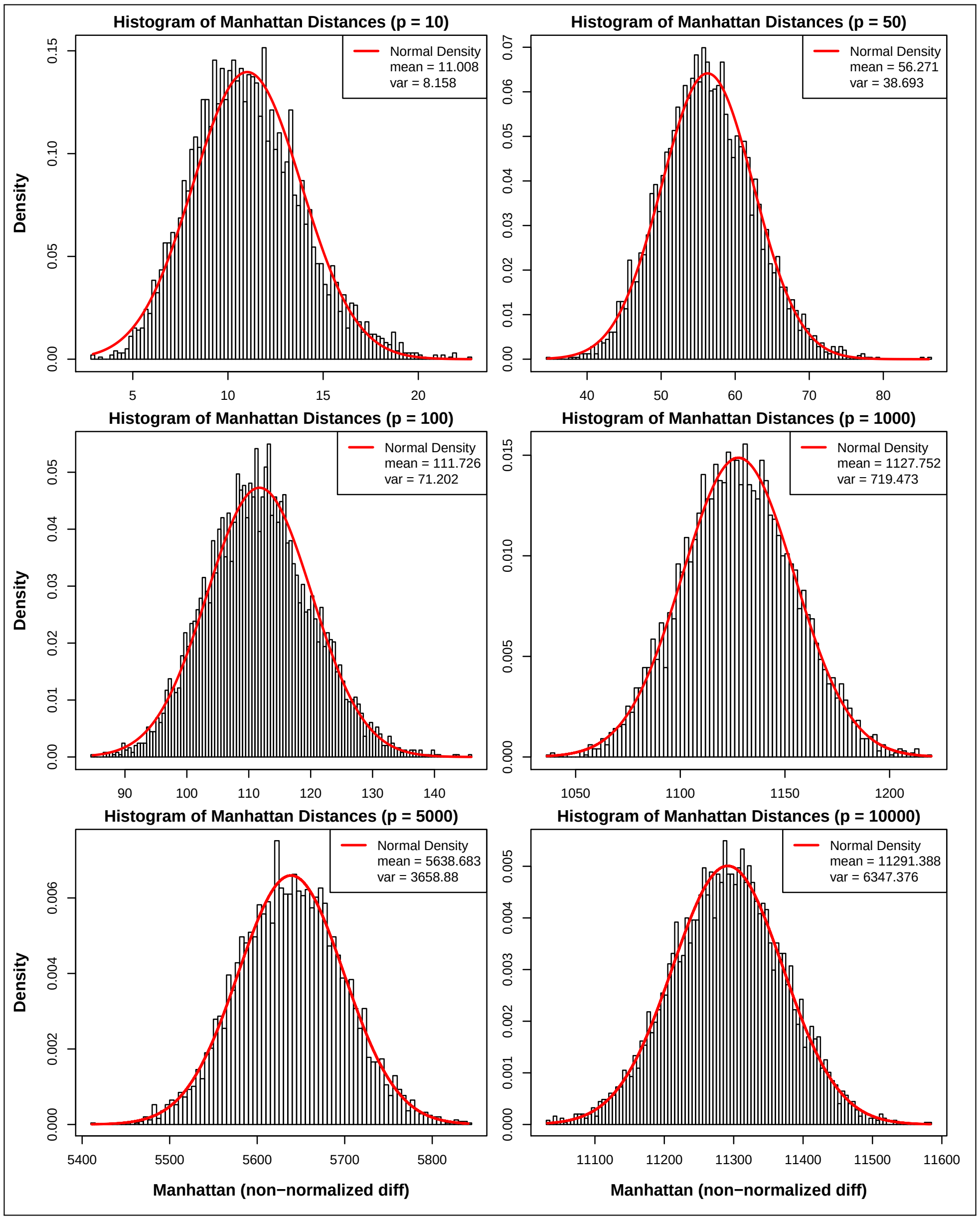

Fig 1. Convergence to normality of Manhattan distances between iid random normal instances. For each simulated distance distribution, we fixed $m = 100$ instances but varied p from 10 to 10000. It is clear that convergence is rapid, and approximate normality can be safely assumed for even $p = 10$.

in a given neighborhood. The total number of possible neighbors for a fixed instance i is always $m - 1$, but this becomes approximately $\lfloor \frac{m-1}{2} \rfloor$ when delineating between possible hits and misses for balanced data.

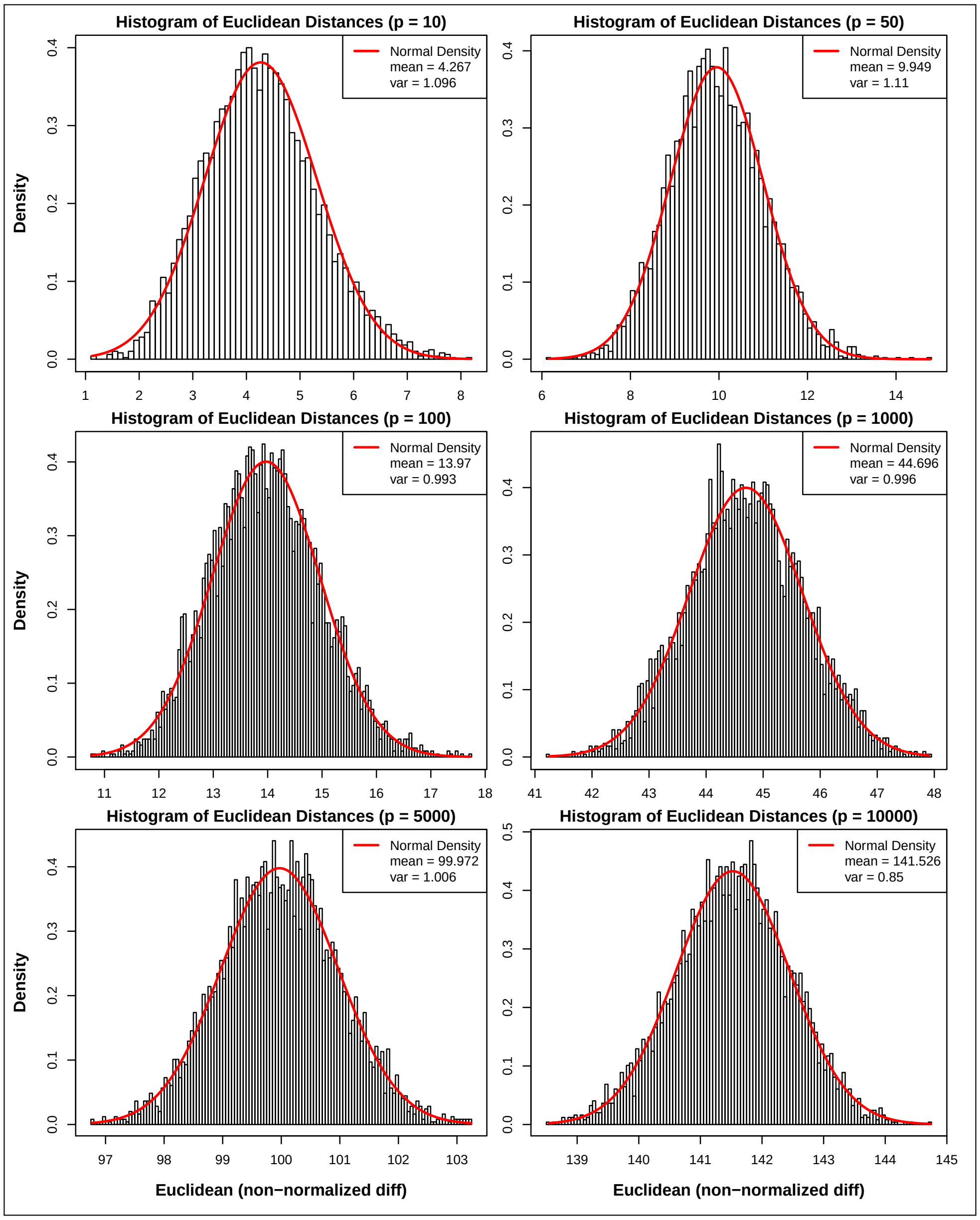

Fig 2. Convergence to normality of Euclidean distances between iid random normal instances. For each simulated distance distribution, we fixed $m = 100$ instances but varied p from 10 to 10000. It is clear that convergence is rapid, and approximate normality can be safely assumed for even $p = 10$.

2 Derivation of means and standard deviations for metrics and data distributions

2.1 Distribution of $|d_{ij}(a)|^q = |X_{ia} - X_{ja}|^q$

Suppose that $X_{ia}, X_{ja} \overset{iid}{\sim} \mathcal{F}_X(\mu_x, \sigma_x^2)$ and define $Z_a^q = |d_{ij}(a)|^q = |X_{ia} - X_{ja}|^q$, where $a \in \mathcal{A}$ and $|\mathcal{A}| = p$. In order to find the distribution of Z_a^q, we will use the following theorem given in [3].

Theorem 2.1 *Let $f(x)$ be the value of the probability density of the continuous random variable X at x. If the function given by $y = u(x)$ is differentiable and either increasing or decreasing for all values within the range of X for which $f(x) \neq 0$, then, for these values of x, the equation $y = u(x)$ can be uniquely solved for x to give $x = w(y)$, and for the corresponding values of y the probability density of $Y = u(X)$ is given by*

$$g(y) = f[w(y)] \cdot |w'(y)| \qquad \textit{provided } u'(x) \neq 0$$

Elsewhere, $g(y) = 0$.

We have the following cases that result from solving for X_{ja} in the equation given by $Z_a^q = |X_{ia} - X_{ja}|^q$.

(i) Suppose that $X_{ja} = X_{ia} - (Z_a^q)^{1/q}$. Based on the iid assumption for X_{ia} and X_{ja}, it follows from Thm. 2.1 that the joint density function $g^{(1)}$ of X_{ia} and Z_a^q is given by

$$\begin{aligned} g^{(1)}(x_{ia}, z_a) &= f_X(x_{ia}, x_{ja}) \left| \frac{\partial x_{ja}}{\partial z_a} \right| \\ &= f_X(x_{ia}) f_X(x_{ja}) \left| \frac{-1}{q} \left(z_a^q\right)^{\frac{1}{q}-1} \right| \\ &= \frac{1}{q \left(z_a^q\right)^{1-\frac{1}{q}}} f_X(x_{ia}) f_X\left(x_{ia} - (z_a^q)^{1/q}\right), \quad z_a > 0 \end{aligned} \tag{8}$$

The density function $f_{Z^q}^{(1)}$ of Z_a^q is then defined as

$$\begin{aligned} f_{Z^q}^{(1)}(z_a^q) &= \int_{-\infty}^{\infty} g^{(1)}(x_{ia}, z_a^q) \mathrm{d}x_{ia} \\ &= \frac{1}{q \left(z_a^q\right)^{1-\frac{1}{q}}} \int_{-\infty}^{\infty} f_X(x_{ia}) f_X\left(x_{ia} - (z_a^q)^{1/q}\right) \mathrm{d}x_{ia}, \quad z_a > 0 \end{aligned} \tag{9}$$

(ii) Suppose that $X_{ja} = X_{ia} + (Z_a^q)^{1/q}$. Based on the iid assumption for X_{ia} and X_{ja}, it follows from Thm. 2.1 that the joint density function $g^{(2)}$ of X_{ia} and Z_a is given by

$$\begin{aligned} g^{(2)}(x_{ia}, z_a) &= f_X(x_{ia}, x_{ja}) \left| \frac{\partial x_{ja}}{\partial z_a} \right| \\ &= f_X(x_{ia}) f_X(x_{ja}) \left| \frac{1}{q} \left(z_a^q\right)^{\frac{1}{q}-1} \right| \\ &= \frac{1}{q \left(z_a^q\right)^{1-\frac{1}{q}}} f_X(x_{ia}) f_X\left(x_{ia} - (z_a^q)^{1/q}\right), \quad z_a > 0 \end{aligned} \tag{10}$$

The density function $f_{Z^q}^{(2)}$ of Z_a^q is then defined as

$$\begin{aligned} f_{Z^q}^{(2)}(z_a^q) &= \int_{-\infty}^{\infty} g^{(2)}(x_{ia}, z_a^q) \mathrm{d}x_{ia} \\ &= \frac{1}{q \left(z_a^q\right)^{1-\frac{1}{q}}} \int_{-\infty}^{\infty} f_X(x_{ia}) f_X\left(x_{ia} + (z_a^q)^{1/q}\right) \mathrm{d}x_{ia}, \quad z_a > 0 \end{aligned} \tag{11}$$

Let F_{Z^q} denote the distribution function of the random variable Z_a^q. Furthermore, we define the events $E^{(1)}$ and $E^{(2)}$ as

$$E^{(1)} = \left\{|X_{ia} - X_{ja}|^q \le z_a^q | X_{ja} = X_{ia} - (Z_a^q)^{1/q}\right\} \tag{12}$$

and

$$E^{(2)} = \left\{|X_{ia} - X_{ja}|^q \le z_a^q | X_{ja} = X_{ia} + (Z_a^q)^{1/q}\right\}. \tag{13}$$

Then it follows from fundamental rules of probability that

$$\begin{aligned}
F_{Z^q}(z_a^q) &= \mathrm{P}\left[Z_a^q \le z_a^q\right] \\
&= \mathrm{P}\left[|X_{ia} - X_{ja}|^q \le z_a^q\right] \\
&= \mathrm{P}\left[E^{(1)} \cup E^{(2)}\right] \\
&= \mathrm{P}\big[E^{(1)}\big] + \mathrm{P}\big[E^{(2)}\big] - \mathrm{P}\big[E^{(1)} \cap E^{(2)}\big] \\
&= \mathrm{P}\big[E^{(1)}\big] + \mathrm{P}\big[E^{(2)}\big] \\
&= \int_{-\infty}^{z_a^q} f_{Z^q}^{(1)}(t)\mathrm{d}t + \int_{-\infty}^{z_a^q} f_{Z^q}^{(2)}(t)\mathrm{d}t \\
&= \int_{-\infty}^{z_a^q} \left(f_{Z^q}^{(1)}(t) + f_{Z^q}^{(2)}(t)\right)\mathrm{d}t \\
&= \frac{1}{q\left(z_a^q\right)^{1-\frac{1}{q}}} \int_{-\infty}^{z_a^q} \left(\int_{-\infty}^{\infty} f_X(x_{ia})\left[f_X(x_{ia} - t) + f_X(x_{ia} + t)\right]\mathrm{d}x_{ia}\right)\mathrm{d}t, \quad z_a > 0
\end{aligned} \tag{14}$$

It follows directly from the result in Eq. 14 that the density function of the random variable Z_a^q is given by

$$\begin{aligned}
f_{Z^q}(z_a^q) &= \frac{\partial}{\partial z_a^q} F_{Z^q}(z_a^q) \\
&= \frac{1}{q\left(z_a^q\right)^{1-\frac{1}{q}}} \int_{-\infty}^{\infty} f_X(x_{ia})\left[f_X\left(x_{ia} - (z_a^q)^{1/q}\right) + f_X\left(x_{ia} + (z_a^q)^{1/q}\right)\right]\mathrm{d}x_{ia}
\end{aligned} \tag{15}$$

where $z_a > 0$.

