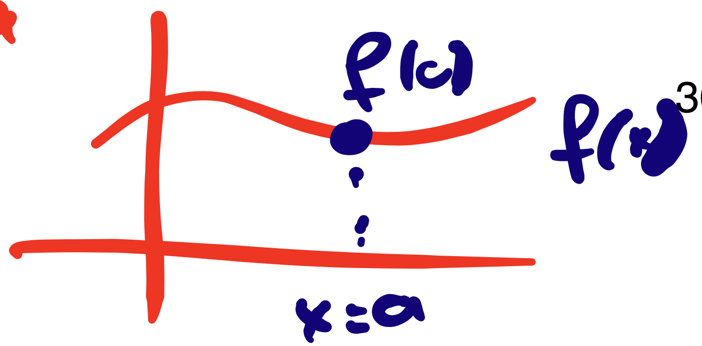
Lecture 3
Convergence

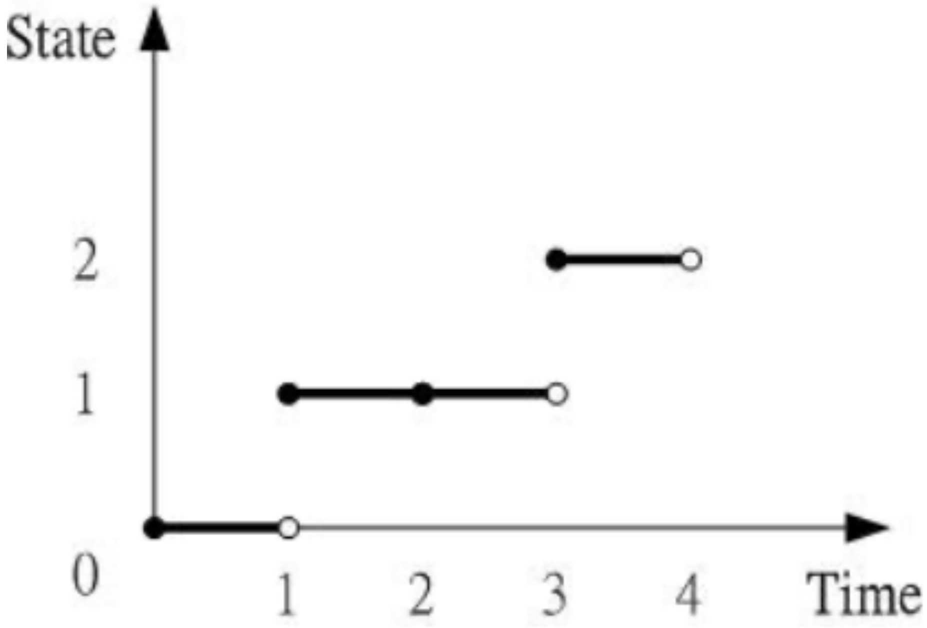
$$\lim_{x \to a} f(x) = f(a)$$



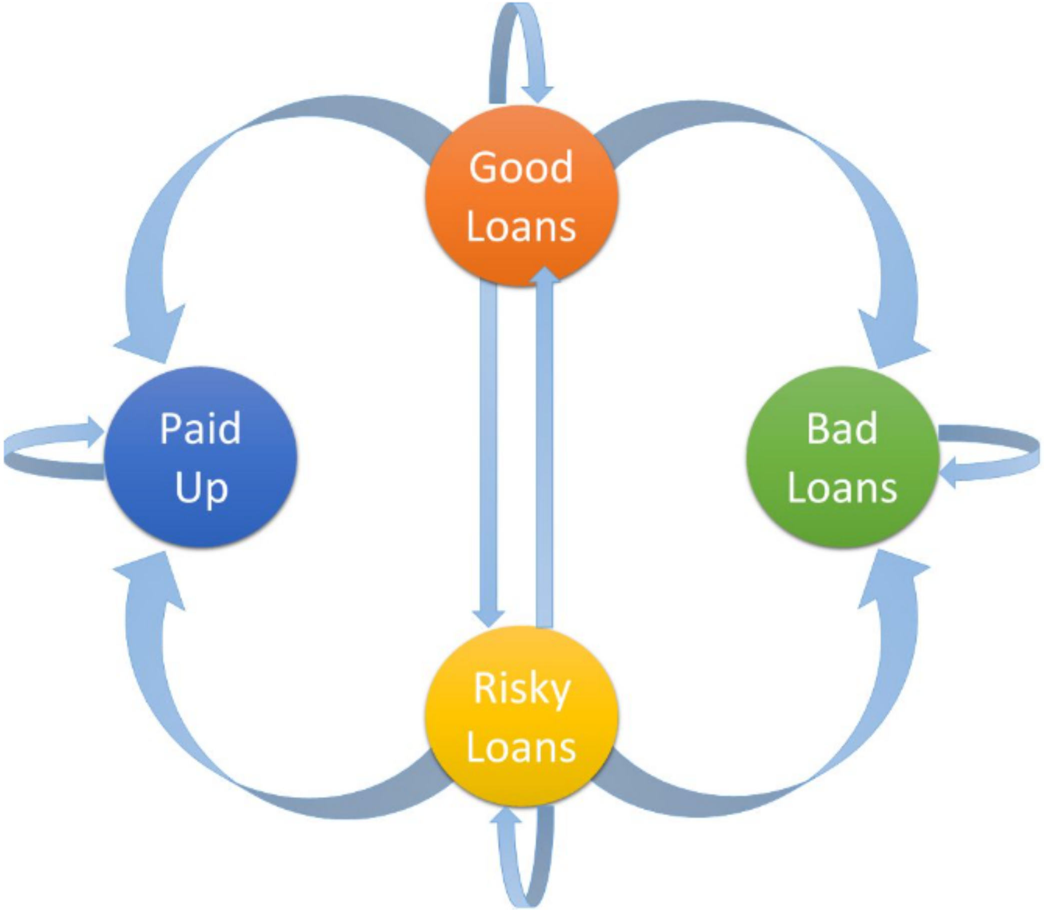
Peter Lukianchenko

30 September 2024

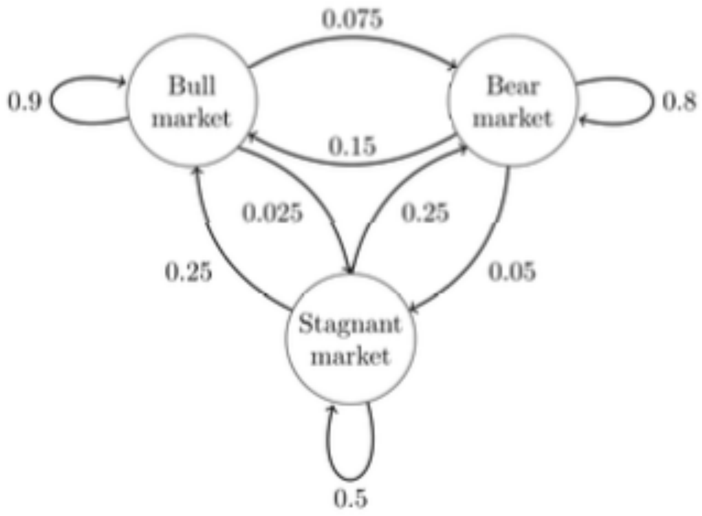
Markov process



Markov process



Markov process



Convergence

Definition *Let $\{x_n, n \geq 1\}$ be a real-valued sequence, i.e., a map from $\mathbb{N}$ to $\mathbb{R}$. We say that the sequence $\{x_n\}$ converges to some $x \in \mathbb{R}$ if there exists an $n_0 \in \mathbb{N}$ such that for all $\epsilon > 0$,*

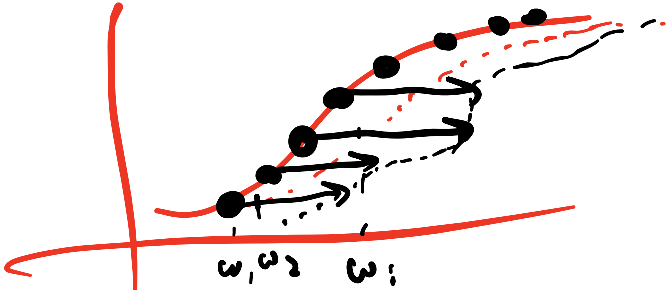
$$|x_n - x| < \epsilon, \ \forall\, n \geq n_0.$$

We say that the sequence $\{x_n\}$ converges to $+\infty$ if for any $M > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n > M$.
We say that the sequence $\{x_n\}$ converges to $-\infty$ if for any $M > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n < -M$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued random variables defined on this probability space.

