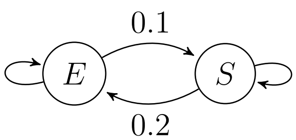
You are more than welcome to openly discuss these problems. You don't need to hand in these problems. The home assignments are graded only through quizzes. Questions with [For Fun] mark will not enter into the quizzes.

Home assignment 1

Enters in the quizzes during the week 2: 8-13 september.

1. The Cat can be only in two states: Sleeping (S) and Eating (E). Cat's mood depends only on the previous state. The transition probabilities are given below:



   (a) Compute the missing probabilities on the graph.

   (b) Write down the transition matrix.

   (c) Compute $\mathbb{P}(X_3 = \text{Eating} \mid X_0 = \text{Eating})$.

2. Cowboy Joe enters the Epsilon Bar and orders one pint of beer. He drinks it and orders one pint more. And so on and so on and so on... The problem is that the barmaid waters down each pint with probability 0.2 independently of other pints. Joe does not like watered down beer. He will blow the Epsilon Bar to hell if two or more out of the last three pints are watered down.

   We point out that Joe never drinks less than 3 pints in a bar.

   Let Y_t be the indicator that the pint number t was watered down. Consider the Markov chain $S_t = (y_{t-2}, y_{t-1}, y_t)$. For example, $S_t = (100)$ means that the pint number $t-2$ was watered down while pints number $t-1$ and t are good.

   (d) What are the possible values of S_3 and their probabilities?

   (e) Write down the transition matrix of this Markov chain.

   (f) [For Fun] What is the expected number of pints of beer Joe will drink?

3. Pavel Durov starts at the point $X_0 = 1$ on the real line. Each minute he moves left with probability 0.4 or right with probability 0.6 independently of past moves. The points 0 and 3 are absorbing. If Pavel reaches 0 or 3 he stays there forever. Let X_t be the coordinate of Pavel after t minutes.

   (a) Write down the transition matrix of this Markov chain.

   (b) Calculate the distribution of X_3 [list all values of the random variable X_3 and estimate the probabilities].

4. Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCD$. Each minute each hedehog simulteneously and independently chooses to go clockwise to the adjacent point, to go counter-clockwise to the adjacent point or to stay at his location.

   Thus the brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.

(a) Draw the graph for the brotherhood Markov chain and calculate all transition probabilities.

(b) Write down the transition matrix of the brotherhood Markov chain.

(c) What is the probability that they will be in one vertex after 3 steps?

Home assignment 2

Enters in the quizzes during the week 3: 15-20 september. During the quiz you may save time by writing only the system of equations and clearly stating which unknown is the answer to the question.

1. I throw a fair dice indefinitely often. If it shows 1 then 1 rouble is added to the hat and the game continues. If it shows 2 then I immediately get 2 roubles [plus the sum in the hat] and the game stops. If it shows 3 then the sum in the hat is burned and the game continues. If it shows 4 then I immediately get 4 roubles and the game continues. It it shows 5 then the sum in the hat is burned and the game stops. If it shows 6 nothing happens and the game continues.

   (a) What is the expected duration of the game?

   (b) What is the expected total payoff?

2. Winnie-the-Pooh starts at zero on the real line. Every minute he moves one step to the left with probability 0.2, one step to the right with probability 0.3 or skips. If he reaches points (-1) or (3) he stays there forever.

   (a) What is the expected duration of the game?

   (b) What is the probability that he will eventually rest at (-1)?

3. Alice throws a fair coin until the sequence HTH appears.

   (a) What is the expected duration of the game?

   (b) What is her expected payoff if she gets 5 roubles for each head and 3 roubles for each tail?

4. [For Fun] Consider three games:

Game A: You toss a biased coin with probability 0.48 of H. You get $+1$ dollar for H and -1 dollar for T.

Game B: If your wellfare is divisible by three you toss a coin that lands on H with probability 0.09. If your wellfare is not divisible by three you toss a coin that lands on H with probability 0.74. You get $+1$ dollar for H and -1 dollar for T.

Game C: You toss an unbiased coin. If it lands on H you play Game A. If it lands on T you play Game B.

Your initial capital is 10000\$.

   (a) Generate and plot two random trajectories of your wellfare if you play Game A 10^6 times.

