Welcome to the Advanced Statistics!

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2 September 2024

Course structure

Course Plan

- Stochastic Processes
- Time Series
- Advanced Statistics UoL

UNIVERSITY OF LONDON

Advanced statistics: statistical inference

J. Penzer

ST2134

2018

Undergraduate study in Economics, Management, Finance and the Social Sciences

This subject guide is for a 200 course offered as part of the University of London undergraduate study in Economics, Management, Finance and the Social Sciences. This is equivalent to Level 5 within the Framework for Higher Education Qualifications in England, Wales and Northern Ireland (FHEQ).

For more information about the University of London, see: london.ac.uk

LSE THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

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Advanced statistics: distribution theory

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LSE THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

Course structure

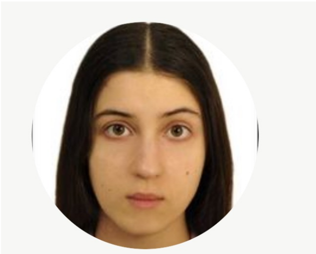
Formula for final grade – Stochastic Processes

$$Final\ Score = 0.35 * \mathrm{FALL} + 0.40 * \mathrm{DEC} + 0.25 * \mathrm{HW(m1 + m2)}$$

Formula for final grade – Time Series Analysis

$$Final\ Score = 0.30 * \mathrm{Midterm\ Mimoza} + 0.45 * \mathrm{Midterm\ Sakura} + 0.25 * \mathrm{HW(m3 + m4)}$$

Course structure



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Syllabus

Lecture 1
Recall basic of probability
MGF to start

Lecture 2
Markov chain

Lecture 3
Convergences

Lecture 4
Conditional Expectations

Lecture 5
Poisson distribution

Lecture 6
G-algebra

Lecture 7
Filtration

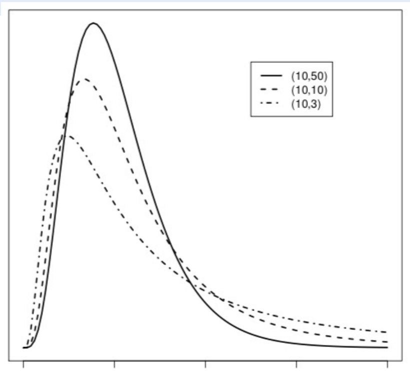
How is about to recall ?

Definition

Let U and V be two independent random variables, where U $\sim \chi^2_p$ and V $\sim \chi^2_k$
Then the distribution of:

$$F = \frac{U/p}{V/k}$$

is the F distribution with degrees of freedom *(p, k)*, denoted F $\sim F_{p,k}$, or F $\sim F(p,k)$

