$\chi^2$ сем-7 ☺

Вспом, утверждение:

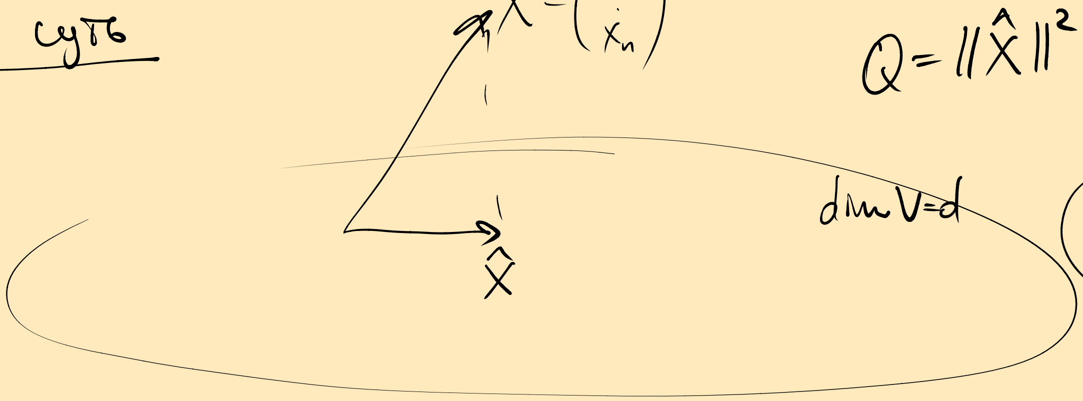
$$Q = X_1^2 + X_2^2 + \dots + X_d^2 \qquad X_i \sim \text{незав } N(0;1)$$

то $Q \sim \chi^2_d$

голая суть

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$Q = \|\hat{X}\|^2$$




$X_i \sim$ незав $N(0;1)$

Упр 3.3 $Z_1, Z_2, \dots \sim$ незав $N(0;1)$

а) $Q = Z_5^2 + Z_6^2 + Z_{32}^2 \qquad Q \sim ?$

е) $T = (Z_7 + Z_{11})^2/2 + (Z_3 + Z_9 + Z_{12})^2/3 \qquad T \sim ?$

а !! $\quad Q \sim \chi^2_3$

$$Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_{32} \end{pmatrix} \qquad Z_i \sim \text{незав и } N(0;1)$$

$$\hat{Z} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Z_5 \\ Z_6 \\ 0 \\ \vdots \\ 0 \\ Z_{32} \end{pmatrix}$$

$$\|\hat{Z}\|^2 = Z_5^2 + Z_6^2 + Z_{32}^2$$

е $\quad T \sim ?$

$$t = \frac{X_1}{\sqrt{Q/d}} \qquad F = \frac{Q_1/d_1}{Q_2/d_2}$$

$$Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_{12} \end{pmatrix}$$

$W = \text{Span}(e_1, e_2) \qquad \dim W = 2$

$$e_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \leftarrow 7\text{ое} \qquad e_2 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} \leftarrow 3\text{е} \\ \\ \leftarrow 9\text{е} \end{matrix}$$

**спроец-ие $z$ на $e_1$?**

$e_1 \perp e_2 \qquad \langle e_1, e_2 \rangle = 0$

проекция $z$ на $e_1$?

$$\begin{pmatrix} 0\\0\\0\\0\\?\\0\\0\\0\\?\\0 \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\0\\(z_7+z_{11})/2\\0\\0\\0\\(z_7+z_{11})/2\\0 \end{pmatrix}$$

проец-ие вектор $X = \begin{pmatrix} x_1\\ \vdots \\ x_n \end{pmatrix}$ на $\mathrm{Span}\left(\begin{pmatrix} 1\\ \vdots \\ 1 \end{pmatrix}\right)$

