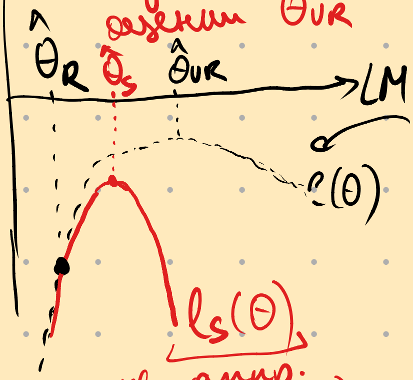
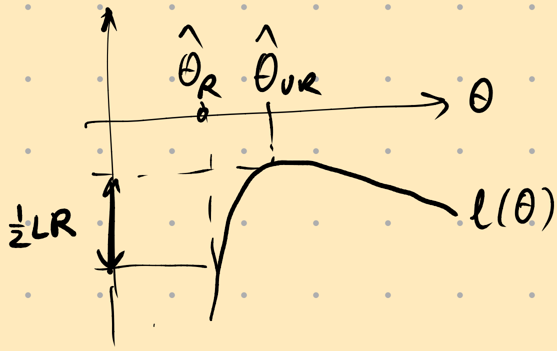
Семинар 16

минус: возможность отказа от оценки $\hat{\theta}_{UR}$



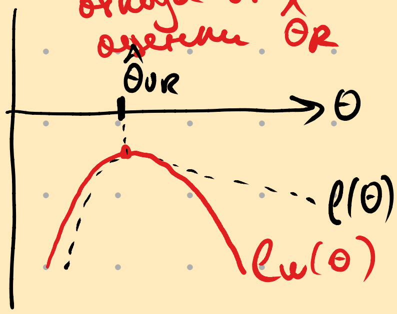
кв. аппр. для $\ell(\theta)$



$$LR = 2\left(\ell(\hat{\theta}_{UR}) - \ell(\hat{\theta}_R)\right)$$

$\xrightarrow{W}$

минус: возможность отказа от оценки $\hat{\theta}_R$



$$W = 2\left(\ell_w(\hat{\theta}_{UR}) - \ell_w(\hat{\theta}_R)\right) = \ldots =$$

$$= (\hat{\theta}_{UR} - \hat{\theta}_R)^T \cdot \hat{I}_{UR} \cdot (\hat{\theta}_{UR} - \hat{\theta}_R)$$

$$= (\hat{\theta}_{UR} - \hat{\theta}_R)^T \cdot \widehat{Var}^{-1}(\hat{\theta}_{UR}) \cdot (\hat{\theta}_{UR} - \hat{\theta}_R)$$

LM-тест (Lagrange Multiplier)

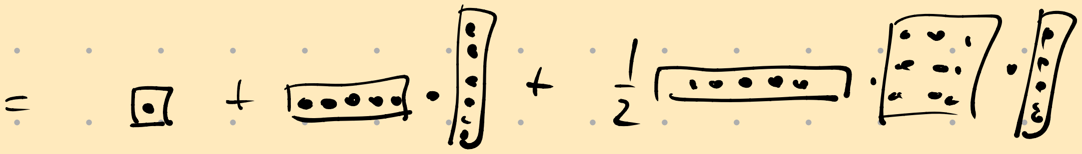
Score-тест $\left(\frac{\partial \ell}{\partial \theta} = \text{score/вклад}\right)$

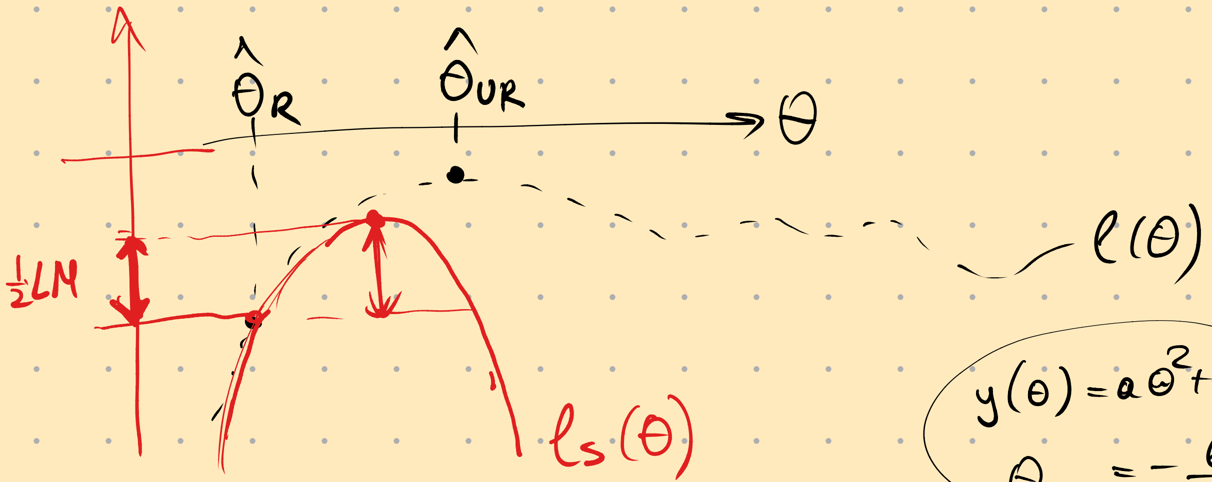
$$LM = 2\cdot\left(\ell_s(\hat{\theta}_s) - \ell_s(\hat{\theta}_R)\right) =$$