Using Eq. 15, we can compute the mean and variance of the random variable Z_a^q as

$$\mu_{z^q} = \int_{-\infty}^{\infty} z_a^q f_{Z^q}(z_a^q)\mathrm{d}z_a^q \tag{16}$$

and

$$\sigma_{z^q}^2 = \int_{-\infty}^{\infty} \left(z_a^q\right)^2 f_{Z^q}(z_a^q)\mathrm{d}z_a^q - \mu_{z^q}^2. \tag{17}$$

It follows immediately from Eqs. 16 and 17 and the Classical Central Limit Theorem (CCLT) that

$$\left(D_{ij}^{(q)}\right)^q = \sum_{a\in\mathcal{A}} Z_a^q = \sum_{a\in\mathcal{A}} |X_{ia} - X_{ja}|^q \dot{\sim} \mathcal{N}\left(\mu_{z^q} p, \sigma_{z^q}^2 p\right) \tag{18}$$

Applying the result given in Eq. 6, the distribution of $D_{ij}^{(q)}$ is given by

$$D_{ij}^{(q)} \mathrel{\dot\sim} \mathcal{N}\left((\mu_{z^q} p)^{1/q}, \frac{\sigma_{z^q}^2 p}{q^2 (\mu_{z^q} p)^{2\left(1-\frac{1}{q}\right)}} \right), \quad \mu_{z^q} > 0 \tag{19}$$

with improved estimate of the mean for $q = 2$ given by Eq. 7.

2.1.1 Standard normal data

If $X_{ia}, X_{ja} \overset{iid}{\sim} \mathcal{N}(0,1)$, then the marginal density functions with respect to X for X_{ia}, $X_{ia} - (Z_a^q)^{1/q}$, and $X_{ia} + (Z_a^q)^{1/q}$ are defined as

$$f_X(x_{ia}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_{ia}^2}, \tag{20}$$

$$f_X\left(x_{ia} - (z_a^q)^{1/q}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x_{ia} - (z_a^q)^{1/q}\right)^2}, \quad z_a > 0, \text{ and} \tag{21}$$

$$f_X\left(x_{ia} + (z_a^q)^{1/q}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x_{ia} + (z_a^q)^{1/q}\right)^2}, \quad z_a > 0 \tag{22}$$

Substituting the results given by Eqs. 20-22 into Eq. 15 and completing the square on x_{ia} in the exponents, we have

$$\begin{aligned}
f_{Z^q}(z_a^q) &= \frac{1}{2q\pi\,(z_a^q)^{1-\frac{1}{q}}} e^{-\frac{1}{4}(z_a^q)^{2/q}} \int_{-\infty}^{\infty} \Bigg(e^{-\frac{1}{2}\left[\sqrt{2}x_{ia} - \frac{\sqrt{2}}{2}(z_a^q)^{1/q}\right]^2} \\
&\qquad + e^{-\frac{1}{2}\left[\sqrt{2}x_{ia} + \frac{\sqrt{2}}{2}(z_a^q)^{1/q}\right]^2} \Bigg) \mathrm{d}x_{ia} \qquad (23) \\
&= \frac{1}{2q\sqrt{\pi}\,(z_a^q)^{1-\frac{1}{q}}} e^{-\frac{1}{4}(z_a^q)^{2/q}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}u^2} + e^{-\frac{1}{2}u^2} \right) \mathrm{d}u \qquad (24) \\
&= \frac{1}{2q\sqrt{\pi}\,(z_a^q)^{1-\frac{1}{q}}} e^{-\frac{1}{4}(z_a^q)^{2/q}} (1+1) \qquad (25) \\
&= \frac{1}{q\sqrt{\pi}} (z_a^q)^{\frac{1}{q}-1} e^{-\frac{1}{4}(z_a^q)^{2/q}} \qquad (26) \\
&= \frac{\frac{2}{q}}{(2^q)^{1/q}\,\Gamma\left(\frac{\frac{1}{q}}{\frac{2}{q}}\right)} (z_a^q)^{\frac{1}{q}-1} e^{-\left(\frac{z_a^q}{2^q}\right)^{2/q}} \qquad (27)
\end{aligned}$$

The density function given by Eq. 23 is a Generalized Gamma density with parameters $b = \frac{2}{q}$, $c = 2^q$, and $d = \frac{1}{q}$. This distribution has mean and variance given by

$$\begin{aligned}
\mu_{z^q} &= \frac{c\Gamma\left(\frac{d+1}{b}\right)}{\Gamma\left(\frac{d}{b}\right)} \\
&= \frac{2^q \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}}
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\sigma_{z^q}^2 &= c^2 \left[\frac{\Gamma\left(\frac{d+2}{b}\right)}{\Gamma\left(\frac{d}{b}\right)} - \left(\frac{\Gamma\left(\frac{d+1}{b}\right)}{\Gamma\left(\frac{d}{b}\right)} \right)^2 \right] \\
&= 4^q \left[\frac{\Gamma\left(q + \frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{2}q + \frac{1}{2}\right)}{\pi} \right]
\end{aligned} \tag{29}$$

By linearity of the expected value and variance operators under the iid assumption, Eqs. 28 and 29 allow the p-dimensional mean and variance of the $D_{ij}^{(q)}$ distribution to be computed directly as

$$\mu_{\left(D_{ij}^{(q)}\right)^q} = \mathrm{E}\left[\left(D_{ij}^{(q)}\right)^q\right] = \mathrm{E}\left(\sum_{a\in\mathcal{A}} Z_a^q\right) = \sum_{a\in\mathcal{A}} \mathrm{E}\left(Z_a^q\right) = \sum_{a\in\mathcal{A}} \frac{2^q\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}} = \frac{2^q\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}}p \tag{30}$$

and

$$\begin{aligned}
\sigma^2_{\left(D_{ij}^{(q)}\right)^q} = \mathrm{Var}\left[\left(D_{ij}^{(q)}\right)^q\right] &= \mathrm{Var}\left(\sum_{a\in\mathcal{A}} Z_a^q\right) \\
&= \sum_{a\in\mathcal{A}} \mathrm{Var}\left(Z_a^q\right) \\
&= \sum_{a\in\mathcal{A}} 4^q\left[\frac{\Gamma\left(q+\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{2}q+\frac{1}{2}\right)}{\pi}\right] \\
&= 4^q\left[\frac{\Gamma\left(q+\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{2}q+\frac{1}{2}\right)}{\pi}\right]p
\end{aligned} \tag{31}$$

Therefore, the asymptotic distribution of $D_{ij}^{(q)}$ for standard normal data is

$$\mathcal{N}\left(\left(2^q\frac{\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}}p\right)^{1/q}, \frac{4^q p}{q^2\left(\frac{2^q\Gamma\left(\frac{1}{2}q+\frac{1}{2}\right)}{\sqrt{\pi}}p\right)^{2\left(1-\frac{1}{q}\right)}}\left[\frac{\Gamma\left(q+\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{2}q+\frac{1}{2}\right)}{\pi}\right]\right) \tag{32}$$

2.1.2 Standard uniform data

If $X_{ia}, X_{ja} \overset{iid}{\sim} \mathcal{U}(0,1)$, then the marginal density functions with respect to X for X_{ia}, $X_{ia} - (Z_a^q)^{1/q}$, and $X_{ia} + (Z_a^q)^{1/q}$ are defined as

$$f_X(x_{ia}) = 1, \quad 0 \le x_{ia} \le 1 \tag{33}$$

$$f_X\left(x_{ia} - (z_a^q)^{1/q}\right) = 1, \quad 0 \le x_{ia} - (z_a^q)^{1/q} \le 1, \text{ and} \tag{34}$$

$$f_X\left(x_{ia} + (z_a^q)^{1/q}\right) = 1, \quad 0 \le x_{ia} + (z_a^q)^{1/q} \le 1. \tag{35}$$

Substituting the results given by Eqs. 33-35 into Eq. 15, we have

$$
\begin{aligned}
f_{Z^q}(z_a^q) &= \frac{1}{q\left(z_a^q\right)^{1-\frac{1}{q}}} \int_{-\infty}^{\infty} f_X(x_{ia}) \left[f_X\left(x_{ia} - (z_a^q)^{1/q}\right) + f_X\left(x_{ia} + (z_a^q)^{1/q}\right)\right] \mathrm{d}x_{ia}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < z_a \leq 1 \\
&= \frac{1}{q\left(z_a^q\right)^{1-\frac{1}{q}}} \int_{0}^{1} [f_X(x_{ia} - (z_a^q) + f_X\left(x_{ia} + (z_a^q)^{1/q}\right)]\mathrm{d}x_{ia}, \quad 0 < z_a \leq 1 \\
&= \frac{1}{q\left(z_a^q\right)^{1-\frac{1}{q}}} \int_{(z_a^q)}^{1} 1\mathrm{d}x_{ia} + \int_{0}^{1-(z_a^q)} 1\mathrm{d}x_{ia}, \quad 0 < z_a \leq 1 \\
&= \frac{1}{q\left(z_a^q\right)^{1-\frac{1}{q}}} \left[(1-(z_a^q)) + (1-(z_a^q))\right], \quad 0 < z_a \leq 1 \\
&= \frac{1}{q} \cdot 2\left(z_a^q\right)^{\frac{1}{q}-1} \left[1 - (z_a^q)^{1/q}\right]^{2-1}, \quad 0 < z_a \leq 1
\end{aligned}
\tag{36}
$$

The density given by Eq. 36 is a Kumaraswamy density with parameters $b = \frac{1}{q}$ and $c = 2$ with moment generating function (MGF) given by

$$
\begin{aligned}
M_n &= \frac{c\Gamma\left(1+\frac{n}{b}\right)\Gamma(c)}{\Gamma\left(1+c+\frac{n}{b}\right)} \\
&= \frac{2}{(nq+2)(nq+1)}
\end{aligned}
\tag{37}
$$

Using the MGF given by Eq. 37, the mean and variance of Z_a^q are computed as

$$
\mu_{z^q} = \frac{2}{(q+2)(q+1)} \tag{38}
$$

and

$$
\sigma_{z^q}^2 = \frac{1}{(q+1)(2q+1)} - \left(\frac{2}{(q+2)(q+1)}\right)^2 \tag{39}
$$

By linearity of the expected value and variance operators under the iid assumption, Eqs. 40 and 41 allow the p-dimensional mean and variance of the $\left(D_{ij}^{(q)}\right)^q$ distribution to be computed directly as

$$
\begin{aligned}
\mu_{\left(D_{ij}^{(q)}\right)^q} = \mathrm{E}\left[\left(D_{ij}^{(q)}\right)^q\right] &= \mathrm{E}\left(\sum_{a\in\mathcal{A}} Z_a^q\right) \\
&= \sum_{a\in\mathcal{A}} \mathrm{E}(Z_a^q) \\
&= \sum_{a\in\mathcal{A}} \frac{2}{(q+2)(q+1)} \\
&= \frac{2p}{(q+2)(q+1)}
\end{aligned}
\tag{40}
$$

and

$$\sigma^2_{\left(D_{ij}^{(q)}\right)^q} = \mathrm{Var}\left[\left(D_{ij}^{(q)}\right)^q\right] = \mathrm{Var}\left(\sum_{a \in \mathcal{A}} Z_a^q\right)$$
$$= \sum_{a \in \mathcal{A}} \mathrm{Var}\left(Z_a^q\right)$$
$$= \sum_{a \in \mathcal{A}} \left[\frac{1}{(q+1)(2q+1)} - \left(\frac{2}{(q+2)(q+1)}\right)^2\right] \tag{41}$$
$$= \left[\frac{1}{(q+1)(2q+1)} - \left(\frac{2}{(q+2)(q+1)}\right)^2\right] p$$

Therefore, the asymptotic distribution of $D_{ij}^{(q)}$ for standard uniform data is

$$\mathcal{N}\left(\left(\frac{2p}{(q+2)(q+1)}\right)^{1/q}, \frac{p}{q^2\left(\frac{2p}{(q+2)(q+1)}\right)^{2\left(1-\frac{1}{q}\right)}}\left[\frac{1}{(q+1)(2q+1)} - \left(\frac{2}{(q+2)(q+1)}\right)^2\right]\right). \tag{42}$$

2.2 Manhattan ($q = 1$)

With our general formulas for the asymptotic mean and variance given by Eqs. 32 and 42 for any value of $q \in \mathbb{Z}^+$, we can simply substitute a particular value of q in order to determine the asymptotic distribution of the corresponding distance metric $D_{ij}^{(q)}$. We demonstrate this with the example of the Manhattan ($q = 1$) metric for standard normal and standard uniform data.

