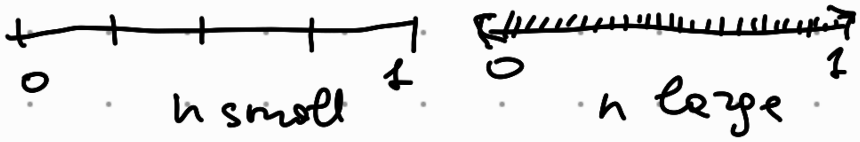
Poisson distribution and process

- Poisson as aprox. of Binomial

$$\mathbb{P}(X=k) = C_k^n \underset{\nearrow \text{suc.}}{p^k} (1-p)^{n-k} = C_k^n p^k q^{n-k}$$

$$(p+q)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!} q^{n-2} p^2 + \dots + p^n = 1.$$

- n - large
- p - small
- $np = \lambda$ - constant



$$(p+q)^n \overset{p\to 0}{\simeq} 1 + np + \frac{n(n-1)}{2!} p^2 + \dots + p^n$$

$$(p+q)^n \overset{n\to\infty}{\approx} 1 + np + \frac{n^2}{2!} p^2 + \frac{n^3}{3!} p^3 + \dots$$

$$(p+q)^n \overset{np=\lambda}{\approx} 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \approx e^{\lambda}$$

$$\frac{(p+q)^n}{e^{\lambda}} \approx 1 \approx e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2!} + \dots + e^{-\lambda}\frac{\lambda^k}{k!} + \dots$$

$$X \sim \mathrm{Pois}(\lambda) \quad \boxed{\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

Task 1 $\quad n=3, \; p=0{,}1$

a) Bin: $\mathbb{P}(X=2) = C_2^3 \, 0{,}1^2 \, 0{,}9 = \frac{3\cdot 2}{2!} 0{,}1^2 \cdot 0{,}9 =$

$= 0{,}027 \approx 2{,}7\%$

b) $Pois(np=0{,}3)$: $\mathbb{P}(X=2) = e^{-0{,}3}\frac{(0{,}3)^2}{2!} = 0{,}033 \approx 3{,}3\%$

c)

d) $n=40$ Bin $\mathbb{P}(X=2) = C_2^{40}\, 0{,}01^2\, 0{,}99^{38} =$

$p = 0{,}01$ $= \frac{40\cdot 39}{2!}\, 0{,}01^2 \cdot 0{,}99^{38} = 0{,}0532 \approx 5{,}3\%$

$Pois(np=0{,}4)$ Pois: $\mathbb{P}(X=2) = e^{-0{,}4}\frac{0{,}4^2}{2!} = 5{,}4\%$

Task 2 $n=100$ $\mathbb{P}(X=0)$ – ?

$p=0{,}005$ $\mathbb{P}(X\geq 2)$ – ?

a) Bin: $\mathbb{P}(X=0)$ – ?

$Pois(np=\lambda=0{,}5)$: $\mathbb{P}(X=0) = e^{-0{,}5}\frac{0{,}5^0}{0!} = e^{-0{,}5} = 0{,}61 \approx$
$\approx 60\%$

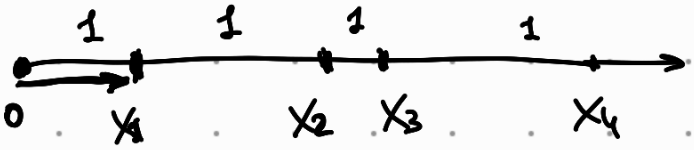
b) $\mathbb{P}(X\geq 2) = \mathbb{P}(X=2) + \mathbb{P}(X=3) + \ldots \mathbb{P}(X=100)$

$\mathbb{P}(X\geq 2) = 1 - \mathbb{P}(X<2) = 1 - \mathbb{P}(X=1) - \mathbb{P}(X=0)$

$$\left[\begin{array}{l} \mathbb{P}(X=0) = 61\% \\ \mathbb{P}(X=1) = e^{-0{,}5}\frac{0{,}5^1}{1!} = \frac{e^{-0{,}5}}{2} \approx 30\% \end{array}\right]$$

$\mathbb{P}(X\geq 2) = 100\% - 61\% - 30\% = 9\%$

Poisson process: X_t



1) $X_0 = 0$

2) X_t has independent increments

3) $X_{t+\tau} - X_t \sim Pois(\lambda\tau)$

Task 3

$\lambda = 3$ tasks per sem

$N_1 \sim Pois\left(\frac{\lambda}{2}\right) \neq Pois\left(\frac{3}{2}\right)$

$N_2 \sim Pois\left(\frac{3}{2}\right)$

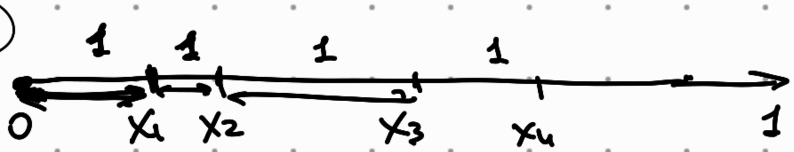
a) $\mathbb{P}(N_1 = 1) = e^{-3/2}\frac{\left(\frac{3}{2}\right)^1}{1!} = 0{,}33 \approx 33\%$

