Convergence

Useful inequalities:

- Markov's inequality : X – nonneg. rand. var.

$$\forall t>0 \quad \mathbb{P}(X \geq t) \leq \frac{EX}{t}$$

- Chebyshev's inequality X - rand. var.

$$\forall t>0 \quad \mathbb{P}(|X - E(X)| \geq t) \leq \frac{\operatorname{Var} X}{t^2}$$

- Triangle inequality

$$|a+b| \leq |a| + |b|$$

almost surely convergence

$$X_n \xrightarrow{a.s.} X$$

$\Downarrow$

$$\mathbb{P}(\{\omega: X_n(\omega) \to X(\omega)\}) = 1$$

- in probability $\qquad X_n \xrightarrow{P} X$

$\Downarrow$

$$\boxed{\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0} \quad \left(\operatorname*{plim}_{n\to\infty} X_n = X\right)$$

- in distribution $\qquad X_n \xrightarrow{d} X$

$$\lim_{n\to\infty} F_n(x) = F_X(x)$$

Task Θ Weak law of large numbers (in prob.)

2 steps: 1) find a limit
2) prove by definition

$X_1, X_2, \ldots, X_N \sim \underline{iid}$

$EX_i = \mu$

$\operatorname{Var} X_i = \sigma^2$

$\overline{X}_1 = X_1$

$\overline{X}_2 = \frac{X_1 + X_2}{2}$

$\vdots$

$\overline{X}_N = \frac{X_1 + X_2 + \ldots X_N}{N}$

1) $\overline{X}_n \xrightarrow{P} EX_i = \mu$

2) $\lim_{n\to\infty} \mathbb{P}(|\overline{X}_n - \mu| > \varepsilon) \overset{\leq}{=}$

$$\leq \lim_{n\to\infty} \frac{\operatorname{Var} \overline{X}_n}{\varepsilon^2} = \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$

$$\mathrm{Var}\,\overline{X_n} = \mathrm{Var}\Big(\frac{1}{n}\sum_{i=1} X_i\Big) = \frac{1}{n^2}\mathrm{Var}\Big(\sum X_i\Big) \underset{\text{(indep)}}{=}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\,X_i = \frac{1}{n^2}\cdot n\sigma^2 = \boxed{\frac{\sigma^2}{n}}$$

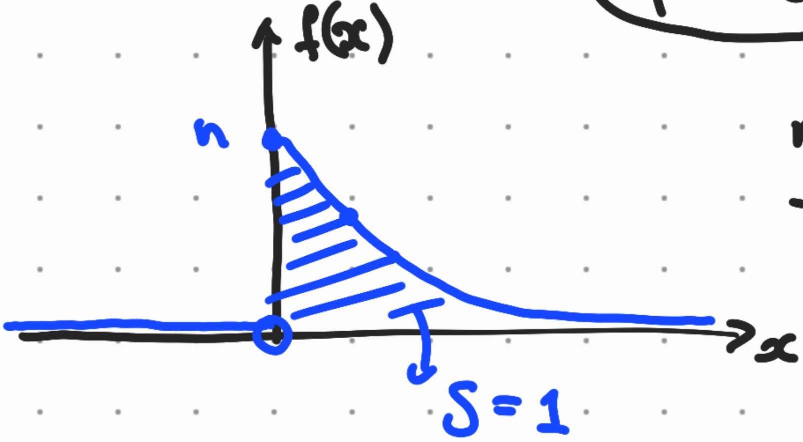
$$\lim_{n\to\infty} \mathbb{P}\big(|\overline{X_n}-\mu| > \varepsilon\big) = 0 \quad \leftarrow \text{доказали}$$

Task 4 $X_n \sim \mathcal{E}xp(n)$

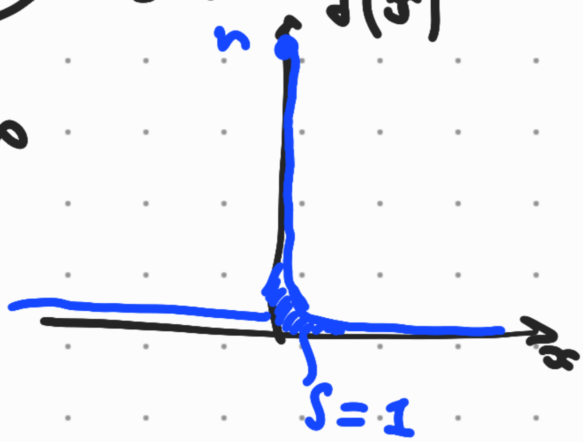
1) find a limit
2) prove it

a) pdf $f_{X_n}(x) = \begin{cases} n\,e^{-nx} & x \geq 0 \\ 0 & x < 0 \end{cases}$

cdf $F_{X_n}(x) = \begin{cases} 1 - e^{-nx} & x \geq 0 \\ 0 & x < 0 \end{cases}$



$\xrightarrow{n\to\infty}$



δ) Let's show that 0 is the limit:

By definition $\forall \varepsilon > 0$ $\lim\limits_{n\to\infty} \mathbb{P}\big(|X_n - 0| > \varepsilon\big) \overset{?}{=} 0$

$$= \lim_{n\to\infty} \mathbb{P}(X_n > \varepsilon) =$$

т.к. $\mathbb{P}(X_n < 0) = 0$ из ф. распр.

$\lim \mathbb{P}(|0-0| > \varepsilon) = 0$ z.mg.

$$\boxed{F_{X_n}(x) = \mathbb{P}(X_n \le x)}$$

$$\Leftrightarrow \lim_{n\to\infty}\left(1 - \mathbb{P}(X_n \le \varepsilon)\right) = \lim_{n\to\infty}\left(1 - (\underbrace{1 - e^{-n\varepsilon}}_{F_{X_n}(\varepsilon)})\right) =$$

$$= \lim_{n\to\infty} e^{-n\varepsilon} = 0$$

$$\boxed{X_n \xrightarrow{P} 0}$$

Task 5 $X_n = X + Y_n$

random variable

Y_n: $EY_n = \frac{1}{n}$

$Var(Y_n) = \frac{\sigma^2}{n}$

1) let's prove that $X_n \xrightarrow{P} X$

2) $\forall \varepsilon > 0$ $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

$\mathbb{P}(|Y_n| > \varepsilon)$ $\quad \mathbb{P}$

- Chebyshev's: $\mathbb{P}(|B-\mathbb{E}(B)|>\varepsilon)\leq$

but we $\leq \dfrac{Var(B)}{\varepsilon^2}$

need $E(Y_n)$ in brackets

so triangle ineq.

- $|Y_n - EY_n + EY_n| < |Y_n - EY_n| + |EY_n|$

$\quad\quad |Y_n| < |Y_n - \frac{1}{n}| + \frac{1}{n}$ no yea.

$$\begin{cases} |Y_n - \frac{1}{n}| > |Y_n| - \frac{1}{n} > \varepsilon - \frac{1}{n} \\ + \quad \text{def. in prob.} \quad |Y_n| > \varepsilon \end{cases}$$

$$\mathbb{P}(|Y_n|>\varepsilon) = \mathbb{P}\left(|Y_n - \tfrac{1}{n}| > \varepsilon - \tfrac{1}{n}\right) \overset{\text{no}}{\leq}$$

Chebyshev's ineq

$$\leq \frac{Var\, Y_n}{(\varepsilon-\frac{1}{n})^2} = \frac{\sigma^2/n}{(\varepsilon-\frac{1}{n})^2} =$$

$$= \frac{\sigma^2}{n(\varepsilon^2+\frac{1}{n^2}-\frac{2\varepsilon}{n})} = \frac{\sigma^2}{n\varepsilon^2+\frac{1}{n}-2\varepsilon}$$

$$\lim_{n\to\infty} \frac{\sigma^2/n}{(\varepsilon-\frac{1}{n})^2} = 0 \implies \lim_{n\to\infty} \mathbb{P}(|Y_n|>0)=0$$

Task 6 $\quad (X_n) \sim \text{iid } U[0,1] \to$ uniform distrib

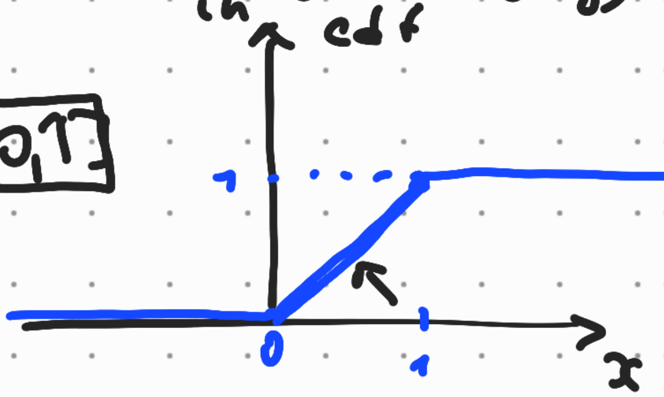
$Y_n = \min\{X_1 \dots X_n\}$

$Y_n \xrightarrow{p,d} 0$

- in distribution : $\lim F_{Y_n}(y) = F_0(y)$

$$F_{X_n}(x) = \begin{cases} 0 & x<0 \\ x & \boxed{x\in[0,1]} \\ 1 & x>1 \end{cases}$$

