Упр $X_1, X_2, \ldots, X_n$ ~ незав, $U[0;1]$ 20.1

ЗБЧ

а) $\operatorname{plim} \dfrac{X_1+X_2+\ldots+X_n}{n} = E(X_1) = \frac{1}{2}$

ф. плотн.

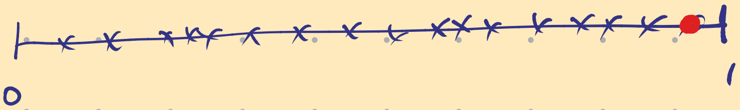
б) $\operatorname{plim} \dfrac{X_1^2+X_2^2+\ldots+X_n^2}{n} = E(X_1^2) = \int_0^1 x^2 \cdot 1\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

в) $\operatorname{plim} \max\{X_1, X_2 \ldots X_n\} = 1$

$R_n = \max\{X_1, \ldots X_n\}$

• догадаться и доказать.

$\boxed{\text{догадка} = 1}$



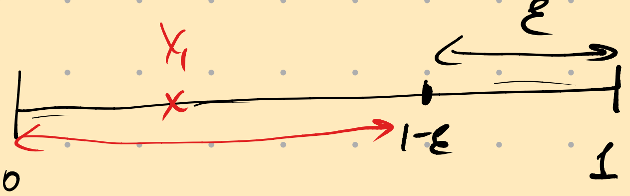
$R_n = \max\{X_1, \ldots, X_n\}$

$$P(|R_n - R| > \varepsilon) \xrightarrow{n\to\infty} 0 \qquad (\forall \varepsilon > 0)$$

догадка: $R=1$

$$P(|R_n - 1| > \varepsilon) = P(1 - R_n > \varepsilon) =$$
$$= P(R_n < 1-\varepsilon) =$$
$$= P(\max(X_1, \ldots, X_n) < 1-\varepsilon) =$$
$$= P(X_1 < 1-\varepsilon, X_2 < 1-\varepsilon, \ldots, X_n < 1-\varepsilon) =$$
$$= P(X_1 < 1-\varepsilon) \cdot P(X_2 < 1-\varepsilon) \cdot \ldots \cdot P(X_n < 1-\varepsilon) \leq$$

$\varepsilon > 0$



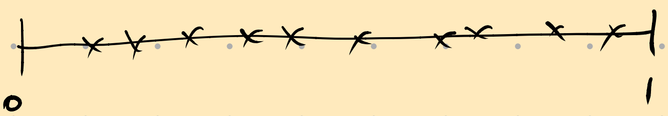
$\boxed{\varepsilon \geq 1} \Rightarrow P(X_1 < 1-\varepsilon) = 0$

$\varepsilon \in (0;1) \Rightarrow P(X_1 < 1-\varepsilon) = \dfrac{1-\varepsilon}{1} = 1-\varepsilon$

$$\leq (1-\varepsilon)^n \xrightarrow{n\to\infty} 0$$

г) $\text{plim}_{n\to\infty} X_1 \cdot X_2 \cdot X_3 \cdot \ldots \cdot X_n =$ догадка 0 ?

$X_1, X_2, \ldots$ ~ незав, $U[0;1]$



$R = 0$ (догадка)

$R_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$

$P(|R_n - R| > \varepsilon) = P(R_n > \varepsilon) =$

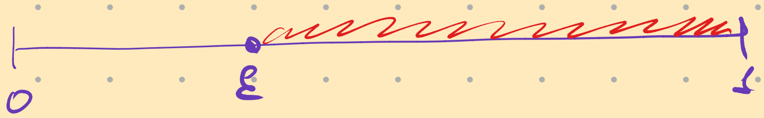
$= P(X_1 \cdot X_2 \cdot X_3 \cdot \ldots \cdot X_n > \varepsilon) \le$

$\varepsilon \geq 1 \Rightarrow$ $\boxed{\text{вер-сть} = 0.}$

$\varepsilon \in (0;1)$

$\le P(X_1 > \varepsilon, X_2 > \varepsilon, X_3 > \varepsilon \ldots X_n > \varepsilon) =$

$= P(X_1 > \varepsilon) \cdot P(X_2 > \varepsilon) \cdot \ldots \cdot P(X_n > \varepsilon) = (1-\varepsilon)^n \xrightarrow{n\to\infty} 0$



Простой критерий

Если $E(|R_n - R|) \xrightarrow{n\to\infty} 0$ то $\text{plim}\, R_n = R$

док-во $P(|R_n - R| > \varepsilon) \le$

СВ СВ

на какое нер-во похоже?

