Asymptotic Syzygies For Products of Projective Spaces

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ASYMPTOTIC SYZYGIES

- Asymptotic syzygies is the study of the algebraic Betti numbers of a variety as the positivity of the embedding increases.

$$X \hookrightarrow \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}$$

n dimensional smooth projective variety

sequence of very ample line bundles

ASYMPTOTIC SYZYGIES

$$X \hookrightarrow \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}$$

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- To this we associate:

$$S(X, L_d) = \bigoplus_{k \in \mathbb{Z}} H^0(X, kL_d)$$

- which we think of as a module over

$$S = \operatorname{Sym} H^0(L_d) \cong \mathbb{C}\left[x_0, x_1, \ldots, x_{r_d}\right]$$

ASYMPTOTIC SYZYGIES

$$0 \longleftarrow S(X, L_d) \longleftarrow \underbrace{\left[F_0 \longleftarrow F_1 \longleftarrow \cdots \qquad \cdots \longleftarrow F_{r_d}\right]} \longleftarrow 0$$

minimal graded
free resolution

$$\beta_{p,q}(X, L_d) = \#\left\{\begin{matrix}\text{minimal generators} \\ \text{of } F_p \text{ of degree } q\end{matrix}\right\} = \begin{matrix}\text{number of syzygies of degree } q \\ \text{and homological degree } p\end{matrix}$$

how do these vary
as a function of d?

ASYMPTOTIC SYZYGIES

$$X \hookrightarrow \mathbb{P}^r$$

- To this we associate:

$$I_X = \langle f_1, \ldots, f_t \rangle \subset S$$

homogeneous defining ideal of X

$$S_X = S/I_X$$

homogeneous coordinate ring of X

- and we are interested in studying the syzygies of

$$S_X^{(d)} = \bigoplus_{k \in \mathbb{Z}} S_{X_{kd}}$$

Veronese subring

ASYMPTOTIC SYZYGIES

$$0 \longleftarrow S_X^{(d)} \longleftarrow \Big[F_0 \longleftarrow F_1 \longleftarrow \cdots \qquad\qquad \cdots \longleftarrow F_{r_d} \Big] \longleftarrow 0$$

minimal graded
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$$\beta(X,d) = \#\left\{ \begin{matrix} \text{minimal generators} \\ \text{of } F_p \text{ of degree } q \end{matrix} \right\} = \begin{matrix} \text{number of syzygies of degree } q \\ \text{and homological degree } p \end{matrix}$$

how do these vary
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ASYMPTOTIC SYZYGIES

- It is often useful to place the Betti numbers into a table:

$$
\beta(X,d) \ = \ \begin{array}{c|cccccc}
 & 0 & 1 & 2 & \cdots & p & \cdots \\
\hline
0 & \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots & \cdots & \\
1 & \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \cdots & \\
q & & & & & \beta_{p,p+q} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

notice the wonky
change of coordinates

ASYMPTOTIC SYZYGIES

- Example: Consider the following resolution:

$$0 \longleftarrow S \longleftarrow \begin{array}{c} S(-2)^3 \\ \oplus \\ S(-3)^3 \end{array} \longleftarrow \begin{array}{c} S(-3)^2 \\ \oplus \\ S(-4)^6 \end{array} \longleftarrow S(-5)^3$$

