Chapter 7

Forwards and Futures

7.1 Basics of forwards and futures

The financial assets −typically stocks− we have been dealing with so far are the so-called spot assets. By a spot asset, we mean a financial asset that is sold and bought for immediate delivery (change of ownership) in exchange for monetary payment. A market in which spot assets are traded is called a spot market.

In contrast to spot assets, forwards and futures are contracts that stipulate the delivery of a financial asset at a future date. They are similar in spirit but differ in details. Typically forward contracts are struck up between two parties over-the-counter, while the futures are bought and sold in and managed and by an officially sanctioned exchange.

7.1.1 Forwards

Forward contract is an agreement to deliver a financial asset at a future day, say, at time T. Suppose this contract is entered into at time $t < T$, and let S_t denote the price process of the underlying spot asset. When this contract is entered into at time t the buyer of this forward contract agrees to pay K at time T to the seller of this forward contract in exchange for the spot asset whose price at time T is obviously S_T. Furthermore this price K, called the *forward price*, is determined at the time when this contract is made, i.e, at time t. The question is what this K has to be in order for it to be fair to both parties. The holder of this contract should have the profit (or loss) at time T given by $S_T - K$ depending on whether S_T is greater or less K. Its risk neutral value at t has to be

$$\begin{aligned} V_t &= B_t E_Q \left[\frac{(S_T - K)}{B_T} \mid \mathcal{F}_t \right] \\ &= S_t - K B_t E_Q \left[\frac{1}{B_T} \mid \mathcal{F}_t \right]. \end{aligned} \tag{7.1}$$

If this contract is fair to both parties, V_t has to be zero. Therefore setting the right hand of (7.1) equal to zero and solving for K, we have

$$K = \frac{S_t}{B_t E_Q \left[\frac{1}{B_T} \mid \mathcal{F}_t \right]}.$$

This K is called the *forward price* and we denote it by G_t or $G(t,T)$ in this Chapter.

Now, $B_t E_Q \left[\frac{1}{B_T} \mid \mathcal{F}_t \right]$ is the value at t of a contingent claim that pays 1 at time T. This contingent claim is called a *zero-coupon bond* and is denoted by $p(t,T)$, If the interest rate r is constant, $p(t,T)$ simply is

$$p(t,T) = e^{-r(T-t)}.$$

Therefore we have the formula for the forward price G_t :

$$\begin{aligned} G_t &= \frac{S_t}{p(t,T)} \\ &= e^{r(T-t)} S_t. \end{aligned} \tag{7.2}$$

Remark 7.1. The forwards and futures are intricately tied with the interest rate model. But since we have not yet developed an adequate model for it, we will later come back to further issues related to forwards and futures when appropriate.

Suppose a forward contract is entered into at a time t_1. Assume the interest r is constant. Then the forward price at time t_1 is

$$G_{t_1} = e^{r(T-t_1)} S_{t_1},$$

which is fixed throughout the duration of this contract. At a later date, say, at $t_2 > t_1$, the holder of this forward contract will face profit or loss depending on whether the price S_{t_2} at time t_2 of the underlying spot asset is greater or less than G_{t1}. First note that the value at t_2 of the money is G_{t1} payable at T is certainly $e^{-r(T-t_2)}G_{t_1} = e^{r(t_2-t_1)}S_{t_1}$. Thus the profit or loss at t_2 has to be

$$S_{t_2} - e^{r(t_2-t_1)}S_{t_1}.$$

It can be also seen by using the risk neutral valuation method. Namely, since the profit or loss at time T of the buyer of this forward contract is $S_T - G_{t_1}$, its value at t_2 has to be

$$\begin{aligned} B_{t_2}E_Q\left[\frac{(S_T - G_{t_1})}{B_T} \mid \mathcal{F}_t\right] &= S_{t_2} - e^{-r(T-t_2)}e^{r(T-t_1)}S_{t_1} \\ &= S_{t_2} - e^{r(t_2-t_1)}S_{t_1}. \end{aligned}$$

