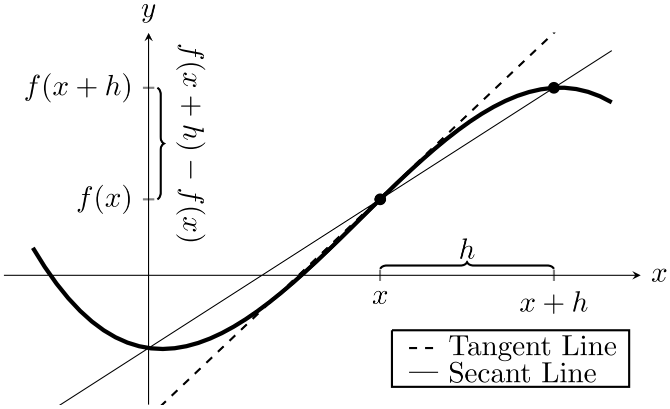
Limit Definition of the Derivative



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: $f'(x)$ represents the instantaneous rate of change of $f(x)$.

Critical Numbers

A number a is a critical number of a function f if (and only if):

1. a is in the domain of f
2. $f'(a)$ is either 0 or does not exist

Second Derivative Test

Given a function f that is twice differentiable at a with $f'(a) = 0$:

- f has a local maximum at a if $f''(a) < 0$
- f has a local minimum at a if $f''(a) > 0$
- The test is inconclusive if $f''(a) = 0$

Limit Laws

Assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. $C \in \mathbb{R}$. $k \in \mathbb{R}^+$.

Substitution	R1	$\lim_{x \to a} x = a$
Constant	R2	$\lim_{x \to a} C = C$
Denom. of ∞	R3	$\lim_{x \to \pm\infty} f(x) = \pm\infty \Rightarrow \lim_{x \to \pm\infty} \frac{C}{f(x)} = 0$
Sum	A1	$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
Difference	A2	$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
Const. Factor	A3	$\lim_{x \to a}(C \cdot f(x)) = C \cdot \lim_{x \to a} f(x)$
Product	A4	$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
Quotient	A5	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \left(\lim_{x \to a} g(x) \neq 0\right)$
Composition	A6	$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$ $\left(\text{with } f \text{ continuous at } \lim_{x \to a} g(x)\right)$
Part Equal	A7	$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ $(f(x) = g(x), x \in (a-k, a) \cup (a, a+k))$

Tangent Lines

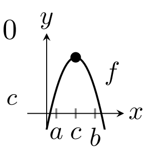
To find the equation of a line tangent to a function f at an input of a:

1. Find $f(a)$. Since the tangent line touches f when $x = a$, the tangent line must go through $(a, f(a))$.
2. Find $f'(a)$. Since the derivative at a is defined to be the slope of a line tangent to f at a, $m = f'(a)$ is the slope.
3. Point-slope formula $(y = m(x - x_1) + y_1)$ gives:
$$y_{\text{tangent}} = f'(a)(x - a) + f(a)$$

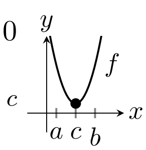
First Derivative Test

Given a function f that is continuous at a critical number $c \in (a, b)$,

- f has a local maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b).
Note: f' is positive "a little" to the left of c and negative "a little" to the right of c.



- f has a local minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b).
Note: f' is negative "a little" to the left of c and positive "a little" to the right of c.



- f has an inflection point at c if $f'(x) < 0$ on $(a, b) \setminus \{c\}$ or $f'(x) > 0$ on $(a, b) \setminus \{c\}$.

Note: Inflection points happen when f' has the same sign "a little" to the left and "a little" to the right of c.

Derivative Relationships

f	f'	f''
Increasing	Positive	
Decreasing	Negative	
Concave Up	Increasing	Positive
Concave Down	Decreasing	Negative

Continuity

A function f is continuous at a if (and only if):

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

Note: A *function* is continuous if (and only if) it is continuous at every point in its domain.