	0	1	2	3
0	-	-	-	-
1	1	-	-	-
2	-	3	-	-
3	-	3	2	-
4	-	-	6	-
5	-	-	-	3

	0	1	2	3
0	1	-	-	-
1	-	3	2	-
2	-	3	6	3

without the change of coordinates the table will always be lower triangular

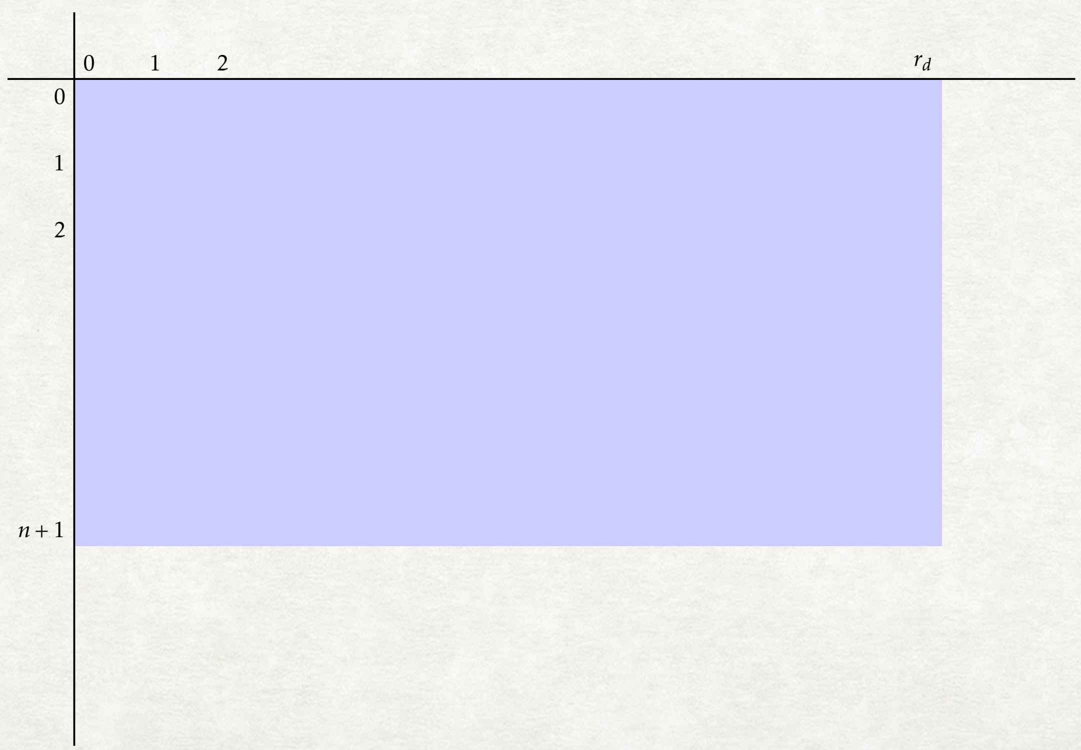
ASYMPTOTIC SYZYGIES

Lemma.

1. $\beta_{p,p+q}(X, L_d) = 0$ for all $p > r_d$.
2. $\beta_{p,p+q}(X, L_d) = 0$ for all $q > n+1$.

this requires a technical assumption on positivity

- So we can picture the Betti table as a box:



ASYMPTOTIC SYZYGIES

- Example: Consider the projective plane:

$$X = \mathbb{P}^2$$

$$S_X = \mathbb{C}[x_0, x_1, x_2]$$

$$S_X^{(d)} = \mathbb{C}\Big[\, x_0^d,\, x_0^{d-1}x_1,\, \dots,\, x_1 x_2^{d-1},\, x_2^d \,\Big]$$

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- The Betti table is then…

$\beta\left(\mathbb{P}^2, 4\right) =$

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	-	-	-	-	-	-	-	-	-	-	-	-
1	-	75	536	1947	4488	7095	7920	6237	3344	1089	120	-	-
2	-	-	-	-	-	-	-	-	-	-	55	24	3

syzygydata.com!

FIRST RESULTS - CURVES

$$X \hookrightarrow \mathbb{P}H^0(L_d) \cong \mathbb{P}^{r_d}$$

smooth projective curve
of genus g

line bundle of
degree d

- For a curve positivity is analogous to the degree.

Theorem (Castelnuovo et al.).

1. If $d \geq 2g+1$ then L_d is defines an embedding into $\mathbb{P}^{r_d}$.
2. If $d \geq 2g+2$ then I_X is generated by quadrics.

