Теор

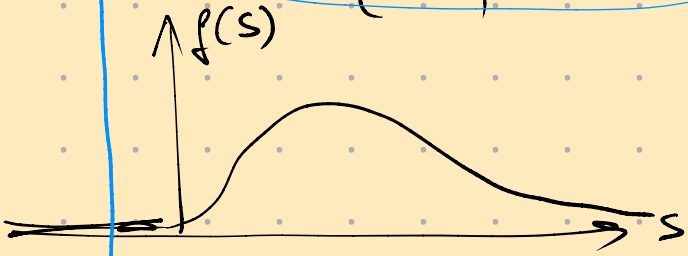
Если $Y_1, Y_2, Y_3, \dots, Y_k$ ~ незав. $\exp(\lambda)$

[или рассмотр. пример $k \in \mathbb{N}$]

$S = Y_1 + Y_2 + \dots + Y_k$

то

$$f_S(s) = \begin{cases} \frac{\lambda^k}{\Gamma(k)} \cdot s^{k-1} \cdot \exp(-\lambda s) & s \geq 0 \\ 0, & s < 0 \end{cases}$$



$$\Gamma(k) = \int_0^\infty t^{k-1} \exp(-t)\, dt$$

гамма-функция

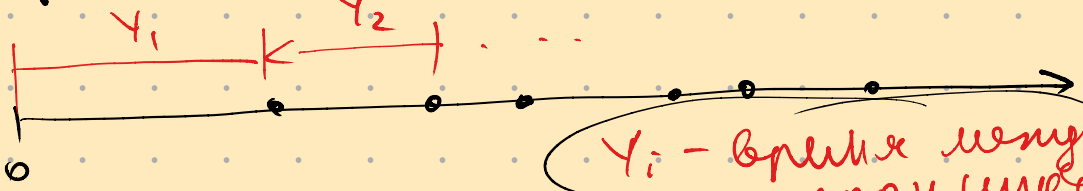
$$E(S) = k \cdot \frac{1}{\lambda} \qquad Var(S) = k \cdot \frac{1}{\lambda^2}$$

Обр

$S \sim Gamma(k, \lambda)$

Ом-ся $f_S(s)$ корректно опр на формулой при $\forall k > 0, \lambda > 0$.

Упр. Пуасс. поток с инт $\lambda = 10$ чел/час



Y_i – время между происшеств

$W_1 = Y_1 + Y_2 + Y_3$

$W_2 = \sum_{i=1}^{5} Y_i$

$W_3 = \sum_{i=1}^{10} Y_i$

$W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$

$f(x)$ – плотность

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

а) $E(W)$, $Var(W)$? !!

б) $\Gamma(1)$, $\Gamma(2)$, $\Gamma(3)$? ?

$$E(W) = E\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} E(W_1) \\ E(W_2) \\ E(W_3) \end{pmatrix} = \begin{pmatrix} E(Y_1+Y_2+Y_3) \\ E(Y_1+Y_2+\dots+Y_5) \\ E(Y_1+Y_2+\dots+Y_{10}) \end{pmatrix}$$

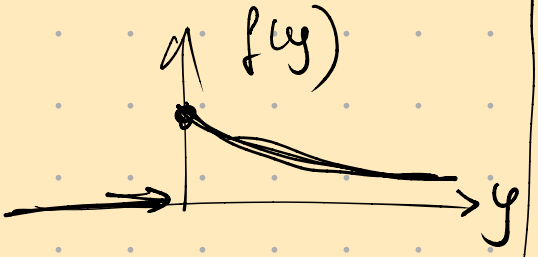
сред. вектор

$$= \begin{pmatrix} 3\cdot E(Y_1) \\ 5\cdot E(Y_1) \\ 10\cdot E(Y_1) \end{pmatrix} = \begin{pmatrix} 3/10 \\ 5/10 \\ 10/10 \end{pmatrix}$$

