1.1 Markov chains

Def: a seq. of random variables $X_0, X_1 \dots X_n \dots$ with values in a countable set of states I

- with initial distributions over $I = \{1, 2, 3 \dots\}$

  $\mathbb{P}(X_0 = i) = p_i$

- transition matrix $P = \begin{pmatrix} p_{11} & p_{12} & \cdots \\ p_{21} & \cdots & \\ \vdots & & \end{pmatrix}$

stochastic matrix

$\left(\forall i, j \quad p_{ij} \geq 0 ; \ \sum_j p_{ij} = 1 \right)$

$$\mathbb{P}(X_{n+1} = j \mid X_0 = \dots, X_1 = \dots, X_n = i) =$$

$$= \mathbb{P}(X_{n+1} = j \mid X_n = i) = p_{ij}$$

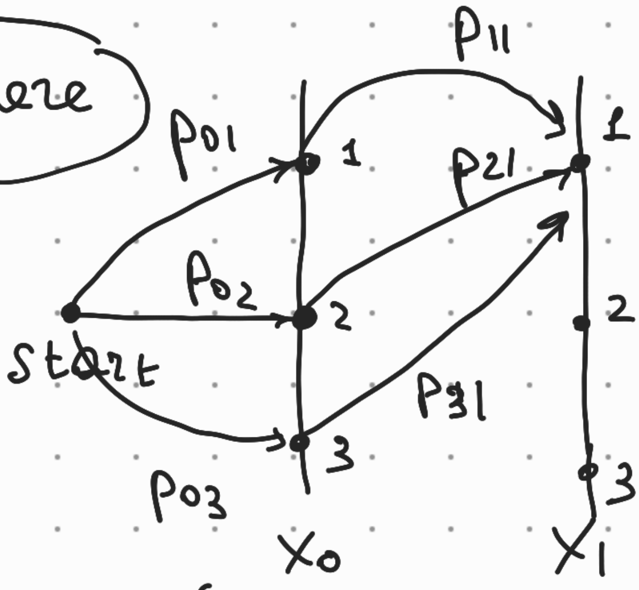
Two main questions:

1) Where will the process be after N steps?

2) How many steps the process need for reaching state i for the first time?

for the first time?

Where



(initial state)

$$P = \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$\vec{p}_0 = \begin{pmatrix} p_{01} \\ p_{02} \\ p_{03} \end{pmatrix} \qquad \begin{pmatrix} p_{01}\cdot p_{11} + p_{02}\, p_{21} + p_{03}\, p_{31} \\ \vdots \end{pmatrix} =$$

$$X_1 : \begin{pmatrix} \sum_i p_{0i}\, p_{i1} \\ \sum_i p_{0i}\, p_{i2} \\ \sum_i p_{0i}\, p_{i3} \end{pmatrix} = \begin{pmatrix} p_{01} & p_{02} & p_{03} \end{pmatrix} \cdot P = \begin{pmatrix} p_1^1 \\ p_2^1 \\ p_3^1 \end{pmatrix}$$

$$X_2 : \begin{pmatrix} p_1^1 \cdot p_{11} + p_2^1 \cdot p_{21} + p_3^1 \cdot p_{31} \\ \dots \\ \dots \end{pmatrix} =$$

$$= \begin{pmatrix} p_1^1 \\ p_2^1 \\ p_3^1 \end{pmatrix} \cdot P = \begin{pmatrix} p_{01} & p_{02} & p_{03} \end{pmatrix} \cdot P^2$$

vector of prob.

—//—//—

$$\boxed{X_n : (p_{01}\ p_{02}\ p_{03}) \cdot P^n}$$

to be in each state after n steps

initial distribution

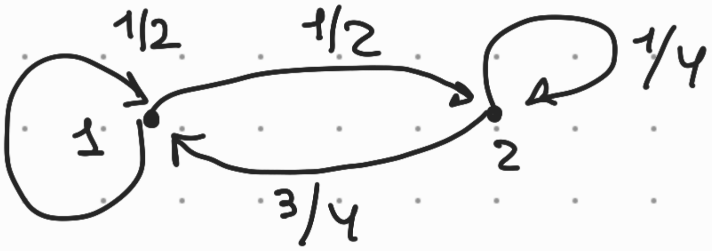
transition matrix

Task 1

a) $1 \equiv$ Study
$2 \equiv$ Sleep

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$$

b)



c) P^{1}_{21} – ? P^{2}_{21} – ?