2.2.1 Standard normal data

Using the mean given by Eq. 32 and substituting $q = 1$, we have the following for standard normal data

$$\mathrm{E}\left(D_{ij}^{(1)}\right) = \left(2\frac{\Gamma\left(\frac{1+1}{2}\right)}{\sqrt{\pi}}p\right)^{1/1}$$
$$= \frac{2p}{\sqrt{\pi}}\Gamma(1) \tag{43}$$
$$= \frac{2p}{\sqrt{\pi}}$$

Similarly, the variance of $D_{ij}^{(1)}$ is given by

$$
\begin{aligned}
\operatorname{Var}\left(D_{ij}^{(1)}\right) &= \frac{4^1 p}{1^2 \left(\frac{2^1 \Gamma\left(\frac{1}{2}(1)+\frac{1}{2}\right)}{\sqrt{\pi}} p\right)^{2\left(1-\frac{1}{1}\right)}} \left[\frac{\Gamma\left(1+\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{2}(1)+\frac{1}{2}\right)}{\pi}\right] \\
&= \frac{4p}{1}\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2(1)}{\pi}\right] \\
&= 4p\left[\frac{1}{2} - \frac{1}{\pi}\right] \\
&= \frac{2(\pi-2)p}{\pi}
\end{aligned} \tag{44}
$$

2.2.2 Standard uniform data

Using the mean given by Eq. 42 and substituting $q = 1$, we have the following for standard uniform data

$$
\begin{aligned}
\mathrm{E}\left(D_{ij}^{(1)}\right) &= \left(\frac{2p}{(1+2)(1+1)}\right)^{1/1} \\
&= \frac{2p}{6} \\
&= \frac{p}{3}
\end{aligned} \tag{45}
$$

Similarly, the variance of $D_{ij}^{(1)}$ is given by

$$
\begin{aligned}
\operatorname{Var}\left(D_{ij}^{(1)}\right) &= \frac{p}{1^2\left(\frac{2p}{(1+2)(1+1)}\right)^{2\left(1-\frac{1}{1}\right)}} \left[\frac{1}{(1+1)(2(1)+1)} - \left(\frac{2}{(1+2)(1+1)}\right)^2\right] \\
&= p\left[\frac{1}{6} - \frac{1}{9}\right] \\
&= \frac{p}{18}
\end{aligned} \tag{46}
$$

2.3 Euclidean ($q = 2$)

Analogous to the previous section, we demonstrate the usage of Eqs. 32 and 42 for the Euclidean ($q = 2$) metric for standard normal and standard uniform data.

2.3.1 Standard normal data

Using the mean given by Eq. 32 and substituting $q = 2$, we have the following for standard normal data

$$
\begin{aligned}
\mathrm{E}\left(D_{ij}^{(2)}\right) &= \left(2\frac{\Gamma\left(\frac{2+1}{2}\right)}{\sqrt{\pi}} p\right)^{1/2} \\
&= \left(\frac{2p}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)\right)^{1/2} \\
&= \sqrt{2p}
\end{aligned} \tag{47}
$$

Similarly, the variance of $D_{ij}^{(2)}$ is given by

$$
\begin{aligned}
\operatorname{Var}\left(D_{ij}^{(1)}\right) &= \frac{4^2 p}{2^2\left(\frac{2^2\Gamma\left(\frac{1}{2}(2)+\frac{1}{2}\right)}{\sqrt{\pi}}p\right)^{2\left(1-\frac{1}{2}\right)}}\left[\frac{\Gamma\left(2+\frac{1}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{1}{2}(2)+\frac{1}{2}\right)}{\pi}\right] \\
&= \frac{16p}{4\left(\frac{4\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}}p\right)}\left[\frac{\Gamma\left(\frac{5}{2}\right)}{\sqrt{\pi}} - \frac{\Gamma^2\left(\frac{3}{2}\right)}{\pi}\right] \\
&= 2\left[\frac{3}{4} - \frac{1}{4}\right] \\
&= 1
\end{aligned}
\tag{48}
$$

For the case in which the number of attributes p is small, an improved estimate of the mean is given by Eq. 7. The lower dimensional estimate of the mean is as follows.

$$
\begin{aligned}
\mathrm{E}\left(D_{ij}^{(2)}\right) &= \left(2\frac{\Gamma\left(\frac{2+1}{2}\right)}{\sqrt{\pi}}p - 1\right)^{1/2} \\
&= \left(\frac{2p}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) - 1\right)^{1/2} \\
&= \sqrt{2p-1}
\end{aligned}
\tag{49}
$$

For high dimensional data sets, such as gene expression, rs-fMRI, or GWAS, it is clear that the magntiude of p will be sufficient to use Eq. 47 since $\sqrt{2p} \approx \sqrt{2p-1}$ in that case.

2.3.2 Standard uniform data

Using the mean given by Eq. 42 and substituting $q = 2$, we have the following for standard uniform data

$$
\begin{aligned}
\mathrm{E}\left(D_{ij}^{(2)}\right) &= \left(\frac{2p}{(2+2)(2+1)}\right)^{1/2} \\
&= \left(\frac{2p}{12}\right)^{1/2} \\
&= \sqrt{\frac{p}{6}}
\end{aligned}
\tag{50}
$$

Similarly, the variance of $D_{ij}^{(2)}$ is given by

$$
\begin{aligned}
\operatorname{Var}\left(D_{ij}^{(2)}\right) &= \frac{p}{2^2\left(\frac{2p}{(2+2)(2+1)}\right)^{2\left(1-\frac{1}{2}\right)}}\left[\frac{1}{(2+1)(2(2)+1)} - \left(\frac{2}{(2+2)(2+1)}\right)^2\right] \\
&= \frac{3p}{2}\left[\frac{1}{15} - \frac{1}{36}\right] \\
&= \frac{7p}{120}
\end{aligned}
\tag{51}
$$

For the case in which the number of attributes p is small, an improved estimate of the mean is given by Eq. 7. The lower dimensional estimate of the mean is as follows.

$$
\begin{aligned}
\mathrm{E}\left(D_{ij}^{(2)}\right) &= \left(\frac{2p}{(2+2)(2+1)} - \frac{7}{120}\right)^{1/2} \\
&= \left(\frac{2p}{12} - \frac{7}{120}\right)^{1/2} \\
&= \sqrt{\frac{p}{6} - \frac{7}{120}}
\end{aligned} \tag{52}
$$

For high dimensional data sets, such as gene expression, rs-fMRI, or GWAS, it is clear that the magntiude of p will be sufficient to use Eq. 47 since $\sqrt{\frac{p}{6}} \approx \sqrt{\frac{p}{6} - \frac{7}{120}}$ in that case.

2.4 Distribution of attribute extremes

For Relief-based methods [4, 5], the standard numeric diff metric is given by

$$
d_{ij}^{\mathrm{num}}(a) = \mathrm{diff}(a, (i, j)) = \frac{|X_{ia} - X_{ja}|}{\max(a) - \min(a)} \tag{53}
$$

where $\max(a) = \max\limits_{k \in \mathcal{I}}\{X_{ka}\}$, $\min(a) = \min\limits_{k \in \mathcal{I}}\{X_{ka}\}$, and $\mathcal{I} = \{1, 2, \ldots, m\}$.

In order to determine moments of asymptotic distance distributions induced by Eq. 53, we must first derive the asymptotic extreme value distributions of the attribute maximum and minimum. Although the exact distribution of the maximum or minimum requires an assumption about the data distribution, the Fisher-Tippett-Gnedenko Theorem allows us to categorize the extreme value distribution for a collection of independent and identically distributed random variables into one of three distributional families. Before stating the theorem, we first need the following definition.

Definition 2.1 *A distribution $\mathcal{F}_X$ is said to be* ***degenerate*** *if its density function f_X is the Dirac delta $\delta(x - c_0)$ centered at a constant $c_0 \in \mathbb{R}$, with corresponding distribution function F_X defined as*

$$
F_X(x) = \begin{cases} 1, & x \geq c_0, \\ 0, & x < c_0. \end{cases}
$$

Theorem 2.2 (Fisher-Tippett-Gnedenko) *Let $X_{1a}, X_{2a}, \ldots, X_{ma} \overset{iid}{\sim} \mathcal{F}_X\left(\mu_x, \sigma_x^2\right)$ and let $X_a^\alpha = \max\limits_{k \in \mathcal{I}}\{X_{ka}\}$. If there exists two non-random sequences $b_m > 0$ and c_m such that*

$$
\lim_{m \to \infty} P\left(\frac{X_a^\alpha - c_m}{b_m} \leq x\right) = G_X(x),
$$

where G_X is a non-degenerate distribution function, then the limiting distribution $\mathcal{G}_X$ is in the Gumbel, Fréchet, or Wiebull family.

The three distribution families given in Thm. 2.2 are actually special cases of the Generalized Extreme Value Distribution. In the context of extreme values, Thm. 2.2 is analogous to the Central Limit Theorem for the distribution of sample mean. We will take advantage of this theorem for the distribution of the maximum for standard normal data to show that the limiting distribution is in the Gumbel family. However, we will derive the distribution of the maximum and minimum for standard uniform data directly. Regardless of data type, the distribution of the sample maximum is derived in Eq. 54.

$$
\begin{aligned}
\mathrm{P}[X_a^\alpha \leq x] &= \mathrm{P}\left[\max_{k\in\mathcal{I}}\{X_{ka}\} \leq x\right] \\
&= \mathrm{P}[X_{1a} \leq x, X_{2a} \leq x, \ldots, X_{ma} \leq x] \\
&= \prod_{k=1}^{m} \mathrm{P}[X_{ka} \leq x] \\
&= \prod_{k=1} F_X(x) \\
&= [F_X(x)]^m
\end{aligned} \tag{54}
$$

Therefore, we have the following expression for the distribution function of the maximum.

$$
F_{\max}(x) = [F_X(x)]^m \tag{55}
$$

Differentiating the distribution function given by Eq. 55 gives us the following density function for the distribution of the maximum.

$$
\begin{aligned}
f_{\max}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_{\max}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} [F_X(x)]^m \\
&= m[F_X(x)]^{m-1} f_X(x)
\end{aligned} \tag{56}
$$

The distribution of the sample minimum, X_a^ω, is derived in Eq. 57.

$$
\begin{aligned}
\mathrm{P}[X_a^\omega \leq x] &= 1 - \mathrm{P}[X_a^\omega \geq x] \\
&= 1 - \mathrm{P}\left[\min_{k\in\mathcal{I}}\{X_{ka}\} \geq x\right] \\
&= 1 - \mathrm{P}[X_{1a} \geq x, X_{2a} \geq x, \ldots, X_{ma} \geq x] \\
&= 1 - \prod_{k=1}^{m} \mathrm{P}[X_{ka} \geq x] \\
&= 1 - [\mathrm{P}[X_{1a} \geq x]]^m \\
&= 1 - [1 - \mathrm{P}[X_{1a} \leq x]]^m \\
&= 1 - [1 - F_X(x)]^m
\end{aligned} \tag{57}
$$

Therefore, we have the following expression for the distribution function of the minimum.

$$
F_{\min}(x) = 1 - [1 - F_X(x)]^m \tag{58}
$$

Differentiating the distribution function given by Eq. 58 gives us the following density function for the distribution of the minimum.

$$
\begin{aligned}
f_{\min}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_{\min}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \left(1 - [1 - F_X(x)]^m\right) \\
&= m\,[1 - F_X(x)]^{m-1}\, f_X(x)
\end{aligned} \tag{59}
$$

Given the densities of the distribution of sample maximum and minimum, we can easily compute moments and the variance. The first and second moment about the origin and the variance of the distribution of the maximum are given by the following.

$$\begin{aligned}\mu_\alpha^{(1)}(m) = \mathrm{E}(X_a^\alpha) &= \int_{-\infty}^{\infty} x f_{\max}(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} x\left(m[F_X(x)]^{m-1} f_X(x)\right)\mathrm{d}x \\ &= m\int_{-\infty}^{\infty} x f_X(x)[F_X(x)]^{m-1}\mathrm{d}x\end{aligned} \tag{60}$$

$$\begin{aligned}\mu_\alpha^{(2)}(m) = \mathrm{E}[(X_a^\alpha)^2] &= \int_{-\infty}^{\infty} x^2 f_{\max}(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} x^2\left(m[F_X(x)]^{m-1} f_X(x)\right)\mathrm{d}x \\ &= m\int_{-\infty}^{\infty} x^2 f_X(x)[F_X(x)]^{m-1}\mathrm{d}x\end{aligned} \tag{61}$$

$$\sigma_\alpha^2(m) = \mu_\alpha^{(2)}(m) - \left[\mu_\alpha^{(1)}(m)\right]^2 \tag{62}$$

