PDE : Lab 3 Report

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Abstract

In this lab session we will consider the following differential equation :

$$((1+x)u'(x))' = -1 \tag{1}$$

Where u is is $C^2(]0,1[)$. We will first analytically find the exact solution with first Dirichlet boundary conditions (i.e $\alpha, \beta \in \mathbb{R}$,$u(0) = \alpha$ and $u(1) = \beta$) then mixed boundary conditions (i.e. with same notation $u(0) = \alpha$ and $u'(1) = \beta$).

Then we will consider a finite difference scheme of order 2 to obtain an approximation of u, scheme that we will implement in python. This implementation will permit us to compute the error between the approximation and the exact solution.

Finally we shall propose and implement a scheme of order 4 for the equation with Dirichlet boundary condition.

1 Exact solution

1.1 General computation

Both side of the equation being continuous, we can integrate 1. We get

$$\begin{aligned}(1+x)u' &= -x + c_1\\ u' &= -\frac{x+c_1}{1+x}\\ u' &= -1 - \frac{c_1 - 1}{1+x}\end{aligned}$$

Where c_1 is a constant and the division by $1+x$ is possible because x lives in $(0,1)$.

Again both sides being continuous on $(0,1)$ we can integrate and we get

$$u = -x + (1-c_1)\ln(1+x) + c_2$$

with $c_1, c_2 \in \mathbb{R}$. We now only have to apply the boundary conditions to these results to get the unique solution to the equation.

1.2 Dirichlet boundary condition

Dirichlet boundary condition ($\alpha, \beta \in \mathbb{R}$,$u(0) = \alpha$ and $u(1) = \beta$) give us through simple calculations :

$$\begin{aligned}c_2 &= \alpha\\ c_1 &= 1 - \frac{\beta - \alpha + 1}{\ln(2)}\end{aligned}$$

Which leads to

$$u(x) = -x + \frac{\beta - \alpha + 1}{\ln(2)}\ln(1+x) + \alpha \tag{2}$$

1.3 Mixed boundary condition

In the same way, Mixed boundary condition ($u(0) = \alpha$ and $u'(1) = \beta$) give us

$$\begin{aligned}c_2 &= \alpha\\ c_1 &= -2\beta - 1\end{aligned}$$

Finally unveiling

$$u(x) = -x + 2(\beta+1)\ln(1+x) + \alpha \tag{3}$$

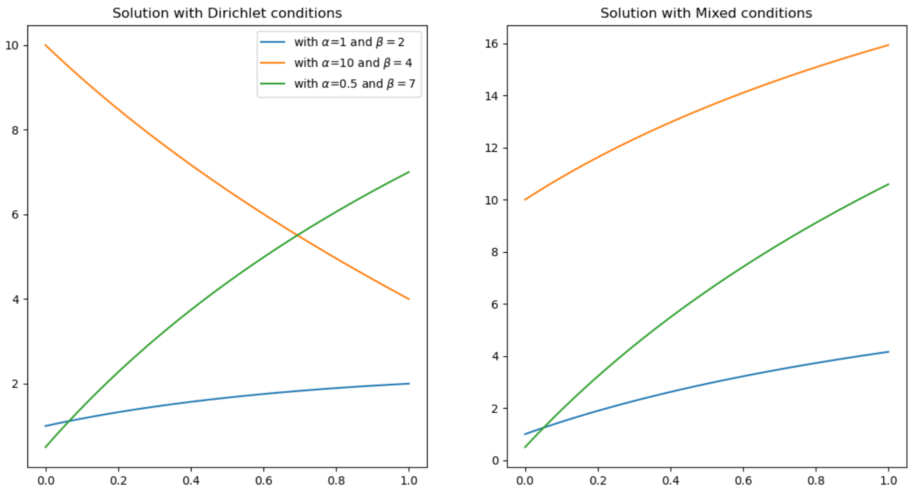

Figure 1: Plot of the exact solutions for certain boundary conditions.

2 Finite differences schemes

We now introduce a subdivision of $[0,1]$, $X = x_0, \dots, x_N$. For clarity we will write $f(x_i) := f_i$ for every function f.