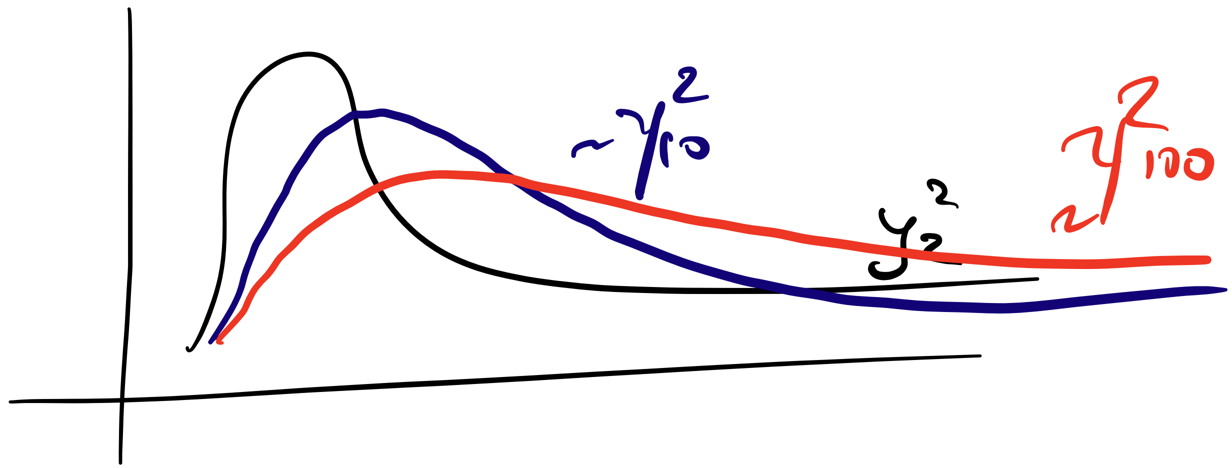


Let's recall rest of distributions

$$X_i \overset{iid}{\sim} N(0,1) \qquad i=\overline{1,n}$$

$$\text{pdf} \quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$$



How is about to recall ?

The (random) interval (L, U) forms an interval estimator of θ:

For estimation to be as precise as possible, intuitively, the width of the interval, U − L, should be small. Then, typically, the coverage probability:

$$P(L(X_{1,} X_2, \ldots X_n)) < \theta < U(X_{1,} X_2, \ldots X_n)) < 1$$

Ideally, L and U should be chosen such that:

- the width of the interval is as small as possible;
- the coverage probability is as large as possible.

How is about to recall ?

Definition

Hence, supposing a 95% coverage probability:

$$
\begin{aligned}
0{,}95 = \; & P\left(\frac{\sqrt{n}|\bar{X}-\mu|}{\sigma} \leq 1{,}96\right) \\
= \; & P\left(|\bar{X}-\mu| \leq 1{,}96 * \frac{\sigma}{\sqrt{n}}\right) \\
= \; & P\left(|\bar{X}-\mu| \leq 1{,}96 * \frac{\sigma}{\sqrt{n}}\right) \\
= \; & P\left(-1{,}96 * \frac{\sigma}{\sqrt{n}} \leq \bar{X}-\mu \leq 1{,}96 * \frac{\sigma}{\sqrt{n}}\right) \\
= \; & P\left(\bar{X}-1{,}96 * \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X}+1{,}96 * \frac{\sigma}{\sqrt{n}}\right)
\end{aligned}
$$

How is about to recall ?

Definition

The likelihood function is defined as:

$$L(\theta) = \prod_{i=1}^{n} f(X_i, \theta)$$

- the likelihood function is the function of θ, while $X_{1,} X_2, \ldots X_n$ are treated as constants (as given observations);
- the likelihood function reflects the information about the unknown parameter θ in the data $X_{1,} X_2, \ldots X_n$.

Moment Generating Function

Definition

For many distributions, all the moments $E(X)$, $E(X^2)$ ... Can be encapsulated in a single function, which is called **moment generation function**. It exists for many commonly used distributions and often provides the most efficient way to calculate moments.

The moment generating function of a random variable X is a function $M_x : R \rightarrow [0; \infty)$ given by:

$$M_{X(t)} = E(e^{tX}) = \begin{cases} \sum_{x} e^{tX} f_X(x) & if\ X\ discrete, \\ \int_{-\infty}^{\infty} e^{tX} f_X(x) dx & if\ X\ continuous \end{cases}$$

Where, to be well-defined, we require some $h > 0$ such that $M_{X(t)} < \infty$ for all $t \in [-h, h]$.

