Option pricing

Discounted price process

Discounted price process: plan

- Discounting in discrete and continuous time.

Discounted price process: plan

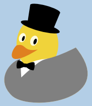
- Discounting in discrete and continuous time.
- Every asset can be replicated.

Discounted price process: plan

- Discounting in discrete and continuous time.
- Every asset can be replicated.
- The pricing formula.

Discounting

Definition in discrete time



If X_t is the price of a claim at time t and r is the interest rate then discounted price is defined as

$$\frac{X_t}{(1+r)^t} = (1+r)^{-t} X_t.$$

Discounting

Definition in discrete time

If X_t is the price of a claim at time t and r is the interest rate then discounted price is defined as

$$\frac{X_t}{(1+r)^t} = (1+r)^{-t} X_t.$$

Definition in continuous time

Discounted price is defined as

$$\frac{X_t}{(\exp r)^t} = \frac{X_t}{\exp(rt)} = \exp(-rt) X_t.$$

Discounting

Definition in discrete time

If X_t is the price of a claim at time t and r is the interest rate then discounted price is defined as

$$\frac{X_t}{(1+r)^t} = (1+r)^{-t} X_t.$$

Definition in continuous time

Discounted price is defined as

$$\frac{X_t}{(\exp r)^t} = \frac{X_t}{\exp(rt)} = \exp(-rt) X_t.$$

For small r the definitions are close as $\exp(r) \approx 1 + r$.

Discounting

Definition in discrete time

If X_t is the price of a claim at time t and r is the interest rate then discounted price is defined as

$$\frac{X_t}{(1+r)^t} = (1+r)^{-t} X_t.$$

Definition in continuous time

Discounted price is defined as

$$\frac{X_t}{(\exp r)^t} = \frac{X_t}{\exp(rt)} = \exp(-rt) X_t.$$

For small r the definitions are close as $\exp(r) \approx 1 + r$.

For $t = 0$ discounted price and price are equal.

Is the discounted share price a martingale?

In short form,

$$
\begin{aligned}
d(\exp(-rt)S_t) = -r\exp(-rt)S_t dt + \exp(-rt)dS_t + \frac{0}{2}\cdot(dS_t)^2 = \\
= - r\exp(-rt)S_t dt + \exp(-rt)(\mu S_t dt + \sigma S_t dW_t) = \\
= \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right).
\end{aligned}
$$

Is the discounted share price a martingale?

In short form,

$$
\begin{aligned}
d(\exp(-rt)S_t) = -r\exp(-rt)S_t dt + \exp(-rt)dS_t + \frac{0}{2}\cdot(dS_t)^2 = \\
= - r\exp(-rt)S_t dt + \exp(-rt)(\mu S_t dt + \sigma S_t dW_t) = \\
= \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right).
\end{aligned}
$$

No, under $\mathbb{P}$ short form has dt term inside!

$$S_0 \neq \mathbb{E}(\exp(-rt)S_t \mid \mathcal{F}_0).$$

Is the discounted share price a martingale?

Let's recall,

$$d(\exp(-rt)S_t) = \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right).$$

Is the discounted share price a martingale?

Let's recall,

$$d(\exp(-rt)S_t) = \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right).$$

But wait, $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$, so

$$d(\exp(-rt)S_t) = \exp(-rt)S_t\sigma dW_t^*.$$

Is the discounted share price a martingale?

Let's recall,

$$d(\exp(-rt)S_t) = \exp(-rt)S_t \left((\mu - r)dt + \sigma dW_t\right).$$

But wait, $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$, so

$$d(\exp(-rt)S_t) = \exp(-rt)S_t \sigma dW_t^*.$$

Yes, under $\mathbb{P}^*$ short form has no dt term inside!

$$S_0 = \mathbb{E}^*(\exp(-rt)S_t \mid \mathcal{F}_0).$$

Replicating strategy

Informal theorem 

In the Black and Scholes model every european type asset can be replicated by a self-financing stategy that trades shares and risk free bonds.

$$X_t = y_t S_t + z_t B_t,$$
$$dX_t = y_t dS_t + z_t dB_t.$$

Replicating strategy

Informal theorem

In the Black and Scholes model every european type asset can be replicated by a self-financing stategy that trades shares and risk free bonds.

$$X_t = y_t S_t + z_t B_t,$$
$$dX_t = y_t dS_t + z_t dB_t.$$

European type asset gives payoff at a fixed time moment T.

