Moment generating function

Task 1 How to calculate probabilities?

X_1, X_2 iid $U\{1,2,3,4\}$ $\quad$ EX_1 – ?

$Y = X_1 + X_2$ $\quad$ EX_1^2 – ?

$\mathbb{P}(Y=5)$ – ? $\quad$ EX_1^k – ? $\quad$ $E(Y)$ – ?

a) $EX_1 = \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 3 + \frac{1}{4}\cdot 4 = \frac{10}{4} = 2{,}5$

$EX_1^2 = \frac{1}{4}(1+4+9+16) = \frac{30}{4} = 7{,}5$

$EX_1^k = \frac{1}{4}(1+2^k+3^k+4^k)$

b) we can find $\mathbb{P}(X_1+X_2=5) = \frac{N_5}{N_{all}}$

$N_5 = 4$ $\quad \left\{\begin{matrix} 1+4, & 4+1 \\ 2+3, & 3+2 \end{matrix}\right\}$ $\quad \Rightarrow \mathbb{P}(Y=5) = \frac{4}{16} = \frac{1}{4}$

$N_{all} = 4\cdot 4 = 16$

c) $E(Y) = E(X_1+X_2) = 2EX_1 = 5.$

d) we can use generating function!

$$G_X(t) = e^{tX} = 1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \dots + \frac{t^kX^k}{k!} + $$

$$\boxed{M_X(t) = Ee^{tX} = 1 + E(X)\cdot\frac{t}{1!} + EX^2\frac{t^2}{2!} + \dots EX^k\cdot\frac{t^k}{k!}}$$

X – random variable

moment generating function — t-parameter

$E(X_1^k)$ can be obtained by hands:

- $Ee^{tX} = \underset{(x=1)}{\frac{1}{4}e^{t}} + \underset{(x=2)}{\frac{1}{4}e^{2t}} + \underset{(x=3)}{\frac{1}{4}e^{3t}} + \underset{(x=4)}{\frac{1}{4}e^{4t}} =$

$$= \frac{1}{4}\left(e^{t} + e^{2t} + e^{3t} + e^{4t}\right) =$$

$$= \frac{1}{4}\Big(1 + t + \frac{t^2}{2!} + \dots \frac{t^k}{k!} \dots$$
$$+ 1 + 2t + \frac{4t^2}{2!} + \dots \frac{2^k t^k}{k!} + \dots$$
$$+ 1 + 3t + \frac{9t^2}{2!} + \frac{3^k t^k}{k!} + \dots$$
$$+ 1 + 4t + \frac{16t^2}{2!} + \frac{4^k t^k}{k!} + \dots \Big)$$

$$E(X_1) = \frac{1}{4}(1+2+3+4)$$

$$E(X_1^k) = \frac{1}{4}(1 + 2^k + 3^k + 4^k)$$

- $M_Y(t) = Ee^{tY} = Ee^{t(X_1+X_2)} = E\left[e^{tX_1}e^{tX_2}\right] \ominus$

Theorem Moment generating function uniquely determines the distribution

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$$\Leftrightarrow \; E e^{tX_1} \cdot E e^{tX_2} = \left(E e^{tX_1}\right)^2 = M_X^2(t)$$

X_1, X_2 independ.

$\Rightarrow$

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theorem If $X_1, X_2 \ldots X_n$ are independent then $M_{X_1+X_2+\ldots X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$

$$\Rightarrow \; M_Y(t) = M_X^2(t) = \left(E e^{tX}\right)^2 =$$

$$= \left(\tfrac{1}{4} e^{t} + \tfrac{1}{4} e^{2t} + \tfrac{1}{4} e^{3t} + \tfrac{1}{4} e^{4t}\right)^2 =$$

$$= \frac{1}{16}\left(\underline{e^{2t}} + \underline{e^{4t}} + e^{6t} + e^{8t} + 2\underline{e^{3t}} + 2\underline{e^{4t}} + 2e^{5t} + 2e^{5t} + 2e^{6t} + 2e^{7t} \right) =$$

$$= \frac{1}{16} e^{2t} + \frac{1}{8} e^{3t} + \frac{3}{16} e^{4t} + \underbrace{\frac{1}{4} e^{5t}}_{\substack{\rotatebox{90}{$=$} \\ \mathbb{P}(Y=5)\cdot e^{5t}}} + \frac{3}{16} e^{6t} + \frac{1}{8} e^{7t} + \frac{1}{16} e^{8t}$$

