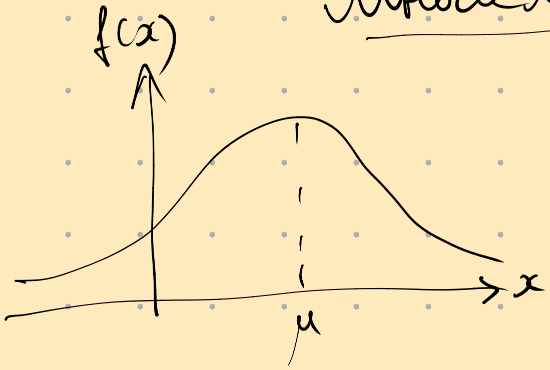
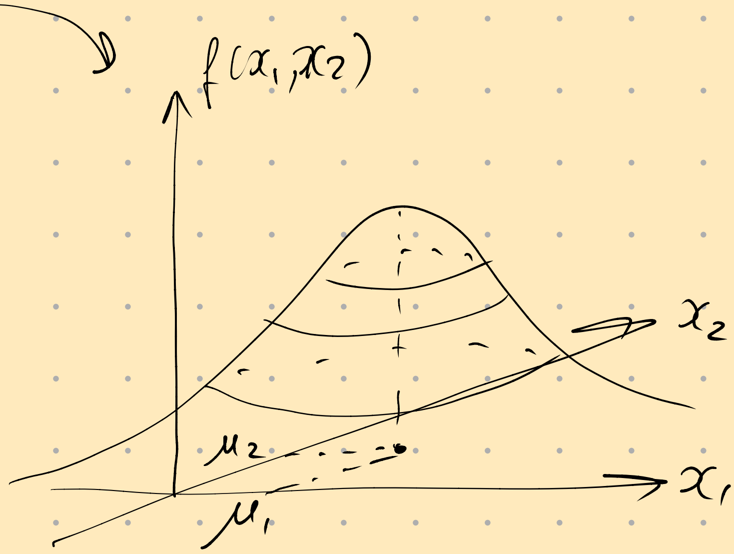
Многомерное нормальное распределение





$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

<u>Опр</u> $W \sim N(0; I)$ многомерное стандартное норм-ое.

$$E(W) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad E(W_i) = 0$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = Var(W)$$

$Var(W_1)$, $Cov(W_1, W_n)$, $Var(W_n)$

E - единичная

I - identity

$$W = \begin{pmatrix} W_1 \\ \vdots \\ W_n \end{pmatrix}$$

$W_1, W_2, \dots, W_n$ - независимы и $W_i \sim N(0;1)$

$$f(w_1, \dots, w_n) = \prod_{i=1}^{n} f(w_i) \qquad f(w_i) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{w_i^2}{2}\right)$$

<u>Опр</u> Если Y представим в виде $Y = \mu + A \cdot W$, где $W \sim N(0; I)$, то мы говорим, что Y имеет многомерное нормальное распр-ие $N(\mu, AA^T)$

$\boxed{\text{Упр}}$ $W \sim N(0; I)$ и $Y = \mu + A \cdot W$

$E(Y)$?

$Var(Y)$?

$$\begin{pmatrix}Y_1\\Y_2\\\vdots\\Y_k\end{pmatrix} = \begin{pmatrix}\mu_1\\\vdots\\\mu_k\end{pmatrix} + \begin{bmatrix}a_{11} & \dots & a_{1n}\\ a_{k1} & \dots & a_{kn}\end{bmatrix}\cdot\begin{pmatrix}W_1\\\vdots\\W_n\end{pmatrix}$$

$$E(Y) = \mu = \begin{pmatrix}\mu_1\\\vdots\\\mu_k\end{pmatrix} \qquad E(W_i) = 0$$

в скал.: a-скал

$E(a\cdot R) = a\cdot E(R)$

$E(R+S) = E(R) + E(S)$

в векторе

$E(A\cdot R) = A\cdot E(R)$!!