Упр. Выразите LM статистику через $\left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\hat{\theta}_R}$, $\left.\frac{\partial \ell}{\partial \theta}\right|_{\hat{\theta}_R}$

а) $\ell_s(\theta) \overset{?}{=}$ квадр. аппр к $\ell(\theta)$ в точке $\hat{\theta}_R$.

$$= \ell(\hat{\theta}_R) + \left.\frac{\partial \ell}{\partial \theta}\right|_{\hat{\theta}_R}^T \cdot (\theta - \hat{\theta}_R) + \frac{1}{2}(\theta - \hat{\theta}_R)^T \cdot \left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\hat{\theta}_R} \cdot (\theta - \hat{\theta}_R)$$





$$y(\theta) = a\theta^2 + b\theta + c$$

$$\theta_{max} = -\frac{b}{2a}$$

$$l_s(\theta) = const + s_R^T\cdot(\theta-\hat\theta_R) + \frac{1}{2}(\theta-\hat\theta_R)^T\cdot H_R\cdot(\theta-\hat\theta_R)$$

$$dl_s = s_R^T\cdot d\theta + (\theta-\hat\theta_R)^T\cdot H_R\cdot d\theta$$

$$s_R = \left(\frac{\partial l}{\partial\theta}\right)\Bigg|_{\hat\theta_R}$$

$$s_R^T + (\theta-\hat\theta_R)^T\cdot H_R = 0$$

$$H_R = \frac{\partial^2 l}{\partial\theta^2}\Bigg|_{\hat\theta_R}$$

условие оптимума $l_s(\theta)$

$$(\hat\theta_s-\hat\theta_R)^T = -s_R^T\cdot H_R^{-1}$$

$$\hat\theta_s-\hat\theta_R = -H_R^{-1}\cdot s_R$$

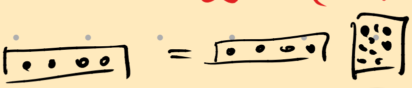
Вектор	школьная
s_R^T	b
$\frac{H_R}{2}$	a
H_R	$2a$
H_R^{-1}	$\frac{1}{2a}$

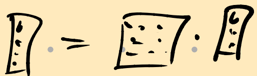
$$LM = 2\left(l_s(\hat\theta_s) - l_s(\hat\theta_R)\right) =$$

$$l_s(\theta) = const + s_R^T\cdot(\theta-\hat\theta_R) + \frac{1}{2}(\theta-\hat\theta_R)^T\cdot H_R\cdot(\theta-\hat\theta_R)$$

- const в разности $l_s(\hat\theta_s) - l_s(\hat\theta_R)$ пропадёт
- $l_s(\hat\theta_R) = const + 0 + 0$

$$= 2\cdot\left(s_R^T\cdot(\hat\theta_s-\hat\theta_R) + \frac{1}{2}(\hat\theta_s-\hat\theta_R)^T\cdot H_R\cdot(\hat\theta_s-\hat\theta_R)\right) =$$





$$(\hat\theta_s-\hat\theta_R)^T = -s_R^T\cdot H_R^{-1}$$

$$\hat\theta_s-\hat\theta_R = -H_R^{-1}\cdot s_R$$

$$= 2\cdot\left(s_R^T\cdot(-H_R^{-1}\cdot s_R) + \frac{1}{2}(-s_R^T\cdot H_R^{-1})\cdot H_R\cdot(-H_R^{-1} s_R)\right) =$$

$$= s_R^T\cdot(-H_R^{-1})\cdot s_R = \frac{\partial l}{\partial\theta}\Bigg|_{\hat\theta_R}^T\cdot\left(-\frac{\partial^2 l}{\partial\theta^2}\Bigg|_{\hat\theta_R}^{-1}\right)\cdot\frac{\partial l}{\partial\theta}\Bigg|_{\hat\theta_R}$$

