Wiener process and martingales

Wiener process

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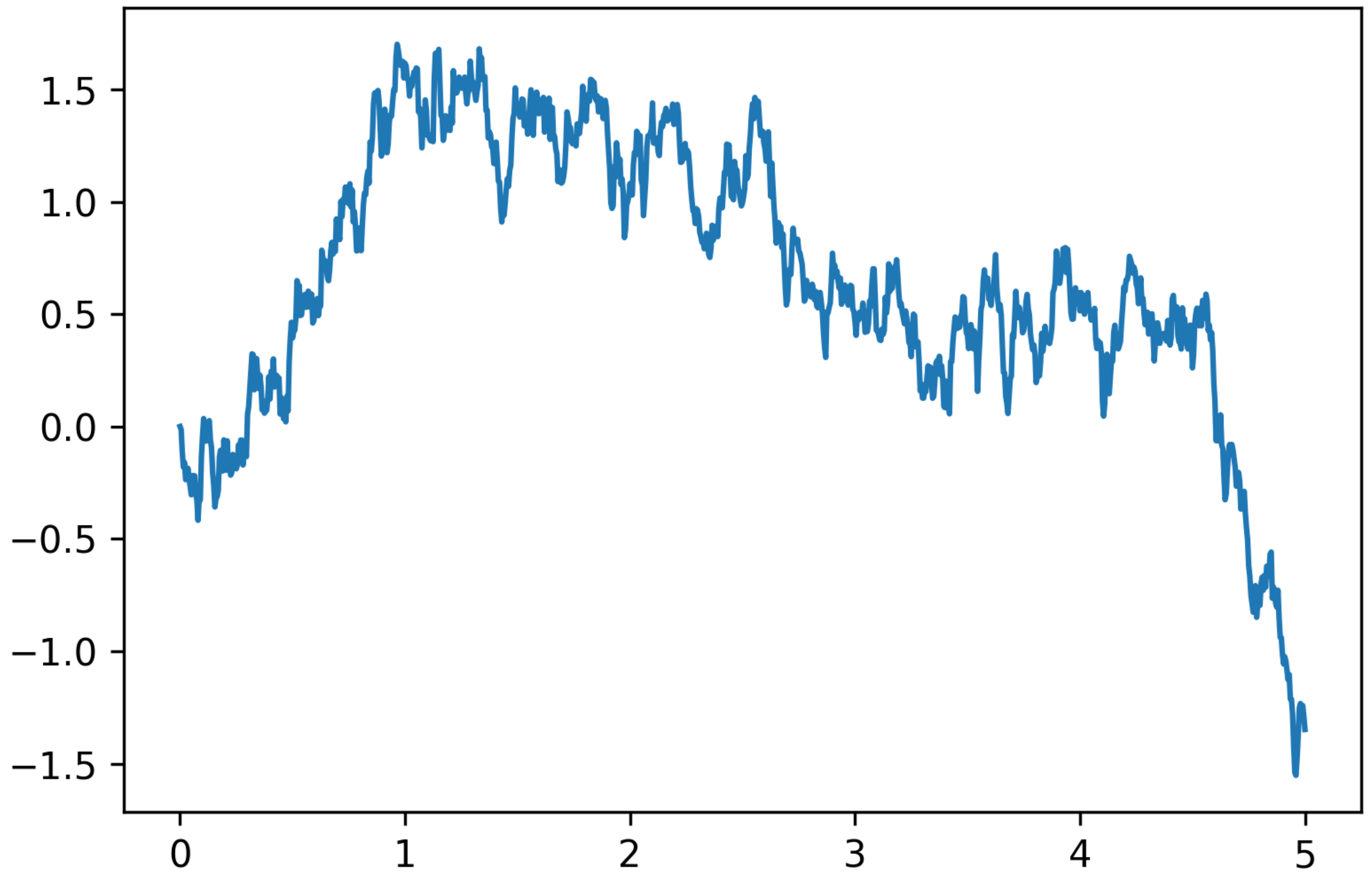

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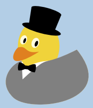
Definition 

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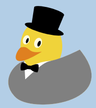
Definition 

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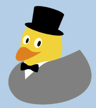
Definition 

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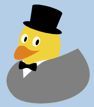
Definition 

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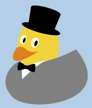
Definition 

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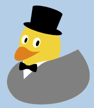
Definition 

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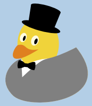
Definition 

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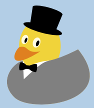
Definition 

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Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

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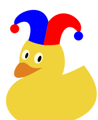
Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

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The increments $W_6 - W_4$, $W_4 - W_3$, $W_{2.5} - W_1$ are independent.