$X_n = 5 + \frac{1}{n}$

Convergence



$\{\Omega, P, \mathcal{F}\}$

Definition **[Definition 0 (Point-wise convergence or sure convergence)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge point-wise or surely to* X *if*

$$X_n(\omega) \to X(\omega), \quad \forall\, \omega \in \Omega.$$

$n = \#$ of rounds

$$X_n = \begin{cases} +1, & p = \frac{1}{2} + \frac{1}{100 \cdot n} \\ -1, & p = \frac{1}{2} - \frac{1}{100 n} \end{cases} \xrightarrow{n \to \infty} X_\infty = \begin{cases} +1, & 1/2 \\ -1, & 1/2 \end{cases}$$

$$X_1 = \begin{cases} +1, & \frac{1}{2} + \frac{1}{100} \\ -1, & \frac{1}{2} - \frac{1}{100} \end{cases} \quad \dots \quad X_{10} = \begin{cases} +1, & \frac{1}{2} + \frac{1}{1000} \\ -1, & \frac{1}{2} - \frac{1}{1000} \end{cases}$$

Convergence

Definition **Definition 1 (Almost sure convergence or convergence with probability 1)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge almost surely or with probability 1 (denoted by a.s. or w.p. 1) to* X *if*

$$\underline{\mathbb{P}\left(\{\omega | X_n(\omega) \to X(\omega)\}\right) = 1}.$$

$$\operatorname*{plim}_{n\to\infty} X_n = X \qquad P(|X_n - X| < \varepsilon) = 1$$

Convergence

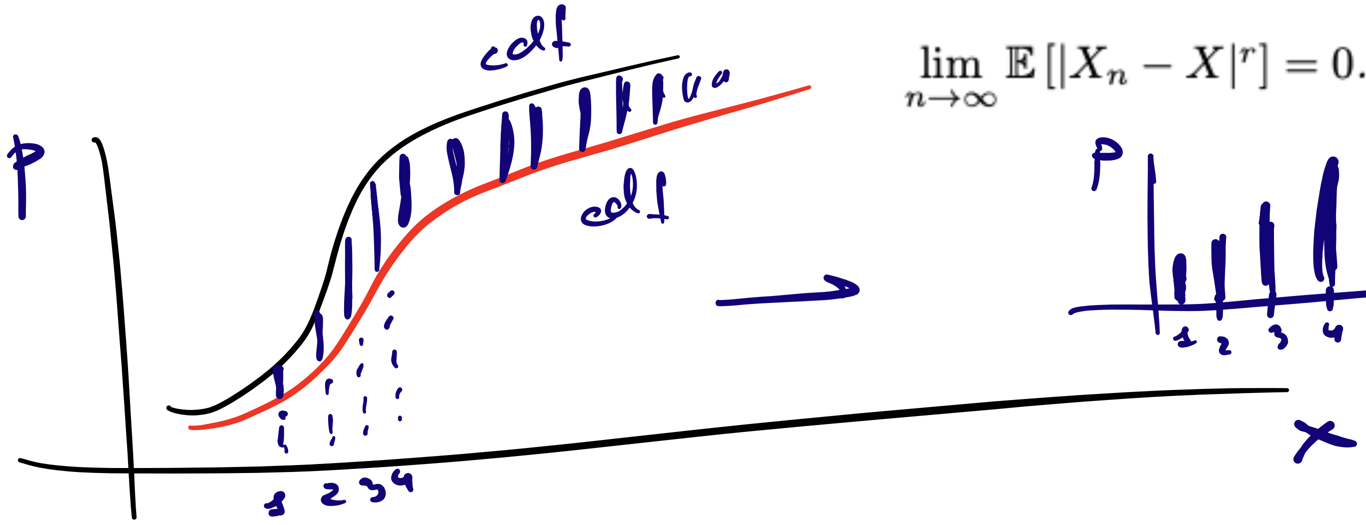
Definition **[Definition 2 (convergence in probability)]**
A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ *is said to converge in probability (denoted by i.p.) to* X *if*

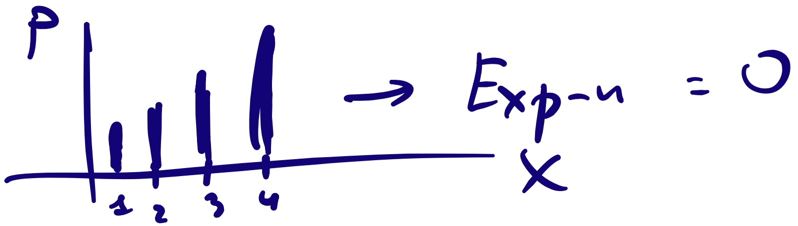
$$\lim_{n \to \infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) = 0, \quad \forall\, \epsilon > 0.$$

Convergence

Definition **[Definition 3 (convergence in r^{th} mean)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to converge in r^{th} mean to X if

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^r\right] = 0.$$





Convergence

Definition **[Definition 4 (convergence in distribution or weak convergence)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge in distribution to* X *if*

