Markov chain structure

Sometimes it is easier to consider the Markov chain as a group of <u>classes</u>, not states:

$i \to j$ means we can get from state i to state j

$i \leftrightarrow j$ we can get from i to j and back (maybe indirectly)

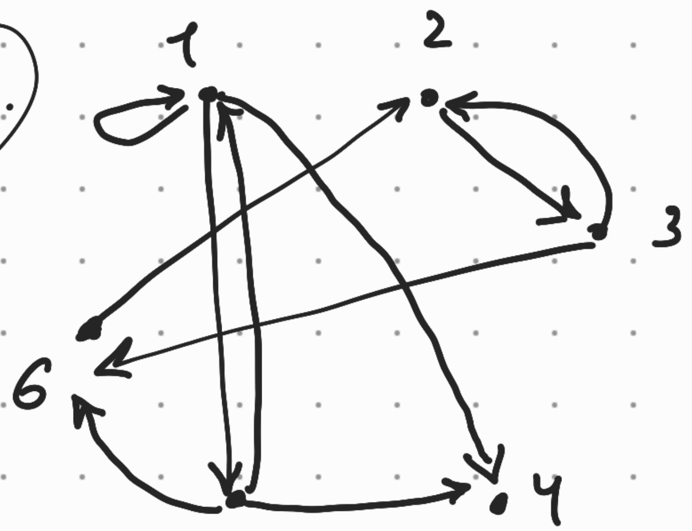
$i \leftrightarrow j \approx$ equivalent $\longrightarrow$ from one <u>class</u>

classes

closed → $\forall i \in C,\ i \to j \Rightarrow j \in C$ (no escape)

open → $\exists\, i \in C, j \notin C : i \to j$ (escape)

Ex 1.



How many classes can you find?

5

$1 \to 4$, $4 \not\to 1$

$1 \to 5$, $5 \to 1$ $\Rightarrow$ $A = \{1, 5\}$ ← open

no ways to get to 1 and 5 from other states!

$2 \leftrightarrow 3$, $2 \leftrightarrow 6$, $3 \leftrightarrow 6$ $\Rightarrow$ $B = \{2, 3, 6\}$

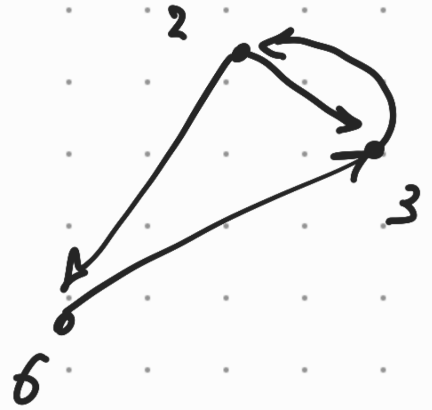
↳ closed

(no escape from B)

$C = \{4\}$ → closed

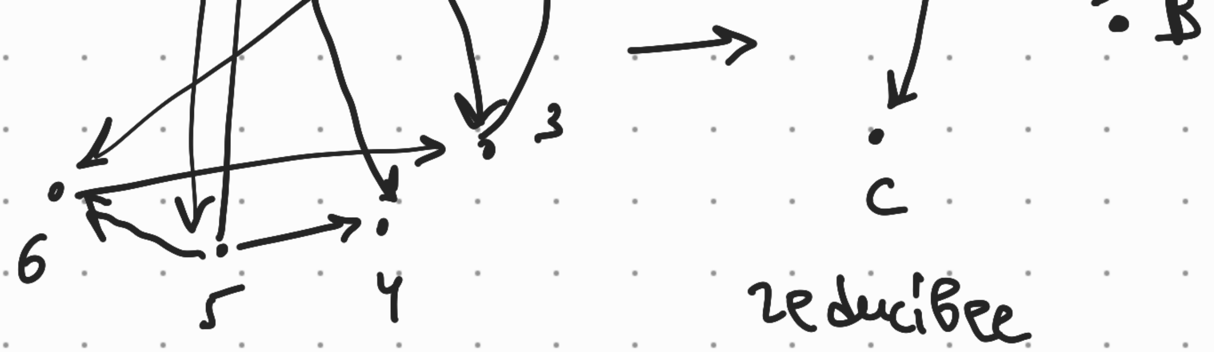
4 – <u>absorbing</u> state – $\{4\}$ is a closed class (no escape from the state)

