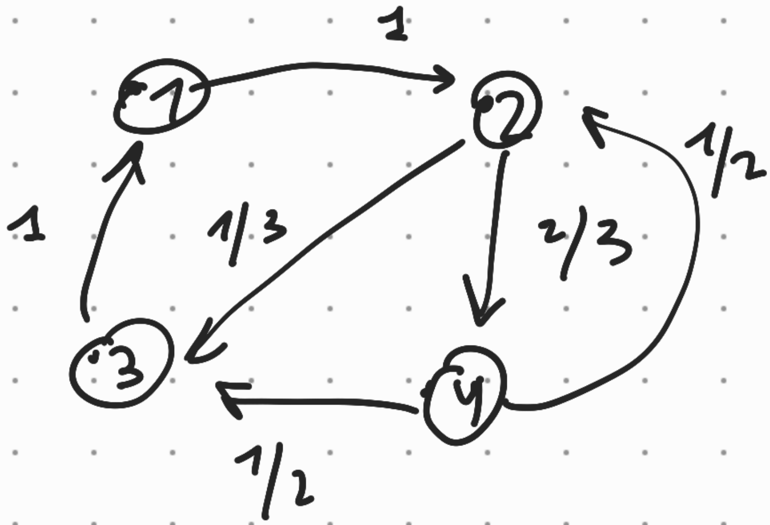
Task 1

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

1) Diagram:



2) Let m_{ij} - average num of steps to get from i to j:

first step analysis

$$\begin{cases} m_1 = m_{21} + 1 \\ m_{21} = \frac{1}{3} m_{31} + \frac{2}{3} m_{41} + 1 \\ m_{31} = 1 \\ m_{41} = \frac{1}{2} m_{21} + \frac{1}{2} m_{31} + 1 \end{cases} \qquad \begin{cases} m_{31} = 1 \\ m_{21} = \frac{4}{3} + \frac{2}{3} m_{4} \\ m_{41} = \frac{1}{2} m_{21} + \frac{3}{2} \\ m_1 = m_{21} + 1 \end{cases}$$

$$\begin{cases} m_{21} = \frac{4}{3} + \frac{2}{3}\left(\frac{1}{2} m_{21} + \frac{3}{2}\right) = \frac{4}{3} + \frac{1}{3} m_{21} + 1 \end{cases}$$

$m_1 = m_{21} + 1$

$\frac{2}{3} m_{21} = \frac{7}{3} \Rightarrow m_{21} = \frac{7}{2} \Rightarrow m_1 = \frac{9}{2}$

$\Rightarrow \pi_1 = \frac{2}{9}$

3) Stationarity :

$$\pi P = \pi : \begin{cases} \pi_1 = \pi_3 \\ \pi_2 = \pi_1 + \frac{1}{2}\pi_4 \\ \pi_3 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_4 \\ \pi_4 = \frac{2}{3}\pi_2 \\ \sum \pi_i = 1 \end{cases} \Rightarrow$$

$$\begin{cases} \pi_1 = \pi_3 \\ \pi_2 = \pi_1 + \frac{1}{3}\pi_2 \\ \pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_2 = \frac{2}{3}\pi_2 \\ 3\pi_1 + \frac{3}{2}\pi_1 = 1 \end{cases} \Rightarrow \begin{cases} \pi_3 = \pi_1 \\ \pi_2 = \frac{3}{2}\pi_3 = \frac{3}{2}\pi_1 \\ \pi_4 = \pi_1 \end{cases}$$

$\frac{9}{2}\pi_1 = 1 \qquad \pi_1 = \frac{2}{9}$

$$\boxed{\begin{cases} \pi_1 = \pi_3 = \pi_4 = \frac{2}{9} \\ \pi_2 = \frac{1}{3} \end{cases}}$$

4) This MC is irreducible, aperiodic and with finite m_{ij} $\Rightarrow$ $P_{ij}^{n} \to \pi_j$

So it is possible to say that $P^{200} \approx \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \dots \\ \vdots \\ \pi_1 & \pi_2 & \pi_3 & \dots \end{pmatrix}$

Are there any other ways to compute P^{200}?

mult. by hands

use $P^n = Q \Lambda^n Q^{-1}$

where Q is the matrix of eigenvectors of P

Λ is the diag. matrix of eigenvalues.

