

Wiener process and martingales

Wiener process

Stochastic calculus course

The goal: price an option in the framework of Black and Scholes model.

- Very short: 4 weeks only.
- Mathematics is hard.
- Informal definitions and theorems.
- Problem solving and computer simulations.

Wiener process

Here goes the plot!

Stochastic process

Definition

Stochastic or **random process** is a collection of random variables indexed by time variable t .

Continuous time: $(X_t, t \geq 0)$.

Discrete time: $(X_t, t \in \{0, 1, 2, 3, \dots\})$.

Notation remark:

- $(X_t, t \geq 0)$ or (X_t) — the collection of random variables;
- X_t — one particular random variable.

Wiener process

Definition

Stochastic process $(W_t, t \geq 0)$ is called **Wiener process** or **Brownian motion** if

1. $W_0 = 0$.
2. Increments $W_t - W_s$ are normally distributed $\mathcal{N}(0; t - s)$.
3. Increment $W_t - W_s$ is independent of the past values $(W_u, u \leq s)$.
4. $\mathbb{P}(\text{trajectory of } (W_t) \text{ is continuous}) = 1$.

Tradition: when we consider two arbitrary moments of time, s and t , we usually assume $s \leq t$.

Divide and conquer

The main trick to study properties:

Future value = Known value + Unpredictable change

Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

Conditional probability exercise

Exercise . Calculate $\mathbb{P}(W_{10} > 2 \mid W_6 = 3)$.

$$\begin{aligned}\mathbb{P}(W_{10} > 2 \mid W_6 = 3) &= \mathbb{P}(W_{10} - W_6 + W_6 > 2 \mid W_6 = 3) = \\ &= \mathbb{P}(W_{10} - W_6 + 3 > 2 \mid W_6 = 3) = \mathbb{P}(W_{10} - W_6 > -1).\end{aligned}$$

$$W_{10} - W_6 \sim \mathcal{N}(0; 4), \text{ hence } \frac{W_{10} - W_6 - 0}{\sqrt{4}} \sim \mathcal{N}(0; 1).$$

We will use standard normal distribution function

$F(u) = \mathbb{P}(Z \leq u)$, where $Z \sim \mathcal{N}(0; 1)$.

$$\begin{aligned}\mathbb{P}(W_{10} - W_6 > -1) &= \mathbb{P}\left(\frac{W_{10} - W_6}{2} > -\frac{1}{2}\right) = \\ &= \mathbb{P}(Z > -0.5) = \mathbb{P}(Z < 0.5) = F(0.5) \approx 0.69.\end{aligned}$$

More gentlemen's agreements

On slides we will follow these agreements:

- s and t denote two arbitrary time moments with $0 \leq s \leq t$.
- (W_t) denotes a Wiener process.
- Z denotes a standard normal random variable, $Z \sim \mathcal{N}(0; 1)$.
- $F(u)$ denotes the standard normal distribution function,
 $F(u) = \mathbb{P}(Z \leq u)$.

Independence of increments: example

Property

Increment $W_t - W_s$ is independent of the past values $(W_u, u \leq s)$.

$W_6 - W_4$ is independent of $W_4, W_3, W_{2.5}, W_1, \dots$

$W_6 - W_4$ is independent of $W_4 - W_3, W_{2.5} - W_1$.

The increments $W_6 - W_4, W_4 - W_3, W_{2.5} - W_1$ are independent.

Independence of increments: full glory


If the time intervals $[s_1, t_1]$, $[s_2, t_2]$, ..., $[s_k, t_k]$ are **non overlapping**,

Here will be a small picture

then the increments $W(t_1) - W(s_1)$, $W(t_2) - W(s_2)$, ..., $W(t_k) - W(s_k)$ are independent.

Remark: the right border of an interval **may touch** the left border of the next one, but **may not exceed** it, $t_j \leq s_{j+1}$.

Expectation and variance

Exercise . Find $\mathbb{E}(W_t)$, $\text{Var}(W_t)$, $\text{Cov}(W_s, W_t)$.

$$\mathbb{E}(W_t) = \mathbb{E}(W_t - W_0) = 0$$

$$\text{Var}(W_t) = \text{Var}(W_t - W_0) = t - 0 = t$$

For $t \geq s$:

$$\text{Cov}(W_s, W_t) = \text{Cov}(W_s, W_s + (W_t - W_s)) = \text{Cov}(W_s, W_s) = s$$

$$\text{Cov}(W_7, W_3) = 3.$$

Two friends

Definition

Stochastic process $(X_t, t \geq 0)$ that may be written as

$$X_t = aW_t + bt,$$

is called **brownian motion with drift and scaling**.

Definition

Stochastic process $(S_t, t \geq 0)$ that may be written as

$$S_t = S_0 \exp(aW_t + bt),$$

is called **geometric brownian motion**.

