Упр

$$f(y_1, y_2) = \text{const} \cdot \exp(-10y_1^2 - 20y_2^2 + 2y_1y_2 + 4y_1)$$

а) $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \overset{?}{\sim} N(\overset{?}{\mu}; \overset{?}{C})$

Ох, это $N(?:?)$

$$f(y) = \frac{1}{\sqrt{(2\pi)^n \cdot C}} \cdot \exp\left(-\frac{1}{2}(y-\mu)^T C^{-1} (y-\mu)\right)$$

$C^{-1} = D$

Внутри по опр.

$$-\frac{1}{2}(y-\mu)^T \cdot D \cdot (y-\mu) = -\frac{1}{2}\left((y_1-\mu_1, \; y_2-\mu_2) \cdot \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \cdot \begin{pmatrix} y_1-\mu_1 \\ y_2-\mu_2 \end{pmatrix}\right)$$

$$= -\frac{1}{2}\left[d_{11} \cdot (y_1-\mu_1)^2 + d_{22} \cdot (y_2-\mu_2)^2 + 2d_{12} \cdot (y_1-\mu_1) \cdot (y_2-\mu_2)\right]$$

$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

У нас внутри: $= -\frac{1}{2}\left[20y_1^2 + 40y_2^2 - 4y_1y_2 - 8y_1\right]$

$c_{ij} = c_{ij} = Cov(Y_i, Y_j)$

$C^T = C \Rightarrow D^T = D$

$d_{ij} = d_{ji}$

Идея! прир-ем коэф-ты!

коэф при y_1^2: $d_{11} = 20$

коэф при y_2^2: $d_{22} = 40$

коэф при $y_1 y_2$: $2d_{12} = -4$

$$D = \begin{bmatrix} 20 & -2 \\ -2 & 40 \end{bmatrix}$$

$$C = D^{-1} = \frac{1}{20 \cdot 40 - 4} \cdot \begin{bmatrix} 40 & +2 \\ +2 & 20 \end{bmatrix}$$

$$= \frac{1}{796} \cdot \begin{bmatrix} 40 & 2 \\ 2 & 20 \end{bmatrix} = \frac{1}{398}\begin{bmatrix} 20 & 1 \\ 1 & 10 \end{bmatrix}$$

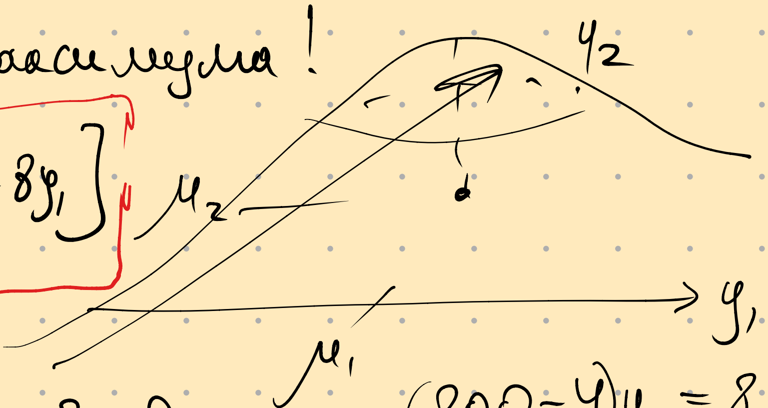
$$C_{11} = \frac{20}{398} = Var(Y_1)$$
$$C_{12} = \frac{1}{398} = Cov(Y_1, Y_2)$$
$$C_{22} = \frac{40}{398} = Var(Y_2)$$

$$C = \boxed{\begin{matrix} Var\, Y_1 & Cov(Y_1, Y_2) \\ Cov(Y_2, Y_1) & Var\, Y_2 \end{matrix}}$$

идея 2

μ – точка максимума!

$$\boxed{\max_{y_1, y_2} -\frac{1}{2}\left[20y_1^2 + 40y_2^2 - 4y_1y_2 - 8y_1\right]}$$



$$\begin{cases} \frac{\partial a(y_1, y_2)}{\partial y_1} = 40y_1 - 4y_2 - 8 = 0 \\ \frac{\partial a(y_1, y_2)}{\partial y_2} = 80y_2 - 4y_1 = 0 \end{cases}$$

