Всё ли ок?

раздел 18

18.1

сложность $P(N(3;5)\in[7;8]) \leftarrow$

$\int_7^8 f(x)\,dx \leftarrow$ не берётся в эл.-х функциях

→ компьютер

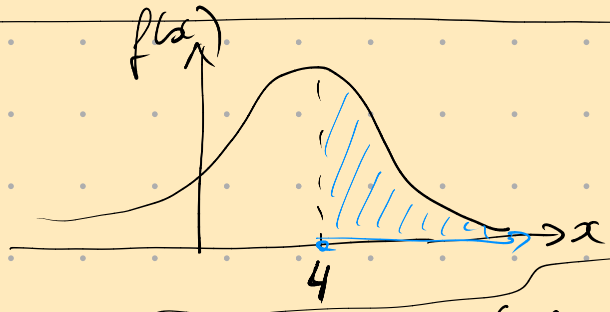
→ для ученых → таблицы

scipy.stats.norm.cdf. $= F(x)$ cumulative distr. function

$\mu = E(X)$, $\sigma^2 = Var(X)$

18.1 $X_1 \sim N(4;100)$

а) $P(X_1 > 4) = \frac{1}{2}$



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$P(X_1 \in [2;20]) =$ → компьют.

$= P(X_1 \le 20) - P(X_1 \le 2) =$

$= F(20) - F(2)$ ← scipy.stats.norm.cdf

табл.

Стандартное нормальное $N(0;1)$

стр 19

Шаг 1 $N(\mu;\sigma^2) \to N(0;1)$ „стандартизация“

$X_1 \sim N(\mu;\sigma^2)$

$X_1 - \mu \sim N(0;\sigma^2)$

$\boxed{\frac{X_1-\mu}{\sqrt{\sigma^2}} \sim N(0;1)}$

$Var\left(\frac{1}{\sigma}X\right) = \left(\frac{1}{\sigma}\right)^2 Var X$

Если $Y \sim N(0;1)$ $P(Y \le 0{,}23) =$

$= F(0{,}23) = 0{,}5910$

3

$0{,}2 \to \boxed{0{,}5910}$

$P(Y \le 0{,}234) =$

$= P(Y \le 0{,}23) + \text{корр} =$

$= 0{,}5910 + 0{,}0015$

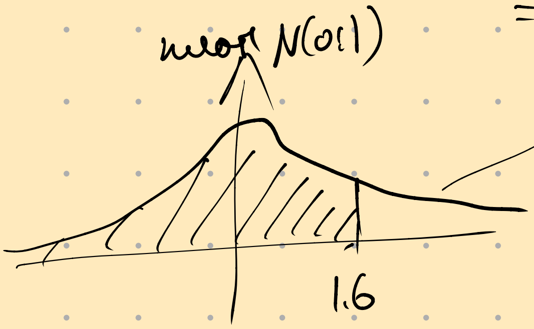
$X_1 \sim N(4;100)$ $\qquad P(X_1 \in [2;20]) =$

$$\boxed{\frac{X_1-4}{\sqrt{100}} \sim N(0;1)}$$

$$= P\left(\frac{X_1-4}{\sqrt{100}} \in \left[\frac{2-4}{\sqrt{100}}; \frac{20-4}{\sqrt{100}}\right]\right) =$$

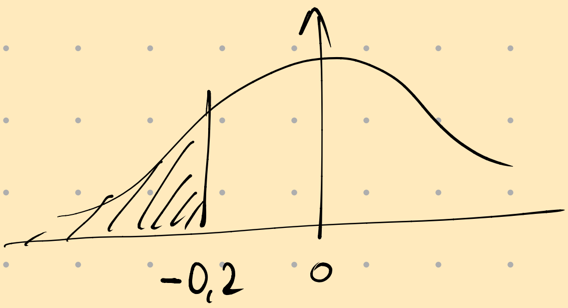
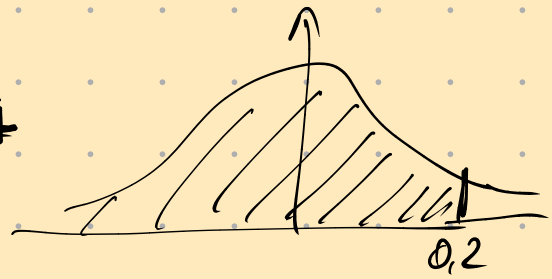
$$= P(N(0;1) \in [-0{,}2; 1{,}6]) =$$

$$= \underbrace{F(1{,}6)}_{=\ 0{,}9452} - F(-0{,}2) = 0{,}9452 - (1 - 0{,}5793)$$



