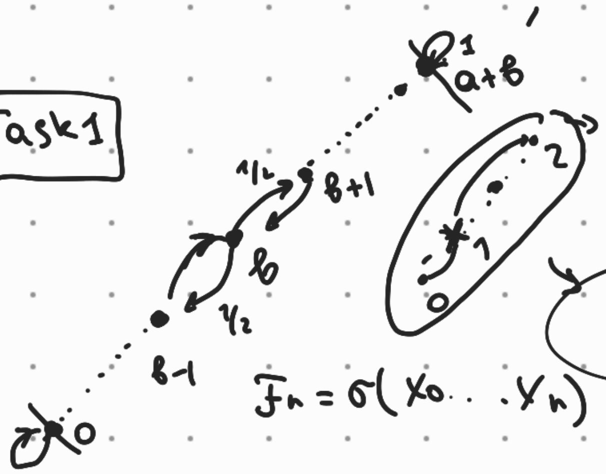
- Filtration: sub-σ-algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_2$ ..
- natural filtration $\mathcal{F}_n = \sigma(X_0, X_1, \dots X_n)$
- X_n $\mathcal{F}_n$-adopted $(\mathcal{F}_n = \sigma(w_0, w_1, w_2 \dots w_n))$

  $X_n = f(w_0, w_1, \dots w_n)$

- X_n - martingal (with respect to $\mathcal{F}_n$)

  1) X_n is adopted
  2) $E|X_n| < \infty$
  3) $E(X_n \mid \mathcal{F}_{n-1}) = X_{n-1}$

Task 1



$0, a, a+b$ - states in markov chain

$P_{b,a+b}$ -? $\quad P_{b,0}$ -?

$\mathcal{F}_n = \sigma(X_0 \dots X_n)$ $\quad X_0 = 0 \to S_0 = 0$

- $S_n = \sum_{i=0}^{n} X_i = X_0 + X_1 + \dots X_n = f(X_0, \dots X_n)$

  + adopted.

$E|S_n| \leq n < \infty$

$E(S_n \mid \mathcal{F}_{n-1}) = S_{n-1}$???

$E(S_n \mid X_0, \dots X_{n-1}) = E(\underbrace{X_0 + \dots X_{n-1}} + X_n \mid X_0 \dots X_{n-1}) =$

$= X_0 + \dots X_{n-1} + \underbrace{E(X_n \mid X_0, \dots X_{n-1})} = S_{n-1} + \underset{\substack{\parallel \\ 0}}{E(X_n)} = S_{n-1}$

$1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$

- Doob's theorem (optional-stopping theorem)

  $\forall Y_n$ - martingal, $\forall t \geq 0$ $\quad E(Y_t) = Y_0$

$S_0 \to S_1 = 1 \to S_2 = 2$

$S_1 = 1 \to S_2 = 0$

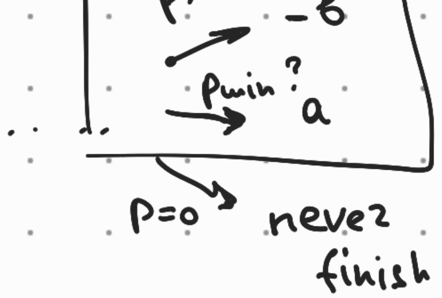
$S_0 \to S_1 = -1 \to S_2 = 0$

$S_1 = -1 \to S_2 = -2$

$\underset{\substack{\parallel \\ 0}}{S_0}$ $\quad E(S_1) = 0 \quad E(S_2) = 0$

Please?

- $P_{never\ finish} = 0$



Doob's

$\exists T$ - finishing of the game $\to$ $E(S_T) = S_0 = 0$

$$\begin{cases} -b \cdot P_{loose} + a \cdot P_{win} = 0 \\ P_{loose} + P_{win} = 1 \end{cases}$$

$$\begin{cases} P_{loose} = \frac{a}{b} P_{win} \\ \frac{a+b}{b} \cdot P_{win} = 1 \end{cases} \Rightarrow \begin{cases} P_{win} = \frac{b}{a+b} \\ P_{loose} = \frac{a}{a+b} \end{cases}$$

$a \to \infty \Rightarrow P_{win} \to 0$.