2.1 FD scheme for the equation

Our equation contains second and first derivative of u. We will use the usual centered fd schemes of order 2 for second and first derivative, namely :

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2h} + \mathcal{O}(h^2)$$
$$u_i'' = \frac{u_{i+1} + u_{i-1} - 2u_i}{2h^2} + \mathcal{O}(h^2)$$

The equation 1 then becomes, when neglecting the $\mathcal{O}(h^2)$:

$$-2(1+x_i)u_i + (h+1+x_i)u_{i+1} + (1-h+x)u_{i-1} = -2h^2 \tag{4}$$

We now write $U = (u_0, \dots, u_N)$ an approximation of u. U is the solution to the system $AU = B$. The considered boundary conditions only influence the first and last line of the system, the rest is defined thusly, $i \in 1, \cdots, N-1$

$$A_i = \begin{bmatrix} 0 & \cdots & 0 & (1-h+x_i) & \overbrace{2(1+x_i)}^{i^{th} column} & h+1+x_i & 0 & \cdots & 0 \end{bmatrix}$$
$$B_i = -2h^2$$

Which is just a linewise translation of (4).

2.1.1 Dirichlet conditions

Dirichlet conditions simply state that $u(0) = u_0 = \alpha$ and $u(1) = u_N = \beta$

So the first and last line of A become

$$\begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}$$

And the first and last line of B

$$\begin{bmatrix} \alpha \end{bmatrix}$$
$$\begin{bmatrix} \beta \end{bmatrix}$$

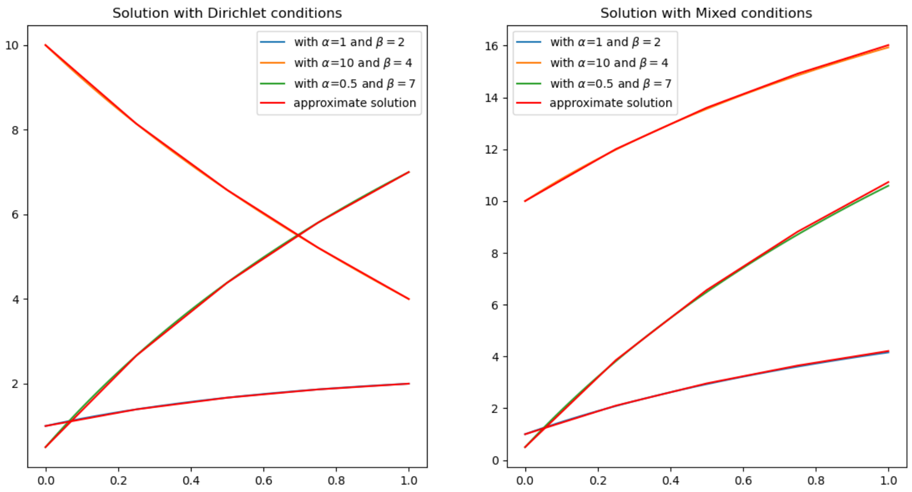

Figure 2: Plot of the approximations, using a grid of only 5 points.

2.2 Mixed conditions

Mixed conditions only change the last condition of Dirichlet so the rest of the system is the same. We now have $u'(1) = u'_N = \beta$. Furthermore, being at the boundaries prevents us from using a centered scheme as u_{i+1} is not defined. We thus use a left-sided scheme for u'_N :

$$u'_N = \frac{u_{N-2} - 4u_{N-1} + 3u_i}{2h} + \mathcal{O}(h^2)$$

The last line of A then becomes

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & -4 & 3 \end{bmatrix}$$

And the corresponding value in B, $B_N = 2h\beta$.

3 Implementation and Error Measurement

See the commented code included with this report for the python implementation.

3.0.1 Implementation

As we can see in Figure 2 the approximation is very close to the exact solution.

3.1 Error Measurement

See Figure 3 with references slopes. The error is clearly on the second order as the graph is parallel to the h^2 slope.

4 Scheme of order 4 for the Dirichlet Boundary conditions

This time we will use the centered schemes of order 4 for u'_i and u''_i when possible and the right-sided and left-sided schemes on the near boundaries values (for 1 and $N-1$).