$y_1 = 20y_2$

$(800 - 4)y_2 = 8$

$199 y_2 = 2$

$y_2^* = \frac{2}{199}$

$y_1^* = \frac{40}{199}$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \frac{2}{199} \\ \frac{40}{199} \end{pmatrix}; \begin{pmatrix} C \end{pmatrix}\right)$$

упр

$$Y = N\left(\begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}; \begin{pmatrix} 10 & 1 & -1 & 0 \\ 1 & 11 & -2 & 3 \\ -1 & -2 & \boxed{12} & 4 \\ 0 & 3 & 4 & 13 \end{pmatrix}\right)$$

$E(Y_2)$, $E(Y_3)$, $Var(Y_3)$, $Cov(Y_3, Y_4)$

а) $E(Y_1 Y_2)$?

$Var(Y_3)$? $= 12$

$Cov(Y_3, Y_4)$? $= 4$

$$E(Y_2 - 3Y_3) \overset{?}{=} E(Y_2) - 3E(Y_3) = 2 - 3\cdot 0 = 2$$

по опр

по нов

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)\cdot E(Y_2)$$

$$1 = E(Y_1 Y_2) - 1\cdot 2$$

$$E(Y_1 Y_2) = 1 + 2 = 3$$

б) $E(Y_1 Y_2 Y_3)$?

$E(Y_1 Y^2)$?

$E(Y_1 Y_2 Y_3 Y_4)$?

$\operatorname{cov}(Y_1, Y_2^2)$?

$E(Y_1 Y_2) = \mu_1\mu_2 + C_{12} = 1\cdot 2 + 1 = 3$

все моменты

- у каждого момента все индексы 1, 2
- μ_i, C_{ij}

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 & 1 \\ 1 & 11 \end{pmatrix}\right)$$

$E(Y_1 Y_2 \boxed{Y_3}) = \mu_1\mu_2\mu_3 + \mu_1 C_{23} + \mu_2 C_{13} + \mu_3 C_{12} =$

$= 1\cdot 2\cdot 0 + 1\cdot(-2) + 2\cdot(-1) + 0\cdot 1$

$E(Y_1 Y_2^2) = E(Y_1\cdot Y_2\cdot \boxed{Y_2}) = \mu_1\mu_2\mu_2 + \mu_1 C_{22} + \mu_2 C_{12} + \mu_2 C_{12} =$

$= 1\cdot 2\cdot 2 + 1\cdot 11 + 2\cdot 1 + 2\cdot 1$

$$C = \begin{pmatrix} 10 & 1 & -1 & 0 \\ 1 & 11 & -2 & 3 \\ -1 & -2 & \boxed{12} & 4 \\ 0 & 3 & 4 & 13 \end{pmatrix}$$

исп-м индекс 2 вместо 3 в пред-ей формуле

$$\mu = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}$$

$E(Y_1 Y_2 Y_3 Y_4) = \mu_1\mu_2\mu_3\mu_4 + \mu_1\mu_2 C_{34} + \mu_1\mu_3 C_{24} + \mu_1\mu_4 C_{23} + \mu_2\mu_3 C_{14} + \mu_2\mu_4 C_{13} + \mu_3\mu_4 C_{12} +$

$+ C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}$

$\operatorname{cov}(Y_1, Y_2^2) = E(Y_1 Y_2^2) - E(Y_1)\cdot E(Y_2^2)$

$E(Y_1 Y_2 Y_3)$, $\boxed{3} \to \boxed{2}$

μ_1

$E(Y_1 Y_2) = \mu_1\mu_2 + C_{12}$

$E(Y_2^2) = \mu_2^2 + C_{22}$

$\boxed{\text{теорем.}}$?

$\boxed{\text{Условное распр-ие.}}$

Упр

$$\begin{pmatrix} Y_1 \\ \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ \end{pmatrix}\right)$$

$$\begin{pmatrix} \ldots \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \ldots \\ 2 \end{pmatrix}; \begin{pmatrix} \ldots & \ldots \\ -1 & 10 \end{pmatrix}\right)$$

a) $P(Y_1 > 2)$?

