

Сегодня по народному календарю Евстратиев день. Нельзя в этот день сквернословить — не то ведьмы с неба упадут прямо на голову. Также нельзя веник на крыльце оставлять — не то ведьмы утащат.

1. The Legendary November Midterm Grader grades tests according to Poisson point process with variable intensity rate. The first 5 weeks the rate is equal to $\lambda_1 = 1$ test per week. Later the rate increases to $\lambda_2 = 100$ tests per week. Let X_t be the number of tests graded after t weeks.

- (a) [2] Sketch the curve $v(t) = \mathbb{V}\text{ar}(X_t)$.
- (b) [4] Using normal approximation find the probability that LNMG will check not less than 120 works at time $t = 6$.
- (c) [4] Now $t = 4.5$ weeks have already passed. Find the expected time for the next test checked.

Note: in point (b) you may use the standard normal cdf $F()$ in your answer.

2. The joint distribution of X and Y is given by the table:

	$Y = -1$	$Y = 0$	$Y = 1$
$X = -1$	1/8	1/16	1/8
$X = 0$	1/16	1/4	1/16
$X = 1$	1/8	1/16	1/8

Three students are building statistical models with different information sets:

- Roman Bokhyan: $\mathcal{F}_1 = \sigma(X, Y)$, the σ -algebra generated by X and Y .
 - Mikhail N: $\mathcal{F}_2 = \sigma(X + Y, X - Y)$.
 - Andrey L: $\mathcal{F}_3 = \sigma(X^2 + Y^2)$.
- (a) [3] For each pair of σ -algebras above determine whether one is contained in the other, they are equal, or neither. Justify your claims.
 - (b) [2] Andrey and Mikhail decide to share their information. Compute the number of elements in the smallest σ -algebra containing both \mathcal{F}_2 and \mathcal{F}_3 .
 - (c) [5] Help Andrey and compute $\mathbb{E}(X | \mathcal{F}_3)$, $\mathbb{V}\text{ar}(X | \mathcal{F}_3)$.
3. The process S_n models the «accumulated wrath of the heavens». Let $S_0 = 0$. Each hour independently of previous history S_n increases by 1 with probability $1/3$, decrease by 1 with probability $1/6$ or does not change.

The witches fall from the sky when S_n first reaches the level of 4. Let $T = \min\{n : S_n = 4\}$.

- (a) [4] For which $\alpha \neq 1$ is the process $M_n = \alpha^{S_n}$ a martingale?
- (b) [2] Check whether $Q_n = S_n - n/6$ is a martingale.
- (c) [4] Find $\mathbb{E}(T)$.

Hint: you may use Doob's theorem without checking technical details.

4. Consider three processes,

$$A_t = \int_0^t W_u^2 dW_u, \quad B_t = \int_0^t W_u^4 dW_u, \quad C_t = \int_0^t W_u^2 du.$$

- (a) [2 + 2] Find $\mathbb{E}(A_t)$, $\mathbb{E}(C_t)$.
- (b) [2 + 2 + 2] Find $\text{Var}(A_t)$, $\text{Cov}(A_t, B_t)$, $\text{Var}(C_t)$.

5. Be brave!

- (a) [2 + 3] Find $\mathbb{E}(W_3^4)$, $\mathbb{E}(W_3^4 \mid W_2 = 10)$.
- (b) [5] Find $\text{Cov}(W_1 W_2, W_5 W_6)$.

6. Consider the process

$$R_t = \exp \left(\int_0^t u^3 dW_u + t^2 \right).$$

- (a) [6] Write R_t as a sum of a constant, an Ito's integral and a Riemann integral.
- (b) [2] Is (R_t) a martingale?
- (c) [2] Provide an ordinary (non-stochastic) differential equation for $h(t) = \mathbb{E}(R_t)$. You don't need to solve this equation.

Hint: you may write R_t as $R_t = \exp(Q_t)$ and apply Ito's lemma :)