1. Unobserved random variables (y_i) are independent and uniform on $[-2a, a]$ with unknown $a > 0$.

   (a) [4] Find $\mathbb{E}(y_i^2)$ and $\mathbb{E}(|y_i|)$.

   (b) [3] Find the method of moments estimator of a if you know the value of $\sum y_i^2$.

   (c) [3] Find the method of moments estimator of a if you know the value of $\sum |y_i|$.

2. Dragon Erik receives gifts from two kingdoms: x_1, x_2, ..., x_{10} from Ex-Kingdom and y_1, y_2, ..., y_{20} from Why-Kingdom. Variables (x_i) are exponentially distributed with rate λ_x and variables (y_i) — with rate λ_y. All variables are independent. Erik has two hypothesis, H_0: $\lambda_x = \lambda_y$ and H_1: $\lambda_x \neq \lambda_y$. In the observed sample $\sum x_i = 300$, $\sum y_i = 500$.

   (a) [4] Find the unrestricted maximum likelihood estimates of λ_x and λ_y.

   (b) [3] Find the maximum likelihood estimate of the rate under H_0.

   (c) [3] Test H_0 against H_1 at 0.05 significance level using likelihood ratio test.

   Note: $\mathbb{P}(\chi^2_1 \geq 3.84) = 0.05$, $\mathbb{P}(\chi^2_2 \geq 5.99) = 0.05$, $\mathbb{P}(\chi^2_3 \geq 7.81) = 0.05$.

3. The variables (X_i) are independent and have exponential distribution with unknown rate λ.

   Yuki-the-Hedgehog would like to estimate the unknown parameter $a = 1/\lambda^2$. He uses the estimator

$$\hat{a} = \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{2n + 1}$$

   (a) [5] Is the estimator unbiased?

   (b) [5] Is the estimator consistent?

4. Princess Lika Pankova would like to test the hypothesis H_0 that y_1 and y_2 are independent and uniform on $[0, 1]$ against the alternative that they have the joint density $f(y_1, y_2) = 2 - y_1 - y_2$ for $y_1, y_2 \in [0, 1]$.

   Lika accepts only the best!

   (a) [7] Construct the test with 0.1 probability of the first type error and lowest probability of the second type error.

   (b) [3] What is the minimal achieved second type error probability?

5. Arina loves all kinds of raisin-studded Easter bread, but her grandmother's is her absolute favorite. Grandmother's Easter bread sizes (y_i) are independent identically distributed with density $f(y) = ab^a/y^{a+1}$ for $y \geq b$ and zero otherwise.

   (a) [2] Draw a typical density for some values of parameters a and b.

   (b) [4] Find a sufficient statistic for the unknown a given that $b = 1$ kilo.

   (c) [4] Find a sufficient statistic for the unknown a and b.

6. The semi-annual (y_t) is modelled by $ETS(ANA)$ process:

$$
\begin{cases}
u_t \sim \mathcal{N}(0; 9) \\
s_t = s_{t-2} + 0.1u_t \\
\ell_t = \ell_{t-1} + 0.3u_t \\
y_t = \ell_{t-1} + s_{t-2} + u_t
\end{cases}
$$

   Let $s_{100} = 2$, $s_{99} = -3$, $\ell_{100} = 200$.

   (a) [6] Find 95% predictive interval for y_{102}.

   (b) [4] Write this model in the form $A(L)y_t = B(L)u_t$, where $A(L)$ and $B(L)$ are lag polynomials.