Itô's lemma

Itô's lemma

Itô's lemma: short plan

- Light version for functions of time and Wiener process.
- Check the martingale property with Itô's lemma.
- More general version.

Itô's lemma: light version

Informal theorem

If $Y_t = f(W_t, t)$ then it may be written as

$$Y_t = Y_0 + \int_0^t f'_W \, dW_u + \int_0^t \left(f'_t + \frac{1}{2} f''_{WW} \right) du.$$

Informal theorem

If $Y_t = f(W_t, t)$ then it may be written in the short form as

$$dY_t = f'_W dW_t + f'_t dt + \frac{1}{2} f''_{WW} dt.$$

Exercise

Express $Y_t = W_t^3 \cdot t^4$ as an Itô process .

$$
\begin{aligned}
dY_t &= 3W_t^2 t^4 \, dW_t + 4W_t^3 t^3 \, dt + \frac{1}{2} 6 W_t t^4 dt = \\
&= 3W_t^2 t^4 \, dW_t + (4W_t^3 t^3 + 3W_t t^4) \, dt.
\end{aligned}
$$

$$
Y_t = 0 + \int_0^t 3W_u^2 u^4 \, dW_u + \int_0^t (4W_u^3 u^3 + 3W_u u^4) \, du.
$$

Exercise

Check whether the process $Y_t = W_t^4 - t^2$ is a martingale .

$$
\begin{aligned}
dY_t &= 4W_t^3\, dW_t - 2t\, dt + \frac{1}{2} 12 W_t^2 dt = \\
&= 4W_t^3\, dW_t + (6W_t^2 - 2t)\, dt.
\end{aligned}
$$

$$
Y_t = 0 + \int_0^t 4W_u^3\, dW_u + \int_0^t (6W_u^2 - 2u)\, du.
$$

(Y_t) is not a martingale!

Exercise

Express $Y_t = W_t^2$ as an Itô process and prove the formula for $\int_0^t W_u \, dW_u$ .

$$dY_t = 2W_t \, dW_t + 0 \, dt + \frac{1}{2} 2dt = 2W_t \, dW_t + \, dt.$$

$$Y_t = 0 + \int_0^t 2W_u \, dW_u + \int_0^t 1 \, du.$$

$$W_t^2 = 0 + 2 \int_0^t W_u \, dW_u + t. \quad \to \quad \int_0^t W_u \, dW_u = \frac{1}{2}(W_t^2 - t).$$

Itô's lemma: general version

Informal theorem 

If $Y_t = f(X_t, t)$ where (X_t) is an Itô process then Y_t may be written in the short form as

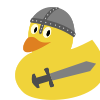
$$dY_t = f'_X dX_t + f'_t dt + \frac{1}{2} f''_{XX} (dX_t)^2,$$

where $(dX_t)^2$ is calculated using symbolic rules

$dt \cdot dW_t = 0$, $dt \cdot dt = 0$, $dW_t \cdot dW_t = dt$.

Exercise

Consider $dS_t = \mu S_t dt + \sigma S_t dW_t$ and $Y_t = S_t^2$.

Find dY_t and recover the full form for Y_t .

$$
\begin{aligned}
dY_t &= 2S_t\, dS_t + 0\, dt + \frac{1}{2}2(dS_t)^2 = \\
&= 2S_t(\mu S_t dt + \sigma S_t dW_t) + 1 \cdot (\mu S_t dt + \sigma S_t dW_t)^2 = \\
&= 2S_t(\mu S_t dt + \sigma S_t dW_t) + \sigma^2 S_t^2 dt = \\
&= (2\mu S_t^2 + \sigma^2 S_t^2) dt + 2\sigma S_t^2 dW_t.
\end{aligned}
$$

$$
Y_t = S_0^2 + \int_0^t 2\sigma S_u^2\, dW_u + \int_0^t (2\mu S_u^2 + \sigma^2 S_u^2)\, du.
$$

Itô's lemma: a way to memorize

Informal theorem 

If $Y_t = f(X_t, Z_t, t)$ where (X_t) and (Z_t) are Itô processes then the short form of Y_t may be obtained in two steps:

1. Calculate the second order Taylor expansion of f.
2. Simplify the result using symbolic rules
   $dt \cdot dW_t = 0$, $dt \cdot dt = 0$, $dW_t \cdot dW_t = dt$.

Reminder: dW_t, dY_t, dX_t, dZ_t do not exist!

It's only a quick way to find the full form.

Itô's lemma: summary

- Basic tool to study stochastic integrals.
- Easy to check whether the process is a martingale.
- Easily written in short form:
  $dt \cdot dW_t = 0$, $dt \cdot dt = 0$, $dW_t \cdot dW_t = dt$.

