Econometrics :

- to estimate smth. ← we are here
- to test hypotheses
- to make forecasts
- to give interpretations

Very simple linear regression models

$y = \beta_1 x + u$

$y = \beta_0 + \beta_1 x + u$

⋮

$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots \beta_k x_k + u$

known (data) ; $\beta_0, \beta_1, \beta_2 \dots$ unknown parameters

Very simple method to estimate unknow Betas is Ordinary Least Squares

(we discuss economical and statistical assumptions later)

- $y = \beta_1 x + u$ → $\boxed{\hat{\beta}_1 = \dfrac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}}$

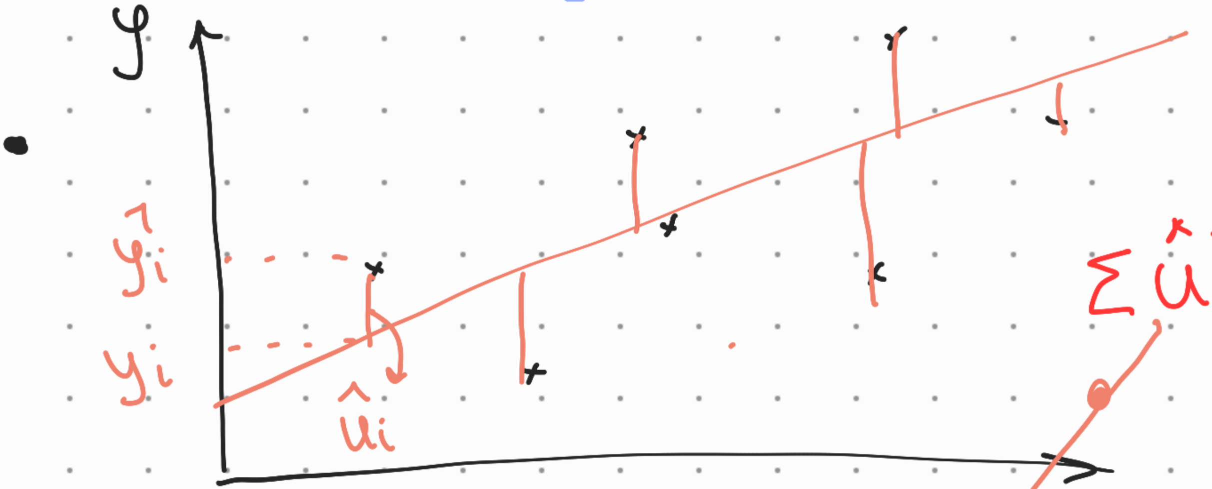
  $\sum \hat{u}_i X_i = 0$

- $y = X \cdot \beta + u$

  $n \times k$ $k \times 1$

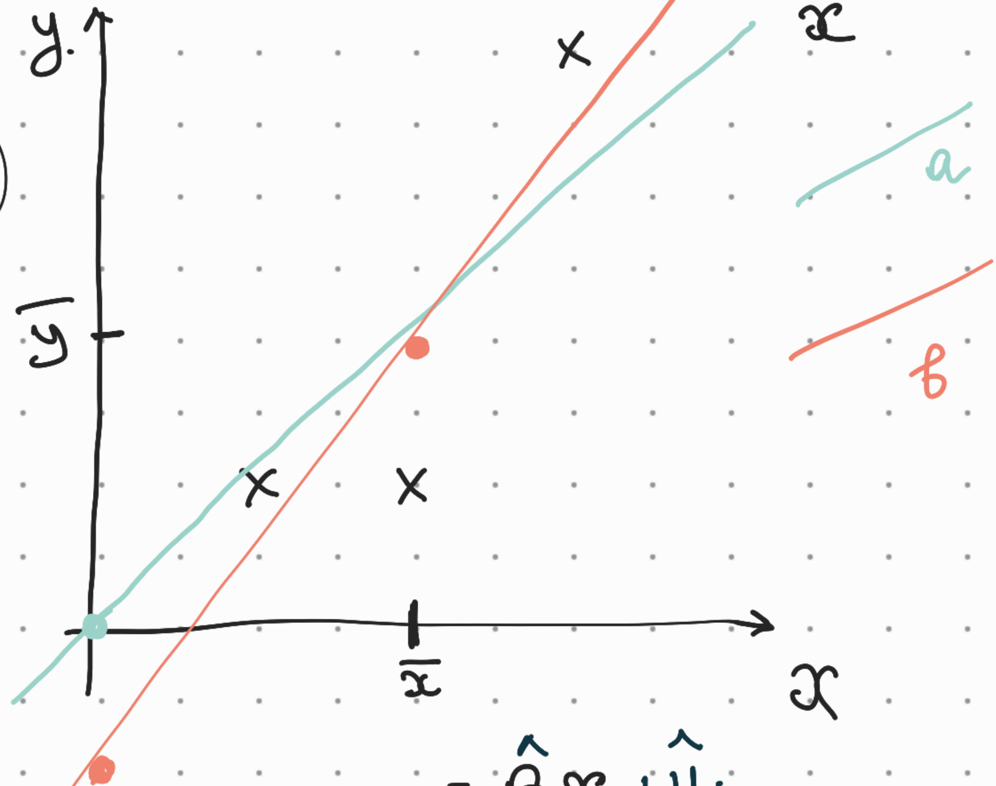
$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{n\times 1} \quad X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1k} \\ x_{21} & & & & \\ \vdots & & & & \\ x_{n1} & & & & \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}$$

$$\begin{cases} \sum \hat{u}_i = 0 \\ \sum \hat{u}_i x_i = 0 \end{cases} \rightarrow \boxed{\hat{\beta} = (X'X)^{-1} X'y}$$