   (b) Generate and plot two random trajectories of your wellfare if you play Game B 10^6 times.

   (c) Generate and plot two random trajectories of your wellfare if you play Game C 10^6 times.

Home assignment 3

Maybe useful for the midterm exam ;)

1. A monkey is randomly typing letters. She types a with probability 0.01, b — with probability 0.02 and c — with probability 0.03. Other probabilities are unknown and not relevant. She stops if starting exactly from the beginning she types one of the words: abc, $bacab$ or $abba$. Let A be the event that she stops.

   Consider the generating function $f(a, b, c) = abc+bacab+abba$. What is the combinatorial or probabilistic meaning of the following quantities:

   (a) $f(1, 1, 1)$;
   (b) $f(0.01, 0.02, 0.03)$;
   (c) $g'(1)$ if $g(t) = f(0.01t, 0.02t, 0.03t)$;
   (d) $g'(1)$ if $g(t) = f(0.01, 0.03, 0.03t)$;
   (e) $g'(1)/g(1)$ if $g(t) = f(0.01t, 0.02t, 0.03t)$;
   (f) $g'(1)/g(1)$ if $g(t) = f(0.01, 0.03, 0.03t)$;
   (g) $g'(0)$ if $g(t) = f(0.01 \exp(t), 0.02 \exp(t), 0.03 \exp(t))$.
   (h) $g'(0)/g(0)$ if $g(t) = f(0.01, 0.02, 0.03 \exp(t))$.

2. A monkey is randomly typing letters. She types a with probability 0.01, b — with probability 0.02 and c — with probability 0.03. Other probabilities are unknown and not relevant. She stops if starting exactly from the beginning she types one of the words: abc, $bacab$, $abba$, ... Let A be the event that she stops.

   You don't know the exact list of words but you can do any calculations with the generating function function $f(a, b, c) = abc + bacab + abba + \ldots$.

   How can you extract the following information:

   (a) the number of words;
   (b) the probability that monkey stops;
   (c) the expected payoff when she gets one dollar for every letter typed in the case of finite game;
   (d) the expected payoff when she gets one dollar for every letter c typed in the case of finite game;
   (e) the conditional expected payoff when she gets one dollar for every letter typed if it is known that the game ended in a finite number of steps;
   (f) the conditional expected payoff when she gets one dollar for every letter c typed if it is known that the game ended in a finite number of steps;
   (g) $\mathbb{E}(N_a N_b I(A))$ where N_a is the number of a letters, N_b — the number of b letters and $I(A)$ — the indicator of a finite game.
   (h) $\mathbb{E}(N_a(N_a - 1)N_b(N_b - 1)(N_b - 2)I(A))$ where N_a is the number of a letters, N_b — the number of b letters and $I(A)$ — the indicator of a finite game.
   (i) $\mathbb{E}(N_a^2 N_b^3 I(A))$ where N_a is the number of a letters, N_b — the number of b letters and $I(A)$ — the indicator of a finite game.

   Hint: in points (c) and (d) try to write the expression with and without exponent.

Home assignment 4

Enters in the quizzes during the week 5: 29 september — 4 october.

1. The moment generating function of a random variable X is $1/(1-2t)$.

   (a) Find the moment generating function of $2X$.

   (b) Find the moment generating function of $X+Y$ where X and Y are independent and identically distributed.

   (c) Do you remember the sum of geometric progression? Find $\mathbb{E}(X^{2025})$.

2. The MGF of the random variable W has a Taylor expansion that starts with $M(t) = 1+2t+7t^2+20t^3+\ldots$.

   (a) Find $\mathbb{E}(W)$, $\mathbb{V}\mathrm{ar}(W)$, $\mathbb{E}(W^3)$.

   (b) Find the starting terms of the Taylor expansion of the moment generating function for the sum $S = W + W^*$, where W and W^* are independent and identically distributed.

3. There are three problems in the home assignment. Time spent on each problem is modelled by independend exponentially distributed random variables with rate λ: X_1, X_2, X_3.

   (a) Find the moment generating function of X_i and hence the moment generating function of $S = X_1 + X_2 + X_3$.

   (b) Find $\mathbb{E}(S^3)$.

