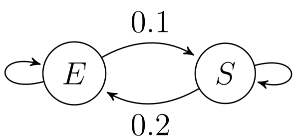
Be brave! You can use python. In this case just provide the code. You can use ChatGPT or any other LLM. In this case just provide the full prompt. Don't panic!

Home assignment 1

Deadline: 2024-09-23, 21:00.

1. The Cat can be only in two states: Sleeping (S) and Eating (E). Cat's mood depends only on the previous state. The transition probabilities are given below:



   (a) Compute the missing probabilities on the graph.
   (b) Write down the transition matrix.
   (c) Compute $\mathbb{P}(X_3 = \text{Eating} \mid X_0 = \text{Eating})$.

2. Cowboy Joe enters the Epsilon Bar and orders one pint of beer. He drinks it and orders one pint more. And so on and so on and so on… The problem is that the barmaid waters down each pint with probability 0.2 independently of other pints. Joe does not like watered down beer. He will blow the Epsilon Bar to hell if two or more out of the last three pints are watered down.

   We point out that Joe never drinks less than 3 pints in a bar.

   (a) What is the expected number of pints of beer Joe will drink?

   Let Y_t be the indicator that the pint number t was watered down. Consider the Markov chain $S_t = (y_{t-2}, y_{t-1}, y_t)$. For example, $S_t = (100)$ means that the pint number $t - 2$ was watered down while pints number $t - 1$ and t are good.

   (b) What are the possible values of S_3 and their probabilities?
   (c) Write down the transition matrix of this Markov chain.

   Note: questions (2b) and (2c) were updated!

3. Pavel Durov starts at the point $X_0 = 3$ on the real line. Each minute he moves left with probability 0.4 or right with probability 0.6 independently of past moves. The points 0 and 5 are absorbing. If Pavel reaches 0 or 5 he stays there forever. Let X_t be the coordinate of Pavel after t minutes.

   (a) Write down the transition matrix of this Markov chain.
   (b) Calculate the distribution of X_7 [list all values of the random variable X_7 and estimate the probabilities].

   Hint: you are free to use python for this problem :)

Home assignment 2

Deadline: 2024-09-27, 21:00.

1. [10 points] Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCD$. Each minute each hedehog simulteneously and independently chooses to go clockwise to the adjacent point, to go counter-clockwise to the adjacent point or to stay at his location.

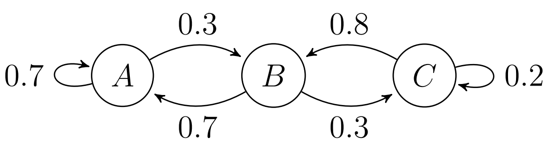
   Thus the brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.

   (a) Draw the graph for the brotherhood Markov chain and calculate all transition probabilities.

   (b) Write down the transition matrix of the brotherhood Markov chain.

   (c) What is the probability that they will be in one vertex after 3 steps?

2. [10 points] Consider the following Markov chain:



   (a) Find the stationary distribution of this Markov chain.

   The Markov chains starts at the vertex A. Let N be the first moment when the state C will be reached.

   (b) Find the expected value $\mathbb{E}(N)$.

   (c) Find the variance $\mathbb{V}\text{ar}(N)$.

3. [10 points] Bonnie and Clyde start at the points $(5, 0)$ and $(-5, 0)$ of the plane. Each minute each of them simulteneously and independently makes one step in one of the four possible directions (south, north, east, west).

   Each of them does n steps. Let X be the number of times they will be at the same point.

   (a) Estimate the probability $\mathbb{P}(X \geq 1)$ for $n = 50$ using $B = 10000$ simulations.

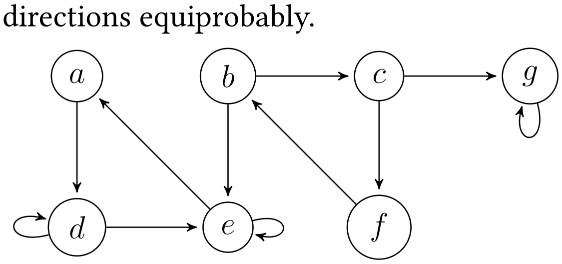
   (b) Estimate $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$ for $n = 50$ using $B = 10000$ simulations.