Remark 7.2. If we look at the forward price process

$$G_t = e^{r(T-t)}S_t = e^{rT}S_t{}^*,$$

it is certainly a Q-martingale as $S_t{}^* = e^{-rt}S_t$ is. However it is a special situation that happens to occur in the case of deterministic interest rate. In general for stochastic interest rate case G_t is *not* a Q-martingale. As we shall see later, G_t is a martingale with respect to some other measure called the *T-forward measure.*

7.1.2 Futures

The futures contract is an agreement to deliver a spot asset at a future date. In this respect, it is similar to forward contract. However, there are many differences. First, each futures contract is a standardized contract that specifies the asset and the delivery date. Second, for such standardized contract, there are buyers and sellers in the market at any time during the trading day with the usual bid and ask prices. When bid and ask prices coincide, a futures contract is traded and a buyer and a seller of the futures contract is established. In this sense, the futures price can be regarded as a price determined by the market. Such futures price changes constantly during the trading day depending on the ebbs and flows of the market.

There is a special price called the "daily" or "daily closing" price that is used to calculate the daily profit and loss settlement. It can be a closing price of the day. But to guard against manipulations, the exchange sets a more elaborate rule. Its detail does not concern us here. But one must remember that there is a well-defined "daily" price. Third, using this daily price as a reference, the buyer or seller of futures contract incurs profit or loss everyday. This profit or loss has to be settled daily by crediting or debiting the appropriate bank account. This daily settlement feature is what really distinguishes futures contract from forward contract.

Let us now look into this daily price. To set up the notation, let $F_t = F(t,T)$ be futures price at t of a futures contract with delivery date T. Let t_i be the ith trading day and let F_{t_i}be the "daily" price of that day and so on. Thus at the close of $(i+1)$st day the buyer of this contract incurs the daily profit or loss by $F_{t_{i+1}} - F_{t_i}$. If the market price should have been determined in such a way that favors neither the buyer nor the seller, F_{t_i} and $F_{t_{i+1}}$ must have been determined so that the risk neutral value at t_i of $F_{t_{i+1}} - F_{t_i}$ has to be zero. Namely

$$B_{t_i} E_Q \left[\frac{F_{t_{i+1}} - F_{t_i}}{B_{t_{i+1}}} \mid \mathcal{F}_{t_i} \right] = 0.$$

Now the interest rate process is always postulated to be predictable. (If r is constant, it is a moot point, anyway.) Thus $B_{t_{i+1}} \in \mathcal{F}_{t_i}$. Therefore the above equality must imply $E_Q \left[F_{t_{i+1}} - F_{t_i} \mid \mathcal{F}_{t_i} \right] = 0$, which again implies that

$$F_{t_i} = E_Q \left[F_{t_{i+1}} \mid \mathcal{F}_{t_i} \right].$$

Extending it to any t, we set that

$$F_t = E_Q \left[F_T \mid \mathcal{F}_t \right].$$

On the other hand, it is obvious that $F_T = S_T$ as T is the delivery day. Therefore we have to following very important

$$F_t = F(t,T) = E_Q \left[S_T \mid \mathcal{F}_t \right]. \tag{7.3}$$

Remark 7.3. Unlike the forwards (7.3) implies that the futures price process F_t is always a Q-martingale even with a stochastic interest rate model.

However, if the interest rate is deterministic, the forward and futures prices coincide. To see it, let $r = r(t)$ be a deterministic function of t. Then the bank account B_t is defined by

$$\begin{aligned} dB_t &= r(t) B_t dt \\ B_0 &= 1, \end{aligned}$$

which implies that

$$B_t = e^{\int_0^t r(u)du}.$$

With this, it is easy to obtain the following:

Proposition 7.4. *Assume the interest rate is deterministic, then forward and futures prices coincide, i.e., $G(t,T) = F(t,T)$*