$1{,}6 \to \boxed{0{,}9452}$

$\boxed{N(0;1)}$

$1 =$  $+$ 

$$F(-t) + F(t) = 1$$

$$F(-t) = 1 - F(t)$$

$$F(-0{,}2) = 1 - F(0{,}2) = 1 - 0{,}5793$$

$F(1{,}65) = P(N(0;1) \leq 1{,}65) =$ $\qquad 1{,}6 \xrightarrow{5} \boxed{0{,}9505}$

Упр. $\boxed{\text{Если } X_1 \sim N(\mu_1; \sigma_1^2) \text{ и } X_2 \sim N(\mu_2; \sigma_2^2) \text{ и } X_1 \text{ и } X_2 \text{ незав, то } X_1 + X_2 \sim N}$

18.2 $\quad W \sim$ ф. плотн $\quad f(w) = c \cdot \exp(5w - 2w^2)$ $\quad \to$ интегр-ть !
$\to$ увидеть знакомое !!

$E(W)$?
$Var(W)$?
c?
$W \sim$?

$$\boxed{f(w) = \frac{1}{\sqrt{2\pi \sigma^2}} \cdot \exp\left(-\frac{(w-\mu)^2}{2\sigma^2}\right)} \to \boxed{W \sim N(\mu; \sigma^2)}$$

Выделим полный квадрат!

$$(5w - 2w^2) = -\frac{1}{2}(4w^2 - 10w) =$$

$$= -\frac{1}{2} \frac{(w^2 - \frac{5}{2}w)}{1/4}$$ до полного квадрата?

2ab

$$= -\frac{1}{2} \frac{(w^2 - \frac{5}{2}w + \frac{25}{16}) - \frac{25}{16}}{1/4}$$

$$f(w) = c \cdot \exp\left(-\frac{1}{2}\frac{(w - 5/4)^2}{(1/4)}\right) \cdot \exp\left(\frac{4}{2} \cdot \frac{25}{16}\right)$$

Var(W)

$$W \sim N\left(\frac{5}{4}; \frac{1}{4}\right)$$

E(W)

$$\frac{1}{\sqrt{2\pi \cdot 1/4}} = c \cdot \exp\left(\frac{25}{8}\right)$$

$$c = \frac{\exp\left(-\frac{25}{8}\right)}{\sqrt{\pi/2}}$$

мораль

Если $f_R(t) = \ldots\ldots \cdot \exp(-?t^2 + ?t)$ то: $R \sim N(?;?)$

Утв. Если $X_1 \sim N(\mu_1; \sigma_1^2)$ и $X_2 \sim N(\mu_2; \sigma_2^2)$ и X_1 и X_2 незав, то $X_1 + X_2 \sim N$

X_1 и X_2 незав

формула свёртки $S = X_1 + X_2$

$$P(S = s) = \sum_t P(X_1 = t) \cdot P(X_2 = s - t)$$

$$f_S(s) = \int_{-\infty}^{\infty} f_1(t) \cdot f_2(s-t)\, dt$$

$$f_S(s) = \int_{-\infty}^{\infty} const \cdot \exp(-?t^2 + ?t) \cdot const \cdot \exp(-?(s-t)^2 + ?(s-t))\, dt$$

$$= \int_{-\infty}^{\infty} const \cdot \left[\exp(-?s^2 + (?t)\cdot s) \cdot \exp(?t^2 + ?t)\right] dt$$

$$\int_{-\infty}^{\infty} \exp(?ts + ?t^2 + ?t)\, dt = ?$$

$$= \exp(?s) \cdot const$$

[доделать]

Ответ: $f_S(s) = const\,(-?s^2 + ?s)$

$S \sim N(\ldots;\ldots)$

18.1 (2)

б?

$$P(\bar{X}_{100} \in [4-b;\ 4+b]) = 0.5$$

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} \qquad \bar{X}_{100} = \frac{X_1 + \ldots + X_{100}}{100}$$

(!) $E(X_1)$

по утр. $S = X_1 + X_2 + \ldots + X_{100} \sim N(?;?)$

$\boxed{X_i \sim N(4;100) \text{ независимы}}$

$$E(S) = E(X_1 + \ldots + X_{100}) \overset{?}{=}$$
$$= E(X_1) + \ldots + E(X_{100}) =$$
$$= 4 + 4 + \ldots + 4 = 400$$

$$Var(S) = Var(X_1) + Var(X_2) + \ldots + Var(X_{100}) + \underbrace{2Cov(X_1, X_2) + 2Cov(X_1, X_3) + \ldots}_{0} =$$

$$= 100\, Var(X_1) = 100 \cdot 100$$

$$\bar{X}_{100} = \frac{S}{100} \qquad E(\bar{X}) = E\left(\frac{S}{100}\right) = \frac{400}{100} = 4$$

