

# Wiener process and martingales

**Itô integral**

# Itô integral: short plan

- Intuitive definition.
- Examples.

# Itô integral

## Definition



The Itô integral  $I_t = \int_0^t A_u dW_u$  is defined as the **total net cash flow** of a strategy if we treat  $W_u$  as the price of the asset and  $A_u$  as the quantity of the asset at time moment  $u$ .

$(W_t)$  is a poor model for the price, but the intuition is ok:

$$\text{Net cash flow} = \int_0^t \text{Quantity}_u d\text{Price}_u$$

# Simple deterministic example

Let's calculate  $\int_2^8 7 du$  

Transaction 1. At time  $u = 2$  we buy 7 units. The price is  $u = 2$ .

Cash flow:  $-7 \cdot 2$ .

Transaction 2. At time  $u = 8$  we sell 7 units. The price is  $u = 8$ .

Cash flow:  $7 \cdot 8$ .

$$\int_2^8 7 du = -14 + 56 = 42.$$

## Example with Wiener process

Let's calculate  $\int_2^8 5 dW_u$  

Transaction 1. At time  $u = 2$  we buy 5 units. The price is  $W_2$ .

Cash flow:  $-5 \cdot W_2$ .

Transaction 2. At time  $u = 8$  we sell 5 units. The price is  $W_8$ .

Cash flow:  $5 \cdot W_8$ .

$$\int_2^8 5 dW_u = -5W_2 + 5W_8.$$

## More gentlemen's agreement

$$I_t = \int_0^t (\text{something}_u) dW_u :$$

$t$  — the upper limit of integration;

$u$  — time variable with range from 0 to  $t$ ;

# Why old integration formulas are wrong?

$$\int_0^t W_u dW_u = \text{🦆}?$$

Why not  $\frac{1}{2}W_t^2 - \frac{1}{2}W_0^2$ ?

The guessed value is non-negative,  $\frac{1}{2}W_t^2 \geq 0$ !

If you buy and sell Wiener process you can have negative cash flow!



## Small table

In most cases Itô integral **can not** be computed explicitly.

$$\int_0^t 1 dW_u = W_t$$

$$\int_0^t W_t dW_u = \frac{W_t^2 - t}{2}$$

$$\int_0^t \exp\left(aW_u - \frac{1}{2}a^2u\right) dW_u = \frac{1}{a} \left( \exp\left(aW_t - \frac{1}{2}a^2t\right) - 1 \right)$$

# Itô integral: summary

- Total net cash flow of a strategy.
- Rarely can be computed explicitly.
- Old rules of integration do not apply.

**Itô integral properties**

# Itô integral properties: short plan

- Common properties with Riemann integral.
- Zero expected value.
- Itô isometry.

## Common properties

$$\int_a^b X_u dW_u + \int_b^c X_u dW_u = \int_a^c X_u dW_u$$

$$\int_a^a X_u dW_u = 0$$

$$\int_0^t c X_u dW_u = c \int_0^t X_u dW_u$$

# Zero expected value

Intuition: we buy and sell Wiener process, hence, expected net cash flow should be zero.

## Informal theorem

For any reasonable process  $(X_t)$  measurable with respect to the natural filtration  $(\mathcal{F}_t)$  of the process  $(W_t)$

$$\mathbb{E} \left( \int_0^t X_u dW_u \right) = 0.$$

# Itô isometry

## Informal theorem



For any reasonable process  $(X_t)$  measurable with respect to the natural filtration  $(\mathcal{F}_t)$  of the process  $(W_t)$

$$\text{Var} \left( \int_0^t X_u dW_u \right) = \int_0^t \mathbb{E}(X_u^2) du.$$

## Exercise

Find  $\mathbb{E}(I_t)$  and  $\text{Var}(I_t)$  for  $I_t = \int_0^t W_u^2 dW_u$  🦆

$$\mathbb{E}(I_t) = 0;$$

$$\text{Var}(I_t) = \int_0^t \mathbb{E}(W_u^4) du = \int_0^t 3u^2 du = t^3.$$



# Itô properties: summary

- Zero expected value.
- Using Itô isometry one may calculate variance.
- Some common properties with Riemann integral.

**Itô process**

# Itô process: short plan

- Definition of an Itô process.
- When Itô process is a martingale?
- Short and full form notation.

# Itô process

## Definition



Stochastic process  $(Y_t, t \geq 0)$  is called **Itô process** if it can be written in the form

$$Y_t = Y_0 + \int_0^t A_u dW_u + \int_0^t B_u du,$$

where  $Y_0$  is a constant.

A wide class of continuous stochastic processes that behave **locally** like a Wiener process with drift.

# Itô integral is a martingale

## Informal theorem



Itô process  $(Y_t)$  is a martingale if and only if it has only Itô integral in the representation

$$Y_t = Y_0 + \int_0^t A_u dW_u.$$

The best guess of a future value of an Itô integral is its current value:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = Y_s \text{ for } s \leq t.$$

# Expected value of Itô process

## Informal theorem



For any reasonable process  $(B_t)$

$$\mathbb{E} \left( \int_0^t B_u du \right) = \int_0^t \mathbb{E}(B_u) du.$$

## Informal theorem



If  $(Y_t)$  is an Itô process with  $Y_t = Y_0 + \int_0^t A_u dW_u + \int_0^t B_u du$ ,  
then

$$\mathbb{E}(Y_t) = Y_0 + \int_0^t \mathbb{E}(B_u) du.$$

# Short form notation

Full form:

$$Y_t = Y_0 + \int_0^t A_u dW_u + \int_0^t B_u du.$$

Short form:

$$dY_t = A_t dW_t + B_t dt.$$

$dW_t$  and  $dY_t$  have **no meaning!**

## Short form in simulations

We need to simulate a path of

$$Y_t = Y_0 + \int_0^t A_u dW_u + \int_0^t B_u du.$$

In short form:

$$dY_t = A_t dW_t + B_t dt.$$

In simulations:

$$Y_{t+\Delta} - Y_t \approx A_t \cdot (W_{t+\Delta} - W_t) + B_t \cdot \Delta,$$

where  $W_{t+\Delta} - W_t \sim \mathcal{N}(0; \Delta)$ .



## Short form: examples

$$dY_t = W_t^4 dW_t \quad Y_t = Y_0 + \int_0^t W_u^4 dW_u.$$

$$dY_t = \cos(W_t) dt \quad Y_t = Y_0 + \int_0^t \cos(W_u) du.$$

**Informal theorem**



Itô process  $(Y_t)$  is a martingale if and only if

$$dY_t = A_t dW_t.$$

# Itô process: summary

- A **sum three terms**: constant, Itô integral and Riemann integral.
- Will be a martingale **without Riemann integral**.
- Often written using **short form** with  $dt$  and  $dW_t$ .