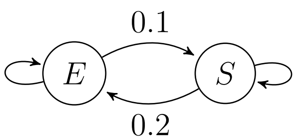
Be brave! You can use python. In this case just provide the code. You can use ChatGPT or any other LLM. In this case just provide the full prompt. Don't panic!

Home assignment 1

Deadline: 2024-09-23, 21:00.

1. The Cat can be only in two states: Sleeping (S) and Eating (E). Cat's mood depends only on the previous state. The transition probabilities are given below:



   (a) Compute the missing probabilities on the graph.

   (b) Write down the transition matrix.

   (c) Compute $\mathbb{P}(X_3 = \text{Eating} \mid X_0 = \text{Eating})$.

2. Cowboy Joe enters the Epsilon Bar and orders one pint of beer. He drinks it and orders one pint more. And so on and so on and so on... The problem is that the barmaid waters down each pint with probability 0.2 independently of other pints. Joe does not like watered down beer. He will blow the Epsilon Bar to hell if two or more out of the last three pints are watered down.

   We point out that Joe never drinks less than 3 pints in a bar.

   (a) What is the expected number of pints of beer Joe will drink?

   Let Y_t be the indicator that the pint number t was watered down. Consider the Markov chain $S_t = (y_{t-2}, y_{t-1}, y_t)$. For example, $S_t = (100)$ means that the pint number $t-2$ was watered down while pints number $t-1$ and t are good.

   (b) What are the possible values of S_3 and their probabilities?

   (c) Write down the transition matrix of this Markov chain.

   Note: questions (2b) and (2c) were updated!

3. Pavel Durov starts at the point $X_0 = 3$ on the real line. Each minute he moves left with probability 0.4 or right with probability 0.6 independently of past moves. The points 0 and 5 are absorbing. If Pavel reaches 0 or 5 he stays there forever. Let X_t be the coordinate of Pavel after t minutes.

   (a) Write down the transition matrix of this Markov chain.

   (b) Calculate the distribution of X_7 [list all values of the random variable X_7 and estimate the probabilities].

   Hint: you are free to use python for this problem :)

Home assignment 2

Deadline: 2024-09-27, 21:00.

1. [10 points] Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCD$. Each minute each hedehog simulteneously and independently chooses to go clockwise to the adjacent point, to go counter-clockwise to the adjacent point or to stay at his location.

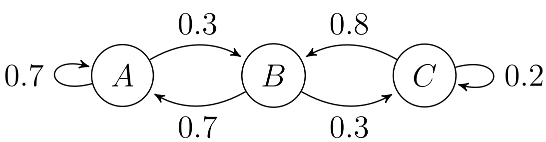
   Thus the brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.

   (a) Draw the graph for the brotherhood Markov chain and calculate all transition probabilities.

   (b) Write down the transition matrix of the brotherhood Markov chain.

   (c) What is the probability that they will be in one vertex after 3 steps?

2. [10 points] Consider the following Markov chain:



   (a) Find the stationary distribution of this Markov chain.

   The Markov chains starts at the vertex A. Let N be the first moment when the state C will be reached.

   (b) Find the expected value $\mathbb{E}(N)$.

   (c) Find the variance $\mathbb{V}\mathrm{ar}(N)$.

3. [10 points] Bonnie and Clyde start at the points $(5, 0)$ and $(-5, 0)$ of the plane. Each minute each of them simulteneously and independently makes one step in one of the four possible directions (south, north, east, west).

   Each of them does n steps. Let X be the number of times they will be at the same point.

   (a) Estimate the probability $\mathbb{P}(X \geq 1)$ for $n = 50$ using $B = 10000$ simulations.

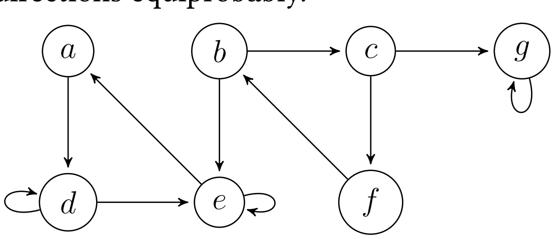
   (b) Estimate $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$ for $n = 50$ using $B = 10000$ simulations.

