

FINAL SALE: Every problem will bring you 10 99.99 points!

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1. Dragon Erik has three towns to kidnap a princess from: A , B and C . He kidnaps the first princess from town A and chooses every next town according to a Markov chain with transition matrix T , where states are written in alphabetical order:

$$T = \begin{pmatrix} 0.1 & 0.2 & ? \\ ? & 0 & 0.7 \\ 0.2 & ? & 0.5 \end{pmatrix}.$$

- (a) [2] Fill in the missing values.
 - (b) [2] Draw the graph of the Markov chain.
 - (c) [3] What is the probability that the third princess will be from town B ?
 - (d) [3] What is the probability that the second princess was from A given that the third princess was from B ?
2. Every minute the cat Tikhon says «meow» with probability $1/3$ or «purr» with probability $2/3$ independently of all other words. For every «purr» that follows «meow» you get 1 PEPE, for every «meow» that follows «purr» you pay 3 PEPEs. For «purr» that follows «purr» or «meow» that follows «meow» you get nothing. You have just earned 1 PEPE. Let T be the time to get the next positive reward.
- (a) [5] Find $\mathbb{E}(T)$.
 - (b) [5] Find $\mathbb{V}\text{ar}(T)$.
3. [10] Gumbatali starts with zero initial sum $S_0 = 0$. Every minute he either wins $X_t = 1$ dollar with probability $1/3$ or pays 1 dollar, $X_t = -1$, with probability $2/3$. He can't go neither below $S_t = -2$ nor above $S_t = 3$, so that

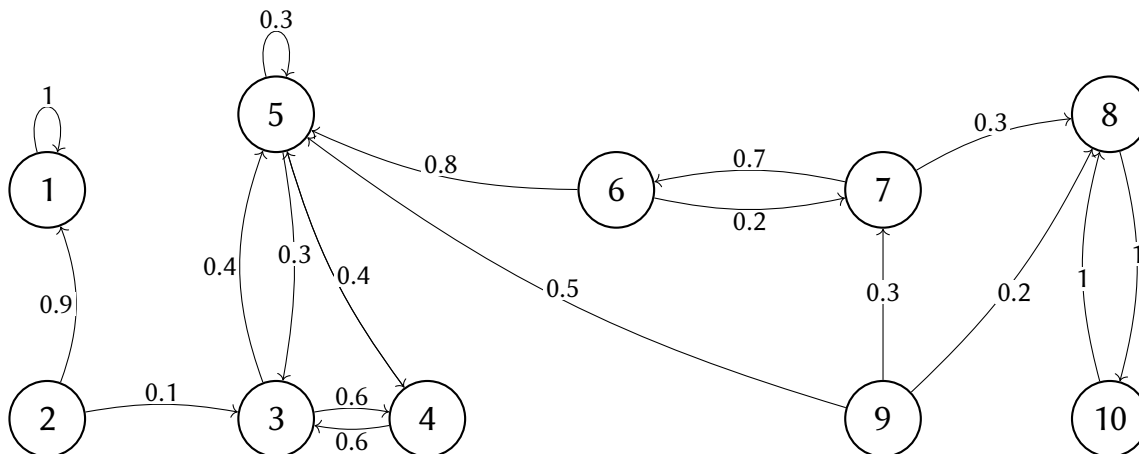
$$S_t = \min\{3, \max\{-2, S_{t-1} + X_t\}\}.$$

The random variables (X_t) are independent.

Find the stationary distribution of S_t .

*Members only deal!

4. Consider the Markov chain:



- [5] Using a chainsaw cut the Markov chain into communicating classes and classify them as transient, positive recurrent or null-recurrent.
 - [1] Is the chain irreducible?
 - [4] Find the period of every state.
5. Let X be a Poisson distributed random variable with intensity $\lambda = 2$. The probability mass function of a Poisson random variable is:

$$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

- [4] Derive the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$.
 - [4] Using moment generating function find the mean and variance of X .
 - [2] Let $X_1 \sim \text{Pois}(\lambda = 2)$ and $X_2 \sim \text{Pois}(\lambda = 3)$ be independent random variables. Find the moment generating function of $S = X_1 + X_2$.
6. By the mayor's decree, the city installs T_n New Year's trees each day, $n = 1, 2, \dots$, starting from October 20th. The random variables T_1, T_2, \dots are independent and take any natural value from 4 to 10 with equal probabilities.

The amounts of snow (W_n) are independent and uniformly distributed on $[0; 1/n]$.

Determine the probability limits for the following sequences:

- [2] mayor's New Year kpi, $X_n = \frac{1}{n} \sum_{i=1}^n T_i$
- [3] the New Year's mood given by $Y_n = \frac{2X_n+1}{X_n+3}$
- [5] The average number of snowmen that can be built each day: $S_n = \frac{1}{n} \sum_{i=1}^n W_i$.