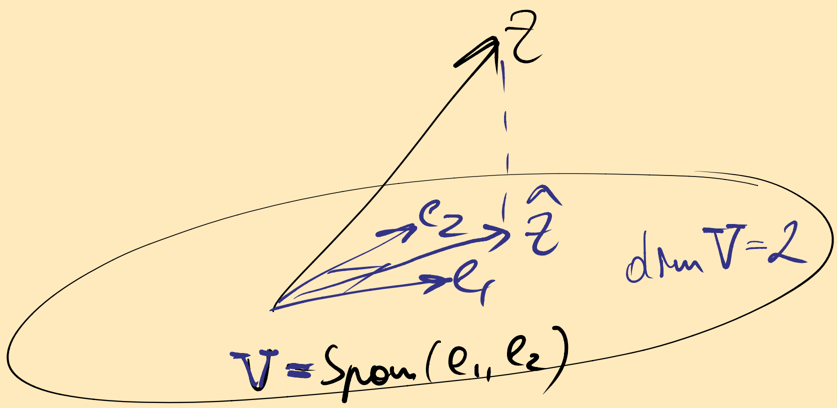
$$\hat{X} = \begin{pmatrix} \bar{X}\\ \bar{X}\\ \vdots \\ \bar{X} \end{pmatrix}$$

$\begin{pmatrix} x_1\\ x_2 \end{pmatrix}$ спр на $\mathrm{Span}\left(\begin{pmatrix} 1\\ 1 \end{pmatrix}\right)$

ответ: $\begin{pmatrix} \frac{x_1+x_2}{2}\\ \frac{x_1+x_2}{2} \end{pmatrix} = \begin{pmatrix} \bar{X}\\ \bar{X} \end{pmatrix}$

проекция $z$ на $e_2$?

$$\begin{pmatrix} 0\\0\\(z_3+z_9+z_{12})/3\\0\\0\\(z_3+z_9+z_{12})/3\\0\\0\\(z_3+z_9+z_{12})/3 \end{pmatrix}$$




$$\mathrm{Proj}(z, V) = \mathrm{Proj}(z, e_1) + \mathrm{Proj}(z, e_2) = \hat{z}$$

вектор

на какое подпр-во

$\hat{z}_i = 0$ ост-ные

$\hat{z}_7 = \hat{z}_{12} = \frac{z_7 + z_{12}}{2}$

$\hat{z}_3 = \hat{z}_9 = \hat{z}_{12} = \frac{z_3 + z_{12} + z_9}{3}$

$$\|\hat{z}\|^2 = \left(\frac{z_7+z_{12}}{2}\right)^2 \cdot 2 + \left(\frac{z_3+z_{12}+z_9}{3}\right)^2 \cdot 3 =$$

$$= \frac{(z_7+z_{12})^2}{2} + \frac{(z_3+z_9+z_{12})^2}{3} \sim \chi^2_2$$

**другое решение:**

$z_7 + z_{12}$ ... $N$

$$\frac{\ldots}{\sqrt{2}} = W_1 \qquad z_i \sim N(0;1)$$

$$E(W_1) = E\left(\frac{z_7+z_{12}}{\sqrt{2}}\right) = 0$$

$$Var(W_1) = Var\left(\frac{z_7+z_{12}}{\sqrt{2}}\right) = \frac{1+1}{(\sqrt{2})^2} = 1 \qquad W_1 \sim N(0;1)$$

$$\frac{z_3+z_9+z_{12}}{\sqrt{3}} = W_2$$

заметим, по аналогии $W_2 \sim N(0;1)$

$$T = W_1^2 + W_2^2 \sim \chi^2_2$$

3.5 $\chi^2_1$ ф. плотности?

о-мальне!!

производ-ую от ф. ции распр-ия?

$Q \sim \chi^2_1 \qquad Q = X_1^2 \qquad X_1 \sim N(0;1)$

$Q \geq 0$

$$F_Q(t) = \begin{cases} ?, & t > 0 \\ 0, & t < 0 \end{cases}$$

ф. распределения

$$\boxed{F_Q(t) = P(Q \leq t)}$$

ф. плотности

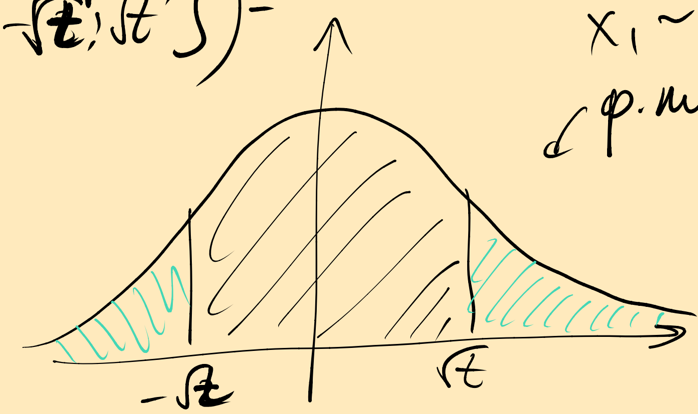
$$f_Q(t) = F_Q'(t) = \begin{cases} ?, & t > 0 \\ 0, & t < 0 \end{cases}$$