$E(R+S) = E(R) + E(S)$ A-мат

Y – вектор

$$Var(Y) = \begin{bmatrix} Var(Y_1) & Cov(Y_1,Y_2) & \dots \\ Cov(Y_2,Y_1) & Var(Y_2) & \dots \\ \vdots & & \ddots \\ & & Var(Y_k) \end{bmatrix}$$

$$Y = \begin{pmatrix}Y_1\\\vdots\\Y_k\end{pmatrix}$$

Y – скаляр

$Var(Y)$ – скаляр дисперсия

скал.:

$Var(R) = E(R^2) - (E(R))^2$

в вектор.

$Var(R) = E(R\cdot R^T) - E(R)\cdot E(R^T)$

$Var(aR) = a^2\cdot Var(R)$

$$Var(A\cdot R) \overset{?}{=} E(AR(AR)^T) - E(AR)\cdot E((AR)^T) =$$
$$= E(A\cdot RR^TA^T) - E(AR)\cdot E(R^TA^T) =$$
$$= A\cdot E(RR^T)A^T -$$
$$- A\cdot E(R)\cdot E(R^T)A^T =$$
$$= A\cdot(E(RR^T) - E(R)E(R^T))A^T$$
$$= A\cdot Var(R)\cdot A^T$$

$R = \begin{pmatrix}R_1\\R_2\end{pmatrix}$ $\quad E(AR) = \dots$ упр. $= A\cdot E(R)$

$A = \begin{pmatrix}5 & 6\\7 & 8\end{pmatrix}$ $\quad E(R^TA^T) = \dots = E(R^T)\cdot A^T$

„выносятся с квадратом"

$$Var(Y) = Var(\mu + A\cdot W) = Var(A\cdot W) = A\cdot Var(W)\cdot A^T = AA^T$$

конст не влияют на $Var(\cdot)$, $Cov(\cdot;\cdot)$

коммент: если $Y = \mu + A \cdot W$, $W \sim N(0; I)$ и $\det A \neq 0$,
то $Y \sim N(\mu; AA^T)$ „многомерное нормальное невырожденное расп-ие"
нез-т

Упр $Y \sim N(\mu; AA^T)$ $Y = \mu + A \cdot W$, $\det A \neq 0$ $W \sim N(0; I)$

ф. плотн. $f(y)$?

$$f_W(w) \xrightarrow{?} f_Y(y)$$

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \xrightarrow[Y = h(W)]{} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$\boxed{f_Y(y) = f_W(w(y)) \cdot \left|\det \frac{\partial w}{\partial y}\right|}$$

$f(y_1, y_2) \xrightarrow{?} f(y_1)$
[выинтегр. y_2]

$$f(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2$$

$$P(Y_1 = y_1) = \sum_{y_2} P(Y_1 = y_1, Y_2 = y_2)$$

Step1 $Y = \mu + A \cdot W$

- $W \overset{?}{=} \boxed{A^{-1}(Y - \mu)}$

$$\text{"}\underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}}_{A^{-1}} \cdot \left(\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right)\text{"}$$

на примере

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 5Y_1 + 6Y_2 + const \\ 7Y_1 + 2Y_2 + const \end{pmatrix}$$

$$\frac{\partial w}{\partial y} = \underbrace{\begin{pmatrix} 5 & 6 \\ 7 & 2 \end{pmatrix}}_{A^{-1}}$$

$$\frac{\partial w}{\partial y} = \begin{pmatrix} \frac{\partial w_1}{\partial y_1} & \frac{\partial w_1}{\partial y_2} \\ \frac{\partial w_2}{\partial y_1} & \frac{\partial w_2}{\partial y_2} \end{pmatrix} =$$

$$\boxed{\frac{\partial w}{\partial y} = A^{-1}}$$

- $\det\left(\frac{\partial w}{\partial y}\right) = \det A^{-1} = \frac{1}{\det A}$

Step2 Подставляем найденное в $f_W(w)$

$$f_W(w_1, w_2 \ldots w_n) = f(w_1) \cdot f(w_2) \cdot \ldots \cdot f(w_n) =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{w_1^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{w_2^2}{2}} \cdot \ldots \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{w_n^2}{2}}$$

$$= \frac{1}{\sqrt{(2\pi)^n}} \cdot \exp\left(-\frac{w^T \cdot w}{2}\right)$$

$$w^T w = \sum w_i^2$$

$$f_Y(y) = f_w(w(y)) \cdot \frac{1}{|\det A|} =$$

$$w = A^{-1} \cdot (Y - \mu)$$
$$w^T = (Y-\mu)^T \cdot (A^{-1})^T$$

$$= \frac{1}{\sqrt{(2\pi)^n (\det A)^2}} \cdot \exp\left(-\frac{(Y-\mu)^T \cdot (A^{-1})^T \cdot A^{-1} \cdot (Y-\mu)}{2}\right)$$

