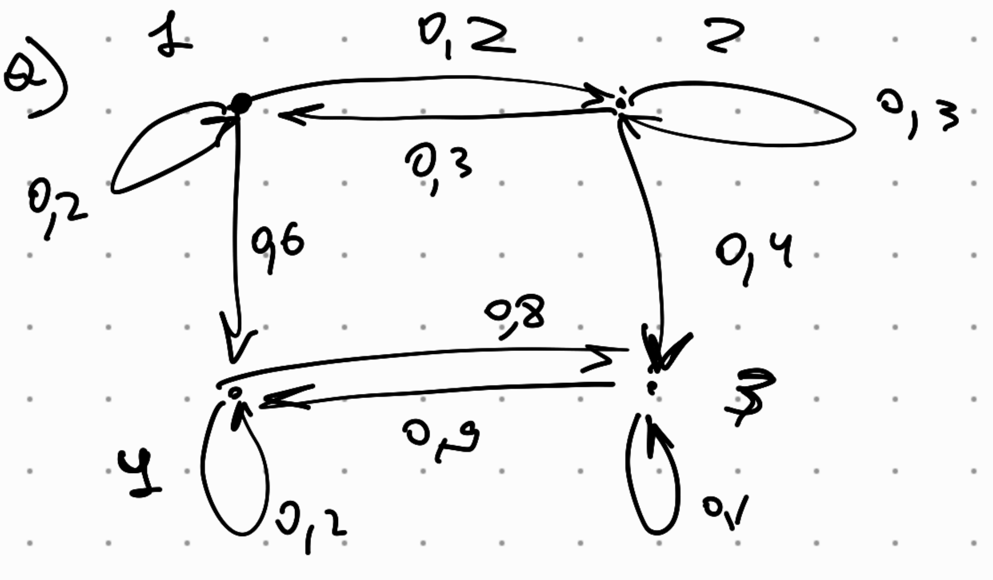
Topics

1) Markov chains (first step analysis, ... stationarity) ..
2) Moment generating functions .
3) Convergence (lim, plim) ..
4) Conditional expectations .
5) Sigma-algebras .
6) Martingales (prove definition)

1

$$P = \begin{pmatrix} 0{,}2 & 0{,}2 & 0 & 0{,}6 \\ 0{,}3 & 0{,}3 & 0{,}4 & 0 \\ 0 & 0 & 0{,}1 & 0{,}9 \\ 0 & 0 & 0{,}8 & 0{,}2 \end{pmatrix}$$

a)



opened
$A = \{1, 2\}$

$B = \{4, 3\}$
closed

b) 1, 2 - trans.. 3, 4 -> recurrent

c) $P_{14}^{(N)} \xrightarrow[N \to \infty]{} \pi_4$ stationary dist.

$$\pi P = \pi$$

$$\pi = (\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4)$$

$$\begin{cases} 0{,}2\pi_1 + 0{,}3\pi_2 = \pi_1 \\ 0{,}2\pi_1 + 0{,}3\pi_2 = \pi_2 \\ 0{,}4\pi_2 + 0{,}1\pi_3 + 0{,}8\pi_4 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

$$\begin{cases} \pi_1 = \pi_2 \\ 0{,}2\pi_1 + 0{,}3\pi_1 = \pi_1 \\ 0{,}4\pi_1 + 0{,}1\pi_3 + 0{,}8\pi_4 = \pi_3 \\ 2\pi_1 + \pi_3 + \pi_4 = 1 \end{cases}$$