   (c) Plot the estimated value of $\mathbb{E}(X)$ as a function of n for n from 1 to 200 using $B = 10000$ simulations.

Home assignment 3

Deadline: 2024-10-04, 23:59.

1. [10 points] We randomly wander on the graph choosing at each moment of time one of the possible directions equiprobably.



(a) Split each Markov chain into communicating classes.
(b) Find the period of every state.
(c) Classify each state as transient or recurrent.
(d) For recurrent states find the expected return time.
(e) Find the stationary distributions.

2. [10 points] Design a Markov chain with 3 states and unique stationary distribution $\pi = (0.1, 0.2, 0.7)$.

3. [10 points] Consider three games:

Game A: You toss a biased coin with probability 0.48 of H. You get $+1$ dollar for H and -1 dollar for T.

Game B: If your wellfare is divisible by three you toss a coin that lands on H with probability 0.09. If your wellfare is not divisible by three you toss a coin that lands on H with probability 0.74. You get $+1$ dollar for H and -1 dollar for T.

Game C: You toss an unbiased coin. If it lands on H you play Game A. If it lands on T you play Game B.

Your initial capital is 10000\$.

(a) Generate and plot two random trajectories of your wellfare if you play Game A 10^6 times.
(b) Generate and plot two random trajectories of your wellfare if you play Game B 10^6 times.
(c) Generate and plot two random trajectories of your wellfare if you play Game C 10^6 times.

Home assignment 4

Deadline: 2024-10-14, 23:59.

1. Recognise the distribution family and its parameters by looking at the moment-generating function:

(a) $0.7 + 0.3 \exp(t)$;
(b) $\exp(2024 \exp(t)) / \exp(2024)$;
(c) $\exp(6t + 2024t^2)$;
(d) $1/(5t - 1)^{2024}$.

You may use the table from the article
https://en.wikipedia.org/wiki/Moment-generating_function.

2. Consider the moment-generating function of a random variable X:

$$g(t) = \frac{\exp(3t) - 1}{3t \exp(-2t)}.$$

(a) Expand the function $g(t)$ as Taylor series up to t^4 included.
(b) Find $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\mathbb{E}(X^3)$, $\mathbb{E}(X^4)$.

3. The moment-generating function of the pair of random variables (X, Y) is given by $\exp(6t_1 + 5t_2 + t_1^2 + 20t_2^2 - 2t_1 t_2)$.

Find $\mathbb{E}(X)$, $\mathbb{V}\text{ar}(Y)$, $\mathbb{E}(XY)$.

Home assignment 5

Deadline: 2024-10-18, 23:59.

1. [10 points] The random variables X_i are independent and exponentially distributed with rate $\lambda = 1$.

   (a) Find the probability limit
   $$\operatorname{plim} \frac{X_1 + X_2 + X_3 + \cdots + X_n}{2n+7}.$$

   (b) Find the probability limit
   $$\operatorname{plim} \frac{X_1^2 + X_2^2 + X_3^2 + \cdots + X_n^2}{2n+7}.$$

   (c) Find the probability limit
   $$\operatorname{plim} \min\{X_1, X_2, X_3, \ldots, X_n\}.$$

   (d) Find the probability limit
   $$\operatorname{plim} \sqrt[n]{\exp(2X_1 + 2X_2 + \cdots + 2X_n)}.$$

2. [10 points] Polina loves sweet chestnuts. She has infinite sequence of baskets before her. In the basket number n there are n chestnuts in total. Unfortunately only one chestnut in every basket is a sweet one.

   She picks chestnuts one by one at random from all the buskets sequentially. First she picks the unique chestnut from the basket number one, than she picks in a random order two chestnuts from the basket number two and so on.

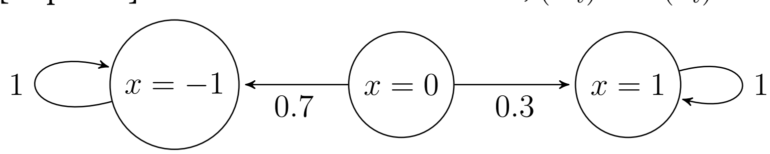
   The random variable S_t indicates whether the chestnut number t was a sweet one.