рассм: $t \geq 0$.

$$F_Q(t) = P(Q \leq t) = P(X_1^2 \leq t) = P(X_1 \in [-\sqrt{t}; \sqrt{t}]) =$$

$X_1 \sim N(0;1)$

ф. плотн.

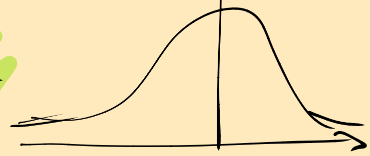



$$F_Q(t) = P(X_1 \leq \sqrt{t}) - \underline{P(X_1 \leq -\sqrt{t})} =$$

$$= P(X_1 \leq \sqrt{t}) - (1 - P(X_1 \leq \sqrt{t})) = 2P(X_1 \leq \sqrt{t}) - 1$$

$$= 2 \cdot F_{X_1}(\sqrt{t}) - 1$$

у $N(0;1)$ $f(u)=\frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

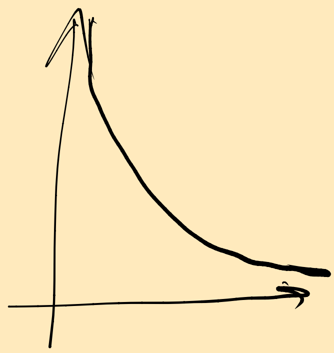


$F(t) = \int_{-\infty}^{t} f(u)\,du$ ← не берётся в элемент. ф-циях

$$F_Q(t) = \begin{cases} 2\int_{-\infty}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du - 1, & t>0 \\ 0, & t\le 0. \end{cases}$$

$$f_Q(t) = F_Q'(t) = \frac{d\left(2F_X(\sqrt{t})-1\right)}{dt} = 2\cdot f_X(\sqrt{t})\cdot\frac{1}{2\sqrt{t}} = f_X(\sqrt{t})\cdot\frac{1}{\sqrt{t}} =$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi t}}\cdot e^{-t/2} & t>0 \\ 0, & t<0 \end{cases}$$



ф. плотн. $\chi^2_1$

$E(\chi^2_1)=1$

$Var(\chi^2_1)=2$

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3.8 выберите ответ через ф-цию распр-ия (или обратную к ней)

from scipy import stats.

stats.t.cdf ← $F$

stats.t.ppf ← $F^{-1}$ /обратная/

$X_i$ ~ незав $N(0;1)$

а) $X_1^2+X_2^2+X_3^2 \sim ?$ $\chi^2_3$

$P(X_1^2+X_2^2+X_3^2>6)=?$

df = dof = degrees of freedom = степени свободы

$= 1 - F_{\chi^2_3}(6) =$

$= 1 -$ stats.chi2.cdf(6, df=3)

ж) $\frac{X_1}{|X_2|} \sim ? \quad \frac{X_1}{\sqrt{X_2^2/1}}$ — $X_1 \sim N(0;1)$, $X_2^2 \sim \chi^2_1$, $\frac{X_1}{|X_2|} \sim t_1$

$P\left(X_1/|X_2| > 2\right) = ?$

$= 1 -$ stats.t.cdf(2, df=1)

$t \in (-\infty;\infty)$ $t_d = \frac{X_1}{\sqrt{Q/d}}$ $\left(X_1 \sim N(0;1),\ Q \sim \chi^2_d\right)$ незав.

$F_{d_1,d_2} = \frac{Q_1/d_1}{Q_2/d_2}$ $\left(Q_1 \sim \chi^2_{d_1},\ Q_2 \sim \chi^2_{d_2}\right)$ незав. $F \geq 0$

и) $R = \frac{2X_1}{\sqrt{X_2^2 + X_3^2 + X_4^2 + X_5^2}} \sim ?\ t_4$ $q?$ $P(R \leq q) = 0{,}95$

$= \frac{X_1}{\sqrt{\frac{X_2^2 + X_3^2 + X_4^2 + X_5^2}{4}}} = \frac{X_1}{\sqrt{Q/4}}$ — $X_1 \sim N(0;1)$, $Q \sim \chi^2_4$, незав.

