Mathematical Statements

(Section 0.2)

Overview

Warm-up

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement:

- Troll 1: If I am a knave, then there are exactly two knights here.
- Troll 2: Troll 1 is lying.
- Troll 3: Either we are all knaves or at least one of us is a knight.

Which troll is which?

Statements: atomic and molecular

- A **statement** is any declarative sentence which is either true or false.
- A statement is **atomic** if it cannot be divided into smaller statements, otherwise it is called **molecular**.

Example (0.2.1)

These are statements (in fact, *atomic* statements):

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.
- $3 + 7 = 12$

Example (0.2.1)

And these are not statements:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $3 + x = 12$

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We can break this down into two smaller statements. The two shorter statements are *connected* by an "and."

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- $\neg P$ is read "not P," and called a **negation**.

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- $\neg P$ is true when P is false.

Implications

An **implication** or **conditional** is a molecular statement of the form

$$P \to Q$$

where P and Q are statements. We say that

- P is the **hypothesis** (or **antecedent**).
- Q is the **conclusion** (or **consequent**).

An implication is *true* provided P is false or Q is true (or both), and *false* otherwise. In particular, the only way for $P \to Q$ to be false is for P to be true *and* Q to be false.

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- Is it true or false? What would it take for the statement to be *false*?
- The only way to be false: Bob does get a 90 on the final *AND* Bob still does not pass the class.
- In particular, if Bob does not get a 90 on the final (P is false), then whether or not he passes the class, the statement is true.

Example (0.2.3)

Decide which of the following statements are true and which are false. Briefly explain.

1. If $1 = 1$, then most horses have 4 legs.
2. If $0 = 1$, then $1 = 1$.
3. If 8 is a prime number, then the 7624th digit of π is an 8.
4. If the 7624th digit of π is an 8, then $2 + 2 = 4$.

Direct Proofs of Implications

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To prove an implication $P \to Q$, it is enough to *assume* P, and from it, *deduce* Q.

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Suppose the numbers a and b are even. This means that $a = 2k$ and $b = 2j$ for some integers k and j. The sum is then $a + b = 2k + 2j = 2(k + j)$. Since $k + j$ is an integer, this means that $a + b$ is even. $\square$

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- The **contrapositive** of an implication $P \to Q$ is the statement $\neg Q \to \neg P$.
- An implication and its contrapositive are logically equivalent (they are either both true or both false).

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Is the converse true?

Example (0.2.6)

Suppose I tell Sue that if she gets a 93% on her final, then she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

1. Sue gets a 93% on her final.
2. Sue gets an A in the class.
3. Sue does not get a 93% on her final.
4. Sue does not get an A in the class.

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Example: Given an integer n, it is true that n is even if and only if n^2 is even. That is, if n is even, then n^2 is even, as well as the converse: if n^2 is even, then n is even.

Example (0.2.7)

Suppose it is true that I sing if and only if I'm in the shower. We know this means both that if I sing, then I'm in the shower, and also the converse, that if I'm in the shower, then I sing. Let P be the statement, "I sing," and Q be, "I'm in the shower." So $P \to Q$ is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

Example (0.2.8)

Rephrase the implication, “if I dream, then I am asleep” in as many different ways as possible. Then do the same for the converse.

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- "P is necessary for Q" means $Q \to P$.
- "P is sufficient for Q" means $P \to Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

Example (0.2.9)

Recall from calculus, if a function is differentiable at a point c, then it is continuous at c, but that the converse of this statement is not true (for example, $f(x) = |x|$ at the point 0). Restate this fact using "necessary and sufficient" language.

Predicates

How could we claim that if n is prime, then $n + 7$ is not prime? This looks like an implication. I would like to write something like

$$P(n) \to \neg P(n + 7)$$

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For one thing, this sentence has a **free variable** (that is, a variable that we have not specified anything about), so it is not a statement. A sentence that contains variables is called a **predicate**.

If we plug in a specific value for n, we do get a statement. What we really want to say is that *for all* values of n, if n is prime, then $n + 7$ is not. We need to *quantify* the variable.

Universal and Existential Quantifiers

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The universal quantifier is $\forall$ and is read "for all" or "every." For example,

$$\forall x(x \geq 0)$$

asserts that every number is greater than or equal to 0.

Quantifiers and Negation

When is a quantified statement false? $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$. $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$.