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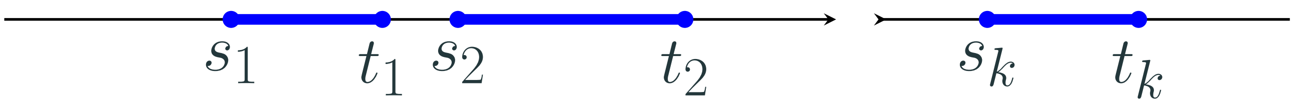

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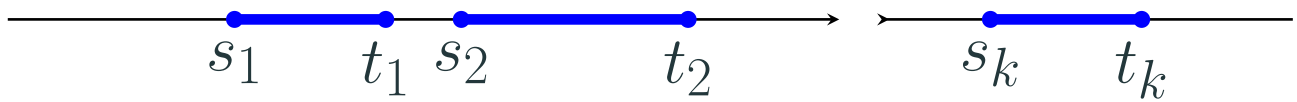

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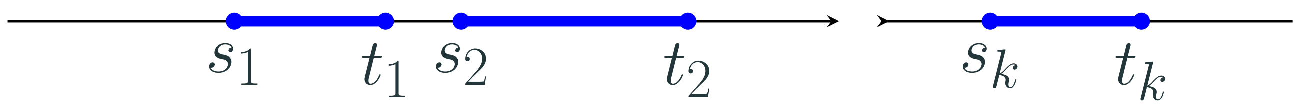

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Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \leq s_{j+1}$.

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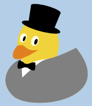
$$\mathrm{Cov}(W_7, W_3) = 3.$$

Two friends

Definition 

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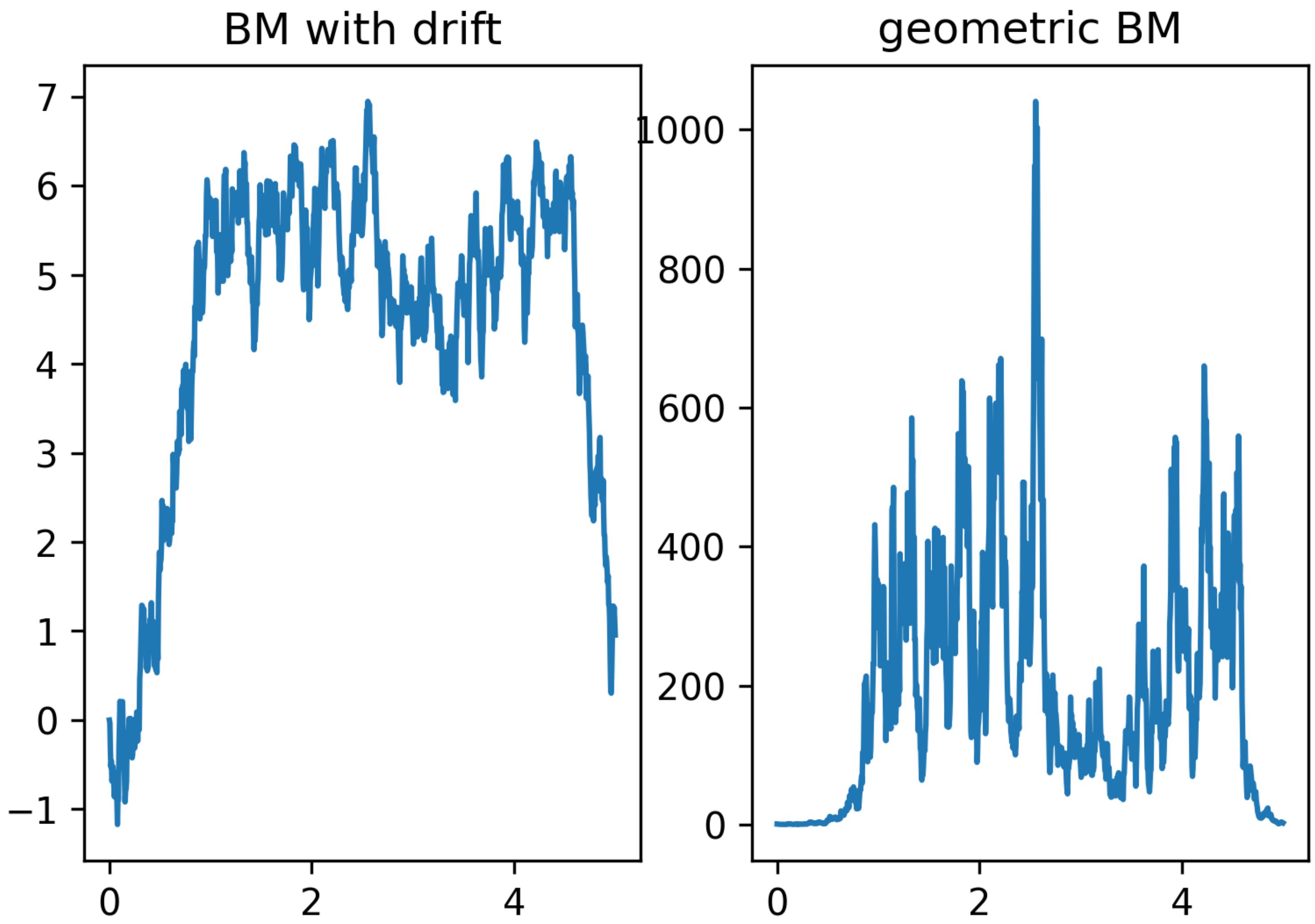

