

Сегодня по народному календарю Евстратиев день. Нельзя в этот день сквернословить — не то ведьмы с неба упадут прямо на голову. Также нельзя веник на крыльце оставлять — не то ведьмы утащат.

1. The Legendary November Midterm Grader grades tests according to Poisson point process with variable intensity rate. The first 5 weeks the rate is equal to  $\lambda_1 = 1$  test per week. Later the rate increases to  $\lambda_2 = 100$  tests per week. Let  $X_t$  be the number of tests graded after  $t$  weeks.

- (a) [2] Sketch the curve  $v(t) = \mathbb{V}\text{ar}(X_t)$ .
- (b) [4] Using normal approximation find the probability that LNMG will check not less than 120 works at time  $t = 6$ .
- (c) [4] Now  $t = 4.5$  weeks have already passed. Find the expected time for the next test checked.

Note: in point (b) you may use the standard normal cdf  $F()$  in your answer.

2. The joint distribution of  $X$  and  $Y$  is given by the table:

	$Y = -1$	$Y = 0$	$Y = 1$
$X = -1$	1/8	1/16	1/8
$X = 0$	1/16	1/4	1/16
$X = 1$	1/8	1/16	1/8

Three students are building statistical models with different information sets:

- Roman Bokhyan:  $\mathcal{F}_1 = \sigma(X, Y)$ , the  $\sigma$ -algebra generated by  $X$  and  $Y$ .
- Mikhail N:  $\mathcal{F}_2 = \sigma(X + Y, X - Y)$ .
- Andrey L:  $\mathcal{F}_3 = \sigma(X^2 + Y^2)$ .

- (a) [3] For each pair of  $\sigma$ -algebras above determine whether one is contained in the other, they are equal, or neither. Justify your claims.
- (b) [2] Andrey and Mikhail decide to share their information. Compute the number of elements in the smallest  $\sigma$ -algebra containing both  $\mathcal{F}_2$  and  $\mathcal{F}_3$ .
- (c) [5] Help Andrey and compute  $\mathbb{E}(X \mid \mathcal{F}_3)$ ,  $\mathbb{V}\text{ar}(X \mid \mathcal{F}_3)$ .

3. The process  $S_n$  models the «accumulated wrath of the heavens». Let  $S_0 = 0$ . Each hour independently of previous history  $S_n$  increases by 1 with probability 1/3, decrease by 1 with probability 1/6 or does not change.

The witches fall from the sky when  $S_n$  first reaches the level of 4. Let  $T = \min\{n : S_n = 4\}$ .

- (a) [4] For which  $\alpha \neq 1$  is the process  $M_n = \alpha^{S_n}$  a martingale?
- (b) [2] Check whether  $Q_n = S_n - n/6$  is a martingale.
- (c) [4] Find  $\mathbb{E}(T)$ .

Hint: you may use Doob's theorem without checking technical details.

4. Consider three processes,

$$A_t = \int_0^t W_u^2 dW_u, \quad B_t = \int_0^t W_u^4 dW_u, \quad C_t = \int_0^t W_u^2 du.$$

(a) [2 + 2] Find  $\mathbb{E}(A_t)$ ,  $\mathbb{E}(C_t)$ .

(b) [2 + 2 + 2] Find  $\mathbb{V}\text{ar}(A_t)$ ,  $\mathbb{C}\text{ov}(A_t, B_t)$ ,  $\mathbb{V}\text{ar}(C_t)$ .

5. Be brave!

(a) [2 + 3] Find  $\mathbb{E}(W_3^4)$ ,  $\mathbb{E}(W_3^4 \mid W_2 = 10)$ .

(b) [5] Find  $\mathbb{C}\text{ov}(W_1 W_2, W_5 W_6)$ .

6. Consider the process

$$R_t = \exp \left( \int_0^t u^3 dW_u + t^2 \right).$$

(a) [6] Write  $R_t$  as a sum of a constant, an Ito's integral and a Riemann integral.

(b) [2] Is  $(R_t)$  a martingale?

(c) [2] Provide an ordinary (non-stochastic) differential equation for  $h(t) = \mathbb{E}(R_t)$ . You don't need to solve this equation.

Hint: you may write  $R_t$  as  $R_t = \exp(Q_t)$  and apply Ito's lemma :)