$\pi_1 = \pi_2$

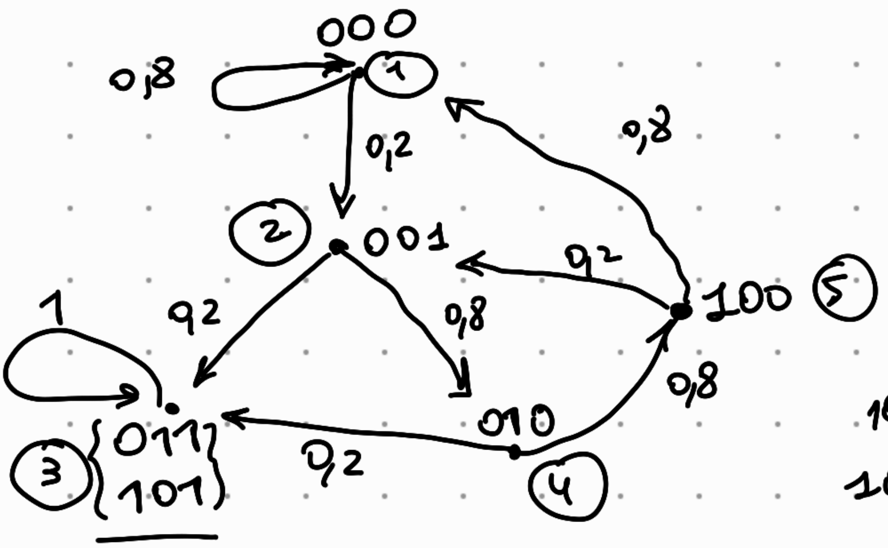
$0{,}5\pi_1 = \pi_1 \rightarrow \boxed{\pi_1 = 0 = \pi_2}$

$$\begin{cases} 0{,}1\pi_3 + 0{,}8\pi_4 = \pi_3 \\ \pi_3 + \pi_4 = 1 \end{cases}$$

$$\begin{cases} \pi_3 + \pi_4 = 1 \\ 0{,}8\pi_4 = 0{,}9\pi_3 \end{cases}$$

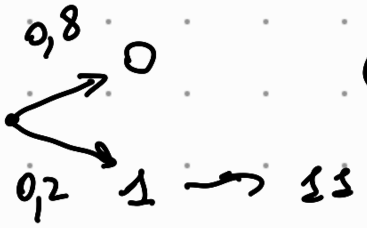
$$\boxed{\begin{aligned} \pi_3 &= \frac{8}{17} \\ \pi_4 &= \frac{9}{17} \end{aligned}}$$

2



$100 \rightarrow 1\boxed{001}$

$100 \rightarrow 1\boxed{000}$



a) $P_{13}^{(3)}$ → $1 \to 2 \to 4 \to 3$ $\quad 0{,}2 \cdot 0{,}8 \cdot 0{,}2$ $= 0{,}4 \cdot 1{,}6 = 0{,}64$

→ $1 \to 1 \to 2 \to 3$ $\quad 0{,}8 \cdot 0{,}2 \cdot 0{,}2$

b)
$$\begin{cases} \mu_{13} = 0{,}8(1 + \mu_{13}) + 0{,}2(1 + \mu_{23}) \\ \mu_{23} = 0{,}2 \cdot 1 + 0{,}8(1 + \mu_{43}) \\ \mu_{43} = 0{,}2 \cdot 1 + 0{,}8(1 + \mu_{53}) \\ \mu_{53} = 0{,}2(1 + \mu_{23}) + 0{,}8(1 + \mu_{13}) \end{cases}$$

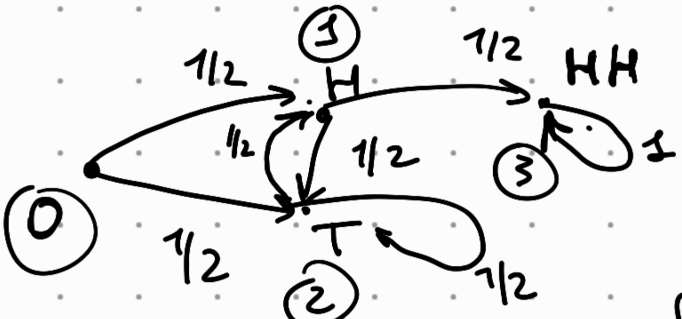
$\to \left(\mu_{53} = 0,2(1+\mu_{23}) + 0,8(1+\mu_{13}) \right).$