$Y_1 \sim Expo(\lambda = 10 \text{ сек}/\text{час})$

Y_1 [час]

$$f(y) = \begin{cases} 10\exp(-10y) & y \geq 0 \\ 0, \text{ иначе} \end{cases}$$



$$E(Y_1) = \int_0^\infty y \cdot f(y)\, dy = \int_0^\infty y\, 10\exp(-10y)\, dy = \ldots =$$

[по частям]

$$= \frac{1}{10} \quad \left[\text{в общем } \frac{1}{\lambda}\right]$$

$Y_1 + Y_2 + Y_3 \sim Gamma(k=3, \lambda=10)$

$Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \sim Gamma(k=5, \lambda=10)$

$$Var(W) = \begin{pmatrix} Var(W_1) & Cov(W_1, W_2) & Cov(W_1, W_3) \\ Cov(W_2, W_1) & Var(W_2) & Cov(W_2, W_3) \\ Cov(W_3, W_1) & Cov(W_3, W_2) & Var(W_3) \end{pmatrix}$$

$Var(W)$ — ковариационная матрица.

$Var(W_i) = Cov(W_i, W_i)$

$$Var(W_1) = Var(Y_1 + Y_2 + Y_3) = 3\,Var(Y_1) = 3 \cdot \frac{1}{100} = \frac{3}{100}$$

(Y_1 + Y_2 + Y_3 — назад.)

$Y_1 \sim Expo(\lambda = 10)$

$$Var(Y_1) = E(Y_1^2) - (E(Y_1))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$E(Y_1^2) = \int_0^\infty y^2 f(y)\, dy = \ldots = \frac{2}{\lambda^2} \quad [\text{по частям}]$$

$$E(Y_1) = \int_0^\infty y f(y)\, dy = \ldots = \frac{1}{\lambda} \quad [\text{по частям}]$$

* производ. ме функции

$$Cov(W_3, W_1) = Cov(Y_1 + Y_2 + \ldots + Y_{10}, Y_1 + Y_2 + Y_3) =$$

= ков-ия линейна по обоим арг-м

$$= \mathrm{Cov}(Y_1, Y_1) + \mathrm{Cov}(Y_1, Y_2) + \mathrm{Cov}(Y_1, Y_3) +$$
$$+ \mathrm{Cov}(Y_2, Y_1) + \mathrm{Cov}(Y_2, Y_2) + \mathrm{Cov}(Y_2, Y_3)$$
$$+ \mathrm{Cov}(Y_{10}, Y_1) + \mathrm{Cov}(Y_{10}, Y_2) + \mathrm{Cov}(Y_{10}, Y_3)$$

$$= \mathrm{Cov}(Y_1, Y_1) + \mathrm{Cov}(Y_2, Y_2) + \mathrm{Cov}(Y_3, Y_3) = 3\,\mathrm{Var}(Y_1) = \frac{3}{100}$$

$C = \mathrm{Var}(W)$ — $[3\times 3]$ — из контекста

$W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$ $[3\times 1]$

$$= \left(\begin{array}{c|c|c} \frac{3}{100} & \frac{3}{100} & \frac{3}{100} \\ \hline \frac{3}{100} & \frac{5}{100} & \frac{5}{100} \\ \hline \frac{3}{100} & \frac{5}{100} & \frac{10}{100} \end{array}\right)$$

$C_{21} = \mathrm{Cov}(W_2, W_1) =$ (нет пары!)
$= \mathrm{Cov}(Y_1 + Y_2 + \ldots + Y_5,\ Y_1 + Y_2 + Y_3) =$
$= 3\,\mathrm{Var}(Y_1) = \frac{3}{100}$

$C_{12} = \mathrm{Cov}(W_1, W_2) = \frac{3}{100} = \mathrm{Cov}(W_2, W_1)$

$C_{22} = \mathrm{Var}(W_2) = \mathrm{Var}(Y_1 + \ldots + Y_5) = \frac{5}{100}$

$C_{33} = \mathrm{Var}(W_3) = \mathrm{Var}(Y_1 + \ldots + Y_{10}) = \frac{10}{100}$

