Wiener process and martingales

Wiener process

Stochastic calculus course

The goal: price an option in the framework of Black and Scholes model.

- Very short: 4 weeks only.
- Mathematics is hard.
- Informal definitions and theorems.
- Problem solving and computer simulations.

Wiener process

Here goes the plot!

Stochastic process

Definition

Stochastic or random process is a collection of random variables indexed by time variable t.

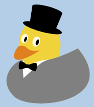
Continuous time: $(X_t, t \geq 0)$.

Discrete time: $(X_t, t \in \{0, 1, 2, 3, \ldots\})$.

Notation remark:

- $(X_t, t \geq 0)$ or (X_t) — the collection of random variables;
- X_t — one particular random variable.

Wiener process

Definition 

Stochastic process $(W_t, t \geq 0)$ is called Wiener process or Brownian motion if

1. $W_0 = 0$.
2. Increments $W_t - W_s$ are normally distributed $\mathcal{N}(0; t - s)$.
3. Increment $W_t - W_s$ is independent of the past values $(W_u, u \leq s)$.
4. $\mathbb{P}(\text{trajectory of } (W_t) \text{ is continuous}) = 1$.

Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

The main trick to study properties:

$$\text{Future value} = \text{Known value} + \text{Unpredictable change}$$

Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

Conditional probability exercise

Exercise . Calculate $\mathbb{P}(W_{10} > 2 \mid W_6 = 3)$.

$$\mathbb{P}(W_{10} > 2 \mid W_6 = 3) = \mathbb{P}(W_{10} - W_6 + W_6 > 2 \mid W_6 = 3) =$$
$$= \mathbb{P}(W_{10} - W_6 + 3 > 2 \mid W_6 = 3) = \mathbb{P}(W_{10} - W_6 > -1).$$

$$W_{10} - W_6 \sim \mathcal{N}(0; 4), \text{ hence } \frac{W_{10} - W_6 - 0}{\sqrt{4}} \sim \mathcal{N}(0; 1).$$

We will use standard normal distribution function

$F(u) = \mathbb{P}(Z \leq u)$, where $Z \sim \mathcal{N}(0; 1)$.

$$\mathbb{P}(W_{10} - W_6 > -1) = \mathbb{P}\left(\frac{W_{10} - W_6}{2} > -\frac{1}{2}\right) =$$
$$= \mathbb{P}(Z > -0.5) = \mathbb{P}(Z < 0.5) = F(0.5) \approx 0.69.$$

More gentlemen's agreements

On slides we will follow these agreements:

- s and t denote two arbitrary time moments with $0 \leq s \leq t$.
- (W_t) denotes a Wiener process.
- Z denotes a standard normal random variable, $Z \sim \mathcal{N}(0; 1)$.
- $F(u)$ denotes the standard normal distribution function, $F(u) = \mathbb{P}(Z \leq u)$.

Independence of increments: example

Property

Increment $W_t - W_s$ is independent of the past values $(W_u, u \leq s)$.

$W_6 - W_4$ is independend of W_4, W_3, $W_{2.5}$, W_1, ...

$W_6 - W_4$ is independent of $W_4 - W_3$, $W_{2.5} - W_1$.

The increments $W_6 - W_4$, $W_4 - W_3$, $W_{2.5} - W_1$ are independent.

Independence of increments: full glory

If the time intervals $[s_1, t_1]$, $[s_2, t_2]$, ..., $[s_k, t_k]$ are non overlapping,

Here will be a small picture

then the increments $W(t_1) - W(s_1)$, $W(t_2) - W(s_2)$, ..., $W(t_k) - W(s_k)$ are independent.

Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \leq s_{j+1}$.

