Lecture 5
Conditional Expectation

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7 October 2024

Lecture

- It all starts with the definition of conditional probability: $P(A|B) = P(AB)/P(B)$.
- If X and Y are jointly discrete random variables, we can use this to define a probability mass function for X *given* $Y = y$.
- That is, we write $p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{p(x,y)}{p_Y(y)}$.
- In words: first restrict sample space to pairs (x, y) with given y value. Then divide the original mass function by $p_Y(y)$ to obtain a probability mass function on the restricted space.
- We do something similar when X and Y are continuous random variables. In that case we write $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.
- Often useful to think of sampling (X, Y) as a two-stage process. First sample Y from its marginal distribution, obtain $Y = y$ for some particular y. Then sample X from its probability distribution *given* $Y = y$.
- Marginal law of X is weighted average of conditional laws.

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- Let X be value on one die roll, Y value on second die roll, and write $Z = X + Y$.
- What is the probability distribution for X given that $Y = 5$?
- Answer: uniform on $\{1, 2, 3, 4, 5, 6\}$.
- What is the probability distribution for Z given that $Y = 5$?
- Answer: uniform on $\{6, 7, 8, 9, 10, 11\}$.
- What is the probability distribution for Y given that $Z = 5$?
- Answer: uniform on $\{1, 2, 3, 4\}$.

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- Now, what do we mean by $E[X|Y=y]$? This should just be the expectation of X in the conditional probability measure for X given that $Y=y$.
- Can write this as $E[X|Y=y] = \sum_x xP\{X=x|Y=y\} = \sum_x xp_{X|Y}(x|y)$.
- Can make sense of this in the continuum setting as well.
- In continuum setting we had $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$. So $E[X|Y=y] = \int_{-\infty}^{\infty} x\frac{f(x,y)}{f_Y(y)}dx$

Lecture

- Let X be value on one die roll, Y value on second die roll, and write $Z = X + Y$.
- What is $E[X|Y = 5]$?
- What is $E[Z|Y = 5]$?
- What is $E[Y|Z = 5]$?

Lecture

- Can think of $E[X|Y]$ as a function of the random variable Y. When $Y = y$ it takes the value $E[X|Y = y]$.
- So $E[X|Y]$ is itself a random variable. It happens to depend only on the value of Y.
- Thinking of $E[X|Y]$ as a random variable, we can ask what *its* expectation is. What is $E[E[X|Y]]$?
- **Very useful fact:** $E[E[X|Y]] = E[X]$.
- In words: what you expect to expect X to be *after learning* Y is same as what you *now* expect X to be.
- Proof in discrete case: $E[X|Y = y] = \sum_x xP\{X = x|Y = y\} = \sum_x x\frac{p(x,y)}{p_Y(y)}$.
- Recall that, in general, $E[g(Y)] = \sum_y p_Y(y)g(y)$.
- $E[E[X|Y = y]] = \sum_y p_Y(y) \sum_x x\frac{p(x,y)}{p_Y(y)} = \sum_x \sum_y p(x,y)x = E[X]$.

Lecture

- Definition:
  $\mathrm{Var}(X|Y) = E\big[(X - E[X|Y])^2|Y\big] = E\big[X^2 - E[X|Y]^2|Y\big]$.
- $\mathrm{Var}(X|Y)$ is a random variable that depends on Y. It is the variance of X in the conditional distribution for X given Y.
- Note $E[\mathrm{Var}(X|Y)] = E[E[X^2|Y]] - E[E[X|Y]^2|Y] = E[X^2] - E[E[X|Y]^2]$.
- If we subtract $E[X]^2$ from first term and add equivalent value $E[E[X|Y]]^2$ to the second, RHS becomes $\mathrm{Var}[X] - \mathrm{Var}[E[X|Y]]$, which implies following:
- **Useful fact:** $\mathrm{Var}(X) = \mathrm{Var}(E[X|Y]) + E[\mathrm{Var}(X|Y)]$.
- One can discover X in two stages: first sample Y from marginal and compute $E[X|Y]$, then sample X from distribution given Y value.
- Above fact breaks variance into two parts, corresponding to these two stages.

Lecture

- Let X be a random variable of variance σ_X^2 and Y an independent random variable of variance σ_Y^2 and write $Z = X + Y$. Assume $E[X] = E[Y] = 0$.
- What are the covariances $\mathrm{Cov}(X, Y)$ and $\mathrm{Cov}(X, Z)$?
- How about the correlation coefficients $\rho(X, Y)$ and $\rho(X, Z)$?
- What is $E[Z|X]$? And how about $\mathrm{Var}(Z|X)$?
- Both of these values are functions of X. Former is just X. Latter happens to be a constant-valued function of X, i.e., happens not to actually depend on X. We have $\mathrm{Var}(Z|X) = \sigma_Y^2$.
- Can we check the formula $\mathrm{Var}(Z) = \mathrm{Var}(E[Z|X]) + E[\mathrm{Var}(Z|X)]$ in this case?

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- Sometimes think of the expectation $E[Y]$ as a "best guess" or "best predictor" of the value of Y.
- It is best in the sense that at among all constants m, the expectation $E[(Y - m)^2]$ is minimized when $m = E[Y]$.
- But what if we allow non-constant predictors? What if the predictor is allowed to depend on the value of a random variable X that we can observe directly?
- Let $g(x)$ be such a function. Then $E[(y - g(X))^2]$ is minimized when $g(X) = E[Y|X]$.

Lecture

- Toss 100 coins. What's the conditional expectation of the number of heads given the number of heads among the first fifty tosses?
- What's the conditional expectation of the number of aces in a five-card poker hand given that the first two cards in the hand are aces?

Lecture

Conditional expectation, $\mathbb{E}(X \mid Y)$, *is a random variable with randomness inherited from* Y, *not* X.

Example: Suppose $Y = \begin{cases} 1 & \text{with probability } 1/8\,, \\ 2 & \text{with probability } 7/8\,, \end{cases}$

and $X \mid Y = \begin{cases} 2Y & \text{with probability } 3/4\,, \\ 3Y & \text{with probability } 1/4\,. \end{cases}$

Conditional variance

The conditional variance is similar to the conditional expectation.

- $\mathrm{Var}(X \mid Y = y)$ is the variance of X, when Y is fixed at the value $Y = y$.
- $\mathrm{Var}(X \mid Y)$ is a random variable, giving the variance of X when Y is fixed at a value to be selected randomly.

Definition: Let X and Y be random variables. The **conditional variance of X, given Y,** is given by

$$\boxed{\mathrm{Var}(X \mid Y) = \mathbb{E}(X^2 \mid Y) - \left\{\mathbb{E}(X \mid Y)\right\}^2 = \mathbb{E}\left\{(X - \mu_{X \mid Y})^2 \mid Y\right\}}$$

Lecture

If all the expectations below are finite, then for ANY random variables X and Y, we have:

i) $\boxed{\mathbb{E}(X) = \mathbb{E}_Y\Big(\mathbb{E}(X\,|\,Y)\Big)}$ *Law of Total Expectation.*

Note that we can pick any r.v. Y, to make the expectation as easy as we can.

ii) $\mathbb{E}(g(X)) = \mathbb{E}_Y\Big(\mathbb{E}(g(X)\,|\,Y)\Big)$ *for any function g.*

iii) $\boxed{\mathrm{Var}(X) = \mathbb{E}_Y\Big(\mathrm{Var}(X\,|\,Y)\Big) + \mathrm{Var}_Y\Big(\mathbb{E}(X\,|\,Y)\Big)}$

Law of Total Variance.

Thank you for your attention!

See next week!