Similarly, we have the first and second moment about the origin and variance of the distribution of sample minimum given by the following.

$$\begin{aligned}\mu_\omega^{(1)}(m) = \mathrm{E}(X_a^\omega) &= \int_{-\infty}^{\infty} x f_{\min}(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} x\left(m[F_X(x)]^{m-1} f_X(x)\right)\mathrm{d}x \\ &= m\int_{-\infty}^{\infty} x f_X(x)[F_X(x)]^{m-1}\mathrm{d}x\end{aligned} \tag{63}$$

$$\begin{aligned}\mu_\omega^{(2)}(m) = \mathrm{E}[(X_a^\omega)^2] &= \int_{-\infty}^{\infty} x^2 f_{\min}(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} x^2\left(m[F_X(x)]^{m-1} f_X(x)\right)\mathrm{d}x \\ &= m\int_{-\infty}^{\infty} x^2 f_X(x)[F_X(x)]^{m-1}\mathrm{d}x\end{aligned} \tag{64}$$

$$\sigma_\omega^2(m) = \mu_\omega^{(2)}(m) - \left[\mu_\omega^{(1)}(m)\right]^2 \tag{65}$$

With the densities of attribute maximum and minimum for sample size m, the expected range is given by the following.

$$\begin{aligned}\mathrm{E}(X_a^\alpha - X_a^\omega) &= \mathrm{E}(X_a^\alpha) - \mathrm{E}(X_a^\omega) \\ &= \mu_\alpha^{(1)}(m) - \mu_\omega^{(1)}(m)\end{aligned} \tag{66}$$

For a data distribution that has zero skewness and has support that is symmetric about 0, the result given by Eq. 66 can be simplified to the following expression.

$$\mathrm{E}(X_a^\alpha - X_a^\omega) = 2\mu_\alpha^{(1)}(m) \tag{67}$$

For large samples ($m >> 1$), the covariance between the sample maximum and minimum is approximately zero [6]. Therefore, the variance of the attribute range of a sample of size m is given by the following.

$$\begin{aligned}\mathrm{Var}(X_a^\alpha - X_a^\omega) &\approx \mathrm{Var}(X_a^\alpha) + \mathrm{Var}(X_a^\omega) \\ &= \sigma_\alpha^2(m) + \sigma_\omega^2(m)\end{aligned} \tag{68}$$

Under the assumption of zero skewness and support that is symmetric about 0, the result given by Eq. 68 becomes the following.

$$\begin{aligned} \mathrm{Var}(X_a^\alpha - X_a^\omega) &= 2\mathrm{Var}(X_a^\alpha) \\ &= 2\sigma_\alpha^2 \end{aligned} \tag{69}$$

Let $\mu_{D_{ij}^{(q)}}$ and $\sigma^2_{D_{ij}^{(q)}}$ denote the mean and variance given by Eq. 19. Furthermore, let $D_{ij}^{(q*)}$ denote the max-min normalized distance between instances i and j that is induced by the metric given by Eq. 53. Then the mean of the max-min normalized distance distribution is given by the following.

$$\begin{aligned} \mu_{D_{ij}^{(q*)}} &= \mathrm{E}\left[\left(\sum_{a\in\mathcal{A}}\left(\frac{|X_{ia}-X_{ja}|}{X_a^\alpha - X_a^\omega}\right)^q\right)^{1/q}\right] \\ &\approx \frac{1}{\mathrm{E}(X_a^\alpha - X_a^\omega)}\mathrm{E}\left[\left(\sum_{a\in\mathcal{A}}|X_{ia}-X_{ja}|^q\right)^{1/q}\right] \\ &= \frac{\mu_{D_{ij}^{(q)}}}{\mathrm{E}(X_a^\alpha) - \mathrm{E}(X_a^\omega)} \\ &= \frac{\mu_{D_{ij}^{(q)}}}{\mu_\alpha^{(1)} - \mu_\omega^{(1)}} \end{aligned} \tag{70}$$

The variance of the max-min normalized distance distribution is given by the following.

$$\begin{aligned} \sigma^2_{D_{ij}^{(q*)}} &= \mathrm{Var}\left[\left(\sum_{a\in\mathcal{A}}\left(\frac{|X_{ia}-X_{ja}|}{X_a^\alpha - X_a^\omega}\right)^q\right)^{1/q}\right] \\ &= \mathrm{E}\left[\left(\sum_{a\in\mathcal{A}}\left(\frac{|X_{ia}-X_{ja}|}{X_a^\alpha - X_a^\omega}\right)^q\right)^{2/q}\right] - \left(\mathrm{E}\left[\left(\sum_{a\in\mathcal{A}}\left(\frac{|X_{ia}-X_{ja}|}{X_a^\alpha - X_a^\omega}\right)^q\right)^{1/q}\right]\right)^2 \\ &\approx \frac{\mathrm{E}\left[\left(\sum_{a\in\mathcal{A}}|X_{ia}-X_{ja}|^q\right)^{2/q}\right]}{\mathrm{E}[(X_a^\alpha - X_a^\omega)^2]} - \frac{\left(\mathrm{E}\left[\left(\sum_{a\in\mathcal{A}}|X_{ia}-X_{ja}|^q\right)^{1/q}\right]\right)^2}{\mathrm{E}[(X_a^\alpha - X_a^\omega)^2]} \\ &= \frac{\sigma^2_{D_{ij}^{(q)}} + \mu^2_{D_{ij}^{(q)}}}{\mathrm{E}[(X_a^\alpha - X_a^\omega)^2]} - \frac{\mu^2_{D_{ij}^{(q)}}}{\mathrm{E}[(X_a^\alpha - X_a^\omega)^2]} \\ &= \frac{\sigma^2_{D_{ij}^{(q)}}}{\mathrm{E}[(X_a^\alpha - X_a^\omega)^2]} \\ &= \frac{\sigma^2_{D_{ij}^{(q)}}}{\mathrm{E}[(X_a^\alpha)^2] - 2\mathrm{E}(X_a^\alpha)\mathrm{E}(X_a^\omega) + \mathrm{E}(X_a^\omega)} \\ &= \frac{\sigma^2_{D_{ij}^{(q)}}}{\mu_\alpha^{(2)}(m) - 2\mu_\alpha^{(1)}(m)\mu_\omega^{(1)}(m) + \mu_\omega^{(2)}(m)} \end{aligned} \tag{71}$$

With the results given by Eqs. 70 and 71, we have the following generalized estimate for the asymptotic distribution of the max-min normalized distance distribution.

$$D_{ij}^{(q*)} \dot{\sim} \mathcal{N}\left(\frac{\mu_{D_{ij}^{(q)}}}{\mu_\alpha^{(1)}(m) - \mu_\omega^{(1)}(m)}, \frac{\sigma^2_{D_{ij}^{(q)}}}{\mu_\alpha^{(2)}(m) - 2\mu_\alpha^{(1)}(m)\mu_\omega^{(1)}(m) + \mu_\omega^{(2)}(m)}\right) \tag{72}$$

For data with zero skewness and support that is symmetric about 0, the expected sample maximum is the additive inverse of the expected sample minimum. This allows us to express the formula given by Eq. 70 exclusively in terms of the expected maximum. This result is given by the following.

$$\mu_{D_{ij}^{(q*)}} \approx \frac{\mu_{D_{ij}^{(q)}}}{2\mu_\alpha^{(1)}(m)} \tag{73}$$

A similar substitution gives us the following expression for the variance of the max-min normalized distance distribution.

$$\begin{aligned} \sigma^2_{D_{ij}^{(q*)}} &\approx \frac{\sigma^2_{D_{ij}^{(q)}}}{2\mu_\alpha^{(2)}(m) + 2\left[\mu_\alpha^{(1)}(m)\right]^2} \\ &= \frac{\sigma^2_{D_{ij}^{(q)}}}{2\left(\sigma^2_\alpha(m) + \left[\mu_\alpha^{(1)}(m)\right]^2\right)} \end{aligned} \tag{74}$$

Therefore, the asymptotic distribution of the max-min normalized distance distribution is given by the following.

$$D_{ij}^{(q*)} \dot{\sim} \mathcal{N}\left(\frac{\mu_{D_{ij}^{(q)}}}{2\mu_\alpha^{(1)}(m)}, \frac{\sigma^2_{D_{ij}^{(q)}}}{2\left(\sigma^2_\alpha(m) + \left[\mu_\alpha^{(1)}(m)\right]^2\right)}\right) \tag{75}$$

2.4.1 Standard Normal Data

Standard normal data has zero skewness and has support that is symmetric about 0. This implies that the mean and variance of the distribution of sample range can be expressed exclusively in terms of the sample maximum. Given the nature of the density function of the sample maximum for sample size m, the integration required to determine the moments given by Eqs. 60 and 61 is not possible. These moments can either be approximated numerically or we can use extreme value theory to determine the form of the asymptotic distribution of the sample maximum. Using the latter method, we will show that the asymptotic distribution of the sample maximum for standard normal data is in the Gumbel family. Let $c_m = -\Phi^{-1}\left(\frac{1}{m}\right)$ and $b_m = \frac{1}{c_m}$. Using Taylor's Theorem, we have the following expansion.

$$\begin{aligned} \log\Phi(-c_m - b_m x) &= \log\Phi(-c_m) - b_m x\frac{\phi(-c_m)}{\Phi(-c_m)} + \mathcal{O}(b_m^2 x^2) \\ &= \log\left(\frac{1}{m}\right) - x\frac{\phi(-c_m)}{c_m\Phi(-c_m)} + \mathcal{O}(b_m^2 x^2) \end{aligned} \tag{76}$$

In order to simplify the right-hand side of Eq. 76, we will use the well known Mills Ratio Bounds [7] given by the following.

$$1 \leq \frac{\phi(x)}{x\Phi(-x)} \leq 1 + \frac{1}{x^2} \quad , x > 0 \tag{77}$$

The inequalities given by Eq. 77 show that $\frac{\phi(x)}{x\Phi(-x)} \to 1$ as $x \to \infty$. This implies that $\frac{\phi(c_m)}{c_m\Phi(-c_m)} \to 1$ as $m \to \infty$ since $c_m = -\Phi^{-1}\left(\frac{1}{m}\right) \to \infty$ as $m \to \infty$. This gives us the following approximation of the right-hand side of Eq. 76.

$$\begin{aligned} \log\Phi(-c_m - b_m x) &\approx \log\left(\frac{1}{m}\right) - x + \mathcal{O}(b_m^2 x^2) \\ \Rightarrow \Phi(-c_m - b_m x) &\approx \frac{1}{m} e^{-x+\mathcal{O}(b_m^2 x^2)} \\ \Rightarrow \Phi(c_m + b_m x) &\approx 1 - \frac{1}{m} e^{-x+\mathcal{O}(b_m^2 x^2)} \end{aligned} \tag{78}$$

Using the result given by Eq. 78, we have the following.

$$\begin{aligned} \mathrm{P}\left(\frac{X_a^\alpha - c_m}{b_m} \le x\right) &= \mathrm{P}(X_a^\alpha \le c_m + b_m x) \\ &= \Phi^m(c_m + b_m x) \\ &\approx \left(1 - \frac{1}{m} e^{-x+\mathcal{O}(b_m^2 x^2)}\right)^m \\ &= \left(1 - \frac{1}{m} e^{-x+\mathcal{O}\left(\frac{1}{c_m^2} x^2\right)}\right)^m \\ &\approx \left(1 - \frac{1}{m} e^{-x}\right)^m \\ \Rightarrow \lim_{m\to\infty} \mathrm{P}\left(\frac{X_a^\alpha - c_m}{b_m} \le x\right) &= \lim_{m\to\infty}\left(1 - \frac{1}{m} e^{-x}\right)^m \\ &= e^{-e^{-x}} \end{aligned} \tag{79}$$

The right-hand side of Eq. 79 is the cumulative distribution function of the standard Gumbel distribution. The mean of the asymptotic distribution is given by the following.

$$\mathrm{E}(X_a^\alpha) = \mu_\alpha^{(1)} = -\Phi^{-1}\left(\frac{1}{m}\right) - \frac{\gamma}{\Phi^{-1}\left(\frac{1}{m}\right)} \tag{80}$$

where γ is the Euler-Mascheroni constant. The median of this distribution is given by the following.

$$\widetilde{\mu}_\alpha = \frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m}\right)} - \Phi^{-1}\left(\frac{1}{m}\right) \tag{81}$$

Finally, the variance of the asymptotic distribution of the sample maximum is given by the following.

$$\mathrm{Var}(X_a^\alpha) = \frac{\pi^2}{6}\left(\frac{1}{-\Phi^{-1}\left(\frac{1}{m}\right)}\right)^2 \tag{82}$$

For typical sample sizes m in high-dimensional spaces, the variance estimate given by Eq. 82 exceeds the variance of the sample maximum significantly. Using the fact that $-\Phi^{-1}\left(\frac{1}{m}\right) \dot{\sim} \sqrt{2\log(m)}$ [8] and $\frac{1}{2\log(m)} \le \left(\frac{1}{-\Phi^{-1}\left(\frac{1}{m}\right)}\right)^2$ for $m \ge 2$, we can get a more accurate approximation of the variance with the following.

$$\begin{aligned}\sigma_{\alpha}^{2}(m) = \mathrm{Var}(X_a^{\alpha}) &\approx \frac{\pi^2}{6}\left(\frac{1}{\sqrt{2\log(m)}}\right)^2 \\ &= \frac{\pi^2}{12\log(m)}\end{aligned} \tag{83}$$