Марков:
если $Y \geq 0$, $a > 0$
то $P(Y \geq a) \le \frac{E(Y)}{a}$

Чебышёв
если $\mu = E(Y)$, $\sigma^2 = \text{Var}\, Y$ то
$P(|Y - \mu| \geq a) \le \frac{\sigma^2}{a^2}$

СВ const

$Y = |R_n - R| \geq 0$

Марков

$\le E(|R_n - R|) / \varepsilon \xrightarrow{n\to\infty} \frac{0}{\varepsilon} = 0$ ☺

$\varepsilon > 0$

Упр. $\Omega = \{a, b, c\}$

	a	b	c
вер	0,1	0,2	0,7

$R_n(a) = 1+\frac{1}{n}$

$R_n(b) = 2-\frac{1}{n}$

$R_n(c) = 0$

t	$1+\frac{1}{1}$	$2-\frac{1}{1}$	0
$P(R_1=t)$	$0{,}1$	$0{,}2$	$0{,}7$

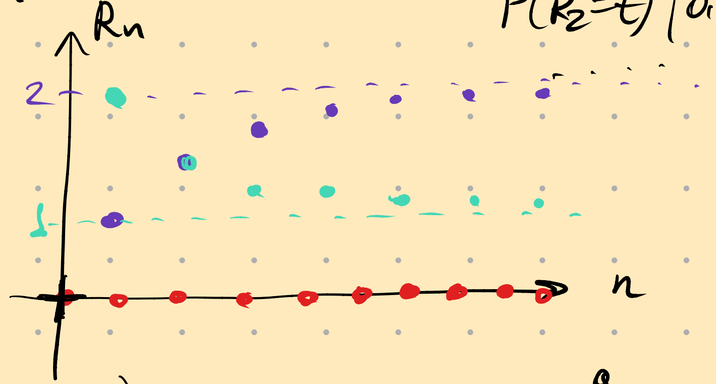
t	$1+\frac{1}{2}$	$2-\frac{1}{2}$	0
$P(R_2=t)$	$0{,}1$	$0{,}2$	$0{,}7$

$\operatorname{plim}_{n\to\infty} R_n \overset{?}{=} R$

$R(a)=1$

$R(b)=2$

$R(c)=0$



$E(|R_n - R|) =$

	a	b	c
$\lvert R_n - R\rvert$	$\frac{1}{n}$	$\frac{1}{n}$	0

$= 0{,}1\cdot\frac{1}{n} + 0{,}2\cdot\frac{1}{n} + 0{,}7\cdot 0 \xrightarrow[n\to\infty]{} 0$

Критерий

Если $E(R_n) \xrightarrow{n\to\infty} E(R)$ и $\operatorname{Var}(R_n - R) \xrightarrow{n\to\infty} 0$, то $\operatorname{plim} R_n = R$

[в частности]

если $E(R_n) \xrightarrow{n\to\infty} \mu$ и $\operatorname{Var}(R_n) \xrightarrow{n\to\infty} 0$, то $\operatorname{plim} R_n = \mu$.

$P(|R_n - R| > \varepsilon) \xrightarrow{?} 0$ $\qquad R_n - R = Y_n$

$\operatorname{plim} R_n = R \iff \operatorname{plim} Y_n = 0$

цель: если $E(Y_n) \to 0$ и $\operatorname{Var}(Y_n) \to 0$ то $\operatorname{plim} Y_n = 0$.

$$|Y_n - 0| \le \underbrace{|E(Y_n)|}_{\downarrow\, 0} + |Y_n - E(Y_n)|$$

$$P(|Y_n - E(Y_n)| \ge a) \le \frac{\operatorname{Var}(Y_n)}{a^2} \to 0$$

$$P(|Y_n - 0| > \varepsilon) \le \underbrace{P\left(|E(Y_n)| > \frac{\varepsilon}{2}\right)}_{n\to\infty\ \ |E(Y_n)|\to 0} + \underbrace{P\left(|Y_n - E(Y_n)| > \frac{\varepsilon}{2}\right)}_{\le \frac{\operatorname{Var}(Y_n)}{(\varepsilon/2)^2} \to 0}$$

$1, 1, 1, 1, 0, 0, 0, 0 \ldots$

Упр.