$\mu_{13} = \mu_{53}$

$$\boxed{\begin{cases} \mu_{13} = 1 + 0,8\,\mu_{13} + 0,2\,\mu_{23} \\ \mu_{23} = 1 + 0,8\,\mu_{43} \\ \mu_{43} = 1 + 0,8\,\mu_{13} \end{cases}} \longrightarrow \mu_{13} \; +$$

MGF: $M_X(t) = Ee^{tX} = 1 + \frac{t}{1!}EX + \frac{t^2}{2!}EX^2 + \dots + \frac{t^k}{k!}EX^3$

- $\left(\frac{d}{dt}\right)^k M_X(t)\Big|_{t=0} = EX^k \leftarrow$



$\boxed{N - \#\ \text{trows}}$

$\mu = EN$

$\mu_{03} = EN_{03}, \quad \mu_{23} = EN_{23}$

first step analysis

$$\begin{cases} EN_{03} = \frac{1}{2}(1+EN_{13}) + \frac{1}{2}(1+EN_{23}) \\ EN_{13} = \frac{1}{2}\cdot 1 + \frac{1}{2}(1+EN_{23}) \\ EN_{23} = \frac{1}{2}(1+EN_{13}) + \frac{1}{2}(1+EN_{23}) \end{cases}$$

we were asked about ↓

$$\begin{cases} M_0 = E(e^{tN_{03}}) = \frac{1}{2}(e^{t(1+EN_{13})}) + \frac{1}{2}e^{t(1+EN_{23})}. \\ M_1 = E(e^{tN_{13}}) = \frac{1}{2}e^{t\cdot 1} + \frac{1}{2}e^{t(1+EN_{23})} \\ \dots \; t(1+EN_{23}) \end{cases} \Rightarrow$$

$$M_2 = E\left(e^{tN_{23}}\right) = \frac{1}{2}e^{t(1+EN_{23})} + \frac{1}{2}e^{t(1+E\ldots)}$$

$$\left[\underset{X \sim N(0,1)}{E\left(e^{tX}\right)} = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \ldots\ldots e^{t/2} \right]$$

$$E(g(x)) = \int_{R_x} g(x) f(x)\, dx$$

$$E(g(x)) = \sum g(x_i) \cdot p(x_i)$$

$$\Rightarrow \begin{cases} M_0 = \frac{1}{2}e^t\left(e^{tEN_{13}} + e^{tEN_{23}}\right) \\ M_1 = \frac{1}{2}e^t\left(1 + e^{tEN_{23}}\right) \\ M_2 = \frac{1}{2}e^t\left(e^{tEN_{13}} + e^{tEN_{23}}\right) \end{cases}$$

$$\begin{cases} M_0 = \frac{1}{2}e^t(M_1 + M_2) \\ M_1 = \frac{1}{2}e^t(1 + M_2) \\ M_2 = \frac{1}{2}e^t(M_1 + M_2) \end{cases} \Rightarrow \begin{matrix} M_0 = ? \\ M_1 = \\ M_2 = \end{matrix}$$

Convergence of x_n

$\swarrow$ lim (usual) $\qquad\qquad \searrow$ plim (in probability)

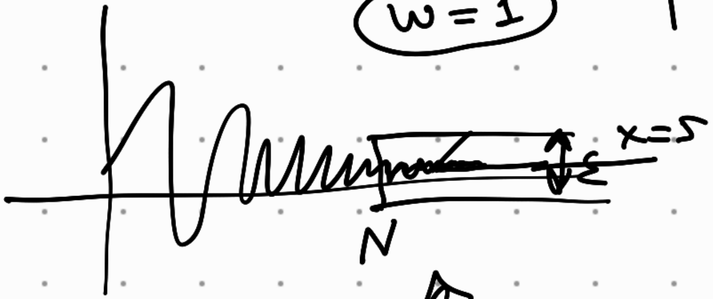
$\lim\limits_{n\to\infty} x_n = x$ | $\operatorname{plim}\limits_{n\to\infty} x_n = x$

$\forall \varepsilon > 0 \;\; \exists N: \;\; \forall n > N$ | $\forall \varepsilon > 0 \quad \lim \mathbb{P}(|x_n - x| > \varepsilon) = 0$