$\Rightarrow$

$$\Rightarrow \mathbb{P}(Y=5) = \frac{1}{4}$$

- Where is $E(Y)$ there??

$$E e^{tY} = 1 + EY \cdot t + EY^2 \cdot \frac{t^2}{2!} + EY^3 \cdot \frac{t^3}{3!} + \ldots$$

$$\Rightarrow M_Y(t) = \quad + \frac{1}{\,} \cdot 2t + \frac{1}{\,} \cdot 3t + \frac{1}{\,} \cdot 5t + \ldots$$

$\Rightarrow \mathbb{E}Y = \frac{2}{16} + \frac{3}{8} + \frac{12}{16} + \frac{5}{4} + \frac{18}{16} + \frac{7}{8} + \frac{8}{16} =$

$= \frac{2+6+12+20+18+14+8}{16} = \frac{80}{16} = 5$

$EY = 5$

Task2 Doesn't it easier to calculate moments by hands?

$X \sim \mathcal{N}(0,1)$ $\quad E(X) - ?$ $\quad E(X^4) - ?$

$E(X^2) - ?$

a) by hands: $\quad E(X^3) - ?$

$E(X) = 0, \quad EX^2 = Var(X) - E^2X = 1 - 0 = 1$

$$\boxed{E(g(X)) = \int_{R_X} g(x) f(x)\, dx}$$

pdf(x)

Eeww:(

$$E(X^4) = \int_{-\infty}^{+\infty} x^4 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx =$$

we don't want to calculate it because even $E(X) = \int^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

is very complicated if we want to be accurate :(

$E(X^3) = 0$ because of the symmetry of normal distribution: $P(X=a) = P(X=-a)$

With the probability density function of the normal distribution, this reads:

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} x \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \mathrm{d}x \,.
\end{aligned} \tag{4}$$

Substituting $z = x - \mu$, we have:

$$\begin{aligned}
\mathrm{E}(X) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty-\mu}^{+\infty-\mu} (z+\mu) \cdot \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right] \mathrm{d}(z+\mu) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} (z+\mu) \cdot \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right] \mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^{+\infty} z \cdot \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right] \mathrm{d}z + \mu \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^2\right] \mathrm{d}z \right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(\int_{-\infty}^{+\infty} z \cdot \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right] \mathrm{d}z + \mu \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right] \mathrm{d}z \right) .
\end{aligned} \tag{5}$$

The general antiderivatives are

$$\begin{aligned}
\int x \cdot \exp\left[-ax^2\right] \mathrm{d}x &= -\frac{1}{2a} \cdot \exp\left[-ax^2\right] \\
\int \exp\left[-ax^2\right] \mathrm{d}x &= \frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot \mathrm{erf}\left[\sqrt{a}x\right]
\end{aligned} \tag{6}$$

where $\mathrm{erf}(x)$ is the error function. Using this, the integrals can be calculated as:

$$\begin{aligned}
\mathrm{E}(X) &= \frac{1}{\sqrt{2\pi}\sigma} \left(\left[-\sigma^2 \cdot \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]\right]_{-\infty}^{+\infty} + \mu \left[\sqrt{\frac{\pi}{2}}\sigma \cdot \mathrm{erf}\left[\frac{1}{\sqrt{2}\sigma} z\right]\right]_{-\infty}^{+\infty} \right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left(\left[\lim_{z\to\infty}\left(-\sigma^2 \cdot \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]\right) - \lim_{z\to-\infty}\left(-\sigma^2 \cdot \exp\left[-\frac{1}{2\sigma^2} \cdot z^2\right]\right)\right] \right. \\
&\quad \left. + \mu \left[\lim_{z\to\infty}\left(\sqrt{\frac{\pi}{2}}\sigma \cdot \mathrm{erf}\left[\frac{1}{\sqrt{2}\sigma} z\right]\right) - \lim_{z\to-\infty}\left(\sqrt{\frac{\pi}{2}}\sigma \cdot \mathrm{erf}\left[\frac{1}{\sqrt{2}\sigma} z\right]\right)\right] \right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left([0-0] + \mu \left[\sqrt{\frac{\pi}{2}}\sigma - \left(-\sqrt{\frac{\pi}{2}}\sigma\right)\right] \right) \\
&= \frac{1}{\sqrt{2\pi}\sigma} \cdot \mu \cdot 2\sqrt{\frac{\pi}{2}}\sigma \\
&= \mu \,.
\end{aligned} \tag{7}$$

https://statproofbook.github.io/P/norm-mean.html

So what if you don't remember moments for normal distribution?

Another way

only for normal distrib.

Stein's lemma

$$\boxed{E\left[g(x)\cdot(x-\mu)\right] = \sigma^2 E\left(g'(x)\right)}$$

$$Ex^3 = E\left[x^2\cdot(x-0)\right] = \underbrace{1}_{=\sigma^2}\cdot E(2x) = 2E(x) = 0$$

$$Ex^4 = E\left[x^3(x-0)\right] = 1\cdot E(3x^2) = 3Ex^2 = 3$$

b) by MGF:

- $M_x(t) = E(e^{tx}) = \int_{-\infty}^{+\infty} e^{tx}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx =$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x^2-2tx)}{2}}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x^2-2tx+t^2)}{2}} e^{t^2/2}dx =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}} e^{t^2/2}dx = e^{t^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}}dx$$

$$= e^{t^2/2}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}dx}_{\text{what is it? pdf } = 1} = e^{t^2/2}$$

- $\boxed{M_x(t) = e^{t^2/2} \text{ for } x \sim N(0,1)}$

is known and can be used with no proofs

- $M_x(t) = 1 + \frac{t^2}{2} + \underbrace{\frac{t^4}{2^2\cdot 2!}}_{=\frac{t^4}{8}} + \frac{t^6}{2^3\cdot 3!} + \dots + \frac{t^{2k}}{2^k\cdot k!} =$

$$= 1 + 0 \cdot t + 1 \cdot \frac{t^2}{2!} + 0 \cdot \frac{t^3}{3!} + 3 \frac{t^4}{4!} + \dots$$

$$\underset{Ex}{\parallel} \quad \underset{Ex^2}{\parallel} \quad \underset{Ex^3}{\parallel} \quad \underset{Ex^4}{\parallel} \quad \underset{24}{\parallel}$$

$\Rightarrow Ex^3 = 0$

$Ex^4 = 3$

c)

Theorem: If $M_x(t)$ is a moment gen. func

then $Ex^k = \frac{d^k}{(dt)^k} M_x(t) \Big|_{t=0}$

$$M_x(t) = e^{t^2/2}$$

$$\frac{d}{dt} M_x(t) = t + \frac{t^3}{2} + \frac{t^5}{8} + \dots$$

$$\Rightarrow EX = \frac{d}{dt} M_x(t) \Big|_{t=0} = 0$$

$$\frac{d^2}{dt^2} M_x(t) = 1 + \frac{3}{2} t^2 + \frac{5}{8} t^4 + \dots$$

$$EX^2 = \Big|_{t=0} = 1$$

$$\frac{d^3}{dt^3} M_x(t) = 3t + \frac{5}{2} t^3 + \dots$$

$\frac{d^k}{dt^k}$ kills all smaller moments

$\Big|_{t=0}$ kills all bigger moments

• $EX^3 = \Big|_{t=0} = 0$

$$\frac{d^4}{dt^4} M_X(t) = 3 + \frac{15}{2} t + \dots$$

• $EX^4 = \Big|_{t=0} = 3$

or • $EX = \frac{d}{dt} M_X(t) \Big|_{t=0} = \left(e^{t^2/2}\right)'_t \Big|_{t=0} =$