$\mathbb{P}(N_2 = 2) = e^{-3/2}\frac{\left(\frac{3}{2}\right)^2}{2!} = 25\%$

b) $\mathbb{P}(N_1 = 1, N_2 = 2) \overset{②}{=} \mathbb{P}(N_1 = 1)\cdot\mathbb{P}(N_2 = 2) \overset{③}{=} 0{,}33\cdot 0{,}25 =$

$= 0{,}08 = 8\%$

Task 4



X_t - pois process

$X_i \to i^{th}$ bulb burn

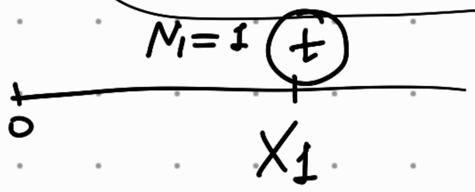
1) $X_0 = 0$

2) $X_1 - X_0$, $X_2 - X_1$, $X_3 - X_2$ .... $\Leftarrow$ independent

3) $X_{t_1} - X_0 \sim Pois(\lambda t_1)$

$X_{t_2} - X_{t_1} \sim Pois(\lambda(t_2 - t_1))$ ...

$\measuredangle$ X_{t_1} 1st bulp was burned at t $k = 1$



$N_1 \sim Pois(\lambda t)$

is the same event

$\mathbb{P}(N_1 = 1) = e^{-\lambda t}\frac{(\lambda t)^1}{1!} = e^{-\lambda t}\cdot\lambda t$

$\mathbb{P}(X_1 = t) = e^{-\lambda t}\cdot\lambda t$

1st bulp was not burned at t

$\mathbb{P}(N_1 = 0) = e^{-\lambda t}\cdot\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$

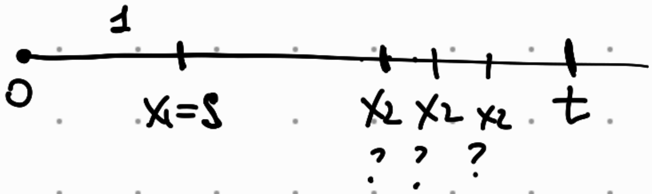
$\mathbb{P}(X_1 > t) = e^{-\lambda t}$

1st bulp was

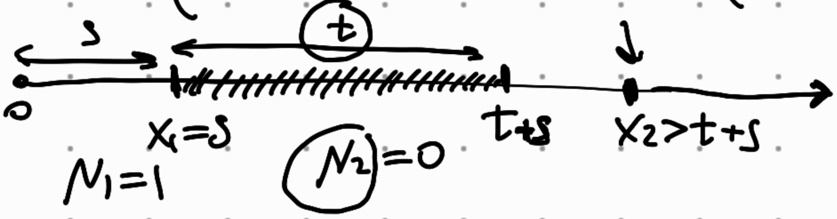
Burned by time t $\quad \mathbb{P}(X_1 \le t) = 1 - \mathbb{P}(X_1 > t) =$

$$= 1 - e^{-\lambda t} \sim \exp(\lambda t)$$

b) $\mathbb{P}(X_2 \le t \mid X_1 = S) = ?$



$\to \mathbb{P}(X_2 > t \mid X_1 = S) = \mathbb{P}\left(0 \text{ suc. on } (S; t+S] \;\middle|\; \begin{matrix} 1 \text{ suc} \\ \text{on } (0, S+t] \end{matrix}\right) \ominus$



N_1 and N_2 independent (by ② from def)

$\ominus$ $\mathbb{P}(0 \text{ suc on } (S, t+S]) = \left[\begin{matrix} \text{by ③ from def} \\ N_2 \sim \text{Pois}(\lambda t) \end{matrix}\right]$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^0}{0!} = e^{-\lambda t}$$

$\Rightarrow \mathbb{P}(X_2 > t \mid X_1 = S) = e^{-\lambda t} \to$

$\Rightarrow \mathbb{P}(X_2 \le t \mid X_1 = S) = 1 - e^{-\lambda t} \leftarrow$ no dependence on S

$$\exp(\lambda t)$$

$\vdots$

$\mathbb{P}(X_k \le t \mid X_1, X_2 \ldots X_{k-1}) = 1 - e^{-\lambda t}$

$X_k \sim \exp(\lambda t)$