$$Y \sim N(\mu; \underbrace{AA^T})$$

$$AA^T = C \qquad c_{ij} = Cov(Y_i, Y_j) \qquad c_{ii} = Var(Y_i)$$

$$\det A \cdot \det A = \det C$$

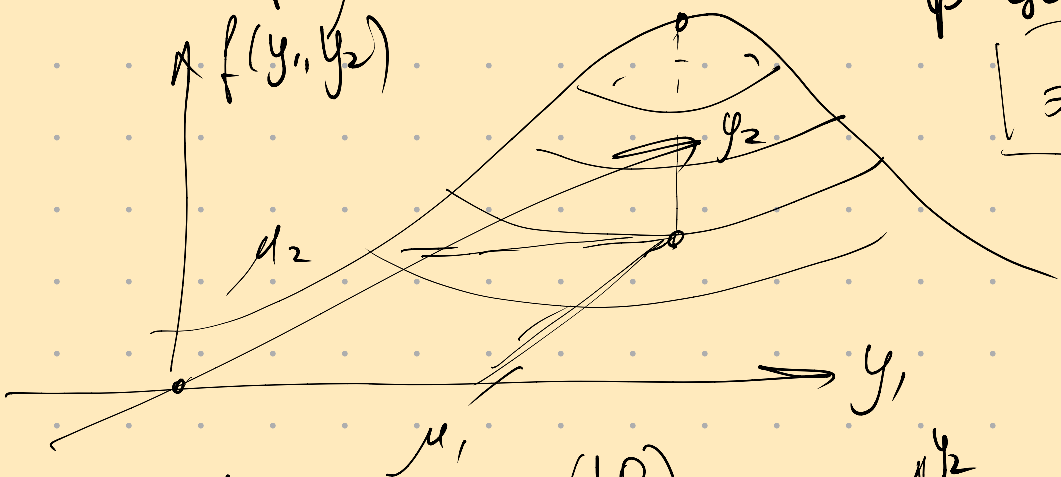
$$Y \sim N(\mu; C)$$

$$f_Y(y) = \frac{1}{\sqrt{(2\pi)^n \cdot \det C}} \cdot \exp\left(-\frac{(y-\mu)^T \cdot C^{-1} \cdot (y-\mu)}{2}\right)$$

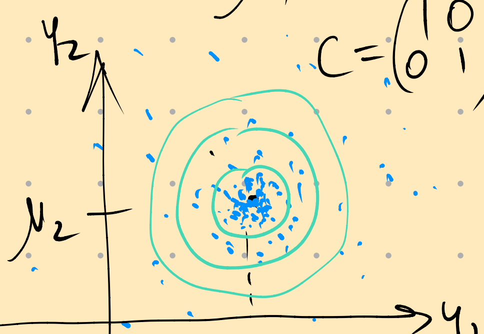
const

вектор μ отвечает положение экстремума ф-ции

$\boxed{\text{экстр в } \mu = \begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}}$



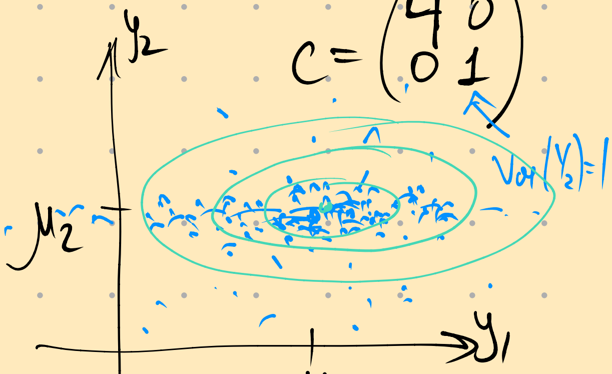
$C = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$



$Var(Y_1) = 4$

$C = \begin{pmatrix}4 & 0 \\ 0 & 1\end{pmatrix}$

$Var(Y_2) = 1$



μ_1 $\qquad\qquad$ μ_1

линии уровня f - эллипсы
[форма опр-ся м-цей C]

$\boxed{\text{Цель}}$ $E(Y_1 \cdot Y_2 \cdot Y_3 \cdot Y_4)$?