4. Let $W = \sum_{i=1}^{N} X_i$, where the X_i's are independent and identically distributed Bernoulli random variables with probability of success p. The random variable N has a binomial distribution with parameter n and probability of success p, and is independent of the X_i's.

   (a) Find the moment generating function for X_i.

   (b) Find the moment generating function for W given that N is fixed.

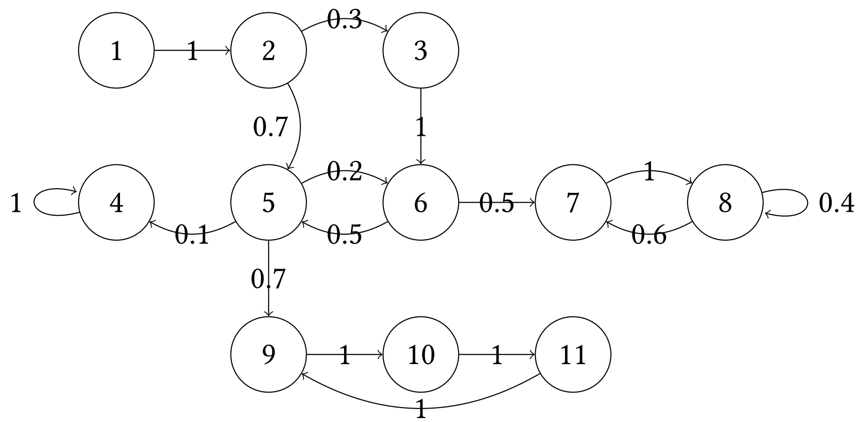
   (c) Find the moment generating function for W.

   (d) State the probability mass function of W.

Home assignment 5

Enters in the quizzes during the week 6: 6–11 october.

1. Consider the following Markov chain:



(a) Cut the Markov chain into communicating classes and classify them as transient, positive recurrent or null-recurrent.

(b) Is the chain irreducible?

(c) Find the period of every state.

Home assignment 6

Live in quizzes: 3–8 October.

1. Let X_n be a discrete time stochastic process that converges in probability to a random variable X as $n \to \infty$.

(a) Does this condition imply that X_n converges to X in mean? Almost surely? In distribution? Support your answers with a proof or counterexample.

(b) Give a definition for each type of convergence.

2. The random variables X_i are independent and exponentially distributed with rate $\lambda = 1$.

   (a) Find the probability limit
$$\operatorname{plim} \frac{X_1 + X_2 + X_3 + \cdots + X_n}{2n+7}.$$

   (b) Find the probability limit
$$\operatorname{plim} \frac{X_1^2 + X_2^2 + X_3^2 + \cdots + X_n^2}{2n+7}.$$

   (c) Find the probability limit
$$\operatorname{plim} \min\{X_1, X_2, X_3, \ldots, X_n\}.$$

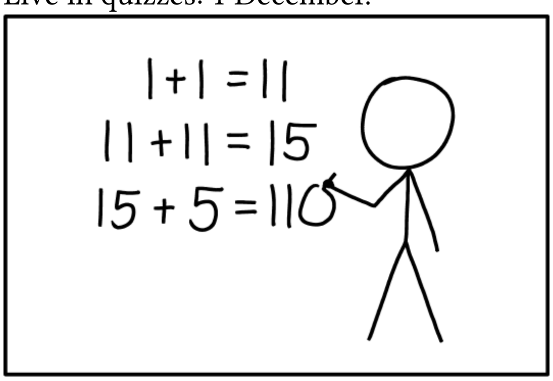
   (d) Find the probability limit
$$\operatorname{plim} \sqrt[n]{\exp(2X_1 + 2X_2 + \cdots + 2X_n)}.$$

3. The random variables X_1, X_2, X_3, ... have the cumulative distribution functions
$$F_n(x) = \begin{cases} 1 - (1 - 1/n)^{nx}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the limit of the sequence $(X_n)_{n=1}^{\infty}$ in distribution.

Home assignment 7

Live in quizzes: 1 December.



REMEMBER, ROMAN NUMERALS ARE ARCHAIC, SO ALWAYS REPLACE THEM WITH MODERN ONES WHEN DOING MATH.

