Семинар 5

Упр. $y_1 \dots y_n \sim Poiss(\lambda)$, незав.

!! а) I_F для $n=100$!!
I_F для $n=1000$!!

$I_F = \frac{n}{\lambda}$

!! б) какое ожидание у ℓ легко найти

$E(\ell'_\lambda)=0$

$E(\ell'_\lambda \cdot \ell'_\lambda) = -E(\ell''_{\lambda\lambda})$!!

$E(y_i)=$?

$Var(y_i)=$

в) $\hat\lambda$ метод. макс. правд. г) CI для λ 95% (оцен.)

данные: $n=1000$
$\Sigma y_i = 4000$

а) ф-ция правд $L(\lambda)$
лог. ф-ция правд $\ell(\lambda)$

$I_F \to Var(\ell')$
$I_F \to E(\ell'\ell')$
$I_F \to -E(\ell'')$ [как правило легче]

$$L(\lambda) = P(y_1^R = y_1,\ y_2^R = y_2 \dots y_n^R = y_n \mid \lambda) =$$

$$P(y_i^R = y_i) = e^{-\lambda} \frac{\lambda^{y_i}}{y_i!}$$

$$= \prod_{i=1}^{n} \left(e^{-\lambda} \cdot \frac{\lambda^{y_i}}{y_i!}\right)$$

$$\ell(\lambda) = \ln L(\lambda) = \sum \left(-\lambda + y_i \ln\lambda - \ln(y_i!)\right)$$

$$\ell'(\lambda) = \sum \left(-1 + \frac{y_i}{\lambda} - 0\right)$$

$$I_F = Var\left(\sum\left(-1 + \frac{y_i}{\lambda}\right)\right)$$

$$\ell''(\lambda) = \sum\left(0 - \frac{y_i}{\lambda^2}\right) = -\frac{\sum y_i}{\lambda^2}$$

$$I_F = -E(\ell''_{\lambda\lambda}) = \frac{E(\sum y_i)}{\lambda^2} =$$

$$= \frac{\sum_{i=1}^{n} E(y_i)}{\lambda^2} = \frac{n \cdot E(y_i)}{\lambda^2} = \frac{n \cdot \lambda}{\lambda^2} =$$

$$= \frac{n}{\lambda}$$

$E(y_i) = \lambda$ [помним]

→ $\sum_{k=0}^{\infty} k \cdot P(y_i = k) =$ [по опр]

$$= \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = \ldots$$

→ $m'(0)$ [$m(u)$ – произв. ф-ция для y_i]

[новый] → $E(\ell') = 0$ (для $\forall n$)

$$E\left(\sum\left(-1 + \frac{y_i}{\lambda}\right)\right) = 0 \quad \forall n$$

$n = 1$

$$E\left(-1 + \frac{y_1}{\lambda}\right) = 0$$

$$E(y_1) = \lambda$$

$Var(y_1) = \lambda$ [помним]

→ $E(y_1^2) - (E(y_1))^2$ [через суммы]

→ [через функцию произв моментов]

→ [новый:] $Var(\ell') = -E(\ell'')$ $\forall n$

$$Var\left(-1 + \frac{y_1}{\lambda}\right) = -E\left(-\frac{y_1}{\lambda^2}\right)$$

$$\frac{1}{\lambda^2} \cdot Var(y_1) = \frac{1}{\lambda^2} \cdot E(y_1) \Rightarrow \lambda$$

$$Var(y_1) = \lambda$$

б/г) $\ell' = 0 \Rightarrow \hat{\lambda}$ метода максимального правдоподобия

$$\ell'(\lambda) = \sum\left(-1 + \frac{y_i}{\lambda} - 0\right) = -n + \frac{\sum y_i}{\lambda}$$

$$-n + \frac{\sum y_i}{\hat{\lambda}} = 0$$

выборка   универс идея:

$$\hat{\lambda} = \frac{\sum y_i}{n} = \frac{4000}{1000} = 4 \quad [\text{Ц.П.Т} \,/\, \text{дельта метод}]$$

