Itô's lemma

Itô's lemma

Itô’s lemma: short plan

- Light version for functions of time and Wiener process.
- Check the martingale property with Itô’s lemma.
- More general version.

Itô's lemma: light version

Informal theorem

If $Y_t = f(W_t, t)$ then it may be written as

$$Y_t = Y_0 + \int_0^t f'_W \, dW_u + \int_0^t \left(f'_t + \frac{1}{2} f''_{WW} \right) du.$$

Informal theorem

If $Y_t = f(W_t, t)$ then it may be written in the short form as

$$dY_t = f'_W dW_t + f'_t dt + \frac{1}{2} f''_{WW} dt.$$

Exercise

Express $Y_t = W_t^3 \cdot t^4$ as an Itô process .

$$dY_t = 3W_t^2 t^4 \, dW_t + 4W_t^3 t^3 \, dt + \frac{1}{2} 6W_t t^4 dt =$$
$$= 3W_t^2 t^4 \, dW_t + (4W_t^3 t^3 + 3W_t t^4) \, dt.$$

$$Y_t = 0 + \int_0^t 3W_u^2 u^4 \, dW_u + \int_0^t (4W_u^3 u^3 + 3W_u u^4) \, du.$$

Exercise

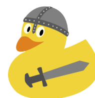
Check whether the process $Y_t = W_t^4 - t^2$ is a martingale .

$$dY_t = 4W_t^3\,dW_t - 2t\,dt + \frac{1}{2}12W_t^2 dt =$$
$$=4W_t^3\,dW_t + (6W_t^2 - 2t)\,dt.$$

$$Y_t = 0 + \int_0^t 4W_u^3\,dW_u + \int_0^t (6W_u^2 - 2u)\,du.$$

(Y_t) is not a martingale!

Exercise

Express $Y_t = W_t^2$ as an Itô process and prove the formula for $\int_0^t W_u \, dW_u$ .

$$dY_t = 2W_t \, dW_t + 0 \, dt + \frac{1}{2} 2dt = 2W_t \, dW_t + \ dt.$$

$$Y_t = 0 + \int_0^t 2W_u \, dW_u + \int_0^t 1 \, du.$$

$$W_t^2 = 0 + 2\int_0^t W_u \, dW_u + t. \quad \to \quad \int_0^t W_u \, dW_u = \frac{1}{2}(W_t^2 - t).$$

Itô's lemma: general version

Informal theorem 

If $Y_t = f(X_t, t)$ where (X_t) is an Itô process then Y_t may be written in the short form as

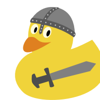
$$dY_t = f'_X dX_t + f'_t dt + \frac{1}{2} f''_{XX} (dX_t)^2,$$

where $(dX_t)^2$ is calculated using symbolic rules

$dt \cdot dW_t = 0$, $dt \cdot dt = 0$, $dW_t \cdot dW_t = dt$.

Exercise

Consider $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $Y_t = S_t^2$.

Find dY_t and recover the full form for Y_t .

$$
\begin{aligned}
dY_t &= 2S_t\, dS_t + 0\, dt + \frac{1}{2} 2 (dS_t)^2 = \\
&= 2S_t(\mu S_t dt + \sigma S_t dW_t) + 1 \cdot (\mu S_t dt + \sigma S_t dW_t)^2 = \\
&= 2S_t(\mu S_t dt + \sigma S_t dW_t) + \sigma^2 S_t^2 dt = \\
&= (2\mu S_t^2 + \sigma^2 S_t^2) dt + 2\sigma S_t^2 dW_t.
\end{aligned}
$$

$$
Y_t = S_0^2 + \int_0^t 2\sigma S_u^2\, dW_u + \int_0^t (2\mu S_u^2 + \sigma^2 S_u^2)\, du.
$$

Itô's lemma: a way to memorize

Informal theorem 

If $Y_t = f(X_t, Z_t, t)$ where (X_t) and (Z_t) are Itô processes then the short form of Y_t may be obtained in two steps:

1. Calculate the second order Taylor expansion of f.
2. Simplify the result using symbolic rules
   $dt \cdot dW_t = 0$, $dt \cdot dt = 0$, $dW_t \cdot dW_t = dt$.

Reminder: dW_t, dY_t, dX_t, dZ_t do not exist!

It's only a quick way to find the full form.

Itô's lemma: summary

- Basic tool to study stochastic integrals.
- Easy to check whether the process is a martingale.
- Easily written in short form:
  $dt \cdot dW_t = 0$, $dt \cdot dt = 0$, $dW_t \cdot dW_t = dt$.

Stochastic differential equations

Stochastic differential equations: short plan

- Common properties with Riemann integral.
- Zero expected value.
- Itô isometry.

Stochastic differential equations: summary

- Zero expected value.
- Using Itô isometry one may calculate variance.
- Some common properties with Riemann integral.

Girsanov theorem

Girsanov theorem: short plan

- Idea of an alternative probability measure.
- Girsanov theorem.
-

Girsanov theorem: summary

- A sum three terms: constant, Itô integral and Riemann integral.
- Will be a martingale without Riemann integral.
- Often written using short form with dt and dW_t.