Option pricing

Discounted price process

Discounted price process: plan

- Discounting in discrete and continuous time.

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- Every asset can be replicated.

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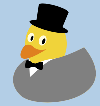
- Discounting in discrete and continuous time.
- Every asset can be replicated.
- The pricing formula.

Discounting

Definition in discrete time 

If X_t is the price of a claim at time t and r is the interest rate then discounted price is defined as

$$\frac{X_t}{(1+r)^t} = (1+r)^{-t} X_t.$$

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$$\frac{X_t}{(\exp r)^t} = \frac{X_t}{\exp(rt)} = \exp(-rt) X_t.$$

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For $t = 0$ discounted price and price are equal.

Is the discounted share price a martingale?

In short form,

$$\begin{aligned} d(\exp(-rt)S_t) &= -r\exp(-rt)S_t dt + \exp(-rt)dS_t + \frac{0}{2}\cdot(dS_t)^2 = \\ &= -\, r\exp(-rt)S_t dt + \exp(-rt)(\mu S_t dt + \sigma S_t dW_t) = \\ &= \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right). \end{aligned}$$

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No, under $\mathbb{P}$ short form has dt term inside!

$$S_0 \neq \mathbb{E}(\exp(-rt)S_t \mid \mathcal{F}_0).$$

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Yes, under $\mathbb{P}^*$ short form has no dt term inside!

$$S_0 = \mathbb{E}^*(\exp(-rt)S_t \mid \mathcal{F}_0).$$

Replicating strategy

Informal theorem 

In the Black and Scholes model every European type asset can be replicated by a self-financing stategy that trades shares and risk free bonds. At time t the portfolio contains y_t shares and z_t bonds and

$$\begin{cases} X_t = y_t S_t + z_t B_t, \\ dX_t = y_t dS_t + z_t dB_t. \end{cases}$$

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European type asset gives payoff at a fixed time moment T.

Self-financing strategy means no exogenous capital flow.

The pricing formula

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In the Black and Scholes model the discounted price of every European type asset is a martingale under probability $\mathbb{P}^*$, hence

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- (W_t^*) is a Wiener process under $\mathbb{P}^*$.
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- Discounted share price $\exp(-rt)S_t$ is a martingale under $\mathbb{P}^*$.

Discounted price process: summary

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$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

Call option price

Call option price: plan

- Definition of call and put options.

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- Definition of call and put options.
- Put-call option parity.

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- Definition of call and put options.
- Put-call option parity.
- The price of a call option.

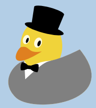
Classic options

Definition

The call option gives a right to buy one share at a specified strike price K on a specified date T.

The put option gives a right to sell one share at a specified strike price K on a specified date T.

Classic options

Definition 

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$$C_T - P_T = S_T - K$$

$$C_0 - P_0 = S_0 - \exp(-rT)K$$

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$$C_T = I \cdot (S_T - K) = I \cdot S_T - I \cdot K.$$

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Let's split into two terms,

$$\begin{aligned}\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \mathbb{E}^*(I \cdot S_T - I \cdot K \mid \mathcal{F}_0) = \\ = \mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) - \mathbb{E}^*(I \cdot K \mid \mathcal{F}_0);\end{aligned}$$

The second term

Strike price K is constant,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{E}^*(I \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0).$$

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Let's go down to W_T^*,

$$\{S_T > K\} = \{\ln S_T > \ln K\} = \{\ln S_0 + (r - \sigma^2/2)T + \sigma W_T^* > \ln K\}$$

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Let's standardise and reverse the inequality,

$$\{S_T > K\} = \left\{ \frac{0 - W_T^*}{\sqrt{T}} < d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right\}.$$

The second term...

We've done one half of the job,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0) = KF(d),$$

where

$$d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

The final answer

The first term,

$$\mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) =$$

$$=\mathbb{E}^*(I(W_T^* < d\sqrt{T}) \cdot S_0 \cdot \exp\Big((r - \sigma^2/2)T + \sigma W_T^*\Big) \mid \mathcal{F}_0) =$$

$$=S_0 \exp\Big((r - \sigma^2/2)T\Big)\mathbb{E}^*(I(W_T^* < d\sqrt{T}) \cdot \exp\big(\sigma W_T^*\big)) =$$

$$=S_0 F(d + \sigma\sqrt{T}).$$

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&\quad = S_0 \exp\left((r - \sigma^2/2)T\right) \mathbb{E}^*\left(I(W_T^* < d\sqrt{T}) \cdot \exp\left(\sigma W_T^*\right)\right) = \\
&\quad = S_0 F(d + \sigma\sqrt{T}).
\end{aligned}$$

The call option price,

$$\begin{aligned}
C_0 &= \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \\
&= \exp(-rT)(S_0 F(d + \sigma\sqrt{T}) - K F(d)),
\end{aligned}$$

where $d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$.

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Delta hedging

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- The receipt for replication.

Claim price as Itô process

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Using Itô's lemma,

$$dX_t = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} (dS_t)^2$$

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Hence,

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Delta hedging rule

Informal theoerm

To replicate a european type claim with price $X(S_t, t)$ we should hold y_t shares and z_t bonds, where

$$\begin{cases} y_t = \frac{\partial X}{\partial S}; \\ z_t = \frac{X_t - y_t S_t}{B_t}. \end{cases}$$

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where y_t is the amount of shares we hold at t.

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Thank you!