$\mathbb{P}(X_n \le 0{,}01) = 0{,}01$



$$F_{Y_n}(y) = \mathbb{P}(Y_n \le y) = \mathbb{P}(\min\{X_1 \dots X_n\} \le y) =$$

$$= 1 - \mathbb{P}(\min\{X_1 \dots X_n\} > y) \;=$$

$\min\{X_i\} > y \Rightarrow$

$\Rightarrow \forall i : \; x_i > y$

$$= 1 - \mathbb{P}(X_1 > y, X_2 > y \dots X_n > y) \overset{\text{indep}}{=}$$

$$= 1 - \prod_{i=1}^{n} \mathbb{P}(X_i > y) = 1 - \prod_{i=1}^{n} (1 - \underbrace{\mathbb{P}(X_i \le y)}_{y}) =$$

$$= 1 - (1-y)^n = F_{Y_n}(y)$$

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} \left(1 - (1-y)^n\right)$$

$\forall y > 0$

but $x_i \in [0,1]$

• If $y > 1$ then $\mathbb{P}(X_i \le y) = 1 \Rightarrow$

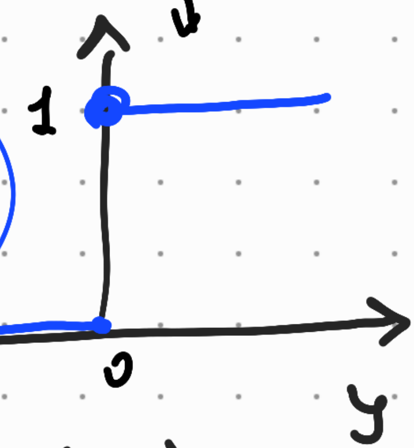
$$\Rightarrow F_{Y_n}(y) = 1 - (1-1) = 1$$

• If $y \in (0;1]$ $\lim\limits_{n\to\infty} F_{Y_n}(y) =$

$$= \lim_{n\to\infty} \Big(1 - (\underbrace{1-y}_{0 \le 1-y < 1})^n \Big) = 1$$

distribution function for zero

$$\lim_{n\to\infty} F_{Y_n}(y) \underset{F_0(y) =}{=} \begin{cases} 0 & y \le 0 \\ 1 & y > 0 \end{cases}$$



• in probability : $\lim\limits_{n\to\infty} \mathbb{P}(|Y_n - 0| > \varepsilon) = 0$

$$\mathbb{P}(|Y_n - 0| > \varepsilon) = \mathbb{P}(Y_n > \varepsilon) = \mathbb{P}(\min\{X_1 \ldots X_n\} > \varepsilon) =$$

$$= \mathbb{P}(X_1 > \varepsilon, X_2 > \varepsilon, X_3 > \varepsilon \ldots) = \prod_{i=1}^{n} \underbrace{\mathbb{P}(X_i > \varepsilon)}_{} =$$

iid

$$= \prod_{i=1}^{n} (1 - F_{X_i}(\varepsilon)) = \prod_{i=1}^{n} (1-\varepsilon) =$$

$$= (1-\varepsilon)^n$$

$$\forall \varepsilon > 0 \quad \lim_{n\to\infty} \mathbb{P}(Y_n > \varepsilon) = \begin{cases} \lim_{n\to\infty} (1-\varepsilon)^n & \varepsilon \in (0,1] \\ \lim_{n\to\infty} 0 & \varepsilon > 1 \end{cases} =$$

$$= \boxed{0}$$

- almost surely

$$\boxed{\textbf{Theorem} \quad \text{Econ } \forall \varepsilon > 0 \ \sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon) < \infty, \quad \text{mo} \quad X_n \xrightarrow{a.s.} X}$$

here $\mathbb{P}(Y_n > \varepsilon) = (1-\varepsilon)^n$

$$\forall \varepsilon > 0 \ \sum_{n=1}^{\infty} (1-\varepsilon)^n = \frac{1-\varepsilon}{1-(1-\varepsilon)} = \frac{1-\varepsilon}{\varepsilon} < \infty$$

as ε is a fixed parameter

$$\Rightarrow \quad Y_n \xrightarrow{a.s.} 0$$