use generating functions

Let's create $G(t) = \underset{\underset{P^0}{\parallel}}{I} + tP + t^2P^2 + \dots t^nP^n + \dots$

we don't know each term but for small t there is a geometric progression so

$$G(t) = \frac{I}{I - tP} = (I - tP)^{-1}$$

$\Leftarrow$ can be computed for diff. t.

$$G(t) = \frac{1}{\det(I-tP)} \operatorname{Adj}(I-tP)$$

5) $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \Rightarrow$

$$\Rightarrow I - tP = \begin{bmatrix} 1-t/2 & -t/2 \\ -t/3 & 1-2t/3 \end{bmatrix}$$

$$(I-tP)^{-1} = \frac{1}{\left(1-\frac{t}{2}\right)\left(1-\frac{2t}{3}\right) - \frac{t}{2}\cdot\frac{t}{3}} \begin{bmatrix} 1-\frac{2t}{3} & \frac{t}{2} \\ \frac{t}{3} & 1-\frac{t}{2} \end{bmatrix}$$

$$\shortparallel$$

$$1 - \frac{t}{2} - \frac{2t}{3} + \frac{2t^2}{6} - \frac{t^2}{6} =$$

$$= 1 - \frac{7t}{6} + \frac{t^2}{6}$$

$$\frac{1}{\det} = 1 + \frac{7t}{6} + \frac{43}{36}t^2 + O(t^3)$$

$$\left(1 + \frac{7t}{6} + \frac{43t^2}{36}\right)\begin{pmatrix} 1-\frac{2t}{3} & \frac{t}{2} \\ \frac{t}{3} & 1-\frac{t}{2} \end{pmatrix} \rightarrow$$

only t^2: $\rightarrow$ $\begin{pmatrix} -\frac{14}{18}t^2 + \frac{43t^2}{36} & \frac{7t^2}{12} \\ \frac{7}{18}t^2 & \frac{43t^2}{36} - \frac{7t^2}{12} \end{pmatrix} \rightarrow$ $\begin{array}{r} 43 \\ -28 \\ \hline 15 \end{array}$

$$\Rightarrow P^2 t^2 = \begin{pmatrix} \frac{15}{36}t^2 & \frac{7}{12}t^2 \\ \frac{7}{18}t^2 & \frac{22}{36}t^2 \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \\ \frac{7}{18} & \frac{11}{18} \end{pmatrix} t^2$$

p^2 :)

Another use of Generation functions and Taylor series

Task 2 (no Markov chains)

a) $X \sim U\{1,2,3,4\}$

$$G(t) = e^{tX} = 1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \dots + \frac{t^kX^k}{k!} + \dots$$

$$\boxed{M(t) = E(G(t)) = E(e^{tX}) = 1 + E(X)\cdot t + E(X^2)\cdot\frac{t^2}{2!} + \dots}$$

$$M(t) = 1 + \underbrace{E(X)}\cdot t + \dots \underbrace{E(X^k)}\frac{t^k}{k!} + \dots$$

unknown coeff. are moments of distribution

so $M(t)$ is a moment generation function

$$M_X(t) = E(e^{tX}) = \frac{1}{4}e^{t\cdot 1} + \frac{1}{4}e^{2t} + \frac{1}{4}e^{3t} + \frac{1}{4}e^{4t}$$

$E(X^2)$? $\quad E(X) = \frac{1}{4}\cdot(1+2+3+4) = \frac{5}{2}$

But also $M_X(t) = \frac{1}{4}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \dots\right) + \frac{1}{4}\left(1 + 2t + \frac{4t^2}{2!} \dots +\right.$

$\left. + \frac{1}{4}\left(1 + 3t + \frac{9t^2}{2!} + \dots\right) + \frac{1}{4}\left(1 + 4t + \frac{16t^2}{2!}\right)\right)$

for $E(x^2)$ we need $\underbrace{E(x^2)}\frac{t^2}{2!} = \frac{1}{4}(1+4+9+16) =$

$= \frac{14}{4} + 4 = 7{,}5$

$E(x^2) = Var(x) + E^2(x) = \frac{1}{4}\left[\left(1-\frac{5}{2}\right)^2 + \left(2-\frac{5}{2}\right)^2 + \left(3-\frac{5}{2}\right)^2 + \left(4-\frac{5}{2}\right)^2\right] + \left(\frac{5}{2}\right)^2 =$

$= \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) + \left(\frac{5}{2}\right)^2 = \frac{10}{8} + \frac{25}{4} = \frac{30}{4} = 7{,}5$

b)