Task1

x_i	y_i
1	1
2	1
3	4



a) $y = \beta x + u \rightarrow y = \hat{\beta} x + \hat{u}_i$

$(0;0) \in$ regression line!

$$\hat{\beta} = \frac{x_i y_i}{\dots} = \frac{1 + 2 + 12}{\dots} = \frac{15}{\dots}$$

$$\beta \quad \frac{\quad}{\sum x_i^2} \quad \frac{\quad}{1+4+9} \quad \frac{\quad}{14}$$

f) $y = \beta_0 + \beta_1 x + u$ $\qquad y = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{u}_i$

$$\sum y_i = \hat{\beta}_0 \cdot n + \beta_1 \sum x_i + \sum \hat{u}_i$$

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x} + 0$$

$(2,2) \in$ regression line

$$\hat{\beta} = (X'X)^{-1} X' y$$

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$n \times k$ $\rightarrow$ two regressors

3×2 (constant and x_i)

$$X'X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

$$(X'X)^{-1} = \begin{pmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{pmatrix}$$

we can invert matrix with some ~~tools~~

$$\hat{\beta} = \begin{pmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \\ \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{3} \\ \end{pmatrix} = \begin{pmatrix} -1 \\ \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots & \cdots \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \cdots \\ 4 \end{pmatrix} = \begin{pmatrix} \cdots \\ -\frac{1}{2}+2 \end{pmatrix} \begin{pmatrix} \cdots \\ \frac{3}{2} \end{pmatrix}$$

$$\boxed{\begin{array}{l} y = -1 + \frac{3}{2}x + \hat{u}_i \\ \text{or} \quad \hat{y} = -1 + \frac{3}{2}x \end{array}}$$

$\ni \boxed{(2;2)}$

$\ni (0;-1)$

c) $y \quad \beta_0 + \beta_1 x + \beta_2 x^2 + u$

it is linear model (according to betas!)

similar to

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

x_{1i}	x_{2i}	y_i
1	1	1
2	4	1
3	9	4

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$X = \begin{array}{c} \begin{array}{ccc} x_{11} & x_{12} & x_{13} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix}$$

$$(X'X)^{-1} = \begin{pmatrix} 19 & -21 & 5 \\ -21 & \frac{49}{2} & -6 \\ 5 & -6 & 3/2 \end{pmatrix} \to \hat{\beta} = \begin{pmatrix} 4 \\ -9/2 \\ 3/2 \end{pmatrix}$$

$$\boxed{y = 4 - \frac{9}{2}x + \frac{3}{2}x^2 + u}$$

$\bar{y} = 2 \qquad \bar{x} = 2 \qquad \overline{x^2} = \frac{14}{3}$

$(\bar{x}, \bar{y}) \in$ regress. line

$2 = 4 - 9 + 7$ + checked!

d) in fact, for $y = \beta_0 + \beta_1 x + u$

you don't need matrix estimations:

we still have minimization problem

$$\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \to \min$$

but 2 unknown parameters: β_0, β_1

$$\min_{\beta_0, \beta_1} \sum (y_i - \beta_0 - \beta_1 x_i)^2 \Rightarrow \text{derivative by } \beta_0, \beta_1 \to$$

new restriction

$$\begin{cases} -2 \cdot \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \\ -2 \sum x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \end{cases} \quad \begin{cases} \sum \hat{u}_i = 0 \\ \sum x_i \hat{u}_i = 0 \end{cases}$$

you know it from the lecture

first

$\sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0 \Rightarrow$

$\sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$

$\Rightarrow n\hat{\beta}_0 = \sum y_i - \hat{\beta}_1 \sum x_i$

$\boxed{\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}}$ $\Rightarrow (\bar{x}, \bar{y}) \in$ regression line

second

$\sum x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0$

$\sum x_i y_i - \hat{\beta}_0 \sum x_i - \hat{\beta}_1 \sum x_i^2 = 0$

$\sum x_i y_i - \bar{y} \sum x_i + \hat{\beta}_1 \bar{x} \sum x_i - \hat{\beta}_1 \sum x_i^2 = 0$