Спец. теория для метода макс. правдоп.:

оценка $\hat{I}_F = -H$

оценка $\hat{Var}(\hat{\lambda}) = \hat{I}_F^{-1}$

$$\frac{\hat{\lambda} - \lambda}{\sqrt{Var(\hat{\lambda})}} \xrightarrow{dist} N(0;1)$$

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\hat{Var}(\hat{\lambda})}} \xrightarrow{dist} N(0;1)$$

• практ

$$\hat{I}_F = -H = -\left(-\frac{\sum y_i}{\lambda^2}\right) = \frac{\sum y_i}{\lambda^2}\Bigg|_{\hat{\lambda}} = \frac{\sum y_i}{(\sum y_i / n)^2} = \frac{n^2}{\sum y_i} = \frac{1000^2}{4000} = 250$$

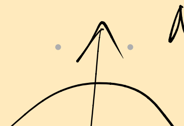
выборка.

$$\ell''(\lambda) = \sum\left(0 - \frac{y_i}{\lambda^2}\right) = -\frac{\sum y_i}{\lambda^2}$$

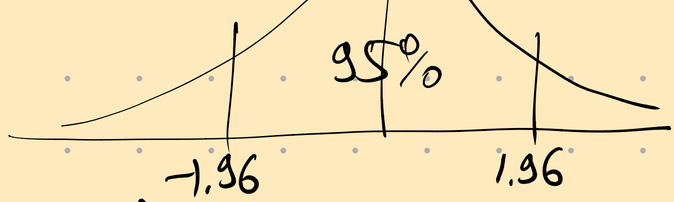
$$\hat{Var}(\hat{\lambda}) = \hat{I}_F^{-1} = \frac{\sum y_i}{n^2} = \frac{1}{250} = 0.004$$

asy CI:

$$\frac{\hat{\lambda} - \lambda}{\sqrt{\hat{Var}(\hat{\lambda})}} \overset{approx}{\sim} N(0;1)$$



N(0;1)



$$P\left(-1{,}96 \le \frac{\hat\lambda - \lambda}{\sqrt{\hat{Var}(\hat\lambda)}} \le 1{,}96\right) \xrightarrow[n\to\infty]{} 0{,}95$$

решаем нер-во относит-но λ.

$$P\left(\lambda \in \left[\hat\lambda - 1{,}96 \cdot \sqrt{\hat{Var}(\hat\lambda)};\ \hat\lambda + 1{,}96\sqrt{\hat{Var}(\hat\lambda)}\right]\right) \to 0{,}95$$

asy CI: $\left[\hat\lambda - 1{,}96 \cdot \sqrt{\frac{\sum y_i}{n^2}};\ \hat\lambda + 1{,}96 \cdot \sqrt{\frac{\sum y_i}{n^2}}\right]$

реализация CI:

$$\left[4 - 1{,}96 \cdot \sqrt{0{,}004};\ 4 + 1{,}96\sqrt{0{,}004}\right]$$

→ классич подход [θ – неизв конст] → метод мом-в; → метод макс. прав. (ЦПТ/дельта; ЦПТ/дельта → I_F) → ...

→ байес. подход [$f(\theta)$ – априорн. распр] → байес. интерв. наив. плотност

метод моментов VS метод максим-го правдоподия

→ для применения дост-но $E(y_i)$

→ для применения надо знать $f(y_i)$, $P(y_i = k)$

→

→ более эффективный /будет позднее/

практика покажет 🙂

пример нарушения регулярности.

Упр. $y_1, y_2, \ldots, y_n \sim$ незав, $U[0;\theta]$

краевые решения не удовл. теореме

(4) а) $\hat\theta$ макс правд? б) $Var(\hat\theta)$?

