Задача. Рассматривается модель регрессии $Y_i = \alpha + \beta x_i + \varepsilon_i$, в которой ошибки $\varepsilon_1, \dots, \varepsilon_n$ независимы и имеют нормальное распределение с нулевым математическим ожиданием и дисперсией σ^2. Для $n = 12$ найдите

(a) $\mathbb{P}\{\hat{\alpha} > \alpha\}$,

(b) $\mathbb{P}\{\alpha > 0\}$,

(c) $\mathbb{P}\left\{|\hat{\alpha} - \alpha| < \sqrt{\hat{\mathrm{D}}(\hat{\alpha})}\right\}$,

(d) $\mathbb{P}\left\{\hat{\beta} > \beta + \sqrt{\hat{\mathrm{D}}(\hat{\beta})}\right\}$,

(e) $\mathbb{P}\left\{\hat{\beta} < \beta - \sqrt{\hat{\mathrm{D}}(\hat{\beta})}\right\}$,

(f) $\mathbb{E}\left[\dfrac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}}\right]$,

(g) $\mathbb{E}\left[\dfrac{\hat{\alpha} + \hat{\beta} - (\alpha + \beta)}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha} + \hat{\beta})}}\right]$,

(h) $\mathrm{D}\left(\dfrac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}}\right)$,

(i) $\mathrm{D}\left(\dfrac{\hat{\alpha} + \hat{\beta} - (\alpha + \beta)}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha} + \hat{\beta})}}\right)$,

(j) $\mathbb{P}\{\hat{\sigma} > \sigma\}$,

(k) $\mathbb{P}\{\hat{\sigma} < \sigma\}$.

Решение. (a) Поскольку $\dfrac{\hat{\alpha} - \alpha}{\sqrt{\mathrm{D}(\hat{\alpha})}} \sim N(0,1)$, то

$$\mathbb{P}\{\hat{\alpha} > \alpha\} = \mathbb{P}\{\hat{\alpha} - \alpha > 0\} = \mathbb{P}\left\{\frac{\hat{\alpha} - \alpha}{\sqrt{\mathrm{D}(\hat{\alpha})}} > 0\right\} =$$

$$= 1 - \mathbb{P}\left\{\frac{\hat{\alpha} - \alpha}{\sqrt{\mathrm{D}(\hat{\alpha})}} \le 0\right\} = 1 - \mathrm{normcdf}(0) = 0.5.$$

(b) $\mathbb{P}\{\alpha > 0\} = \begin{cases} 0, & \text{если } \alpha \le 0, \\ 1, & \text{если } \alpha > 0. \end{cases}$

(c) Поскольку $\dfrac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то

$$\mathbb{P}\left\{|\hat{\alpha} - \alpha| < \sqrt{\hat{\mathrm{D}}(\hat{\alpha})}\right\} = \mathbb{P}\left\{\left|\frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}}\right| < 1\right\} = \mathbb{P}\left\{-1 < \frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} < 1\right\} =$$

$$= \mathbb{P}\left\{\frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} < 1\right\} - \mathbb{P}\left\{\frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} \le -1\right\} = \mathbb{P}\left\{\frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} \le 1\right\} - \mathbb{P}\left\{\frac{\hat{\alpha} - \alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} \le -1\right\} =$$

$$= \mathrm{tcdf}(1, 10) - \mathrm{tcdf}(-1, 10) = 0.8296 - 0.1704 = 0.6592.$$

(d) Поскольку $\dfrac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{D}}(\hat{\beta})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то

$$\mathbb{P}\left\{\hat{\beta} > \beta + \sqrt{\hat{\mathrm{D}}(\hat{\beta})}\right\} = \mathbb{P}\left\{\frac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{D}}(\hat{\beta})}} > 1\right\} = 1 - \mathbb{P}\left\{\frac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{D}}(\hat{\beta})}} \le 1\right\} =$$

$$= 1 - \mathrm{tcdf}(1, 10) = 1 - 0.8296 = 0.1704.$$

(e) Поскольку $\dfrac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{D}}(\hat{\beta})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то

$$\mathbb{P}\left\{\hat{\beta} < \beta - \sqrt{\hat{\mathrm{D}}(\hat{\beta})}\right\} = \mathbb{P}\left\{\frac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{D}}(\hat{\beta})}} < -1\right\} = \mathbb{P}\left\{\frac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{D}}(\hat{\beta})}} \le -1\right\} = \mathrm{tcdf}(-1, 10) = 0.1704.$$

(f) Напомним, что если $\xi \sim t(m)$ и $m \ge 2$, то $\mathbb{E}[\xi] = 0$. Далее, поскольку $\dfrac{\hat{\alpha}-\alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то $\mathbb{E}\left[\dfrac{\hat{\alpha}-\alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}}\right] = 0$.

(g) Напомним, что если $\xi \sim t(m)$ и $m \ge 2$, то $\mathbb{E}[\xi] = 0$. Далее, поскольку $\dfrac{\hat{\alpha}+\hat{\beta}-(\alpha+\beta)}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha}+\hat{\beta})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то $\mathbb{E}\left[\dfrac{\hat{\alpha}+\hat{\beta}-(\alpha+\beta)}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha}+\hat{\beta})}}\right] = 0$.

(h) Напомним, что если $\xi \sim t(m)$ и $m \ge 3$, то $\mathrm{D}(\xi) = \dfrac{m}{m-2}$. Далее, поскольку $\dfrac{\hat{\alpha}-\alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то $\mathrm{D}\left(\dfrac{\hat{\alpha}-\alpha}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha})}}\right) = \dfrac{10}{10-2} = 1.25$.

(i) Напомним, что если $\xi \sim t(m)$ и $m \ge 3$, то $\mathrm{D}(\xi) = \dfrac{m}{m-2}$. Далее, поскольку $\dfrac{\hat{\alpha}+\hat{\beta}-(\alpha+\beta)}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha}+\hat{\beta})}} \sim t(n-k-1) = t(12-1-1) = t(10)$, то $\mathrm{D}\left(\dfrac{\hat{\alpha}+\hat{\beta}-(\alpha+\beta)}{\sqrt{\hat{\mathrm{D}}(\hat{\alpha}+\hat{\beta})}}\right) = \dfrac{10}{10-2} = 1.25$.

(j) Напомним, что $RSS/\sigma^2 \sim \chi^2(n-k-1)$. Стало быть, $\dfrac{\widehat{\sigma^2}}{\sigma^2}(n-k-1) = \dfrac{RSS}{\sigma^2} \sim \chi^2(n-k-1)$. Значит,

$$\mathbb{P}\{\hat{\sigma} > \sigma\} = \mathbb{P}\left\{\frac{\hat{\sigma}}{\sigma} > 1\right\} = \mathbb{P}\left\{\frac{\widehat{\sigma^2}}{\sigma^2} > 1\right\} = \mathbb{P}\left\{\frac{\widehat{\sigma^2}}{\sigma^2}(n-k-1) > (n-k-1)\right\} =$$

$$= \mathbb{P}\left\{\frac{\widehat{\sigma^2}}{\sigma^2} \cdot 10 > 10\right\} = 1 - \mathbb{P}\left\{\frac{\widehat{\sigma^2}}{\sigma^2} \cdot 10 \le 10\right\} = 1 - \mathrm{chi2cdf}(10, 10) = 1 - 0.5595 = 0.4405.$$

(k) $\mathbb{P}\{\hat{\sigma} < \sigma\} = 1 - \mathbb{P}\{\hat{\sigma} > \sigma\} = 0.5595$. □