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \quad \forall\, x \in \mathbb{R} \textit{ where } F_X(\cdot) \textit{ is continuous.}$$

$$X_n = \begin{cases} +1, & \frac{1}{2} + \frac{1}{10n} \\ -1, & \frac{1}{2} - \frac{1}{10n} \end{cases} \qquad Y_n = \begin{cases} -1, & \frac{1}{2} + \frac{1}{1000n} \\ +1, & \frac{1}{2} - \frac{1}{1000n} \end{cases}$$

$$E\left[(X_n - Y_n)^2\right] = \underset{1}{EX_n^2} + EY_n^2 - 2\underset{E X_n \cdot E Y_n}{E(X_n Y_n)} =$$

$$= \overbrace{1^2\left(\tfrac{1}{2} + \tfrac{1}{10n}\right) + 1^2\left(\tfrac{1}{2} - \tfrac{1}{10n}\right)}^{=1} + \overbrace{(-1)^2\left(\tfrac{1}{2} + \tfrac{1}{1000n}\right) + 1^2\left(\tfrac{1}{2} - \tfrac{1}{1000n}\right)}^{=1} -$$

$$-2\cdot\left[\cancel{\tfrac{1}{2}}+\frac{1}{10n}-\cancel{\tfrac{1}{2}}+\frac{1}{10n}\right]\cdot\left[-\cancel{\tfrac{1}{2}}-\frac{1}{1000n}+\cancel{\tfrac{1}{2}}-\frac{1}{1000n}\right]$$

Convergence

$$= 1+1-2\cdot\left[\frac{2}{10n}\right]\cdot\left[\frac{-2}{1000n}\right]$$

$$= 2+\frac{8}{10000n^2} \xrightarrow[n\to\infty]{} 2$$

(1) *Point-wise Convergence:* $X_n \xrightarrow{\text{p.w.}} X$.

(2) *Almost sure Convergence:* $X_n \xrightarrow{\text{a.s.}} X$ or $X_n \xrightarrow{\text{w.p. } 1} X$.

(3) *Convergence in probability:* $X_n \xrightarrow{\text{i.p.}} X$.

(4) *Convergence in r^{th} mean:* $X_n \xrightarrow{r} X$. When $r = 2$, $X_n \xrightarrow{\text{m.s.}} X$.

(5) *Convergence in Distribution:* $X_n \xrightarrow{\text{D}} X$.

Convergence

Example: Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ and a sequence of random variables $\{X_n, n \geq 1\}$ defined by

$$X_n(\omega) = \begin{cases} n, & \text{if } \omega \in \left[0, \frac{1}{n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

$$E\left[|X_n - X_\infty|^2\right] = EX_n^2 = n^2 \cdot \frac{1}{n} + 0^2 \cdot \left(1 - \frac{1}{n}\right) =$$

Convergence

pw P

iP

$$X_n = \begin{cases} n, & \text{with probability } \frac{1}{n}, \quad = n \\ 0, & \text{with probability } 1 - \frac{1}{n}. \quad = 0 \end{cases}$$

$p \to 0$

$p \to 1$

$X_\infty = \begin{cases} n, & 0 \\ 0, & 1 \end{cases}$

$X_\infty = \{ 0, \ p = 1$

$P[|X_n - X_\infty| > \varepsilon] = P[|X_n| > \varepsilon] = P[X_n = n] = \frac{1}{n} \to 0$

$E[|X_n - X_\infty|] = E[|X_n|] = EX_n = n \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) = 1$

Clearly, when $\omega \neq 0$, $\lim_{n\to\infty} X_n(\omega) = 0$ but it diverges for $\omega = 0$. This suggests that the limiting random variable must be the constant random variable 0. Hence, except at $\omega = 0$, the sequence of random variables converges to the constant random variable 0. Therefore, this sequence does not converge surely, but converges almost surely.

Convergence

For some $\epsilon > 0$, consider

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(|X_n| > \epsilon\right) &= \lim_{n\to\infty} \mathbb{P}\left(X_n = n\right), \\ &= \lim_{n\to\infty} \left(\frac{1}{n}\right), \\ &= 0.\end{aligned}$$

Hence, the sequence converges in probability.

Convergence

Consider the following two expressions:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[|X_n|^2\right] &= \lim_{n\to\infty}\left(n^2 \times \frac{1}{n} + 0\right), \\ &= \infty.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[|X_n|\right] &= \lim_{n\to\infty}\left(n \times \frac{1}{n} + 0\right), \\ &= 1.\end{aligned}$$

Convergence

$$
\begin{tikzcd}
\text{p.w.} \arrow[r, Rightarrow] & \boxed{\text{a.s.}}^{=1} \arrow[dr, Rightarrow] & & \\
& & \text{i.p.} \arrow[r, Rightarrow] & \text{D} \\
r^{th} \text{ mean } (r \geq 1) \arrow[urr, Rightarrow] \arrow[rrr, Rightarrow] & & & s^{th} \text{ mean } (r > s \geq 1)
\end{tikzcd}
$$

!