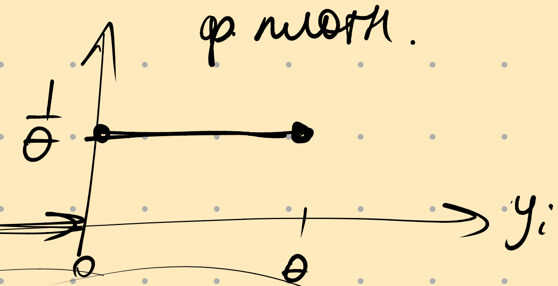
$\hat\theta - E(\hat\theta)$ dist

(11) б) $Var(\hat\theta) \approx (-H)$. г) $\frac{\hat\theta - \dots}{\sqrt{Var(\hat\theta)}} \xrightarrow{dist} N(0;1)$?

увы, нет — нет

а) $\hat\theta$?

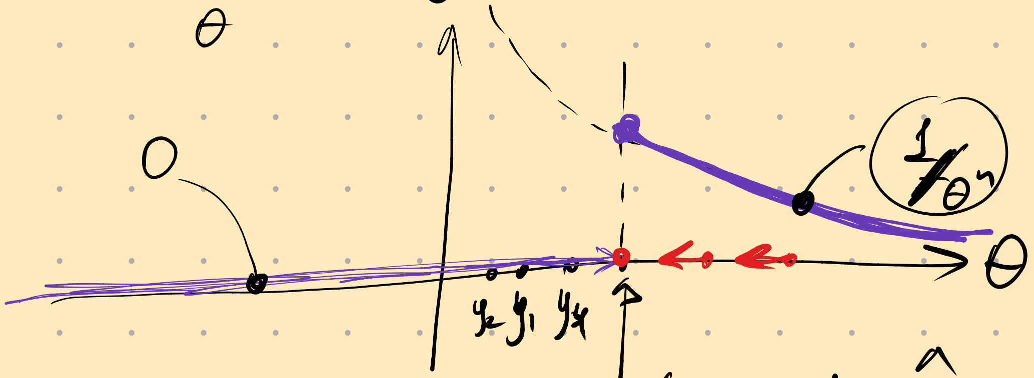
$$f(y_i|\theta)=\begin{cases} 1/\theta & y_i\in[0;\theta] \\ 0, & \text{иначе}\end{cases}$$



$$L(y_1 \dots y_n|\theta)=\begin{cases} \frac{1}{\theta}\cdot\frac{1}{\theta}\cdot\frac{1}{\theta}\cdots\frac{1}{\theta} & y_1\in[0;\theta],\ y_2\in[0;\theta] \dots y_n\in[0;\theta] \\ 0, & \text{иначе}\end{cases}$$

$$L=\begin{cases} 1/\theta^n, & \forall y_i\in[0;\theta] \\ 0, & \text{иначе}\end{cases}$$

$$\max_\theta L(y_1 \dots y_n|\theta)$$



$\max\{y_1 \dots y_n\} = \hat\theta$

$$\boxed{\hat\theta = \max\{y_1, y_2 \dots y_n\}}$$

$$\ell(\theta)=\begin{cases} \ln\left(\frac{1}{\theta^n}\right), & \forall y_i\in[0;\theta] \\ -\infty, & \text{иначе}\end{cases}$$