- average time of the game $\longrightarrow$ $E(T)$

we need a martingale

$M_n = S_n^2 - n$

$$E(M_n \mid \mathcal{F}_{n-1}) = M_{n-1} = \underline{S_{n-1}^2 - (n-1)}$$

$$E(S_n^2 - n \mid X_0 \dots X_{n-1}) = E\left((S_{n-1} + X_n)^2 - n \mid X_0 \dots X_{n-1}\right) =$$

$$= E(S_{n-1}^2 + \underline{2 S_{n-1} X_n} + X_n^2 - n \mid X_0 \dots X_{n-1}) =$$

$$= S_{n-1}^2 + 2 S_{n-1} \underbrace{E(X_n \mid X_0 \dots X_{n-1})}_{=0} + \underbrace{E(X_n^2 \mid X_0 \dots X_{n-1})}_{=\frac{1}{2} \cdot 1^2 + \frac{1}{2}(-1)^2 = 1} - n =$$

$$= S_{n-1}^2 + 0 + 1 - n = \underline{S_{n-1}^2 - (n-1)} \to \text{martingale}$$

Doob's theorem $E(M_T) = M_0 = S_0^2 - 0 = 0$

$E(M_T) = 0$

$$E(S_T^2 - T) = E(S_T^2) - E(T) = 0 \quad \Rightarrow E(T) = E(S_T^2) \circleddash$$

$S_T \xrightarrow{P_{loose}} -b$, $S_T \xrightarrow{P_{win}} a$

$S_T^2 \xrightarrow{P_{loose}} b^2$, $S_T^2 \xrightarrow{P_{win}} a^2$

$\circleddash$ $b^2 \cdot P_{loose} + a^2 P_{win} = E(T)$

Из предыдущ. пункта

$E(T) = b^2 \cdot \frac{a}{\dots} + a^2 \cdot \frac{b}{\dots} =$

$$= \frac{ab(a+b)}{a+b} = ab$$

$\boxed{\text{Task2}}$ $\mathbb{P}(X_i = 1) = p$ $\quad \mathbb{P}(X_i = -1) = q$ $\quad p+q=1$, $p \neq q$

- $S_n = X_0 + \dots X_n$

$\underline{E(S_n \mid \mathcal{F}_{n-1})} = E(S_{n-1} + X_n \mid \mathcal{F}_{n-1}) = S_{n-1} +$

$+ E(X_n \mid \mathcal{F}_{n-1}) = S_{n-1} + E(X_n) = S_{n-1} + p \cdot 1 + q \cdot (-1) =$

$= S_{n-1} + (p-q) \underset{p \neq q}{\neq} \underline{S_{n-1}}$ not a martingale

$E(S_n \mid \mathcal{F}_{n-1}) \underset{p>q}{>} S_{n-1}$ sup | super martingale

$\underset{p<q}{<} S_{n-1}$

- $K_n = \left(\frac{q}{p}\right)^{S_n}$ $\quad E(K_n \mid \mathcal{F}_{n-1}) = K_{n-1} = \left(\frac{q}{p}\right)^{S_{n-1}}$?

$E(K_n \mid \mathcal{F}_{n-1}) = E\left(\underbrace{\left(\frac{q}{p}\right)^{S_{n-1}}} \cdot \left(\frac{q}{p}\right)^{X_n} \mid \mathcal{F}_{n-1}\right) = \left(\frac{q}{p}\right)^{S_{n-1}} \times$

$\times E\left(\left(\frac{q}{p}\right)^{X_n} \mid \mathcal{F}_{n-1}\right) = \left(\frac{q}{p}\right)^{S_{n-1}} E\left(\left(\frac{q}{p}\right)^{X_n}\right) = \left(\frac{q}{p}\right)^{S_{n-1}} \left(\frac{q}{p}^{1} p + \left(\frac{q}{p}\right)^{-1} q\right) =$

$= \left(\frac{q}{p}\right)^{S_{n-1}} \underset{\substack{\parallel \\ 1}}{(q+p)} = \left(\frac{q}{p}\right)^{S_{n-1}} \Rightarrow K_n$ - martingale

- Doob's theorem : $E(K_T) = K_0 = \left(\frac{q}{p}\right)^{S_0} = 1$

$E(K_T) = 1$

$P_w \nearrow \left(\frac{q}{p}\right)^{S_T | w} = \left(\frac{q}{p}\right)^{a}$

$P_{loos} \searrow \left(\frac{q}{p}\right)^{S_T | l} = \left(\frac{q}{p}\right)^{-b}$

$$\Rightarrow \begin{cases} \left(\frac{q}{p}\right)^{a} \mathbb{P}_{win} + \left(\frac{q}{p}\right)^{-b} \mathbb{P}_{loose} = 1 \\ \mathbb{P}_{win} + \mathbb{P}_{loose} = 1 \end{cases}$$

