January 9, 2020

Overview

Warm-up

While walking through a fictional forest, you encounter three trolls guarding a bridge. Each is either a *knight*, who always tells the truth, or a *knave*, who always lies. The trolls will not let you pass until you correctly identify each as either a knight or a knave. Each troll makes a single statement: Which troll is which?

Statements: atomic and molecular

- A **statement** is any declarative sentence which is either true or false.
- A statement is **atomic** if it cannot be divided into smaller statements, otherwise it is called **molecular**.

Example

g:example:idm36

These are statements (in fact, *atomic* statements):

- Telephone numbers in the USA have 10 digits.
- The moon is made of cheese.
- 42 is a perfect square.
- Every even number greater than 2 can be expressed as the sum of two primes.
- $3 + 7 = 12$

Example

g:example:idm53

And these are not statements:

- Would you like some cake?
- The sum of two squares.
- $1 + 3 + 5 + 7 + \cdots + 2n + 1$.
- Go to your room!
- $3 + x = 12$

Molecular statements

You can build more complicated (molecular) statements out of simpler (atomic or molecular) ones using **logical connectives**. For example, this is a molecular statement:

> *Telephone numbers in the USA have 10 digits and 42 is a perfect square.*

we can break this down into two smaller statements. The two shorter statements are *connected* by an “and.”

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- $P \leftrightarrow Q$ is read "P if and only if Q," and called a **biconditional**.
- $\neg P$ is read "not P," and called a **negation**.

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- $\neg P$ is true when P is false.

Implications

An **implication** or **conditional** is a molecular statement of the form

$$P \to Q$$

where P and Q are statements. We say that

- P is the **hypothesis** (or **antecedent**).
- Q is the **conclusion** (or **consequent**).

An implication is *true* provided P is false or Q is true (or both), and *false* otherwise. In particular, the only way for $P \to Q$ to be false is for P to be true *and* Q to be false.

Example

g:example:idm163

Consider the statement:

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- Is it true or false? What would it take for the statement to be *fasle*?
- The only way to be false: Bob does get a 90 on the final *AND* Bob still does not pass the class.
- In particular, if Bob does not get a 90 on the final (P is false), then whether or not he passes the class, the statement is true.

Example

g:example:idm186

Decide which of the following statements are true and which are false. Briefly explain.

1. If $1 = 1$, then most horses have 4 legs.
2. If $0 = 1$, then $1 = 1$.
3. If 8 is a prime number, then the 7624th digit of π is an 8.
4. If the 7624th digit of π is an 8, then $2 + 2 = 4$.

Direct Proofs of Implications

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g:example:idm215

Prove: If two numbers a and b are even, then their sum $a + b$ is even.

Proof:

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Prove: If two numbers a and b are even, then their sum $a + b$ is even.

Proof:Suppose the numbers a and b are even. This means that $a = 2k$ and $b = 2j$ for some integers k and j. The sum is then $a + b = 2k + 2j = 2(k + j)$. Since $k + j$ is an integer, this means that $a + b$ is even. □

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How does $P \to Q$ relate to $Q \to P$?

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- The **contrapositive** of an implication $P \to Q$ is the statement $\neg Q \to \neg P$.
- An implication and its contrapositive are logically equivalent (they are either both true or both false).

Example

g:example:idm256

True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit.

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True or false: If you draw any nine playing cards from a regular deck, then you will have at least three cards all of the same suit. Try looking at the contrapositive! Is the converse true?

Example

g:example:idm262

Suppose I tell Sue that if she gets a 93% on her final, then she will get an A in the class. Assuming that what I said is true, what can you conclude in the following cases:

1. Sue gets a 93% on her final.
2. Sue gets an A in the class.
3. Sue does not get a 93% on her final.
4. Sue does not get an A in the class.

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Example: Given an integer n, it is true that n is even if and only if n^2 is even. That is, if n is even, then n^2 is even, as well as the converse: if n^2 is even, then n is even.

Example

g:example:idm290

Suppose it is true that I sing if and only if I'm in the shower. We know this means both that if I sing, then I'm in the shower, and also the converse, that if I'm in the shower, then I sing. Let P be the statement, "I sing," and Q be, "I'm in the shower." So $P \to Q$ is the statement "if I sing, then I'm in the shower." Which part of the if and only if statement is this?

Example

g:example:idm300

Rephrase the implication, “if I dream, then I am asleep” in as many different ways as possible. Then do the same for the converse.

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- "P is necessary for Q" means $Q \to P$.
- "P is sufficient for Q" means $P \to Q$.
- If P is necessary and sufficient for Q, then $P \leftrightarrow Q$.

Example

g:example:idm324

Recall from calculus, if a function is differentiable at a point c, then it is continuous at c, but that the converse of this statement is not true (for example, $f(x) = |x|$ at the point 0). Restate this fact using "necessary and sufficient" language.

Predicates

How could we claim that if n is prime, then $n+7$ is not prime? This looks like an implication. I would like to write something like

$$P(n) \to \neg P(n+7)$$

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where $P(n)$ means "n is prime." But this is not quite right. For one thing, this sentence has a **free variable** (that is, a variable that we have not specified anything about), so it is not a statement. A sentence that contains variables is called a **predicate**. If we plug in a specific value for n, we do get a statement. What we really want to say is that *for all* values of n, if n is prime, then $n+7$ is not. We need to *quantify* the variable.

Universal and Existential Quantifiers

The existential quantifier is $\exists$ and is read "there exists" or "there is." For example,

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$$\exists x(x < 0)$$

asserts that there is a number less than 0. The universal quantifier is $\forall$ and is read "for all" or "every." For example,

$$\forall x(x \geq 0)$$

asserts that every number is greater than or equal to 0.

Quantifiers and Negation

When is a quantified statement false? $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$. $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$.