$\sum (y_i - \bar{y}) x_i = \hat{\beta}_1 \left(\sum x_i (x_i - \bar{x}) \right)$

interesting fact: $\sum (x_i - \bar{x}) \cdot \bar{x} = 0 =$

$= \sum x_i \bar{x} - \sum \bar{x}^2 =$

$= n\bar{x}^2 - n\bar{x}^2$

$\sum (y_i - \bar{y}) \cdot \bar{x} = 0 =$

$= \sum y_i \bar{x} - \sum \bar{y} \bar{x} = n\bar{y}\bar{x} -$

$- n\bar{y}\bar{x}$

all this points to equation

+ add this parts to equation

$$\sum(y_i-\bar{y})x_i - \sum(y_i-\bar{y})\bar{x} =$$

$$= \hat{\beta}_1\left(\sum x_i(x_i-\bar{x}) - \sum(x_i-\bar{x})\bar{x}\right)$$

$$\boxed{\hat{\beta}_1 = \frac{\sum(y_i-\bar{y})(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}}$$

check the result ! $\bar{y}=2$

$\bar{x}=2$

$$\hat{\beta}_0 = 2-\hat{\beta}_1\cdot 2 = 2-3=-1$$

$$\hat{\beta}_1 = \frac{(1-2)(1-2)+(2-2)(1-2)+(3-2)(4-2)}{(1-2)^2+(2-2)^2+(3-2)^2} =$$

$$= \frac{1+2}{1+1} = \frac{3}{2}$$

$$\boxed{y = -1+\frac{3}{2}x+u}$$

Task 2 find OLS estimations of unknown coefficients

a) $y_i = \theta + \theta x_i + u_i$

do we need to minimize for 2 param.?

No

$\downarrow z_i = x_i + 1$

$y_i = \theta z_i + u_i$

one unknown

$$\boxed{\hat{\theta} = \frac{\sum y_i z_i}{\sum z_i^2}}$$

b) $y_i = 1 + \theta x_i + u_i$

$\downarrow z_i = y_i - 1$

$z_i = \theta x_i + u_i \Rightarrow \boxed{\hat{\theta} = \frac{\sum z_i x_i}{\sum x_i^2}}$

c) $y_i = \frac{\theta}{x_i} + u_i$ $\qquad z_i = \frac{1}{x_i}$

$$\boxed{\begin{gathered} y_i = \theta z_i + u_i \\ \hat{\theta} = \frac{\sum y_i / x_i}{\sum \frac{1}{x_i^2}} \end{gathered}}$$

d) $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + u_i$

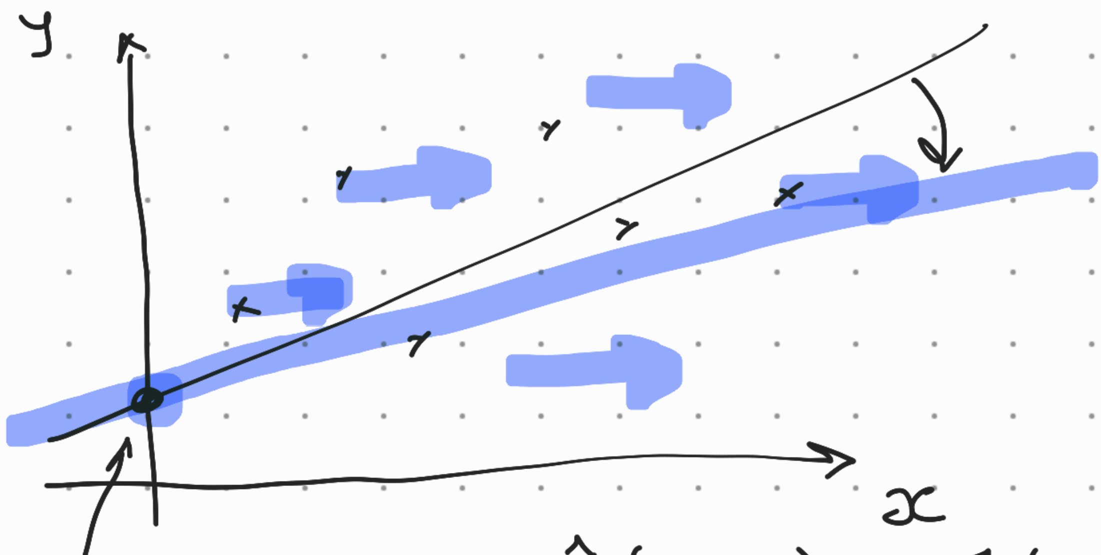
replacement : $2x_i = z_i$

$$y_i = \hat{\alpha}_0 + \hat{\alpha}_1 z_i + v_i \longrightarrow \text{connection between } \alpha s, \beta s?$$

$$\hat{\alpha}_1 = \frac{\sum (y_i - \bar{y})(z_i - \bar{z})}{\sum (z_i - \bar{z})^2} = \frac{2\sum (y_i - \bar{y})(x_i - \bar{x})}{4\sum (x_i - \bar{x})^2} =$$

$$= \frac{1}{2}\hat{\beta}_1$$

$$\hat{\alpha}_0 = \bar{y} - \hat{\alpha}_1 \bar{z} = \bar{y} - \frac{1}{2}\hat{\beta}_1 \cdot 2\bar{x} = \bar{y} - \hat{\beta}_1 \bar{x} = \hat{\beta}_0$$



no changes: $\hat{y}(x=0) = \hat{y}(z=0)$