$$\ell=\begin{cases} -n\ln\theta \\ -\infty\end{cases}$$

$\ell' = -\frac{n}{\theta}$ если $\forall y_i\in[0;\theta]$

в) $-H = -\ell'' = \frac{n}{\theta^2}$ если $\forall y_i\in[0;\theta]$

б) $Var(\hat\theta) = E(\hat\theta^2) - (E(\hat\theta))^2$

б₁) $F_{\hat{\theta}}(t)$? ф. распр-ия

б₂) $f_{\hat{\theta}}(t)$? ф. плотности

б₃) $E(\hat{\theta}^2) = \int_{\mathbb{R}} u^2 \cdot f_{\hat{\theta}}(u)\, du$

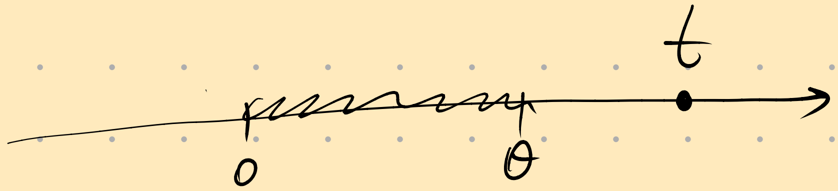
$E(\hat{\theta}) = \int_{\mathbb{R}} u \cdot f_{\hat{\theta}}(u)\, du$

план.

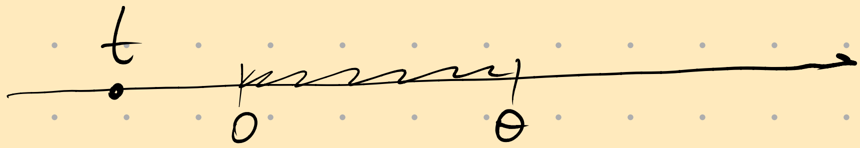
$$F_{\hat{\theta}}(t) = \int_{-\infty}^{t} f_{\hat{\theta}}(u)\, du \quad (1) \quad = P(\hat{\theta} \le t)$$

$$P(\hat{\theta} \le t) = P(\max(y_1 \ldots y_n) \le t) = P(y_1 \le t, y_2 \le t, \ldots y_n \le t)$$

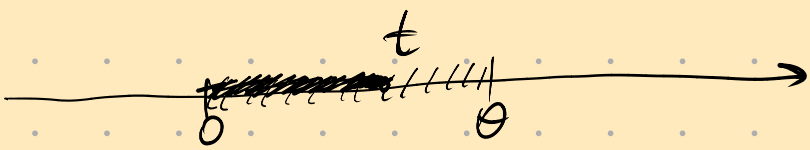
$$= P(y_1 \le t) \cdot \ldots \cdot P(y_n \le t) = (P(y_1 \le t))^n = (*)$$



$P(y_1 \le t) = 1$



$P(y_1 \le t) = 0$

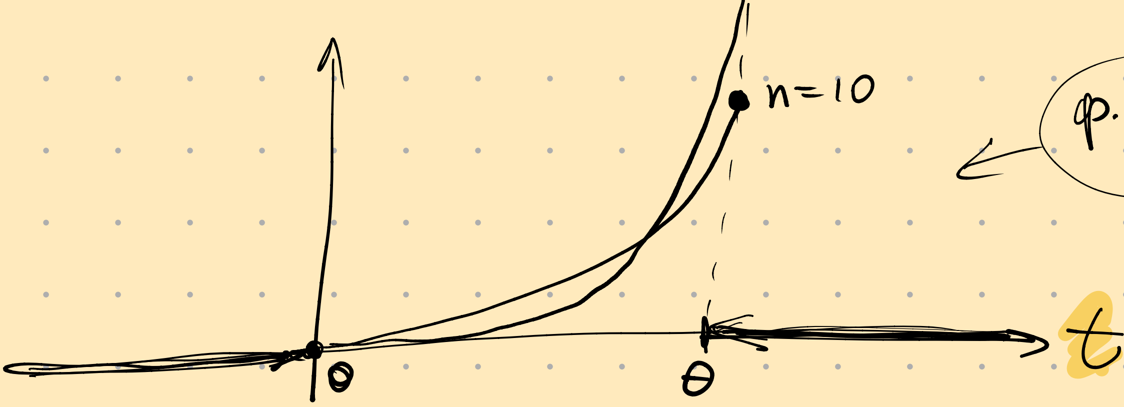


$P(y_1 \le t) = \dfrac{t}{\theta}$

$$F(t) = \begin{cases} 1^n, & t > \theta \\ 0^n, & t < \theta \\ (t/\theta)^n, & t \in [0; \theta] \end{cases}$$

$$f(t) = F'(t) = \begin{cases} n \cdot (t/\theta)^{n-1} \cdot \frac{1}{\theta}, & t \in [0; \theta] \\ 0, & t \notin [0; \theta] \end{cases}$$

