

Itô's lemma

Itô's lemma

Itô's lemma: short plan

- Light version for functions of **time** and **Wiener process**.
- Check the **martingale property** with Itô's lemma.
- More **general** version.

Itô's lemma: light version

Informal theorem



If $Y_t = f(W_t, t)$ then it may be written as

$$Y_t = Y_0 + \int_0^t f'_W dW_u + \int_0^t \left(f'_t + \frac{1}{2} f''_{WW} \right) du.$$

Informal theorem



If $Y_t = f(W_t, t)$ then it may be written in the short form as

$$dY_t = f'_W dW_t + f'_t dt + \frac{1}{2} f''_{WW} dt.$$


Exercise

Express $Y_t = W_t^3 \cdot t^4$ as an Itô process 🦆.

$$\begin{aligned} dY_t &= 3W_t^2 t^4 dW_t + 4W_t^3 t^3 dt + \frac{1}{2} 6W_t t^4 dt = \\ &= 3W_t^2 t^4 dW_t + (4W_t^3 t^3 + 3W_t t^4) dt. \end{aligned}$$

$$Y_t = 0 + \int_0^t 3W_u^2 u^4 dW_u + \int_0^t (4W_u^3 u^3 + 3W_u u^4) du.$$

Exercise


Check whether the process $Y_t = W_t^4 - t^2$ is a martingale .

$$\begin{aligned} dY_t &= 4W_t^3 dW_t - 2t dt + \frac{1}{2}12W_t^2 dt = \\ &= 4W_t^3 dW_t + (6W_t^2 - 2t) dt. \end{aligned}$$

$$Y_t = 0 + \int_0^t 4W_u^3 dW_u + \int_0^t (6W_u^2 - 2u) du.$$

(Y_t) is **not** a martingale!

Exercise

Express $Y_t = W_t^2$ as an Itô process and prove the formula for $\int_0^t W_u dW_u$ .

$$dY_t = 2W_t dW_t + 0 dt + \frac{1}{2}2dt = 2W_t dW_t + dt.$$

$$Y_t = 0 + \int_0^t 2W_u dW_u + \int_0^t 1 du.$$

$$W_t^2 = 0 + 2 \int_0^t W_u dW_u + t. \quad \rightarrow \quad \int_0^t W_u dW_u = \frac{1}{2}(W_t^2 - t).$$

Itô's lemma: general version

Informal theorem



If $Y_t = f(X_t, t)$ where (X_t) is an Itô process then Y_t may be written in the short form as

$$dY_t = f'_X dX_t + f'_t dt + \frac{1}{2} f''_{XX} (dX_t)^2,$$

where $(dX_t)^2$ is calculated using symbolic rules

$$dt \cdot dW_t = 0, \quad dt \cdot dt = 0, \quad dW_t \cdot dW_t = dt.$$

Exercise

Consider $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $Y_t = S_t^2$.

Find dY_t and recover the full form for Y_t .

$$\begin{aligned} dY_t &= 2S_t dS_t + 0 dt + \frac{1}{2} 2(dS_t)^2 = \\ &= 2S_t(\mu S_t dt + \sigma S_t dW_t) + 1 \cdot (\mu S_t dt + \sigma S_t dW_t)^2 = \\ &= 2S_t(\mu S_t dt + \sigma S_t dW_t) + \sigma^2 S_t^2 dt = \\ &= (2\mu S_t^2 + \sigma^2 S_t^2) dt + 2\sigma S_t^2 dW_t. \end{aligned}$$

$$Y_t = S_0^2 + \int_0^t 2\sigma S_u^2 dW_u + \int_0^t (2\mu S_u^2 + \sigma^2 S_u^2) du.$$

Itô's lemma: a way to memorize

Informal theorem



If $Y_t = f(X_t, Z_t, t)$ where (X_t) and (Z_t) are Itô processes then the short form of Y_t may be obtained in two steps:

1. Calculate the **second order Taylor expansion** of f .
2. **Simplify** the result using symbolic rules

$$dt \cdot dW_t = 0, dt \cdot dt = 0, dW_t \cdot dW_t = dt.$$

Reminder: dW_t, dY_t, dX_t, dZ_t **do not exist!**

It's only a **quick way** to find the full form.

Itô's lemma: summary

- Basic tool to study stochastic integrals.
- Easy to check whether the process is a martingale.
- Easily written in short form:

$$dt \cdot dW_t = 0, \quad dt \cdot dt = 0, \quad dW_t \cdot dW_t = dt.$$

Stochastic differential equations

Stochastic differential equations: short plan

- Common properties with Riemann integral.
- Zero expected value.
- Itô isometry.

Stochastic differential equations: summary

- Zero expected value.
- Using Itô isometry one may calculate variance.
- Some common properties with Riemann integral.

Girsanov theorem

Girsanov theorem: short plan

- Idea of an alternative probability measure.
- Girsanov theorem.
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Girsanov theorem: summary

- A **sum three terms**: constant, Itô integral and Riemann integral.
- Will be a martingale **without Riemann integral**.
- Often written using **short form** with dt and dW_t .