<u>irreducible</u> chain is a Markov chain with a single state



→ irreducible

A



reducible

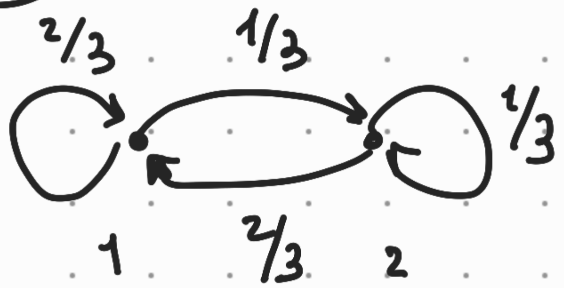
classes are very important in infinitivly large games!

Another important feature is

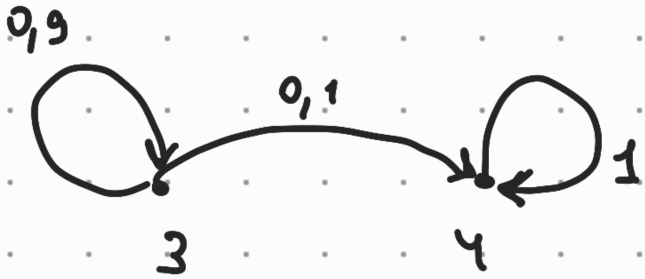
<u>recurrence</u> and <u>transience</u> of the state

State i is recurrent if $\mathbb{P}_i(\text{num. of visits to } i = \infty) = 1$

Ex.2



1, 2 – recurrent states



3 – transient

4 – recurrent (and absorbing)

transient ~ not recurrent, you can leave it forever

<u>Theorem 1</u>

a) state i is recurrent $\Longleftrightarrow$ $\mathbb{P}_i$(num. of steps to return to $i < \infty) = f_i = 1$.

$$\iff \sum_{n=0}^{\infty} p_{ii}^{n} = \infty$$

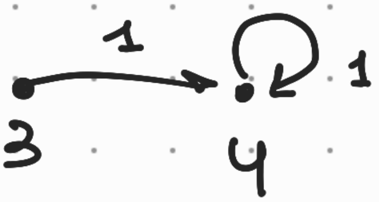
b) i is transinet $\iff f_i < 1, \sum_{n=0}^{\infty} p_{ii}^{n} < \infty$

1 and 2 states : yes

3 : is it possible not to return? yes

4 is it possible not to return? no

So in infinitively large game:



Theorem 2 Let C be a communicating class.

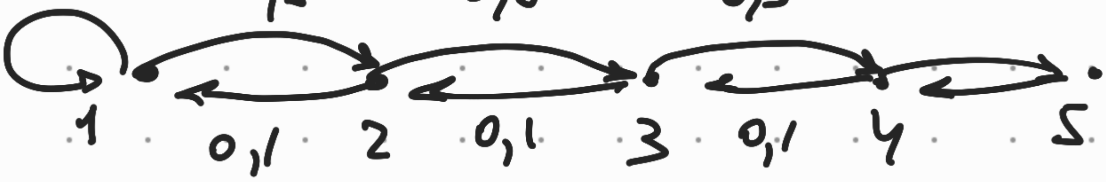
- Then either all states in C are transient or all are recurrent.
- Every recurrent class is closed
- Every finite closed class is recurrent.

Ex3

why finite

0,1 0.9 0.9 0.9

is important?