   (a) Find $\lim S_t$ or prove that the limit does not exist.

   (b) Find $\operatorname{plim} S_t$ or prove that the limit does not exist.

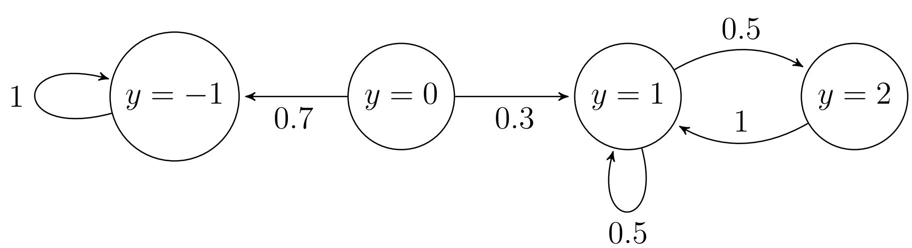
   (c) Find mean square limit of S_t or prove that the limit does not exist.

3. [10 points] Consider two Markov chains, (X_t) and (Y_t):



with $X_0 = 0$;

and



with $Y_0 = 0$.

   (a) Find $\mathbb{P}(\lim X_n \text{ exists})$ and $\mathbb{P}(\lim Y_n \text{ exists})$.

   (b) Find the limiting distribution of (X_n) and the limiting distribution of (Y_n).
   Hint: here you need to calculate all limits $\lim \mathbb{P}(X_n = k)$, $\lim \mathbb{P}(Y_n = k)$.

   (c) Does (X_n) converges almost surely? In distribution? In probability?

   (d) Does (Y_n) converges almost surely? In distribution? In probability?

Home assignment 6

Deadline: 2024-11-01, 23:59.

1. [10 points] Albert Nikolayevich Shiryaev randomly selects a natural number N from 1 to 7. Let Y be the remainder after division of N by 2 and X be the remainder after division of N by 3.

   (a) Write the joint probability table for (X, Y).

   (b) Find $\mathbb{E}(Y \mid X)$. Is it linear in X?

   (c) Find $\mathbb{E}(\mathbb{E}(Y \mid X))$ and $\mathbb{V}\mathrm{ar}(\mathbb{E}(Y \mid X))$.

   (d) Find $\mathbb{V}\mathrm{ar}(Y \mid X)$.

   (e) Find $\mathbb{E}(\mathbb{V}\mathrm{ar}(Y \mid X))$.

2. [10 points] Albert Nikolayevich selects a random point uniformly inside a quadrilateral $ABCD$ where $A = (0, 0)$, $B = (0, 2)$, $C = (4, 4)$, $D = (4, 0)$.

   (a) Find $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

   (b) Find $\mathbb{V}\mathrm{ar}(Y \mid X)$ and $\mathbb{V}\mathrm{ar}(X \mid Y)$.

   Hint: you may use the formula for the variance of uniform distribution :)

3. [10 points] Albert Nikolayevich selects a random point (X, Y) with joint probability density

$$f(x, y) = \begin{cases} (3x^2 + 4y^3)/2, \text{ if } x \in [0; 1], y \in [0; 1]; \\ 0, \text{ otherwise.} \end{cases}$$

   (a) For the random variable x find the marginal probability density function $f(x)$.

   (b) Find the conditional density $f(y \mid x)$.

   (c) Find the conditional expected value $\mathbb{E}(Y \mid X)$. Is it linear in X?

   (d) Find $\mathbb{V}\mathrm{ar}(Y \mid X)$. Is it constant?

   (e) Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{C}\mathrm{ov}(X, Y)$ and $\mathbb{V}\mathrm{ar}(X)$.

Home assignment 7

Deadline: 2024-11-03, 23:59.

1. Experiment may end by one of the six outcomes:

	$X = -2$	$X = 0$	$X = 2$
$Y = -1$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

   (a) Find expicitely the sigma-algebras $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$.

   (b) How many elements are there in $\sigma(X + Y)$, $\sigma(X - Y)$?