Then the mean of the range of m iid standard normal random variables are given by the following.

$$\mathrm{E}(X_a^{\alpha} - X_a^{\omega}) = 2\mu_{\alpha}^{(1)}(m) = 2\left[-\Phi^{-1}\left(\frac{1}{m}\right) - \frac{\gamma}{\Phi^{-1}\left(\frac{1}{m}\right)}\right] \tag{84}$$

It is well known that the sample extremes from the standard normal distribution are approximately uncorrelated for large sample size m [6]. This implies that we can approximate the variance of the range of m iid standard normal random variables with the following result.

$$\begin{aligned}\mathrm{Var}(X_a^{\alpha} - X_a^{\omega}) &\approx \mathrm{Var}(X_a^{\alpha}) + \mathrm{Var}(X_a^{\omega}) \\ &= \sigma_{\alpha}^2(m) + \sigma_{\omega}^2(m) \\ &= 2\sigma_{\alpha}^2(m) \\ &\approx 2\left(\frac{\pi^2}{2\log(m)}\right) \\ &= \frac{\pi^2}{6\log(m)}\end{aligned} \tag{85}$$

For the purpose of approximating the mean and variance of the max-min normalized distance distribution, the formula for the median of the distribution of the attribute maximum yields more accurate results. That is, the approximation of the expected maximum given by Eq. 80 overestimates the sample maximum. The formula for the median of the sample maximum, given by Eq. 81, provides a more accurate estimate of this sample extreme. Therefore, the following estimate for the mean of the attribute range will be used instead.

$$\mathrm{E}(X_a^{\alpha} - X_a^{\omega}) = 2\mu_{\alpha}^{(1)}(m) \approx 2\left[\frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m}\right)} - \Phi^{-1}\left(\frac{1}{m}\right)\right] \tag{86}$$

We have already determined that $\mu_{D_{ij}^{(q)}}$ and $\sigma^2_{D_{ij}^{(q)}}$ are given by Eq. 32. Using the results given by Eqs. 86 and 85 and the general formulas for the mean and variance of the max-min normalized distance distribution given in Eq. 75, this leads us to the following asymptotic estimate for the distribution of the max-min normalized distances for standard normal data.

$$D_{ij}^{(q*)} \dot\sim \mathcal{N}\left(\frac{\mu_{D_{ij}^{(q)}}}{2\mu_{\alpha}^{(1)}(m)}, \frac{6\log(m)\sigma^2_{D_{ij}^{(q)}}}{\pi^2 + 24\left[\mu_{\alpha}^{(1)}(m)\right]^2\log(m)}\right) \tag{87}$$

2.4.2 Standard Uniform Data

Standard uniform data does not have support that is symmetric about 0. Due to the simplicity of the density function, however, we can derive the distribution of the maximum and minimum of a sample of size m explicitly. Using the general forms of the distribution functions of the maximum and minimum given by Eqs. 55 and 58, we have the following distribution functions for standard uniform data.

$$F_{\max}(x) = x^m \tag{88}$$

$$F_{\min}(x) = 1 - (1-x)^m \tag{89}$$

Using the general forms of the density functions of the maximum and minimum given by Eqs. 56 and 59, we have the following density functions for standard uniform data.

$$f_{\max}(x) = mx^{m-1} \tag{90}$$

$$f_{\min}(x) = m(1-x)^{m-1} \tag{91}$$

Then the expected maximum and minimum are computed through straightforward integration as follows.

$$\begin{aligned}
\mathrm{E}(X_a^\alpha) = \mu_\alpha^{(1)}(m) &= \int_0^1 x f_{\max}(x)\mathrm{d}x \\
&= \int_0^1 x[mx^{m-1}]\mathrm{d}x \\
&= \frac{m}{m+1}
\end{aligned} \tag{92}$$

$$\begin{aligned}
\mathrm{E}(X_a^\omega) = \mu_\omega^{(1)}(m) &= \int_0^1 x f_{\min}(x)\mathrm{d}x \\
&= \int_0^1 x[m(1-x)^{m-1}]\mathrm{d}x \\
&= \frac{1}{m+1}
\end{aligned} \tag{93}$$

We can compute the second moment about the origin of the sample range as follows.

$$\begin{aligned}
\mathrm{E}[(X_a^\alpha - X_a^\omega)^2] &= \mathrm{E}[(X_a^\alpha)^2 - 2X_a^\alpha X_a^\omega + (X_a^\omega)^2] \\
&= \mathrm{E}[(X_a^\alpha)^2] - 2\mathrm{E}(X_a^\alpha)\mathrm{E}(X_a^\omega) + \mathrm{E}[(X_a^\omega)^2] \\
&= \mu_\alpha^{(2)}(m) - 2\mu_\alpha^{(1)}(m)\mu_\omega^{(1)}(m) + \mu_\omega^{(2)}(m) \\
&= \int_0^1 x^2[mx^{m-1}]\mathrm{d}x - 2\left(\frac{m}{m+1}\right)\left(\frac{1}{m+1}\right) + \int_0^1 x^2[m(1-x)^{m-1}]\mathrm{d}x \\
&= \frac{m}{m+2} - \frac{2m}{(m+1)^2} + \frac{2}{(m+1)(m+2)} \\
&= \frac{m^3 - m + 2}{(m+2)(m+1)^2}
\end{aligned} \tag{94}$$

Using the general formulas given in Eq. 72 and the mean ($\mu_{D_{ij}^{(q)}}$) and variance ($\sigma^2_{D_{ij}^{(q)}}$) given by Eq. 42, we have the following asymptotic estimate for the max-min normalized distance distribution for standard uniform data.

$$D_{ij}^{(q*)} \dot\sim \mathcal{N}\left(\frac{(m+1)\mu_{D_{ij}^{(q)}}}{m-1}, \frac{(m+2)(m+1)^2\sigma^2_{D_{ij}^{(q)}}}{m^3 - m + 2}\right) \tag{95}$$

2.5 GWAS Distance Distributions

Consider a GWAS data set, which has the following encoding based on minor allele frequency.

$$X_{ia} = \begin{cases} 0 & \text{if there are no minor alleles at locus } a \\ 1 & \text{if there is 1 minor allele at locus } a \\ 2 & \text{if there are 2 minor alleles at locus } a \end{cases} \tag{96}$$

A minor allele at a particular locus a is the least frequent of the two alleles at that particular locus a. For random GWAS data sets, we can think X_{ia} as the number of successes in two Bernoulli trials. That is, $X_{ia} \sim \mathcal{B}(2, f_a)$ where f_a is the probability of success. The success probability f_a is the probability of a minor allele occurring at X_{ia}. Furthermore, the minor allele probabilities are assumed to be independent and identically distributed. Two commonly known types of metrics for GWAS data are the Genotype Mismatch (GM) and Allele Mismatch (AM) metrics. The GM and AM metrics are defined as follows.

$$d_{ij}^{\text{GM}}(a) = \begin{cases} 0 & \text{if } X_{ia} \neq X_{ja} \\ 1 & \text{otherwise} \end{cases} \tag{97}$$

$$d_{ij}^{\text{AM}}(a) = \frac{1}{2}\left|X_{ia} - X_{ja}\right| \tag{98}$$

A more informative metric must take into account whether differences in allele frequency at a particular locus a result in transitions or transversions. A metric that accounts for transitions (Ti) and transversions (Tv) was introduced in [9]. This metric is given by the following.

$$d_{ij}^{\text{TiTv}}(a) = \begin{cases} 0 & \text{if } X_{ia} = X_{ja} \text{ and Ti/Tv} \\ 1/4 & \text{if } |X_{ia} - X_{ja}| = 1 \text{ and Ti} \\ 1/2 & \text{if } |X_{ia} - X_{ja}| = 1 \text{ and Tv} \\ 3/4 & \text{if } |X_{ia} - X_{ja}| = 2 \text{ and Ti} \\ 1 & \text{if } |X_{ia} - X_{ja}| = 2 \text{ and Tv} \end{cases} \tag{99}$$

With any of the three metrics given by Eqs. 97 - 99, we compute the pairwise distance between two instances i and j using Eq. 1 with $q = 1$. Assuming that all data entries X_{ia} are independent and identically distributed, we have already shown that the distribution of pairwise distances is asymptotically normal regardless of data distribution and value of q. Therefore, the distance distributions induced by each of the GWAS metrics given by Eqs. 97 - 99 are asymptotically normal. Thus, we will proceed by deriving the mean and variance for each distance distribution induced by these three GWAS metrics.

2.5.1 GM Distance Distribution

The expected value of the GM metric is given by the following.

$$
\begin{aligned}
\mathrm{E}\left[d_{ij}^{\mathrm{GM}}(a)\right] &= \sum_{k=0}^{1} k \cdot \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(a) = k\right] \\
&= 0 \cdot \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(a) = 0\right] + 1 \cdot \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(a) = 1\right] \\
&= \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(a) = 1\right] \\
&= 2\mathrm{P}[X_{ia} = 0, X_{ja} = 1] + 2\mathrm{P}[X_{ia} = 1, X_{ja} = 2] + 2\mathrm{P}[X_{ia} = 0, X_{ja} = 2] \\
&= 4(1 - f_a)^3 f_a + 4(1 - f_a) f_a^3 + 2(1 - f_a)^2 f_a^2 \\
&= 2\left[2(1 - f_a)^3 f_a + 2(1 - f_a) f_a^3 + (1 - f_a)^2 f_a^2\right] \\
&= 2F(a)
\end{aligned}
\tag{100}
$$

where $F(a) = 2(1 - f_a)^3 f_a + 2(1 - f_a) f_a^3 + (1 - f_a)^2 f_a^2$.

Then the expected pairwise GM distance between instances i and j is computed as follows.

$$
\begin{aligned}
\mathrm{E}\left(\sum_{a \in \mathcal{A}} d_{ij}^{\mathrm{GM}}(a)\right) &= \sum_{a \in \mathcal{A}} \mathrm{E}\left[d_{ij}^{\mathrm{GM}}(a)\right] \\
&= 2 \sum_{a \in \mathcal{A}} F(a)
\end{aligned}
\tag{101}
$$

The second moment about the origin for the GM distance is computed as follows.

$$
\begin{aligned}
\mathrm{E}\left[(D_{ij})^2\right] &= \mathrm{E}\left[\left(\sum_{a \in \mathcal{A}} d_{ij}^{\mathrm{GM}}(a)\right)^2\right] \\
&= \mathrm{E}\left[\sum_{a \in \mathcal{A}} \left(d_{ij}^{\mathrm{GM}}(a)\right)^2\right] + 2\mathrm{E}\left[\sum_{r \in \mathcal{A}} \sum_{s \leq r-1} d_{ij}^{\mathrm{GM}}(r) \cdot d_{ij}^{\mathrm{GM}}(s)\right] \\
&= \sum_{a \in \mathcal{A}} \left(\sum_{k=0}^{1} k^2 \cdot \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(a) = k\right]\right) \\
&+ 2 \sum_{a \in \mathcal{A}} \sum_{s \leq r-1} \left(\sum_{k=0}^{1} k \cdot \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(r) = k\right]\right) \cdot \left(\sum_{k=0}^{1} k \cdot \mathrm{P}\left[d_{ij}^{\mathrm{GM}}(s) = k\right]\right) \\
&= 2 \sum_{a \in \mathcal{A}} F(a) + 8 \sum_{r \in \mathcal{A}} \sum_{s \leq r-1} \prod_{\lambda \in \{r,s\}} F(\lambda)
\end{aligned}
\tag{102}
$$

where $F(a) = 2(1 - f_a)^3 f_a + 2(1 - f_a) f_a^3 + (1 - f_a)^2 f_a^2$.

Using the moments given by Eqs. 101 and 102, the variance is computed as follows.

$$
\begin{aligned}
\mathrm{Var}(D_{ij}) &= \mathrm{E}\left[(D_{ij})^2\right] - \left[\mathrm{E}(D_{ij})\right]^2 \\
&= 2 \sum_{a \in \mathcal{A}} F(a) + 8 \sum_{r \in \mathcal{A}} \sum_{s \leq r-1} \prod_{\lambda \in \{r,s\}} F(\lambda) - 4 \left(\sum_{a \in \mathcal{A}}\right)^2 \\
&= 2 \sum_{a \in \mathcal{A}} F(a) - 4 \sum_{a \in \mathcal{A}} F^2(a) \\
&= 2 \sum_{a \in \mathcal{A}} F(a)[1 - 2F(a)]
\end{aligned}
\tag{103}
$$

where $F(a) = 2(1-f_a)^3 f_a + 2(1-f_a)f_a^3 + (1-f_a)^2 f_a^2$.

With the mean and variance estimates given by Eqs. 101 and 103, the asymptotic GM distance distribution is given by the following.

$$D_{ij} \dot{\sim} \mathcal{N}\left(2\sum_{a\in\mathcal{A}} F(a), 2\sum_{a\in\mathcal{A}} F(a)[1-2F(a)]\right) \tag{104}$$

where $F(a) = 2(1-f_a)^3 f_a + 2(1-f_a)f_a^3 + (1-f_a)^2 f_a^2$.