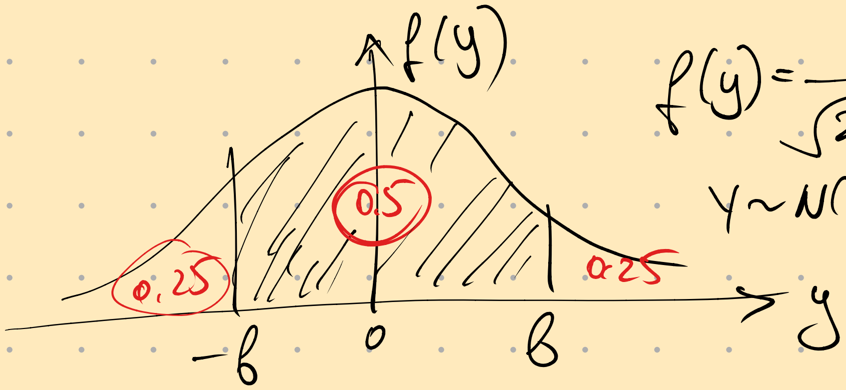
$$Var(\bar{X}) = Var\left(\frac{S}{100}\right) = \frac{1}{100^2} Var\, S = \frac{100 \cdot 100}{100^2}$$

$\overline{X} \sim N(4; 1)$ $\xrightarrow{\text{станд.}}$ $\boxed{\text{табл.}}$ $=1$

$$P(\overline{X} \in [4-b; 4+b]) = 0.5 \Rightarrow b?$$

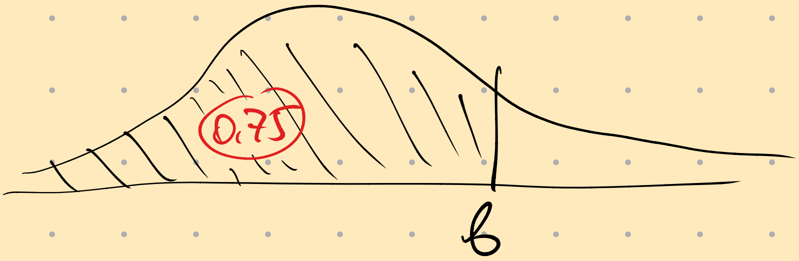
$$\frac{\overline{X}-4}{\sqrt{1}} \sim N(0;1) \qquad P\left(\frac{\overline{X}-4}{\sqrt{1}} \in [-b; b]\right) = 0.5$$

$$P(N(0;1) \in [-b; b]) = 0.5$$



$$f(y) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{y^2}{2}\right)$$

$Y \sim N(0;1)$



$$\boxed{F_Y(b) = P(Y \le b) = 0.75}$$

$b = F^{-1}(0.75)$ квант

scipy.stats.norm.ppf

percentile prob-ty function

таблица сколько?

десят? $\leftarrow$ $\boxed{0.75}$ $\uparrow$ сколько?

по табл.

$b \simeq 0.67$ $b \simeq 0.675$

18.5 $X = \ln Y \sim N(\mu; \sigma^2)$ Y имеет лог-норм. распр.

- !! $f_Y(y)$?
- !! $E(Y)$?
- !! $E(Y^2)$?
- $E(Y^2) - E(Y)^2$ = $Var(Y)$?

- !! медиану Y?
- моду Y?

$$\int_{-\infty}^{\infty} f_W(w)\,dw = 1$$

$$f_W(w) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{w^2}{2}\right) \quad W \sim N(0;1)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad X \sim N(\mu;\sigma^2)$$

Способ перейти от $f_X(x)$ к $f_Y(y)$

$$f_X(x)\cdot dx = \ldots\ldots\ldots\ldots = f_Y(y)\cdot dy$$

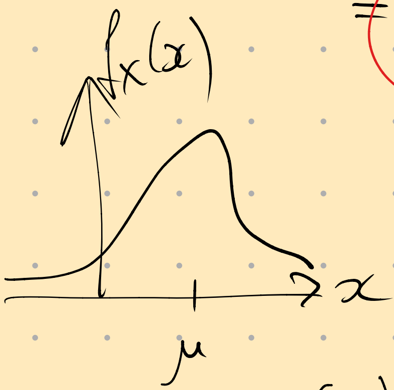
подставить
x как ф-цию от y

$$\boxed{Y = \exp(X)} \qquad X = \ln Y$$

$$f_X(x)\,dx = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(\ln y-\mu)^2}{2\sigma^2}\right) d(\ln y) =$$