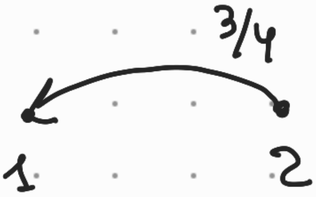
step

states $2 \to 1$

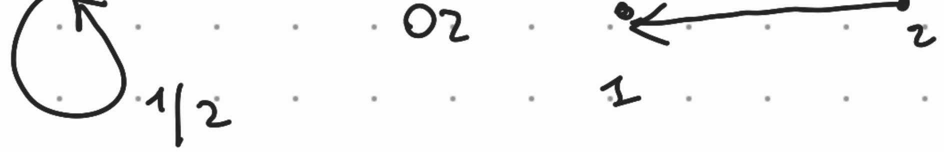
Two ways: by hands and with matrices

sometimes easier

- by hands



$p^{1}_{21} = \frac{3}{4}$ 1 step



$$p_{21}^{2} = \frac{3}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{3}{4} = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}$$ <u>2 steps</u>

- with matrices : initial state $\begin{pmatrix} 0 & 1 \end{pmatrix}$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$$

$$p_{21}^{1} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 3/4 & 1/4 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \end{pmatrix}$$

we need only 1 number

$$p_{22}^{1} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 3/4 & 1/4 \end{pmatrix}^2 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 3/4 & 1/4 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{3}{8} + \frac{3}{16} & \frac{3}{8} + \frac{1}{16} \end{pmatrix} = \begin{pmatrix} \frac{9}{16} & \frac{7}{16} \end{pmatrix}$$

you don't need to calcul. all elements in matrix

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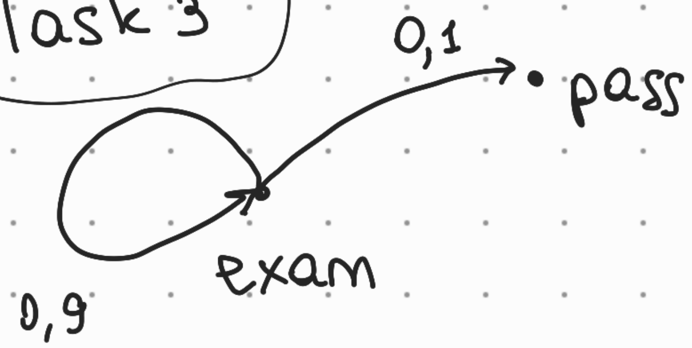
when How many steps we need to get from i to j ?

$\downarrow$

First step analysis

Task 3

a)



b) $\mathbb{P}^n_{pass} = 0{,}1 + 0{,}9 \cdot 0{,}1 + 0{,}9^2 \cdot 0{,}1 + \dots \xrightarrow[n\to\infty]{} \frac{0{,}1}{1-0{,}9} = 1$

(convergency)

all students pass exams with $p = 1$

c) first step analysis:

$\mu_R = 0{,}1 \cdot 0 + 0{,}9(1 + \mu_R)$

we need one more retake

pass

no more retakes

av. num of retakes

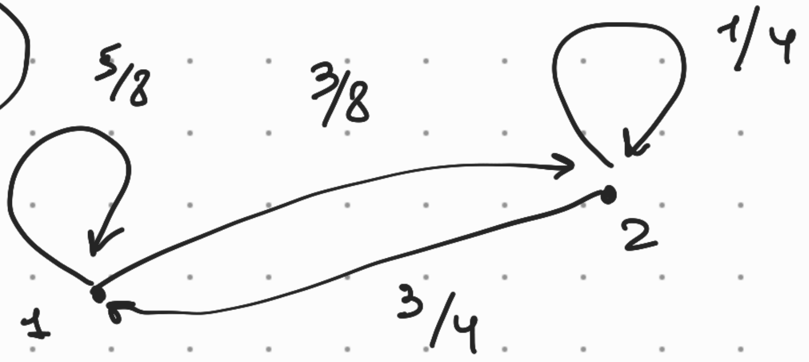
$\mu_R = 0{,}9 + 0{,}9\mu_R \Rightarrow \mu_R = 9$

Task 4

1 - commiss.

