Sigma-algebras

$\mathcal{F}$ - all logical operations which you can do with available information

Def

- $\Omega \in \mathcal{F}$
- if $A \in \mathcal{F}$ th $\bar{A} \in \mathcal{F}$ $(A^c \in \mathcal{F})$
- if $A_1 \dots A_i \in \mathcal{F}$ $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Task 1 toss a dice $\Omega = \{1, 2, 3, 4, 5, 6\}$

- $\mathcal{F}_1 = \sigma($ see the toss, but don't know the result $) = \sigma\{\Omega; \varnothing\}$ $2^1 = 2$

trivial σ-alg.

impossible events

- $\mathcal{F}_2 = \sigma($ result of one toss $) = \sigma\{\varnothing, \underbrace{\{1\}, \{2\}, \dots, \{6\}}, \dots$ (the remaining 2, 4, 6)

a partition (minimal info to build $\mathcal{F}$)

$A_i \in \mathcal{F}$

- $\Omega = \bigcup_{i \in J} A_i$
- disjoint $A_i \cup A_j = \varnothing$
- $\forall A \in \mathcal{F} \ \exists J \subseteq I : A = \bigcup_{i \in J} A_i$

How many elements?

$$\begin{array}{ccccc} \{1\} & \{2\} & \dots & \{6\} & \\ \pm & \pm & & \pm & \\ 2 & \times 2 & \times \dots \times & 2 & = 2^6 = 64 \end{array}$$

- $\tilde{\mathcal{F}}_2 = \sigma($ odd or even number $) =$

$= \sigma(\varnothing; \underbrace{\underset{A_1}{\{2, 4, 6\}}, \underset{A_2}{\{1, 3, 5\}}}_{\text{partition}}, \Omega)$ $2^2 = 4$

$\text{card}(\tilde{\mathcal{F}}_2) = \underline{2^2 = 4}$ num of el

- $\mathcal{F}_3 = \sigma(1 / 2 / \text{„a lot"}) = \sigma\{\varnothing, \underbrace{\{1\}, \{2\}, \{3, 4, 5, 6\}}_{A_1 \ A_2 \ A_3}, \{1, 2\},$

$\{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \Omega\}$

$2^3 = 8 = \text{card}(\mathcal{F}_3)$

$\sigma(\{1\}, \{2\}, \{3, 4, 5, 6\})$

Task 2

Ω:	1	2	3
$\mathbb{P}$	0,5	0,2	0,3
X	1	5	10

1) $\mathcal{F}_1 = \{\emptyset; \underbrace{\{1\},\{2\},\{3\}}_{\text{partition}} \; \dots\dots \; \Omega\}$ $\quad 2^3 = 8 = \text{card}$

$$E(X \mid \mathcal{F}_1) = \begin{cases} 1 & \{1\} \\ 5 & \{2\} \\ 10 & \{3\} \end{cases} = X$$

best guess

$$E(X|Y) = E(X \mid Y, Y^2, 2Y)$$

5 $\qquad$ 5, 25

2) $\mathcal{F}_2 = \{\emptyset; \{1,2,3\}\}$, $\{1,2,3\} = \Omega$ $\qquad E(X \mid \mathcal{F}_2) = E(X) =$

no information

$$= 1 \cdot \frac{1}{2} + 5 \cdot \frac{2}{10} + 10 \cdot \frac{3}{10} = \underline{4{,}5}$$

3) $\mathcal{F}_3 = \sigma(\{1\}$ ~~unu~~ $\text{ne}\{1\}) = \{\emptyset, \underbrace{\{1\}, \{2,3\}}_{\text{partition}}, \overset{\Omega}{\{1,2,3\}}\}$

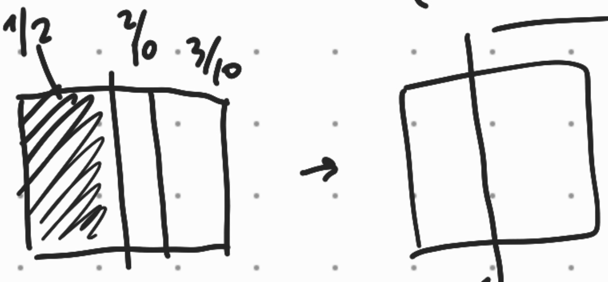
$\{2,3\} = \Omega \setminus \{1\}$

$\{XY = 1\}$

$\{XY \neq 1\}$

$= \{XY = 2, \; XY = 3, \; XY = 4 \; \dots\dots \; XY = 100\}$

$$E(X \mid \mathcal{F}_3) = \begin{cases} 1 & \{1\} \rightarrow p = \frac{1}{2} \\ 8 & \{2,3\} \quad p = \frac{1}{2} \end{cases}$$