интерпретация полученной записи

$$I_F = Var\left(\frac{\partial \ell}{\partial \theta}\right) = E\left(-\frac{\partial^2 \ell}{\partial \theta^2}\right) = E\left(\left(\frac{\partial \ell}{\partial \theta}\right)\cdot\left(\frac{\partial \ell}{\partial \theta}\right)^T\right)$$

$$\hat{I}_F = -\frac{\partial^2 \ell}{\partial \theta^2}$$

$$\boxed{\text{похожесть W и LM}}$$

$$\Downarrow \quad s_R = \left.\frac{\partial \ell}{\partial \theta}\right|_{\hat{\theta}_R}$$

$$0 = \left.\frac{\partial \ell}{\partial \theta}\right|_{\hat{\theta}_{UR}}$$

$$W = (\hat{\theta}_{UR} - \hat{\theta}_R)^T \cdot \widehat{Var}^{-1}(\hat{\theta}_{UR}) \cdot (\hat{\theta}_{UR} - \hat{\theta}_R)$$

$$LM = (s_R - 0)^T \cdot \widehat{Var}^{-1}\left(\frac{\partial \ell}{\partial \theta}\right)\Big|_{\hat{\theta}} \cdot (s_R - 0) = s_R^T \cdot \hat{I}_R^{-1} \cdot s_R$$

Упр.

$n = 100$ $\quad \Sigma y_i = 60$

$P(y_i = 1) = \theta$ $\quad y_i$ незав.

$P(y_i = 0) = 1 - \theta$

$H_0: \theta = \frac{1}{2}$

$H_1: \theta \neq \frac{1}{2}$

проверьте H_0 с помощью LM теста на ур. знач. 5%.

применен. результата с лекции

$$\ell(\theta) = \Sigma y_i \cdot \ln\theta + (n - \Sigma y_i)\cdot \ln(1-\theta)$$

$$s(\theta) = \ell'(\theta) = \Sigma y_i \cdot \frac{1}{\theta} - \frac{n - \Sigma y_i}{1-\theta}$$

$$\ell''(\theta) = -\frac{\Sigma y_i}{\theta^2} - \frac{n - \Sigma y_i}{(1-\theta)^2}$$

$$\hat{\theta}_R = \frac{1}{2}$$

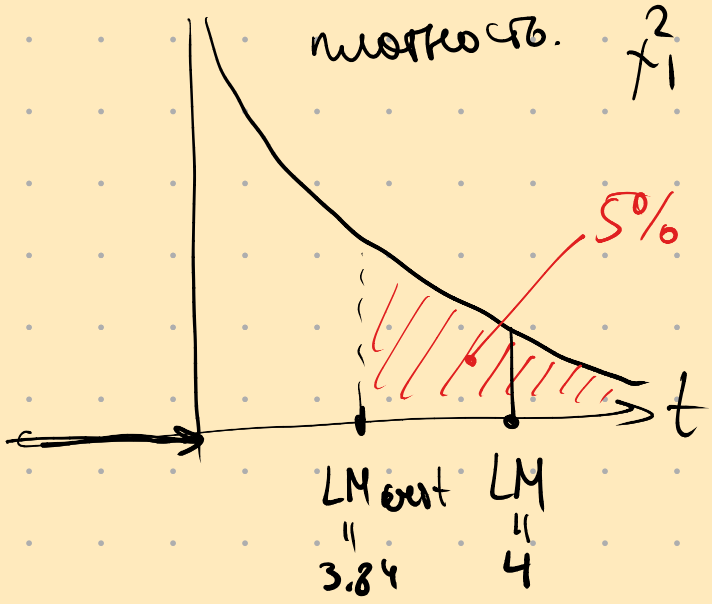
$$\hat{\theta}_{UR} = 0.6$$

нашли на лекции, но для LM она не нужна.

$$s_R = \ell'(\hat{\theta}_R) = 60 \cdot \frac{1}{1/2} - \frac{40}{1/2} = 120 - 80 = 40$$

$$\hat{I}_R = -\ell''(\hat{\theta}_R) = -\left(-\frac{60}{(1/2)^2} - \frac{40}{(1/2)^2}\right) = 240 + 160 = 400$$

$$L = 40 \cdot 400)^{-1} \cdot 40 = \frac{40 \cdot 40}{400} = 4$$



$W = 4.17 \quad LR = 4.02$

$$LM \xrightarrow[H_0\ dist]{n \to \infty} \chi^2_d$$

d → число нул-х ур-ий в H_0

d → $p_{UR} - p_R$

p_{UR}, p_R — число свобод. пар-в в UR и R модели

Вывод: $LM > LM_{crit}$, следов-но, H_0 отвергается.

$$\text{p-значение} = P(LM_{new} > LM \mid H_0, LM) =$$
$$= P(\chi^2_1 > 4) = 0.0455$$

Вывод: $0{,}0455 < \alpha = 5\%$, след-но, H_0 отвергается.