$$M_x(t) = \sum_{x \in X} e^{t \cdot x} \cdot f_x(x) = E[e^{tx}]$$

$$(M_x)'_t = (E[e^{tx}])'_t = E[x \cdot e^{tx}]$$

$(M_x)''_{tt} = \text{—} \;\; '' \;\; \text{—} = E[x^2 e^{tx}]$

Moment Generating Function

$$\boxed{E(x) = (M_x)'_t \big|_{t=0}}$$

Moment generating function

$M_X(t) = \mathrm{E}[e^{tX}]$

$\dfrac{d^n M_X(0)}{dt^n} = \mathrm{E}[X^n]$

$M_X(t)$ uniquely determines probability distribution of X

expectation EX

variance $E[X^2] - E^2(X)$

skewness

kurtosis

Moment Generating Function

$S_\infty = \frac{b_1}{1-q}$

Discrete

r.v. X $f_X = \lambda e^{-\lambda x}$, $\lambda > 0$, $x \geq 0$

$M_X(t) = E[e^{tX}] = \sum_{x \geq 0} e^{tx} \cdot \lambda e^{-\lambda x} =$

$= \lambda \sum_{x \geq 0} e^{(t-\lambda)x} = \lambda[1 + e^{t-\lambda} + e^{(t-\lambda)2} + \dots]$

if $t > \lambda$ $\nexists$

if $t \leq \lambda$ $= \lambda \cdot \frac{1}{1-e^{t-\lambda}} = \boxed{\frac{\lambda}{1-e^{t-\lambda}}}$

$E[X] = \frac{\partial M}{\partial t}\Big|_{t=0} = -\frac{\lambda}{(1-e^{t-\lambda})^2} \cdot (-e^{t-\lambda})$

$= \frac{\lambda e^{t-\lambda}}{(1-e^{t-\lambda})^2} = \frac{\lambda e^{-\lambda}}{(1-e^{-\lambda})^2}$.

Example Let X be a continuous random variable with support

$$R_X = [0, \infty)$$

and probability density function

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \in R_X \\ 0 & \text{if } x \notin R_X \end{cases}$$

where λ is a strictly positive number. The expected value $E[\exp(tX)]$ can be computed as follows:

$$\begin{aligned} E[\exp(tX)] &= \int_{-\infty}^{\infty} \exp(tx) f_X(x) dx \\ &= \int_0^{\infty} \exp(tx) \lambda \exp(-\lambda x) dx \\ &= \lambda \int_0^{\infty} \exp((t-\lambda)x) dx \qquad \text{(which is finite only if } t < \lambda) \\ &= \lambda \left[\frac{1}{t-\lambda} \exp((t-\lambda)x) \right]_0^{\infty} \\ &= \lambda \left[0 - \frac{1}{t-\lambda} \right] \\ &= \frac{\lambda}{\lambda - t} \end{aligned}$$

Furthermore, the above expected value exists and is finite for any $t \in [-h, h]$, provided $0 < h < \lambda$. As a consequence, X possesses a mgf:

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

Moment Generating Function – Linear Transformation

Let X be a random variable possessing a mgf $M_X(t)$.

Define

$$Y = a + bX$$

where $a, b \in \mathbb{R}$ are two constants and $b \neq 0$.

Then, the random variable Y possesses a mgf $M_Y(t)$ and

$$M_x(t) = E[e^{tx}]$$

$$M_Y(t) = \exp(at)M_X(bt)$$

$$M'_t = a e^{at} \cdot M_x(bt) + e^{at} \cdot (M_x(bt))'_t \cdot b$$

$$M'_t\Big|_{t=0} = a \cdot \overbrace{M_x(0)}^{=1} + 1 \cdot (M_x(bt))'_t\Big|_{t=0} \cdot b$$

$$= a + E[X] \cdot b$$

Proof

By the very definition of mgf, we have

$$\begin{aligned} M_Y(t) &= \mathrm{E}[\exp(tY)] \\ &= \mathrm{E}[\exp(at + btX)] \\ &= \mathrm{E}[\exp(at)\exp(btX)] \\ &= \exp(at)\mathrm{E}[\exp(btX)] \\ &= \exp(at)M_X(bt) \end{aligned}$$

Obviously, if $M_X(t)$ is defined on a closed interval $[-h, h]$, then $M_Y(t)$ is defined on the interval $\left[-\frac{h}{b}, \frac{h}{b}\right]$.

Moment Generating Function

Let $X_1, \ldots, X_n$ be n mutually independent random variables.