- probably, I will never return to

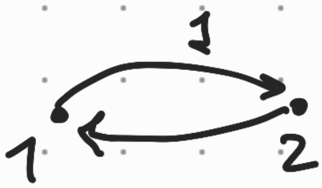
1: $1 \to 2 \to 3 \to 4 \cdots$ $\Rightarrow \mathbb{P}_1($time to ret. to 1 state $< \infty) < 1$ $\Rightarrow$ 1 is transient. (Theorem 1a)

State i is aperiodic if $p_{ii}^{n} > 0$ for large n, otherwise i is periodic

Theorem 3 A state is aperiodic if there is no common divisor for $n_1, n_2 \ldots : p_{ii}^{n_j} > 0$

Ex. 4

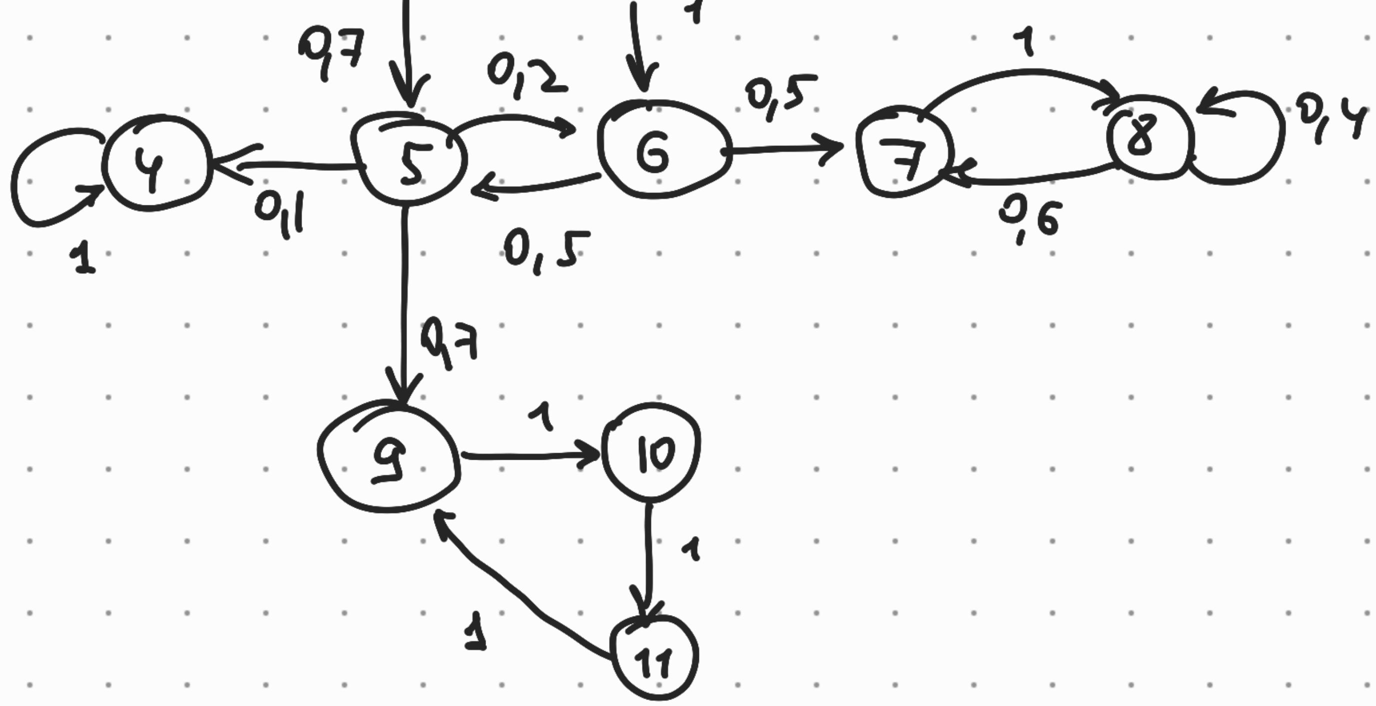
$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