BM with drift
geometric BM
7
6
5
4
3
2
1
0
−1
1000
800
600
400
200
0
0
2
4

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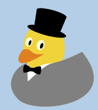
How these are obtained? 

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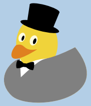
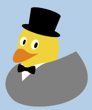
How these are obtained? 

$$\mathbb{E}(\exp(aZ)) = \int_{-\infty}^{+\infty} \exp(az) f(z)\, dz =$$
$$= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz$$

Moment generating function

Definition 

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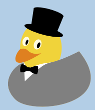
Definition 

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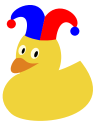
MGF is a funny  way to calculate expected value!

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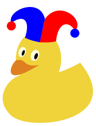
MGF is a funny  way to calculate expected value!

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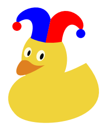
MGF is a funny  way to calculate expected value!

$$\begin{aligned} M''(0) &= \mathbb{E}(X^2) \\ M'''(0) &= \mathbb{E}(X^3) \\ &\vdots \\ M^{(k)}(0) &= \mathbb{E}(X^k) \end{aligned}$$

Wiener process: summary

- Stochastic process with normal and independent increments.

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Wiener process: summary

- Stochastic process with normal and independent increments.
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Conditional expectation

Conditional expectation: short plan

- Modeling information using sigma-algebras;

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- Modeling information using sigma-algebras;
- Properties of conditional expected value;
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Modeling information

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John knows the value of X.

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John knows the value of X.

Maria knows the value of X and Y.

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Maria knows more!

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How to model this mathematically?

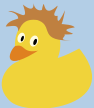
Sigma-algebra

Sigma-algebra

Informal definition 

Sigma-algebra (σ-algebra) generated by random variables X and Y is the collection of all events that can be stated in terms of these random variables.

Sigma-algebra

Informal definition 

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Notation: $\sigma(X, Y)$.

Sigma-algebra

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Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

Modeling information

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John knows the value of X, $\mathcal{F}_J = \sigma(X)$.

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Maria knows more: $\mathcal{F}_J \subset \mathcal{F}_M$.

Measurability

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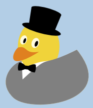
Definition

The random variable Z is measurable with respect to σ-algebra $\mathcal{F}$ if $\sigma(Z) \subset \mathcal{F}$.

Measurability

Definition 

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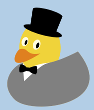
Definition 

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Informal theorem 

The random variable Z is measurable with respect to $\sigma(X, Y)$ if and only if Z is a deterministic function of X and Y.