(c) Calculate conditional expected values $\mathbb{E}(X \mid \sigma(Y))$, $\mathbb{E}(X \mid \sigma(X+Y))$.

2. We throw a coin infinitely many times. Let X_n be the indicator that the coin landed on Head at toss number n. Consider a pack of σ-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, \ldots, X_n)$, $\mathcal{H}_n := \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$.

   (a) Simplify experessions: $\mathcal{F}_{11} \cap \mathcal{F}_{25}$, $\mathcal{F}_{11} \cup \mathcal{F}_{25}$, $\mathcal{H}_{11} \cup \mathcal{H}_{25}$.

   (b) For each case provide two examples of σ-algebras that contain the corresponding event

      i. $\{X_{37} > 0\}$;

      ii. $\{X_{37} > X_{2024}\}$;

      iii. $\{X_{37} > X_{2024} > X_{12}\}$;

   (c) For each case provide two non-trivial examples (different from Ω and $\emptyset$) of an event A such that

      i. $A \in \mathcal{F}_{2024}$;

      ii. $A \notin \mathcal{F}_{2025}$;

      iii. $A \in \mathcal{H}_n$ for all possible n;

3. Consider a fair dice. In the experiment we throw the dice until the first six appears.

   (a) Simulate $B = 100000$ experiments. For every experiment number i record the total number of throws, y_i, and the number of even faces appeared, x_i.

   (b) For all values of x where you have more than 100 records estimate $\hat{\mu}(x) = \hat{\mathbb{E}}(y_i \mid x_i = x)$ and $\hat{v}(x) = \widehat{\mathbb{V}\mathrm{ar}}(y_i \mid x_i = x)$.

   (c) Explain intuitively why $\hat{\mu}(0)$ is less than 3.

   (d) Randomly select 100 experiments out of all B experiments. Draw the scatter plot (x_i, y_i) for randomly selected experiments. Add the line $\hat{\mu}(x)$ with bands $\hat{\mu}(x) \pm 2\sqrt{\hat{v}(x)}$ to the scatter plot.

   (e) Is it reasonable to assume that $\hat{\mu}(x)$ is linear?

   (f) Is it reasonable to assume that $\hat{v}(x)$ is constant?

   No formal tests are required for the last two questions, graphical analysis is sufficient.

Home assignment 8

Deadline: 2024-11-16, 23:59.

1. The random variables X_n are independent and take values $+1$ with probability 0.7 or 2 with probability 0.3. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the cumulative sum.

   (a) Find the constant a such that $M_n = S_n - an$ is a martingale.

   (b) Find all constants b such that $K_n = \exp(bS_n)$ is a martingale.

2. The population starts with one microbe Eve. So the size of the initial generations is $G_0 = 1$. After one minute every microbe either dies with probability 0.2, remains alive with probability 0.5 or splits in two copies with probability 0.3. Let G_n be the size of microbe population after n minutes.

   (a) Draw a pretty picture of Eve :)

(b) Find the distribution of G_2.

(c) Find a constant a such that $M_n = G_n/a^n$ is a martingale.

(d) Let D be the event of eventual death of the microbe civilization. Check whether the process $K_n = \mathbb{E}(I_D \mid G_n, G_{n-1}, \ldots, G_0)$ is martingale. Here I_D is the indicator of the event D.

(e) Using first step analysis find $\mathbb{P}(D)$.

Hint: you may obtain a quadratic equation for $\mathbb{P}(D)$, the smallest root is your friend :)

3. Consider a symmetric random walk $S_t = X_1 + X_2 + \cdots + X_t$ where (X_t) are independent identically distributed and take values $+1$ or -1 with equal probabilities. We start at zero, $S_0 = 0$.

Let N_k the number of visits of point k before the first return of a random walk to 0.

(a) What is the probability that a random walk will eventually return to 0?

(b) What is the probability $\mathbb{P}(N_k = 0)$?

(c) What is the probability $\mathbb{P}(N_k = n)$ for arbitrary natural number n?

(d) Compare $\mathbb{E}(N_2)$ and $\mathbb{E}(N_{\text{undecillion}})$.