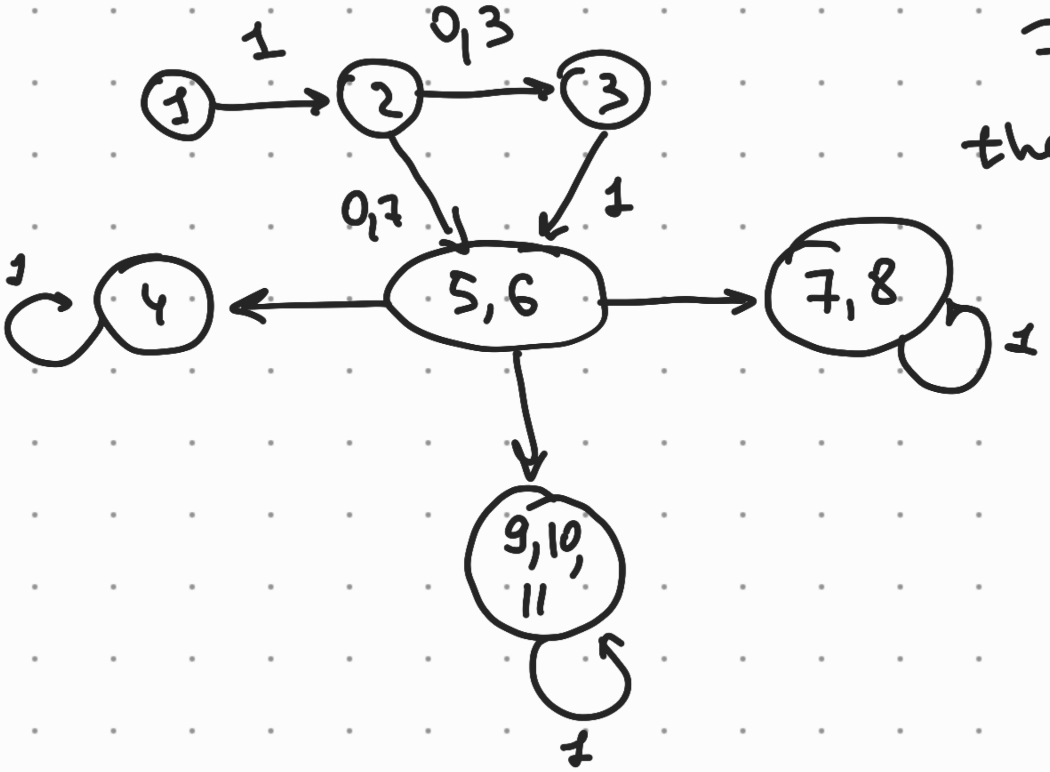
period = 2

$$P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad P^3 = P \quad \boxed{P^{2n} = I, \ P^{2n+1} = P}$$

Task 1



1) Find all classes. Is the chain irreducible?



7 classes, the chain is NOT irreducible

2) Find transient \ recurrent \ absorbing states

transient : 1, 2, 3 & 5, 6

absorbing : 4

recurrent: 7,8   8   9,10,11   8 4

3) which recurr. states are periodic and aperiodic ?

- period of 4 : $p_{11}^{1} = p_{11}^{2} = p_{11}^{3} = \ldots = 1 > 0$

1,2,3,4... no common divisor (Theorem 3)

- periods of 7,8 $\begin{bmatrix} 0 & 1 \\ 0{,}6 & 0{,}4 \end{bmatrix} = P$

$$\begin{bmatrix} 0 & 1 \\ 0{,}6 & 0{,}4 \end{bmatrix}^2 = \begin{bmatrix} 0{,}6 & 0{,}4 \\ 0{,}24 & 0{,}16 \end{bmatrix}$$

$n = 1,2,3,4\ldots$

no common divisors (Theorem 3)

$P^3 = \begin{bmatrix} 0{,}6 & \cdots \\ \cdots & \end{bmatrix}$ no zeros ever

$\Rightarrow$ 4,7,8 aperiodic

- periods of 9,10,11 $\quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

not zeroes + + $P^3 = I$

9: 3
10 : 3 $\rightarrow$ periodic
11 : 3

(theorem 3)

$p^4 = I \cdot P = P$ ← period 3 for 9, 10, 11

4) Will your answer change if

9, 10, 11 $\quad P = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ z & 0 & 0 \end{bmatrix}$ - ?

no, because the zero positions will be the same!

Why do we need all these features?

Stationary distribution!