Proof. Since $B_T = e^{\int_0^T r(u)du}$ is a constant,

$$\begin{aligned}
F(t,T) &= E_Q\left[S_T \mid \mathcal{F}_t\right] \\
&= B_T E_Q\left[\frac{S_T}{B_T} \mid \mathcal{F}_t\right] \\
&= B_T \frac{S_t}{B_t} && (\because \tfrac{S_t}{B_t} \text{ is a } Q\text{-Martingale.}) \\
&= e^{\int_0^T r(u)du} S_t e^{-\int_0^t r(u)du} \\
&= e^{\int_t^T r(u)du} S_t
\end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned}
p(t,T) &= B_t E_Q\left[\frac{1}{B_T} \mid \mathcal{F}_t\right] \\
&= e^{-\int_t^T r(u)du}.
\end{aligned}$$

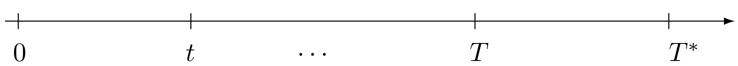
Therefore

$$G(t,T) = \frac{S_t}{p(t,T)} = e^{\int_t^T r(u)du} S_t.$$

$\square$

7.2 Futures option

In this subsection, we study the European options on the futures price, especially the call and put options. In many respects, futures options are quite similar to the usual options on the spot assets we have been studying. But there are subtle differences coming from the fact that no cash is to be tied up to maintain the futures position. For the sake of simplicity and clarity, we assume the interest rate r is constant in this subsection.



Let T^* be the delivery date of a futures contract and let F_t be its price at time t, i.e $F_t = F(t, T^*)$. We have shown that

$$F_t = e^{r(T^*-t)} = e^{rT^*} {S_t}^*,$$

where ${S_t}^* = e^{-rt} S_t$ is the discounted spot asset price at time t. Since ${S_t}^*$ is a Q-martingale, so is F_t. In particular, as in (5.5) of Chapter 5, ${S_t}^*$ is known to satisfy

$$dS_t^* = \sigma S_t^* d\widetilde{W}_t.$$

Therefore F_t must also satisfy

$$dF_t = \sigma F_t d\widetilde{W}_t. \tag{7.4}$$

It is to be expected in view of the fact that F_t is a Q-martingale. Let X be a European option on F_t with the expiry T. What we are mostly interested in is the one of the form $\varphi(F_T)$ where φ is a continuous piecewise C^1-function. Typical of such X is a call option $X = (F_T - K)^+$ or a put option $X = (K - F_T)^+$.

7.2.1 Existence of self-financing replicating portfolio

We show that there is a portfolio(ζ_t, ξ_t) consisting of ζ_t futures contracts and ξ_t units of bank account which is self-financing and which also replicates X.

But unlike the portfolio of spot assets, the futures contract itself requires no money to be tied up to maintain the position. Thus the portfolio's value is simply $V_t = \xi_t B_t$. On the other hand, the change of V_t comes from the change of the futures price F_t as well as the bank account itself. Thus

$$dV_t = \zeta_t dF_t + \xi_t dB_t.$$

It is certainly a self-financing condition. The question is how to find such ζ_t and ξ_t. Here we follow the method of section 5.3 with obvious modification adapted to the situation of futures market.

Define

$$V_t = B_t E_Q\left[\frac{X}{B_T} \mid \mathcal{F}_t\right].$$

This V_t will be shown to be equal to the value of the portfolio. To do so we first define

$$\xi_t = \frac{V_t}{B_t} = V_t^*$$

Now obviously V_t^* is a a Q-martingale. Thus by the Martingale Representation Theorem, there exists predictable α_t such that

$$dV_t^* = \alpha_t d\widetilde{W}_t.$$

Combining this with (7.4), we get

$$dV_t^* = \beta_t dF_t,$$

where

$$\beta_t = \frac{\alpha_t}{\sigma F_t}.$$

Note that the denominator in the expression of β_t never vanishes. So β_t is well-defined. To define ζ_t, look at

$$\begin{aligned} dV_t &= d(B_t V_t{}^*) \\ &= B_t dV_t{}^* + V_t{}^* dB_t \\ &= B_t \beta_t dF_t + \xi_t dB_t. \end{aligned}$$

If we define

$$\zeta_t = B_t \beta_t = e^{rt} \beta_t,$$

then

$$dV_t = \zeta_t dF_t + \xi_t dB_t,$$

which is certainly a self-financing condition. Finally, check that

$$V_T = B_T E_Q \left[\frac{X}{B_T} \mid \mathcal{F}_T \right] = X.$$

There $(\zeta_t, \xi_t)$is a self-financing replicating portfolio of X.