$= t e^{t^2/2} \Big|_{t=0} = 0$

• $EX^2 = \left(t e^{t^2/2}\right)'_t \Big|_{t=0} = e^{t^2/2} + t^2 e^{t^2/2} \Big|_{t=0} =$

$= 1 + 0 = 1$

• $EX^3 = \left(e^{t^2/2} + t^2 e^{t^2/2}\right)'_t \Big|_{t=0} =$

$= t e^{t^2/2} + 2t e^{t^2/2} + t^3 e^{t^2/2} \Big|_{t=0} =$

$= 0$

• $EX^4 = \Big(e^{t^2/2} + t^2 e^{t^2/2} + 2 e^{t^2/2} + 2t^2 e^{t^2/2} +$

$+ 3t^2 e^{t^2/2} + t^4 e^{t^2/2} \Big)\Big|_{t=0} = 1 + 2 = 3$

$t=0$

Task 3 $X \sim \text{Pois}(\lambda) \Rightarrow \mathbb{P}(X) = e^{-\lambda}\dfrac{\lambda^x}{x!}$

λ — intensity $x = 0, 1, 2, 3 \ldots$

$EX = \lambda$

- by definition

$E(X) = 0 \cdot \mathbb{P}(X=0) + 1 \cdot \mathbb{P}(X=1) + 2 \cdot \mathbb{P}(X=2) + \ldots$

$= 1 \cdot e^{-\lambda}\dfrac{\lambda}{1!} + 2e^{-\lambda}\dfrac{\lambda^2}{2!} + \ldots ke^{-\lambda}\dfrac{\lambda^k}{k!} + \ldots$

$= e^{-\lambda} \cdot \lambda \left(1 + \dfrac{2\lambda}{2!} + \dfrac{3\lambda^2}{3!} + \ldots\right) =$

$$= e^{-\lambda} \cdot \lambda \underbrace{\left(1 + \lambda + \dfrac{\lambda^2}{2!} + \ldots \dfrac{\lambda^k}{k!} + \ldots\right)}_{e^{\lambda}} =$$

(for small λ around 0)

$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$

- by MGF

$M_X(t) = E\,e^{tX}$ … $e^{t\cdot 0}$ …

$$M_X(t) = \mathbb{E}\, e^{tX} = e^{0t} \cdot \mathbb{P}(X=0) + e^{t} \cdot \mathbb{P}(X=1) +$$

$$+ e^{2t} \cdot \mathbb{P}(X=2) + \dots =$$

$$= e^{-\lambda} \frac{\lambda^0}{0!} + e^{-\lambda} \frac{\lambda}{1!} e^{t} + e^{-\lambda} \frac{\lambda^2}{2!} \cdot e^{2t} + e^{-\lambda} \frac{\lambda^3}{3!} \cdot e^{3t} \dots$$

$$= e^{-\lambda} + \lambda e^{-\lambda} e^{t} + \frac{\lambda^2}{2!} e^{2t} e^{-\lambda} + \frac{\lambda^3}{3!} e^{3t} e^{-\lambda} + \dots =$$

$$= e^{-\lambda} \left(1 + \lambda e^{t} + \frac{\lambda^2}{2!} e^{2t} + \frac{\lambda^3}{3!} e^{3t} + \dots \right) =$$

$$= e^{-\lambda} \left(\underbrace{1 + \lambda e^{t} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \dots}_{e^{\lambda e^t} \text{ for small } \lambda, t} \right) =$$

$$= e^{-\lambda} e^{\lambda e^t} = \boxed{e^{\lambda (e^t - 1)}}$$

$$E(X) = \frac{d}{dt} e^{\lambda(e^t - 1)} \Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)} \Big|_{t=0} =$$

$$= \lambda e^0 e^{\lambda(e^0 - 1)} = \lambda$$