Convergence

Theorem $\quad X_n \xrightarrow{r} X \implies X_n \xrightarrow{\text{i.p.}} X, \quad \forall\, r \geq 1.$

Proof: Consider the quantity $\lim\limits_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right)$. Applying Markov's inequaliy, we get

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) \quad &\leq \quad \lim_{n\to\infty} \frac{\mathbb{E}\left[|X_n - X|^r\right]}{\epsilon^r},\ \forall \epsilon > 0, \\ &\stackrel{(a)}{=} \quad 0,\end{aligned}$$

where (a) follows since $X_n \xrightarrow{r} X$. Hence proved.

Convergence

Theorem $X_n \xrightarrow{\text{i.p.}} X \implies X_n \xrightarrow{\text{D}} X.$

Proof: Fix an $\epsilon > 0$.

$$\begin{aligned} F_{X_n}(x) &= \mathbb{P}(X_n \le x), \\ &= \mathbb{P}(X_n \le x, X \le x+\epsilon) + \mathbb{P}(X_n \le x, X > x+\epsilon), \\ &\le F_X(x+\epsilon) + \mathbb{P}(|X_n - X| > \epsilon). \end{aligned}$$

Similarly,

$$\begin{aligned} F_X(x-\epsilon) &= \mathbb{P}(X \le x-\epsilon), \\ &= \mathbb{P}(X \le x-\epsilon, X_n \le x) + \mathbb{P}(X \le x-\epsilon, X_n > x), \\ &\le F_{X_n}(x) + \mathbb{P}(|X_n - X| > \epsilon). \end{aligned}$$

Thus,

$$F_X(x-\epsilon) - \mathbb{P}(|X_n - X| > \epsilon) \le F_{X_n}(x) \le F_X(x+\epsilon) + \mathbb{P}(|X_n - X| > \epsilon).$$

As $n \to \infty$, since $X_n \xrightarrow{\text{i.p.}} X$, $\mathbb{P}(|X_n - X| > \epsilon) \to 0$. Therefore,

$$F_X(x-\epsilon) \le \liminf_{n\to\infty} F_{X_n}(x) \le \limsup_{n\to\infty} F_{X_n}(x) \le F_X(x+\epsilon), \ \forall \epsilon > 0.$$

If F is continuous at x, then $F_X(x-\epsilon) \uparrow F_X(x)$ and $F_X(x+\epsilon) \downarrow F_X(x)$ as $\epsilon \downarrow 0$. Hence proved.

Convergence

Theorem $X_n \xrightarrow{r} X \implies X_n \xrightarrow{s} X$, *if* $r > s \geq 1$.

$$(\mathbb{E}[|X_n - X|^s])^{1/s} \leq (\mathbb{E}[|X_n - X|^r])^{1/r}\,,$$

Convergence

Theorem $X_n \xrightarrow{\text{i.p.}} X \not\Rightarrow X_n \xrightarrow{\text{r}} X$ *in general.*

Proof: Proof by counter-example:
Let X_n be an independent sequence of random variables defined as

$$\underline{X_n = \begin{cases} n^3, & \text{w.p. } \frac{1}{n^2}, \\ 0, & \text{w.p. } 1\text{-}\frac{1}{n^2}. \end{cases}} \rightarrow X_\infty = \begin{cases} 0 \end{cases}, p=1$$

Then, $\mathbb{P}(|X_n| > \epsilon) = \frac{1}{n^2}$ for large enough n, and hence $X_n \xrightarrow{\text{i.p.}} 0$. On the other hand, $\mathbb{E}[|X_n|] = n$, which diverges to infinity as n grows unbounded. ■

$$P(|X_n - 0| > \varepsilon) = \frac{1}{n^2} \to 0 \rightarrow \lim P(|X_n - X_\infty| > \varepsilon) = 0$$

$$E[|X_n - X_\infty|] = E[|X_n|] = E[X_n] = n^3 \cdot \frac{1}{n^2} + 0 \cdot \left(1 - \frac{1}{n^2}\right) = n$$

Convergence

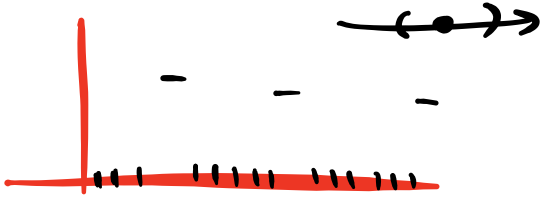
Theorem $X_n \xrightarrow{\text{D}} X \;\not\Rightarrow\; X_n \xrightarrow{\text{i.p.}} X$ *in general.*

