Wiener process and martingales

The goal: price an option in the framework of Black and Scholes model.

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- Problem solving and computer simulations.

Here goes the plot!

Definition



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Notation remark:

• $(X_t, t \ge 0)$ or (X_t) — collection of random variables;

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Definition 6



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1.
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- 1. $W_0 = 0$.
- Increments $W_t W_s$ are normally distributed $\mathcal{N}(0; t-s)$.
- Increment $W_t W_s$ is independent of the past values $(W_u, u \leq s)$.

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Brownian motion if

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Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

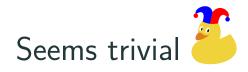
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Future value = Known value + Unpredictable change

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$$W_t = W_s + (W_t - W_s)$$

Exercise
$$\ref{eq:width}$$
. Calculate $\mathbb{P}(W_{10} > 2 \mid W_6 = 3)$.

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$$\mathbb{P}(W_{10} - W_6 > -1) = \mathbb{P}\left(\frac{W_{10} - W_6}{2} > -\frac{1}{2}\right) =$$
$$= \mathbb{P}(Z > -0.5) = \mathbb{P}(Z < 0.5) = F(0.5) \approx 0.69.$$

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- s and t denote two arbitrary time moments with $0 \le s \le t$.
- (W_t) denotes a Wiener process.
- Z denotes a standard normal random variable, $Z \sim \mathcal{N}(0; 1)$.
- F(u) denotes the standard normal distribution function, $F(u) = \mathbb{P}(Z \le u)$.

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 W_6-W_4 is independent of W_4-W_3 , $W_{2.5}-W_1$.

The increments W_6-W_4 , W_4-W_3 , $W_{2.5}-W_1$ are independent.

Independence of increments: full glory

If the time intervals $[s_1,t_1]$, $[s_2,t_2]$, ..., $[s_k,t_k]$ are non overlapping, Here will be a small picture

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If the time intervals $[s_1,t_1]$, $[s_2,t_2]$, ..., $[s_k,t_k]$ are non overlapping, Here will be a small picture then the increments $W(t_1)-W(s_1)$, $W(t_2)-W(s_2)$, ..., $W(t_k)-W(s_k)$ are independent.

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Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \leq s_{j+1}$.

Exercise \bullet . Find $\mathbb{E}(W_t)$, $Var(W_t)$, $Cov(W_s, W_t)$.

$$\mathbb{E}(W_t) = \mathbb{E}(W_t - W_0) = 0$$

Exercise $\ref{eq:lemma:eq:lem$

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For $t \geq s$:

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 $Cov(W_7, W_3) = 3.$

Two friends

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Stochastic process $(X_t, t \ge 0)$ that may be written as

$$X_t = aW_t + bt,$$

is called brownian motion with drift and scaling.

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Stochastic process $(S_t, t \ge 0)$ that may be written as

$$S_t = S_0 \exp(aW_t + bt),$$

is called geometric brownian motion.



here will be the plots of BM with drift and geometric BM

BM with drift and scaling

Exercise \blacksquare . Find $\mathbb{E}(5W_t + 6t)$ and $\operatorname{Var}(5W_t + 6t)$.

BM with drift and scaling

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BM with drift and scaling

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$$\mathbb{E}(5W_t + 6t) = 0 + 6t = 6t$$

$$Var(5W_t + 6t) = Var(5W_t) = 25t$$

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$$\mathbb{E}(\exp(aZ)) = \int_{-\infty}^{+\infty} \exp(az)f(z) dz =$$

$$= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp(-z^2/2) dz$$

Moment generating function

Definition 🕹

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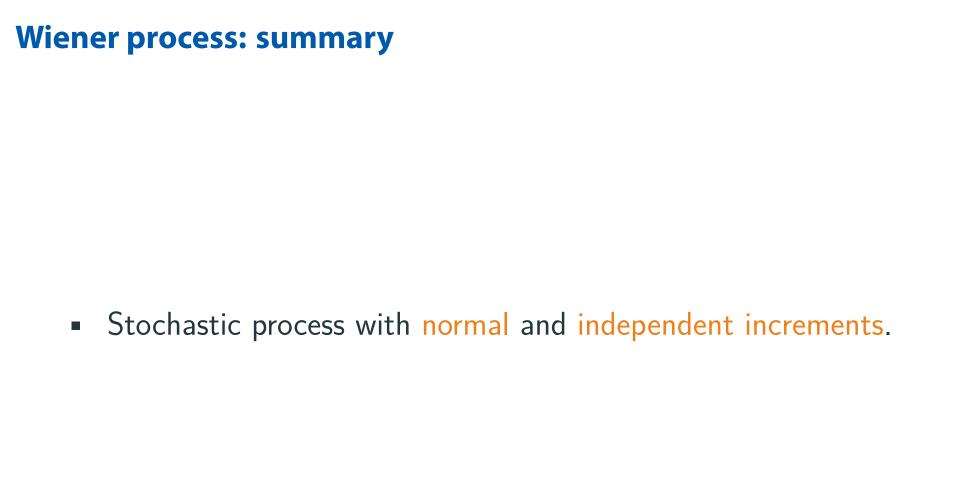
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$$M^{(k)}(0) = \mathbb{E}(X^k)$$



Wiener process: summary

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Conditional expectation

Conditional expectation: short plan

Modeling information using sigma-algebras;

Conditional expectation: short plan

- Modeling information using sigma-algebras;
- Properties of conditional expected value;

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- Modeling information using sigma-algebras;
- Properties of conditional expected value;
- Conditional variance.