Let Z be their sum:

$$Z = \sum_{i=1}^{n} X_i$$

Then, the mgf of Z is the product of the mgfs of $X_1, \ldots, X_n$:

$$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

Proof

This is easily proved by using the definition of mgf and the properties of mutually independent variables:

$$\begin{aligned} M_Z(t) &= \mathrm{E}[\exp(tZ)] \\ &= \mathrm{E}\left[\exp\left(t\sum_{i=1}^{n} X_i\right)\right] \\ &= \mathrm{E}\left[\exp\left(\sum_{i=1}^{n} tX_i\right)\right] \\ &= \mathrm{E}\left[\prod_{i=1}^{n} \exp(tX_i)\right] \\ &= \prod_{i=1}^{n} \mathrm{E}[\exp(tX_i)] && \text{(by mutual independence)} \\ &= \prod_{i=1}^{n} M_{X_i}(t) && \text{(by the definition of mgf)} \end{aligned}$$

Moment Generating Function - Problem

Let X be a random variable with moment generating function

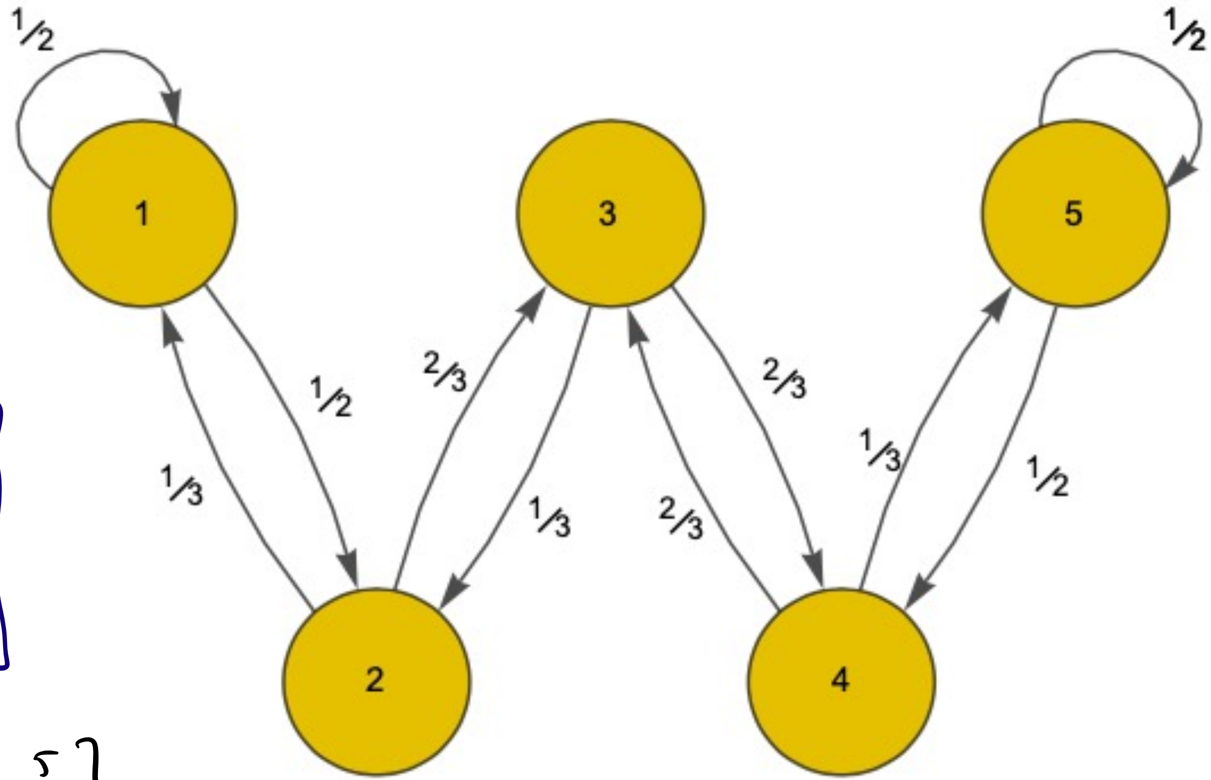
$$M_X(t) = \frac{1}{2}(1 + \exp(t))$$

Derive the variance of X.