7.2.2 Black's equation

We now derive a variant of Black-Scholes equation that describes the value of futures option. Let $X = \varphi(F_T)$ is a given European option on the futures price, where φ is a continuous, piecewise C^1-function. We are looking for a C^2-function $u(t, x)$ of two deterministic variables t and x such that the value V_t of the option at time t is given by

$$V_t = u(t, F_t)$$

when the futures price at time t is $F_t = F(t, T^*)$.

Upon taking the stochastic differential, we have

$$\begin{aligned} dV_t &= \frac{\partial u}{\partial t}(t, F_t)dt + \frac{\partial u}{\partial x}(t, F_t)dF_t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t, F_t)(dF_t)^2 \\ &= \frac{\partial u}{\partial t}(t, F_t)dt + \sigma F_t \frac{\partial u}{\partial x}(t, F_t)d\widetilde{W}_t + \frac{1}{2}\sigma^2 F_t{}^2 \frac{\partial^2 u}{\partial x^2}(t, F_t)dt \end{aligned} \tag{7.5}$$

On the other hand, we have earlier shown that

$$\begin{aligned} dV_t &= \zeta_t dF_t + \xi_t dB_t \\ &= \zeta_t \sigma F_t d\widetilde{W}_t + re^{rt}\xi_t dt \end{aligned} \tag{7.6}$$

Thus equating the random terms of (7.5) and (7.6) we have

$$\zeta_t = \frac{\partial u}{\partial x}(t, F_t),$$

and equating the coefficient of dt, we have

$$\frac{\partial u}{\partial t}(t, F_t) + \frac{1}{2}\sigma^2 {F_t}^2 \frac{\partial^2 u}{\partial x^2}(t, F_t) = re^{rt}\xi_t$$

But,

$$\xi_t = \frac{V_t}{B_t} = e^{-rt}u(t, F_t)$$

Therefore the above equation becomes

$$\frac{\partial u}{\partial t}(t, F_t) + \frac{1}{2}\sigma^2 {F_t}^2 \frac{\partial^2 u}{\partial x^2}(t, F_t) = ru(t, F_t)$$

In other words,

$$\left.\frac{\partial u}{\partial t}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - ru(t, x)\right|_{x=F_t} = 0,$$

for any value of F_t. Since F_t can take all positive values, we conclude that

$$\frac{\partial u}{\partial t}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - ru(t, x) = 0$$

which is called *Black's equation.* The boundary(terminal) condition is obviously

$$u(T, F_T) = \varphi(F_T)$$

In other words,

$$u(T, x) = \varphi(x)$$

is the boundary condition that goes with Black's equation.

7.2.3 Black's formula

Here we derive a formula for the call option. As we did in Chapter5, we can directly solve Black's equation to derive it, which is, however left to the reader. Instead, we take a shortcut of taking advantage of the known Black-Scholes formula. Note that

$$\begin{aligned}(F_t - K)^+ &= (e^{r(T^*-T)}S_T - K)^+ \\ &= e^{r(T^*-T)}[S_T - Ke^{-r(T^*-T)}]^+. \qquad (7.7)\end{aligned}$$

Therefore the call option on the futures price is really a call option on the spot asset with appropriate adjustment in the strike price and the quantity of the option as spelled out in (7.7)

Therefore by the Black-Scholes formula, the value C_t of the call option at time t when the futures price F_t is given by

$$C_t = e^{r(T^*-T)}\{S_t N(d_1) - Ke^{-r(T^*-T)}e^{-r(T-t)}N(d_2)\}$$

where

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_t}{Ke^{-r(T^*-T)}}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(\frac{F_t}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \end{aligned}$$

and similarly

$$d_2 = \frac{\log(\frac{F_t}{K}) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

