Conditional expectations

$E(X)$ - num; $\underbrace{E(Y|X)}_{\text{the best guess about } Y \text{ with info } X.} = g(X)$ ← random variable

Task 0:

	w_1	w_2	
p	$1/3$	$1/3$	$1/3$
X weather	0	1	1
Y mood	-1	0	1

0 - bad
1 - good
-1 - awful

$$E(y|x=0) = -1\cdot(p=1) = -1$$

$$E(y|x=1) = \underbrace{\tfrac{1}{2}}_{\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}}}\cdot 1 + \tfrac{1}{2}\cdot 0 = \tfrac{1}{2} \quad \Bigg| \quad E(y|x) = \begin{cases} -1 & p=\frac{1}{3} \\ \frac{1}{2} & p=\frac{2}{3} \end{cases}$$

$$E(y) = \tfrac{1}{3}\cdot(-1) + \tfrac{1}{3}\cdot 0 + \tfrac{1}{3}\cdot 1 = 0$$

$$E(E(y|x)) = -1\cdot\underset{w_1}{\tfrac{1}{3}} + \tfrac{1}{2}\cdot\underset{w_2}{\tfrac{2}{3}} = 0$$

tower prop.

$\star$ $\underset{\text{info 1}}{E(y|x)} = \underset{\text{info 2}}{E(y|x^3)}$ info 1 = info 2

$E(y|x) \neq E(y|x^2)$ info 1 ≠ info 2

$E(y|x) = E(y|x, x^2)$ info 1 = info 2

* $E(aX+b\mid Y) = b + aE(X\mid Y)$

$E(\underbrace{E(X\mid Y)}) = EX$ tower prob.

$E(f(Y)\mid Y) = f(Y)$

$E(f(Y)X\mid Y) = f(Y)\cdot E(X\mid Y)$

$Var(X\mid Y) = E(X^2\mid Y) - E^2(X\mid Y)$

$Var(aX+b\mid Y) = a^2 Var(X\mid Y)$

$Var(X) = Var(E(X\mid Y)) + E(Var(X\mid Y))$

↑ total variance

Task1 $X \sim Pois(1)$, $Y \sim U[1,2]$

$\underline{X,Y}$ indep.

$E(XY\mid X)$, $cov(XY,X)$, $Var(XY\mid X)$, $Var(XY)$

* $E(XY\mid X) = XE(Y\mid X) \underset{indep}{=} X\,EY \overset{EY=\frac{3}{2}}{=} X\cdot\frac{1+2}{2} = \frac{3X}{2}$

* $cov(XY, X) = E(XY\cdot X) - E(XY)\cdot E(X) =$

$= E(X^2Y) - E(XY)\cdot E(X) \underset{indep}{=} \underbrace{EX^2}\cdot EY - \underbrace{E^2X}\,EY =$

$= (\underbrace{EX^2} - \underbrace{E^2X})EY = VarX\cdot EY = VarX\cdot\frac{3}{2}$

$VarX - ?$ $X \sim Pois(1)$

↑ λ

$\boxed{Mgf\ (t) = e^{\lambda(e^t-1)}}$ $EX^k = \frac{d^k}{}M(t)\Big|$

$$\boxed{M_{g+X}(\ldots)} \qquad EX^k = \frac{d^k}{dt^k} \ldots \Big|_{t=0}$$

$$EX = \frac{d}{dt} e^{\lambda(e^t-1)}\Big|_{t=0} = \lambda e^{\lambda(e^t-1)} \cdot e^t \Big|_{t=0} = \lambda$$

$$EX^2 = \frac{d}{dt}\left(\lambda e^{\lambda e^t - \lambda + t}\right)\Big|_{t=0} = \lambda e^{\lambda e^t - \lambda + t}(\lambda e^t + 1)\Big|_{t=0} =$$

$$= \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$\operatorname{cov}(xy, x) = (EX^2 - E^2X)\,Ey = (\lambda^2 + \lambda - \lambda^2)\,\frac{3}{2} =$$

$$= \underset{\operatorname{Var} X}{\underset{\parallel}{\lambda}} \cdot \frac{3}{2} = \frac{3}{2}$$

- $\operatorname{Var}(xy \mid X) = X^2 \operatorname{Var}(y \mid x) \underset{\text{indep}}{=} X^2 \operatorname{Var} y = X^2(Ey^2 - E^2y) =$

$$Eg(y) = \int_{\mathbb{R}} g(y) \cdot f(y)\,dy$$

$$\rightarrow Ey^2 = \int_1^2 y^2 \cdot 1 \cdot dy = \frac{y^3}{3}\Big|_1^2 = \frac{8-1}{3} = \frac{7}{3}$$

$$= X^2\left(\frac{7}{3} - \left(\frac{3}{2}\right)^2\right) = X^2\left(\frac{7}{3} - \frac{9}{4}\right) = \frac{1}{12}X^2$$

