Option pricing

Discounted price process

Discounted price process: plan

- Discounting in discrete and continuous time.

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- Discounting in discrete and continuous time.
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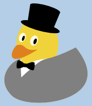
- The pricing formula.

Discounting

Definition in discrete time



If X_t is the price of a claim at time t and r is the interest rate then the value

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For $t = 0$ discounted price and price are equal.

Is the discounted share price a martingale?

$$d(\exp(-rt)S_t) = -r\exp(-rt)S_t dt + \exp(-rt)dS_t + \frac{1}{2}\cdot 0 \cdot (dS_t)^2 =$$

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No, under $\mathbb{P}$ short form has dt term inside!

$$S_0 \neq \mathbb{E}(\exp(-rt)S_t \mid \mathcal{F}_0).$$

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$$d(\exp(-rt)S_t) = \exp(-rt)S_t\sigma dW_t^*.$$

Yes, under $\mathbb{P}^*$ short form has no dt term inside!

$$S_0 = \mathbb{E}^*(\exp(-rt)S_t \mid \mathcal{F}_0).$$

Replicating strategy

Informal theorem 

In the Black and Scholes model every european type asset can be replicated by a self-financing stategy that trades shares and risk free bonds.

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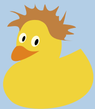
Self-financing strategy means no exogenous capital inflow or outflow.

The pricing formula

Informal theorem 

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- (W_t^*) is a Wiener process under $\mathbb{P}^*$.
- $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$.
- Discounted share price $\exp(-rt)S_t$ is also a martingale under $\mathbb{P}^*$.

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