Stochastic processes and applications: Seminar #8

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HSE — November 1, 2024

Info: Do not be afraid to solve unfinished tasks at home!

Martingales

Task 1: Fair gambler's ruin

A gambler wins or looses one rouble in each round of the game in the casino ($X_i = -1$ or $X_i = -1$) with equal chances and independently of the past events. The gambler will stop gambling when either she broke the bank (a roubles) or lost all her money (b roubles).

Question 1

(a) prove that $S_n = \sum_{i=0}^{n} X_i$ is a martingale;

(b) use **optional-stopping theorem** to find probabilities to win and to loose;

(c) prove that $M_n = S_n^2 - n$ is a martingale;

(d) find the expected number of rounds before she will stop gambling.

Task 2: Unfair gambler's ruin

A gambler wins or looses one rouble in each round of the game in the casino ($X_i = -1$ or $X_i = -1$) independently of the past events. But the chances are **not equal**, the probability of winning is p. The gambler will stop gambling when either she broke the bank (a roubles) or lost all her money (b roubles).

Question 2

(a) prove that $S_n = \sum_{i=0}^{n} X_i$ is not a martingale;

(b) prove that $K_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale;

(c) use **optional-stopping theorem** to find probabilities to win and to loose;

(d) prove that $M_n = S_n - (p - q)n$ is a martingale;

(e) find the expected number of rounds before she will stop gambling.

Task 3

Let X_n be a simple symmetric random walk and $\mathcal{F}_n$ its natural filtration. Find a deterministic sequence a_n such that $Z_n = X_n^3 + a_n X_n$ be a martingale with respect to $\mathcal{F}_n$.

Sources:

1. Demeshev B., Problems on stochastic analysis https://github.com/bdemeshev
2. D. Williams, Probability with Martingales. Cambridge University Press, 1991