$|x_n - x| < \varepsilon$

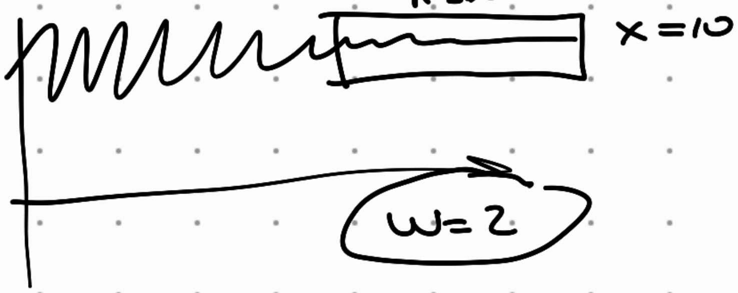
x_n not too far from x

$w = 1$



+ a.s limit

$\forall w_i : X_n(w_i) \xrightarrow[n\to\infty]{} X(w_i)$

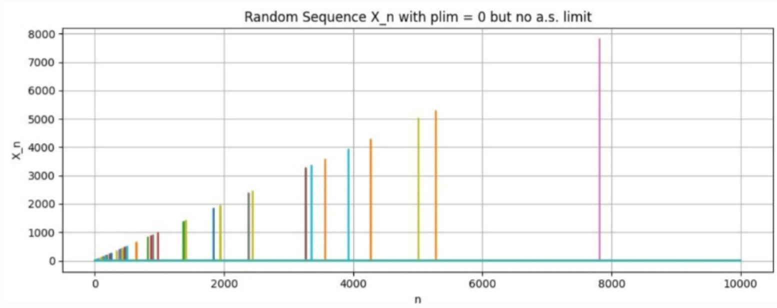


$n \to \infty$

$\forall \delta > 0 \quad \exists N: \forall n > N$

$\mathbb{P}(|X_n - X| > \varepsilon) < \delta$

probability to be too far from $X \to 0$

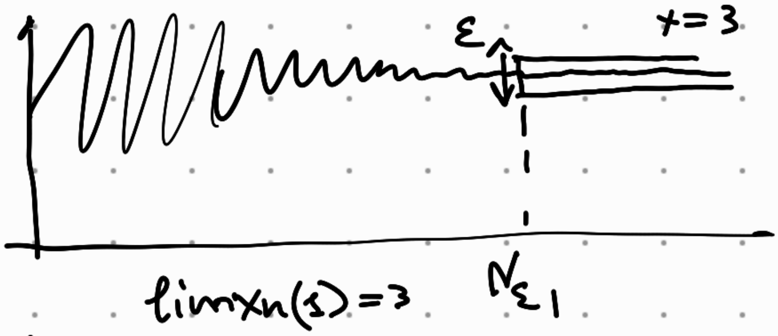


diff. colors $\equiv$ diff w_i

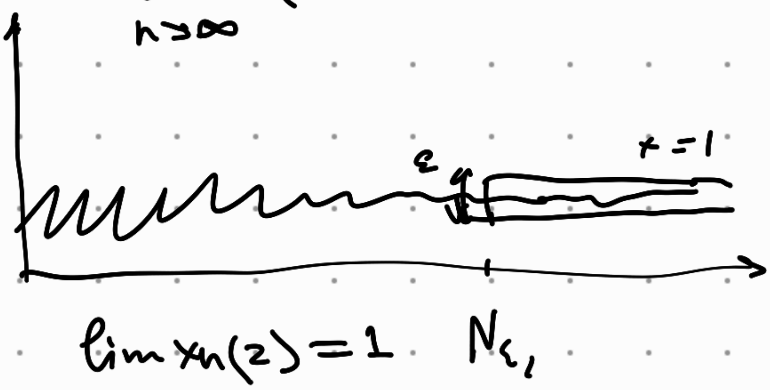
$$X_n = \begin{cases} 0 & p = 1 - \frac{1}{n} \\ n & p = \frac{1}{n} \end{cases}$$

a.s $\Rightarrow$ plim $\Rightarrow$ distrib.