Упр: $y_1 \dots y_n \sim Exp(\lambda)$ независимы.

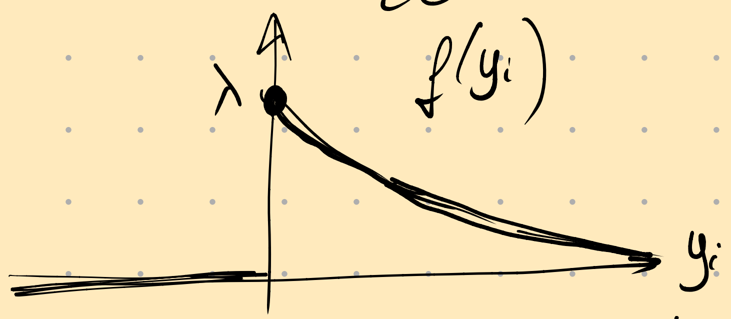
$H_0: \lambda = 1$ против $H_1: \lambda \neq 1$ $\qquad \alpha = 5\%$

с помощью LR, LM, W. $\qquad n = 1000 \quad \sum y_i = 1200$

а) LM $\quad$ б) LR $\quad$ в) W

$$LM = S_R^T \cdot \hat{I}_R^{-1} \cdot S_R$$

$$f(y_i) = \begin{cases} \lambda \cdot e^{-\lambda y_i}, & y_i \geq 0 \\ 0, & y_i < 0 \end{cases}$$



$$f(y_1, \dots, y_n) = \lambda e^{-\lambda y_1} \cdot \lambda \cdot e^{-\lambda y_2} \cdot \dots \cdot \lambda e^{-\lambda y_n}$$
$$= \lambda^n \cdot e^{-\lambda \sum y_i}$$

$$\ell(\lambda) = \ln\left(f(y_1 \ldots y_n)\right) = n \ln\lambda - \lambda \cdot \Sigma y_i$$

$$s = \frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \Sigma y_i \qquad \hat{\lambda}_R = 1$$

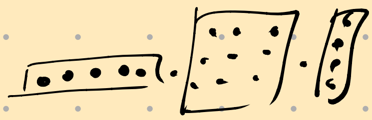
$$H = \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} \qquad s_R = \frac{1000}{1} - 1200 = -200$$