$F(q) = 0{,}95$

$q = F^{-1}(0{,}95)$

stats.t.ppf(0.95, df=4)

л) $L = \frac{(X_1^2 + X_2^2 + X_3^2 + X_4^2) \cdot 2}{X_5^2 + X_6^2 + \ldots + X_{12}^2} \sim ?$ — $Q_1 \sim \chi^2_4$, $\chi^2_8$

$P(L \geq q) = 0{,}03$ $q?$

$P(L \leq q) = 0{,}97$

$F(q) = 0{,}97$

$q = F^{-1}(0{,}97)$

stats.f.ppf(0.97, df1=4, df2=8)

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$L \sim F_{4,8}$

$\frac{Q_1 \cdot 2}{Q_2} = \frac{Q_1/4}{Q_2/8}$

Упр. а) $\operatorname{plim}\limits_{k\to\infty} \frac{\chi^2_k}{k} \overset{?}{=} \operatorname{plim} \frac{X_1^2+X_2^2+\ldots+X_k^2}{k} \underset{\text{ЗБЧ}}{=} E(X_1^2) = 1$

$X_1 \sim N(0;1)$

$E(X_1)=0$

$\underbrace{Var(X_1)}_{E(X_1^2)-(E(X_1))^2} = 1$

б) $t_k \xrightarrow[k\to\infty]{dist} ?$

$t_k = \frac{\overset{N(0;1)}{X_1}}{\sqrt{\chi^2_k/k}} = X_1 \cdot \frac{1}{\sqrt{\chi^2_k/k}}$ $\xrightarrow[\text{по т. Слуцкого}]{dist} N(0;1)$

$\frac{1}{\sqrt{\chi^2_k/k}} \xrightarrow[dist]{prob} 1$

в) $10 \cdot F_{10,k} \xrightarrow[k\to\infty]{dist} ?$

$10 \cdot \frac{Q_1/10}{Q_2/k} = Q_1 \cdot \frac{1}{Q_2/k}$

$Q_1 \xrightarrow{dist} \chi^2_{10}$, $Q_2 \sim \chi^2_k$

$\frac{1}{Q_2/k} \xrightarrow[dist]{prob} 1$

Упр.

а) $(t_{42})^2 \sim ?$

$\boxed{\left(\frac{X_1}{\sqrt{Q/k}}\right)^2} = \frac{X_1^2/1}{Q/k}$ $\sim F_{1,k}$.

$\chi^2_1/1$, $\chi^2_k/k$ незав.

б) $1/F_{5,7} \sim ?$ $F_{7,5}$

в) $Q_1 \sim \chi^2_5$, $Q_2 \sim \chi^2_7$ $Q_1$ и $Q_2$ незав.

$Q_1 + Q_2 \sim ?$ $\chi^2_{12}$

5 штук

$Q_1 = X_1^2 + \dots + X_5^2$

$Q_2 = X_6^2 + \dots + X_{12}^2$

7 штук

$X_i \sim N(0;1)$ незав

$Q_1 + Q_2 = X_1^2 + \dots + X_{12}^2 \sim \chi^2_{12}$

Упр.

$X_1, X_2 \dots \sim$ незав $\quad X_i \sim N(0; \sigma^2)$

$Y_1, Y_2 \dots \sim$ незав $\quad Y_i \sim N(0; \sigma^2)$

$$R = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2} =$$

$R \sim ?$ док-ка $F$?

$$= \frac{\sum_{i=1}^n \left(\frac{X_i}{\sigma} - \frac{\bar{X}}{\sigma}\right)^2}{\sum_{i=1}^n \left(\frac{Y_i}{\sigma} - \frac{\bar{Y}}{\sigma}\right)^2} =$$

в лекции: $X_i \sim N(0;1)$

$$\sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2_{n-1}$$

$X_i^* = \frac{X_i}{\sigma} \sim N(0;1)$

$Y_i^* = \frac{Y_i}{\sigma} \sim N(0;1)$

$$= \frac{\sum (X_i^* - \bar{X}^*)^2}{\sum (Y_i^* - \bar{Y}^*)^2} \sim F_{n-1, n-1}$$

!!