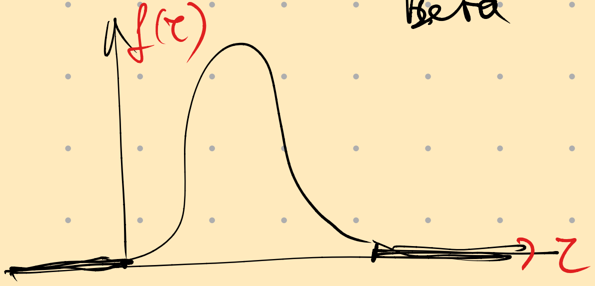
→ распределения с плотностями:

→ Beta (a, b)

→ Gamma (a, λ)

Beta



$$f(r) = \frac{r^{a-1}(1-r)^{b-1}}{B(a,b)}$$

$a>0$
$b>0$

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

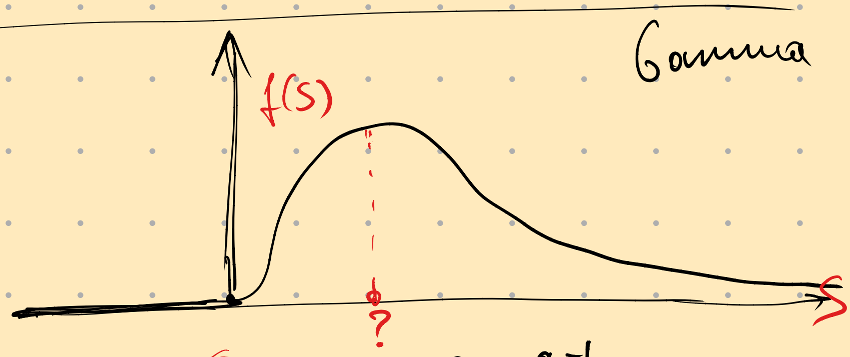
для $a, b \in \mathbb{N}$

$$B(a,b) = \frac{(a-1)!\,(b-1)!}{(a+b-1)!}$$

$$E(R) = \frac{a}{a+b}$$

$$Var(R) = \frac{ab}{(a+b)^2(a+b+1)}$$

Gamma



$$f(s) = \frac{\lambda^a \cdot \exp(-\lambda s) \cdot s^{a-1}}{\Gamma(a)}$$

для $a \in \mathbb{N}$ $\Gamma(a) = (a-1)!$

$$\boxed{\Gamma(a) = \int_0^\infty e^{-s} \cdot s^{a-1}\, ds}$$

$$E(S) = \frac{a}{\lambda}$$

$$Var(S) = \frac{a}{\lambda^2}$$

Упр $S \sim Gamma(5, \lambda = 3)$

Найдите $\boxed{\text{моду}}$ СВ S?

$\boxed{\text{точка максимума плотности}}$

$$\max_s f(s) \to \max_s \exp(-\lambda s) \cdot s^{a-1}$$

$$\to \max_s \ln\left(\exp(-\lambda s) \cdot s^{a-1}\right) = \boxed{-\lambda s + (a-1) \cdot \ln s}$$