идея: !!
$$\int\int\int\int\ldots\int_{\mathbb{R}^n} y_1 \cdot y_2 \cdot y_3 \cdot y_4 \cdot f(y_1, \ldots, y_n)\,dy_1 \ldots dy_n$$

не пойдём !!

поехали!!

на примере

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}$$

$$E(☺) = \int_{\mathbb{R}} ☺ \cdot f(☺)\,d☺$$

$\boxed{\text{шаг 1}}$
$$m(u) = E(\exp(u_1 \cdot Y_1 + u_2 \cdot Y_2 + \ldots + u_3 Y_3)) = E(\exp(u^T Y))$$

функция производящая моменты.

$m(u)$?

$\boxed{\text{шаг 2!}}$
$$E(Y_1) = \frac{\partial m}{\partial u_1} = E(Y_1 \cdot \exp(u^T Y)) \underset{u=0}{=} E(Y_1 \cdot 1) = E(Y_1)$$

$$E(Y_1 Y_2) = \frac{\partial^2 m}{\partial u_2 \partial u_1} = E(Y_1 \cdot Y_2 \cdot \exp(u^T Y)) \underset{u=0}{=} E(Y_1 \cdot Y_2)$$

$$E(Y_1 Y_2 Y_3 Y_4) = \frac{\partial^4 m}{\partial u_4 \partial u_3 \partial u_2 \partial u_1} = E(Y_1 Y_2 Y_3 \cdot Y_4 \cdot \exp(u^T Y)) = E(Y_1 \cdot Y_2 \cdot \ldots \cdot Y_4)$$

шаг 1
$$E(\exp(u^T Y)) = E(\exp(u^T(\mu + A \cdot W))) =$$
$$= E(\underbrace{\exp(u^T \mu)}_{\text{const}} \cdot \exp(u^T A \cdot W)) =$$

случ. вектор

$$= \exp(u^T \cdot \mu) \cdot E(\exp(u^T A \cdot W))$$

$$E(\exp(\underbrace{u^T A}_{\text{вектор конст}} \cdot \underbrace{w}_{\text{СВ}})) =$$

$u^T = (u_1\ u_2\ u_3\ u_4)$ $\quad A = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ $\quad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$

$$u^T \cdot A = \boxed{\text{строка}} = b^T = (b_1\ b_2\ b_3\ b_4)$$

$$= E(\exp(b^T \cdot w)) = E(\exp(b_1 w_1 + b_2 w_2 \ldots))$$

$$= \int_{\mathbb{R}^n} \exp(b^T w) \cdot f_w(w)\, dw = \int_{\mathbb{R}^n} \exp(b^T w) \cdot \frac{1}{\sqrt{(2\pi)^n}} \cdot \exp\left(-\frac{w^T w}{2}\right) dw$$

$$= \int_{\mathbb{R}^n} \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\left(w^T w - 2b^T w + b^T b - b^T b\right)\right) dw$$

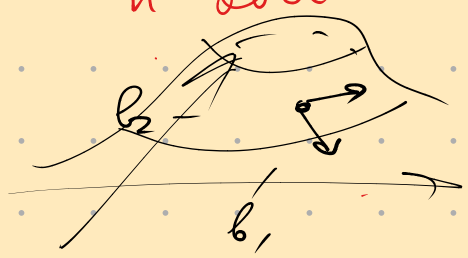
$$(w-b)^T (w-b) = \underbrace{w^T w}_{\sum w_i^2} + \underbrace{b^T b}_{\sum b_i} - \underbrace{b^T w - w^T b}_{\sum b_i w_i}$$

$$= \int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi^n}} \exp\left(-\frac{1}{2}(w-b)^T (w-b)\right) \cdot \exp\left(\frac{b^T b}{2}\right) dw$$

$$= \exp\left(\frac{b^T b}{2}\right) \cdot \boxed{\int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(w-b)^T (w-b)\right) dw}$$

! инт. не зависит от b

n-объём



$$= \exp\left(\frac{b^T b}{2}\right) \cdot \underbrace{\boxed{\int_{\mathbb{R}^n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} w^T w\right) dw}}_{1}$$

