

Itô's lemma

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Itô's lemma: short plan

- Light version for functions of **time** and **Wiener process**.

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- Check the **martingale property** with Itô's lemma.

Itô's lemma: short plan

- Light version for functions of **time** and **Wiener process**.
- Check the **martingale property** with Itô's lemma.
- More **general** version.

Itô's lemma: light version

Informal theorem



If $Y_t = f(W_t, t)$ then it may be written as

$$Y_t = Y_0 + \int_0^t f'_W dW_u + \int_0^t \left(f'_t + \frac{1}{2} f''_{WW} \right) du.$$

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$$dY_t = f'_W dW_t + f'_t dt + \frac{1}{2} f''_{WW} dt.$$

Exercise

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$$dY_t = 3W_t^2 t^4 dW_t + 4W_t^3 t^3 dt + \frac{1}{2}6W_t t^4 dt =$$

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$$Y_t = 0 + \int_0^t 3W_u^2 u^4 dW_u + \int_0^t (4W_u^3 u^3 + 3W_u u^4) du.$$

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Check whether the process $Y_t = W_t^4 - t^2$ is a martingale .

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$$Y_t = 0 + \int_0^t 4W_u^3 dW_u + \int_0^t (6W_u^2 - 2u) du.$$

(Y_t) is **not** a martingale!

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$$W_t^2 = 0 + 2 \int_0^t W_u dW_u + t. \quad \rightarrow \quad \int_0^t W_u dW_u = \frac{1}{2}(W_t^2 - t).$$

Itô's lemma: general version

Informal theorem



If $Y_t = f(X_t, t)$ where (X_t) is an Itô process then Y_t may be written in the short form as

$$dY_t = f'_X dX_t + f'_t dt + \frac{1}{2} f''_{XX} (dX_t)^2,$$

where $(dX_t)^2$ is calculated using symbolic rules

$$dt \cdot dW_t = 0, \quad dt \cdot dt = 0, \quad dW_t \cdot dW_t = dt.$$

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Consider $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $Y_t = S_t^2$.

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$$\begin{aligned} dY_t &= 2S_t dS_t + 0 dt + \frac{1}{2} 2(dS_t)^2 = \\ &= 2S_t(\mu S_t dt + \sigma S_t dW_t) + 1 \cdot (\mu S_t dt + \sigma S_t dW_t)^2 = \end{aligned}$$

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$$Y_t = S_0^2 + \int_0^t 2\sigma S_u^2 dW_u + \int_0^t (2\mu S_u^2 + \sigma^2 S_u^2) du.$$

Itô's lemma: a way to memorize

Informal theorem



If $Y_t = f(X_t, Z_t, t)$ where (X_t) and (Z_t) are Itô processes then the short form of Y_t may be obtained in two steps:

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1. Calculate the **second order Taylor expansion** of f .

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$$dt \cdot dW_t = 0, \quad dt \cdot dt = 0, \quad dW_t \cdot dW_t = dt.$$

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$$dt \cdot dW_t = 0, dt \cdot dt = 0, dW_t \cdot dW_t = dt.$$

Reminder: dW_t, dY_t, dX_t, dZ_t **do not exist!**

It's only a **quick way** to find the full form.

Itô's lemma: summary

- Basic tool to study stochastic integrals.

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- Basic tool to study stochastic integrals.
- Easy to check whether the process is a martingale.
- Easily written in short form:

$$dt \cdot dW_t = 0, \quad dt \cdot dt = 0, \quad dW_t \cdot dW_t = dt.$$

Black and Scholes model

Black and Scholes model: short plan

- Assumptions of the model.

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- The main question.

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- Assumptions of the model.
- The main question.
- Solving the price stochastic differential equation.

BS-model: assumptions

- **Unique** share type is traded. The share price S_t satisfies

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- **No arbitrage** opportunities.
- The parameters r, σ, μ **are known**.

The Question

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What is the fair price John should pay at $t = 0$?

SDE for the price

Stochastic differential equation for the price:

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In full form:

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u.$$

Solving the SDE

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$$\begin{aligned} dY_t &= \frac{1}{S_t} dS_t + 0 dt + \frac{1}{2} \left(\frac{-1}{S_t^2} \right) (dS_t)^2 = \\ &= \mu dt + \sigma dW_t - \frac{1}{2} \sigma^2 dt. \end{aligned}$$

Solving the SDE...

The log-price $Y_t = \ln S_t$ in **short** form,

$$dY_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$

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Without integrals,

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t.$$

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The log-price $Y_t = \ln S_t$ in **short** form,

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In **full** form,

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Without integrals,

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t.$$

Finally,

$$S_t = S_0 \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

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- **Risk free rate** r is constant.
- There are **no arbitrage** opportunities.

Girsanov theorem

Girsanov theorem: short plan

- The **wrong answer** to the pricing problem.

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- Idea of **alternative probability**.

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- The **wrong answer** to the pricing problem.
- Idea of **alternative probability**.
- **Girsanov theorem**.

The wrong answer

Wrong intuition



The future payoff X_T is random, we just need to calculate the expected payoff given all available information,

$$X_0 \stackrel{???}{=} \mathbb{E}(X_T \mid \mathcal{F}_0).$$

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Wrong intuition



The future payoff X_T is random, we just need to calculate the expected payoff given all available information,

$$X_0 \stackrel{???}{=} \mathbb{E}(X_T \mid \mathcal{F}_0).$$

This is true for a martingale, but the claim price is **not a martingale**.

Alternative probability

A	$X = -2$	$X = 0$	$X = 4$
$\mathbb{P}(A)$	0.3	0.4	0.3
$\mathbb{P}^*(A)$	0.4	0.1	0.5

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$$\mathbb{E}(X) = 0.6$$

$$\mathbb{E}^*(X) = 1.2$$

Idea

We will introduce a **new probability** \mathbb{P}^* in the Black and Scholes model to simplify the calculation of prices.

Girsanov theorem

Theorem

If (W_t) is a Wiener process under probability \mathbb{P} and $W_t^* = b \cdot t + W_t$, then there is a probability \mathbb{P}^* such that (W_t^*) is a Wiener process under \mathbb{P}^* .

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$$\mathbb{E}(W_t) = 0, \quad \mathbb{E}(W_t^*) = b \cdot t,$$

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Theorem

If (W_t) is a Wiener process under probability \mathbb{P} and $W_t^* = b \cdot t + W_t$, then there is a probability \mathbb{P}^* such that (W_t^*) is a Wiener process under \mathbb{P}^* .

$$\mathbb{E}(W_t) = 0, \quad \mathbb{E}(W_t^*) = b \cdot t, \quad \mathbb{E}^*(W_t^*) = 0.$$

Girsanov theorem in BS model

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In the Black and Scholes model there is an alternative probability \mathbb{P}^* such that (W_t^*) is a Wiener process under \mathbb{P}^* and

$$S_t = S_0 \exp \left(\left(\textcolor{brown}{r} - \frac{\sigma^2}{2} \right) t + \sigma W_t^* \right).$$

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Old formula is still valid,

$$S_t = S_0 \exp \left(\left(\textcolor{brown}{\mu} - \frac{\sigma^2}{2} \right) t + \sigma W_t \right),$$

where (W_t) is a Wiener process under probability \mathbb{P} .

Link between (W_t) and (W_t^*)

Equivalent formula for share price means that

$$\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t = \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t^*$$

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In short form

$$(\mu - r)dt + \sigma dW_t = \sigma dW_t^*.$$

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- (W_t^*) is a Wiener process under **artificial** probability \mathbb{P}^* ,

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