Wiener process and martingales

Itô integral

Itô integral: short plan

- Intuitive definition.
- Examples.

Itô integral

Definition

The Itô integral $I_t = \int_0^t A_u \, dW_u$ is defined as the total net cash flow of a strategy if we treat W_u as the price of the asset and A_u as the quantity of the asset at time moment u.

(W_t) is a poor model for the price, but the intuition is ok:

$$\text{Net cash flow} = \int_0^t \text{Quantity}_u \, d\text{Price}_u$$

Simple deterministic example

Let's calculate $\int_2^8 7\,du$ 

Transaction 1. At time $u = 2$ we buy 7 units. The price is $u = 2$.

Cash flow: $-7 \cdot 2$.

Transaction 2. At time $u = 8$ we sell 7 units. The price is $u = 8$.

Cash flow: $7 \cdot 8$.

$\int_2^8 7\,du = -14 + 56 = 42$.

Example with Wiener process

Let's calculate $\int_2^8 5\,dW_u$ 

Transaction 1. At time $u = 2$ we buy 5 units. The price is W_2.

Cash flow: $-5 \cdot W_2$.

Transaction 2. At time $u = 8$ we sell 5 units. The price is W_8.

Cash flow: $5 \cdot W_8$.

$\int_2^8 5\,dW_u = -5W_2 + 5W_8$.

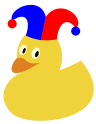
More gentlemen's agreement

$$I_t = \int_0^t (\text{something}_u)\, dW_u :$$

t — the upper limit of integration;

u — time variable with range from 0 to t;

Why old integration formulas are wrong?

$$\int_0^t W_u \, dW_u =$$  ?

Why not $\frac{1}{2}W_t^2 - \frac{1}{2}W_0^2$?

The guessed value is non-negative, $\frac{1}{2}W_t^2 \geq 0$!

If you buy and sell Wiener process you can have negative cash flow!

Small table

In most cases Itô integral can not be computed explicitely.

$$\int_0^t 1 \, dW_u = W_t$$

$$\int_0^t W_u \, dW_u = \frac{W_t^2 - t}{2}$$

$$\int_0^t \exp\left(aW_u - \frac{1}{2}a^2 u\right) dW_u = \frac{1}{a}\left(\exp\left(aW_t - \frac{1}{2}a^2 t\right) - 1\right)$$

Itô integral: summary

- Total net cash flow of a strategy.
- Rarely can be computed explicitely.
- Old rules of integration do not apply.

Itô integral properties

Itô integral properties: short plan

- Common properties with Riemann integral.
- Zero expected value.
- Itô isometry.

Common properties

$$\int_a^b X_u \, dW_u + \int_b^c X_u \, dW_u = \int_a^c X_u \, dW_u$$

$$\int_a^a X_u \, dW_u = 0$$

$$\int_0^t c X_u \, dW_u = c \int_0^t X_u \, dW_u$$

Zero expected value

Intuition: we buy and sell Wiener process, hence, expected net cash flow should be zero.

Informal theorem 

For any reasonable process (X_t) measurable with respect to the natural filtration $(\mathcal{F}_t)$ of the process (W_t)

$$\mathbb{E}\left(\int_0^t X_u\, dW_u\right) = 0.$$

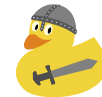
Itô isometry

Informal theorem

For any reasonable process (X_t) measurable with respect to the natural filtration $(\mathcal{F}_t)$ of the process (W_t)

$$\mathrm{Var}\left(\int_0^t X_u \, dW_u\right) = \int_0^t \mathbb{E}(X_u^2) \, du.$$

Exercise

Find $\mathbb{E}(I_t)$ and $\mathrm{Var}(I_t)$ for $I_t = \int_0^t W_u^2 \, dW_u$ 

$\mathbb{E}(I_t) = 0;$

$$\mathrm{Var}(I_t) = \int_0^t \mathbb{E}(W_u^4) \, du = \int_0^t 3u^2 \, du = t^3.$$

Itô properties: summary

- Zero expected value.
- Using Itô isometry one may calculate variance.
- Some common properties with Riemann integral.

Itô process

Itô process: short plan

- Definition of an Itô process.
- When Itô process is a martingale?
- Short and full form notation.

Itô process

Definition

Stochastic process $(Y_t, t \geq 0)$ is called Itô process if it can be written in the form

$$Y_t = Y_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du,$$

where Y_0 is a constant.

A wide class of continuous stochastic processes that behave locally like a Wiener process with drift.

Itô integral is a martingale

Informal theorem

Itô process (Y_t) is a martingale if and only if it has only Itô integral in the representation

$$Y_t = Y_0 + \int_0^t A_u \, dW_u.$$

The best guess of a future value of an Itô integral is its current value:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = Y_s \text{ for } s \leq t.$$

Expected value of Itô process

Informal theorem

For any reasonable process (B_t)

$$\mathbb{E}\left(\int_0^t B_u\,du\right) = \int_0^t \mathbb{E}(B_u)\,du.$$

Informal theorem

If (Y_t) is an Itô process with $Y_t = Y_0 + \int_0^t A_u\,dW_u + \int_0^t B_u\,du$, then

$$\mathbb{E}(Y_t) = Y_0 + \int_0^t \mathbb{E}(B_u)\,du.$$

Short form notation

Full form:

$$Y_t = Y_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du.$$

Short form:

$$dY_t = A_t \, dW_t + B_t \, dt.$$

dW_t and dY_t have no meaning!

Short form in simulations

We need to simulate a path of

$$Y_t = Y_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du.$$

In short form:

$$dY_t = A_t \, dW_t + B_t \, dt.$$

In simulations:

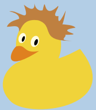
$$Y_{t+\Delta} - Y_t \approx A_t \cdot (W_{t+\Delta} - W_t) + B_t \cdot \Delta,$$

where $W_{t+\Delta} - W_t \sim \mathcal{N}(0; \Delta)$.

Short form: examples

$$dY_t = W_t^4 dW_t \quad Y_t = Y_0 + \int_0^t W_u^4 \, dW_u.$$

$$dY_t = \cos(W_t) dt \quad Y_t = Y_0 + \int_0^t \cos(W_u) \, du.$$

Informal theorem 

Itô process (Y_t) is a martingale if and only if

$$dY_t = A_t \, dW_t.$$

Itô process: summary

- A sum three terms: constant, Itô integral and Riemann integral.
- Will be a martingale without Riemann integral.
- Often written using short form with dt and dW_t.