Wiener process and martingales

Wiener process

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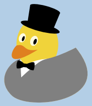
Wiener process

Here goes the plot!

Stochastic process

Definition 

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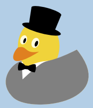
Definition 

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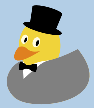
Definition 

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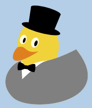
Definition 

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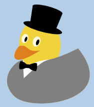
Definition 

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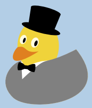
Definition 

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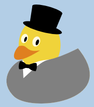
Definition 

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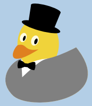
Definition 

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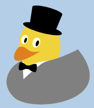
Definition 

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Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

The main trick to study properties:

$$\text{Future value} = \text{Known value} + \text{Unpredictable change}$$

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Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

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More gentlemen's agreements

On slides we will follow these agreements:

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The increments $W_6 - W_4$, $W_4 - W_3$, $W_{2.5} - W_1$ are independent.

Independence of increments: full glory

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Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \leq s_{j+1}$.

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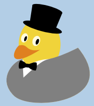
$$\mathrm{Cov}(W_7, W_3) = 3.$$

Two friends

Definition 

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Stochastic process $(S_t, t \geq 0)$ that may be written as

$$S_t = S_0 \exp(aW_t + bt),$$

is called geometric brownian motion.

Two plots

here will be the plots of BM with drift and geometric BM

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$$\mathrm{Var}(5W_t + 6t) = \mathrm{Var}(5W_t) = 25t$$

Frequently used expected values

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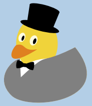
How these are obtained? 

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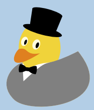
$$\mathbb{E}(\exp(aZ)) = \int_{-\infty}^{+\infty} \exp(az) f(z)\, dz =$$

$$= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz$$

Moment generating function

Definition 

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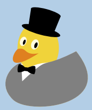
Definition 

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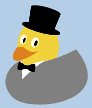
Definition 

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- $M_{W_t}(a) = \exp\left(a^2 t/2\right)$ for a Wiener process W_t.

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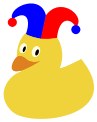
MGF is a funny  way to calculate expected value!

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How to model this mathematically?

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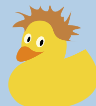
Definition 

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Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

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Maria knows more: $\mathcal{F}_J \subset \mathcal{F}_M$.

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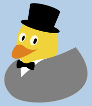
Definition 

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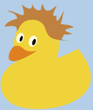
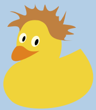
Theorem 

The random variable Z is measurable with respect to $\sigma(X, Y)$ if and only if Z is a deterministic function of X and Y.

Best prediction

Definition 

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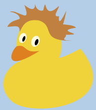
Definition 

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Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y.

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a random variable.

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- $\mathbb{E}(\mathbb{E}(Z \mid \mathcal{F})) = \mathbb{E}(Z)$:
  The average of best guess is the average of predicted variable.

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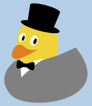
Definition 

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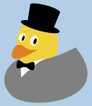
Definition 

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Properties of conditional variance

- Irrelevant information may be omitted:
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Martingales

Martingales: short plan

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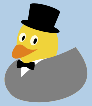
Definition 

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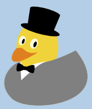
Definition 

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Reminder: Sigma-algebra $\mathcal{F}_t$ is the collection of events.

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The filtration $(\mathcal{F}_t, t \geq 0)$ is called a natural filtration of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

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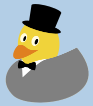
Definition 

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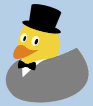
Definition 

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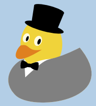
Definition 

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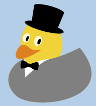
Definition 

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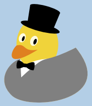
Definition 

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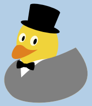
Definition 

Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

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Usually we consider natural filtration $(\mathcal{F}_t)$ of the process (M_t).

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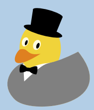
Theorem 

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