Expectation and variance

Exercise . Find $\mathbb{E}(W_t)$, $\mathrm{Var}(W_t)$, $\mathrm{Cov}(W_s, W_t)$.

$$\mathbb{E}(W_t) = \mathbb{E}(W_t - W_0) = 0$$

$$\mathrm{Var}(W_t) = \mathrm{Var}(W_t - W_0) = t - 0 = t$$

For $t \geq s$:

$$\mathrm{Cov}(W_s, W_t) = \mathrm{Cov}(W_s, W_s + (W_t - W_s)) = \mathrm{Cov}(W_s, W_s) = s$$

$$\mathrm{Cov}(W_7, W_3) = 3.$$

Two friends

Definition

Stochastic process $(X_t, t \geq 0)$ that may be written as

$$X_t = aW_t + bt,$$

is called brownian motion with drift and scaling.

Definition

Stochastic process $(S_t, t \geq 0)$ that may be written as

$$S_t = S_0 \exp(aW_t + bt),$$

is called geometric brownian motion.

Two plots

here will be the plots of BM with drift and geometric BM

BM with drift and scaling

Exercise . Find $\mathbb{E}(5W_t + 6t)$ and $\mathrm{Var}(5W_t + 6t)$.

$$\mathbb{E}(5W_t + 6t) = 0 + 6t = 6t$$

$$\mathrm{Var}(5W_t + 6t) = \mathrm{Var}(5W_t) = 25t$$

Frequently used expected values

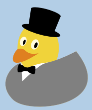
Expected values of exponents:

- $\mathbb{E}(\exp(aZ)) = \exp\left(a^2/2\right)$ for $Z \sim \mathcal{N}(0;1)$.
- $\mathbb{E}(\exp(aW_t)) = \exp\left(a^2 t/2\right)$ for Wiener process W_t.

How these are obtained?

$$\begin{aligned}
\mathbb{E}(\exp(aZ)) &= \int_{-\infty}^{+\infty} \exp(az) f(z)\, dz = \\
&= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz
\end{aligned}$$

Moment generating function

Definition 

The moment generating function (MGF) of a random variable X is defined as

$$M_X(a) = \mathbb{E}(\exp(aX)).$$

- $M_Z(a) = \exp\left(a^2/2\right)$ for a normal $Z \sim \mathcal{N}(0;1)$.
- $M_{W_t}(a) = \exp\left(a^2t/2\right)$ for a Wiener process W_t.

Why may we need MGF?

$$M'(u) = \frac{d}{du}\mathbb{E}(\exp(uX)) = \mathbb{E}(X\exp(uX))$$

$$M'(0) = \mathbb{E}(X)$$

MGF is a funny  way to calculate expected value!

$$\begin{aligned} M''(0) &= \mathbb{E}(X^2) \\ M'''(0) &= \mathbb{E}(X^3) \\ &\vdots \\ M^{(k)}(0) &= \mathbb{E}(X^k) \end{aligned}$$

Wiener process: summary

- Stochastic process with normal and independent increments.
- Wiener process with drift and geometric Wiener process.
- Moment generating function.

Conditional expectation

Conditional expectation: short plan

- Modeling information using sigma-algebras;
- Properties of conditional expected value;
- Conditional variance.

Modeling information

John knows the value of X.

Maria knows the value of X and Y.

Maria knows more!

How to model this mathematically?

Sigma-algebra

Definition 

Sigma-algebra (σ-algebra) generated by random variables X and Y is the collection of all events that can be stated in terms of these random variables.

Notation: $\sigma(X, Y)$.

Example

The sigma-algebra $\sigma(X, Y)$ contains the events $\{X < 5\}$, $\{X > 2Y\}$, $\{\sin Y > \cos X\}$, ...

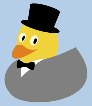
Modeling information

John knows the value of X, $\mathcal{F}_J = \sigma(X)$.

Maria knows the value of X and Y, $\mathcal{F}_M = \sigma(X, Y)$

Maria knows more: $\mathcal{F}_J \subset \mathcal{F}_M$.

Measurability

Definition 

The random variable Z is measurable with respect to σ-algebra $\mathcal{F}$ if $\sigma(Z) \subset \mathcal{F}$.