↳ $\max_s$

$h(s)$

$$h'(s) = \boxed{-\lambda + \frac{a-1}{s} = 0}$$

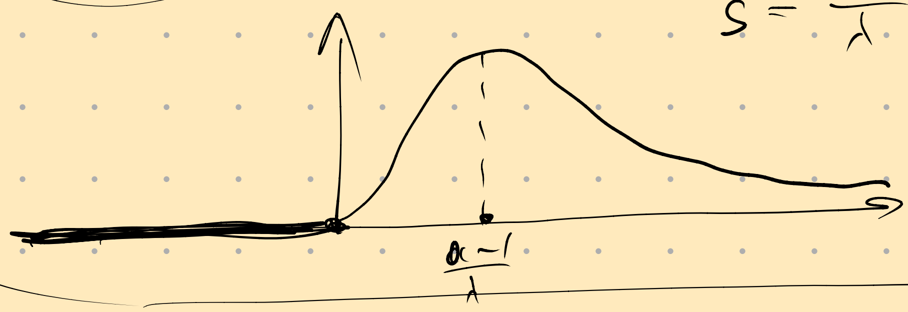
$$h''(s) = -\frac{(a-1)}{s^2}$$

разные случаи

$a = 1$

$a < 1$

$a > 1$ $\Rightarrow$ есть экстр в точке $s^* = \frac{a-1}{\lambda}$

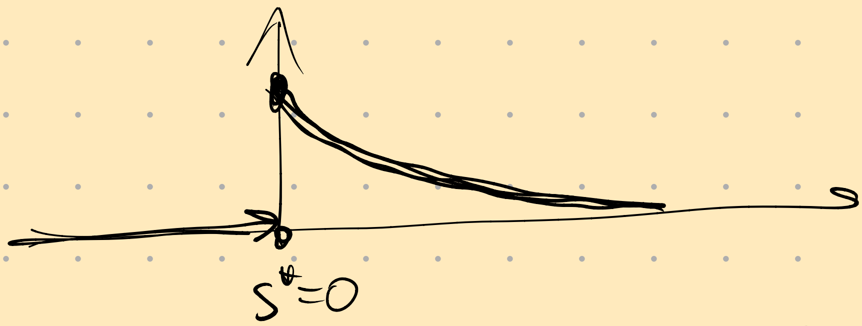


$\boxed{a=1}$

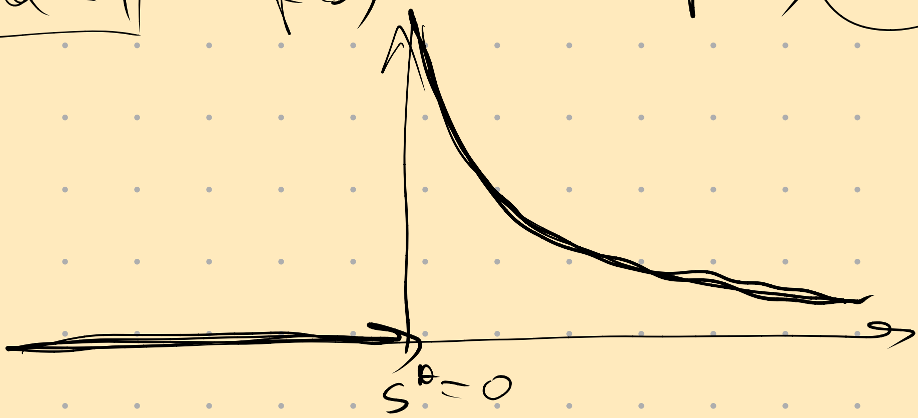
$$f(s) = \begin{cases} \lambda \cdot \exp(-\lambda s), & s \geq 0 \\ 0, & s < 0 \end{cases}$$

$\Gamma(1) = 0! = 1$

$$\text{Гамма}(1, \lambda) = \text{Expo}(\lambda)$$



$\boxed{a<1}$ $f(s) = \text{const} \cdot \exp(-s) \cdot s^{a-1}$



Упр $R \sim \text{Beta}(a, b)$ Где нах-ся мода R?

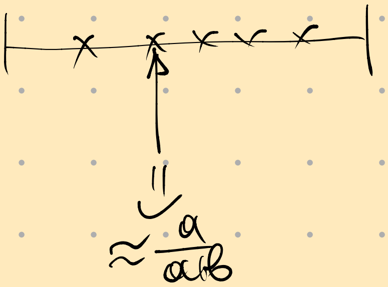
$\max_r f(r)$

$$f(\tau) = \frac{\tau^{a-1}\cdot(1-\tau)^{b-1}}{B(a,b)} \longrightarrow \max_\tau ?$$

$$\max_\tau \tau^{a-1}\cdot(1-\tau)^{b-1}$$

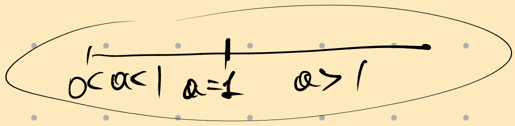
$$\max_\tau \ln\left(\tau^{a-1}(1-\tau)^{b-1}\right)$$

$$\max_\tau \underbrace{(a-1)\ln\tau + (b-1)\cdot\ln(1-\tau)}_{\varphi(\tau)}$$



$$\varphi'(\tau) = \frac{a-1}{\tau} - \frac{b-1}{1-\tau} = 0$$

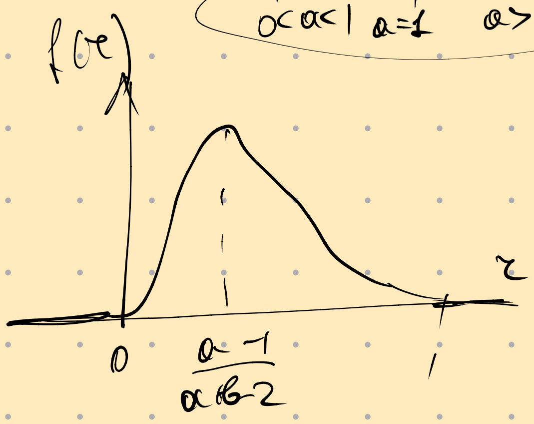
$\boxed{\text{Случай „основной"}}$ $\boxed{a>1 \quad b>1}$