Best prediction

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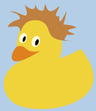
The best prediction of a random variable Y given σ-algebra $\mathcal{F}$ is called conditional expected value $\mathbb{E}(Z \mid \mathcal{F})$.

Best prediction

Informal definition 

The best prediction of a random variable Y given σ-algebra $\mathcal{F}$ is called conditional expected value $\mathbb{E}(Z \mid \mathcal{F})$.

Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y.

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a random variable.

Notation

- $\mathbb{E}(Z \mid \mathcal{F})$:
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  for a general σ-algebra $\mathcal{F}$;
- $\mathbb{E}(Z \mid \sigma(X, Y))$ or $\mathbb{E}(Z \mid X, Y)$:
  for σ-algebra generated by X and Y.

When we may omit conditioning?

- If Z is independend of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$: If I know nothing useful about Z then I can drop my information.

When we may omit conditioning?

- If Z is independend of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$:
  If I know nothing useful about Z then I can drop my information.
- $\mathbb{E}(\mathbb{E}(Z \mid \mathcal{F})) = \mathbb{E}(Z)$:
  The average of best guess is the average of predicted variable.

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$$\mathbb{E}(2ZR + Z^2 \mid \mathcal{F}) = 2Z\mathbb{E}(R \mid \mathcal{F}) + Z^2.$$

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Theorem

$$\mathrm{Var}(Z \mid \mathcal{F}) = \mathbb{E}(Z^2 \mid \mathcal{F}) - (\mathbb{E}(Z \mid \mathcal{F}))^2.$$

Properties of conditional variance

- Irrelevant information may be omitted:
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Martingales

Martingales: short plan

- Filtration models the information acquisition.

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- Definition of a martingale.

Martingales: short plan

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- Examples of martingales.

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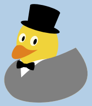
Definition 

The family of sigma-algebras $(\mathcal{F}_t, t \geq 0)$ is called filtration if it grows in time, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Reminder: Sigma-algebra $\mathcal{F}_t$ is the collection of events.

Natural filtration

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Definition

The filtration $(\mathcal{F}_t, t \geq 0)$ is called a natural filtration of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

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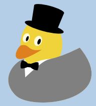
Definition 

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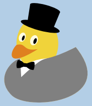
Definition 

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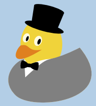
Definition 

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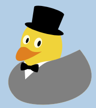
Martingale

Definition 

Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a martingale with respect to the filtration $(\mathcal{F}_t)$.

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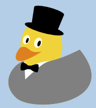
Definition 

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Usually we consider natural filtration $(\mathcal{F}_t)$ of the process (M_t).

Simple examples

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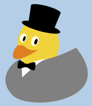
More examples

Theorem 

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&= W_s^2 + \mathbb{E}((W_t - W_s)^2) + 2W_s\mathbb{E}(W_t - W_s) - t = \\
&= W_s^2 + (t - s) + 2W_s \cdot 0 - t = W_s^2 - s
\end{aligned}
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More examples

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The process $Z_t = \exp\left(aW_t - a^2 t/2\right)$ is a martingale for every constant a.

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This martingale is very useful in Black and Scholes model.

Martingales in discrete time

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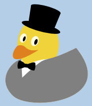
Theorem 

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In discrete time the condition

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s \text{ for all } s \leq t$$

is completely equivalent to

$$\mathbb{E}(M_{t+1} \mid \mathcal{F}_t) = M_t.$$

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Consider independent and identically distributed Z_1, Z_2, ... with $\mathbb{E}(Z_t) = 0$.

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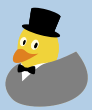
Theorem 

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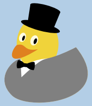
Theorem 

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$$\mathcal{F}_t = \sigma(Z_1, Z_2, Z_3, \ldots, Z_t)$$

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Theorem

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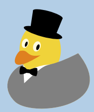
Theorem 

The random walk process is a martingale.

$$\mathcal{F}_t = \sigma(Z_1, Z_2, Z_3, \ldots, Z_t)$$

$$\mathbb{E}(S_{t+1} \mid \mathcal{F}_t) = \mathbb{E}(S_t + Z_{t+1} \mid \mathcal{F}_t) = S_t + \mathbb{E}(Z_{t+1}) = S_t.$$

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- Martingales related to Wiener process, random walk.