We can rewrite the above formula for C_t to get

$$C_t = e^{-r(T-t)}\big[F_t N(d_1) - KN(d_2)\big] \tag{7.8}$$

This is the celebrated *Black's formula* for the futures call option. To derive the formula for the put option, we use the put-call parity. The put-call parity for the futures option is as follow,

$$\begin{aligned} C(T, F_T) - P(T, F_T) &= (F_T - K)^+ - (K - F_T)^+ \\ &= F_T - K. \end{aligned}$$

Thus

$$\begin{aligned} C_t - P_t &= B_t E_Q\left[\frac{F_T - K}{B_T} \mid \mathcal{F}_s\right] \\ &= e^{-r(T-t)}\big\{E_Q[F_T \mid \mathcal{F}_s] - K\big\} \\ &= e^{-r(T-t)}[F_t - K] \qquad\qquad (\because F_t \text{ is a } Q\text{-martingale.}) \end{aligned}$$

therefore

$$\begin{aligned} P_t &= C_t - e^{-r(T-t)}[F_t - K] \\ &= e^{-r(T-t)}\big\{-F_t N(-d_1) + KN(-d_2)\big\} \end{aligned}$$

This is the Black's formula for the futures put option.

7.3 Option on forward contract

An option in forward contract, sometimes simply called a forward option, is an option on the value of the forward contract itself.

Suppose a forward contract with the delivery date T is made and entered into at $t = 0$. From Section (7.1), we know that the forward price at $t = 0$ is $e^{rT}S_0$. The value at T of this forward contract is then $S_T - e^{rT}S_0$. We now consider a call option on the value at T of this forward contract. namely we consider it as an option whose payoff at T is $(S_T - e^{rT}S_0)^+$. Let C_t be the value at time t of this call option. Assuming S_t is a geometric Brownian motion with volatility σ, we can use the usual Black-Scholes formula to derive the formula for C_t. Namely,

$$\begin{aligned} C_t &= B_t E_Q\left[\frac{(S_T - S_0e^{rT})^+}{B_T} \mid \mathcal{F}_t\right] \\ &= S_tN(d_1) - S_0e^{rT}e^{-r(T-t)}N(d_2) \\ &= S_tN(d_1) - S_0e^{rt}N(d_2) \\ &= e^{-r(T-t)}[G_tN(d_1) - G_0N(d_2)], \end{aligned}$$

where $G_t = G(t,T) = e^{r(T-t)S_t}$ is the forward price;

$$\begin{aligned} d_1 &= \frac{\log(\frac{S_t}{S_0e^{rT}}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log(\frac{G_t}{G_0}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}; \end{aligned}$$

and similarly

$$d_2 = \frac{\log(\frac{G_t}{G_0}) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

The put-call parity at time T is written as

$$\begin{aligned} C_T - P_T &= (S_T - S_0e^{rT})^+ - (S_0e^{rT} - S_T)^+ \\ &= S_T - S_0e^{rT}. \end{aligned}$$

The usual risk neutral valuation method gives the put-call parity at time t as

$$C_t - P_t = S_t - S_0e^{rT}.$$

From this, we can easily conclude that the following formula for the put option on forward contract:

$$P_t = e^{-r(T-t)}\big[-G_tN(-d_1) + G_0N(-d_2)\big].$$

Exercises

7.1.

One wants to buy a forward contract that stipulates the delivery of a stock in one year. Suppose that the stock is traded at 100 at time $t = 0$; and assume that the zero-coupon bond that pays 1 in one year is traded at 0.8 at time $t = 0$

(a) What is the forward price at $t = 0$?

(b) Assume the stock is traded at 130 in one year. What will be the profit or loss of the holder(buyer) of this forward contract? Calculate its present value at time $t = 0$.

7.2.

(a) Write down the Black-Scholes formula for the put option on the futures price.

(b) A trader sells this put option and wants to create a hedging portfolio using the above formula in order to neutralizing the risk, answer the following questions.

  (1) How many units of futures contract does the trader have to buy or sell?

  (2) How much money should the trader holds in the bank?