Stochastic Processes: Basic Definitions

Stochastic process	The value of a variable changes in an uncertain way
Discrete vs. continuous time	When can a variable change? What values can a variable take?
Markov property	Only the current value of a variable is relevant for future predictions No information from past prices or path

A Chain



$$P: \quad \begin{array}{c|c} & P_{i+1} \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

$$P: \quad \begin{array}{c|c} & P_{i+2} \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 5/12 & 1/4 & 1/3 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

$$P(i+2=1, i=1) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3}$$

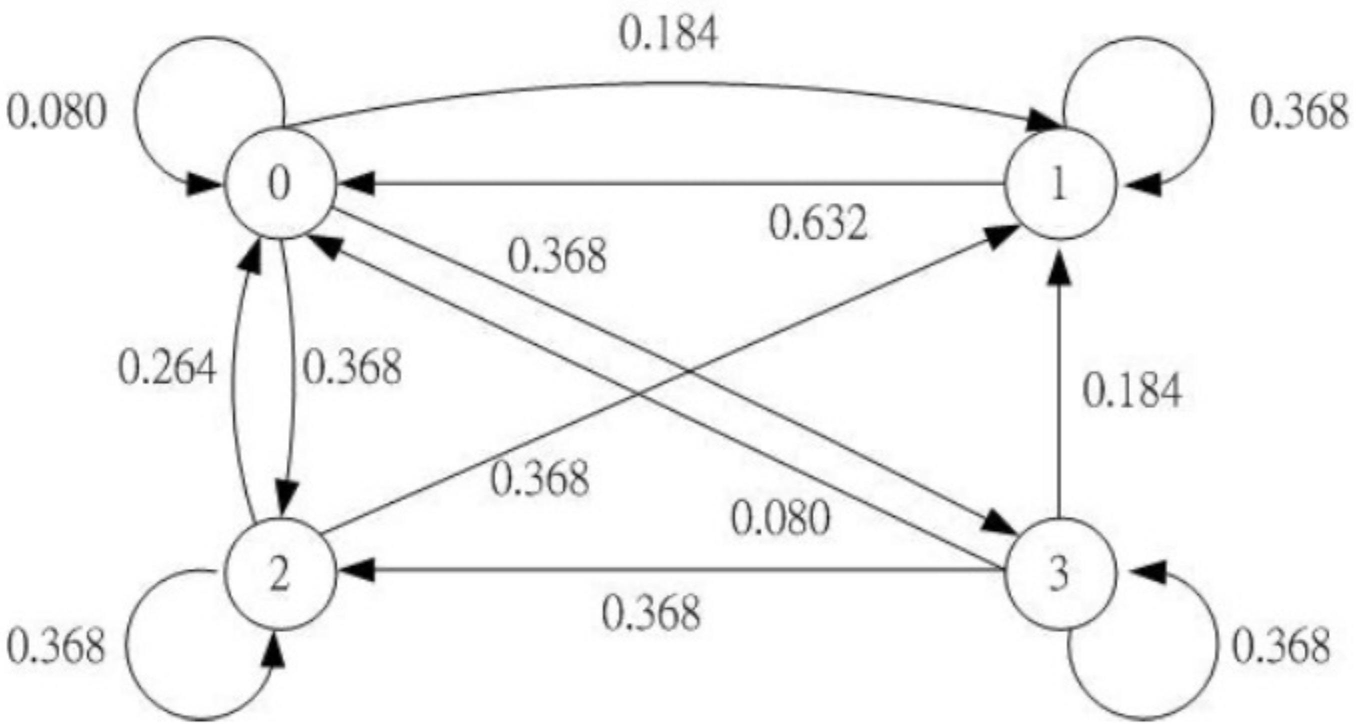
$$P(i+2=2, i=1) = 1/4$$

$$P_{i,i+2} = \left(P_{i,i+1}\right)^2$$

$P_{i,i+2} = P_{i,i+1} \times P_{i+1,i+2}$

Markov Chain

- The state transition diagram:



Markov Chain

- Consider time index $n = 0, 1, 2, \ldots$ & time dependent random state X_n
- State X_n takes values on a countable number of states
  - In general denotes states as $i = 0, 1, 2, \ldots$
  - Might change with problem
- Denote the history of the process $\mathbf{X}_n = [X_n, X_{n-1}, \ldots, X_0]^T$
- Denote stochastic process as $\mathbf{X}_{\mathbb{N}}$
- The stochastic process $X_{\mathbb{N}}$ is a Markov chain (MC) if

$$\mathrm{P}\left[X_{n+1} = j \,\middle|\, X_n = i, \mathbf{X}_{n-1}\right] = \mathrm{P}\left[X_{n+1} = j \,\middle|\, X_n = i\right] = P_{ij}$$

- Future depends only on current state X_n

Observations

- Process's history $\mathbf{X}_{n-1}$ irrelevant for future evolution of the process
- Probabilities P_{ij} are constant for all times (time invariant)
- From the definition we have that for arbitrary m

$$\mathrm{P}\left[X_{n+m} \,\middle|\, X_n, \mathbf{X}_{n-1}\right] = \mathrm{P}\left[X_{n+m} \,\middle|\, X_n\right]$$

- X_{n+m} depends only on X_{n+m-1}, which depends only $on X_{n+m-2}$, ... which depends only on X_n
- Since P_{ij}'s are probabilities they're positive and sum up to 1

$$P_{ij} \geq 0 \qquad \sum_{j=1}^{\infty} P_{ij} = 1$$

Matrix Representation

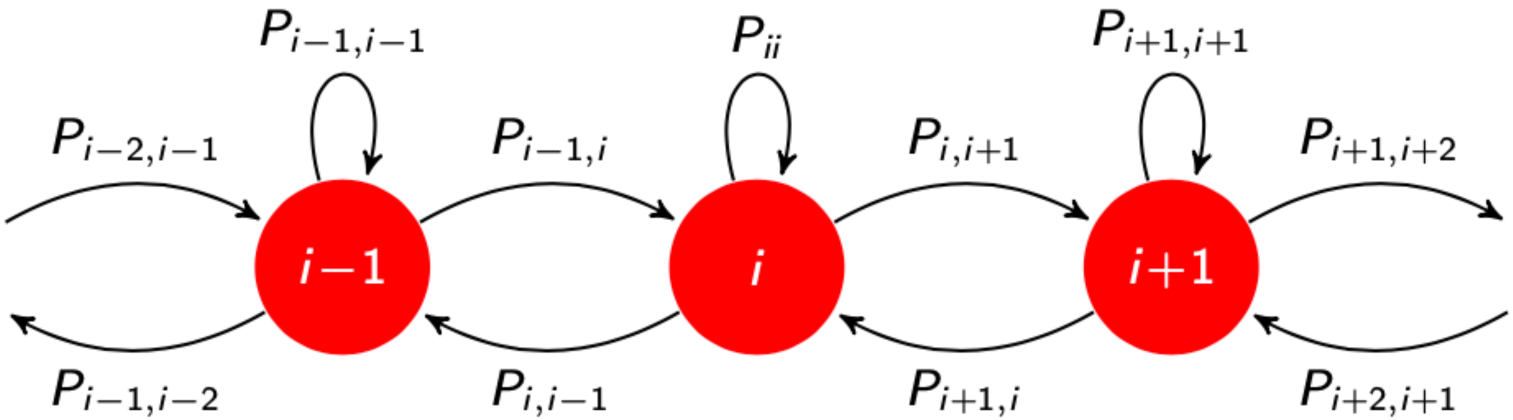
- Group transition probabilities P_{ij} in a "matrix" $\mathbf{P}$

$$\mathbf{P} := \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Not really a matrix if number of states is infinite

Graph Representation

- A graph representation is also used



- Useful when number of states is infinite

Thank you for your attention!

See next week!