2.5.2 AM Distance Distribution

The expected value of the AM metric is given by the following.

$$\begin{aligned}
\mathrm{E}\left[d_{ij}^{\mathrm{AM}}(a)\right] &= \sum_{k\in\mathcal{D}} k\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(a)=k\right] \\
&= 0\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(a)=0\right] + \frac{1}{2}\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(a)=\frac{1}{2}\right] + 1\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(a)=1\right] \\
&= \frac{1}{2}\left(2\mathrm{P}\left[X_{ia}=0, X_{ja}=1\right] + 2\mathrm{P}\left[X_{ia}=1, X_{ja}=2\right]\right) \\
&+ 2\mathrm{P}\left[X_{ia}=0, X_{ja}=2\right] \\
&= \mathrm{P}\left[X_{ia}=0, X_{ja}=1\right] + \mathrm{P}\left[X_{ia}=1, X_{ja}=2\right] + 2\mathrm{P}\left[X_{ia}=0, X_{ja}=2\right] \\
&= 2(1-f_a)^3 f_a + 2(1-f_a)f_a^3 + 2(1-f_a)^2 f_a^2 \\
&= 2\left[(1-f_a)^3 f_a + (1-f_a)f_a^3 + (1-f_a)^2 f_a^2\right] \\
&= 2F(a)
\end{aligned} \tag{105}$$

where $F(a) = (1-f_a)^3 f_a + (1-f_a)f_a^3 + (1-f_a)^2 f_a^2$ and $\mathcal{D} = \{0, \frac{1}{2}, 1\}$.

Then the expected pairwise AM distance between instances i and j is computed as follows.

$$\begin{aligned}
\mathrm{E}\left(\sum_{a\in\mathcal{A}} d_{ij}^{\mathrm{AM}}(a)\right) &= \sum_{a\in\mathcal{A}} \mathrm{E}\left[d_{ij}^{\mathrm{AM}}(a)\right] \\
&= 2\sum_{a\in\mathcal{A}} F(a)
\end{aligned} \tag{106}$$

The second moment about the origin for the AM distance is computed as follows.

$$\begin{aligned}
\mathrm{E}\left[(D_{ij})^2\right] &= \mathrm{E}\left[\left(\sum_{a\in\mathcal{A}} d_{ij}^{\mathrm{AM}}(a)\right)^2\right] \\
&= \mathrm{E}\left[\sum_{a\in\mathcal{A}}\left(d_{ij}^{\mathrm{AM}}(a)\right)^2\right] + 2\mathrm{E}\left[\sum_{r\in\mathcal{A}}\sum_{s\leq r-1} d_{ij}^{\mathrm{AM}}(r)\cdot d_{ij}^{\mathrm{AM}}(s)\right] \\
&= \sum_{a\in\mathcal{A}}\left(\sum_{k\in\mathcal{D}} k^2\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(a)=k\right]\right) \\
&+ 2\sum_{a\in\mathcal{A}}\sum_{s\leq r-1}\left(\sum_{k\in\mathcal{D}} k\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(r)=k\right]\right)\cdot\left(\sum_{k\in\mathcal{D}} k\cdot \mathrm{P}\left[d_{ij}^{\mathrm{AM}}(s)=k\right]\right) \\
&= \sum_{a\in\mathcal{A}} G(a) + 8\sum_{r\in\mathcal{A}}\sum_{s\leq r-1}\prod_{\lambda\in\{r,s\}} F(\lambda)
\end{aligned} \tag{107}$$

where $G(a) = (1-f_a)^3 f_a + f_a^3(1-f_a) + 2(1-f_a)^2 f_a^2$ and $F(\lambda) = (1-f_\lambda)^3 f_\lambda + f_\lambda^3(1-f_\lambda) + (1-f_\lambda)^2 f_\lambda^2$.

Using the moments given by Eqs. 106 and 107, the variance is computed as follows.

$$\begin{aligned}
\mathrm{Var}(D_{ij}) &= \mathrm{E}\left[(D_{ij})^2\right] - \left[\mathrm{E}(D_{ij})\right]^2 \\
&= \sum_{a\in\mathcal{A}} G(a) + 8\sum_{r\in\mathcal{A}}\sum_{s\leq r-1}\prod_{\lambda\in\{r,s\}} F(\lambda) - 4\left(\sum_{a\in\mathcal{A}}\right)^2 \\
&= \sum_{a\in\mathcal{A}} G(a) - 4\sum_{a\in\mathcal{A}} F^2(a) \\
&= \sum_{a\in\mathcal{A}} \left[G(a) - 4F^2(a)\right]
\end{aligned} \tag{108}$$

where $G(a) = (1-f_a)^3 f_a + f_a^3(1-f_a) + 2(1-f_a)^2 f_a^2$ and $F(\lambda) = (1-f_\lambda)^3 f_\lambda + f_\lambda^3(1-f_\lambda) + (1-f_\lambda)^2 f_\lambda^2$.

With the mean and variance estimates given by Eqs. 106 and 108, the asymptotic AM distance distribution is given by the following.

$$D_{ij} \dot{\sim} \mathcal{N}\left(2\sum_{a\in\mathcal{A}} F(a), \sum_{a\in\mathcal{A}}\left[G(a) - 4F^2(a)\right]\right) \tag{109}$$

where $G(a) = (1-f_a)^3 f_a + f_a^3(1-f_a) + 2(1-f_a)^2 f_a^2$ and $F(\lambda) = (1-f_\lambda)^3 f_\lambda + f_\lambda^3(1-f_\lambda) + (1-f_\lambda)^2 f_\lambda^2$.

2.5.3 TiTv Distance Distribution

The TiTv metric allows for one to account for both genotype mismatch, allele mismatch, transition, and transversion. However, this added dimension of information requires knowledge of the nucleotide makeup at a particular locus. A sufficient conditions to compute the TiTv metric between instances i an j is that we know whether the nucleotides associated with a particular locus a are both purines (PuPu), purine and pyrimidine (PuPy), or both pyrimidines (PyPy). This information is always given in a particular data set. Let γ_0, γ_1, and γ_2 denote the probabilities of PuPu, PuPy, and PyPy, respectively, for the p loci of data matrix X. In real data, there are approximately twice as many transitions as there are transversions. Thas is, the probability of a transition P(Ti) is approximately twice the probability of transversion P(Tv). In order to enforce this in simulated data, we sample γ_0, γ_1, and γ_2 subject to the following constraints.

$$\gamma_0 + \gamma_1 + \gamma_2 = 1 \tag{110}$$

$$\mathrm{P(Ti)} - 2\mathrm{P(Tv)} = 0 \tag{111}$$

$$1 - \mathrm{P(Ti)} - \mathrm{P(Tv)} = 0 \tag{112}$$

Let y represent a random sample of size p from $\{\mathrm{PuPu}, \mathrm{PuPy}, \mathrm{PyPy}\}$. Let Y be a matrix with the following entries.

$$Y_{ia} = \begin{cases} 0 & \text{if } X_{ia} \text{ is PuPu} \\ 1 & \text{if } X_{ia} \text{ is PuPy} \\ 2 & \text{if } X_{ia} \text{ is PyPy} \end{cases} \tag{113}$$

Using the PuPu, PuPy, and PyPy encoding given previously, we compute the probability of a transversion occuring at a position X_{ia} as follows.

$$
\begin{aligned}
\mathrm{P(Tv)} &= \mathrm{P}\left[Y_{ia}=1, Y_{ja}=0\right] + \mathrm{P}\left[Y_{ia}=0, Y_{ja}=1\right] \\
&+ \mathrm{P}\left[Y_{ia}=2, Y_{ja}=1\right] + \mathrm{P}\left[Y_{ia}=1, Y_{ja}=2\right] \\
&+ \mathrm{P}\left[Y_{ia}=2, Y_{ja}=0\right] + \mathrm{P}\left[Y_{ia}=0, Y_{ja}=2\right] \\
&= \gamma_0\gamma_1 + \gamma_1\gamma_0 + \gamma_1\gamma_2 + \gamma_2\gamma_1 + \gamma_0\gamma_2 + \gamma_2\gamma_0 \\
&= 2(\gamma_0\gamma_1 + \gamma_1\gamma_2 + \gamma_0\gamma_2)
\end{aligned} \tag{114}
$$

Using the constraint given by Eq. 111, the probability of a transition occuring at a position X_{ia} is computed as follows.

$$
\begin{aligned}
\mathrm{P(Ti)} &= 1 - \mathrm{P(Tv)} \\
&= 1 - 2(\gamma_0\gamma_1 + \gamma_1\gamma_2 + \gamma_0\gamma_2)
\end{aligned} \tag{115}
$$

Based on the constraint given by Eq. 112, it is clear that we have $\mathrm{P(Tv)} = \frac{1}{3}$. This implies that $\mathrm{P(Ti)} = \frac{2}{3}$. In order to satisfy the constraint given by Eq. 110, we can sample γ_0 so that we have the following.

$$
\begin{aligned}
&\gamma_2 = \frac{6(1-\gamma_0) + 2\sqrt{9(1-\gamma_0)^2 - 6(6\gamma_0^2 - 6\gamma_0 + 1)}}{12} \\
&\gamma_1 = 1 - \gamma_0 - \gamma_2 \\
&\ 0 < \gamma_1 < 1 \\
&\ 0 < \gamma_2 < 1
\end{aligned} \tag{116}
$$

The first equality given in Eq. 116 comes from the fact that we have a known probability γ_0, which gives two equations in two unknowns (γ_1 and γ_2) that come from the constraint given by Eq. 110 and the fact that $\mathrm{P(Tv)} = 2(\gamma_0\gamma_1 + \gamma_1\gamma_2 + \gamma_0\gamma_2) = \frac{1}{3}$. The solution $(\gamma_0, \gamma_1, \gamma_2)$ to the constraint equations 116 allows one to simulate a SNP data set that approximately satisfies constraint 111.

We proceed by computing $\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a) = k\right]$ for each $k \in \mathcal{D} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$. We derive $\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a) = 0\right]$ as follows.

$$
\begin{aligned}
\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a) = 0\right] &= \mathrm{P}\left[y_a = 0, X_{ia} = X_{ja}\right] \\
&+ \mathrm{P}\left[y_a = 1, X_{ia} = X_{ja}\right] \\
&+ \mathrm{P}\left[y_a = 2, X_{ia} = X_{ja}\right] \\
&= \gamma_0\left[(1-f_a)^2 + 2f_a(1-f_a) + f_a^2\right] \\
&+ \gamma_1\left[(1-f_a)^2 + 2f_a(1-f_a) + f_a^2\right] \\
&+ \gamma_2\left[(1-f_a)^2 + 2f_a(1-f_a) + f_a^2\right] \\
&= (\gamma_0 + \gamma_1 + \gamma_2)\left[(1-f_a)^2 + 2f_a(1-f_a) + f_a^2\right] \\
&= (1-f_a)^2 + 2f_a(1-f_a) + f_a^2
\end{aligned} \tag{117}
$$

We derive $\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a) = \frac{1}{4}\right]$ as follows.

$$
\begin{aligned}
\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=\frac{1}{4}\right] &= 2\mathrm{P}\left[y_a=0, X_{ia}=0, X_{ja}=1\right] \\
&+ 2\mathrm{P}\left[y_a=0, X_{ia}=1, X_{ja}=2\right] \\
&+ 2\mathrm{P}\left[y_a=2, X_{ia}=0, X_{ja}=1\right] \\
&+ 2\mathrm{P}\left[y_a=2, X_{ia}=1, X_{ja}=2\right] \\
&= 4\gamma_0(1-f_a)^3 f_a + 4\gamma_0 f_a^3(1-f_a) + 4\gamma_2(1-f_a)^3 f_a \\
&+ 4\gamma_2 f_a^3(1-f_a) \\
&= 4\gamma_0\left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right] \\
&+ 4\gamma_2\left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right] \\
&= 4(\gamma_0+\gamma_2)\left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right]
\end{aligned} \tag{118}
$$

We derive $\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=\frac{1}{2}\right]$ as follows. 389

$$
\begin{aligned}
\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=\frac{1}{2}\right] &= 2\mathrm{P}\left[y_a=1, X_{ia}=0, X_{ja}=1\right] \\
&+ 2\mathrm{P}\left[y_a=1, X_{ia}=1, X_{ja}=2\right] \\
&= 4\gamma_1(1-f_a)^3 f_a + 4\gamma_1 f_a^3(1-f_a) \\
&= 4\gamma_1\left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right]
\end{aligned} \tag{119}
$$

We derive $\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=\frac{3}{4}\right]$ as follows. 390

$$
\begin{aligned}
\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=\frac{3}{4}\right] &= 2\mathrm{P}\left[y_a=0, X_{ia}=0, X_{ja}=2\right] \\
&+ 2\mathrm{P}\left[y_a=2, X_{ia}=0, X_{ja}=2\right] \\
&= 2\gamma_0(1-f_a)^2 f_a^2 + 2\gamma_2(1-f_a)^2 f_a^2 \\
&= 2(\gamma_0+\gamma_2)(1-f_a)^2 f_a^2
\end{aligned} \tag{120}
$$

We derive $\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=1\right]$ as follows. 391

$$
\begin{aligned}
\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=1\right] &= 2\mathrm{P}\left[y_a=1, X_{ia}=0, X_{ja}=2\right] \\
&= 2\gamma_1(1-f_a)^2 f_a^2
\end{aligned} \tag{121}
$$

