(13) Stel de vergelijking op van de raaklijnen aan de ruimtekromme $\begin{cases} x^2+y^2=z^2+1 \\ x=y \end{cases}$ evenwijdig met de rechte $A: 2x=2y=z$.

Cursus p. 28:

Raaklijn R aan ruimtekromme $\begin{cases} \varphi(x,y,z)=0 \\ \psi(x,y,z)=0 \end{cases}$ in punt p: ← raakpunt

$\vec{v}_R \,/\!/\, \vec\nabla\varphi_p \times \vec\nabla\psi_p$ $\quad \hookrightarrow \begin{cases} x^2+y^2-z^2-1=0 \\ x-y=0 \end{cases}$

(1) * $\vec\nabla\varphi_p = \{2x_p, 2y_p, -2z_p\} \,/\!/\, \{x_p, y_p, -z_p\}$

* $\vec\nabla\psi_p = \{1,-1,0\}$

$$\Rightarrow \boxed{\vec{v}_R \,/\!/\, \vec\nabla\varphi_p \times \vec\nabla\psi_p} \,/\!/\, \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ x_p & y_p & -z_p \\ 1 & -1 & 0 \end{vmatrix} = \{-z_p, -z_p, -x_p-y_p\}$$

$$/\!/\, \boxed{\{z_p, z_p, x_p+y_p\}}$$

(2) $A: \dfrac{x}{\frac{1}{2}} = \dfrac{y}{\frac{1}{2}} = z \qquad \vec{v}_A = \{\frac{1}{2}, \frac{1}{2}, 1\}$

$\boxed{\vec{v}_R \,/\!/\, \vec{v}_A} \,/\!/\, \boxed{\{1,1,2\}}$

(3) $\{z_p, z_p, x_p+y_p\} \,/\!/\, \{1,1,2\}$

$\Rightarrow \begin{cases} z_p = k \\ x_p+y_p = 2k \end{cases} \Rightarrow$ raakpunt $p(x_p, y_p, z_p) = p(l, 2k-l, k)$ $\quad k, l \in \mathbb{R}$

* $p \in$ ruimtekromme

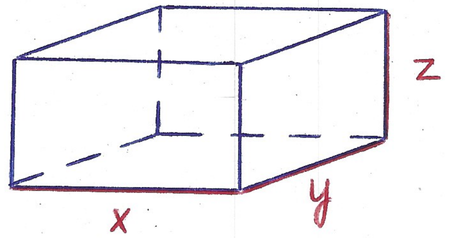
→ $p \in \psi(x,y,z)=0$: $l = 2k-l \Rightarrow l = k \Rightarrow p(k,k,k)$

→ $p \in \varphi(x,y,z)=0$: $\cancel{k^2} + k^2 = \cancel{k^2} + 1$

* $k_1 = 1 \quad \Rightarrow p_1(1,1,1) \Rightarrow$ raaklijn R_1: $x-1 = y-1 = \dfrac{z-1}{2}$

* $k_2 = -1 \quad \Rightarrow p_2(-1,-1,-1) \Rightarrow$ raaklijn R_2: $x+1 = y+1 = \dfrac{z+1}{2}$

(15) Een rechthoekige doos, bovenaan gesloten, heeft een inhoud van 27dm³. Bepaal de dimensies voor een minimale oppervlakte A.



inhoud: $V = 27 dm^3$

$x.y.z = 27 \Rightarrow \boxed{z = \frac{27}{x.y}}$ (*)

oppervlakte: $A = 2xy + 2xz + 2yz$

$\Downarrow$ (*)

$A = 2xy + 2.\frac{27}{y} + 2.\frac{27}{x}$

STAP 1: kandidaat extrema zoeken

$\begin{cases} \partial A/\partial x = 0 \\ \partial A/\partial y = 0 \end{cases} \Rightarrow \begin{cases} 2y - 2.\frac{27}{x^2} = 0 \\ 2x - 2.\frac{27}{y^2} = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{27}{x^2} & (2) \\ x = \frac{x^4}{27} & (1) \end{cases}$

(1) $\cancel{x=0}$ (geen doos) of $x^3 = 27 \Rightarrow x_1 = 3dm \overset{(2)}{\Rightarrow} y_1 = 3dm \overset{(*)}{\Rightarrow} z_1 = 3dm$

$\Rightarrow p_1(x_1, y_1) = p_1(3,3)$

STAP 2: aantonen dat de oppervlakte A minimaal is als $x = y = z = 3dm$ (p_1)