$$H_R = -\frac{1000}{1^2} = -1000$$

$$\hat{I} = -H = 1000$$

$$\hat{I}_R^{-1} = \frac{1}{1000}$$

$$LM = -200 \cdot \frac{1}{1000} \cdot (-200) = \frac{40000}{1000} = 40$$

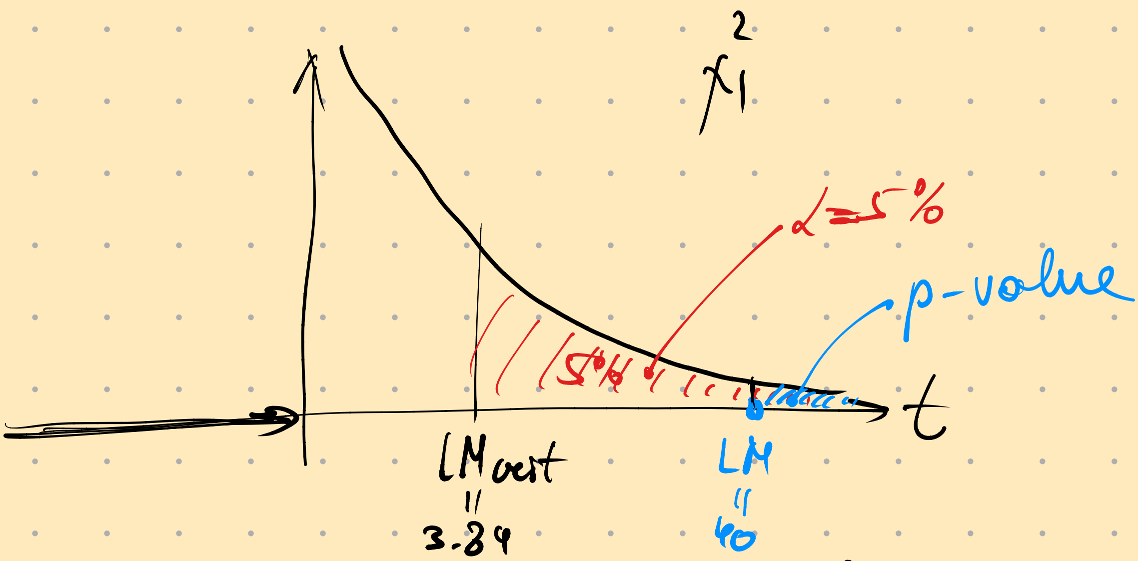


$H_0: \lambda = 1$ ← одно ур-ие. $(df = 1)$

R. модели : 0 пар-в надо оценивать $(\lambda = 1)$

UR- модели : 1 пар-р — надо оценивать (λ)

$df = p_{UR} - p_R = 1 - 0 = 1$



Вывод: $LM > LM_{crit}$, след-но, H_0 отвергается

δ) LR

$$\ell(\lambda) = n \ln\lambda - \lambda \cdot \Sigma y_i$$

$\operatorname{argmax} \ell(\lambda)$
UR-модель

$$s = \ell'(\lambda) = \frac{n}{\lambda} - \sum y_i$$

$$\frac{n}{\hat\lambda_{UR}} - \sum y_i = 0.$$

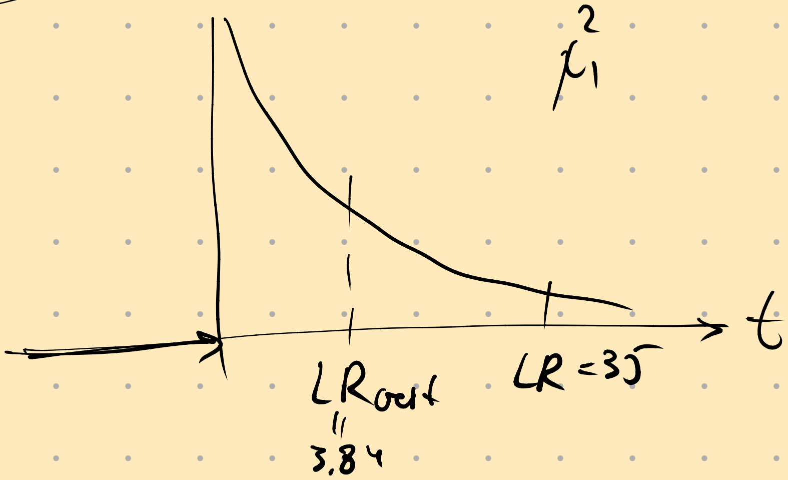
$$\hat\lambda_{UR} = \frac{n}{\sum y_i} = \frac{1000}{1200} = \frac{5}{6} \approx 0{,}83$$

$\operatorname{argmax} \ell(\lambda)$
R-модель

$H_0: \lambda = 1$

$$\hat\lambda_R = 1$$

$$LR = 2\left(\ell(\hat\lambda_{UR}) - \ell(\hat\lambda_R)\right) =$$
$$= 2\left(\left(1000 \cdot \ln\frac{5}{6} - \frac{5}{6}\cdot 1200\right) - \left(1000 \ln 1 - 1 \cdot 1200\right)\right) =$$
$$= \ldots = 35$$



Вывод: $LR > LR_{крит}$, следовательно, H_0 отвергается

б) W

$$W = (\hat\theta_{UR} - \hat\theta_R)^T \cdot \left(-\frac{\partial^2 \ell}{\partial\theta^2}\Big|_{\hat\theta_{UR}}\right) \cdot (\hat\theta_{UR} - \hat\theta_R)$$

$$\hat\lambda_{UR} = \frac{5}{6} \qquad \hat\lambda_{UR} - \hat\lambda_R = -\frac{1}{6}$$
$$\hat\lambda_R = 1$$

$$H = \frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n}{\lambda^2} \qquad \frac{\partial^2 \ell}{\partial\lambda^2}\Big|_{\hat\lambda_{UR}} = -\frac{1000}{(5/6)^2}$$

$$W = -\frac{1}{6} \cdot \frac{1000}{(5/6)^2} \cdot \left(-\frac{1}{6}\right) = \frac{1000}{25} = 40$$

Вывод, $W > \underbrace{W_{crit}}_{3{,}84}$, след-но, H_0 отвергается.