Option pricing

Discounted price process

Discounted price process: plan

- Discounting in discrete and continuous time.
- Every asset can be replicated.
- The pricing formula.

Discounting

Definition in discrete time

If X_t is the price of a claim at time t and r is the interest rate then discounted price is defined as

$$\frac{X_t}{(1+r)^t} = (1+r)^{-t} X_t.$$

Definition in continuous time

Discounted price is defined as

$$\frac{X_t}{(\exp r)^t} = \frac{X_t}{\exp(rt)} = \exp(-rt) X_t.$$

For small r the definitions are close as $\exp(r) \approx 1 + r$.

For $t = 0$ discounted price and price are equal.

Is the discounted share price a martingale?

In short form,

$$
\begin{aligned}
d(\exp(-rt)S_t) = -r\exp(-rt)S_t dt + \exp(-rt)dS_t + \frac{0}{2}\cdot(dS_t)^2 = \\
= -\, r\exp(-rt)S_t dt + \exp(-rt)(\mu S_t dt + \sigma S_t dW_t) = \\
= \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right).
\end{aligned}
$$

No, under $\mathbb{P}$ short form has dt term inside!

$$S_0 \neq \mathbb{E}(\exp(-rt)S_t \mid \mathcal{F}_0).$$

Is the discounted share price a martingale?

Let's recall,

$$d(\exp(-rt)S_t) = \exp(-rt)S_t\left((\mu - r)dt + \sigma dW_t\right).$$

But wait, $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$, so

$$d(\exp(-rt)S_t) = \exp(-rt)S_t \sigma dW_t^*.$$

Yes, under $\mathbb{P}^*$ short form has no dt term inside!

$$S_0 = \mathbb{E}^*(\exp(-rt)S_t \mid \mathcal{F}_0).$$

Replicating strategy

Informal theorem 

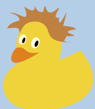
In the Black and Scholes model every European type asset can be replicated by a self-financing stategy that trades shares and risk free bonds. At time t the portfolio contains y_t shares and z_t bonds and

$$\begin{cases} X_t = y_t S_t + z_t B_t, \\ dX_t = y_t dS_t + z_t dB_t. \end{cases}$$

European type asset gives payoff at a fixed time moment T.

Self-financing strategy means no exogenous capital flow.

The pricing formula

Informal theorem 

In the Black and Scholes model the discounted price of every European type asset is a martingale under probability $\mathbb{P}^*$, hence

$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

- (W_t^*) is a Wiener process under $\mathbb{P}^*$.
- $(\mu - r)dt + \sigma dW_t = \sigma dW_t^*$.
- Discounted share price $\exp(-rt)S_t$ is a martingale under $\mathbb{P}^*$.

Discounted price process: summary

- European claim gives payoff at a fixed moment of time T.
- The discounted price of any European type claim is a martingale under $\mathbb{P}^*$.
- Every European claim may be replicated.
- The pricing formula is

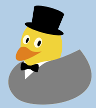
$$X_0 = \mathbb{E}^*(\exp(-rt)X_t \mid \mathcal{F}_0) = \exp(-rt)\mathbb{E}^*(X_t \mid \mathcal{F}_0).$$

Call option price

Call option price: plan

- Definition of call and put options.
- Put-call option parity.
- The price of a call option.

Classic options

Definition 

The call option gives a right to buy one share at a specified strike price K on a specified date T.

The put option gives a right to sell one share at a specified strike price K on a specified date T.

$$C_T = \begin{cases} S_T - K, & \text{if } S_T > K; \\ 0, & \text{otherwise.} \end{cases} \qquad P_T = \begin{cases} K - S_T, & \text{if } S_T < K; \\ 0, & \text{otherwise.} \end{cases}$$

Put-call parity

$$C_T = \begin{cases} S_T - K, & \text{if } S_T > K; \\ 0, & \text{otherwise.} \end{cases} \qquad P_T = \begin{cases} K - S_T, & \text{if } S_T < K; \\ 0, & \text{otherwise.} \end{cases}$$

$$C_T - P_T = S_T - K$$

$$C_0 - P_0 = S_0 - \exp(-rT)K$$

Call option price

The pricing formula,

$$C_0 = \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0).$$

We rewrite C_T using indicator $I = I(S_T > K)$,

$$C_T = I \cdot (S_T - K) = I \cdot S_T - I \cdot K.$$

Let's split into two terms,

$$\begin{aligned}\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \mathbb{E}^*(I \cdot S_T - I \cdot K \mid \mathcal{F}_0) =& \\ =&\mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) - \mathbb{E}^*(I \cdot K \mid \mathcal{F}_0);\end{aligned}$$