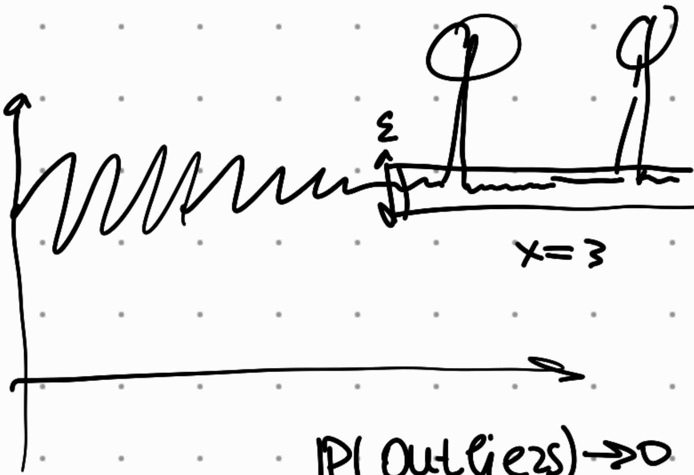
$\forall i \quad \lim_{n\to\infty} X_n(w_i) = X(w)$



$\lim_{n\to\infty} x_n(1) = 3$



$\lim_{n\to\infty} x_n(2) = 1$



$\mathbb{P}(\text{outliers}) \xrightarrow[n\to\infty]{} 0$

$\operatorname{plim}_{n\to\infty} x_n = 3$

$X_n \xrightarrow{P} X$

$\operatorname{plim} X_n = X$

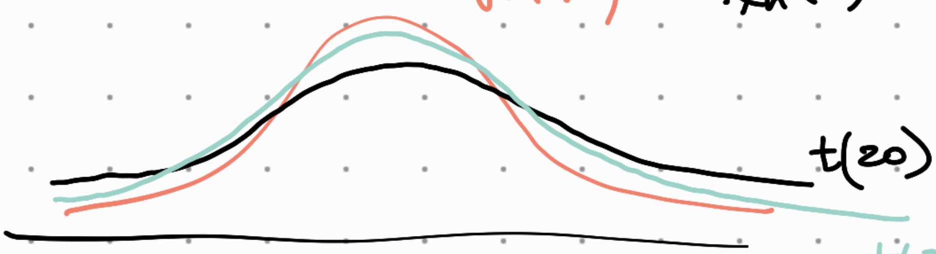
$\operatorname{plim}_{n\to\infty} X_n$

$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0$

in distribution

$\lim_{n\to\infty} F_{X_n} = F_X$

$\lim p X_n = X$???



- $\operatorname{plim}_{n\to\infty} \dfrac{(X_1-\overline{X})^3 + \dots (X_n-\overline{X})^3}{n+2022} = \operatorname{plim}_{n\to\infty} \dfrac{\sum (X_i-\overline{X})^3}{n} \circleddash$

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[P]{} EX_i$ WLN

$\frac{1}{n-1}\sum (X_i-\overline{X})^2 \xrightarrow[P]{} \mathrm{Var}X_i$ WLN

$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow[P]{} EX_i^2$ WLN

$\circleddash \operatorname{plim}_{n\to\infty} \dfrac{X_1^3 - \overline{X}^3 + 3X_1\overline{X}^2 - 3X_1^2\overline{X} + \dots}{n} =$

$= \operatorname{plim}_{n\to\infty} \dfrac{\sum X_i^3 - n\overline{X}^3 - 3\overline{X}\sum X_i^2 + 3\overline{X}^2 \sum X_i}{n} =$

$= \operatorname{plim}_{n\to\infty} \left(\underbrace{\frac{\sum X_i^3}{n}}_{\downarrow\, 2} - \left(\frac{1}{n}\sum X_i\right)^3 - 3\left(\frac{1}{n}\sum X_i\right)\underbrace{\left(\frac{1}{n}\sum X_i^2\right)}_{\downarrow\, \frac{4}{3}} + 3\underbrace{\left(\frac{1}{n}\sum X_i\right)}_{\downarrow\, 1}\underbrace{\frac{\sum X_i}{n}}_{\downarrow\, 1} \right)$