$C_{23} = C_{32} = \mathrm{Cov}(Y_1 + \ldots + Y_5,\ Y_1 + Y_2 + \ldots + Y_5 + Y_6 + \ldots + Y_{10})$ (5 пар слева)
$= 5 \cdot \mathrm{Var}(Y_1) = \frac{5}{100}$

Упр $Y_1 \sim \mathrm{Expo}(\lambda)$

а) $m(t)$ производящую функцию моментов?

б) $E(Y_1^3) = ?$

$$m(t) = E(\exp(tY_1)) = \int_0^\infty \exp(ty) \cdot \lambda \cdot \exp(-\lambda y)\, dy$$

$$= \int_0^\infty \lambda \exp((t-\lambda)\cdot y)\, dy = \lambda \cdot \frac{\exp((t-\lambda)\cdot y)}{t-\lambda}\Bigg|_{y=0}^{+\infty} =$$

$= \begin{cases} t \geq \lambda & \text{не сущ.} \\ t < \lambda & (t-\lambda)\cdot y < 0 \end{cases} = 0 - \frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda - t}$

$E(Y_1^3) = m'''(0)$

как найти много производных св вдав ни одну?

$m(t) = m(0) + m'(0)\cdot t + \frac{1}{2!}m''(0)\cdot t^2 + \frac{1}{3!}\cdot m'''(0)\cdot t^3 + \dots$

$|\frac{t}{\lambda}|<1$

$\frac{\lambda}{\lambda - t} = \frac{1}{1-\left(\frac{t}{\lambda}\right)} = 1 + \frac{t}{\lambda} + \left(\frac{t}{\lambda}\right)^2 + \left(\frac{t}{\lambda}\right)^3 + \left(\frac{t}{\lambda}\right)^4 + \dots$

$|q|<1$

$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \dots$

коэф при	t	$m'(0)$	$\frac{1}{\lambda}$
коэф при	t^2	$\frac{m''(0)}{2!}$	$\frac{1}{\lambda^2}$
коэф при	t^3	$\frac{m'''(0)}{3!}$	$\frac{1}{\lambda^3}$
...		...	...

$E(Y_1) = m'(0) = \frac{1}{\lambda}$

$E(Y_1^2) = m''(0) = 2!\cdot\frac{1}{\lambda^2}$

$E(Y_1^3) = m'''(0) = 3!\cdot\frac{1}{\lambda^3}$

$E(Y_1^k) = m^{(k)}(0) = k!\cdot\frac{1}{\lambda^k}$ //

Упр. $W \sim \text{Gamma}(3, \lambda=10)$

$E(W^3) \stackrel{?}{=} E((Y_1+Y_2+Y_3)^3) =$

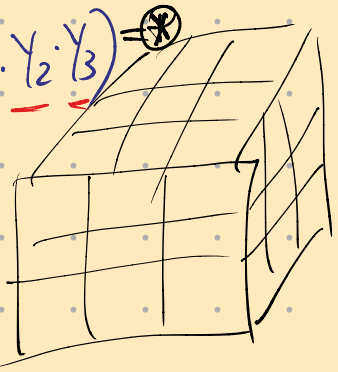
при таких слаг.:

$\rightarrow Y_1^3, Y_2^3, Y_3^3$

$\rightarrow Y_1^2\cdot Y_2, Y_2\cdot Y_3^2, \dots$

$= 3 \cdot E(Y_1^3) + 18\, E(Y_1^2 Y_2) +$ $\quad Y_1 \cdot Y_2 \cdot Y_3$

$+ 6\, E(Y_1 \cdot Y_2 \cdot Y_3) = (*)$



Всего слаг: 27

$(Y_1 + Y_2 + Y_3) \cdot (Y_1 + Y_2 + Y_3) \cdot (Y_1 + Y_2 + Y_3)$

$Y_1 \cdot Y_2 \cdot Y_3$ — 6 вар-в.