б) $P(Y_1 > 2 \mid Y_2 = 3)$?

$$Y_1 \sim N(\underbrace{1}_{E(Y_1)}; \underbrace{4}_{Var(Y_1)})$$ Step 1

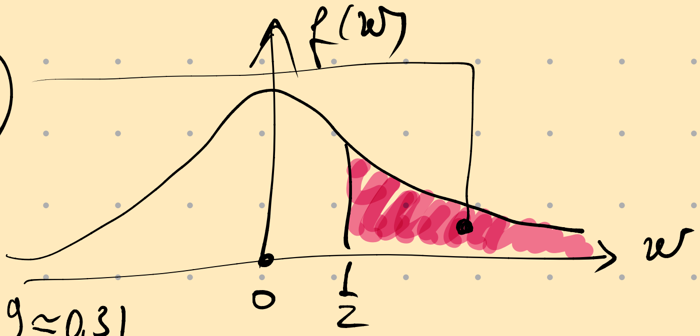
$$\downarrow$$

$$W_1 \sim N(0;1)$$

$$W_1 = \frac{Y_1 - 1}{\sqrt{4}} \sim N(0;1)$$ Step 2

$$P(Y_1 > 2) = P\left(\frac{Y_1 - 1}{\sqrt{4}} > \frac{2-1}{\sqrt{4}}\right) =$$

$$= P\left(W_1 > \tfrac{1}{2}\right)$$



$$= 1 - P\left(W_1 \le \tfrac{1}{2}\right) =$$

$$= \boxed{1 - F\left(\tfrac{1}{2}\right)} \approx 1 - 0{,}69 \approx 0{,}31$$

F – ф. расп-ия → stats. norm. cdf ‼ → табл. ‼

$$\to F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)\, du$$

б) Step 1

$$(Y_1 \mid Y_2 = 3) \sim N(?;?)$$

Step 2

$$\frac{Y_1 - ?}{\sqrt{?}} \sim N(0;1)$$

можно

$$f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f(y_2)} = \frac{const \cdot \exp(\ldots)}{const \cdot \exp(\ldots)} = \ldots$$

Верно, но возни много

$$Y_1 = X_1 + \mu_1$$
$$Y_2 = X_2 + \mu_2$$
$$X_1 = Y_1 - 1$$
$$X_2 = Y_2 - 2$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \underbrace{\begin{pmatrix} 4 & -1 \\ -1 & 10 \end{pmatrix}}_{C}\right)$$

$C_{22} = 10$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}; \begin{pmatrix} 4 & -1 \\ -1 & 10 \end{pmatrix}\right)$$

$Y_2 = 3 \Longleftrightarrow X_2 = 3 - 1 = 1$

цель: $(Y_1 \mid Y_2 = 3) \sim N(?;?)$

$(X_1 \mid X_2 = 1) \sim N(?;?)$

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f(x_2)} = const \cdot \frac{\exp\left(-\frac{1}{2} x^T \cdot C^{-1} x\right)}{\exp\left(-\frac{1}{2} x_2 \cdot C_{22}^{-1} x_2\right)}$$

$$f(x_1|x_2) = const \exp\left(-\frac{1}{2}\left[x^T C^{-1} x - x_2 \cdot C_{22}^{-1} x_2\right]\right)$$

$$= const \cdot \exp\left(-\frac{1}{2} g(x)\right)$$

конст

$$g(x) = (x_1\ x_2) \cdot \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}^{-1} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - x_2^2 \cdot C_{22}^{-1}$$

нас интересует как зависимость x_1

$$= (x_1\ x_2) \cdot \frac{1}{d} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + const \qquad d = \det C$$

$$= \frac{1}{d} \cdot \left[C_{22} \cdot x_1^2 + C_{11} \cdot x_2^2 - 2 C_{12} \cdot x_1 x_2\right] + const =$$

мне не инт-но!

$$= \frac{1}{d} C_{22} \cdot x_1^2 - 2 C_{12} \cdot x_1 x_2 + Const.$$

$$= \frac{C_{22}}{d}\left[x_1^2 - 2 x_1 \cdot \frac{C_{12}}{C_{22}} x_2 + (\text{до полн. кв.})\right] + const$$

$$= \frac{C_{22}}{d}\left[x_1 - \frac{C_{12}}{C_{22}} x_2\right]^2 + const$$

$\boxed{\text{итог:}}$

$$\boxed{f(x_1|x_2)} = const \cdot \exp\left(-\frac{1}{2} \frac{\left(x_1 - \frac{C_{12}}{C_{22}} x_2\right)^2}{d / C_{22}}\right)$$

одномерное $N(\mu; \sigma^2)$

$$const \cdot \exp\left(-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}\right)$$