FIRST RESULTS - CURVES

- Part (1) tells us the Betti table of X eventually looks like:

	0	1	2	$\cdots$	p	$\cdots$	r_d
0	1	-	-	$\cdots$	-	$\cdots$	
1	-	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{p,p+1}$	$\cdots$	
2	-	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$	$\beta_{p,p+2}$	$\cdots$	

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- Part (2) tells us the Betti table of X eventually looks like:

	0	1	2	$\cdots$	p	$\cdots$	r_d
0	1	-	-	$\cdots$	-	$\cdots$	
1	-	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$	$\beta_{p,p+1}$	$\cdots$	
2	-	-	$\beta_{2,4}$	$\cdots$	$\beta_{p,p+2}$	$\cdots$	

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Theorem (Green, 1984)**.** If X is a smooth curve and $\deg(L_d) = d$ then

$$\beta_{p,p+2}(X, L_d) = 0 \qquad \text{for all} \qquad p \in [0, d-(2g+1)].$$

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percentage of non-zero
syzygies in 2nd row

FIRST RESULTS - AMPLE GROWTH

Theorem (Ein-Lazarsfeld, 2012). Let $n \geq 2$ and fix $1 \leq q \leq n$. If $L_{d+1} - L_d$ is constant and ample then there exist functions $P_-(d)$ and $P_+(d)$ such that

$$\beta_{p,p+q}(X, L_d) \neq 0 \qquad \text{for all} \quad p \in [P_-(d), P_+(d)],$$

where

(a) $P_-(d) = O(d^q)$,

(b) $P_+(d) = r_d - O\left(d^{n+1}\right)$, and

(c) $\displaystyle\lim_{d\to\infty} \frac{P_+(d) - P_-(d)}{r_d} = 1.$

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as functions of d the endpoints have specific asymptotic behavior

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as functions of d the endpoints have specific asymptotic behavior

asymptotically syzygies occur in every possible homological degree

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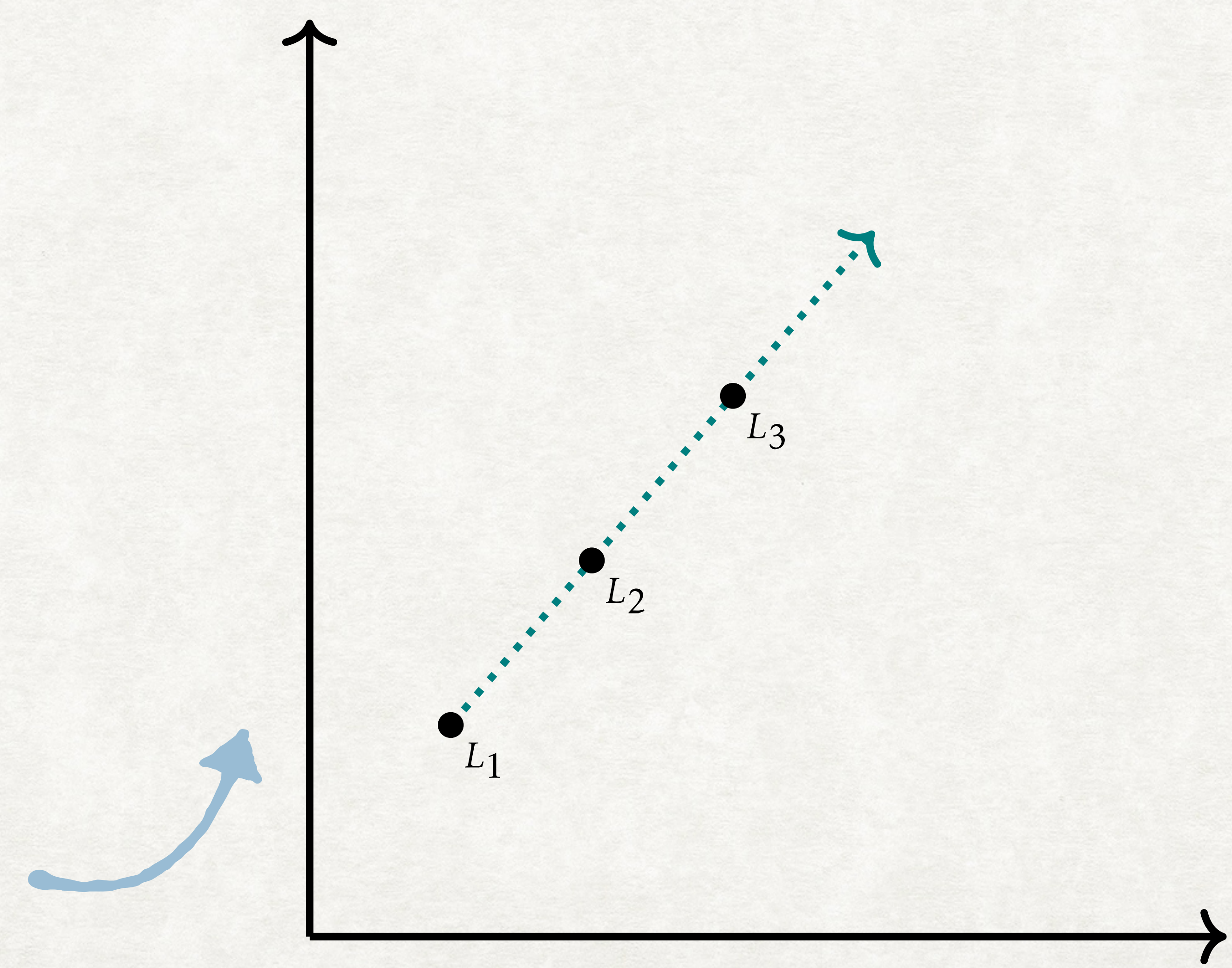

