

Сегодня по народному календарю Евстратиев день. Нельзя в этот день сквернословить — не то ведьмы с неба упадут прямо на голову. Также нельзя веник на крыльце оставлять — не то ведьмы утащат.

1. [10] Consider the Black and Scholes model. The risk-free interest rate is equal to $r = 0.1$. The volatility of the share is equal to $\sigma = 2$. You have an option that pays you 100\$ three years later if the percentage change of price of the share during the first year, S_1/S_0 , is less than the percentage change after the first year, S_3/S_1 .

What is the fair price of this option?

Hint: you may use standard normal cdf $F()$ in your answer.

2. The joint distribution of X and Y is given by the table:

	$Y = -1$	$Y = 0$	$Y = 1$
$X = -1$	1/8	1/16	1/8
$X = 0$	1/16	1/4	1/16
$X = 1$	1/8	1/16	1/8

Three students are building statistical models with different information sets:

- Roman Bokhyan: $\mathcal{F}_1 = \sigma(X, Y)$, the σ -algebra generated by X and Y .
- Mikhail N: $\mathcal{F}_2 = \sigma(X + Y, X - Y)$.
- Andrey L: $\mathcal{F}_3 = \sigma(X^2 + Y^2)$.

- (a) [3] For each pair of σ -algebras above determine whether one is contained in the other, they are equal, or neither. Justify your claims.
- (b) [2] Andrey and Mikhail decide to share their information. Compute the number of elements in the smallest σ -algebra containing both \mathcal{F}_2 and \mathcal{F}_3 .
- (c) [5] Help Andrey and compute $\mathbb{E}(X \mid \mathcal{F}_3)$, $\mathbb{V}\text{ar}(X \mid \mathcal{F}_3)$.

3. The process S_n models the «accumulated wrath of the heavens». Let $S_0 = 0$. Each hour independently of previous history S_n increases by 1 with probability 1/3, decrease by 1 with probability 1/6 or does not change.

The witches fall from the sky when S_n first reaches the level of 4. Let $T = \min\{n : S_n = 4\}$.

- (a) [4] For which $\alpha \neq 1$ is the process $M_n = \alpha^{S_n}$ a martingale?
- (b) [2] Check whether $W_n = S_n - n/6$ is a martingale.
- (c) [4] Find $\mathbb{E}(T)$.

Hint: you may use Doob's theorem without checking technical details.

4. Consider three processes,

$$A_t = \int_0^t W_u^2 dW_u, \quad B_t = \int_0^t W_u^4 dW_u, \quad C_t = \int_0^t W_u^2 du.$$

(a) [2 + 2] Find $\mathbb{E}(A_t), \mathbb{E}(C_t)$.

(b) [2 + 2 + 2] Find $\mathbb{V}\text{ar}(A_t), \mathbb{C}\text{ov}(A_t, B_t), \mathbb{V}\text{ar}(C_t)$.

5. Be brave!

(a) [2 + 3] Find $\mathbb{E}(W_3^4), \mathbb{E}(W_3^4 \mid W_2 = 10)$.

(b) [5] Find $\mathbb{C}\text{ov}(W_1 W_2, W_5 W_6)$.

6. Consider the process

$$R_t = \exp \left(\int_0^t u^3 dW_u + t^2 \right).$$

(a) [6] Write R_t as a sum of a constant, an Ito's integral and a Riemann integral.

(b) [2] Is (R_t) a martingale?

(c) [2] Provide an ordinary (non-stochastic) differential equation for $h(t) = \mathbb{E}(R_t)$. You don't need to solve this equation.

Hint: you may write R_t as $R_t = \exp(Q_t)$ and apply Ito's lemma :)