

**FINAL SALE:** Every problem will bring you 10 99.99 points!

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1. Dragon Erik has three towns to kidnap a princess from:  $A$ ,  $B$  and  $C$ . He kidnaps the first princess from town  $A$  and chooses every next town according to a Markov chain with transition matrix  $T$ , where states are written in alphabetical order:

$$T = \begin{pmatrix} 0.1 & 0.2 & ? \\ ? & 0 & 0.7 \\ 0.2 & ? & 0.5 \end{pmatrix}.$$

- (a) [2] Fill in the missing values.
  - (b) [2] Draw the graph of the Markov chain.
  - (c) [3] What is the probability that the third princess will be from town  $B$ ?
  - (d) [3] What is the probability that the second princess was from  $A$  given that the third princess was from  $B$ ?
2. Every minute the cat Tikhon says «meow» with probability  $1/3$  or «purr» with probability  $2/3$  independently of all other words. For every «purr» that follows «meow» you get 1 PEPE, for every «meow» that follows «purr» you pay 3 PEPEs. For «purr» that follows «purr» or «meow» that follows «meow» you get nothing. You have just earned 1 PEPE. Let  $T$  be the time to get the next positive reward.
- (a) [5] Find  $\mathbb{E}(T)$ .
  - (b) [5] Find  $\text{Var}(T)$ .
3. [10] Gumbatali starts with zero initial sum  $S_0 = 0$ . Every minute he either wins  $X_t = 1$  dollar with probability  $1/3$  or pays 1 dollar,  $X_t = -1$ , with probability  $2/3$ . He can't go neither below  $S_t = -2$  nor above  $S_t = 3$ , so that

$$S_t = \min\{3, \max\{-2, S_{t-1} + X_t\}\}.$$

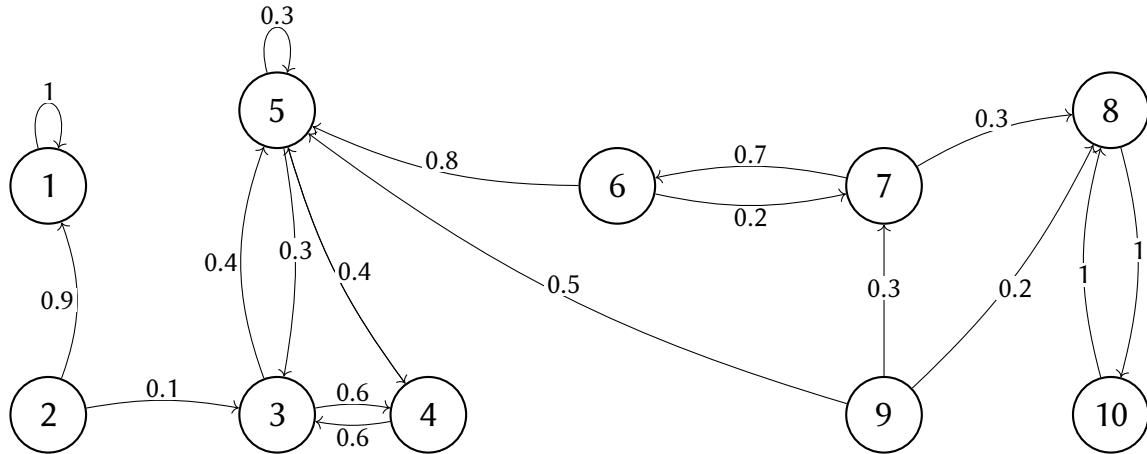
The random variables  $(X_t)$  are independent.

Find the stationary distribution of  $S_t$ .

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\*Members only deal!

4. Consider the Markov chain:



- (a) [5] Using a chainsaw cut the Markov chain into communicating classes and classify them as transient, positive recurrent or null-recurrent.
  - (b) [1] Is the chain irreducible?
  - (c) [4] Find the period of every state.
5. Let  $X$  be a Poisson distributed random variable with intensity  $\lambda = 2$ . The probability mass function of a Poisson random variable is:
- $$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$
- (a) [4] Derive the moment generating function  $M_X(t) = \mathbb{E}(e^{tX})$ .
  - (b) [4] Using moment generating function find the mean and variance of  $X$ .
  - (c) [2] Let  $X_1 \sim \text{Pois}(\lambda = 2)$  and  $X_2 \sim \text{Pois}(\lambda = 3)$  be independent random variables. Find the moment generating function of  $S = X_1 + X_2$ .
6. By the mayor's decree, the city installs  $T_n$  New Year's trees each day,  $n = 1, 2, \dots$ , starting from October 20th. The random variables  $T_1, T_2, \dots$  are independent and take any natural value from 4 to 10 with equal probabilities.

The amounts of snow ( $W_n$ ) are independent and uniformly distributed on  $[0; 1/n]$ .

Determine the probability limits for the following sequences:

- (a) [2] mayor's New Year kpi,  $X_n = \frac{1}{n} \sum_{i=1}^n T_i$
- (b) [3] the New Year's mood given by  $Y_n = \frac{2X_n+1}{X_n+3}$
- (c) [5] The average number of snowmen that can be built each day:  $S_n = \frac{1}{n} \sum_{i=1}^n W_i$ .