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# Itô integral: short plan

- Intuitive definition.

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- Examples.

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$$\int_2^8 7 du = -14 + 56 = 42.$$

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$$\int_2^8 5 dW_u = -5W_2 + 5W_8.$$

## More gentlemen's agreement

$$I_t = \int_0^t (\text{something}_u) dW_u :$$

$t$  — the upper limit of integration;

$u$  — time variable with range from 0 to  $t$ ;

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If you buy and sell Wiener process you can have negative cash flow!



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$$\int_0^t \exp\left(aW_u - \frac{1}{2}a^2u\right) dW_u = \frac{1}{a} \left( \exp\left(aW_t - \frac{1}{2}a^2t\right) - 1 \right)$$

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- Total net cash flow of a strategy.
- Rarely can be computed explicitly.
- Old rules of integration do not apply.

**Itô integral properties**



# Itô integral properties: short plan

- Common properties with Riemann integral.

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- Zero expected value.

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- Common properties with Riemann integral.
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- Itô isometry.

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Intuition: we buy and sell Wiener process, hence, expected net cash flow should be zero.

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## Informal theorem



For any reasonable process  $(X_t)$  measurable with respect to the natural filtration  $(\mathcal{F}_t)$  of the process  $(W_t)$

$$\mathbb{E} \left( \int_0^t X_u dW_u \right) = 0.$$



# Itô isometry

## Informal theorem



For any reasonable process  $(X_t)$  measurable with respect to the natural filtration  $(\mathcal{F}_t)$  of the process  $(W_t)$

$$\text{Var} \left( \int_0^t X_u dW_u \right) = \int_0^t \mathbb{E}(X_u^2) du.$$

## Exercise

Find  $\mathbb{E}(I_t)$  and  $\text{Var}(I_t)$  for  $I_t = \int_0^t W_u^2 dW_u$  🦆

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**Itô process**

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- Definition of an Itô process.
- When Itô process is a martingale?
- Short and full form notation.

# Itô process

## Definition



Stochastic process  $(Y_t, t \geq 0)$  is called **Itô process** if it can be written in the form

$$Y_t = Y_0 + \int_0^t A_u dW_u + \int_0^t B_u du,$$

where  $Y_0$  is a constant.



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where  $Y_0$  is a constant.

A wide class of continuous stochastic processes that behave **locally** like a Wiener process with drift.

# Itô integral is a martingale

## Informal theorem



Itô process  $(Y_t)$  is a martingale if and only if it has only Itô integral in the representation

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The best guess of a future value of an Itô integral is its current value:

$$\mathbb{E}(Y_t \mid \mathcal{F}_s) = Y_s \text{ for } s \leq t.$$

# Expected value of Itô process

## Informal theorem



For any reasonable process  $(B_t)$

$$\mathbb{E} \left( \int_0^t B_u du \right) = \int_0^t \mathbb{E}(B_u) du.$$

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## Informal theorem



If  $(Y_t)$  is an Itô process with  $Y_t = Y_0 + \int_0^t A_u dW_u + \int_0^t B_u du$ , then

$$\mathbb{E}(Y_t) = Y_0 + \int_0^t \mathbb{E}(B_u) du.$$

## Short form notation

Full form:

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$dW_t$  and  $dY_t$  have **no meaning!**

## Short form in simulations

We need to simulate a path of

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In simulations:

$$Y_{t+\Delta} - Y_t \approx A_t \cdot (W_{t+\Delta} - W_t) + B_t \cdot \Delta,$$

where  $W_{t+\Delta} - W_t \sim \mathcal{N}(0; \Delta)$ .

## Short form: examples

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$$dY_t = \cos(W_t) dt$$

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$$dY_t = \cos(W_t) dt \quad Y_t = Y_0 + \int_0^t \cos(W_u) du.$$



## Short form: examples

$$dY_t = W_t^4 dW_t \quad Y_t = Y_0 + \int_0^t W_u^4 dW_u.$$

$$dY_t = \cos(W_t) dt \quad Y_t = Y_0 + \int_0^t \cos(W_u) du.$$

**Informal theorem**



Itô process  $(Y_t)$  is a martingale if and only if

$$dY_t = A_t dW_t.$$

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- Will be a martingale **without Riemann integral**.
- Often written using **short form** with  $dt$  and  $dW_t$ .