, n=20



ф. плотн. $\hat{\theta}$

$$E(\hat\theta) = \int_0^\theta t \cdot f(t)\,dt = \int_0^\theta t \cdot n\left(\frac{t}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\,dt =$$

$$= \frac{n}{\theta^n} \int_0^\theta t^n\,dt = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} = \theta \cdot \frac{n}{n+1}$$

$$E(\hat\theta^2) = \int_0^\theta t^2 f(t)\,dt = \frac{n}{\theta^n} \int_0^\theta t^{n+1}\,dt = \frac{n}{\theta^n} \cdot \frac{\theta^{n+2}}{n+2}$$

$$= \theta^2 \cdot \frac{n}{n+2}$$

$$\mathrm{Var}(\hat\theta) = E(\hat\theta^2) - (E(\hat\theta))^2 = \theta^2 \cdot \frac{n}{n+2} - \left(\theta \cdot \frac{n}{n+1}\right)^2 =$$

$$= \theta^2 \cdot \left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)$$

$$-H = -\ell'' = \frac{n}{\theta^2}$$

$$-H^{-1} = \theta^2 \cdot \frac{1}{n}$$

похожи ли?

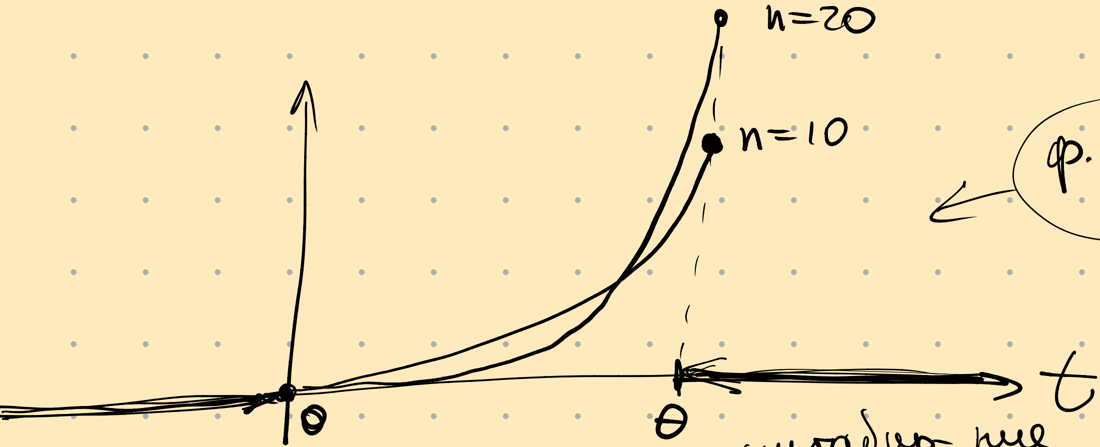
$$\lim \frac{\theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)}{\theta^2 \cdot \frac{1}{n}} =$$