Using Eqs. 117 - 121, we compute the expected TiTv distance between instances i and j as follows. 392 393

$$
\begin{aligned}
\mathrm{E}(D_{ij}) &= \sum_{a\in\mathcal{A}}\left(\sum_{k\in\mathcal{D}} k\cdot\mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=k\right]\right) \\
&= (\gamma_0+\gamma_2+2\gamma_1)\sum_{a\in\mathcal{A}}\left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right] \\
&+ \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}}(1-f_a)^2 f_a^2 \\
&= (\gamma_0+\gamma_2+2\gamma_1)\sum_{a\in\mathcal{A}} F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a)
\end{aligned} \tag{122}
$$

where $F(a)=(1-f_a)^3 f_a + f_a^3(1-f_a)$ and $G(a)=(1-f_a)^2 f_a^2$. 394

The second moment about the origin for the TiTv distance is computed as follows. 395

$$
\begin{aligned}
\mathrm{E}\left[\left(D_{ij}\right)^2\right] &= \mathrm{E}\left[\left(\sum_{a\in\mathcal{A}} d_{ij}^{\mathrm{TiTv}}(a)\right)^2\right] \\
&= \mathrm{E}\left[\sum_{a\in\mathcal{A}} \left(d_{ij}^{\mathrm{TiTv}}(a)\right)^2\right] + 2\mathrm{E}\left[\sum_{r\in\mathcal{A}} \sum_{s\le r-1} d_{ij}^{\mathrm{TiTv}}(r)\cdot d_{ij}^{\mathrm{TiTv}}(s)\right] \\
&= \sum_{a\in\mathcal{A}} \left(\sum_{k\in\mathcal{D}} k^2 \cdot \mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(a)=k\right]\right) \\
&+ 2\sum_{a\in\mathcal{A}} \sum_{s\le r-1} \left(\sum_{k\in\mathcal{D}} k\cdot \mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(r)=k\right]\right)\cdot\left(\sum_{k\in\mathcal{D}} k\cdot \mathrm{P}\left[d_{ij}^{\mathrm{TiTv}}(s)=k\right]\right) \\
&= \left[\frac{1}{4}(\gamma_0+\gamma_2)+\gamma_1\right]\sum_{a\in\mathcal{A}} F(a) + \left[\frac{9}{8}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a) \\
&+ 2\sum_{r\in\mathcal{A}} \sum_{s\le r-1} \prod_{\lambda\in\{r,s\}} \left([\gamma_0+\gamma_2+2\gamma_1]F(\lambda) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]G(\lambda)\right)
\end{aligned}
\tag{123}
$$

where $F(\lambda) = (1-f_\lambda)^3 f_\lambda + f_\lambda^3(1-f_\lambda)$ and $G(\lambda) = (1-f_\lambda)^2 f_\lambda^2$.

Using the moments given by Eqs. 122 and 123, the variance is computed as follows.

$$
\begin{aligned}
\mathrm{Var}(D_{ij}) &= \mathrm{E}\left[(D_{ij})^2\right] - \left[\mathrm{E}(D_{ij})\right]^2 \\
&= \left[\frac{1}{4}(\gamma_0+\gamma_2)+\gamma_1\right]\sum_{a\in\mathcal{A}} F(a) + \left[\frac{9}{8}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a) \\
&+ 2\sum_{r\in\mathcal{A}} \sum_{s\le r-1} \prod_{\lambda\in\{r,s\}} \left([\gamma_0+\gamma_2+2\gamma_1]F(\lambda) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]G(\lambda)\right) \\
&- \left([\gamma_0+\gamma_2+2\gamma_1]\sum_{a\in\mathcal{A}} F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a)\right)^2 \\
&= \left[\frac{1}{4}(\gamma_0+\gamma_2)+\gamma_1\right]\sum_{a\in\mathcal{A}} F(a) + \left[\frac{9}{8}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a) \\
&+ \sum_{a\in\mathcal{A}} \left([\gamma_0+\gamma_2+2\gamma_1]F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]G(a)\right)^2
\end{aligned}
\tag{124}
$$

where $F(a) = (1-f_a)^3 f_a + f_a^3(1-f_a)$ and $G(a) = (1-f_a)^2 f_a^2$.

With the mean and variance estimates given by Eqs. 122 and 124, the asymptotic TiTv distance distribution is given by the following.

$$
\begin{aligned}
D_{ij} \dot\sim \mathcal{N}\Bigg(&(\gamma_0+\gamma_2+2\gamma_1)\sum_{a\in\mathcal{A}} F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a), \\
&\left[\frac{1}{4}(\gamma_0+\gamma_2)+\gamma_1\right]\sum_{a\in\mathcal{A}} F(a) + \left[\frac{9}{8}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a) \\
&+ \sum_{a\in\mathcal{A}} \left([\gamma_0+\gamma_2+2\gamma_1]F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]G(a)\right)^2\Bigg)
\end{aligned}
\tag{125}
$$

where $F(a) = (1 - f_a)^3 f_a + f_a^3(1 - f_a)$ and $G(a) = (1 - f_a)^2 f_a^2$.

2.6 Resting-State fMRI Distance Distribution

For resting-state fMRI (rs-fMRI), the data consists of correlation matrices for each instance. These correlations are between different ROIs for a particular brain atlas. We would like the attributes to be the ROIs themselves, which leads us to the following metric.

$$d_{ij}^{\mathrm{ROI}}(a) = \sum_{k \neq a} \left| A_{ka}^{(i)} - A_{ka}^{(j)} \right| \tag{126}$$

where $A_{ka}^{(i)}$ and $A_{ka}^{(j)}$ are the correlations between ROI a and ROI k for instances i and j, respectively. In order for comparisons between different correlations to be possible, we first perform a Fisher r-to-z transform on the correlations. We then load all of the transformed correlations into a $p(p-1) \times m$ matrix X with the following form.

$$X = \begin{bmatrix} \hat{A}_{12}^{(1)} & \hat{A}_{12}^{(2)} & \hat{A}_{12}^{(3)} & \ldots & \hat{A}_{12}^{(m)} \\ \hat{A}_{13}^{(1)} & \hat{A}_{13}^{(2)} & \hat{A}_{13}^{(3)} & \ldots & \hat{A}_{13}^{(m)} \\ \hat{A}_{14}^{(1)} & \hat{A}_{14}^{(2)} & \hat{A}_{14}^{(3)} & \ldots & \hat{A}_{14}^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{1p}^{(1)} & \hat{A}_{1p}^{(2)} & \hat{A}_{1p}^{(3)} & \ldots & \hat{A}_{1p}^{(m)} \\ \hat{A}_{21}^{(1)} & \hat{A}_{21}^{(2)} & \hat{A}_{21}^{(3)} & \ldots & \hat{A}_{21}^{(m)} \\ \hat{A}_{23}^{(1)} & \hat{A}_{23}^{(2)} & \hat{A}_{23}^{(3)} & \ldots & \hat{A}_{23}^{(m)} \\ \hat{A}_{24}^{(1)} & \hat{A}_{24}^{(2)} & \hat{A}_{24}^{(3)} & \ldots & \hat{A}_{24}^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{2p}^{(1)} & \hat{A}_{2p}^{(2)} & \hat{A}_{2p}^{(3)} & \ldots & \hat{A}_{2p}^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{p1}^{(1)} & \hat{A}_{p1}^{(2)} & \hat{A}_{p1}^{(3)} & \ldots & \hat{A}_{p1}^{(m)} \\ \hat{A}_{p2}^{(1)} & \hat{A}_{p2}^{(2)} & \hat{A}_{p2}^{(3)} & \ldots & \hat{A}_{p2}^{(m)} \\ \hat{A}_{p3}^{(1)} & \hat{A}_{p3}^{(2)} & \hat{A}_{p3}^{(3)} & \ldots & \hat{A}_{p3}^{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{A}_{p,(p-1)}^{(1)} & \hat{A}_{p,(p-1)}^{(2)} & \hat{A}_{p,(p-1)}^{(3)} & \ldots & \hat{A}_{p,(p-1)}^{(m)} \end{bmatrix} \tag{127}$$

where $\hat{A}_{ka}^{(i)}$ is the r-to-z transformed correlation between ROIs a and k for instance i.

We further transform the data matrix X by standardizing so that each of the m columns has zero mean and unit variance. Therefore, the data in matrix X are standard normal. Recall from Eqs. 43 and 44, that the mean and variance of the Manhattan ($q = 1$) distance distribution for standard normal data are $\frac{2p}{\sqrt{\pi}}$ and $\frac{2(\pi-2)p}{\pi}$, respectively. This allows us to easily derive the expected pairwise distance between instances i and j in rs-fMRI data as follows.

$$
\begin{aligned}
\mathrm{E}(D_{ij}) &= \mathrm{E}\left(\sum_{a\in\mathcal{A}}\sum_{k\neq a}\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|\right)\\
&= \sum_{a\in\mathcal{A}}\sum_{k\neq a}\mathrm{E}\left(\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|\right)\\
&= \sum_{a\in\mathcal{A}}\sum_{k\neq a}\frac{2}{\sqrt{\pi}}\\
&= \frac{2p(p-1)}{\sqrt{\pi}}
\end{aligned}
\tag{128}
$$

Due to the dependencies that exist between terms in the double sum when computing the rs-fMRI distance, linearity no longer applies to the variance operator. We proceed by writing the form of the variance as follows.

$$
\begin{aligned}
\mathrm{Var}(D_{ij}) &= \mathrm{Var}\left(\sum_{a\in\mathcal{A}}\sum_{k\neq a}\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|\right)\\
&= \sum_{a=1}^{p-1}\mathrm{Var}\left(\sum_{k=a+1}^{p}2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|\right)\\
&+ 2\sum_{a=1}^{p-1}\sum_{r=a+1}^{p-1}\mathrm{Cov}\left(\sum_{k=a+1}^{p}2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|,\ \sum_{s=r+1}^{p}2\left|\hat{A}_{rs}^{(i)}-\hat{A}_{rs}^{(j)}\right|\right)\\
&= \sum_{a=1}^{p-1}\sum_{k=a+1}^{p}\mathrm{Var}\left(2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|\right)\\
&+ 2\sum_{a=1}^{p-1}\sum_{r=a+1}^{p-1}\mathrm{Cov}\left(\sum_{k=a+1}^{p}2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|,\ \sum_{s=r+1}^{p}2\left|\hat{A}_{rs}^{(i)}-\hat{A}_{rs}^{(j)}\right|\right)\\
&= \sum_{a=1}^{p-1}\sum_{k=a+1}^{p-1}\frac{4(\pi-2)}{\pi}\\
&+ 2\sum_{a=1}^{p-1}\sum_{r=a+1}^{p-1}\mathrm{Cov}\left(\sum_{k=a+1}^{p}2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|,\ \sum_{s=r+1}^{p}2\left|\hat{A}_{rs}^{(i)}-\hat{A}_{rs}^{(j)}\right|\right)\\
&= \frac{2p(\pi-2)(p-1)}{\pi}\\
&+ 2\sum_{a=1}^{p-1}\sum_{r=a+1}^{p-1}\mathrm{Cov}\left(\sum_{k=a+1}^{p}2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|,\ \sum_{s=r+1}^{p}2\left|\hat{A}_{rs}^{(i)}-\hat{A}_{rs}^{(j)}\right|\right)
\end{aligned}
\tag{129}
$$

In order to have a formula in terms of the number of ROIs p only, we must estimate the double sum on the right-hand side of Eq. 129. Through simulation, it can be seen that the difference between the sample variance $S_{D_{ij}}^2$ and $\frac{2p(\pi-2)(p-1)}{\pi}$ has a quadratic relationship with p. More explicitly, we have the following relationship.

$$
S_{D_{ij}}^2 - \frac{2p(\pi-2)(p-1)}{\pi} = \beta_1 p^2 + \beta_0 p \tag{130}
$$

The coefficient estimates found through least squares fitting are $\beta_0 = -\beta_1 \approx 0.08$. These estimates allow one to infer a functional form for the double sum in the right-hand side of Eq. 129 that is actually proportional to $\frac{2p(\pi-2)(p-1)}{\pi}$. That is, we have the following formula for approximating the double sum.

$$2\sum_{a=1}^{p-1}\sum_{r=a+1}^{p-1}\mathrm{Cov}\left(\sum_{k=a+1}^{p}2\left|\hat{A}_{ak}^{(i)}-\hat{A}_{ak}^{(j)}\right|,\sum_{s=r+1}^{p}2\left|\hat{A}_{rs}^{(i)}-\hat{A}_{rs}^{(j)}\right|\right)=\frac{p(\pi-2)(p-1)}{4\pi} \tag{131}$$

Therefore, the variance of the rs-fMRI distances is approximated well by the following.

$$\mathrm{Var}(D_{ij})=\frac{9p(\pi-2)(p-1)}{4\pi} \tag{132}$$

With the mean and variance estimates given by Eqs. 128 and 132, we have the following asymptotic distribution for rs-fMRI distances.

$$D_{ij}\mathrel{\dot\sim}\mathcal{N}\left(\frac{2p(p-1)}{\sqrt{\pi}},\frac{9p(\pi-2)(p-1)}{4\pi}\right) \tag{133}$$

Consider the max-min normalized rs-fMRI distance given by the following equation.

$$D_{ij}^{*}=\sum_{a\in\mathcal{A}}\sum_{k\neq a}\frac{\left|A_{ak}^{(i)}-A_{ak}^{(j)}\right|}{\max(a)-\min(a)} \tag{134}$$

Assuming that the data X has been r-to-z transformed and standardized, we can easily compute the expected attribute range and variance of the attribute range. The expected maximum of a given attribute in data matrix X is estimated by the following.

$$\mathrm{E}\left(X_{a}^{\alpha}-X_{a}^{\omega}\right)=2\mu_{\alpha}^{(1)}(m,p)=2\left[\frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m(p-1)}\right)}-\Phi^{-1}\left(\frac{1}{m(p-1)}\right)\right] \tag{135}$$

The variance can be esimated with the following.

$$\mathrm{Var}\left(X_{a}^{\alpha}-X_{a}^{\omega}\right)=\frac{\pi^{2}}{6\log[m(p-1)]} \tag{136}$$

Let $\mu_{D_{ij}}$ and $\sigma_{D_{ij}}^{2}$ denote the mean and variance of the rs-fMRI distance distribution given by Eqs. 128 and 132. Using the formulas for the mean and variance of the max-min normalized distance distribution given in Eq. 87, we have the following asymptotic distribution for the max-min normalized rs-fMRI distances.

$$D_{ij}^{*}\mathrel{\dot\sim}\mathcal{N}\left(\frac{\mu_{D_{ij}}}{2\mu_{\alpha}^{(1)}(m,p)},\frac{6\sigma_{D_{ij}}^{2}\log[m(p-1)]}{\pi^{2}+24\left[\mu_{\alpha}^{(1)}(m,p)\right]^{2}\log[m(p-1)]}\right) \tag{137}$$

3 Expected maximum and minimum distances

Table 1. Summary of asymptotic distance distributions for common data types. Metrics with subscripts M and E represent Manhattan and Euclidean, respectively. Metrics with superscript * represent a deviation from the standard metric by attribute range normalization. The function $\Phi^{-1}(x)$ denotes the standard normal quantile function, where $x \in (0, 1)$.