Proof: Proof by counter-example:
Let X be a Bernoulli random variable with parameter 0.5, and define a sequence such that $X_i{=}X\ \forall\ i$. Let $Y{=}1 - X$. Clearly, $X_i \xrightarrow{\text{D}} Y$. But, $|X_i - Y|{=}1,\ \forall\ i$. Hence, X_i does not converge to Y in probability. ■

Convergence

Theorem $X_n \xrightarrow{\text{i.p.}} X \not\Rightarrow X_n \xrightarrow{\text{a.s.}} X$ *in general.*

$=0$ $\quad$ $=1$



Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent random variables defined as

$$X_n = \begin{cases} 1, & \text{w.p. } \frac{1}{n}, \\ 0, & \text{w.p. } 1\text{-}\frac{1}{n}. \end{cases}$$

$$X_\infty = \begin{cases} 1 & p = \varepsilon \\ 0, & p \to 1 \end{cases}$$

$\lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) = \lim_{n\to\infty} \mathbb{P}(X_n = 1) = \lim_{n\to\infty} \frac{1}{n} = 0$. So, $X_n \xrightarrow{\text{i.p.}} 0$.

Let A_n be the event that $\{X_n = 1\}$. Then, A_n's are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. By Borel-Cantelli Lemma 2, w.p. 1 infinitely many A_n's will occur, *i.e.*, $\{X_n = 1\}$ i.o.. So, X_n does not converge to 0 almost surely. ■

$$P(|X_n - X_\infty| > \varepsilon) \overset{ip}{=} P(|X_n| > \varepsilon) = \frac{1}{n} \to 0$$

$$P(|X_n - X_\infty| < \varepsilon) = P(|X_n - 0| < \varepsilon) \not\to 1$$

Convergence

Theorem $X_n \xrightarrow{s} X \not\Rightarrow X_n \xrightarrow{r} X$ *if* $r > s \geq 1$ *in general.*

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent random variables defined as

$$X_n = \begin{cases} n, & \text{w.p. } \frac{1}{n^{\frac{r+s}{2}}}, \\ 0, & \text{w.p. } 1\text{-} \frac{1}{n^{\frac{r+s}{2}}}. \end{cases}$$

Hence, $\mathbb{E}[|X_n^s|] = n^{\frac{s-r}{2}} \to 0$. But, $\mathbb{E}[|X_n^r|] = n^{\frac{r-s}{2}} \to \infty$.

Convergence

$P(|X_n - X_\infty| < \varepsilon) = 1$

E $r=2$

Theorem $X_n \xrightarrow{\text{m.s.}} X \;\not\Rightarrow\; X_n \xrightarrow{\text{a.s.}} X$ *in general.*

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent random variables defined as

$$X_n = \begin{cases} 1, & \text{w.p. } \frac{1}{n}, \\ 0, & \text{w.p. } 1\text{-}\frac{1}{n}. \end{cases}$$

$X_\infty = \{0, p=1$

$\mathbb{E}[X_n^2]=\frac{1}{n}$. So, $X_n \xrightarrow{\text{m.s.}} 0$. X_n does not converge to 0 almost surely.

$r=2$

$E[(X_n - X_\infty)^2] = E(X_n^2) = 1^2 \cdot \frac{1}{n} + 0^2 \cdot (1-\frac{1}{n}) = \frac{1}{n} \to 0$

1 $E[|X_n - X_\infty|] = E[|X_n|] = 1 \cdot \frac{1}{n} + 0 \cdot (1-\frac{1}{n}) = \frac{1}{n} \to 0$

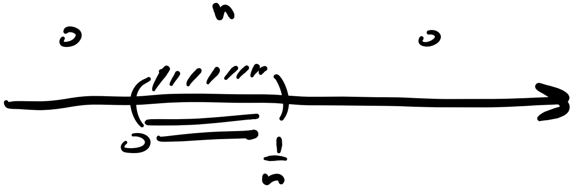
Convergence

Theorem $X_n \xrightarrow{\text{a.s.}} X \not\Rightarrow X_n \xrightarrow{\text{m.s.}} X$ *in general.*

$X_\infty = \{0, \quad p = 1$

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent of random variables defined as

$$X_n(\omega) = \begin{cases} n, & \omega \in (0, \frac{1}{n}), \\ 0, & \text{otherwise.} \end{cases}$$



We know that X_n converges to 0 almost surely. $\mathbb{E}[X_n^2]=n \longrightarrow \infty$. So, X_n does not converge to 0 in the mean-squared sense. ■

$E[|X_n - X_\infty|^2] = E(X_n^2) = n^2 \cdot \frac{1}{n} = n \neq 0$

Before proving the implication $X_n \xrightarrow{\text{a.s.}} X \implies X_n \xrightarrow{\text{i.p.}} X$, we derive a sufficient condition followed by a necessary and sufficient condition for almost sure convergence.

