1. [10] I have two pockets: left and right. I also have two indistinguishable coins. Initially they are both in the left pocket. Each moment of time I randomly select one of my pockets. If it is empty I do nothing. It it is not empty I move a coin from the selected pocket to another one.

   Consider the Markov chain where the state is the location of the two coins.

   (a) [5] Draw the diagram and find the transition matrix.

   (b) [2] Classify the states.

   (c) [3] Which proportion of my eternal life the coins are split in both pockets?

2. [10] The random variable X, Y and Z are independent and normally distributed. Consider the sigma-algebras

   $$\mathcal{F}_1 = \sigma(X, Y), \quad \mathcal{F}_2 = \sigma(Y, Z), \quad \mathcal{F}_3 = \sigma(X+Z, Y), \quad \mathcal{F}_4 = \sigma(X+Y, X-Y), \quad \mathcal{F}_5 = \sigma(X, Y, X+Y).$$

   (a) [4] For each sigma-algebra provide two examples of non-trivial (different from $\emptyset$ and Ω) events that belong to it.

   (b) [3] Which of the sigma-algebras are always equal?

   (c) [3] Which sigma-algebra is always a subset of another one?

3. [10] The random variables (X_k) are independent and uniform on $[0; 2]$ and $Y = X_1 + 2X_2 + \cdots + 5X_5 + 10$.

   (a) [4] Find the moment generating function of X_1.

   (b) [3] Find the moment generating function of Y.

   (c) [3] Find $\mathbb{V}\mathrm{ar}(Y \mid X_2)$.

4. [10] Gleb Zheglov catches one criminal every day. With probability 0.2 the catched criminal is replaced by 2 new criminals. Initially there is 1 criminal in the town.

   Let T be the day of the ultimate crime eradication in the town.

   (a) [4] Find $\mathbb{E}(T)$.

   (b) [6] Find $\mathbb{V}\mathrm{ar}(T)$.

5. [10] The random varible (X_k) are independend and uniform on $[0; 1]$. Let $Y_n = X_1 \cdot X_2 \cdot \cdots \cdot X_k$.

   (a) [5] Does (X_n) converge in probability? In distribution? Explain.

   (b) [5] Does (Y_n) converge in probability? In distribution? Explain.

6. [10] The random variables (X_n) are independent and they have exponential distribution with rate $\lambda = 2$. Consider the cumulative sum $S_n = X_1 + X_2 + \cdots + X_n$ with $S_0 = 0$ and the natural filtration $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

   (a) [4] Is $\mathcal{F} = \mathcal{F}_9 \backslash \mathcal{F}_7$ a sigma-algebra? Why?

   (b) [6] Find all constants a and b such that $M_n = S_n + a + b \cdot n$ is a martingale.

   Hint: if $R \sim \mathrm{Expo}(\lambda)$ then $\mathbb{E}(R) = 1/\lambda$ and $\mathbb{V}\mathrm{ar}(R) = 1/\lambda^2$.