Мораль: Если $E(R_n) \to \mu$ $\quad Var(R_n) \to 0$ то $\operatorname{plim} R_n = \mu$

Если $E(R_n) \to E(R)$ $\quad Var(R_n - R) \to 0$ то $\operatorname{plim} R_n = R$

$X_1, X_2, \ldots, X_n \sim U[0;1]$ незав

догадка 0?

$$R_n = \frac{X_1 + 2X_2 + 3X_3 + 4X_4 + \ldots + nX_n}{n^4}$$

$$E(R_n) = \frac{E(X_1) + 2E(X_2) + \ldots + nE(X_n)}{n^4} = \frac{\frac{1}{2}(1+2+\ldots+n)}{n^4}$$

$$= \frac{\frac{1}{2}\cdot n \cdot \frac{n+1}{2}}{n^4} \to 0$$

$$Var(R_n) = \frac{Var(X_1) + 2^2 Var(X_2) + 3^2 Var(X_3) + \ldots + n^2 Var(X_n)}{n^8} =$$

$$= \frac{\frac{1}{12}(1 + 2^2 + \ldots n^2)}{n^8} = \frac{1}{12}\cdot\frac{n\cdot\frac{n+1}{2}\cdot(2n+1)\cdot\frac{1}{3}}{n^8} \to 0$$

$E(R_n) \to 0 \quad Var(R_n) \to 0 \quad \Rightarrow \quad \operatorname{plim} R_n = 0$

$$n\bar{X} = X_1 + \ldots + X_n = \sum_{i=1}^{n} X_i$$

Упр. $X_1, X_2, \ldots, X_n \sim$ незав $U[0;1]$

$$\operatorname{plim}_{n\to\infty} \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n} \ ?$$

$$\bar{X} = \frac{X_1 + \ldots + X_n}{n}$$

интуиция ≠ не ноль!

$$(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \ldots + (X_n - \bar{X})^2$$

$$\sum (X_i - \bar{X})^2 = \sum (X_i^2 - 2X_i\bar{X} + \bar{X}^2) =$$

$$= \sum X_i^2 - 2\cdot\sum_{i=1}^{n} X_i \cdot \bar{X} + \sum_{i=1}^{n} \bar{X}^2 =$$

$2\bar{X}\cdot\sum X_i \qquad n\cdot\bar{X}^2$

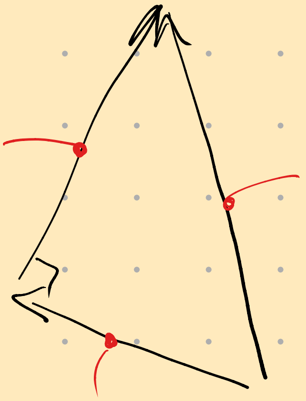
$$= \sum X_i^2 - 2n\cdot\bar{X}^2 + n\bar{X}^2 = \sum X_i^2 - n\cdot\bar{X}^2$$

т. Пифагора.

$$n\cdot\bar{X}^2 + \sum (X_i - \bar{X})^2 = \sum X_i^2$$

$$b = \begin{pmatrix} x_1-\bar{x} \\ x_2-\bar{x} \\ \vdots \\ x_n-\bar{x} \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \|x\|^2 = \sum x_i^2$$

$$\|b\|^2 = \sum (x_i-\bar{x})^2$$



$$a \perp b$$
$$\sum \bar{x}\cdot(x_i-\bar{x}) = 0$$

$$a = \begin{pmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \end{pmatrix} \qquad \|a\|^2 = n\cdot\bar{x}^2$$

