Itô integral

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Itô integral: short plan

- Intuitive definition.

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- Examples.

Itô integral

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$$\text{Net cash flow} = \int_0^t \text{Quantity}_u \, d\text{Price}_u$$

Simple deterministic example

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$\int_2^8 7\,dt = -14 + 56 = 42$.

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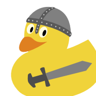
Let's calculate $\int_2^8 5\,dW_u$ 

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$\int_2^8 5\,dW_u = -5W_2 + 5W_8$.

More gentlemen's agreement

$$I_t = \int_0^t (\text{something}_u)\, dW_u :$$

t — the upper limit of integration;

u — time variable with range from 0 to t;

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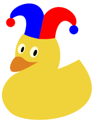
$$\int_0^t W_u \, dW_u =$$  ?

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If you buy and sell Wiener process you can have negative cash flow!

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$$\int_0^t 1\, dW_u = W_t$$

$$\int_0^t W_t\, dW_u = \frac{W_t^2 - t}{2}$$

$$\int_0^t \exp\left(aW_u - \frac{1}{2}a^2 u\right) dW_u = \frac{1}{a}\left(\exp\left(aW_t - \frac{1}{2}a^2 t\right) - 1\right)$$

Itô integral: summary

- Total net cash flow of a strategy.

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- Total net cash flow of a strategy.
- Rarely can be computed explicitely.
- Old rules of integration do not apply.

Itô integral properties

Itô integral properties: short plan

- Common properties with Riemann integral.

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- Common properties with Riemann integral.
- Zero expected value.

Itô integral properties: short plan

- Common properties with Riemann integral.
- Zero expected value.
- Itô isometry.

Common properties

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Intuition: we buy and sell Wiener process, hence, expected net cash flow should be zero.

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Informal theorem 

For any reasonable process (X_t) measurable with respect to the natural filtration $(\mathcal{F}_t)$ of the process (W_t)

$$\mathbb{E}\left(\int_0^t X_u \, dW_u\right) = 0.$$

Itô isometry

Informal theorem 

For any reasonable process (X_t) measurable with respect to the natural filtration $(\mathcal{F}_t)$ of the process (W_t)

$$\mathrm{Var}\left(\int_0^t X_u\,dW_u\right) = \int_0^t \mathbb{E}(X_u^2)\,du.$$

Exercise

Find $\mathbb{E}(I_t)$ and $\mathrm{Var}(I_t)$ for $I_t = \int_0^t W_u^2 \, dW_u$ 

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- Some common properties with Riemann integral.

Itô process

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- Short and full form notation.

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- Definition.
- Short and full form notation.
- When Itô process is a martingale?

Itô process

Definition

Stochastic process $(Y_t, t \geq 0)$ is called Itô process if it can be written in the form

$$Y_t = Y_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du,$$

where Y_0 is a constant.

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where Y_0 is a constant.

A wide class of continuous stochastic processes that behave locally like a Wiener process with drift.

Itô integral is a martingale

Informal theorem 

Itô process (Y_t) is a martingale if and only if it has only Itô integral in the representation

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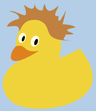
The best guess of a future value of an Itô integral is its current value:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = Y_t \text{ for } s \leq t.$$

Expected value of Itô process

Informal theorem 

For any reasonable process (B_t)

$$\mathbb{E}\left(\int_0^t B_u \, du\right) = \int_0^t \mathbb{E}(B_u) \, du.$$

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Informal theorem

If (Y_t) is an Itô process with $Y_t = Y_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du$, then

$$\mathbb{E}(Y_t) = Y_0 + \int_0^t \mathbb{E}(B_u) \, du.$$

Short form notation

Full form:

$$Y_t = Y_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du.$$

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dW_t and dY_t have no meaning!

Short form in simulations

We need to simulate a path of

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In simulations:

$$Y_{t+\Delta} - Y_t \approx A_t \cdot (W_{t+\Delta} - W_t) + B_t \cdot \Delta,$$

where $W_{t+\Delta} - W_t \sim \mathcal{N}(0; \Delta)$.

Short form: examples

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$$dY_t = \cos(W_t) dt$$

Short form: examples

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$$dY_t = \cos(W_t) dt \quad Y_t = Y_0 + \int_0^t \cos(W_u) \, du.$$

Short form: examples

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$$dY_t = \cos(W_t) dt \quad Y_t = Y_0 + \int_0^t \cos(W_u) \, du.$$

Informal theorem 

Itô process (Y_t) is a martingale if and only if

$$dY_t = A_t \, dW_t.$$

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- Will be a martingale without Riemann integral.
- Often written using short form with dt and dW_t.