Mean Value Theorem

If a function f is continuous on $[a, b]$ and differentiable on (a, b) with $a < b$, then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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Derivative Rules	$f(x)$	$f'(x)$
Sum	$f(x)+g(x)$	$f'(x)+g'(x)$
Difference	$f(x)-g(x)$	$f'(x)-g'(x)$
Product	$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$
Quotient	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain	$f(g(x))$	$f'(g(x))g'(x)$
Constant Factor	$kf(x)$	$kf'(x)$
Constant	k	0
Linear	$mx+b$	m
Power	x^n	nx^{n-1}
Logarithm (General)	$\log_b(x)$	$\frac{1}{x\ln(b)}$
Logarithm (Natural)	$\ln(x)$	$\frac{1}{x}$
Exponential (General)	b^x	$b^x\ln(b)$
Exponential (Natural)	e^x	e^x
Sine	$\sin(x)$	$\cos(x)$
Cosine	$\cos(x)$	$-\sin(x)$
Tangent	$\tan(x)$	$\sec^2(x)$
Cotangent	$\cot(x)$	$-\csc^2(x)$
Secant	$\sec(x)$	$\sec(x)\tan(x)$
Cosecant	$\csc(x)$	$-\csc(x)\cot(x)$
Inverse Sine	$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
Inverse Cosine	$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
Inverse Tangent	$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
Inverse Cotangent	$\cot^{-1}(x)$	$\frac{-1}{1+x^2}$
Inverse Secant	$\sec^{-1}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
Inverse Cosecant	$\csc^{-1}(x)$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
Hyperbolic Sine	$\sinh(x)$	$\cosh(x)$
Hyperbolic Cosine	$\cosh(x)$	$\sinh(x)$
Hyperbolic Tangent	$\tanh(x)$	$\mathrm{sech}^2(x)$
Hyperbolic Cotangent	$\coth(x)$	$-\mathrm{csch}^2(x)$
Hyperbolic Secant	$\mathrm{sech}(x)$	$-\mathrm{sech}(x)\tanh(x)$
Hyperbolic Cosecant	$\mathrm{csch}(x)$	$-\mathrm{csch}(x)\coth(x)$
Inv. Hyper. Sine	$\sinh^{-1}(x)$	$\frac{1}{\sqrt{x^2+1}}$
Inv. Hyper. Cosine	$\cosh^{-1}(x)$	$\frac{1}{\sqrt{x^2-1}}$
Inv. Hyper. Tangent	$\tanh^{-1}(x)$	$\frac{1}{1-x^2}$
Inv. Hyper. Cotangent	$\coth^{-1}(x)$	$\frac{1}{1-x^2}$
Inv. Hyper. Secant	$\mathrm{sech}^{-1}(x)$	$\frac{-1}{x\sqrt{1-x^2}}$
Inv. Hyper. Cosecant	$\mathrm{csch}^{-1}(x)$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Related Rates

1. Construct a pre-calculus equation (or system of equations) that model the problem.
   (a) In the case of a system, reduce to a single equation with only the relevant variables.
2. Implicitly differentiate the equation.
3. Substitute all variables with values except the one to be solved for, including the other rate(s).
4. Solve using algebra & arithmetic.
5. Interperate the result in terms of the word problem. (Write a sentence or two.)

Anti-Derivative

F is called an anti-derivative of f if $F'(x)=f(x)$.

Note: f may have more than one anti-derivative. For example, consider $F_1(x)=x^2$, $F_2(x)=x^2+5$, and $f(x)=2x$.

Intermediate Value Theorem

If a function f is continuous on $[a,b]$ with a number u between $f(a)$ and $f(b)$, then there exists a $c\in(a,b)$ such that $f(c)=u$.

Note: Here "between" means $f(a)<u<f(b)$ or $f(b)<u<f(a)$.

Squeeze Theorem

Given an interval I and functions f,g, and h defined on $I\setminus\{a\}$ with:

1. $g(x)\le f(x)\le h(x)$
2. $\lim_{x\to a} g(x)=\lim_{x\to a} h(x)=L$

The squeeze theorem states,

$$\lim_{x\to a} f(x)=L$$