Wiener process and martingales

Wiener process

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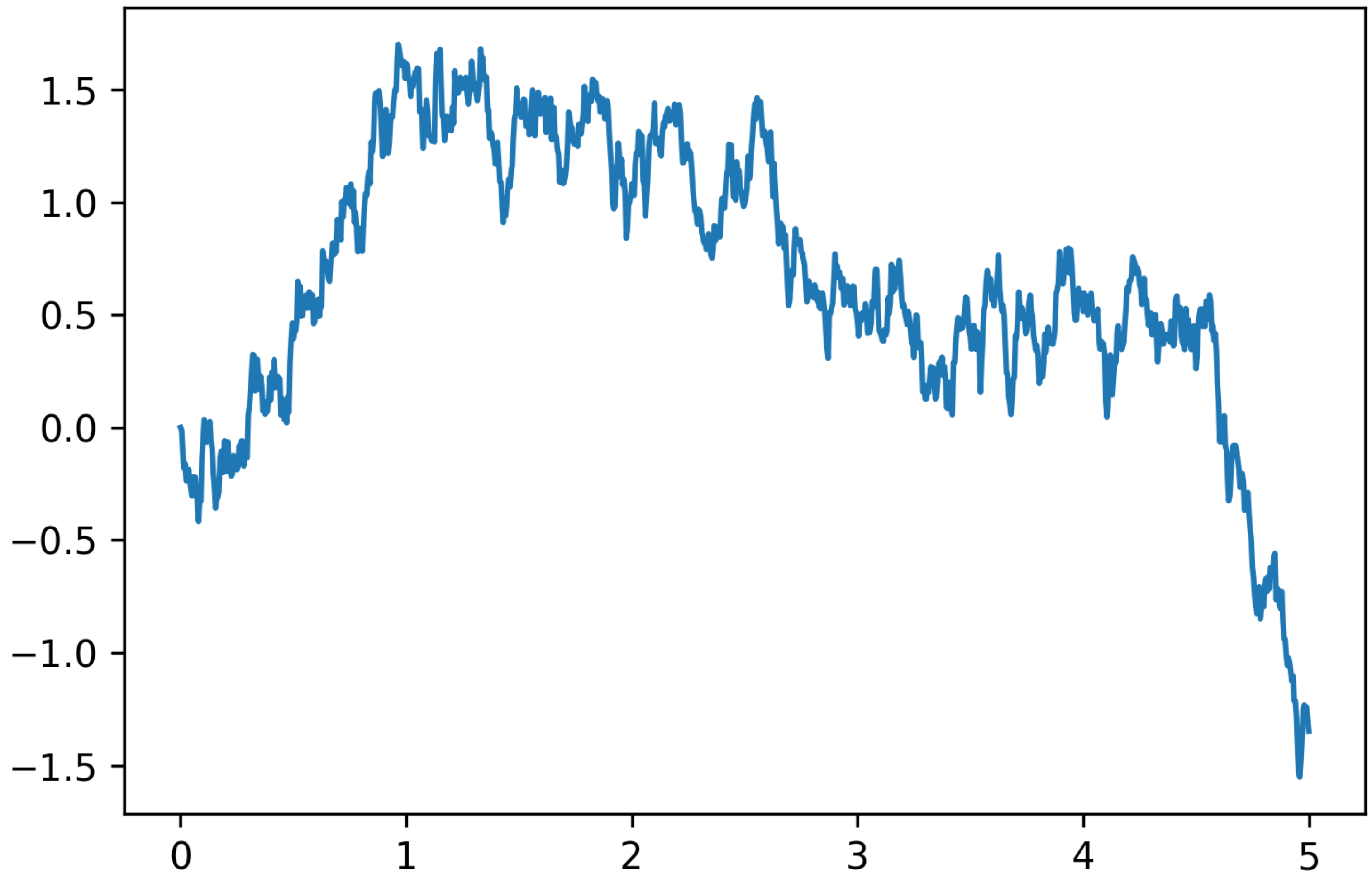

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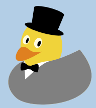
Definition 

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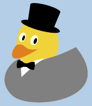
Definition 

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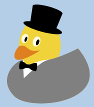
Definition 

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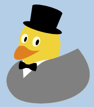
Definition 

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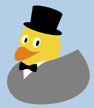
Definition 

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Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

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$$\text{Future value} = \text{Known value} + \text{Unpredictable change}$$

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Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

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On slides we will follow these agreements:

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The increments $W_6 - W_4$, $W_4 - W_3$, $W_{2.5} - W_1$ are independent.