Итог: $E\left(\exp(u_1 Y_1 + u_2 Y_2 + \dots u_n Y_n)\right) =$

$$= \exp(u^T \mu) \cdot \exp\left(\frac{b^T b}{2}\right) =$$

$$= \exp(u^T \mu) \cdot \exp\left(\frac{u^T A \cdot A^T u}{2}\right)$$

$b^T = u^T A$

$b = A^T u$

$A A^T = C$

$$= \exp\left(u^T \mu + \frac{u^T C u}{2}\right)$$

$\boxed{\text{Пример.}}$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \begin{pmatrix} 10 & 0 & 1 \\ 0 & 12 & -1 \\ 1 & -1 & 13 \end{pmatrix}\right)$$

$$m(u) = E\left(\exp(u^T Y)\right) = \exp\left(\boxed{1 \cdot u_1} + 2 \cdot u_2 + 3 \cdot u_3 + \frac{1}{2}(u_1\ u_2\ u_3) \cdot \begin{pmatrix} 10 & 0 & 1 \\ 0 & 12 & -1 \\ 1 & -1 & 13 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}\right)$$

$E(Y_1) = 1 = \mu_1$

$E(Y_2) = 2 = \mu_2$

$E(Y_3) = 3 = \mu_3$

Упр. $m(u) = \exp(Q(u))$

$Q(u) = u^T \cdot \mu + \frac{1}{2} u^T \cdot C \cdot u$ ← то, что внутри exp

$$\frac{\partial Q}{\partial u_1} = 1 + u_1 \cdot 10 + 0 \cdot u_2 + 1 \cdot u_3$$

$Q(0) = 0$

$\exp(Q(0)) = 1$

$$Q_1' = \frac{\partial Q}{\partial u_1} = \mu_1 + \overset{0}{u^T \cdot c_1} = \mu_1$$

c_1 — первый столб C

$$Q_{12}'' = \frac{\partial^2 Q}{\partial u_1 \partial u_2} = c_{12} \qquad Q''' = 0$$

$$E(Y_1) = \frac{\partial m}{\partial u_1} = \exp(Q(u)) \cdot Q_1' = \exp(Q(u)) \cdot (\mu_1 + u^T c_1) \overset{u=0}{=}$$

$$= 1 \cdot \mu_1$$

↓ по u_2

$$E(Y_1 \cdot Y_2) = \frac{\partial^2 m}{\partial u_2 \partial u_1} = \exp(Q) \cdot \left[Q_2' \cdot Q_1' + Q_{12}''\right] \underset{u=}{=} 1 \cdot (\mu_1 \mu_2 + c_{12})$$

по u_3

$\boxed{Q''' = 0}$

$$E(Y_1 \cdot Y_2 \cdot Y_3) = \frac{\partial^3 m}{\partial u_3 \partial u_2 \partial u_1} = \exp(Q) \cdot \left[Q_3' \cdot [Q_2' Q_1' + Q_{12}''] + Q_{23}'' \cdot Q_1' + Q_2' \cdot Q_{13}'' \right]$$

$u = 0$

$$= 1 \cdot (\mu_3 \cdot \mu_2 \mu_1 + \mu_3 C_{12} + C_{23} \cdot \mu_1 + \mu_2 \cdot C_{13})$$

по u_4

$E(Y_1 Y_2 \cdot Y_3 \cdot Y_4) =$

$Q_1' \cdot Q_2' \cdot Q_{34}''$

$$= \mu_1 \mu_2 \mu_3 \mu_4 + \mu_1 \mu_2 C_{34} +$$
$$+ \mu_4 \mu_3 \cdot C_{12} + \mu_1 \mu_4 C_{23} +$$
$$+ \mu_1 \mu_3 C_{24} + \mu_2 \mu_4 C_{13} +$$
$$+ \mu_2 \mu_3 C_{14} + \underbrace{C_{12} C_{34}}_{Q_{12}'' Q_{34}''} + C_{13} C_{24} + C_{14} C_{23}$$

! в каждый моном содержит все индексы по 1-му разу. $\boxed{1,2,3}$

! Все мономы с коэф 1!

! все вар-ты! $\boxed{\mu_i}$ $\boxed{C_{ij}}$