2 - rent

a)



b)

$$P = \begin{pmatrix} 5/8 & 3/8 \\ 3/4 & 1/4 \end{pmatrix} \qquad \mu_{2\to 2} - ?$$

c)

$$\begin{cases} \mu_{22} = \frac{1}{4}\cdot 1 + \frac{3}{4}(1 + \mu_{21}) \\ \mu_{21} = \frac{3}{8}\cdot 1 + \frac{5}{8}(1 + \mu_{21}) \end{cases}$$

we reach 2 with 1 step

two unknown variables

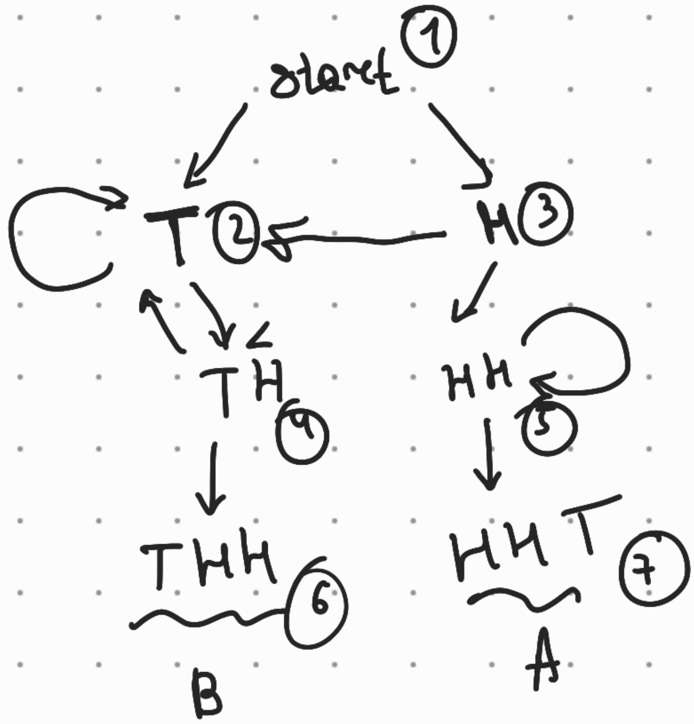
we need additional equation

$$\begin{cases} \mu_{22} = 1 + \frac{3}{4}\mu_{21} \\ \mu_{21} = 1 + \frac{5}{8}\mu_{21} \end{cases} \qquad \mu_{21} = \frac{8}{3} \qquad \mu_{22} = 1 + 2 = 3$$

Task 5 Alice and Bob toss the coin

Alice wins HHT, Bob THH

a)



THT ≡ T

HHH ≡ HH

HHT

c) Probability „Alice wins" $\rightarrow p_{17}$

first step analysis

$$p_{17} = \frac{1}{2}\overset{0}{\cancel{p_{27}}} + \frac{1}{2}p_{37} = \frac{1}{2}p_{37}$$

$$p_{37} = \frac{1}{2}\overset{0}{\cancel{p_{27}}} + \frac{1}{2}p_{57} = \frac{1}{2}p_{57}$$

$$p_{57} = \frac{1}{2}p_{57} + \frac{1}{2}\cdot 1$$

$$p_{57} = 1 \rightarrow p_{17} = \frac{1}{4}$$

Alice wins with 100% prob.

d) Finish the game in 4 steps:

S → T → T → TH → THH

S → H → HH → HH → HHT

S → H → T → TH → THH

$$\frac{3}{2^4} = \frac{3}{16}$$