$P(|X_n - X_\infty| > \varepsilon) \leq \frac{E[X_n - X_\infty]}{\varepsilon} = \frac{EX_n}{\varepsilon} \to 0$

Theorem 28.20 [Skorokhod's Representation Theorem]
Let $\{X_n, n \geq 1\}$ and X be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that X_n converges to X in distribution. Then, there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and random variables $\{Y_n, n \geq 1\}$ and Y on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that,

a) $\{Y_n, n \geq 1\}$ and Y have the same distributions as $\{X_n, n \geq 1\}$ and X respectively.

b) $Y_n \overset{a.s.}{\to} Y$ as $n \to \infty$.

Convergence

Theorem 28.21 **[Continuous Mapping Theorem]**
If $X_n \overset{D}{\to} X$, *and* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous, then* $g(X_n) \overset{D}{\to} g(X)$.

Proof: By Skorokhod's Representation Theorem, there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and $\{Y_n, n \geq 1\}$, Y on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that, $Y_n \overset{a.s.}{\to} Y$. Further, from continuity of g,
$\{\omega \in \Omega' \mid g(Y_n(\omega)) \to g(Y(\omega))\} \supseteq \{\omega \in \Omega' \mid Y_n(\omega) \to Y(\omega)\}$,
$\Rightarrow \mathbb{P}(\{\omega \in \Omega' \mid g(Y_n(\omega)) \to g(Y(\omega))\}) \geq \mathbb{P}(\{\omega \in \Omega' \mid Y_n(\omega) \to Y(\omega)\})$,
$\Rightarrow \mathbb{P}(\{\omega \in \Omega' \mid g(Y_n(\omega)) \to g(Y(\omega))\}) \geq 1$,
$\Rightarrow g(Y_n) \overset{a.s.}{\to} g(Y)$,
$\Rightarrow g(Y_n) \overset{D}{\to} g(Y)$.
This completes the proof since, $g(Y_n)$ has the same distribution as $g(X_n)$, and $g(Y)$ has the same distribution as $g(X)$. ■

Convergence

Theorem 28.23 *If* $X_n \xrightarrow{D} X$, *then* $C_{X_n}(t) \longrightarrow C_X(t)$, $\forall t$.

Proof: If $X_n \xrightarrow{D} X$, from Skorokhod's Representation Theorem, there exist random variables $\{Y_n\}$ and Y such that $Y_n \xrightarrow{a.s.} Y$.
So,

$$\cos(Y_n t) \longrightarrow \cos(Yt),\ \cos(X_n t) \longrightarrow \cos(Xt),\ \ \forall t.$$

As $\cos(\cdot)$ and $\sin(\cdot)$ are bounded functions,

$$\mathbb{E}[\cos(Y_n t)] + i\mathbb{E}[\sin(Y_n t)] \longrightarrow \mathbb{E}[\cos(Yt)] + i\mathbb{E}[\sin(Yt)],\ \ \forall t.$$

$$\Rightarrow C_{Y_n}(t) \longrightarrow C_Y(t),\ \ \forall t.$$

We get,

$$C_{X_n}(t) \longrightarrow C_X(t),\ \ \forall t,$$

since distributions of $\{X_n\}$ and X are same as those of $\{Y_n\}$ and Y respectively, from Skorokhod's Representation Theorem. ■

Convergence

Example 1: Let the random variable U be uniformly distributed on $[0,1]$. Consider the sequence defined as:

$$X(n) = \frac{(-1)^n U}{n}.$$

1. *Almost sure convergence:* Suppose

   $$U = a.$$

   The sequence becomes

   $$\begin{aligned} X_1 &= -a, \\ X_2 &= \frac{a}{2}, \\ X_3 &= -\frac{a}{3}, \\ X_4 &= \frac{a}{4}, \\ &\vdots \end{aligned}$$

   In fact, for any $a \in [0,1]$

   $$\lim_{n\to\infty} X_n = 0,$$

   therefore, $X_n \xrightarrow{a.s.} 0$.