не сами

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(\ln y-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{y}\,dy.$$



$$f_Y(y) = \begin{cases} \ldots & y > 0 \\ 0 & y \le 0 \end{cases}$$

$$E(Y) = E(\exp(X)) = \boxed{\begin{array}{l} X \to W \sim N(0;1) \\ \dfrac{X-\mu}{\sqrt{\sigma^2}} = W \\ X = \mu + \sigma \cdot W \end{array}} =$$

$$= E(\exp(\mu + \sigma W)) = E(\exp(\mu)\cdot\exp(\sigma W)) =$$

$$= \exp(\mu) \cdot E(\exp(\sigma \cdot W)) =$$

$$f(w) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{w^2}{2}\right)$$

$$= \exp(\mu) \cdot \int^{\infty} \exp(\sigma \cdot w) \cdot \overbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right)}^{f(w)} dw$$

$$= \exp(\mu) \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\left(w^2 - 2\sigma w + \sigma^2 - \sigma^2\right)\right) dw$$

$$= \exp(\mu) \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(w - \sigma)^2\right) \cdot \exp\left(\frac{\sigma^2}{2}\right) dw$$

$$= \exp(\mu) \cdot \exp\left(\frac{\sigma^2}{2}\right) \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(w - \sigma)^2\right) dw}_{=1}$$

сдвиг на σ $f_W(w)$

- $E(Y) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \leftarrow E(\exp(\mu + \sigma \cdot W))$

- $E(Y^2) = E(\exp(X) \cdot \exp(X)) =$

$$= E(\exp(2X)) =$$

$$= E(\exp(2 \cdot (\mu + \sigma \cdot W))) =$$

$$= E(\exp(2\mu + 2\sigma \cdot W)) \overset{?}{=} \text{(по аналогии)}$$

$$= \exp\left(2\mu + \frac{(2\sigma)^2}{2}\right) =$$

$$= \exp(2\mu + 2\sigma^2)$$

медиана Y $\quad$ m — такое число, что:

$$P(Y \le m) = \boxed{P(Y \ge m) = \frac{1}{2}} \qquad m?$$

$$P(\exp(X) \ge m) = \frac{1}{2}$$

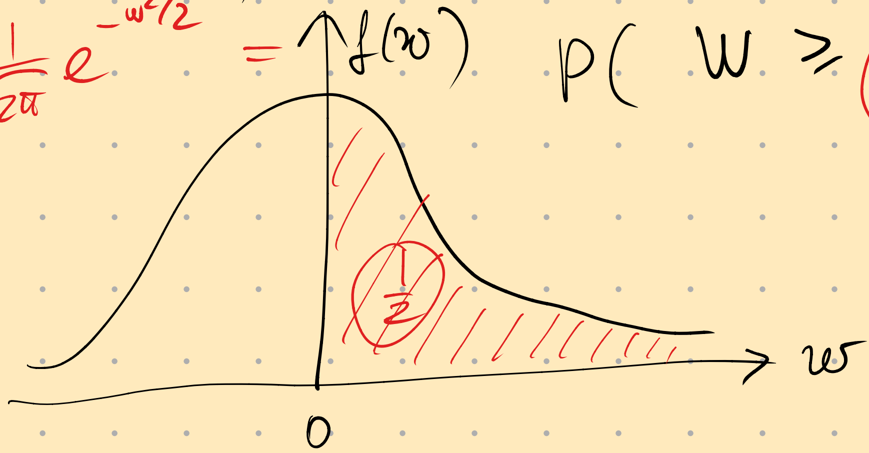
$$P(\exp(\mu + \sigma \cdot W) \ge m) = \frac{1}{2}$$

$$P(\mu + \sigma W \ge \ln m) = \frac{1}{2}$$

$W \sim N(0;1)$

$\frac{1}{\sqrt{2\pi}} e^{-w^2/2} = f(w)$

$$P\left(W \ge \frac{\ln m - \mu}{\sigma}\right) = \frac{1}{2}$$



$$\frac{\ln m - \mu}{\sigma} = 0$$

$$m = \boxed{\exp(\mu)}$$

медиана Y

$$P(Y \ge \exp(\mu)) = \frac{1}{2}$$

$$P(Y \le \exp(\mu)) = \frac{1}{2}$$

мода СВ Y $\leftarrow$ точка максимума $f_Y(y)$ $[P(Y=y)]$

$$\max_y \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right) \cdot \frac{1}{y}$$

$$\max_y \ln\left[\exp\left(-\frac{(\ln y-\mu)^2}{2\sigma^2}\right)\cdot\frac{1}{y}\right]$$

$$\max_y \; -\frac{(\ln y-\mu)^2}{2\sigma^2} - \ln y \qquad \ln y = t$$

$$\max_t \left[-\frac{(t-\mu)^2}{2\sigma^2} - t\right] = \frac{-t^2+2\mu t-2\sigma^2 t \ldots}{2\sigma^2}$$

верш. параболы.

$$t^* = \boxed{-\frac{b}{2a}} = \frac{2\mu-2\sigma^2}{2} = \mu-\sigma^2$$

$$\ln y^* = \mu-\sigma^2 \qquad y^* = \exp(\mu-\sigma^2)$$

$f_Y(y)$

$Y=\exp(X)$