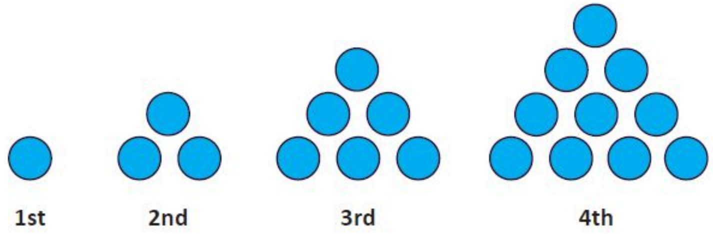
Module 10A: Integer sequences

MTH 225
11 November 2020

The triangular numbers are the integers in the sequence 1, 3, 6, 10, 15, 21, ... A closed formula for this sequence is $T(n) = \frac{n(n+1)}{2}$. What is the next (seventh) term of the sequence?

- 27
- 28
- 36
- 42

The triangular numbers are *also* the number of dots in each stage of the visual pattern you see here. Based on this visual pattern, a recursive definition for $T(n)$ would be



- $T(1) = 1$ and $T(2) = 3$, and $T(n) = T(n-1) + T(n-2)$ when $n > 2$
- $T(1) = 1$, and $T(n) = T(n-1)$ when $n > 1$
- $T(1) = 1$, and $T(n) = T(n-1) + n$ when $n > 1$
- $T(1) = 1$, and $T(n) = n \cdot T(n-1)$ when $n > 1$

To 0

A sequence b_n is defined recursively by $b_1 = 2$ and $b_n = 2^n \cdot b_{n-1}$ when $n > 1$. The value of b_5 is

- 32
- 64
- 1024
- 2048
- 32768

Discovered by accident:

$$b_1 = 1, b_n = 2^n b_{n-1}$$

2	$2^2 * 2 = 2^3$	$2^3 * 2^3 = 2^6$	$2^4 * 2^6 = 2^{10}$	$2^5 * 2^{10} = 2^{15}$
n=1	n=2	n=3	n=4	n=5

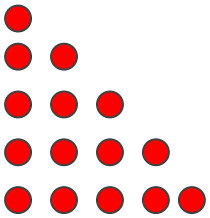
Exponents are 1, 3, 6, 10, 15...

$$b_n = 2^{T(n)}$$

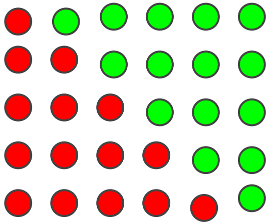
$$\boxed{b_n = 2^{\frac{n(n+1)}{2}}}$$

Why does the closed formula for T(n) work?

Combinatorial proof: T(n) is the number of dots in a right triangle with side lengths equal to n:



← Make a “green” copy of this and stack it on the right of the red copy:



← This gives a rectangle with dimensions n(n+1). The number of red dots is half of this, so T(n) = n(n+1)/2.

Partial sums

Sequences of partial sums

Given any sequence (a_i), we can form a new sequence (b_n) by adding up the first n terms of (a_i): The **sequence of partial sums**.

Example: The triangular numbers 1, 3, 6, 10, 15, …

n	1	2	3	4	5
Nth term of original	1	3	6	10	15
Nth partial sum	1	1+3 = 4	1+ 3 + 6 = 10	1 + 3 + 6 + 10 = 20	1 + 3 + 6 + 10 + 15 = 35

Jamboard:

$a(1)=3$, $a(n) = 0.5*a(n-1)$ when $n > 1$

Find the first 10 terms and the first 10 partial sums.

Notice anything?