Wiener process and martingales

Wiener process

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Here goes the plot!

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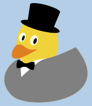
Definition 

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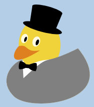
Definition 

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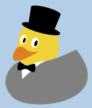
Definition 

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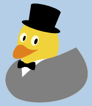
Definition 

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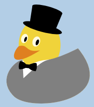
Definition 

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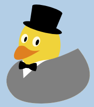
Definition 

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Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

The main trick to study properties:

$$\text{Future value} = \text{Known value} + \text{Unpredictable change}$$

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Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

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$$= \mathbb{P}(Z > -0.5) = \mathbb{P}(Z < 0.5) = F(0.5) \approx 0.69.$$

More gentlemen's agreements

On slides we will follow these agreements:

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- $F(u)$ denotes the standard normal distribution function, $F(u) = \mathbb{P}(Z \leq u)$.

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The increments $W_6 - W_4$, $W_4 - W_3$, $W_{2.5} - W_1$ are independent.

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$W(t_k) - W(s_k)$ are independent.

Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \leq s_{j+1}$.

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$$\operatorname{Cov}(W_7, W_3) = 3.$$

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Stochastic process $(S_t, t \geq 0)$ that may be written as

$$S_t = S_0 \exp(aW_t + bt),$$

is called geometric brownian motion.

Two plots

here will be the plots of BM with drift and geometric BM

BM with drift and scaling

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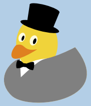
How these are obtained? 

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$$\mathbb{E}(\exp(aZ)) = \int_{-\infty}^{+\infty} \exp(az) f(z)\, dz =$$

$$= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz$$

Moment generating function

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$$M_X(a) = \mathbb{E}(\exp(aX)).$$

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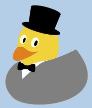
Definition 

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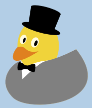
Definition 

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- $M_{W_t}(a) = \exp\left(a^2 t/2\right)$ for a Wiener process W_t.

Why may we need MGF?

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$$\begin{aligned} M''(0) &= \mathbb{E}(X^2) \\ M'''(0) &= \mathbb{E}(X^3) \\ &\vdots \\ M^{(k)}(0) &= \mathbb{E}(X^k) \end{aligned}$$

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How to model this mathematically?

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Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

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Maria knows more: $\mathcal{F}_J \subset \mathcal{F}_M$.

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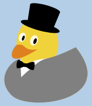
Definition 

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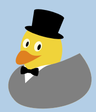
Definition 

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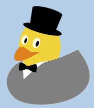
Definition 

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Theorem 

The random variable Z is measurable with respect to $\sigma(X, Y)$ if and only if Z is a deterministic function of X and Y.

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Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y.

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a random variable.

Notation

- $\mathbb{E}(Z \mid \mathcal{F})$:
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- $\mathbb{E}(Z \mid \sigma(X, Y))$ or $\mathbb{E}(Z \mid X, Y)$:
  for σ-algebra generated by X and Y.

When we may omit conditioning?

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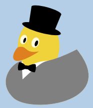
Conditional variance

Definition

The conditional variance $\mathrm{Var}(Z \mid \mathcal{F})$ is the conditional expected value of the squared error of the best prediction,

Conditional variance

Definition



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Theorem

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Properties of conditional variance

- Irrelevant information may be omitted:
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Martingales

Martingales: short plan

- Filtration models the information acquisition.

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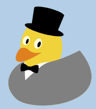
Definition 

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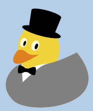
Definition 

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Reminder: Sigma-algebra $\mathcal{F}_t$ is the collection of events.

Natural filtration

Natural filtration

Definition

The filtration $(\mathcal{F}_t, t \geq 0)$ is called a natural filtration of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

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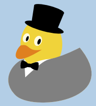
Definition



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Let $(\mathcal{F}_t)$ be a natural filtration of a Wiener process (W_t).

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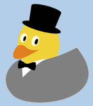
Definition 

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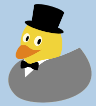

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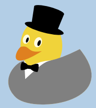
Definition 

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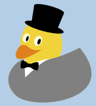

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Martingale

Definition

Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a martingale with respect to the filtration $(\mathcal{F}_t)$.

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Usually we consider natural filtration $(\mathcal{F}_t)$ of the process (M_t).

Simple examples

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$$\mathbb{E}(W_t \mid \mathcal{F}_s) = \mathbb{E}(W_s + (W_t - W_s) \mid \mathcal{F}_s) = W_s + \mathbb{E}(W_t - W_s) = W_s.$$

More examples

Theorem

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\mathbb{E}(W_t^2 - t \mid \mathcal{F}_s) &= \mathbb{E}((W_s + (W_t - W_s))^2 \mid \mathcal{F}_s) - t = \\
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=& W_s^2 + (t - s) + 2W_s \cdot 0 - t = W_s^2 - s
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This martingale is very useful in Black and Scholes model.

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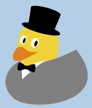
Theorem 

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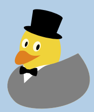
Theorem 

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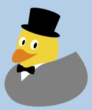
Theorem 

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- Martingales related to Wiener process, random walk.