$\Delta p = \left(\frac{\partial^2 A}{\partial x \partial y}\right)_p^2 - \left(\frac{\partial^2 A}{\partial x^2}\right)_p \cdot \left(\frac{\partial^2 A}{\partial y^2}\right)_p$

* $\frac{\partial^2 A}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial x}\right) = 4.\frac{27}{x^3}$
* $\frac{\partial^2 A}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial A}{\partial y}\right) = 4.\frac{27}{y^3}$
* $\frac{\partial^2 A}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A}{\partial y}\right) = 2$

$\Rightarrow \left(\frac{\partial^2 A}{\partial x^2}\right)_{p_1} = \left(\frac{\partial^2 A}{\partial y^2}\right)_{p_1} = 4$ en $\left(\frac{\partial^2 A}{\partial x \partial y}\right)_{p_1} = 2$

$\Rightarrow \Delta p_1 = 4 - 16 < 0 \Rightarrow$ extremum

$\Rightarrow \left(\frac{\partial^2 A}{\partial x^2}\right)_{p_1} = 4 > 0 \Rightarrow$ minimum

$\boxed{\text{minimale oppervlakte als } x = y = z = 3dm}$

17a Bepaal de gebonden extrema van $z = x + 2y$ met $x^2 + y^2 = 5$

Gebonden extrema van $\underbrace{z = f(x,y)}_{z = x+2y}$ met $\underbrace{g(x,y) = 0}_{x^2+y^2-5=0}$ bepalen.

$\boxed{\text{STAP 1}}$: vergelijking van Lagrange

$$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$$

$$L(x,y,\lambda) = x + 2y + \lambda(x^2 + y^2 - 5)$$

$\boxed{\text{STAP 2}}$: kandidaat extrema zoeken

$$\begin{cases} \partial L/\partial x = 0 \\ \partial L/\partial y = 0 \\ \partial L/\partial \lambda = 0 \end{cases} \Rightarrow \begin{cases} 1 + 2\lambda x = 0 \\ 2 + 2\lambda y = 0 \\ x^2 + y^2 = 5 \end{cases} \Rightarrow \begin{cases} x = -1/2\lambda & (2) \\ y = -1/\lambda & (3) \\ \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} \cdot \frac{4}{4} = 5 & (1) \end{cases}$$

$$(1)\ \lambda^2 = 5/20 = 1/4 \begin{cases} \nearrow \lambda_1 = \frac{1}{2} \xRightarrow[(2),(3)]{} p_1(-1,-2) \\ \searrow \lambda_2 = -\frac{1}{2} \Longrightarrow p_2(1,2) \end{cases}$$

$\boxed{\text{STAP 3}}$:

$$\Delta_p = \left(\frac{\partial^2 L}{\partial x \partial y}\right)_p^2 - \left(\frac{\partial^2 L}{\partial x^2}\right)_p \cdot \left(\frac{\partial^2 L}{\partial y^2}\right)_p$$

$* \ \frac{\partial^2 L}{\partial x^2} = 2\lambda \qquad * \ \frac{\partial^2 L}{\partial y^2} = 2\lambda \qquad * \ \frac{\partial^2 L}{\partial x \partial y} = 0$

$$\Rightarrow \left(\frac{\partial^2 L}{\partial x^2}\right)_{p_1} = \left(\frac{\partial^2 L}{\partial y^2}\right)_{p_1} = 2 \cdot \frac{1}{2} = 1, \quad \left(\frac{\partial^2 L}{\partial x^2}\right)_{p_2} = \left(\frac{\partial^2 L}{\partial y^2}\right)_{p_2} = 2 \cdot \left(\frac{-1}{2}\right) = -1$$

en $\left(\frac{\partial^2 L}{\partial x \partial y}\right)_{p_1} = \left(\frac{\partial^2 L}{\partial x \partial y}\right)_{p_2} = 0$

$\Rightarrow \underline{\Delta_{p_1} = \Delta_{p_2}} = 0 - 1 < 0 \Rightarrow p_1$ en p_2 zijn gebonden extrema $\left(\lambda_1 = \frac{1}{2}\right)$ $\left(\lambda_2 = \frac{-1}{2}\right)$

$\Rightarrow \underline{\left(\frac{\partial^2 L}{\partial x^2}\right)_{p_1}} = 1 > 0 \Rightarrow p_1(-1,-2)$ is een gebonden minimum $(\lambda_1 = 1/2)$

$\Rightarrow \underline{\left(\frac{\partial^2 L}{\partial x^2}\right)_{p_2}} = -1 < 0 \Rightarrow p_2(1,2)$ is een gebonden maximum $(\lambda_2 = -1/2)$