$\exists\ T_i = \inf\{n \geq 1, X_n = i\}$ first passage of i

$m_i = E(T_i)$ average return time

$V_i(n)$ - num. # visits to i before n

$V_i^k = V_i(T_k)$ - num of visits to i before return to k.

invariant distribution π :

- $\forall i \in I \quad \pi_i \geq 0$
- $\sum \pi_i = 1$

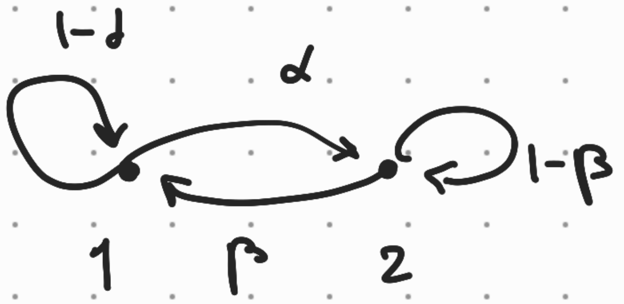
$$\cdot \quad \pi P = \pi$$

Sometimes we can say that

$$m_i = \frac{1}{\pi_i}, \quad \mathbb{P}\left(\frac{V_j(n)}{n} \xrightarrow[n\to\infty]{} \pi_j\right) = 1.$$

Ex 5

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$



How can we find invariant distribution:

1) Left hand equation:

$$\pi = \pi P \Rightarrow (\pi_1 \ \pi_2) = (\pi_1 \ \pi_2)\begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\begin{cases} \pi_1 = (1-\alpha)\pi_1 + \beta\pi_2 \\ \pi_2 = \alpha\pi_1 + (1-\beta)\pi_2 \end{cases} \Rightarrow \begin{cases} \alpha\pi_1 = \beta\pi_2 \\ \beta\pi_2 = \alpha\pi_1 \end{cases} \Rightarrow$$

$$\Rightarrow \pi_1 = \frac{\alpha}{\beta}\pi_2 \quad + \quad \pi_1 + \pi_2 = 1 \quad \Rightarrow$$

$$\Rightarrow \begin{cases} \pi_1 = \frac{\beta}{\alpha}\pi_2 \\ \pi_1 = 1 - \pi_2 \end{cases} \qquad 1 = \frac{\alpha+\beta}{\alpha}\pi_2 \Rightarrow$$

$\pi_1 = 1 - \pi_2$

$$\Rightarrow (\pi_1 \ \pi_2) = \left(\frac{\beta}{\alpha+\beta} \ \ \frac{\alpha}{\alpha+\beta} \right)$$

2) $$P^n = \begin{bmatrix} p_{11}^n & p_{12}^n \\ p_{21}^n & p_{22}^n \end{bmatrix} \xrightarrow[n\to\infty]{} \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{bmatrix}$$

$$\Rightarrow P^\infty = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

Invariant distributions can not be unique!

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \forall \pi : \pi P = \pi \ !$$

Theorem 4

a) Let P be the transition matrix of an irreducible, $\boxed{\text{aperiodic}}$ Markov chain with finite avarage num. of steps to return to state i (m_i) then $\forall i j$: $p_{ij}^n \xrightarrow[n\to\infty]{} \pi_j$

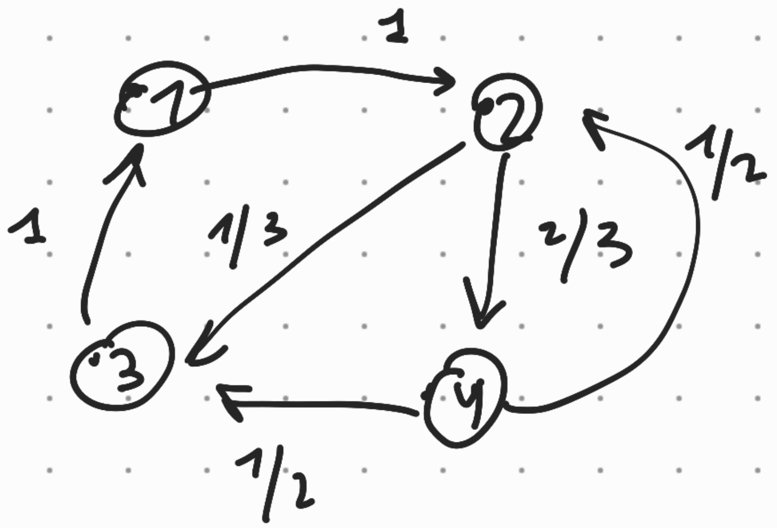
b) for $\boxed{\text{periodic}}$ $P\left(\frac{V_i(n)}{n} \to \frac{1}{m_i}\right) = 1$ will be true

$n\to\infty$

Task 2

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

1) Diagram:



2) Which classes can you find?