- $\operatorname{Var}(XY) = E(\operatorname{Var}(xy \mid x)) + \operatorname{Var}(E(xy \mid x)) =$

$$= E\left(\frac{X^2}{12}\right) + \operatorname{Var}\left(\frac{3X}{2}\right) = \frac{EX^2}{12} + \frac{9}{4}\operatorname{Var}(x) =$$

$$= \frac{1+1}{12} + \frac{9}{4} = \frac{1}{6} + \frac{9}{4} = \frac{29}{12}$$

we used previous points

Task2

$$f(x,y) = \begin{cases} x+y & \text{if } x,y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$f(x)$, $f(y|x)$, $E(y|x)$, $Var(y|x)$ $-?$

- $f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = \int_0^1 x\,dy + \int_0^1 y\,dy =$

$= xy\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 = x + \frac{1}{2}$

$\left(P(A|B) = \frac{P(AB)}{P(B)}\right)$

- $f(y|x) \underset{\text{by def}}{=} \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}}$

- $E(y) = \int_0^1 y \cdot f(y)\,dy$

$E(y|x) = \int_0^1 y \cdot f(y|x)\,dy = \int_0^1 y\,\frac{x+y}{x+\frac{1}{2}}\,dy =$

$= \frac{1}{x+\frac{1}{2}} \int_0^1 (yx+y^2)\,dy = \frac{1}{x+\frac{1}{2}}\left(\frac{xy^2}{2}\Big|_0^1 + \frac{y^3}{3}\Big|_0^1\right) =$

$= \frac{1}{x+\frac{1}{2}}\left(\frac{x}{2} + \frac{1}{3}\right)$

- $Var(y|x) = E(y^2|x) - \underbrace{E^2(y|x)}_{\text{is known from prev.}}$ $=$

$\Big[E(y^2|x) = \int_0^1 y^2 f(y|x)\,dy = \int_0^1 y^2\,\frac{x+y}{x+\frac{1}{2}}\,dy =$

$= \frac{1}{x+\frac{1}{2}} \int_0^1 (xy^2+y^3)\,dy = \frac{1}{x+\frac{1}{2}}\left(\frac{xy^3}{3}\Big|_0^1 + \frac{y^4}{4}\Big|_0^1\right) =$

$= \frac{1}{x+\frac{1}{2}}\left(\frac{x}{3} + \frac{1}{4}\right) \Big]$

$= \frac{1}{x+\frac{1}{2}}\left(\frac{x}{3}+\frac{1}{4}\right) - \frac{1}{(x+\frac{1}{2})^2}\left(\frac{x}{2}+\frac{1}{3}\right)^2$ ∎

-n-

Task 3 $\quad X, Y$ - iid $\sim \exp(n) \quad \mathbb{P}(X=x) = ne$

$\rightarrow \mathbb{P}(X \mid X+Y), \rightarrow \mathbb{E}(X \mid X+Y)$

- $X \in \{0,1\}$, $Y \in \{0,1\}$ $\rightarrow$ $\mathbb{E}(X \mid 0) = 0$, $\mathbb{E}(X \mid 1) = \frac{1}{2} \rightarrow \mathbb{E}(X \mid X+Y) = \frac{X+Y}{2}$, $\mathbb{E}(X \mid 2) = 1$

- $\mathbb{P}(X = x \mid X+Y = t) = \dfrac{\mathbb{P}(X=x, X+Y=t)}{\mathbb{P}(X+Y=t)} =$

$$= \frac{\mathbb{P}(X=x, Y=t-x)}{\mathbb{P}(X+Y=t)} = \frac{\mathbb{P}(x)\cdot \mathbb{P}(t-x)}{\mathbb{P}(X+Y=t)} = \frac{ne^{-nx}\, ne^{-n(t-x)}}{\mathbb{P}(X+Y=t)} =$$

$$= \frac{n^2 e^{-nt}}{\mathbb{P}(X+Y=t)} \;=$$

$$\mathbb{P}(X+Y=t) = \int_0^t ne^{-na} \cdot ne^{-n(t-a)}\, da \;=$$

$[\;\mathbb{P}(X=a, Y=t-a)$ for all a

$$= \; n^2 \int_0^t e^{-nt}\, da = n^2 e^{-nt} \cdot a \Big|_0^t = n^2 t e^{-nt} \;]$$

$$= \; \frac{n^2 e^{-nt}}{n^2 t e^{-nt}} = \frac{1}{t} = \boxed{\frac{1}{X+Y}}$$

- $\mathbb{E}(X=x \mid X+Y) = \displaystyle\int_0^{X+Y} x \frac{1}{X+Y}\, dx = \frac{1}{X+Y} \int_0^{X+Y} x\, dx = \frac{x^2}{2}\Big|_0^{X+Y} \cdot \frac{1}{X+Y} =$

$$= \frac{(x+y)^2}{2(x+y)} = \boxed{\frac{x+y}{2}}$$