$$(a-1)\cdot(1-\tau) = (b-1)\cdot\tau$$

$$(a-1) = (b-1+a-1)\cdot\tau$$

$$\tau = \frac{a-1}{a+b-2}$$



Упр

$$\Gamma(3) \overset{?}{=} (3-1)! = 2! = 2$$

$$B(2,3) \overset{?}{=} \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)} = \frac{1!\,2!}{4!} = \frac{2}{4\cdot3\cdot2\cdot1} = \frac{1}{12}$$

Упр Форм-но докажите, что $\Gamma(a+1) = a\cdot\Gamma(a)$ $\forall a>0$.

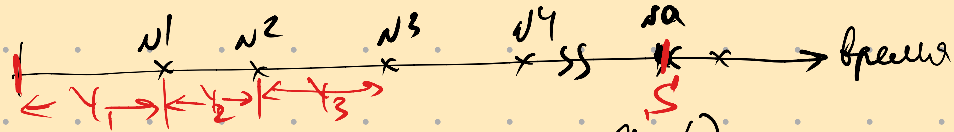
$$\Gamma(a) = \int_0^\infty \exp(-s)\cdot s^{a-1}\,ds$$

$$\Gamma(a+1) = \int_0^\infty \underbrace{\exp(-s)}_{u'}\cdot\underbrace{s^a}_{v}\,ds = \left.\underbrace{\frac{\exp(-s)}{-1}}_{u}\cdot\underbrace{s^a}_{v}\right|_{s=0}^{s=+\infty} - \int_0^\infty \underbrace{\frac{\exp(-s)}{-1}}_{u}\cdot\underbrace{a\cdot s^{a-1}}_{v'}\,ds$$

$$= 0 + a\cdot\int_0^\infty \exp(-s)\cdot s^{a-1}\,ds = a\cdot\Gamma(a)$$

$$\Gamma(1.5) = 0.5\cdot\Gamma(0.5)$$

Упр. $a \in \mathbb{N}$



$(X_t) \sim PP(\lambda)$

λ посет/час

$E(X_t) = \lambda \cdot t$

$X_t \sim Poiss(\lambda \cdot t)$

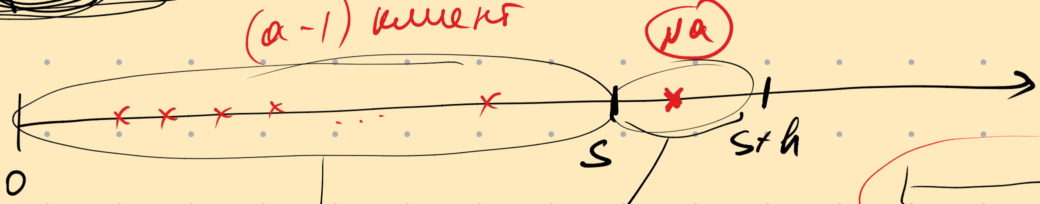
исходя из $X_t \sim Poiss(\lambda t)$ найдите $f_S(s)$ для

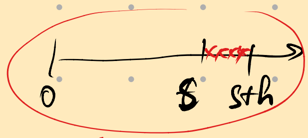
$$S = Y_1 + Y_2 + Y_3 + \ldots + Y_a$$

Y_i – время между клиентами

„а-ый по счёту клиент пришел в $[s; s+h]$"

$$\boxed{f_S(s) \cdot h} = P(S \in [s; s+h]) + o(h) =$$



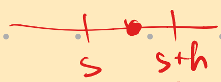


вер-сть $o(h)$

$$= o(h) + P(X_s = a-1) \cdot (\lambda h + o(h)) =$$

$$= \exp(-\lambda s) \cdot \frac{(\lambda s)^{a-1}}{(a-1)!} \cdot \lambda h + o(h)$$

$$f_S(s) = \begin{cases} \frac{\lambda^a \cdot s^{a-1} \cdot \exp(-\lambda s)}{(a-1)!} & s \geq 0 \\ 0, & \text{иначе.} \end{cases}$$



$$P(X_{s+h} - X_s = 1) = \lambda h + o(h)$$

$$P(X_{s+h} - X_s = 0) = 1 - \lambda h + o(h)$$

$$P(X_{s+h} - X_s \geq 2) = o(h)$$

Упр. а) функцию произв-ую моментов для $Gamma(a,1)$

б) — // — $Gamma(a,\lambda)$

в) как выразить $E(S^3)$ через ф. произв. моментов?