Two plots

here will be the plots of BM with drift and geometric BM

BM with drift and scaling

Exercise . Find $\mathbb{E}(5W_t + 6t)$ and $\text{Var}(5W_t + 6t)$.

$$\mathbb{E}(5W_t + 6t) = 0 + 6t = 6t$$

$$\text{Var}(5W_t + 6t) = \text{Var}(5W_t) = 25t$$

Frequently used expected values

Expected values of exponents:

- $\mathbb{E}(\exp(aZ)) = \exp(a^2/2)$ for $Z \sim \mathcal{N}(0; 1)$.
- $\mathbb{E}(\exp(aW_t)) = \exp(a^2t/2)$ for Wiener process W_t .

How these are obtained?



$$\begin{aligned}\mathbb{E}(\exp(aZ)) &= \int_{-\infty}^{+\infty} \exp(az) f(z) dz = \\ &= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz\end{aligned}$$

Moment generating function

Definition

The **moment generating function** (MGF) of a random variable X is defined as

$$M_X(a) = \mathbb{E}(\exp(aX)).$$

- $M_Z(a) = \exp(a^2/2)$ for a normal $Z \sim \mathcal{N}(0; 1)$.
- $M_{W_t}(a) = \exp(a^2 t/2)$ for a Wiener process W_t .

Why may we need MGF?

$$M'(u) = \frac{d}{du} \mathbb{E}(\exp(uX)) = \mathbb{E}(X \exp(uX))$$

$$M'(0) = \mathbb{E}(X)$$

MGF is a funny  way to calculate expected value!

$$M''(0) = \mathbb{E}(X^2)$$

$$M'''(0) = \mathbb{E}(X^3)$$

⋮

$$M^{(k)}(0) = \mathbb{E}(X^k)$$

Wiener process: summary

- Stochastic process with normal and independent increments.
- Wiener process with drift and geometric Wiener process.
- Moment generating function.

Conditional expectation

Conditional expectation: short plan

- Modeling information using **sigma-algebras**;
- **Properties** of conditional expected value;
- Conditional **variance**.

Modeling information

John knows the value of X .

Maria knows the value of X and Y .

Maria knows **more!**

How to model this **mathematically?**

Sigma-algebra

Informal definition



Sigma-algebra (σ -algebra) generated by random variables X and Y is the collection of all events that can be stated in terms of these random variables.

Notation: $\sigma(X, Y)$.

Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

Modeling information

John knows the value of X , $\mathcal{F}_J = \sigma(X)$.

Maria knows the value of X and Y , $\mathcal{F}_M = \sigma(X, Y)$

Maria knows **more**: $\mathcal{F}_J \subset \mathcal{F}_M$.

Measurability

Definition



The random variable Z is measurable with respect to σ -algebra \mathcal{F} if $\sigma(Z) \subset \mathcal{F}$.

Information in \mathcal{F} is sufficient to calculate the value of Z .

Informal theorem



The random variable Z is measurable with respect to $\sigma(X, Y)$ if and only if Z is a deterministic function of X and Y .

Best prediction

Informal definition



The **best prediction** of a random variable Y given σ -algebra \mathcal{F} is called **conditional expected value** $\mathbb{E}(Z \mid \mathcal{F})$.

Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y .

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a **random variable**.

Notation

- $\mathbb{E}(Z \mid \mathcal{F})$:
for a general σ -algebra \mathcal{F} ;
- $\mathbb{E}(Z \mid \sigma(X, Y))$ or $\mathbb{E}(Z \mid X, Y)$:
for σ -algebra generated by X and Y .

When we may omit conditioning?

- If Z is **independend** of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$:
If I know **nothing useful** about Z then I can drop my information.
- $\mathbb{E}(\mathbb{E}(Z \mid \mathcal{F})) = \mathbb{E}(Z)$:
The **average of best guess** is the average of predicted variable.

The case of known variable

If Z is **known** (measurable with respect to \mathcal{F}), then we may treat Z *like* a constant:

$$\mathbb{E}(Z \mid \mathcal{F}) = Z;$$

$$\mathbb{E}(2 \exp(5W_t) \mid W_t) = 2 \exp(5W_t);$$

$$\mathbb{E}(2ZR + Z^2 \mid \mathcal{F}) = 2Z\mathbb{E}(R \mid \mathcal{F}) + Z^2.$$

Conditional variance

Definition



The **conditional variance** $\text{Var}(Z \mid \mathcal{F})$ is the conditional expected value of the squared error of the best prediction,

$$\text{Var}(Z \mid \mathcal{F}) = \mathbb{E}(\Delta^2 \mid \mathcal{F}), \text{ where } \Delta = Z - \mathbb{E}(Z \mid \mathcal{F}).$$

Theorem



$$\text{Var}(Z \mid \mathcal{F}) = \mathbb{E}(Z^2 \mid \mathcal{F}) - (\mathbb{E}(Z \mid \mathcal{F}))^2.$$

