

Wiener process and martingales

Wiener process

Stochastic calculus course

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- Problem solving and computer simulations.

Wiener process

Here goes the plot!

Stochastic process

Definition

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- $(X_t, t \geq 0)$ or (X_t) — collection of random variables;

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- $(X_t, t \geq 0)$ or (X_t) — collection of random variables;
- X_t — one particular random variable.

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Tradition: when we consider two arbitrary moments of time, s and t , we usually assume $s \leq t$.

Divide and conquer

The main trick to study properties:

$$\text{Future value} = \text{Known value} + \text{Unpredictable change}$$

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Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

Conditional probability exercise

Exercise . Calculate $\mathbb{P}(W_{10} > 2 \mid W_6 = 3)$.

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$$F(u) = \mathbb{P}(Z \leq u), \text{ where } Z \sim \mathcal{N}(0; 1).$$

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$$\begin{aligned}\mathbb{P}(W_{10} - W_6 > -1) &= \mathbb{P}\left(\frac{W_{10} - W_6}{2} > -\frac{1}{2}\right) = \\ &= \mathbb{P}(Z > -0.5) = \mathbb{P}(Z < 0.5) = F(0.5) \approx 0.69.\end{aligned}$$

More gentlemen's agreements

On slides we will follow these agreements:

- s and t denote two arbitrary time moments with $0 \leq s \leq t$.

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- (W_t) denotes a Wiener process.
- Z denotes a standard normal random variable, $Z \sim \mathcal{N}(0; 1)$.
- $F(u)$ denotes the standard normal distribution function,
 $F(u) = \mathbb{P}(Z \leq u)$.

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$W_6 - W_4$ is independent of $W_4 - W_3, W_{2.5} - W_1$.

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The increments $W_6 - W_4, W_4 - W_3, W_{2.5} - W_1$ are independent.

Independence of increments: full glory

If the time intervals $[s_1, t_1]$, $[s_2, t_2]$, ..., $[s_k, t_k]$ are non overlapping,

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then the increments $W(t_1) - W(s_1)$, $W(t_2) - W(s_2)$, ..., $W(t_k) - W(s_k)$ are independent.

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If the time intervals $[s_1, t_1]$, $[s_2, t_2]$, ..., $[s_k, t_k]$ are **non overlapping**,

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Remark: the right border of an interval **may touch** the left border of the next one, but **may not exceed** it, $t_j \leq s_{j+1}$.

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$$\text{Cov}(W_7, W_3) = 3.$$

Two friends

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Stochastic process $(S_t, t \geq 0)$ that may be written as


$$S_t = S_0 \exp(aW_t + bt),$$

is called **geometric brownian motion**.

Two plots

here will be the plots of BM with drift and geometric BM

BM with drift and scaling


Exercise . Find $\mathbb{E}(5W_t + 6t)$ and $\text{Var}(5W_t + 6t)$.

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$$\mathbb{E}(5W_t + 6t) = 0 + 6t = 6t$$

$$\text{Var}(5W_t + 6t) = \text{Var}(5W_t) = 25t$$

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How these are obtained?



$$\begin{aligned}\mathbb{E}(\exp(aZ)) &= \int_{-\infty}^{+\infty} \exp(az) f(z) dz = \\ &= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz\end{aligned}$$

Moment generating function

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- $M_{W_t}(a) = \exp(a^2 t/2)$ for a Wiener process W_t .

Why may we need MGF?

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$$M^{(k)}(0) = \mathbb{E}(X^k)$$

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- Moment generating function.

Conditional expectation

Conditional expectation: short plan

- Modeling information using **sigma-algebras**;

Conditional expectation: short plan

- Modeling information using **sigma-algebras**;
- **Properties** of conditional expected value;

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- Modeling information using **sigma-algebras**;
- **Properties** of conditional expected value;
- Conditional **variance**.

Modeling information

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John knows the value of X .

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Maria knows the value of X and Y .

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John knows the value of X .

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Maria knows **more!**

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How to model this **mathematically?**

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Informal definition



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Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

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Maria knows **more**: $\mathcal{F}_J \subset \mathcal{F}_M$.

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Informal theorem



The random variable Z is measurable with respect to $\sigma(X, Y)$ if and only if Z is a deterministic function of X and Y .

Best prediction

Informal definition



The **best prediction** of a random variable Y given σ -algebra \mathcal{F} is called **conditional expected value** $\mathbb{E}(Z \mid \mathcal{F})$.

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Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y .

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a **random variable**.

Notation

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Notation

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- $\mathbb{E}(Z \mid \sigma(X, Y))$ or $\mathbb{E}(Z \mid X, Y)$:
for σ -algebra generated by X and Y .

When we may omit conditioning?

- If Z is **independend** of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$:
If I know **nothing useful** about Z then I can drop my information.

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- If Z is **independend** of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$:
If I know **nothing useful** about Z then I can drop my information.
- $\mathbb{E}(\mathbb{E}(Z \mid \mathcal{F})) = \mathbb{E}(Z)$:
The **average of best guess** is the average of predicted variable.

The case of known variable

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$$\mathbb{E}(2ZR + Z^2 \mid \mathcal{F}) = 2Z\mathbb{E}(R \mid \mathcal{F}) + Z^2.$$

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Theorem



$$\text{Var}(Z \mid \mathcal{F}) = \mathbb{E}(Z^2 \mid \mathcal{F}) - (\mathbb{E}(Z \mid \mathcal{F}))^2.$$

Properties of conditional variance

- Irrelevant information may be omitted:

If Z is **independent** of \mathcal{F} then $\mathbb{E}(Z \mid \mathcal{F}) = \mathbb{E}(Z)$ and $\text{Var}(Z \mid \mathcal{F}) = \text{Var}(Z)$.

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If Z is **independent** of \mathcal{F} then $\mathbb{E}(Z \mid \mathcal{F}) = \mathbb{E}(Z)$ and $\text{Var}(Z \mid \mathcal{F}) = \text{Var}(Z)$.

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Martingales

Martingales: short plan

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Reminder: Sigma-algebra \mathcal{F}_t is the collection of events.

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The filtration $(\mathcal{F}_t, t \geq 0)$ is called a **natural filtration** of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

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Martingale

Definition



Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a **martingale** with respect to the filtration (\mathcal{F}_t) .

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Usually we consider natural filtration (\mathcal{F}_t) of the process (M_t) .

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The process $Z_t = \exp(aW_t - a^2t/2)$ is a **martingale** for every constant a .

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This martingale is very useful in Black and Scholes model.

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is completely equivalent to

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- Martingales related to **Wiener process, random walk**.