$= \operatorname{plim}_{n\to\infty}$

$X_i \sim U[0,2]$ $\qquad \bar{x} = 1$

$$EX_i^2 = \int_0^2 x^2 \cdot \frac{1}{2}\, dx = \frac{1}{6} x^3 \Big|_0^2 = \frac{4}{3}$$

$$EX_i^3 = \int_0^2 x^3 \frac{1}{2}\, dx = \frac{1}{8} x^4 \Big|_0^2 = 2$$

$$\operatorname*{plim}_{n\to\infty} (\ ) = 2 - 1 - 3 \cdot \frac{4}{3} + 3 = 0$$

$$\operatorname*{plim}_{n\to\infty} \max\left\{ \frac{\sum X_i}{n} ; 2\frac{\sum X_i^2}{n} \right\} \qquad X_i \sim U[0,1]$$

$$\frac{\sum X_i}{n} \xrightarrow{p} EX_i = \frac{1}{2} \qquad 2\frac{\sum X_i^2}{n} \to 2EX_i^2 = 2\int_0^1 x^2 \cdot 1\, dx = 2 \cdot \frac{1}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

$$\operatorname*{plim}_{n\to\infty} \max\{\ \} = \max\left\{ \frac{1}{2} ; \frac{2}{3} \right\} = \frac{2}{3}.$$

Sigma-algebra.

	p	$1-p$
X_i	$+1$	-1

	$p^2 + (1-p)^2$	$2p(1-p)$
$X_1 X_2$	1	-1

$$\mathcal{F}_1 = \sigma(\underbrace{X_1 \cdot X_2}_{+1,\ -1}) = \Big\{ \emptyset, \overset{A}{\begin{Bmatrix} X_1 = 1, X_2 = 1 \\ X_1 = -1, X_2 = -1 \end{Bmatrix}}; \overset{A^c}{\begin{Bmatrix} X_1 = 1, X_2 = -1 \\ X_1 = -1, X_2 = 1 \end{Bmatrix}}; \Omega \Big\}$$

$$\operatorname{card} \mathcal{F}_1 = 4 = 2^2$$

$$\sigma(X_1, X_2, X_3) = \mathcal{F}_2 \Rightarrow \operatorname{card} \mathcal{F} = 2^p = 2^8 = 256$$

b) $\sigma(X_1, X_2, X_3) = \mathcal{F}_3$ card $\mathcal{F}_3 = 2 = 2 \; 256$

$(1,1,1)$

$(1,1,-1)$ $\quad 8 = 2^3 = P$

$\vdots$

$(-1,-1,-1)$

c) $\sigma(X_1, X_1+X_2, X_1+X_2+X_3) = \mathcal{F}_3$ $\qquad \underline{\mathcal{F}_2 = \mathcal{F}_3}$

$(1,2,3) \to X_1 = 1$
$X_2 = 1$
$X_3 = 1$

$$E(X_1+X_2+X_3 \mid \mathcal{F}_3) = E\left(\sum_{i=1}^{3} X_i \mid X_1, X_2, X_3\right) = X_1+X_2+X_3$$

$$E\left(\sum_{i=1}^{4} X_i \mid \mathcal{F}_3\right) = X_1+X_2+X_3 + \overbrace{E(X_4)}^{=0} = X_1+X_2+X_3$$

$\sum_{i=1}^{n} X_i$ – martingale

Martingale

$S_0 = 0 \quad S_t = \underbrace{X_1 + \dots + X_n}$

-1	0	1
0,2	0,2	0,6

1) R_t is adapted ($\mathcal{F}_{t-1}$)