John knows the value of X.

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Maria knows the value of X and Y.

John knows the value of X.

Maria knows the value of X and Y.

Maria knows more!

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How to model this mathematically?

Informal definition



Sigma-algebra (σ -algebra) generated by random variables X and Y is the collection of all events that can be stated in terms of these random variables.

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Example

The sigma-algebra $\sigma(X,Y)$ contains the events $\{X<5\}$, $\{X>2Y\}$, $\{\sin Y>\cos X\}$, ...



John knows the value of X, $\mathcal{F}_J = \sigma(X)$.

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Maria knows the value of X and Y, $\mathcal{F}_M = \sigma(X,Y)$

Maria knows more: $\mathcal{F}_J \subset \mathcal{F}_M$.

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Informal theorem



The random variable Z is measurable with respect to $\sigma(X,Y)$ if and only if Z is a deterministic function of X and Y.

Best prediction

Informal definition



The best prediction of a random variable Y given σ -algebra \mathcal{F} is called conditional expected value $\mathbb{E}(Z \mid \mathcal{F})$.

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Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y.

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a random variable.

Notation

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Notation

- $\mathbb{E}(Z \mid \mathcal{F})$: for a general σ -algebra \mathcal{F} ;
- $\mathbb{E}(Z \mid \sigma(X, Y))$ or $\mathbb{E}(Z \mid X, Y)$: for σ -algebra generated by X and Y.

When we may omit conditioning?

• If Z is independend of X and Y then $\mathbb{E}(Z\mid X,Y)=\mathbb{E}(Z)$: If I know nothing useful about Z then I can drop my information.

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- If Z is independend of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$: If I know nothing useful about Z then I can drop my information.
- $\mathbb{E}(\mathbb{E}(Z\mid\mathcal{F}))=\mathbb{E}(Z)$: The average of best guess is the average of predicted variable.

$$\mathbb{E}(Z \mid \mathcal{F}) = Z;$$

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$$\mathbb{E}(2\exp(5W_t) \mid W_t) = 2\exp(5W_t);$$

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$$\mathbb{E}(2ZR + Z^2 \mid \mathcal{F}) = 2Z\mathbb{E}(R \mid \mathcal{F}) + Z^2.$$

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Theorem



$$\operatorname{Var}(Z \mid \mathcal{F}) = \mathbb{E}(Z^2 \mid \mathcal{F}) - (\mathbb{E}(Z \mid \mathcal{F}))^2.$$

• Irrelevant information may be omitted:

If Z is independent of \mathcal{F} then $\mathbb{E}(Z \mid \mathcal{F}) = \mathbb{E}(Z)$ and $\mathrm{Var}(Z \mid \mathcal{F}) = \mathrm{Var}(Z)$.

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$$Var(2\exp(5W_t) \mid W_t) = 0;$$

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- If Z is known (measurable with respect to \mathcal{F}), then we may treat Z like a constant:

$$Var(2\exp(5W_t) \mid W_t) = 0;$$

$$Var(Z^3 + 3ZR \mid \mathcal{F}) = 0 + (3Z)^2 Var(R \mid \mathcal{F}).$$

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- Conditional variance $\mathrm{Var}(Z\mid X,Y)$ is the conditional expected value of the squared error of the best prediction.

Martingales

Martingales: short plan

• Filtration models the information acquisition.

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- Filtration models the information acquisition.
- Definition of a martingale.

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- Filtration models the information acquisition.
- Definition of a martingale.
- Examples of martingales.

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Reminder: Sigma-algebra \mathcal{F}_t is the collection of events.

Definition



The filtration $(\mathcal{F}_t, t \geq 0)$ is called a natural filtration of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

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Examples

$$\{W_2 < 5\} \in \mathcal{F}_2$$
,

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$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

Examples

$$\{W_2 < 5\} \in \mathcal{F}_2, \{W_2 > W_5\} \in \mathcal{F}_6,$$

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Martingale

Definition



Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a martingale with respect to the filtration (\mathcal{F}_t) .

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Usually we consider natural filtration (\mathcal{F}_t) of the process (M_t) .

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This martingale is very useful in Black and Scholes model.

Martingales in discrete time

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is completely equivalent to

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- Martingales related to Wiener process, random walk.