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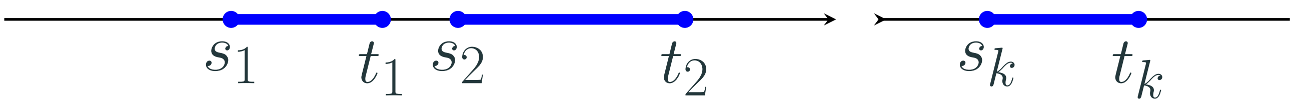
If the time intervals $[s_1, t_1]$, $[s_2, t_2]$, ..., $[s_k, t_k]$ are non overlapping,



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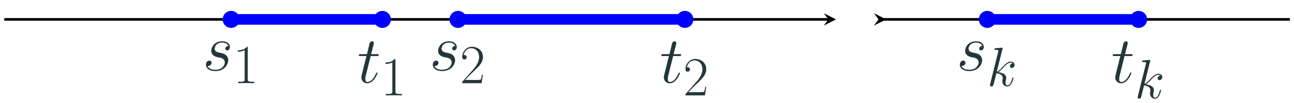

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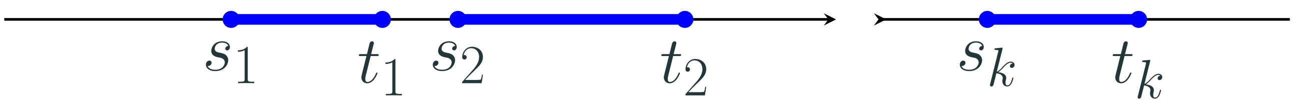

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Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \le s_{j+1}$.

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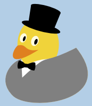
$$\mathrm{Cov}(W_7, W_3) = 3.$$

Two friends

Definition 

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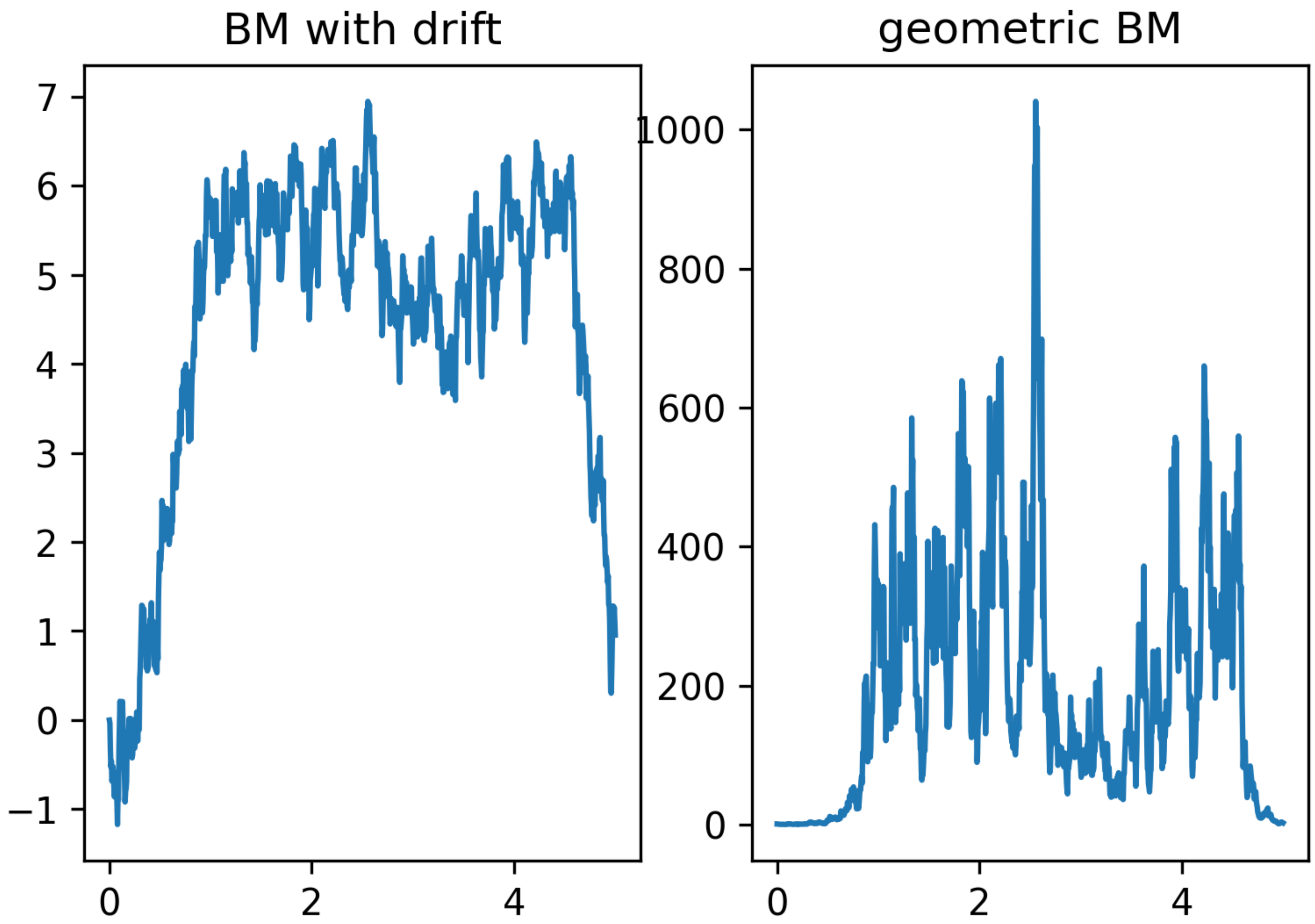
Two plots



BM with drift
geometric BM
7
6
5
4
3
2
1
0
−1
0
2
4
1000
800
600
400
200
0
0
2
4

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$$\mathrm{Var}(5W_t + 6t) = \mathrm{Var}(5W_t) = 25t$$

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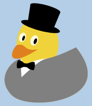
How these are obtained? 