The second term

Strike price K is constant,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{E}^*(I \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0).$$

Let's go down to W_T^*,

$$\{S_T > K\} = \{\ln S_T > \ln K\} = \{\ln S_0 + (r - \sigma^2/2)T + \sigma W_T^* > \ln K\}$$

Or,

$$\{S_T > K\} = \left\{W_T^* > \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma}\right\}$$

Let's standardise and reverse the inequality,

$$\{S_T > K\} = \left\{\frac{0 - W_T^*}{\sqrt{T}} < d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right\}.$$

The second term…

We've done one half of the job,

$$\mathbb{E}^*(I \cdot K \mid \mathcal{F}_0) = K\mathbb{P}^*(S_T > K \mid \mathcal{F}_0) = KF(d),$$

where

$$d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

The final answer

The first term,

$$
\begin{aligned}
&\mathbb{E}^*(I \cdot S_T \mid \mathcal{F}_0) = \\
&\quad = \mathbb{E}^*\left(I(W_T^* < d\sqrt{T}) \cdot S_0 \cdot \exp\left((r - \sigma^2/2)T + \sigma W_T^*\right) \mid \mathcal{F}_0\right) = \\
&\quad = S_0 \exp\left((r - \sigma^2/2)T\right) \mathbb{E}^*\left(I(W_T^* < d\sqrt{T}) \cdot \exp\left(\sigma W_T^*\right)\right) = \\
&\quad = S_0 F(d + \sigma\sqrt{T}).
\end{aligned}
$$

The call option price,

$$
\begin{aligned}
C_0 &= \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0) = \\
&= \exp(-rT)(S_0 F(d + \sigma\sqrt{T}) - K F(d)),
\end{aligned}
$$

where $d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$.

Call option price: summary

- Call option is the right to buy a share, put option is the right to sell a share.
- Put-call parity between their prices,

$$C_0 - P_0 = S_0 - \exp(-rT)K.$$

- The call option price is

$$C_0 = \exp(-rT)\mathbb{E}^*(C_T \mid \mathcal{F}_0) =$$
$$= \exp(-rT)(S_0 F(d + \sigma\sqrt{T}) - KF(d)),$$

where $d = \frac{\ln S_0 - \ln K + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$.

Delta hedging

Delta hedging: plan

- dX_t using Itô's lemma.
- dX_t using replicating strategy.
- The receipt for replication.

Claim price as Itô process

The price of a claim X_t is a function of S_t and t,

$$X_t = X(S_t, t).$$

We need to do some investments to start replicating strategy.

Using Itô's lemma,

$$dX_t = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} (dS_t)^2$$

The structure of the answer is

$$dX_t = (\dots)dt + (\dots)dW_t.$$

Focus on dW_t

Using Itô's lemma,

$$dX_t = \frac{\partial X}{\partial t} dt + \frac{\partial X}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 X}{\partial S^2} (dS_t)^2$$

The structure of the answer is

$$dX_t = (\dots) dt + (\dots) dW_t.$$

There are no dW_t in $(dS_t)^2$, only in $dS_t = \mu S_t dt + \sigma S_t dW_t$.

Hence,

$$dX_t = (\dots) dt + \frac{\partial X}{\partial S} \sigma S_t dW_t.$$

Replicating portfolio idea

We have y_t shares and z_t bonds in portfolio,

$$X_t = y_t S_t + z_t B_t.$$

Stochastic integral represents the net cash-flow,

$$X_t = X_0 + \int_0^t y_u \, dS_u + \int_0^t z_u dB_u$$

In short form,

$$dX_t = y_t dS_t + z_t dB_t.$$

In Black and Scholes model,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad dB_t = B_t dt.$$

Focus on dW_t again

In short form,

$$dX_t = y_t dS_t + z_t dB_t.$$

In Black and Scholes model,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad dB_t = B_t dt.$$

Hence,

$$dX_t = (\dots)dt + y_t \sigma S_t dW_t.$$

Delta hedging rule

Informal theoerm

To replicate a european type claim with price $X(S_t, t)$ we should hold y_t shares and z_t bonds, where

$$\begin{cases} y_t = \frac{\partial X}{\partial S}; \\ z_t = \frac{X_t - y_t S_t}{B_t}. \end{cases}$$

Delta hedging: summary

- From Itô's lemma

$$dX_t = \ldots \cdot dt + \frac{\partial X}{\partial S} \sigma S_t \, dW_t.$$

- From self-financing assumptions

$$dX_t = \ldots \cdot dt + y_t \sigma S_t \, dW_t,$$

  where y_t is the amount of shares we hold at t.

- The delta-hedging rule

$$y_t = \frac{\partial X}{\partial S}.$$

Thank you!