Replicating strategy

Informal theorem

In the Black and Scholes model every european type asset can be replicated by a self-financing stategy that trades shares and risk free bonds.

$$X_t = y_t S_t + z_t B_t,$$
$$dX_t = y_t dS_t + z_t dB_t.$$

European type asset gives payoff at a fixed time moment T.

Self-financing strategy means no exogenous capital flow.

The pricing formula

Informal theorem 

In the Black and Scholes model the discounted price of every european type asset is a martingale under probability $\mathbb{P}^*$, hence

$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

The pricing formula

Informal theorem 

In the Black and Scholes model the discounted price of every european type asset is a martingale under probability $\mathbb{P}^*$, hence

$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

- (W_t^*) is a Wiener process under $\mathbb{P}^*$.

The pricing formula

Informal theorem

In the Black and Scholes model the discounted price of every european type asset is a martingale under probability $\mathbb{P}^*$, hence

$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

- (W_t^*) is a Wiener process under $\mathbb{P}^*$.
- $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$.

The pricing formula

Informal theorem

In the Black and Scholes model the discounted price of every european type asset is a martingale under probability $\mathbb{P}^*$, hence

$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

- (W_t^*) is a Wiener process under $\mathbb{P}^*$.
- $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$.
- Discounted share price $\exp(-rt)S_t$ is a martingale under $\mathbb{P}^*$.

Discounted price process: summary

- European claim gives payoff at a fixed moment of time T.

Discounted price process: summary

- European claim gives payoff at a fixed moment of time T.
- The discounted price of any european type claim is a martingale under $\mathbb{P}^*$.

Discounted price process: summary

- European claim gives payoff at a fixed moment of time T.
- The discounted price of any european type claim is a martingale under $\mathbb{P}^*$.
- Every European claim may be replicated.

Discounted price process: summary

- European claim gives payoff at a fixed moment of time T.
- The discounted price of any european type claim is a martingale under $\mathbb{P}^*$.
- Every European claim may be replicated.
- The pricing formula is

$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

Call option price

Call option price: plan

- Definition of call and put options.

Call option price: plan

- Definition of call and put options.
- Put-call option parity.

Call option price: plan

- Definition of call and put options.
- Put-call option parity.
- The price of a call option.

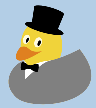
Classic options

Definition

The call option gives a right to buy one share at a specified strike price K on a specified date T.

The put option gives a right to sell one share at a specified strike price K on a specified date T.

Classic options

Definition 

The call option gives a right to buy one share at a specified strike price K on a specified date T.

The put option gives a right to sell one share at a specified strike price K on a specified date T.

$$C_T = \begin{cases} S_T - K, & \text{if } S_T > K; \\ 0, & \text{otherwise.} \end{cases} \qquad P_T = \begin{cases} K - S_T, & \text{if } S_T < K; \\ 0, & \text{otherwise.} \end{cases}$$

Put-call parity

$$C_T = \begin{cases} S_T - K, & \text{if } S_T > K; \\ 0, & \text{otherwise.} \end{cases} \qquad P_T = \begin{cases} K - S_T, & \text{if } S_T < K; \\ 0, & \text{otherwise.} \end{cases}$$

Put-call parity

$$C_T = \begin{cases} S_T - K, & \text{if } S_T > K; \\ 0, & \text{otherwise.} \end{cases} \qquad P_T = \begin{cases} K - S_T, & \text{if } S_T < K; \\ 0, & \text{otherwise.} \end{cases}$$

$$C_T - P_T = S_T - K$$

Put-call parity

$$C_T = \begin{cases} S_T - K, & \text{if } S_T > K; \\ 0, & \text{otherwise.} \end{cases} \qquad P_T = \begin{cases} K - S_T, & \text{if } S_T < K; \\ 0, & \text{otherwise.} \end{cases}$$