NEW RESULTS - SEMI-AMPLE GROWTH

Theorem (Juliette Bruce). Fix $1 \leq q \leq n+m$. There exist functions $P_-(d_1, d_2)$ and $P_+(d_1, d_2)$ such that

$$\beta_{p,p+q}(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{O}(d_1, d_2)) \neq 0 \qquad \text{for all} \quad p \in [P_-(d_1,d_2),\, P_+(d_1,d_2)],$$

where

(a) $P_-(d_1, d_2) \;=\; \min \left\{ O\left(d_1^a d_2^b\right) \;\middle|\; \begin{matrix} a+b=q \\ 0 \leq a \leq n \\ 0 \leq b \leq m \end{matrix} \right\}$

(b) $P_+(d_1, d_2) \;=\; r_{d_1,d_2} - \min \left\{ O\left(d_1^{n-a} d_2^{m-b}\right) \;\middle|\; \begin{matrix} a+b=q \\ 0 \leq a \leq n \\ 0 \leq b \leq m \end{matrix} \right\}.$

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Theorem (Juliette Bruce). Fix $1 \leq q \leq n+m$. There exist functions $P_{-}(d_1,d_2)$ and $P_{+}(d_1,d_2)$ such that

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there is some interval where we are guaranteed non-zero syzygies

where

(a) $P_{-}(d_1,d_2) \;=\; \min\left\{ O\left(d_1^a d_2^b\right) \;\middle|\; \begin{array}{c} a+b=q \\ 0 \leq a \leq n \\ 0 \leq b \leq m \end{array} \right\}$

(b) $P_{+}(d_1,d_2) \;=\; r_{d_1,d_2} - \min\left\{ O\left(d_1^{n-a} d_2^{m-b}\right) \;\middle|\; \begin{array}{c} a+b=q \\ 0 \leq a \leq n \\ 0 \leq b \leq m \end{array} \right\}.$

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$$\text{(a)} \quad P_{-}(d_1, d_2) \;=\; \min \left\{ O\left(d_1^a d_2^b\right) \;\middle|\; \begin{matrix} a+b=q \\ 0 \leq a \leq n \\ 0 \leq b \leq m \end{matrix} \right\}$$

$$\text{(b)} \quad P_{+}(d_1, d_2) \;=\; r_{d_1, d_2} - \min \left\{ O\left(d_1^{n-a} d_2^{m-b}\right) \;\middle|\; \begin{matrix} a+b=q \\ 0 \leq a \leq n \\ 0 \leq b \leq m \end{matrix} \right\}.$$

as functions of two variables the endpoints have specific asymptotic behavior

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as functions of two variables the endpoints have specific asymptotic behavior

where's part (c)?

