Wiener process and martingales

Wiener process

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Wiener process

Here goes the plot!

Stochastic process

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Stochastic or random process is a collection of random variables indexed by time variable t.

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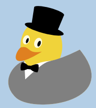
Definition 

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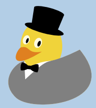
Definition 

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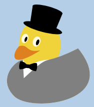
Definition 

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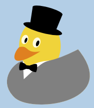
Definition 

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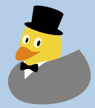
Definition 

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Tradition: when we consider two arbitrary moments of time, s and t, we usually assume $s \leq t$.

Divide and conquer

The main trick to study properties:

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Seems trivial 

$$W_t = W_s + (W_t - W_s)$$

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$$= \mathbb{P}(Z > -0.5) = \mathbb{P}(Z < 0.5) = F(0.5) \approx 0.69.$$

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On slides we will follow these agreements:

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Remark: the right border of an interval may touch the left border of the next one, but may not exceed it, $t_j \leq s_{j+1}$.

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$$\mathrm{Cov}(W_7, W_3) = 3.$$

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Stochastic process $(S_t, t \geq 0)$ that may be written as

$$S_t = S_0 \exp(aW_t + bt),$$

is called geometric brownian motion.

Two plots

here will be the plots of BM with drift and geometric BM

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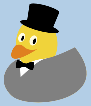
How these are obtained? 

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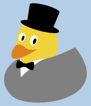
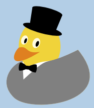
How these are obtained? 

$$\mathbb{E}(\exp(aZ)) = \int_{-\infty}^{+\infty} \exp(az) f(z)\, dz =$$
$$= \int_{-\infty}^{+\infty} \exp(az) \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz$$

Moment generating function

Definition 

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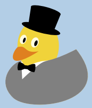
Definition 

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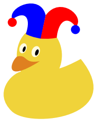
MGF is a funny  way to calculate expected value!

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