Module 11A: Solutions to recurrence relations

MTH 225
18 November 2020

Agenda

- Review of Daily Prep
- Minilecture: Is it a solution?
- Activity: From sequence to solution
- Minilecture: Telescoping and iteration
- Wrap up and look ahead

Hold for Daily Prep debrief

What is a solution to a recurrence relation?

$$a_n = 3a_{n-1} + 4a_{n-2}, \; a_0 = 2, a_1 = 3$$

2, 3, 17, 63, 257, 1023, 4097, 16383, ...

A **solution** to this recurrence relation is a **closed formula** a(n) (domain = **N**) that produces the same sequence as the recurrence relation with these initial conditions

NOT a solution: a(n) = n + 2
2, 3, 4, 5, ...

$$a(n) = 4^n + (-1)^n$$

=4^C2+(-1)^C2

A	B	C	D
		n	a(n)
2		0	2
3		1	3
17		2	17
63		3	63
257		4	257
1023		5	1023
4097		6	4097
16383		7	16383
65537		8	65537
262143		9	262143
1048577		10	1048577

Q: Does this spreadsheet mean that this a(n) is a solution to the recurrence relation?

A: No. It only means that it produces the first 11 terms correctly.

We need to know that EVERY term is produced correctly. $\rightarrow$ Need more than a list of data!

How do we show that ALL terms are produced correctly?

A: You don't. You show instead that **the proposed formula satisfies the recurrence relation and the initial conditions.**

$$a_n = 3a_{n-1} + 4a_{n-2},\ a_0 = 2, a_1 = 3$$

$$\boxed{a(n-1) = 4^{n-1} + (-1)^{n-1}} \qquad \boxed{a(n-2) = 4^{n-2} + (-1)^{n-2}}$$

Jamboard: Working out the algebra to show this is a solution

Jamboard: From sequence to solution

Setup

Consider the sequence: 5, 7, 11, 19, 35, 67, 131, 259, …

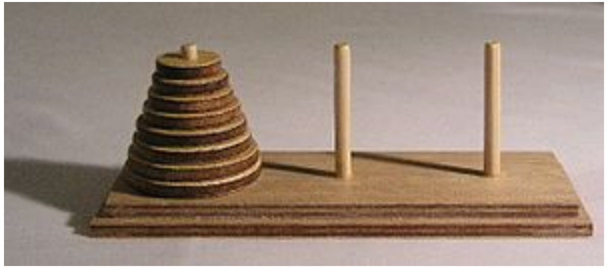
- Start indexing at 0, so $a_0 = 5$, $a_1 = 7$, $a_2 = 11$, etc.
- Come up with a recurrence relation that produces this sequence.
  - **Hint**: Look at the differences between the terms: $a_1 - a_0$, $a_2 - a_1$, $a_3 - a_2$, etc. Is there a pattern that's emerging that tells you what $a_n - a_{n-1}$ equals?
  - Use that pattern to get the recurrence relation for a_n.
- Then show that an = 3 + 2n+1 is a closed-formula solution to that recurrence relation.

Debrief

- The technique of looking at differences between terms is called **telescoping.** Look at differences → Find a pattern → turn into a recurrence relation.
- Where did the closed formula even come from? → What kind of sequence is the difference of terms?
- Related technique: **Iteration** → Writing out terms and then adding them up

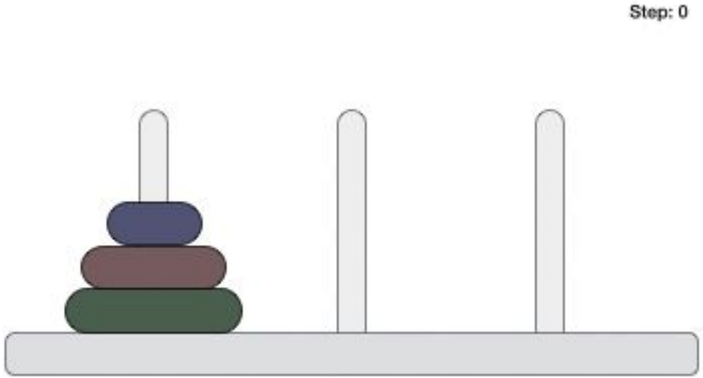
Example: Solve the recurrence relation $a_n = 2a_{n-1} + 1$, $a_1 = 1$

Where that last sequence is applied



Towers of Hanoi: Start with *n* discs as shown. Move the disks from the peg on the left to another peg, one at a time, without ever placing a large disc on top of a smaller one.

What’s the minimum number of moves required?



$$m_n = 2m_{n-1} + 1, m_1 = 1$$

$$\text{So } m(n) = 2^n - 1.$$

What we learned/what's next

- A **solution** to a recurrence relation is a closed formula that produces the same sequence (including initial conditions)
- Given a proposed solution, we can check it by seeing if it **algebraically satisfies the recurrence relation**
- *Coming up with* a solution can be done sometimes using **telescoping** or **iteration**

Next:

- Using more algebra to find solutions, if the recurrence relation is “linear”