Сегодня по народному календарю Евстратиев день. Нельзя в этот день сквернословить — не то ведьмы с неба упадут прямо на голову. Также нельзя веник на крыльце оставлять — не то ведьмы утащат.

1. [10] Consider the Black and Scholes model. The risk-free interest rate is equal to $r = 0.1$. The volatility of the share is equal to $\sigma = 2$. You have an option that pays you 100$ three years later if the percentage change of price of the share during the first year, $S_1/S_0$, is less than the percentage change after the first year, $S_3/S_1$.

   What is the fair price of this option?

   Hint: you may use standard normal cdf $F()$ in your answer.

2. The joint distribution of $X$ and $Y$ is given by the table:

| | $Y=-1$ | $Y=0$ | $Y=1$ |
|---|---|---|---|
| $X=-1$ | $1/8$ | $1/16$ | $1/8$ |
| $X=0$ | $1/16$ | $1/4$ | $1/16$ |
| $X=1$ | $1/8$ | $1/16$ | $1/8$ |

   Three students are building statistical models with different information sets:

   - Roman Bokhyan: $\mathcal{F}_1 = \sigma(X, Y)$, the $\sigma$-algebra generated by $X$ and $Y$.
   - Mikhail N: $\mathcal{F}_2 = \sigma(X+Y, X-Y)$.
   - Andrey L: $\mathcal{F}_3 = \sigma(X^2+Y^2)$.

   (a) [3] For each pair of $\sigma$-algebras above determine whether one is contained in the other, they are equal, or neither. Justify your claims.

   (b) [2] Andrey and Mikhail decide to share their information. Compute the number of elements in the smallest $\sigma$-algebra containing both $\mathcal{F}_2$ and $\mathcal{F}_3$.

   (c) [5] Help Andrey and compute $\mathbb{E}(X \mid \mathcal{F}_3)$, $\mathbb{V}\text{ar}(X \mid \mathcal{F}_3)$.

3. The process $S_n$ models the «accumulated wrath of the heavens». Let $S_0 = 0$. Each hour independently of previous history $S_n$ increases by $1$ with probability $1/3$, decrease by $1$ with probability $1/6$ or does not change.

   The witches fall from the sky when $S_n$ first reaches the level of $4$. Let $T = \min\{n : S_n = 4\}$.

   (a) [4] For which $\alpha \neq 1$ is the process $M_n = \alpha^{S_n}$ a martingale?

   (b) [2] Check whether $Q_n = S_n - n/6$ is a martingale.

   (c) [4] Find $\mathbb{E}(T)$.

   Hint: you may use Doob's theorem without checking technical details.

In this exam $(W_t)$ always denotes a standard Wiener process.

4. Consider three processes,

$$A_t = \int_0^t W_u^2 \, dW_u, \quad B_t = \int_0^t W_u^4 \, dW_u, \quad C_t = \int_0^t W_u^2 \, du.$$

   (a) [2 + 2] Find $\mathbb{E}(A_t)$, $\mathbb{E}(C_t)$.
   (b) [2 + 2 + 2] Find $\mathbb{V}\mathrm{ar}(A_t)$, $\mathbb{C}\mathrm{ov}(A_t, B_t)$, $\mathbb{V}\mathrm{ar}(C_t)$.

5. Be brave!

   (a) [2 + 3] Find $\mathbb{E}(W_3^4)$, $\mathbb{E}(W_3^4 \mid W_2 = 10)$.
   (b) [5] Find $\mathbb{C}\mathrm{ov}(W_1 W_2, W_5 W_6)$.

6. Consider the process

$$R_t = \exp\left(\int_0^t u^3 dW_u + t^2\right).$$

   (a) [6] Write $R_t$ as a sum of a constant, an Ito's integral and a Riemann integral.
   (b) [2] Is $(R_t)$ a martingale?
   (c) [2] Provide an ordinary (non-stochastic) differential equation for $h(t) = \mathbb{E}(R_t)$. You don't need to solve this equation.

   Hint: you may write $R_t$ as $R_t = \exp(Q_t)$ and apply Ito's lemma :)