4.0.1 Center values

We have

$$\begin{aligned} u'_i &= \frac{u_{i-2} - u_{i+2} - 8u_{i-1} + 8u_{i+1}}{12h} && + \mathcal{O}(h^4) \\ u''_i &= \frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12h^2} && + \mathcal{O}(h^4) \end{aligned}$$

Applying those to (1) neglecting the error, we obtain the following equation

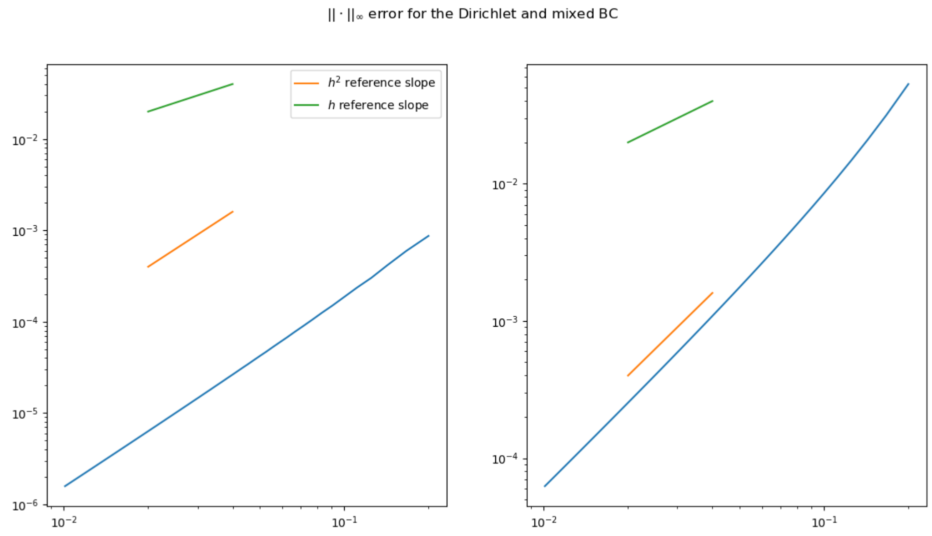

Figure 3: plot of the error with respect to the number of points in the subdivition

$$(1-)u_{i-2} + (16-8)u_{i-1} - 30u_i + (16+8)u_{i-1} + (-1-)u_{i-2} = -12h^2$$

for $i \in 2, \ldots, N-2$.

4.1 Near center values

We consider here the values where a centered scheme is not possible. Those cases are $i = 1$ and $i = N-1$.

4.1.1 $i = 1$

$$\begin{aligned}
u_1' &= \frac{-25u_1 + 48u_2 - 36u_3 + 16u_4 - 3u_5}{12h} && + \mathcal{O}(h^4) \\
u_1'' &= \frac{45u_1 - 154u_2 + 214u_3 - 156u_4 + 61u_5 - 10u_6}{12h^2} && + \mathcal{O}(h^4)
\end{aligned}$$

Applying those to (1) we get

$$\begin{aligned}
u_1(45(1+x_1) - 25) + u_2(-154(1+x_1) + 48) + u_3(214(1+x_1) - 36) \\
+u_4(-156(1+x_1) + 16) + u_5(61(1+x_1) - 3) - 10(1+x_1)u_6 = -12h^2
\end{aligned}$$

4.1.2 $i = N-1$

$$\begin{aligned}
u_{N-1}' &= \frac{-25u_{N-1} + 48u_{N-2} - 36u_{N-3} + 16u_{N-4} - 3u_{N-5}}{12h} && + \mathcal{O}(h^4) \\
u_{N-1}'' &= \frac{45u_{N-1} - 154u_{N-2} + 214u_{N-3} - 156u_{N-4} + 61u_{N-5} - 10u_{N-6}}{12h^2} && + \mathcal{O}(h^4)
\end{aligned}$$

Applying those to (1) we get

$$\begin{aligned}
u_{N-1}(45(1+x_{N-1}) - 25) + u_{N-2}(-154(1+x_{N-1}) + 48) + u_3(214(1+x_{N-1}) - 36) \\
+u_{N-4}(-156(1+x_{N-1}) + 16) + u_{N-5}(61(1+x_{N-1}) - 3) - 10(1+x_{N-1})u_{N-6} = -12h^2
\end{aligned}$$

4.2 System and implementation

To save on space and clarity we shall not explicitly write the system. We implement the Dirichlet boundary condition the same way we did before.