$\boxed{m(t) = E(e^{tS})}$ ф-ция произв-ая моменты

b) $E(S) = m'(0)$

$E(S^2) = m''(0)$

$E(S^3) = m'''(0)$

$\vdots$

Док-во $\left(E \leftrightarrow \frac{d}{dt} \right)$

$m'(t) = E(S \cdot e^{tS})$

$m'(0) = E(S \cdot e^0) = E(S)$

$m''(t) = E(S^2 \cdot e^{tS})$

$m''(0) = E(S^2 \cdot 1)$

...

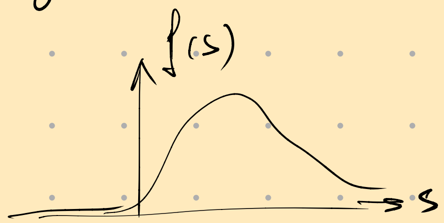
c) Gamma$(a, 1)$

$$E(e^{tS}) = \int_0^\infty e^{ts} f(s)\, ds =$$

$$= \int_0^\infty e^{ts} \cdot \frac{s^{a-1} \cdot \exp(-s)}{\Gamma(a)}\, ds =$$

Gamma(a, λ)

$$f(s) = \begin{cases} \dfrac{\lambda^a \cdot s^{a-1} \cdot \exp(-\lambda s)}{\Gamma(a)} \\ 0 \end{cases}$$



$$= \int_0^\infty \frac{s^{a-1} \cdot \exp(-s \cdot (1-t))}{\Gamma(a)}\, ds =$$

$$= \frac{1}{\Gamma(a)} \cdot \int_0^\infty \left(\frac{u}{1-t} \right)^{a-1} \cdot \exp(-u) \cdot \frac{du}{1-t} =$$

$s \cdot (1-t) = u$

$s^{a-1} = \left(\frac{u}{1-t} \right)^{a-1}$

$ds = \frac{du}{1-t}$

$$= \frac{1}{\Gamma(a)} \cdot \frac{1}{(1-t)^a} \boxed{\int_0^\infty u^{a-1} \cdot \exp(-u)\, du} = \frac{\Gamma(a)}{\Gamma(a)} \cdot \frac{1}{(1-t)^a} = \frac{1}{(1-t)^a}$$

опр

$\Gamma(a)$

для Gamma$(a, 1)$

$$\boxed{m(t) = (1-t)^{-a}}$$

$S \sim$ Gamma$(a, 1)$ $\quad E(S^2), E(S^3), E(S^4), \ldots$!!

$$m(t) = (1-t)^{-a} = 1 - a \cdot (-t) + \frac{(-a) \cdot (-a-1)}{2!} \cdot (-t)^2 + \frac{(-a) \cdot (-a-1) \cdot (-a-2)}{3!} (-t)^3 \ldots$$

$$m(t) = 1 + m'(0) \cdot t + \frac{m''(0) \cdot t^2}{2!} + \frac{m'''(0) \cdot t^3}{3!} + \ldots$$

$m'(0) = E(S)$ $\quad m''(0) = E(S^2)$ $\quad m'''(0) = E(S^3)$ $\quad \ldots$

$$m(t) = 1 + E(S)\cdot t + \frac{E(S^2)}{2!}\cdot t^2 + \frac{E(S^3)}{3!}\cdot t^3 + \dots$$

$$\boxed{S' \sim \text{Gamma}(a,1)} \quad E(S') = a \qquad E(S^2) = a\cdot(a+1)$$

$$E(S^3) = a\cdot(a+1)\cdot(a+2)$$

б) $Q \sim \text{Gamma}(a,\lambda)$

$E(Q)$? $\quad E(Q^2)$? $\quad E(Q^3)$? . . . . .

мащабиране!

$\lambda = 2$

$$\boxed{Q = \frac{S'}{\lambda}} \qquad E(Q) = \frac{a}{\lambda} \qquad E(Q^2) = \frac{a(a+1)}{\lambda^2} \dots$$