FINAL SALE: Every problem will bring you 10 ~~99.99~~ points!

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1. Dragon Erik has three towns to kidnap a princess from: A, B and C. He kidnaps the first princess from town A and chooses every next town according to a Markov chain with transition matrix T, where states are written in alphabetical order:
$$T = \begin{pmatrix} 0.2 & 0.1 & ? \\ ? & 0 & 0.6 \\ 0.5 & ? & 0.3 \end{pmatrix}.$$

   (a) [2] Fill in the missing values.

   (b) [2] Draw the graph of the Markov chain.

   (c) [3] What is the probability that the third princess will be from town B?

   (d) [3] What is the probability that the second princess was from A given that the third princess was from B?

2. Every minute the cat Tikhon says «meow» with probability $1/3$ or «purr» with probability $2/3$ independently of all other words.

   Let T be the first moment of time when he says the sequence «meow-purr-meow».

   (a) [5] Find $\mathbb{E}(T)$.

   (b) [5] Find $\mathbb{V}\mathrm{ar}(T)$.

3. [10] Gumbatali starts with zero initial sum $S_0 = 0$. Every minute he either wins $X_t = 1$ dollar with probability $1/4$ or pays 1 dollar, $X_t = -1$, with probability $3/4$. He can't go neither below $S_t = -3$ nor above $S_t = 2$, so that
$$S_t = \min\{2, \max\{-3, S_{t-1} + X_t\}\}.$$
   The random variables (X_t) are independent.

   Find the stationary distribution of S_t.

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[*]Members only deal!

4. Consider the Markov chain:
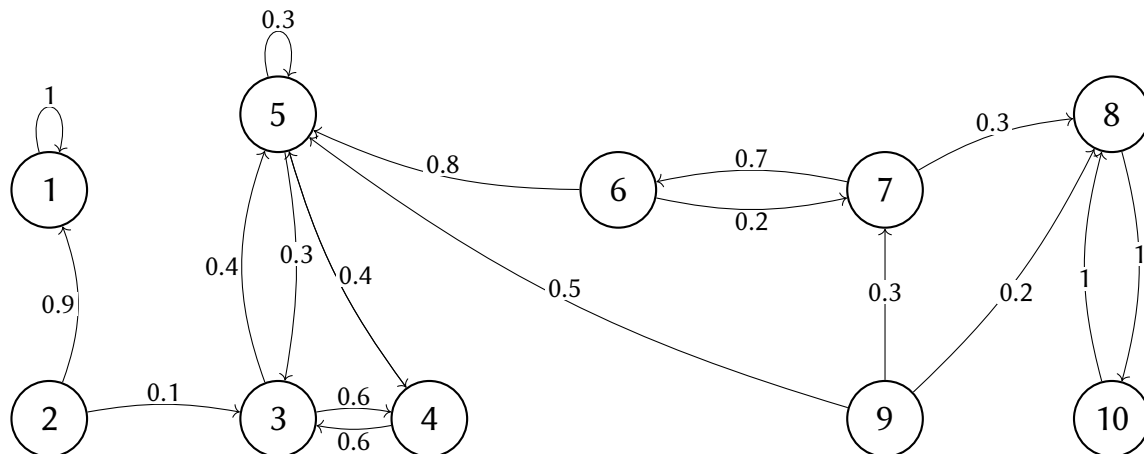
   (a) [5] Using a chainsaw cut the Markov chain into communicating classes and classify them as transient, positive recurrent or null-recurrent.

   (b) [1] Is the chain irreducible?

   (c) [4] Find the period of every state.

5. Let X be exponentially distributed random variable with intensity $\lambda = 2$. The density for $x \geq 0$ is given by:

$$f(x) = \lambda \exp(-\lambda x).$$

   (a) [4] Derive the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$.

   (b) [3] Using moment generating function find the mean and variance of X.

   (c) [3] Let $X_1 \sim \text{Expo}(\lambda = 2)$ and $X_2 \sim \text{Expo}(\lambda = 3)$ be independent random variables. Find the moment generating function of $S = X_1 + X_2$. Is S exponentially distributed?

6. By the mayor's decree, the city installs T_n New Year's trees each day, $n = 1, 2, \ldots$, starting from October 20th. The random variables $T_1, T_2, \ldots$ are independent and take any natural value from 0 to 4 with equal probabilities.

   The amounts of snow (W_n) are independent and uniformly distributed on $[0; 2/n]$.

   Determine the probability limits for the following sequences:

   (a) [2] mayor's New Year kpi, $X_n = \frac{1}{n} \sum_{i=1}^{n} T_i$

   (b) [3] the New Year's mood given by $Y_n = \frac{2X_n + 1}{X_n + 10}$

   (c) [5] The average number of snowmen that can be built each day: $S_n = \frac{1}{n} \sum_{i=1}^{n} W_i^2$.