Black and Scholes model

Black and Scholes model: short plan

- Assumptions of the model.
- The main question.
- Solving the price stochastic differential equation.

BS-model: assumptions

- Unique share type is traded. The share price S_t satisfies

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

- The risk free interest rate r is constant and unique for all horizons.

$$dB_t = r B_t \, dt \quad B_t = B_0 \exp(rt), \; B_0 = 1.$$

- No taxes, no dividends, no transaction costs.
- Time is continuous, shares are infinitely divisible.
- Short selling is allowed.
- No arbitrage opportunities.
- The parameters r, σ, μ are known.

The Question

How can we calculate the price of a particular financial claim?

Example. At time moment $T = 2$ John would like to get 10 if the $S_2 > 100$ and nothing otherwise.

What is the fair price John should pay at $t = 0$?

SDE for the price

Stochastic differential equation for the price:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

In full form:

$$S_t = S_0 + \int_0^t \mu S_u \, du + \int_0^t \sigma S_u \, dW_u.$$

Solving the SDE

Stochastic differential equation for the price:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t.$$

By mnemonic rules,

$$(dS_t)^2 = (\mu S_t\, dt + \sigma S_t\, dW_t)^2 = 0 + 0 + \sigma^2 S_t^2 dt.$$

Let's consider $Y_t = \ln S_t$. Accordign to Itô's lemma,

$$dY_t = \frac{1}{S_t}\, dS_t + 0\, dt + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)(dS_t)^2 =$$

$$=\mu\, dt + \sigma dW_t - \frac{1}{2}\sigma^2\, dt.$$

Solving the SDE...

The log-price $Y_t = \ln S_t$ in short form,

$$dY_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t.$$

In full form,

$$Y_t = Y_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) du + \int_0^t \sigma\, dW_u$$

Without integrals,

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

Finally,

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

Itô's lemma: summary

- With BS model we will calculate the fair price.
- The share price follows geometric brownian motion,
$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$
- Risk free rate r is constant.
- There are no arbitrage opportunities.

Girsanov theorem

Girsanov theorem: short plan

- The wrong answer to the pricing problem.
- Idea of alternative probability.
- Girsanov theorem.

The wrong answer

Wrong intuition 

The future payoff X_T is random, we just need to calculate the expected payoff given all available information,

$$X_0 \overset{???}{=} \mathbb{E}(X_T \mid \mathcal{F}_0).$$

This is true for a martingale, but the claim price is not a martingale.

Alternative probability

A	$X = -2$	$X = 0$	$X = 4$
$\mathbb{P}(A)$	0.3	0.4	0.3
$\mathbb{P}^*(A)$	0.4	0.1	0.5

$$\mathbb{E}(X) = 0.6$$

$$\mathbb{E}^*(X) = 1.2$$

Idea

We will introduce a new probability $\mathbb{P}^*$ in the Black and Scholes model to simplify the calculation of prices.

Girsanov theorem

Theorem

If (W_t) is a Wiener process under probability $\mathbb{P}$ and $W_t^* = b \cdot t + W_t$, then there is a probability $\mathbb{P}^*$ such that (W_t^*) is a Wiener process under $\mathbb{P}^*$.

$$\mathbb{E}(W_t) = 0, \quad \mathbb{E}(W_t^*) = b \cdot t, \quad \mathbb{E}^*(W_t^*) = 0.$$

Girsanov theorem in BS model

Theorem

In the Black and Scholes model there is an alternative probability $\mathbb{P}^*$ such that (W_t^*) is a Wiener process under $\mathbb{P}^*$ and

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^*\right).$$

Old formula is still valid,

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right),$$

where (W_t) is a Wiener process under probability $\mathbb{P}$.

Link between (W_t) and (W_t^*)

Equivalent formula for share price means that

$$\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^*$$

We simplify,

$$(\mu - r)t + \sigma W_t = \sigma W_t^*$$

In short form

$$(\mu - r)dt + \sigma dW_t = \sigma dW_t^*.$$

The meaning of probabilities:

- $\mathbb{P}$ — real world probability;
- $\mathbb{P}^*$ — artificial probability to simplify formulas.

Girsanov theorem: summary

- Fair price is not a simple expected value.
- Girsanov theorem gives equivalent formula for S_t:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^*\right) = $$
$$= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

- (W_t^*) is a Wiener process under artificial probability $\mathbb{P}^*$,

$$(\mu - r)dt + \sigma dW_t = \sigma dW_t^*.$$