1. [10] Consider an acrobat walking on the vertices of a triangle $\triangle ABC$. The acrobat is starting in vertex A and the flea is glued to vertex B. Each minute the acrobat moves to an adjacent vertex with the following transition probabilities: $\mathbb{P}(A \to B) = 0.4,\qquad \mathbb{P}(A \to C) = 0.6,\qquad \mathbb{P}(B \to C) = 0.2,\qquad \mathbb{P}(B \to A) = 0.5,\qquad \mathbb{P}(C \to B) = \mathbb{P}(C \to A) = 0.5.$

   (a) [2] Draw the graph of the Markov chain.

   (b) [2] Write down its transition matrix.

   (c) [3] Compute the probability that the acrobat steps on the flea exactly twice in three steps.

   (d) [3] Compute the expected number of steps needed for the acrobat to return to vertex A for the first time.

2. The creature cycles through internal behavioural states. At each minute it is in one of three states: Dormant (D), Active (A), or Feeding (F). The transition mechanism is governed by the following rules:

   - If the creature is Dormant (D), it becomes Active with probability 0.6 and begins Feeding with probability 0.4.
   - If the creature is Active (A), it remains Active with probability 0.5, becomes Dormant with probability 0.3.
   - If the creature is Feeding (F), it becomes Dormant with probability 0.4.

   (a) [2] Write the transition matrix P of the Markov chain.

   (b) [5] Find the stationary distribution of this chain.

   (c) [3] Using the stationary distribution find the average time between two Feeding states.

3. [10] The acrobat is making his way along the real line now. He begins at the point $x = 1$. Each minute he moves one step to the left with probability 0.3 or one step to the right with probability 0.7. This rule has two exceptions: the absorbing point $x = 0$ (where he is pulled in and **never** move again) and the reflecting barrier at $x = 3$, where a strange force always pushes him back with probability 1. Let T be the time to be pulled in.

   (a) [1] Draw the corresponding Markov chain.

   (b) [1] Write down the transition matrix.

   (c) [4] Find $\mathbb{E}(T)$.

   (d) [4] Find $\mathbb{V}\mathrm{ar}(T)$.

4. Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) [3] Split the chain in classes and classify them into closed or not closed.
   (b) [3] Classify the states into recurrent or transient.
   (c) [4] Find the period of each state.

5. [10] Let the moment generating function (MGF) of X is

$$M_X(t) = \frac{1}{1-2t}, \qquad t < \frac{1}{2}.$$

   (a) [5] Compute the mean and variance of X using the MGF.
   (b) [5] Let $S_n = X_1 + X_2 + \cdots + X_n$, where the X_i are independent and identically distributed. Find the MGF of S_n.

6. By the mayor's decree, the city installs the total mass of T_n New Year's trees each day, $n = 1, 2, \ldots$, starting from October 20th. The random variables $T_1, T_2, \ldots$ are independent and uniform on $[0; 4]$.

   The amounts of snow (W_n) are independent and exponentially distributed with rate $\lambda_n = n$.

   Determine the probability limits for the following sequences:

   (a) [2] mayor's New Year kpi, $X_n = \frac{1}{n}\sum_{i=1}^{n} T_i$
   (b) [3] the New Year's mood given by $Y_n = \frac{2X_n+1}{X_n+10}$
   (c) [5] The average number of snowmen that can be built each day: $S_n = \frac{1}{n}\sum_{i=1}^{n} W_i$.