   (c) Plot the estimated value of $\mathbb{E}(X)$ as a function of n for n from 1 to 200 using $B = 10000$ simulations.

Home assignment 3

Deadline: 2024-10-04, 23:59.

1. [10 points] We randomly wander on the graph choosing at each moment of time one of the possible directions equiprobably.



(a) Split each Markov chain into communicating classes.
(b) Find the period of every state.
(c) Classify each state as transient or recurrent.
(d) For recurrent states find the expected return time.
(e) Find the stationary distributions.

2. [10 points] Design a Markov chain with 3 states and unique stationary distribution $\pi = (0.1, 0.2, 0.7)$.

3. [10 points] Consider three games:

Game A: You toss a biased coin with probability 0.48 of H. You get $+1$ dollar for H and -1 dollar for T.

Game B: If your wellfare is divisible by three you toss a coin that lands on H with probability 0.09. If your wellfare is not divisible by three you toss a coin that lands on H with probability 0.74. You get $+1$ dollar for H and -1 dollar for T.

Game C: You toss an unbiased coin. If it lands on H you play Game A. If it lands on T you play Game B.

Your initial capital is 10000\$.

(a) Generate and plot two random trajectories of your wellfare if you play Game A 10^6 times.
(b) Generate and plot two random trajectories of your wellfare if you play Game B 10^6 times.
(c) Generate and plot two random trajectories of your wellfare if you play Game C 10^6 times.

Home assignment 4

Deadline: 2024-10-14, 23:59.

1. Recognise the distribution family and its parameters by looking at the moment-generating function:

(a) $0.7 + 0.3\exp(t)$;
(b) $\exp(2024\exp(t))/\exp(2024)$;
(c) $\exp(6t + 2024t^2)$;
(d) $1/(5t-1)^{2024}$.

You may use the table from the article https://en.wikipedia.org/wiki/Moment-generating_function.

2. Consider the moment-generating function of a random variable X:

$$g(t) = \frac{\exp(3t) - 1}{3t\exp(-2t)}.$$

(a) Expand the function $g(t)$ as Taylor series up to t^4 included.
(b) Find $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\mathbb{E}(X^3)$, $\mathbb{E}(X^4)$.

3. The moment-generating function of the pair of random variables (X, Y) is given by $\exp(6t_1 + 5t_2 + t_1^2 + 20t_2^2 - 2t_1t_2)$.

Find $\mathbb{E}(X)$, $\mathbb{V}\text{ar}(Y)$, $\mathbb{E}(XY)$.

Home assignment 5

Deadline: 2024-10-18, 23:59.

1. [10 points] The random variables X_i are independent and exponentially distributed with rate $\lambda = 1$.

   (a) Find the probability limit
   $$\operatorname{plim} \frac{X_1 + X_2 + X_3 + \dots + X_n}{2n + 7}.$$

   (b) Find the probability limit
   $$\operatorname{plim} \frac{X_1^2 + X_2^2 + X_3^2 + \dots + X_n^2}{2n + 7}.$$

   (c) Find the probability limit
   $$\operatorname{plim} \min\{X_1, X_2, X_3, \dots, X_n\}.$$

   (d) Find the probability limit
   $$\operatorname{plim} \sqrt[n]{\exp(2X_1 + 2X_2 + \dots + 2X_n)}.$$

2. [10 points] Polina loves sweet chestnuts. She has infinite sequence of baskets before her. In the basket number n there are n chestnuts in total. Unfortunately only one chestnut in every basket is a sweet one.

   She picks chestnuts one by one at random from all the buskets sequentially. First she picks the unique chestnut from the basket number one, than she picks in a random order two chestnuts from the basket number two and so on.