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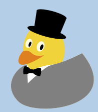
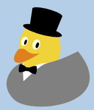
How these are obtained? 

$$\begin{aligned}\mathbb{E}(\exp(aZ)) &= \int_{-\infty}^{+\infty} \exp(az) f(z)\, dz = \\ &= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz\end{aligned}$$

Moment generating function

Definition 

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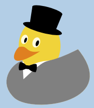
Definition 

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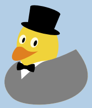
Definition 

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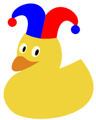
MGF is a funny  way to calculate expected value!

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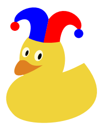
MGF is a funny  way to calculate expected value!

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Conditional expectation

Conditional expectation: short plan

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- Modeling information using sigma-algebras;
- Properties of conditional expected value;
- Conditional variance.

Modeling information

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John knows the value of X.

Maria knows the value of X and Y.

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How to model this mathematically?

Sigma-algebra

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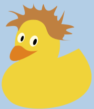
Informal definition 

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Notation: $\sigma(X, Y)$.

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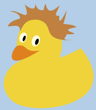
Informal definition 

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Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

Modeling information

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Maria knows more: $\mathcal{F}_J \subset \mathcal{F}_M$.

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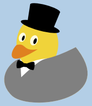
Definition 

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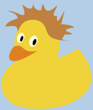
Informal theorem 

The random variable R is measurable with respect to $\sigma(X, Y)$ if and only if R is a deterministic function of X and Y.

Best prediction

Informal definition 

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Difference of $\mathbb{E}(R \mid \mathcal{F})$ and $\mathbb{E}(R)$

If I know X and Y then my best prediction of R may depend on X and Y.

In general: $\mathbb{E}(R \mid \mathcal{F})$ is a random variable.

Notation

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  for a general σ-algebra $\mathcal{F}$;
- $\mathbb{E}(R \mid \sigma(X, Y))$ or $\mathbb{E}(R \mid X, Y)$:
  for σ-algebra generated by X and Y.

When we may omit conditioning?

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- If R is independend of X and Y then $\mathbb{E}(R \mid X, Y) = \mathbb{E}(R)$:
  If I know nothing useful about R then I can drop my information.
- $\mathbb{E}(\mathbb{E}(R \mid \mathcal{F})) = \mathbb{E}(R)$:
  The average of best guess is the average of predicted variable.

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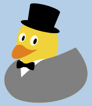
Conditional variance

Definition 

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Theorem

$$\mathrm{Var}(R \mid \mathcal{F}) = \mathbb{E}(R^2 \mid \mathcal{F}) - (\mathbb{E}(R \mid \mathcal{F}))^2.$$

Properties of conditional variance

- Irrelevant information may be omitted:
  If R is independent of $\mathcal{F}$ then $\mathbb{E}(R \mid \mathcal{F}) = \mathbb{E}(R)$ and $\mathrm{Var}(R \mid \mathcal{F}) = \mathrm{Var}(R)$.

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Martingales

Martingales: short plan

- Filtration models the information acquisition.

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- Examples of martingales.

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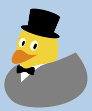
Definition 

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Reminder: Sigma-algebra $\mathcal{F}_t$ is the collection of events.

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The filtration $(\mathcal{F}_t, t \geq 0)$ is called a natural filtration of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

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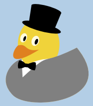
Definition 

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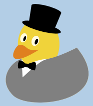
Definition 

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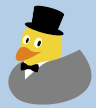
Martingale

Definition 

Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a martingale with respect to the filtration $(\mathcal{F}_t)$.

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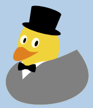
Definition 

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Usually we consider natural filtration $(\mathcal{F}_t)$ of the process (M_t).

Simple examples

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\end{aligned}
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=&W_s^2 + (t - s) + 2W_s \cdot 0 - t = W_s^2 - s
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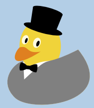
This martingale is very useful in Black and Scholes model.

Martingales in discrete time

Theorem 

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is completely equivalent to

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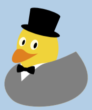
Theorem 

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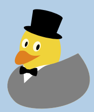
Theorem 

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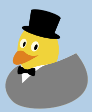
Theorem 

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- Martingales related to Wiener process, random walk.