$27 - 3 - 6 = 18$ вар-в $Y_1^2 \cdot Y_3$

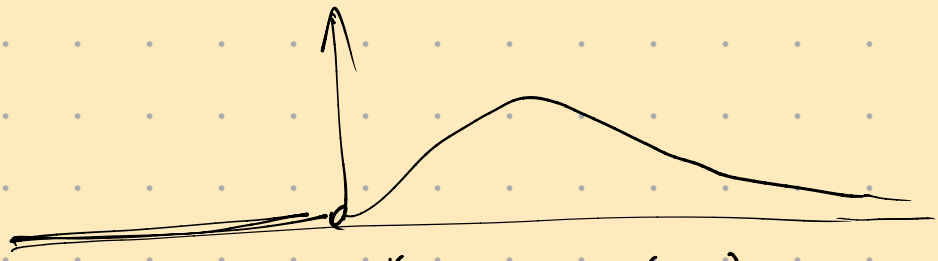
$= (*)\ 3 \cdot 3! \cdot \frac{1}{\lambda^3} + 18 \cdot E(Y_1^2) \cdot E(Y_2) +$

$+ 6 \cdot E(Y_1) \cdot E(Y_2) \cdot E(Y_3) =$

Если R_1 и R_2 нерав то $E(R_1 \cdot R_2) = E(R_1) \cdot E(R_2)$

$= 3 \cdot \frac{3!}{\lambda^3} + 18 \cdot \frac{2!}{\lambda^2} \cdot \frac{1}{\lambda} + 6 \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda}$;

$S \sim \text{Gamma}(k, \lambda)$ $\quad k > 0$, $\lambda > 0$



$$f(s) = \begin{cases} \frac{\lambda^k}{\Gamma(k)} \cdot s^{k-1} \cdot \exp(-\lambda s), & s \geq 0. \\ 0, & s < 0 \end{cases}$$

Упр. экстремум — один у $f(s)$

$\Gamma(1), \Gamma(2), \Gamma(3)$?

$S \sim \text{Gamma}(k, \lambda = 1)$ $\quad f(s) = \begin{cases} \frac{1}{\Gamma(k)} \cdot s^{k-1} \cdot \exp(-s), & s \geq 0 \\ 0, & s < 0. \end{cases}$

$\Gamma(1)$ $\quad k = 1$ $\quad S = Y_1 \sim \text{Expo}(\lambda = 1)$

$$f(s) = \begin{cases} 1 \cdot \exp(-1 \cdot s), & s \geq 0 \\ 0, & s < 0 \end{cases}$$

$\Gamma(1) \overset{?}{=} 1$

$\Gamma(2)$? $k=2$ $S = Y_1 + Y_2$ $f(s) = \frac{1}{\Gamma(2)} \cdot s^{2-1} \cdot \exp(-s)$ $s \geq 0$

$S \sim \text{Gamma}(2, \lambda=1)$ $= \frac{1}{\Gamma(2)} \cdot s \cdot \exp(-s)$

$$\int_0^\infty \frac{1}{\Gamma(2)} \cdot s \cdot \exp(-s)\, ds = 1 \rightarrow$$

путь преобразования по свойствам $\Gamma(2) = \dots$

путь умная

$$\Gamma(2) = \int_0^\infty s \exp(-s)\, ds$$

доп. множитель s. — ф. плотн-сти $\text{Expo}(\lambda=1)$

$\Gamma(2) =$ вер-ятный смысл $= E(Y) =$

$= \frac{1}{\lambda} = 1$

$$E(h(S)) = \int_0^\infty h(s) \cdot (\text{ф. плотн.})\, ds$$

$S \sim \text{Gamma}(k=3, \lambda=1)$

$$f(s) = \frac{1}{\Gamma(3)} \cdot s^{3-1} \cdot \exp(-s)$$

$$\int_0^\infty f(s)\, ds = 1$$

$$\Gamma(3) = \int_0^\infty s^2 \exp(-s)\, ds$$

$$\Gamma(3) = \int_0^\infty y^2 \exp(-y)\, dy = E(Y^2) = \frac{2!}{\lambda^2} = 2!$$