Information in $\mathcal{F}$ is sufficient to calculate the value of Z.

Theorem 

The random variable Z is measurable with respect to $\sigma(X, Y)$ if and only if Z is a deterministic function of X and Y.

Best prediction

Definition 

The best prediction of a random variable Y given σ-algebra $\mathcal{F}$ is called conditional expected value $\mathbb{E}(Z \mid \mathcal{F})$.

Difference of $\mathbb{E}(Z \mid \mathcal{F})$ and $\mathbb{E}(Z)$

If I know X and Y then my best prediction of Z may depend on X and Y.

In general: $\mathbb{E}(Z \mid \mathcal{F})$ is a random variable.

Notation

- $\mathbb{E}(Z \mid \mathcal{F})$:
  for a general σ-algebra $\mathcal{F}$;
- $\mathbb{E}(Z \mid \sigma(X, Y))$ or $\mathbb{E}(Z \mid X, Y)$:
  for σ-algebra generated by X and Y.

When we may omit conditioning?

- If Z is independend of X and Y then $\mathbb{E}(Z \mid X, Y) = \mathbb{E}(Z)$:
  If I know nothing useful about Z then I can drop my information.
- $\mathbb{E}(\mathbb{E}(Z \mid \mathcal{F})) = \mathbb{E}(Z)$:
  The average of best guess is the average of predicted variable.

The case of known variable

If Z is known (measurable with respect to $\mathcal{F}$), then we may treat Z *like* a constant:

$$\mathbb{E}(Z \mid \mathcal{F}) = Z;$$

$$\mathbb{E}(2\exp(5W_t) \mid W_t) = 2\exp(5W_t);$$

$$\mathbb{E}(2ZR + Z^2 \mid \mathcal{F}) = 2Z\mathbb{E}(R \mid \mathcal{F}) + Z^2.$$

Conditional variance

Definition

The conditional variance $\mathrm{Var}(Z \mid \mathcal{F})$ is the conditional expected value of the squared error of the best prediction,

$$\mathrm{Var}(Z \mid \mathcal{F}) = \mathbb{E}(\Delta^2 \mid \mathcal{F}), \text{ where } \Delta = Z - \mathbb{E}(Z \mid \mathcal{F}).$$

Theorem

$$\mathrm{Var}(Z \mid \mathcal{F}) = \mathbb{E}(Z^2 \mid \mathcal{F}) - (\mathbb{E}(Z \mid \mathcal{F}))^2.$$

Properties of conditional variance

- Irrelevant information may be omitted:
  If Z is independent of $\mathcal{F}$ then $\mathbb{E}(Z \mid \mathcal{F}) = \mathbb{E}(Z)$ and $\mathrm{Var}(Z \mid \mathcal{F}) = \mathrm{Var}(Z)$.
- If Z is known (measurable with respect to $\mathcal{F}$), then we may treat Z *like* a constant:

$$\mathrm{Var}(2\exp(5W_t) \mid W_t) = 0;$$

$$\mathrm{Var}(Z^3 + 3ZR \mid \mathcal{F}) = 0 + (3Z)^2\,\mathrm{Var}(R \mid \mathcal{F}).$$

Conditioning: summary

- Sigma-algebra $\sigma(X, Y)$ is the collection of all events that can be stated using X and Y.
- Conditional expected value $\mathbb{E}(Z \mid X, Y)$ is the best prediction of Z using X and Y.
- Conditional variance $\mathrm{Var}(Z \mid X, Y)$ is the conditional expected value of the squared error of the best prediction.

Martingales

Martingales: short plan

- Filtration models the information acquisition.
- Definition of a martingale.
- Examples of martingales.

Filtration

The σ-algebra $\mathcal{F}_t$ describes all the information available at time t.

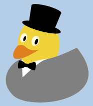
Definition

The family of sigma-algebras $(\mathcal{F}_t, t \geq 0)$ is called filtration if it grows in time, $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

Reminder: Sigma-algebra $\mathcal{F}_t$ is the collection of events.