2) $E|R_t| < \infty$

3) $E(R_t \mid \mathcal{F}_{t-1}) = R_{t-1}$

$R_t = \beta^{S_t}$

$$E(\beta^{S_t} \mid \mathcal{F}_{t-1}) = E(\beta^{S_{t-1}} \cdot \beta^{X_t} \mid \mathcal{F}_{t-1}) =$$

$$= E(\beta^{S_{t-1}} \cdot \beta^{X_t} \mid X_1, X_2 \dots X_{t-1}) =$$

$$= \beta^{S_{t-1}} \cdot E\beta^{X_t} = \beta^{S_{t-1}}$$ by def of martingale

$E\,b^{X_t} = 1$ $\qquad E(g(X))$

$0{,}2\, b^{-1} + 0{,}2\, b^{0} + 0{,}6\, b^{1} = 1$

$0{,}2 + 0{,}2 b + 0{,}6 b^2 = b$

$0{,}6 b^2 - 0{,}8 b + 0{,}2 = 0$

$b^2 - \frac{4}{3} b + \frac{1}{3} = 0$ $\qquad b_1 + b_2 = \frac{4}{3}$

$b_1 = 1 \quad b_2 = \frac{1}{3}$ $\qquad b_1 b_2 = \frac{1}{3}$

$R_t = 1^{S_t} = 1$ trivial

$\boxed{R^t = 3^{-S_t}}$ martingale

---

(2)

	1/2	1/2
X_i	1	0

A $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$

B $\mathcal{H}_n = \sigma(X_n, X_{n+1} \dots)$

b) $\{X_{37} > 0\} \equiv \{X_{37} = 1\}$ $\qquad \mathcal{F}_{37}$

$\mathcal{F}_{38}$

$\{X_{37} > X_{2024}\} \equiv \left\{ \begin{array}{l} X_{37} = 1 \\ X_{2024} = 0 \end{array} \right\}$ $\qquad \mathcal{F}_{2024}$

$\mathcal{F}_{2024}$ $\quad \mathcal{H}_{37}$

$\mathcal{F}_{2025}$

$\sigma(X_{37}, X_{2024})$

$A = \{X_{37} = 1\}$ $\quad B = \{X_{2024} = 0\}$

$A \cup B \in \mathcal{F}_{2024}$

$\{X_{37} > X_{2024} > X_{12}\} \to$ false $\to \emptyset$ in all sigma

c) $A \notin \mathcal{F}_{2025}$

$\{X_{2026} = 1\} \notin \emptyset, \sigma(X_1 \dots X_{2025})$

$\{X_{2026} \cdot X_{2027} = 0\}$

$\{X_{37} > 0\}$

$\sigma(X_{37}) = \{\emptyset, \{X_{37}=1\}, \{X_{37}=0\}, \{X_{37}=0/1\}\}$

0 1 $p=2 \Rightarrow \text{card} = 2^2 = 4$

X	0	1	2
	0,3	0,4	0,3

$X: 2$

$\mathcal{F} = \sigma(X_1) = \{\emptyset, \{X=0\}, \{X=1\}, \{X=2\}, \{X=0, X=2\}, \{X=0, X=1\}\}$ +

$\boxed{\begin{matrix} X=0 \\ X=1 \\ X=2 \end{matrix}}$ $p=3$ $\text{card}\,\mathcal{F} = 2^3 = 8$

0	1	2
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$E(X_1 \mid \sigma(X_1)) = E(X_1 \mid X_1) = X_1$

$E(X_2 \mid \sigma(X_1+X_2)) = E(X_2 \mid X_1+X_2)$

$\mathcal{F} = \sigma(X_1+X_2) \Rightarrow \text{card}\,\mathcal{F} = 2^4$

$\sigma(X_1+X_2)$	0	1	2	3	4
X_1+X_2	0+0	0+1, 1+0	0+2, 2+0	1+2, 2+1	2+2, 2+2

$$E(X_2 \mid X_1+X_2) = \begin{cases} 0 & X_1+X_2=0 \\ \frac{1}{2} & X_1+X_2=1 \\ 1 & X_1+X_2=2 \end{cases}$$

$$E(X \mid \sigma(Y)) = E(X \mid Y) = \begin{cases} E(X \mid Y=1) = \ldots \\ E(X \mid Y=-1) = \ldots \end{cases}$$