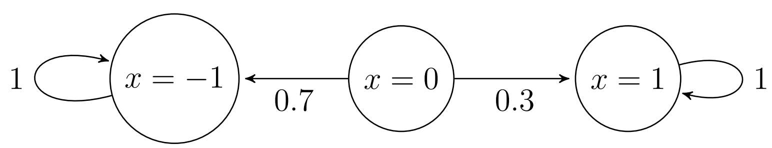
   The random variable S_t indicates whether the chestnut number t was a sweet one.

   (a) Find $\lim S_t$ or prove that the limit does not exist.

   (b) Find $\operatorname{plim} S_t$ or prove that the limit does not exist.

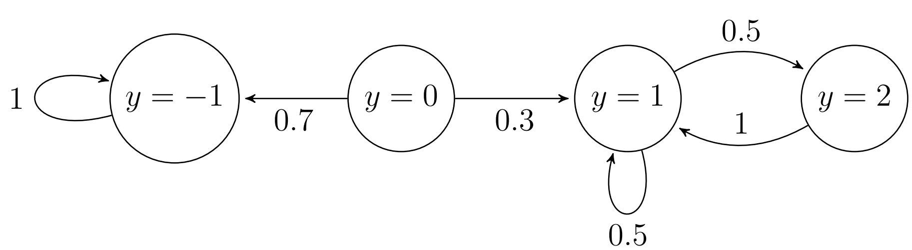
   (c) Find mean square limit of S_t or prove that the limit does not exist.

3. [10 points] Consider two Markov chains, (X_t) and (Y_t):



with $X_0 = 0$;

and



with $Y_0 = 0$.

   (a) Find $\mathbb{P}(\lim X_n \text{ exists})$ and $\mathbb{P}(\lim Y_n \text{ exists})$.

   (b) Find the limiting distribution of (X_n) and the limiting distribution of (Y_n).
   Hint: here you need to calculate all limits $\lim \mathbb{P}(X_n = k)$, $\lim \mathbb{P}(Y_n = k)$.

   (c) Does (X_n) converges almost surely? In distribution? In probability?

   (d) Does (Y_n) converges almost surely? In distribution? In probability?

Home assignment 6

Deadline: 2024-11-01, 23:59.

1. [10 points] Albert Nikolayevich Shiryaev randomly selects a natural number N from 1 to 7. Let Y be the remainder after division of N by 2 and X be the remainder after division of N by 3.

   (a) Write the joint probability table for (X, Y).

   (b) Find $\mathbb{E}(Y \mid X)$. Is it linear in X?

   (c) Find $\mathbb{E}(\mathbb{E}(Y \mid X))$ and $\mathbb{V}\text{ar}(\mathbb{E}(Y \mid X))$.

   (d) Find $\mathbb{V}\text{ar}(Y \mid X)$.

   (e) Find $\mathbb{E}(\mathbb{V}\text{ar}(Y \mid X))$.

2. [10 points] Albert Nikolayevich selects a random point uniformly inside a quadrilateral $ABCD$ where $A = (0, 0)$, $B = (0, 2)$, $C = (4, 4)$, $D = (4, 0)$.

   (a) Find $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

   (b) Find $\mathbb{V}\text{ar}(Y \mid X)$ and $\mathbb{V}\text{ar}(X \mid Y)$.

   Hint: you may use the formula for the variance of uniform distribution :)

3. [10 points] Albert Nikolayevich selects a random point (X, Y) with joint probability density

$$f(x, y) = \begin{cases} (3x^2 + 4y^3)/2, & \text{if } x \in [0; 1], y \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) For the random variable x find the marginal probability density function $f(x)$.

   (b) Find the conditional density $f(y \mid x)$.

   (c) Find the conditional expected value $\mathbb{E}(Y \mid X)$. Is it linear in X?

   (d) Find $\mathbb{V}\text{ar}(Y \mid X)$. Is it constant?