Type	Mean	Variance
$\mathcal{N}(0,1) - \mathbf{d_M}$	$\frac{2p}{\sqrt{\pi}}$	$\frac{2p(\pi-2)}{\pi}$
$\mathcal{N}(0,1) - \mathbf{d}^*_\mathbf{M}$	$\boxed{\frac{p}{\sqrt{\pi}\mu(m)}}$ where $\mu(m) = \frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m}\right)} - \Phi^{-1}\left(\frac{1}{m}\right)$	$\boxed{\frac{p(\pi-2)}{2\pi\mu^2(m)}}$ where $\mu(m) = \frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m}\right)} - \Phi^{-1}\left(\frac{1}{m}\right)$
$\mathcal{N}(0,1) - \mathbf{d_E}$	$\sqrt{2p-1}$	1
$\mathcal{N}(0,1) - \mathbf{d}^*_\mathbf{E}$	$\boxed{\frac{\sqrt{2p-1}}{2\mu(m)}}$ where $\mu(m) = \frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m}\right)} - \Phi^{-1}\left(\frac{1}{m}\right)$	$\boxed{\frac{2\log(m)}{\pi^2 + 12\mu^2(m)\log(m)}}$ where $\mu(m) = \frac{\log(\log(2))}{\Phi^{-1}\left(\frac{1}{m}\right)} - \Phi^{-1}\left(\frac{1}{m}\right)$
$\mathcal{U}(0,1) - \mathbf{d_M}$	$\frac{p}{3}$	$\frac{p}{18}$
$\mathcal{U}(0,1) - \mathbf{d}^*_\mathbf{M}$	$\frac{(m+1)p}{3(m-1)}$	$\frac{(m^3-18m^2-5m+2)p}{18(m^3+m^2+2)(m-1)^2}$
$\mathcal{U}(0,1) - \mathbf{d_E}$	$\sqrt{\frac{p}{6} - \frac{7}{120}}$	$\frac{7}{120}$
$\mathcal{U}(0,1) - \mathbf{d}^*_\mathbf{E}$	$\sqrt{\frac{p}{6} - \frac{7}{120}}\left(\frac{m+1}{m-1}\right)$	$\frac{7(m+1)^2(m+2)}{120(m^3+m^2+2)}$

Table 2. Summary of asymptotic distance distributions for rs-fMRI and GWAS data. Metrics with superscript * represent a deviation from the standard metric by attribute range normalization. The function $\Phi^{-1}(x)$ denotes the standard normal quantile function, where $x \in (0, 1)$.

Type	Mean	Variance
rs-fMRI ($\mathbf{d_{ROI}}$)	$\dfrac{2p(p-1)}{\sqrt{\pi(p-3)}}$	$\dfrac{4(\pi-2)p(p-1)}{\pi(p-3)}$
rs-fMRI ($\mathbf{d^*_{ROI}}$)	$\boxed{\dfrac{2p(p-1)}{\mu(m,p)\sqrt{\pi(p-3)}}}$ where $\mu(m,p) = \dfrac{1}{\sqrt{p-3}}\Phi^{-1}\left(1 - \dfrac{1}{m(p-1)}\right)$	$\boxed{\dfrac{2[6(p-3)\mu^2(m,p)\log[m(p-1)](\pi-2)-\pi^2]p(p-1)}{\pi(p-3)\mu^2(m,p)(\pi^2+12(p-3)\mu^2(m,p)\log[m(p-1)])}}$ where $\mu(m,p) = \dfrac{1}{\sqrt{p-3}}\Phi^{-1}\left(1 - \dfrac{1}{m(p-1)}\right)$
GWAS ($\mathbf{d_{GM}}$)	$\boxed{2\sum_{a=1}^{p} F(a)}$ where $F(a) = \left[2(1-f_a)^3 f_a + 2f_a^3(1-f_a) + (1-f_a)^2 f_a^2\right]$, and f_a is the probability of a minor allele at locus a.	$\boxed{2\sum_{a=1}^{p} F(a)[1-2F(a)]}$ where $F(a) = \left[2(1-f_a)^3 f_a + 2f_a^3(1-f_a) + (1-f_a)^2 f_a^2\right]$, and f_a is the probability of a minor allele at locus a.
GWAS ($\mathbf{d_{AM}}$)	$\boxed{2\sum_{a=1}^{p} F(a)}$ where $F(a) = \left[(1-f_a)^3 f_a + f_a^3(1-f_a) + (1-f_a)^2 f_a^2\right]$, and f_a is the probability of a minor allele at locus a.	$\boxed{\sum_{a=1}^{p}\left[G(a) - 4F^2(a)\right]}$ where $F(a) = \left[(1-f_a)^3 f_a + f_a^3(1-f_a) + f_a^3(1-f_a) + (1-f_a)^2 f_a^2\right]$, $G(a) = \left[(1-f_a)^3 f_a + f_a^3(1-f_a) + 2(1-f_a)^2 f_a^2\right]$, and f_a is the probability of a minor allele at locus a.
GWAS ($\mathbf{d_{TiTv}}$)	$\boxed{(\gamma_0+\gamma_2+2\gamma_1)\sum_{a=1}^{p} F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a=1}^{p} G(a)}$ where $F(a) = \left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right]$ and $G(a) = (1-f_a)^2 f_a^2$, f_a is the probability of a minor allele at locus a, and γ_0, γ_1, and γ_2 are probabilities of PuPu, PuPy, and PyPy, respectively, at locus a.	$\boxed{\left[\frac{1}{4}(\gamma_0+\gamma_2)+\gamma_1\right]\sum_{a=1}^{p} F(a) + \left[\frac{9}{8}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a=1}^{p} G(a) + \sum_{a=1}^{p}\left[(\gamma_0+\gamma_2+2\gamma_1)F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]G(a)\right]^2}$ where $F(a) = \left[(1-f_a)^3 f_a + f_a^3(1-f_a)\right]$ and $G(a) = (1-f_a)^2 f_a^2$, f_a is the probability of a minor allele at locus a, and γ_0, γ_1, and γ_2 are probabilities of PuPu, PuPy, and PyPy, respectively, at locus a.

Table 3. Summary of distance distribution derivations for standard normal and standard uniform data.

***q*-Metric**	**Data**	**Stat**	**Formula (Eq. #)**
standard (**Eq. 2**)	$\mathcal{N}(0,1)$	mean	$\left(\frac{2^q\Gamma\left(\frac{q+1}{2}\right)p}{\sqrt{\pi}}\right)^{1/q}$ (**38**)
	$\mathcal{N}(0,1)$	variance	$\frac{4^q p}{q^2\left(\frac{2^q\Gamma\left(\frac{1}{2}q+\frac{1}{2}\right)}{\sqrt{\pi}}p\right)^{2\left(1-\frac{1}{q}\right)}}\left[\frac{\Gamma\left(q+\frac{1}{2}\right)}{\sqrt{\pi}}-\frac{\Gamma^2\left(\frac{1}{2}q+\frac{1}{2}\right)}{\pi}\right]$ (**38**)
	$\mathcal{U}(0,1)$	mean	$\left(\frac{2p}{(q+2)(q+1)}\right)^{1/q}$ (**48**)
	$\mathcal{U}(0,1)$	variance	$\frac{p}{q^2\left(\frac{2p}{(q+2)(q+1)}\right)^{2\left(1-\frac{1}{q}\right)}}\left[\frac{1}{(q+1)(2q+1)}-\left(\frac{2}{(q+2)(q+1)}\right)^2\right]$ (**48**)
max-min normalized (**Eq. 59**)	$\mathcal{N}(0,1)$	mean	$\boxed{\frac{\mu_{D_{ij}^{(q)}}}{2\mu_{\alpha}^{(1)}(m)}}$ (**93**) where $\mu_{D_{ij}^{(q)}}$ and $\mu_{\alpha}^{(1)}(m)$ are given by Eqs. 38 and 87, respectively.
	$\mathcal{N}(0,1)$	variance	$\boxed{\frac{6\log(m)\sigma^2_{D_{ij}^{(q)}}}{\pi^2+24\left[\mu_{\alpha}^{(1)}(m)\right]^2\log(m)}}$ (**93**) where $\sigma^2_{D_{ij}^{(q)}}$ and $\mu_{\alpha}^{(1)}(m)$ are given by Eqs. 38 and 87, respectively.
	$\mathcal{U}(0,1)$	mean	$\boxed{\frac{(m+1)\mu_{D_{ij}^{(q)}}}{m-1}}$ (**101**) where $\mu_{D_{ij}^{(q)}}$ is given by Eq. 48
	$\mathcal{U}(0,1)$	variance	$\boxed{\frac{(m+2)(m+1)^2\sigma^2_{D_{ij}^{(q)}}}{m^3-m+2}}$ (**101**) where $\sigma^2_{D_{ij}^{(q)}}$ is given by Eq. 48

Table 4. Summary of distance distribution derivations for GWAS data.

GWAS-Metric	Stat	Formula (Eq. #)
GM (**Eq. 103**)	mean	$\boxed{2\sum_{a\in\mathcal{A}} F(a)}$ (**110**) where $F(a) = 2(1-f_a)^3 f_a + 2f_a^3(1-f_a) + (1-f_a)^2 f_a^2$
	variance	$\boxed{2\sum_{a\in\mathcal{A}} F(a)[1-2F(a)]}$ (**110**) where $F(a) = 2(1-f_a)^3 f_a + 2f_a^3(1-f_a) + (1-f_a)^2 f_a^2$
AM (**Eq. 104**)	mean	$\boxed{2\sum_{a\in\mathcal{A}} F(a)}$ (**115**) where $F(a) = (1-f_a)^3 f_a + f_a^3(1-f_a) + (1-f_a)^2 f_a^2$
	variance	$\boxed{\sum_{a\in\mathcal{A}}[G(a) - 4F^2(a)]}$ (**115**) where $F(a) = 2(1-f_a)^3 f_a + 2f_a^3(1-f_a) + (1-f_a)^2 f_a^2$ and $G(a) = (1-f_a)^3 f_a + f_a^3(1-f_a) + 2(1-f_a)^2 f_a^2$
TiTv (**Eq. 105**)	mean	$\boxed{(\gamma_0+\gamma_2+2\gamma_1)\sum_{a\in\mathcal{A}} F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a)}$ (**131**) where $F(a) = (1-f_a)^3 f_a + f_a^3(1-f_a)$ and $G(a) = (1-f_a)^2 f_a^2$
	mean	$\boxed{\begin{aligned}&\left[\frac{1}{4}(\gamma_0+\gamma_2)+\gamma_1\right]\sum_{a\in\mathcal{A}} F(a) + \left[\frac{9}{8}(\gamma_0+\gamma_2)+2\gamma_1\right]\sum_{a\in\mathcal{A}} G(a)\\ &+\sum_{a\in\mathcal{A}}\left([\gamma_0+\gamma_2+2\gamma_1]F(a) + \left[\frac{3}{2}(\gamma_0+\gamma_2)+2\gamma_1\right]G(a)\right)^2\end{aligned}}$ (**131**) where $F(a) = (1-f_a)^3 f_a + f_a^3(1-f_a)$ and $G(a) = (1-f_a)^2 f_a^2$

Table 5. Summary of distance distribution derivations for rs-fMRI data.

rs-fMRI - Metric	**Stat**	**Formula** (**Eq.** #)
standard (**Eq. 132**)	mean	$\dfrac{2p(p-1)}{\sqrt{\pi}}$ (**139**)
	variance	$\dfrac{9p(\pi-2)(p-1)}{4\pi}$ (**139**)
max-min normalized (**Eq. 140**)	mean	$\boxed{\dfrac{\mu_{D_{ij}}}{2\mu_{\alpha}^{(1)}(m,p)}}$ (**143**) where $\mu_{D_{ij}}$ and $\mu_{\alpha}^{(1)}(m,p)$ are given by Eqs. 140 and 142
	variance	$\boxed{\dfrac{6\sigma^2_{D_{ij}}\log[m(p-1)]}{\pi^2+24\left[\mu_{\alpha}^{(1)}(m,p)\right]^2\log[m(p-1)]}}$ (**143**) where $\sigma^2_{D_{ij}}$ and $\mu_{\alpha}^{(1)}(m,p)$ are given by Eqs. 140 and 142

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