Если $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; C\right)$

$d = \det C$

то $(X_1|X_2) \sim N\left(\frac{C_{12}}{C_{22}} X_2 ; \frac{d}{C_{22}}\right)$

$$d = \det C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} =$$

$C_{11} C_{22} - C_{12}^2$

$$\boxed{(X_1 | X_2) \sim N\left(\frac{C_{12}}{C_{22}} \cdot X_2 ; C_{11} - \frac{C_{12}^2}{C_{22}}\right)}$$

$Y_1 = X_1 + \mu_1$

$Y_2 = X_2 + \mu_2$

$$(\cancel{X_1} | Y_2) \sim N\left(\frac{C_{12}}{C_{22}} \cdot (Y_2 - \mu_2) ; C_{11} - \frac{C_{12}^2}{C_{22}}\right)$$

$$\boxed{(Y_1 | Y_2) \sim N\left(\mu_1 + \frac{C_{12}}{C_{22}} (Y_2 - \mu_2) ; C_{11} - \frac{C_{12}^2}{C_{22}}\right)}$$

$$(Y_1 | Y_2 = 3) \sim N\left(1 + \frac{-1}{10} \cdot (3 - 2) ; 4 - \frac{(-1)^2}{10}\right)$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} ; \begin{pmatrix} 4 & -1 \\ -1 & 10 \end{pmatrix}\right)$$

$$(Y_1 | Y_2 = 3) \sim N(0.9 ; 3.9)$$

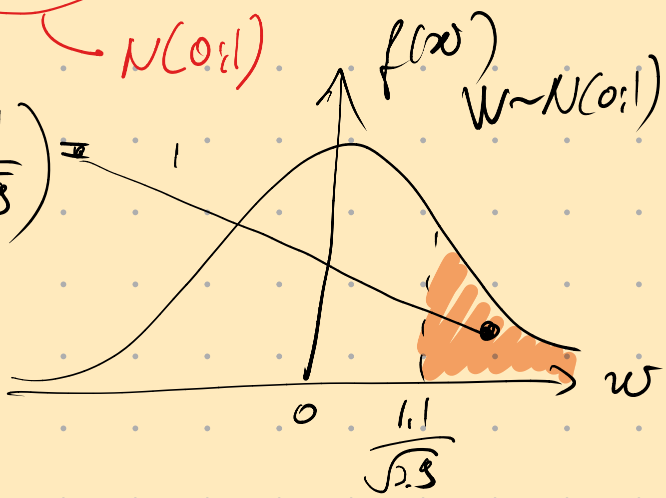
б) $P(Y_1 > 2 | Y_2 = 3)$?

$$P(Y_1 > 2 | Y_2 = 3) =$$

$$= P\left(\underbrace{\frac{Y_1 - 0.9}{\sqrt{3.9}}}_{N(0;1)} > \frac{2 - 0.9}{\sqrt{3.9}} \,\Big|\, Y_2 = 3\right) =$$

$$= P\left(N(0;1) > \frac{1.1}{\sqrt{3.9}}\right) =$$

$$= \boxed{1 - F\left(\frac{1.1}{\sqrt{3.9}}\right)}$$



$1 - \text{stats.norm.cdf}\left(\frac{1.1}{\sqrt{3.9}}\right)$

таблицы

24.10

$X \sim N(0; 4)$ Var(X)

$(Y|X) \sim N(2X - 1, 9)$

$\begin{pmatrix} X \\ Y \end{pmatrix} \sim$

$\begin{pmatrix} Y \\ X \end{pmatrix} \overset{?}{\sim} N\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}; \begin{pmatrix} 25 & 8 \\ 8 & 4 \end{pmatrix}\right)$

$$(Y_1|Y_2) \sim N\left(\mu_1 + \frac{C_{12}}{C_{22}}(Y_2 - \mu_2); C_{11} - \frac{C_{12}^2}{C_{22}}\right)$$

$\boxed{2 = \frac{C_{12}}{C_{22}}}$ $\quad 2 = \frac{C_{12}}{4} \quad C_{12} = 8$

$9 = C_{11} - \frac{C_{12}^2}{C_{22}} \quad C_{11} = Var(Y) = 9 + \frac{8^2}{4} = 9 + 16 = 25$

$\mu_1 - \frac{C_{12}}{C_{22}} \cdot \mu_2 = -1$