   (e) Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{C}\text{ov}(X, Y)$ and $\mathbb{V}\text{ar}(X)$.

Home assignment 7

Deadline: 2024-11-03, 23:59.

1. Experiment may end by one of the six outcomes:

	$X = -2$	$X = 0$	$X = 2$
$Y = -1$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

   (a) Find expicitely the sigma-algebras $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$.

   (b) How many elements are there in $\sigma(X + Y)$, $\sigma(X - Y)$?

(c) Calculate conditional expected values $\mathbb{E}(X \mid \sigma(Y))$, $\mathbb{E}(X \mid \sigma(X+Y))$.

2. We throw a coin infinitely many times. Let X_n be the indicator that the coin landed on Head at toss number n. Consider a pack of σ-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, \ldots, X_n)$, $\mathcal{H}_n := \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$.

   (a) Simplify experessions: $\mathcal{F}_{11} \cap \mathcal{F}_{25}$, $\mathcal{F}_{11} \cup \mathcal{F}_{25}$, $\mathcal{H}_{11} \cup \mathcal{H}_{25}$.

   (b) For each case provide two examples of σ-algebras that contain the corresponding event

      i. $\{X_{37} > 0\}$;
      ii. $\{X_{37} > X_{2024}\}$;
      iii. $\{X_{37} > X_{2024} > X_{12}\}$;

   (c) For each case provide two non-trivial examples (different from Ω and $\emptyset$) of an event A such that

      i. $A \in \mathcal{F}_{2024}$;
      ii. $A \notin \mathcal{F}_{2025}$;
      iii. $A \in \mathcal{H}_n$ for all possible n;

3. Consider a fair dice. In the experiment we throw the dice until the first six appears.

   (a) Simulate $B = 100000$ experiments. For every experiment number i record the total number of throws, y_i, and the number of even faces appeared, x_i.

   (b) For all values of x where you have more than 100 records estimate $\hat{\mu}(x) = \hat{\mathbb{E}}(y_i \mid x_i = x)$ and $\hat{v}(x) = \widehat{\mathbb{V}\mathrm{ar}}(y_i \mid x_i = x)$.

   (c) Explain intuitively why $\hat{\mu}(0)$ is less than 3.

   (d) Randomly select 100 experiments out of all B experiments. Draw the scatter plot (x_i, y_i) for randomly selected experiments. Add the line $\hat{\mu}(x)$ with bands $\hat{\mu}(x) \pm 2\sqrt{\hat{v}(x)}$ to the scatter plot.

   (e) Is it reasonable to assume that $\hat{\mu}(x)$ is linear?

   (f) Is it reasonable to assume that $\hat{v}(x)$ is constant?

   No formal tests are required for the last two questions, graphical analysis is sufficient.

Home assignment 8

Deadline: 2024-11-16, 23:59.

1. The random variables X_n are independent and take values $+1$ with probability 0.7 or 2 with probability 0.3. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the cumulative sum.

   (a) Find the constant a such that $M_n = S_n - an$ is a martingale.

   (b) Find all constants b such that $K_n = \exp(bS_n)$ is a martingale.

2. The population starts with one microbe Eve. So the size of the initial generations is $G_0 = 1$. After one minute every microbe either dies with probability 0.2, remains alive with probability 0.5 or splits in two copies with probability 0.3. Let G_n be the size of microbe population after n minutes.

   (a) Draw a pretty picture of Eve :)

(b) Find the distribution of G_2.

(c) Find a constant a such that $M_n = G_n/a^n$ is a martingale.

(d) Let D be the event of eventual death of the microbe civilization. Check whether the process $K_n = \mathbb{E}(I_D \mid G_n, G_{n-1}, \ldots, G_0)$ is martingale. Here I_D is the indicator of the event D.