$\dfrac{p_2}{p_2 + p_3} = 0{,}4 \qquad \dfrac{p_3}{p_2 + p_3} = 0{,}6$

1 2:3 5 10

$E(X|\{2,3\}) = 0{,}4\cdot 5 + 0{,}6\cdot 10 =$
$= 8$

$E(E(X|\mathcal{F}_3)) = E(X) = 4{,}5$

$E(E(X|\mathcal{F}_3)) = 1\cdot\frac{1}{2} + 8\cdot\frac{1}{2} = 4{,}5$

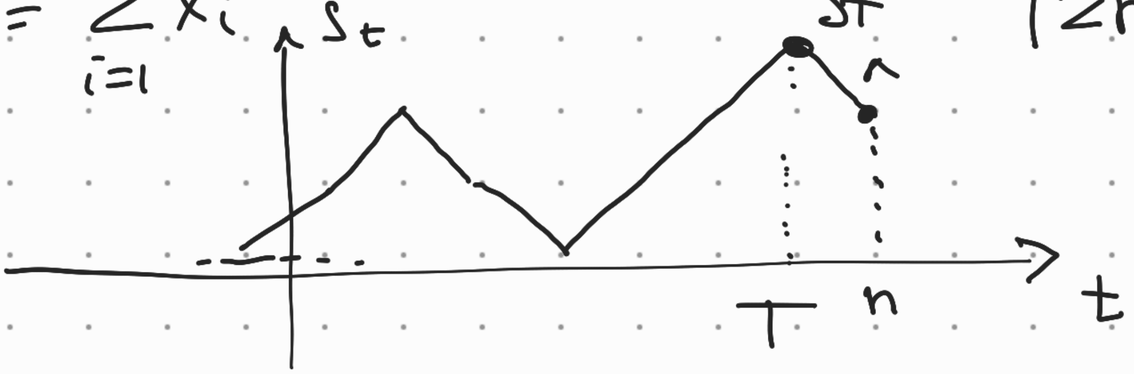
Task 3

$X_t = \begin{cases} 1 & p = \frac{1}{2} \\ -1 & p = \frac{1}{2} \end{cases}$

local max

$S_n = \sum_{i=1}^{n} X_i$

$T < n$



- $\mathcal{F}_n = \sigma(n) = (X_1, X_2 \ldots X_n)$

$\mathcal{F}_n$: σ-algebra

$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \ldots \ldots \mathcal{F}_n$ ← filtration

- $\sigma(n) = (X_1 \ldots X_n)$ ← natural filtration (just remember all outcomes)

- Find $E(T \mid \sigma(T))$ — ?

$E(X_{T-1} \mid \sigma(T))$ — ?

$$E(X_{T+1} \mid \sigma(T)) - ?$$

$$E(S_T \mid \sigma(T)) - ?$$

- $E(T \mid \sigma(T)) = T$ (you can count elem. in $\sigma(T)$)
- $E(X_{T-1} \mid \sigma(T)) = X_{T-1}$ $\quad X_{T-1} \in \sigma(T)$
- $E(X_{T+1} \mid \sigma(T)) = -1$ (T - time of local max)
- $E(S_T \mid \sigma(T)) = \sum_{i=1}^{T} X_i$

Task 4 $\quad X \in \{1, 2, 3, 4, 5, 6\}$ $\quad Y = I(X = 2, 4, 6)$

$Z = I(X > 2) = I(X = 3, 4, 5, 6)$

- $\mathcal{F}_1 : \sigma(Z) = \sigma\{\varnothing, \boxed{\{Z=0\}, \{Z=1\}}, \Omega\} =$

$= \sigma\{\varnothing, \boxed{\{1,2\}, \{3,4,5,6\}}, \Omega\}$

- $\mathcal{F}_2 : \sigma(Y \cdot Z) = \sigma\{\varnothing, \boxed{\{Y \cdot Z = 0\}, \{Y \cdot Z = 1\}}, \Omega\} =$

$$= \sigma\left\{\varnothing, \left\{\begin{matrix} Y=0, Z=0 \\ Y=0, Z=1 \\ Y=1, Z=0 \end{matrix}\right\}, \left\{\begin{matrix} Y=1 \\ Z=1 \end{matrix}\right\}, \Omega\right\} =$$

$$= \sigma\left\{\varnothing, \left\{\begin{matrix} \{4,6\} \\ \{3,5\} \\ 2 \end{matrix}\right\}, \left\{\{4,6\}\right\}, \Omega\right\} =$$

$$= \sigma\{ \emptyset, \underbrace{\{2,3,4,5,6\}, \{4,6\}}, \Omega \}$$

- $\mathcal{F}_3 = \sigma(Y, Z) = \sigma\{ \emptyset, \begin{Bmatrix} Y=0 \\ Z=1 \end{Bmatrix}, \begin{Bmatrix} Y=0 \\ Z=0 \end{Bmatrix}, \begin{Bmatrix} Y=1 \\ Z=0 \end{Bmatrix}, \begin{Bmatrix} Y=1 \\ Z=1 \end{Bmatrix}, \Omega + \ldots\ldots \}$

$$\text{card}(\sigma(Y,Z)) = 2^4 = 16$$

- $\mathcal{F}_2$ vs $\mathcal{F}_3$ $\quad \mathcal{F}_2 \subseteq \mathcal{F}_3$