Convergence

Convergence in mean square sense:

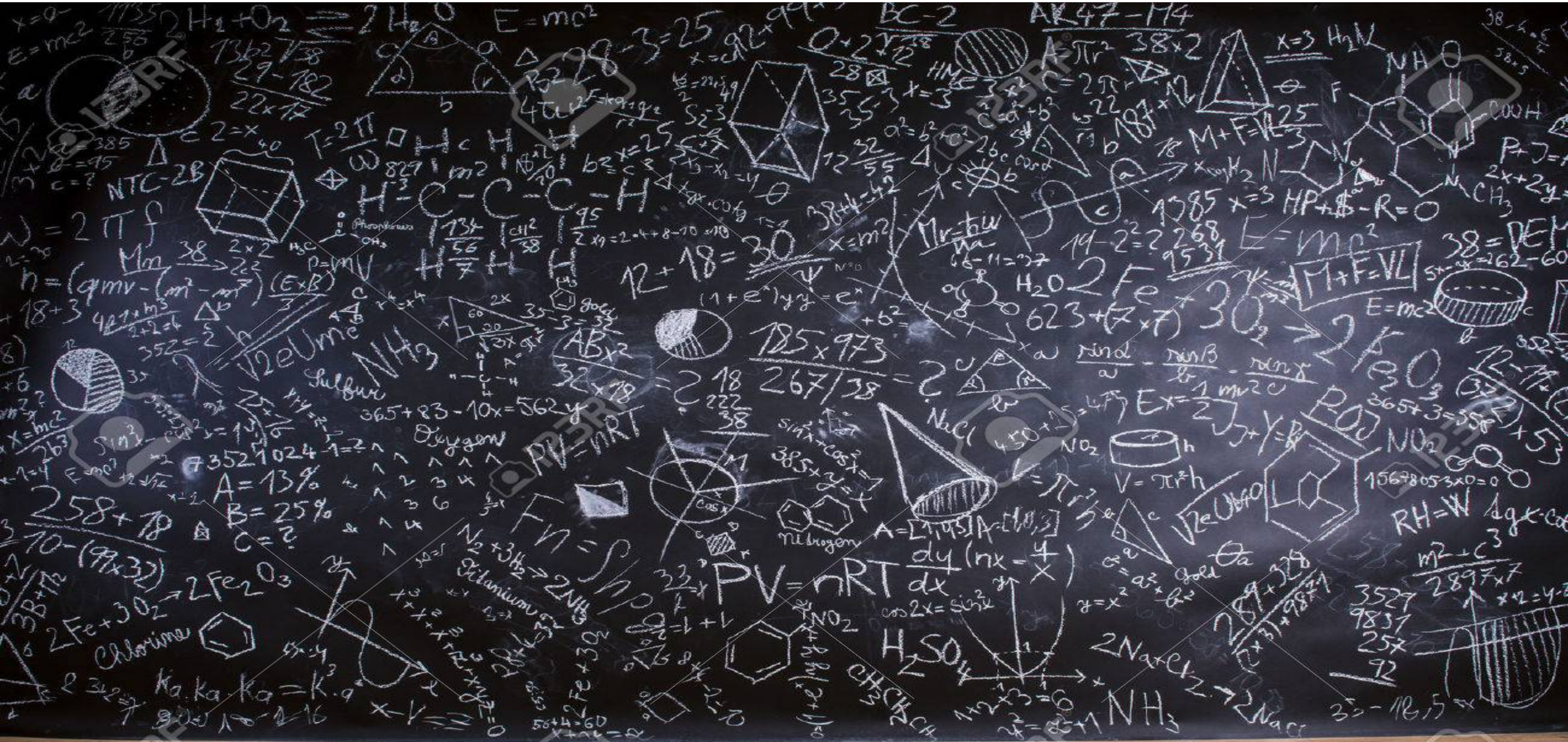
In order to answer this question, we need to prove that

$$\lim_{n\to\infty} E\left[|X_n - 0|^2\right] = 0.$$

We know that,

$$\begin{aligned}
\lim_{n\to\infty} E\left[|X_n - 0|^2\right] &= \lim_{n\to\infty} E\left[X_n^2\right], \\
&= \lim_{n\to\infty} E\left[\frac{U^2}{n^2}\right], \\
&= \lim_{n\to\infty} \frac{1}{n^2} E\left[U^2\right], \\
&= \lim_{n\to\infty} \frac{1}{n^2} \int_0^1 u^2 du, \\
&= \lim_{n\to\infty} \frac{1}{n^2} \left.\frac{u^3}{3}\right]_0^1, \\
&= \lim_{n\to\infty} \frac{1}{3n^2}, \\
&= 0.
\end{aligned}$$

Hence, $X_n \xrightarrow{m.s.} 0$.



Thank you for your attention!

See next week!