Properties of conditional variance

- Irrelevant information may be omitted:
If Z is **independent** of \mathcal{F} then $\mathbb{E}(Z \mid \mathcal{F}) = \mathbb{E}(Z)$ and $\text{Var}(Z \mid \mathcal{F}) = \text{Var}(Z)$.
- If Z is **known** (measurable with respect to \mathcal{F}), then we may treat Z *like* a constant:

$$\text{Var}(2 \exp(5W_t) \mid W_t) = 0;$$

$$\text{Var}(Z^3 + 3ZR \mid \mathcal{F}) = 0 + (3Z)^2 \text{Var}(R \mid \mathcal{F}).$$

Conditioning: summary

- Sigma-algebra $\sigma(X, Y)$ is the collection of all events that **can be stated** using X and Y .
- Conditional expected value $\mathbb{E}(Z \mid X, Y)$ is the **best prediction** of Z using X and Y .
- Conditional variance $\text{Var}(Z \mid X, Y)$ is the conditional expected value of the **squared error** of the best prediction.

Martingales

Martingales: short plan

- **Filtration** models the information acquisition.
- Definition of a **martingale**.
- **Examples** of martingales.

Filtration

The σ -algebra \mathcal{F}_t describes all the information available at time t .

Definition

The family of sigma-algebras $(\mathcal{F}_t, t \geq 0)$ is called **filtration** if it grows in time, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Reminder: Sigma-algebra \mathcal{F}_t is the collection of events.

Natural filtration

Definition



The filtration $(\mathcal{F}_t, t \geq 0)$ is called a **natural filtration** of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

Examples

Let (\mathcal{F}_t) be a natural filtration of a Wiener process (W_t) .

$$\{W_2 < 5\} \in \mathcal{F}_2, \{W_2 > W_5\} \in \mathcal{F}_6,$$

$$\{W_2 < 5\} \notin \mathcal{F}_1, \{W_2 > W_5\} \notin \mathcal{F}_2.$$

Martingale

Definition



Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a **martingale** with respect to the filtration (\mathcal{F}_t) .

Usually we consider natural filtration (\mathcal{F}_t) of the process (M_t) .

Simple examples

Constant process:

If $M_t = 777$ for all t then $\mathbb{E}(M_t \mid \mathcal{F}_s) = 777 = M_s$.

Wiener process:

$$\mathbb{E}(W_t \mid \mathcal{F}_s) = \mathbb{E}(W_s + (W_t - W_s) \mid \mathcal{F}_s) = W_s + \mathbb{E}(W_t - W_s) = W_s.$$

More examples

Theorem



The process $Z_t = W_t^2 - t$ is a **martingale**.

Proof



$$\begin{aligned}\mathbb{E}(W_t^2 - t \mid \mathcal{F}_s) &= \mathbb{E}((W_s + (W_t - W_s))^2 \mid \mathcal{F}_s) - t = \\ &= \mathbb{E}(W_s^2 + (W_t - W_s)^2 + 2W_s(W_t - W_s) \mid \mathcal{F}_s) - t = \\ &= W_s^2 + \mathbb{E}((W_t - W_s)^2 \mid \mathcal{F}_s) + 2W_s\mathbb{E}(W_t - W_s \mid \mathcal{F}_s) - t = \\ &= W_s^2 + \mathbb{E}((W_t - W_s)^2) + 2W_s\mathbb{E}(W_t - W_s) - t = \\ &= W_s^2 + (t - s) + 2W_s \cdot 0 - t = W_s^2 - s\end{aligned}$$

More examples

Theorem



The process $Z_t = \exp(aW_t - a^2t/2)$ is a **martingale** for every constant a .

This martingale is very useful in Black and Scholes model.

Martingales in discrete time

Theorem

Consider a filtration $(\mathcal{F}_t, t \in \{0, 1, 2, \dots\})$ and a process $(M_t, t \in \{0, 1, 2, \dots\})$.

In discrete time the condition

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s \text{ for all } s \leq t$$

is completely equivalent to

$$\mathbb{E}(M_{t+1} \mid \mathcal{F}_t) = M_t.$$

Random walk

Consider independent and identically distributed Z_1, Z_2, \dots with $\mathbb{E}(Z_t) = 0$. The cumulative sum

$$S_t = Z_1 + Z_2 + \dots + Z_t, \text{ with } S_0 = 0$$

is called a **random walk**.

Theorem



The random walk process is a martingale.

$$\mathcal{F}_t = \sigma(Z_1, Z_2, Z_3, \dots, Z_t)$$

$$\mathbb{E}(S_{t+1} \mid \mathcal{F}_t) = \mathbb{E}(S_t + Z_{t+1} \mid \mathcal{F}_t) = S_t + \mathbb{E}(Z_{t+1}) = S_t.$$

Martingales: summary

- **Filtration** models the information acquisition.
- The best prediction of a **martingale** is its current value.
- Martingales related to **Wiener process, random walk**.