$$\operatorname{plim} \frac{\sum (X_i-\bar{X})^2}{n} = \operatorname{plim} \frac{\sum X_i^2 - n\bar{X}^2}{n} =$$
$$= \operatorname{plim} \frac{\sum X_i^2}{n} - \operatorname{plim} \bar{X}^2 =$$
$$= \operatorname{plim}\left(\frac{X_1^2+\ldots+X_n^2}{n}\right) - \operatorname{plim}\left(\frac{X_1+X_2+\ldots+X_n}{n}\right)^2 =$$
$$= \operatorname{plim}\left(\frac{X_1^2+\ldots+X_n^2}{n}\right) - \left(\operatorname{plim} \frac{X_1+\ldots+X_n}{n}\right)^2 \overset{3Б4}{=}$$
$$= E(X_1^2) - (E(X_1))^2 = \operatorname{Var}(X_1) = \frac{1}{12}$$
$$X_1 \sim U[0;1]$$

Упр. $X_1, X_2, \ldots, X_n \sim$ незав $U[0;1]$

нет n внутр!

$$\operatorname{plim}_{n\to\infty} \frac{X_1+X_2}{1+\bar{X}} = \frac{\operatorname{plim}(X_1+X_2)}{\operatorname{plim}(1+\bar{X})} \overset{3Б4}{=} \frac{X_1+X_2}{1+E(X_1)} = \frac{X_1+X_2}{1+\frac{1}{2}} = \frac{X_1+X_2}{1.5}$$

proof $\to$ $X_1+X_2 = R$

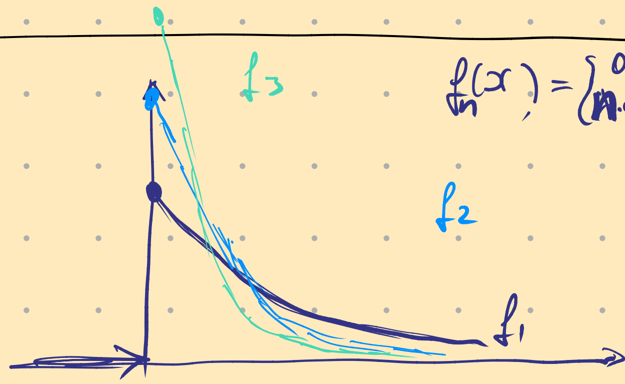
$R_1 = X_1+X_2 \quad R_2 = X_1+X_2 \quad R_3 = X_1+X_2 \quad \ldots$

Упр. $X_n \sim Expo(\lambda = n)$

а) $\operatorname{plim} X_n \overset{?}{=} 0$!! ☺

б) $X_n \xrightarrow{dist} \overset{?}{=} 0$

догадка



$$f_n(x) = \begin{cases} 0, & x < 0 \\ n\cdot\exp(-nx), & x \geq 0 \end{cases}$$

$\operatorname{plim} X_n = 0$

$$E(X_n) = \frac{1}{\lambda} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$
$$Var(X_n) = \frac{1}{\lambda^2} = \frac{1}{n^2} \xrightarrow{n\to\infty} 0$$
$\Rightarrow \operatorname{plim} X_n = 0$

Упр

$X_n \sim Bin(n, p = 0{,}7)$

$Y_n = \ln\left(\frac{1+X_n}{n}\right)$

$W_n = \frac{X_n/n}{(n+1-X_n)/n}$ $\to 0{,}7$

а) $\operatorname{plim} X_n \overset{?}{=}$ не существует

б) $\operatorname{plim} Y_n \overset{?}{=} \ln 0{,}7$

в) $\operatorname{plim} W_n \overset{?}{=} \frac{0{,}7}{1-0{,}7}$

$$E(X_n) = np = n\cdot 0{,}7 \longrightarrow \infty$$
$$Var\, X_n = np(1-p) = n\cdot 0{,}21 \longrightarrow \infty$$

$X_n = S_1 + S_2 + \dots + S_n$ $\qquad S_i \sim$ незав

$P(S_i = 1) = 0{,}7$

$P(S_i = 0) = 0{,}3$

ЗБЧ: $\operatorname{plim} \frac{S_1 + \dots + S_n}{n} = E(S_1) = 0{,}7$

$$\operatorname{plim} \frac{X_n}{n} = 0{,}7$$