$\mathbb{P}win = 1 - \mathbb{P}\,loose$

$$\left(\frac{q}{p}\right)^{a} \cdot (1 - \mathbb{P}\,loose) + \left(\frac{q}{p}\right)^{-b} \cdot \mathbb{P}\,loose = 1$$

$$\mathbb{P}\,loose \left(\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^{a} \right) = 1 - \left(\frac{q}{p}\right)^{a} \Rightarrow \boxed{P_{loose} = \frac{1-\left(\frac{q}{p}\right)^{a}}{\left(\frac{q}{p}\right)^{-b} - \left(\frac{q}{p}\right)^{a}} \quad P_{win} = \frac{1-\left(\frac{q}{p}\right)^{-b}}{\left(\frac{q}{p}\right)^{a} - \left(\frac{q}{p}\right)^{-b}}}$$

- $E(T) \to$ average time to finish the game

Момент номоrо: $L_n = S_n - (p-q)n$ why? because

$$E(S_n - (p-q)n \mid \mathcal{F}_{n-1}) = S_{n-1} - (p-q)(n-1)$$

$$E(S_n - (p-q)n \mid \mathcal{F}_{n-1}) = E(S_{n-1} + X_n - (p-q)n \mid \mathcal{F}_{n-1}) =$$

$$= S_{n-1} - (p-q)n + \underline{\underline{E(X_n \mid \mathcal{F}_{n-1})}} = S_{n-1} - (p-q)n + \underline{\underline{p-q}} =$$

$$= S_{n-1} - (p-q)(n-1) \quad \to \text{martingale}$$

- Doob's theorem $E(L_T) = L_0 = S_0 - (p-q)\cdot 0 = 0$

$$E(S_T - (p-q)T) = 0 \Rightarrow ES_T = (p-q)\cdot E(T)$$

$$E(T) = \frac{E(S_T)}{p-q} = \frac{\left(\frac{q}{p}\right)^{a} \cdot P_{win}(p,q,a,b) + \left(\frac{q}{p}\right)^{-b} P_{loose}(p,q,a,b)}{p-q}$$

and we can substitute p_{win} and p_{loose} with $\square$ finded in prev. point

$\boxed{\text{Task 3}}$

$$X_n \begin{cases} \nearrow -1 & p = \frac{1}{2} \\ \searrow 1 & p = \frac{1}{2} \end{cases} \qquad S_n = \sum X_n$$

$$Z_n = S_n^3 + a_n S_n \qquad a_n : Z_n - \text{a martingale?}$$

$E(\underline{Z_{n+1}} \mid \mathcal{F}_n) = Z_n$ -?

$X_1, X_2, \ldots\ldots X_n$

$S_{n+1} = \underbrace{X_1 + \ldots + X_{n+1}}$

$\mathcal{F}_n$

$$E\Big(\underbrace{(S_n + X_{n+1})}_{S_{n+1}}{}^3 + a_{n+1}(S_n + X_{n+1}) \,\Big|\, \mathcal{F}_n \Big) =$$

$$= E\Big(\underline{S_n^3} + X_{n+1}^3 + 3 S_n X_{n+1}^2 + 3 S_n^2 X_{n+1} + \underline{a_{n+1} \cdot S_n} + a_{n+1} X_{n+1} \,\Big|\, \mathcal{F}_n \Big)$$

$$= S_n^3 + \underbrace{E(X_{n+1}^3 \mid \cancel{\mathcal{F}_n})}_{0} + 3 S_n \overbrace{E(X_{n+1}^2 \mid \mathcal{F}_n)}^{1} + 3 S_n^2 \underbrace{E(X_{n+1} \mid \cancel{\mathcal{F}_n})}_{0} +$$

$$+ a_{n+1} \cdot S_n + a_{n+1} \underbrace{E(X_{n+1} \mid \cancel{\mathcal{F}_n})}_{0} =$$

$$= S_n^3 + 3 S_n + a_{n+1} S_n = S_n^3 + a_n \cdot S_n$$

$$S_n^3 + (3 + a_{n+1}) S_n = S_n^3 + a_n S_n$$

$$\underline{3 + a_{n+1} = a_n}$$

$a_0 = 0$ $\qquad a_{n+1} = a_n - 3$

$a_1 = -3$ $\qquad Z_1 = S_1^3 - 3 S_1$

$a_2 = -6$ $\rightarrow$ $Z_2 = S_2^3 - 6 S_2$

etc