$1 \leftrightarrow 2,\ 2 \leftrightarrow 3,\ 3 \leftrightarrow 4 \rightarrow \{1,2,3,4\}$

Is the chain irreducible? yes!

3) Recurrent, transient, absorbing?

Recurrent $\downarrow$ yes

transient $\downarrow$ no

absorbing $\downarrow$ NO

4) What are the periods? (I computed P^N :))

$$P^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 1/3 & 1/6 & 1/6 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/6 & 1/6 \end{pmatrix} \quad P^3 = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/9 & 2/9 \\ 0 & 0 & 1/3 & 2/3 \\ 1/6 & 2/3 & 1/6 & 0 \end{pmatrix}$$

1: $n = 3, 4 \ldots$

2: $n = 2, 3, 4 \ldots$

3: $n = 3, 4 \ldots$

4: $n = 2, 4, 5 \ldots$

P^4 (... zeroes) P^5 (no zeroes)

$P = \begin{pmatrix} \text{no zeroes} \\ \text{on diag.} \end{pmatrix}$ $P^2 = \begin{pmatrix} \text{no zeros} \\ \cdots \end{pmatrix}$

for all states $i = 1,2,3,4$ for $n \to$ no common divisor $\Rightarrow$ Theorem 3 $\Rightarrow$ aperiodic

5) Irreducible, aperiodic. $\Rightarrow$ Theorem 4a

$$\begin{cases} m_1 = m_{21} + 1 \\ m_{21} = \frac{1}{3} m_{31} + \frac{2}{3} m_{41} + 1 \\ m_{31} = 1 \\ m_{41} = \frac{1}{2} m_{21} + \frac{1}{2} m_{31} + 1 \end{cases} \qquad \begin{cases} m_{31} = 1 \\ m_{21} = \frac{4}{3} + \frac{2}{3} m_{41} \\ m_{41} = \frac{1}{2} m_{21} + \frac{3}{2} \\ m_1 = m_{21} + 1 \end{cases}$$

$$\begin{cases} m_{21} = \frac{4}{3} + \frac{2}{3}\left(\frac{1}{2} m_{21} + \frac{3}{2}\right) = \frac{4}{3} + \frac{1}{3} m_{21} + 1 \\ m_1 = m_{21} + 1 \end{cases}$$

$$\frac{2}{3} m_{21} = \frac{7}{3} \Rightarrow m_{21} = \frac{7}{2} \Rightarrow m_1 = \frac{9}{2}$$

Theorem 4a $\Rightarrow$ $\Rightarrow \pi_1 = \frac{2}{9}$?

at least finite

c) Stationarity:

$\pi P = \pi$ $\begin{cases} \pi_1 = \pi_3 \\ \pi_2 = \pi_1 + \frac{1}{2}\pi_4 \end{cases}$

$$\begin{cases} \pi_3 = \frac{1}{3}\pi_2 + \frac{1}{2}\pi_4 \\ \pi_4 = \frac{2}{3}\pi_2 \\ \sum \pi_i = 1 \end{cases} \Rightarrow$$

$$\begin{cases} \pi_1 = \pi_3 \\ \pi_2 = \pi_1 + \frac{1}{3}\pi_2 \\ \pi_3 = \frac{1}{3}\pi_2 + \frac{1}{3}\pi_2 = \frac{2}{3}\pi_2 \\ 3\pi_1 + \frac{3}{2}\pi_1 = 1 \end{cases} \Rightarrow \begin{cases} \pi_3 = \pi_1 \\ \pi_2 = \frac{3}{2}\pi_3 = \frac{3}{2}\pi_1 \\ \pi_4 = \pi_1 \end{cases}$$

$$\frac{9}{2}\pi_1 = 1 \qquad \pi_1 = \frac{2}{9} \Rightarrow$$

$$\boxed{\begin{cases} \pi_1 = \pi_3 = \pi_4 = \frac{2}{9} \\ \pi_2 = \frac{1}{3} \end{cases}}$$