(e) Using first step analysis find $\mathbb{P}(D)$.
Hint: you may obtain a quadratic equation for $\mathbb{P}(D)$, the smallest root is your friend :)

3. Consider a symmetric random walk $S_t = X_1 + X_2 + \cdots + X_t$ where (X_t) are independent identically distributed and take values $+1$ or -1 with equal probabilities. We start at zero, $S_0 = 0$.

Let N_k the number of visits of point k before the first return of a random walk to 0.

(a) What is the probability that a random walk will eventually return to 0?

(b) What is the probability $\mathbb{P}(N_k = 0)$?

(c) What is the probability $\mathbb{P}(N_k = n)$ for arbitrary natural number n?

(d) Compare $\mathbb{E}(N_2)$ and $\mathbb{E}(N_{\text{undecillion}})$.

Home assignment 9

Deadline: 2024-12-03, 23:59.

1. It's $X_0 = -35°C$ today in Oymyakon. Each day the daily temperature (X_t) may go up or down by $2°$ degrees with equal probability independently of the previous days.

Initially my piggy-bank is empty, $B_t = 0$. Each day I add in my piggy-bank $|X_t|$ undecillion roubles, $B_t = B_{t-1} + |X_t|$. I will do my final investment in the piggy-bank when the temperature in Oymyakon will reach $-45°C$ or $-25°C$. At the end of this day τ I will break the piggy-bank and go for vacations in Oymyakon.

(a) Is (X_t) a martingale?

(b) Using (X_t) and Doob's theorem or otherwise find $\mathbb{P}(X_\tau = -25)$.

(c) Find a constant a such that $Y_t = X_t^2 - at$ is a martingale.

(d) Using (Y_t) and Doob's theorem find $\mathbb{E}(\tau)$.

(e) Find a constant b such that $W_t = B_t + bX_t t$ is a martingale.

(f) Find my expected final wellfare $\mathbb{E}(B_\tau)$.

Comment: you may not check technical conditions of the Doob's theorem.

2. Consider a Poisson point process (X_t) of snowflakes falling in my palm with intensity $\lambda = 0.5$ snowflakes per second.
   (a) What is the probability that in 5 I will catch two or more snowflakes?
   (b) I've just opened my palm. What is the probability that the next two snowflakes will fall in less than three seconds?
   (c) Find conditional probability $\mathbb{P}(X_{10} = 5 \mid X_4 = 1)$.
   (d) Find conditional expected value $\mathbb{E}(X_{10} \mid X_4 = 1)$ and variance $\mathbb{V}\text{ar}(X_{10} \mid X_4 = 1)$.

3. Insurance claims arrive according to Poisson process with intensity 100 claims per month. The payments for each claim are independent random variables uniformly distributed between 0 and 1 undecillion roubles.

   Simulate 10^4 trajectories of this process with monthly duration.
   (a) Draw the histogram of total payments for 10 days.
   (b) Estimate the probability that the total payments for 10 will be more than 12 undecillion roubles.
   (c) Estimate reserves necessary for the company that are sufficient to cover all the claims with probability 0.05.
   (d) Recalculate estimates in (b) and (c) if every saturday and sunday the intensity drops from 100 to 10 and the month starts from monday.

Home assignment 10

Deadline: 2024-12-14, 23:59.

1. Let (W_t) be a standard Wiener process.
   (a) Calculate $\mathbb{E}(W_9 - 2W_6)$, $\mathbb{V}\text{ar}(W_9 - 2W_6)$, $\mathbb{P}(W_9 - 2W_6 > 1)$.
   (b) Calculate $\mathbb{E}(W_9 - 2W_6 \mid W_1 = 1)$, $\mathbb{V}\text{ar}(W_9 - 2W_6 \mid W_1 = 1)$, $\mathbb{P}(W_9 - 2W_6 > 1 \mid W_1 = 1)$.
   (c) Calculate $\mathbb{C}\text{ov}(W_9, W_6 - W_1)$ and $\mathbb{C}\text{ov}(W_9, W_6 - W_1 \mid W_1 = 1)$.

   Remark: we will include a similar problem in the exam.