NEW RESULTS - SEMI-AMPLE GROWTH

$$\mathbb{P}^n \times \mathbb{P}^m \xhookrightarrow{\mathcal{O}(d_1,d_2)} \mathbb{P}^{r_d}$$

- To this we associate:

$$S = \mathbb{C}[x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m]$$

which we consider as a bi-graded ring

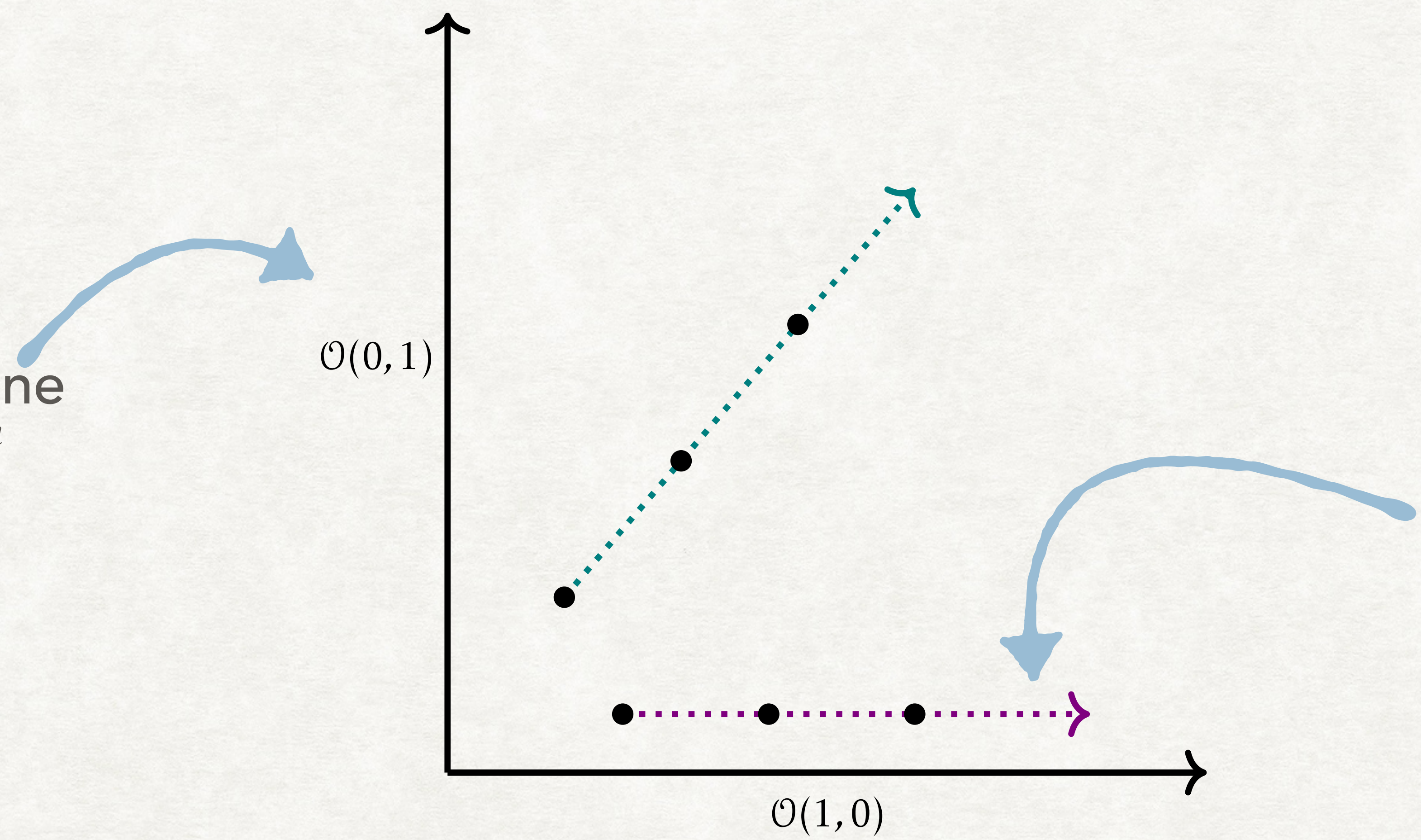
- and are interested in the syzygies of

$$S^{(d_1,d_2)} = \bigoplus_{k \in \mathbb{Z}} S_{k(d_1,d_2)}$$

bi-graded Veronese ring

NEW RESULTS - SEMI-AMPLE GROWTH

- My result does not require an assumption of ample growth.



Definition. A line bundle L is semi-ample if $|kL|$ is base point free for some k.