1. Let $\Omega = \{a, b, c\}$ and the distribution of Y is given by the table

w	a	b	c
$Y(w)$	5	4	3
$\mathbb{P}(\{w\})$	0.2	0.3	0.5

   The sigma-algebra $\mathcal{F}$ is defined by $\mathcal{F} = \sigma(A)$ where $A = \{Y \neq 4\}$, and $\mathcal{H} = \sigma(Y)$.

(a) Find $\mathbb{E}(Y \mid \mathcal{F})$, $\mathbb{V}\text{ar}(Y \mid \mathcal{F})$, $\mathbb{P}(Y = 3 \mid \mathcal{F})$.

(b) Find $\mathbb{E}(Y \mid \mathcal{H})$, $\mathbb{V}\text{ar}(Y \mid \mathcal{H})$, $\mathbb{P}(Y = 3 \mid \mathcal{H})$.

2. A Hedgehog in the fog starts in $(0, 0)$ at $t = 0$ and moves randomly with equal probabilities in four directions (north, south, east, west) by one unit every minute.

   Let X_t and Y_t be his coordinates after t minutes and $S_t = X_t + Y_t$.

   (a) Find $\mathbb{E}(X_2 \mid S_2)$;

   (b) Find $\mathbb{V}\text{ar}(X_2 \mid S_2)$.

3. The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

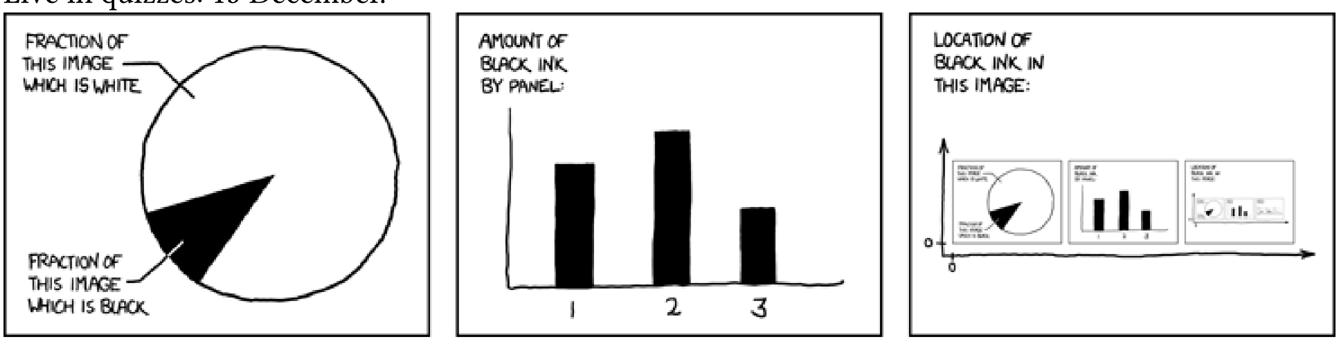
   (a) Explicitely list all the events in sigma-algebra $\sigma(X_1)$ and $\sigma(X_1 \cdot X_2)$.

   (b) Pavel says that he knows only whether X_1 and X_3 are equal. How will you describe his knowledge with sigma-algebra?

   (c) How many events are in the sigma-algebra $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3)$?

Last dance

Live in quizzes: 15 December.



1. Let (W_t) be a standard Wiener process.

   (a) Find $\mathbb{E}(W_5 W_6 \mid W_5)$, $\mathbb{V}\text{ar}(W_5 W_6 \mid W_5)$.

   (b) Find $\mathbb{E}(W_7^6)$.

   (c) Find $\mathbb{E}(W_3^2 W_5^2)$.

   (d) What is the distribution of $R = 5W_2 + 6W_7$?

   Hint: Stein lemma, Isserlis theorem.

2. Let (W_t) be a standard Wiener process. Consider the process Q_t:

$$Q_t = \begin{cases} 1, & \text{if } t < 2, \\ W_1^2, & \text{if } t \geq 2. \end{cases}$$

   (a) Find the integral $I = \int_0^5 Q_s \, dW_s$ in terms of values of (W_t).

   (b) Find $\mathbb{E}(I)$ and $\mathbb{V}\text{ar}(I)$.