Natural filtration

Definition



The filtration $(\mathcal{F}_t, t \geq 0)$ is called a natural filtration of a process $(X_t, t \geq 0)$ if at time t you have only the information about past values of the process,

$$\mathcal{F}_t = \sigma(X_u, u \in [0; t]).$$

Examples

Let $(\mathcal{F}_t)$ be a natural filtration of a Wiener process (W_t).

$\{W_2 < 5\} \in \mathcal{F}_2$, $\{W_2 > W_5\} \in \mathcal{F}_6$,

$\{W_2 < 5\} \notin \mathcal{F}_1$, $\{W_2 > W_5\} \notin \mathcal{F}_2$.

Martingale

Definition

Consider a filtration $(\mathcal{F}_t, t \geq 0)$ and a process $(M_t, t \geq 0)$.

If the best prediction of the future value M_t of a process is its current value M_s for $s \leq t$,

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s,$$

then (M_t) is called a martingale with respect to the filtration $(\mathcal{F}_t)$.

Usually we consider natural filtration $(\mathcal{F}_t)$ of the process (M_t).

Simple examples

Constant process:

If $M_t = 777$ for all t then $\mathbb{E}(M_t \mid \mathcal{F}_s) = 777 = M_s$.

Wiener process:

$$\mathbb{E}(W_t \mid \mathcal{F}_s) = \mathbb{E}(W_s + (W_t - W_s) \mid \mathcal{F}_s) = W_s + \mathbb{E}(W_t - W_s) = W_s.$$

More examples

Theorem

The process $Z_t = W_t^2 - t$ is a martingale.

Proof

$$
\begin{aligned}
\mathbb{E}(W_t^2 - t \mid \mathcal{F}_s) = \mathbb{E}((W_s + (W_t - W_s))^2 \mid \mathcal{F}_s) - t = \\
&= \mathbb{E}(W_s^2 + (W_t - W_s)^2 + 2W_s(W_t - W_s) \mid \mathcal{F}_s) - t = \\
&= W_s^2 + \mathbb{E}((W_t - W_s)^2 \mid \mathcal{F}_s) + 2W_s \mathbb{E}(W_t - W_s \mid \mathcal{F}_s) - t = \\
&= W_s^2 + \mathbb{E}((W_t - W_s)^2) + 2W_s \mathbb{E}(W_t - W_s) - t = \\
&= W_s^2 + (t - s) + 2W_s \cdot 0 - t = W_s^2 - s
\end{aligned}
$$

More examples

Theorem

The process $Z_t = \exp\left(aW_t - a^2t/2\right)$ is a martingale for every constant a.

This martingale is very useful in Black and Scholes model.

Martingales in discrete time

Theorem

Consider a filtration $(\mathcal{F}_t, t \in \{0, 1, 2, \ldots\})$ and a process $(M_t, t \in \{0, 1, 2, \ldots\})$.

In discrete time the condition

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = M_s \text{ for all } s \leq t$$

is completely equivalent to

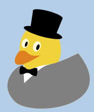
$$\mathbb{E}(M_{t+1} \mid \mathcal{F}_t) = M_t.$$

Random walk

Consider independent and identically distributed Z_1, Z_2, ... with $\mathbb{E}(Z_t) = 0$. The cumulative sum

$$S_t = Z_1 + Z_2 + \ldots + Z_t, \text{ with } S_0 = 0$$

is called a random walk.

Theorem 

The random walk process is a martingale.

$$\mathcal{F}_t = \sigma(Z_1, Z_2, Z_3, \ldots, Z_t)$$

$$\mathbb{E}(S_{t+1} \mid \mathcal{F}_t) = \mathbb{E}(S_t + Z_{t+1} \mid \mathcal{F}_t) = S_t + \mathbb{E}(Z_{t+1}) = S_t.$$

Martingales: summary

- Filtration models the information acquisition.
- The best prediction of a martingale is its current value.
- Martingales related to Wiener process, random walk.