2. Let (W_t) be a standard Wiener process. Consider three more processes, $R_t = W_t^6 \cos t$, $X_t = Y_t^3 + t^2 Y_t$ and $dY_t = W_t^2 dW_t + tW_t dt$ with $Y_0 = 1$.
   (a) Find dR_t and dX_t.
   (b) Represent the process R_t as a sum of an Ito integral and a Riemann integral.
   (c) Check whether X_t is a martingale.
   (d) Find $\mathbb{E}(Y_t)$.
   (e) Find $\mathbb{V}\text{ar}(Y_t)$. Hint: find $d(Y_t^2)$ and calculate $\mathbb{E}(Y_t^2)$.

3. In the framework of Black and Scholes model find the price at $t = 0$ of the following two financial assets, $dS_t = \mu S_t dt + \sigma S_t dW_t$ is the share price equation.
   (a) The asset pays you at time T exactly one dollar if $S_T < K$ where K is a constant specified in the contract.
   (b) The asset pays you at time T exactly S_T^2 dollars.

Home assignment 11

Deadline: 2025-02-04, 23:59.

1. Consider a process $y_t = 6 + u_t - 0.5u_{t-1} + 0.06u_{t-2}$ where (u_t) is a white noise process with variance σ_u^2.
   (a) Find the expected value $\mathbb{E}(y_t)$ and the autocorrelation function ρ_k of this process.
   (b) Is (y_t) weakly stationary? Is (y_t) invertable with respect to (u_t)?
   (c) If (y_t) is invertable with respect to (u_t) then find α, δ_1 and δ_2 in the expression
   $$u_t = \alpha + y_t + \delta_1 y_{t-1} + \delta_2 y_{t-2} + \ldots.$$
2. Consider the equation $y_t = 2 + 0.8y_{t-1} + u_t$ where (u_t) is a white noise with variance σ_u^2.
   (a) Find the stationary solution (y_t) in terms of u_t, u_{t-1}, u_{t-2}, ...
   (b) Find the expected value $\mathbb{E}(y_t)$ and autocorrelation function of the stationary solution.
   (c) Find at least one initial value y_0 such that $\mathbb{E}(y_2) = -4$. Is the process with such y_0 stationary?
   (d) How many non-stationary solution are there?
3. Consider the processes $a_t = u_1 \cos(t) + u_2 \sin(t)$ and $b_t = 7 + u_t + (-1)^t u_{t-1}$ where (u_t) is a white noise with variance σ_u^2.
   (a) Check whether each process, (a_t) and (b_t), is stationary.
   (b) For a stationary process find the autocorrelation function and expected value.

Home assignment 12

Deadline: 2025-02-20, 23:59.

1. Consider the equation $y_t = 2 - 0.25y_{t-2} + u_t$, where (u_t) is a white noise process.
   - How many non-stationary solutions does the equation have?
   - How many stationary solutions does this equation have?
   - If there is a stationary solution, can it be presented in $MA(\infty)$ form with respect to u_t?
   - Find the coefficients before u_{t-1} and u_{t-2} of the stationary solution if it exists.
2. Consider the stationary solution of the equation $y_t = 2 + 0.7y_{t-1} - 0.12y_{t-2} + u_t$, where (u_t) is a white noise process with unit variance.
   - Find $\mathbb{E}(y_t)$ and $\mathbb{V}\text{ar}(y_t)$.
   - Find the first two values of the autocorrelation function, ρ_1 and ρ_2.
   - Find the recurrence equation for the values of the autocorrelation function ρ_k for $k \geq 2$.
3. Consider the stationary solution of the equation $y_t = 2 + 0.7y_{t-1} + u_t - 0.5u_{t-1}$, where (u_t) is a white noise process with unit variance.
   - Find $\mathbb{E}(y_t)$ and $\mathbb{V}\text{ar}(y_t)$.
   - Find all values of the autocorrelation function ρ_k.