$$C_T - P_T = S_T - K$$

$$C_0 - P_0 = S_0 - \exp(-rT)K$$

Call option price

The pricing formula,

$$C_0 = \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0).$$

Call option price

The pricing formula,

$$C_0 = \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0).$$

We rewrite C_T using indicator $I = I(S_T > K)$,

$$C_T = I \cdot (S_T - K) = I \cdot S_T - I \cdot K.$$

Call option price

The pricing formula,

$$C_0 = \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0).$$

We rewrite C_T using indicator $I = I(S_T > K)$,

$$C_T = I \cdot (S_T - K) = I \cdot S_T - I \cdot K.$$

Let's split into two terms,

$$\begin{aligned}\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \mathbb{E}^*(I \cdot S_T - I \cdot K \mid \mathcal{F}_0) = \\ = \mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) - \mathbb{E}^*(I \cdot K \mid \mathcal{F}_0);\end{aligned}$$

The second term

Strike price K is constant,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{E}^*(I \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0).$$

The second term

Strike price K is constant,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{E}^*(I \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0).$$

Let's go down to W_T^*,

$$\{S_T > K\} = \{\ln S_t > \ln K\} = \{\ln S_0 + (r - \sigma^2/2)T + \sigma W_T^* > \ln K\}$$

The second term

Strike price K is constant,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{E}^*(I \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0).$$

Let's go down to W_T^*,

$$\{S_T > K\} = \{\ln S_t > \ln K\} = \{\ln S_0 + (r - \sigma^2/2)T + \sigma W_T^* > \ln K\}$$

Or,

$$\{S_T > K\} = \left\{ W_T^* > \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma} \right\}$$

The second term

Strike price K is constant,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{E}^*(I \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0).$$

Let's go down to W_T^*,

$$\{S_T > K\} = \{\ln S_t > \ln K\} = \{\ln S_0 + (r - \sigma^2/2)T + \sigma W_T^* > \ln K\}$$

Or,

$$\{S_T > K\} = \left\{ W_T^* > \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma} \right\}$$

Let's standardise and reverse the inequality,

$$\{S_T > K\} = \left\{ \frac{0 - W_T^*}{\sqrt{T}} < d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right\}.$$

The second term…

We've done one half of the job,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0) = KF(d),$$

where

$$d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

The final answer

The first term,

$$\begin{aligned}\mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) = & \mathbb{E}^*(I(W_T^* < d\sqrt{T}) \cdot S_0 \cdot \exp\Big((r - \sigma^2/2)T + \sigma W \\ = & S_0 \exp\Big((r - \sigma^2/2)T\Big)\mathbb{E}^*(I(W_T^* < d\sqrt{T}) \cdot \exp\big(\sigma W_T^* \\ = & S_0 F(d + \sigma\sqrt{T}).\end{aligned}$$

The final answer

The first term,

$$\begin{aligned}\mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) = \mathbb{E}^*(I(W_T^* < d\sqrt{T}) \cdot S_0 \cdot \exp\Big((r - \sigma^2/2)T + \sigma W \\ &= S_0 \exp\Big((r - \sigma^2/2)T\Big)\mathbb{E}^*(I(W_T^* < d\sqrt{T}) \cdot \exp\big(\sigma W_T^* \\ &= S_0 F(d + \sigma\sqrt{T}).\end{aligned}$$

The call option price,

$$\begin{aligned}C_0 &= \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \\ &= \exp(-rT)(S_0 F(d + \sigma\sqrt{T}) - KF(d)),\end{aligned}$$

where $d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$.

Call option price: summary

- Call option is the right to buy a share, put option is the right to sell a share.

Call option price: summary

- Call option is the right to buy a share, put option is the right to sell a share.
- Put-call parity between their prices,

$$C_0 - P_0 = S_0 - \exp(-rT)K.$$

Call option price: summary

- Call option is the right to buy a share, put option is the right to sell a share.
- Put-call parity between their prices,

$$C_0 - P_0 = S_0 - \exp(-rT)K.$$

- The call price is The call option price,

$$\begin{aligned} C_0 &= \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \\ &= \exp(-rT)(S_0 F(d + \sigma\sqrt{T}) - KF(d)), \end{aligned}$$

where $d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$.