Module 12B: Setting up and analyzing mathematical induction proofs

MTH 225
2 December 2020

Agenda

- Review of the steps of a mathematical induction proof
- Practice with constructing the framework
- Critical analysis of written induction proofs

"For all natural numbers n, $1+2+4+8+\cdots+2^n = 2^{n+1}-1$" To prove this conjecture using mathematical induction, we first

- Prove that the statement above holds just for $n = 0$
- Prove that the statement above holds just for $n = 1$
- Assume that the statement above holds for some natural number n
- Prove that the statement above holds for some natural number n

To 0

"**For all natural numbers n,**
$1+2+4+8+\cdots+2^n = 2^{n+1}-1$" **To prove this conjecture using mathematical induction, after establishing the base case, we then**

- Assume that the statement above holds for all natural numbers n
- Prove that the statement above holds for $n = 1$
- Assume that the statement above holds for $n-1$, where n is some natural number
- Prove that the statement above holds for some natural number n

To 0

Proof by (“weak”) mathematical induction

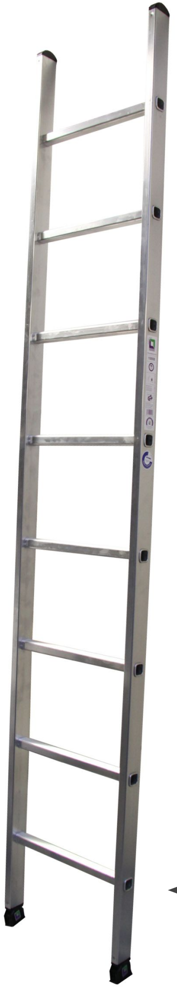
To be used when proving a conjecture claimed to be true for all natural numbers (or all natural numbers of a certain pattern), involving recursively-defined objects.

Let P(n) be the predicate involved without the quantifier.

Step 1 (Base case): Show that P(n) is true for the initial value of n.

Step 2 (Induction hypothesis): Assume that P(n-1) is true for some n.

Step 3 (Proof step): Prove that P(n) is true, making use of the induction hypothesis.



Proof step: Assuming you made it that far, show you can get to the next rung.

Induction hypothesis: Assume you've made it this far up the ladder.

Base case: Show that you can get on the bottom rung of the ladder.

A completed proof

To prove: For all natural numbers n, $1 + 2 + 4 + 8 + \ldots + 2^n = 2^{n+1} - 1$.

Proof: We will use mathematical induction. For the base case, look at n = 0. The left side above becomes just 1. The right side is $2^{0+1} - 1$, which equals $2 - 1 = 1$. Since the left and right sides are equal, the statement holds in the base case.

Now let n be some natural number and assume that $\mathbf{1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1}$.

Take just the left side of the main statement to prove:

$$
\begin{aligned}
&1 + 2 + 4 + 8 + \ldots + 2^n \\
&= (1 + 2 + 4 + 8 + \ldots + 2^{n-1}) + 2^n \\
&= \mathbf{2^n - 1} + 2^n \quad (\leftarrow \textit{By the induction hypothesis}) \\
&= 2(2^n) - 1 \\
&= 2^{n+1} - 1.
\end{aligned}
$$

Therefore the inductive step works, and the statement is proven.

Practice with induction proof setup

Conjecture: Every set of n elements has 2^n subsets. To prove this by induction, first:

- Assume that a set with 0 elements has 1 subset
- Assume that a set with 1 element has 2 subsets
- Prove (by demonstration) that a set with 0 elements has 1 subset
- Prove (by demonstration) that a set with 1 elements has 2 subsets

Tc 0

Conjecture: Every set of elements has subsets. To prove this by induction, after establishing the base case:

- Prove (by demonstration) that a set with 2 elements has 4 subsets
- Assume that for all natural numbers n, every set with $n-1$ elements has 2^{n-1} subsets
- Prove (by demonstration) that a set with 1 elements has 2 subsets
- Assume that for some natural number n, every set with $n-1$ elements has 2^{n-1} subsets

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Conjecture: Every set of n elements has ^{n} subsets. To prove this by induction, after assuming the induction hypothesis:

- Prove (by demonstration) that a set with 3 elements has 8 subsets
- Assume that for some natural number n, every set with n elements has 2^n subsets
- Assume that for all natural numbers n, every set with n elements has 2^n subsets
- Prove that every set with n elements has 2^n subsets

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Critical analysis of a proof

Given a written proof of a conjecture, three options:

1. The conjecture itself is false (there is a “counterexample”) and therefore the proof cannot be correct.
2. The conjecture is true, but the proof has a significant/fatal mistake or omission in the logic or the mathematics.
3. The conjecture is true and the proof has no significant/fatal mistakes or omissions.

Proposition 1

For all natural numbers n, f_{3n+2} is even.

https://docs.google.com/document/d/1J3lQabY7P3NpvxL5iUamtdLutYGcbXXm7lF3ptALB-M/edit?usp=sharing

How would you rate the proposed proof of Proposition 1?

- The proposition itself is false, so the proof can't be right.
- The proposition is true, but the proof has a significant/fatal error or omission.
- The proposition is true, and the proof has no significant/fatal errors or omissions.

To 0

"For all natural numbers n, f_{3n+2} is even." To establish the base case,

- Prove (by demonstration) that f_0 is even
- Prove (by demonstration) that f_1 is even
- Prove (by demonstration) that f_2 is even
- None of the above

"For all natural numbers n, f_{3n+2} is even." After proving the base case,

- We prove that f_5 is even
- We assume that f_{3n-2} is even for some n
- We assume that f_{3n-1} is even for some n
- We assume that f_{3n} is even for some n
- We assume that f_{3n+1} is even for some n

To 0

"For all natural numbers n, f_{3n+2} is even." After assuming the induction hypothesis,

We prove that f_8 is even

We prove that f_{3n-2} is even for some n

We assume that f_{3n+1} is even for some n

We prove that f_{3n+1} is even

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Corrected proof

To prove the base case, let n = 0. Then 3n + 2 = 2, and notice that $f_2 = 2$ which is even.

Now assume that $f_{3(n-1)+2} = f_{3n-1}$ is even for some n. We want to show that f_{3n+2} is even. To do this, start with f_{3n+2} and use the Fibonacci definition:

$$\mathbf{f_{3n+2} = f_{3n+1} + f_{3n} = f_{3n} + f_{3n-1} + f_{3n} = 2f_{3n} + f_{3n-1}}$$

The first term is even because it's 2 times an integer, and the second term is even because of the induction hypothesis. Therefore f_{3n+2} is even.

Proposition 2

Every set with n elements has 2^n subsets.

https://docs.google.com/document/d/1J3lQabY7P3NpvxL5iUamtdLutYGcbXXm7lF3ptALB-M/edit?usp=sharing

How would you rate the proposed proof of Proposition 2?

- The proposition itself is false, so the proof can't be right.
- The proposition is true, but the proof has a significant/fatal error or omission.
- The proposition is true, and the proof has no significant/fatal errors or omissions.

What we learned/what's next

- Proof by induction has a **framework** of three parts:
  - Proving that the proposition is true in the base case (by demonstration)
  - Assuming the inductive hypothesis (the proposition is true for n-1)
  - Proving the main proposition using the inductive hypothesis
- Proofs have to be critically analyzed in order to learn anything from them.

NEXT:

- Additional practice with setup and analysis of induction proofs
- Brief review of week 15 and final exams