NEW RESULTS - SEMI-AMPLE GROWTH

Corollary (Juliette Bruce). For $\mathbb{P}^n \times \mathbb{P}^m$ and an index $1 \leq q \leq n+m$

$$\lim_{d_1 \to \infty} \frac{P_+(d_1, d_2) \; - \; P_-(d_1, d_2)}{r_{d_1, d_2}} \;\; \geq \;\; 1 \; - \sum_{\substack{i+j=q \\ 0 \leq i \leq n_1 \\ 0 \leq j \leq n_2}} \left(\frac{C_{i,j}}{d_1^i d_2^j} + \frac{D_{i,j}}{d_1^{n_1 - i} d_2^{n_2 - j}} \right) \; + \; O\left(\begin{matrix} \text{lower order} \\ \text{terms} \end{matrix} \right).$$

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the percent of possible degrees
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the main specific
asymptotic behavior

the percent of possible degrees
with non-zero syzygies

NEW RESULTS - SEMI-AMPLE GROWTH

Conjecture. For $\mathbb{P}^n \times \mathbb{P}^m$ if d_2 is fixed and $q = n + m$ then

$$\lim_{d_1 \to \infty} \frac{P_+(d_1, d_2) \; - \; P_-(d_1, d_2)}{r_{d_1, d_2}} \;=\; 1 - \frac{m}{m + d_2}.$$

NEW RESULTS - SEMI-AMPLE GROWTH

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last interesting q

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the percent of possible degrees
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neither 0 nor 1

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the percent of possible degrees
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neither 0 nor 1

- Syzygies in the setting of semi-ample growth exhibit behavior that is substantially different from previous cases!

- curves (Green)
- ample growth (Ein-Lazarsfeld)

APPROACH OF PROOF

- The proof generalizes of the monomial methods of Ein, Erman, Lazarsfeld to explicitly produce non-trivial syzygies.

this requires an Artinian reduction, but there are no monomial regular sequences

- If n = 2 and m = 4 the regular sequence we work with is:

$$\left\langle \begin{array}{c} x_0^{d_1} y_0^{d_2} \\ x_0^{d_1} y_1^{d_2} + x_1^{d_1} y_0^{d_2} \\ x_0^{d_1} y_2^{d_2} + x_1^{d_1} y_1^{d_2} + x_2^{d_1} y_0^{d_2} \\ x_0^{d_1} y_3^{d_2} + x_1^{d_1} y_2^{d_2} + x_2^{d_1} y_1^{d_2} \\ x_0^{d_1} y_4^{d_2} + x_1^{d_1} y_3^{d_2} + x_2^{d_1} y_2^{d_2} \\ x_1^{d_1} y_4^{d_2} + x_2^{d_1} y_3^{d_2} \\ x_2^{d_1} y_4^{d_2} \end{array} \right\rangle$$

$$x_0^{d_1-1} x_1^{d_1-1} x_2^{3d_1+2} y_0^{3d_2-1} y_1^{d_2-1} y_2^{d_2-1} y_3^{3}$$

is not in this ideal, while

$$x_0^{d_1-1} x_1^{d_1-1} x_2^{3d_1+2} y_0^{3d_2} y_1^{d_2-1} y_2^{d_2-1} y_3^{3}$$

is in this ideal…

APPROACH OF PROOF

- I exploit the fact that this regular sequence has a lot of symmetries:

i.e. homogenous with respect to a grading

the sum of the indices is constant

$$\left\langle \begin{array}{c} x_0^{d_1} y_0^{d_2} \\ x_0^{d_1} y_1^{d_2} + x_1^{d_1} y_0^{d_2} \\ x_0^{d_1} y_2^{d_2} + x_1^{d_1} y_1^{d_2} + x_2^{d_1} y_0^{d_2} \\ x_0^{d_1} y_3^{d_2} + x_1^{d_1} y_2^{d_2} + x_2^{d_1} y_1^{d_2} \\ x_0^{d_1} y_4^{d_2} + x_1^{d_1} y_3^{d_2} + x_2^{d_1} y_2^{d_2} \\ x_1^{d_1} y_4^{d_2} + x_2^{d_1} y_3^{d_2} \\ x_2^{d_1} y_4^{d_2} \end{array} \right\rangle$$

the power on each x is 0 mod d_1

- These symmetries enable me to use spectral sequence arguments to deeply understand this ideal.