$$= \lim \left(\frac{n^2}{n+2} - \frac{n^3}{(n+1)^2}\right) =$$

$$\lim \left(\frac{n^2 \cdot (n+1)^2 - n^3 \cdot (n+2)}{\ldots} \right)$$

$$= \lim \left(\frac{\quad}{(n+2)\cdot(n+1)^2} \right) =$$

$$= \lim \frac{n^4+2n^3+n^2-n^3-2n^3}{n^3+(\text{..меньш.})} = 0$$

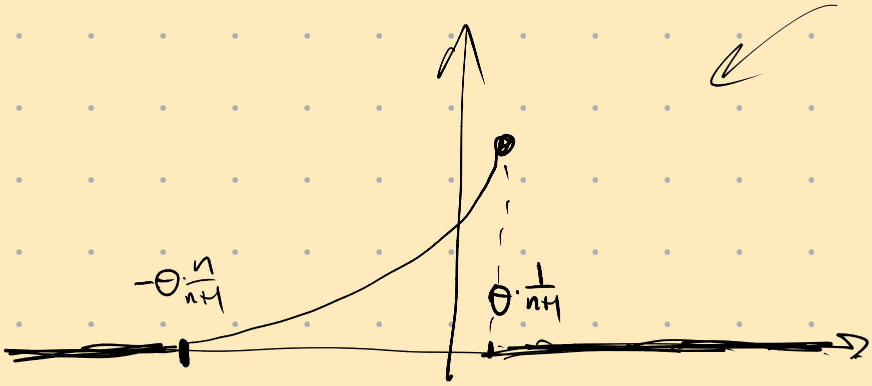


ф. плотн. $\hat{\theta}$

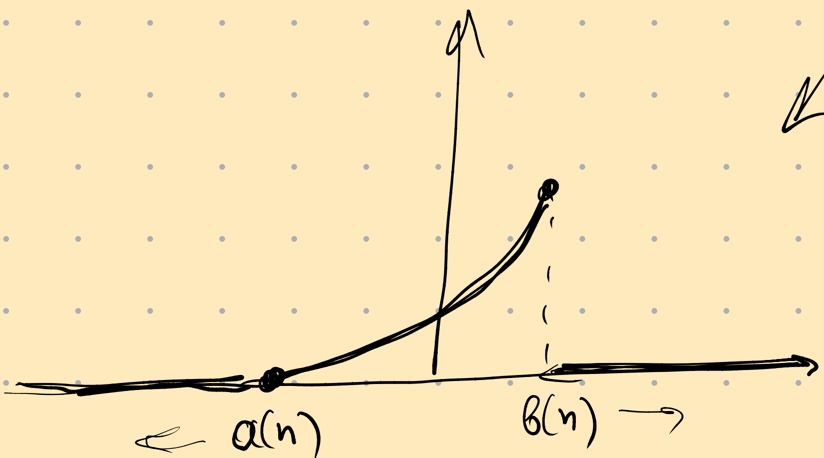
масштабирование

$$R_n = \frac{\hat{\theta} - E(\hat{\theta})}{\sqrt{Var(\hat{\theta})}} = \frac{\hat{\theta} - \theta\cdot\frac{n}{n+1}}{\sqrt{\theta^2\cdot\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)}} \xrightarrow[?]{dist} N(0;1)$$

ф. плотн $\hat{\theta} - \theta\cdot\frac{n}{n+1}$

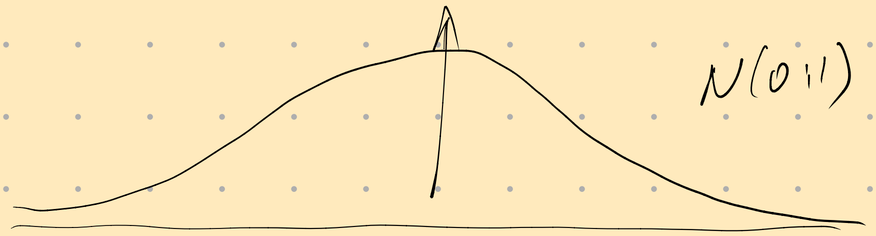


ф. плотн. $\dfrac{\hat{\theta} - \theta\frac{n}{n+1}}{\sqrt{\theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)}}$



$$a(n) = -\frac{n}{n+1} \Big/ \sqrt{\frac{n}{n+2} - \frac{n^2}{(n+1)^2}}$$

$b(n) = \ldots$



$N(0,1)$?

сходимости
к $N(0,1)$
нет

!!