

January 9, 2020

# Overview

# Warm-up

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement: Which troll is which?

# Statements: atomic and molecular

- A **statement** is any declarative sentence which is either true or false.
- A statement is **atomic** if it cannot be divided into smaller statements, otherwise it is called **molecular**.

## Example

g:example:idm36

These are statements (in fact, *atomic* statements):

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.
- $3 + 7 = 12$

## Example

g:example:idm53

And these are not statements:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1.$
- Go to your room!
- $3 + x = 12$

# Molecular statements

You can build more complicated (molecular) statements out of simpler (atomic or molecular) ones using **logical connectives**. For example, this is a molecular statement:

*Telephone numbers in the USA have 10 digits and 42 is a perfect square.*

we can break this down into two smaller statements. The two shorter statements are *connected* by an “and.”

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- $\neg P$  is read “not  $P$ ,” and called a **negation**.

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- $\neg P$  is true when  $P$  is false.

# Implications

An **implication** or **conditional** is a molecular statement of the form

$$P \rightarrow Q$$

where  $P$  and  $Q$  are statements. We say that

- $P$  is the **hypothesis** (or **antecedent**).
- $Q$  is the **conclusion** (or **consequent**).

An implication is *true* provided  $P$  is false or  $Q$  is true (or both), and *false* otherwise. In particular, the only way for  $P \rightarrow Q$  to be false is for  $P$  to be true *and*  $Q$  to be false.

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g:example:idm163

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- Is it true or false? What would it take for the statement to be *false*?
- The only way to be false: Bob does get a 90 on the final *AND* Bob still does not pass the class.
- In particular, if Bob does not get a 90 on the final ( $P$  is false), then whether or not he passes the class, the statement is true.

## Example

g:example:idm186

Decide which of the following statements are true and which are false. Briefly explain.

- ① If  $1 = 1$ , then most horses have 4 legs.
- ② If  $0 = 1$ , then  $1 = 1$ .
- ③ If 8 is a prime number, then the 7624th digit of  $\pi$  is an 8.
- ④ If the 7624th digit of  $\pi$  is an 8, then  $2 + 2 = 4$ .

# Direct Proofs of Implications

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g:example:idm215

Prove: If two numbers  $a$  and  $b$  are even, then their sum  $a + b$  is even.

Proof:

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g:example:ldm215

Prove: If two numbers  $a$  and  $b$  are even, then their sum  $a + b$  is even.

Proof: Suppose the numbers  $a$  and  $b$  are even. This means that  $a = 2k$  and  $b = 2j$  for some integers  $k$  and  $j$ . The sum is then  $a + b = 2k + 2j = 2(k + j)$ . Since  $k + j$  is an integer, this means that  $a + b$  is even.  $\square$

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- The converse is *NOT* logically equivalent to the original implication. That is, whether the converse of an implication is true is independent of the truth of the implication.
- The **contrapositive** of an implication  $P \rightarrow Q$  is the statement  $\neg Q \rightarrow \neg P$ .
- An implication and its contrapositive are logically equivalent (they are either both true or both false).

## Example

g:example:idm256

True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit.

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True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit. Try looking at the contrapositive! Is the converse true?

## Example

g:example:idm262

Suppose I tell Sue that if she gets a 93% on her final, then she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

- 1 Sue gets a 93% on her final.
- 2 Sue gets an A in the class.
- 3 Sue does not get a 93% on her final.
- 4 Sue does not get an A in the class.

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Example: Given an integer  $n$ , it is true that  $n$  is even if and only if  $n^2$  is even. That is, if  $n$  is even, then  $n^2$  is even, as well as the converse: if  $n^2$  is even, then  $n$  is even.

## Example

g:example:idm290

Suppose it is true that I sing if and only if I'm in the shower. We know this means both that if I sing, then I'm in the shower, and also the converse, that if I'm in the shower, then I sing. Let  $P$  be the statement, "I sing," and  $Q$  be, "I'm in the shower." So  $P \rightarrow Q$  is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

## Example

g:example:idm300

Rephrase the implication, “if I dream, then I am asleep” in as many different ways as possible. Then do the same for the converse.

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- “ $P$  is sufficient for  $Q$ ” means  $P \rightarrow Q$ .
- If  $P$  is necessary and sufficient for  $Q$ , then  $P \leftrightarrow Q$ .

## Example

g:example:idm324

Recall from calculus, if a function is differentiable at a point  $c$ , then it is continuous at  $c$ , but that the converse of this statement is not true (for example,  $f(x) = |x|$  at the point 0). Restate this fact using “necessary and sufficient” language.



# Predicates

How could we claim that if  $n$  is prime, then  $n + 7$  is not prime? This looks like an implication. I would like to write something like

$$P(n) \rightarrow \neg P(n + 7)$$

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# Universal and Existential Quantifiers

The existential quantifier is  $\exists$  and is read “there exists” or “there is.” For example,

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asserts that there is a number less than 0. The universal quantifier is  $\forall$  and is read “for all” or “every.” For example,

$$\forall x(x \geq 0)$$

asserts that every number is greater than or equal to 0.

# Quantifiers and Negation

When is a quantified statement false?  $\neg\forall xP(x)$  is equivalent to  $\exists x\neg P(x)$ .  
 $\neg\exists xP(x)$  is equivalent to  $\forall x\neg P(x)$ .