$\exp(-y)$ — функц. плотн. для $Y \sim \text{Expo}(\lambda=1)$

$$\Gamma(4) = \int_0^\infty s^3 \exp(-s)\, ds = \int_0^\infty y^3 \cdot \exp(-y)\, dy = E(Y^3) = \frac{3!}{1^3} = 3!$$

$\exp(-y)$ — ф плотн

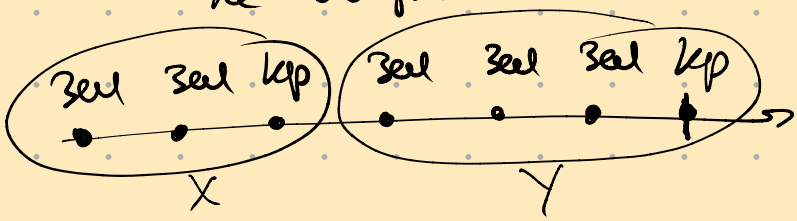
$\Gamma(k) = (k-1)!$

Про яблоки

2 красн
8 зеленых

красн - сразу сведем
зел — кладем обратно.

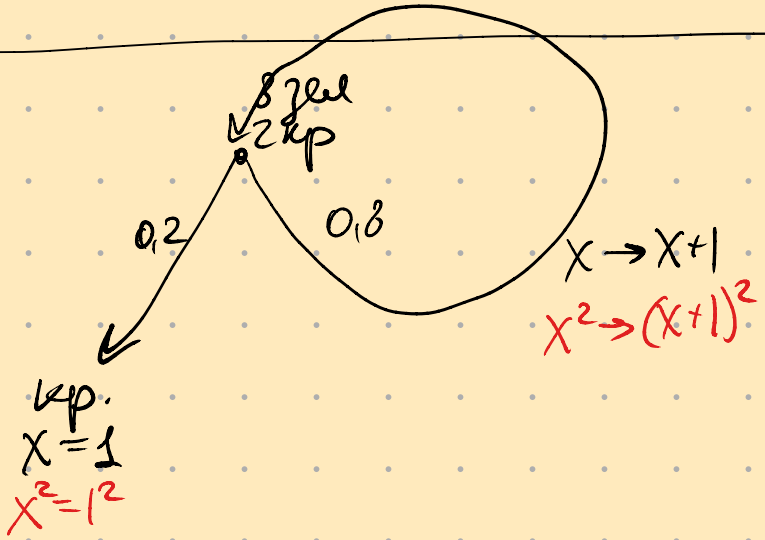
N - кол-во извлечений пока не соберем оба красных.



$N = X + Y$ незав.

$E(N) = E(X) + E(Y)$

$\mathrm{Var}(N) = \mathrm{Var}\,X + \mathrm{Var}\,Y$



$$\begin{cases} E(X) = 0{,}2 \cdot 1 + 0{,}8 \cdot E(X+1) \\ E(X^2) = 0{,}2 \cdot 1^2 + 0{,}8 \cdot E((X+1)^2) \end{cases}$$

$\Rightarrow E(X) = \frac{1}{0{,}2} = 5$

$E(X^2) = 0{,}2 + 0{,}8 \cdot E(X^2 + 2X + 1)$

$E(X^2) = 0{,}2 + 0{,}8\,E(X^2) + 0{,}8 \cdot 2E(X) + 0{,}8$

$0{,}2\,E(X^2) = 0{,}2 + 0{,}8 \cdot 10 + 0{,}8 = 9$

$$E(X^2) = 45$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 45 - 5^2 = 20$$