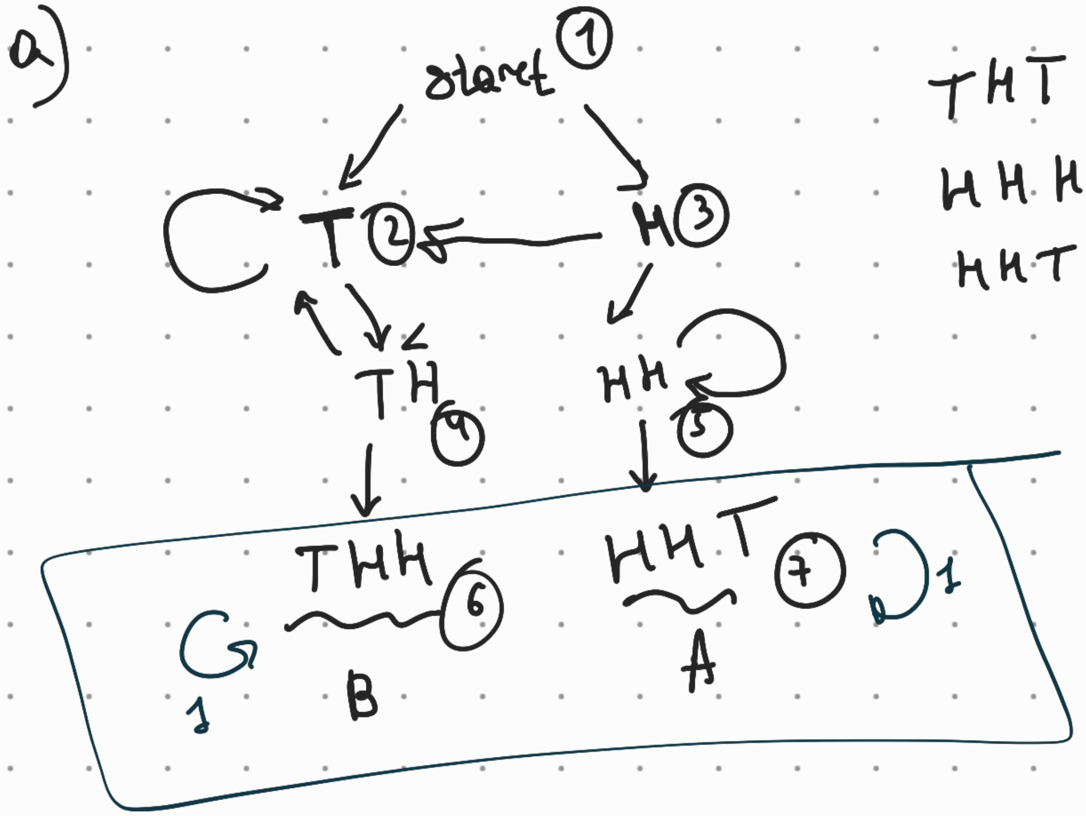
From the prev. seminar

Task 5 Alice and Bob toss the coin

Alice wins HHT, Bob THH

a)



THT ≡ T

HHH ≡ HH

HHT

8 ≡ finish

c) Probability „Alice wins" → p_{17}

first step analysis

$$p_{17} = \frac{1}{2}\overset{0}{p_{27}} + \frac{1}{2}p_{37} = \frac{1}{2}p_{37}$$

$$p_{37} = \frac{1}{2}\overset{0}{p_{27}} + \frac{1}{2}p_{57} = \frac{1}{2}p_{57}$$

$$p_{57} = \frac{1}{2}p_{57} + \frac{1}{2}\cdot 1$$

$$p_{57} = 1 \rightarrow p_{17} = \frac{1}{4}$$

d) N - random variable for the number of tosses

$E(N),\ Var(N)$ — ?

- first step analysis: $\mu_{18} = E(N)$

$$\begin{cases} \mu_{18} = \frac{1}{2}(1+\mu_{28}) + \frac{1}{2}(1+\mu_{38}) \\ \mu_{28} = 1 + \frac{1}{2}\mu_{28} + \frac{1}{2}\mu_{48} \\ \mu_{48} = 1 + \frac{1}{2}\mu_{28} + \frac{1}{2}\cdot 0 \\ \mu_{38} = 1 + \frac{1}{2}\mu_{28} + \frac{1}{2}\mu_{58} \\ \mu_{58} = 1 + \frac{1}{2}\mu_{58} + \frac{1}{2}\cdot 0 \end{cases} \Rightarrow \mu_{58} = 2$$

$$\begin{cases} \mu_{18} = 1 + \frac{1}{2}\mu_{28} + \frac{1}{2}\mu_{38} \\ \left.\begin{array}{l} \frac{1}{2}\mu_{28} = 1 + \frac{1}{2}\mu_{48} \\ \frac{1}{2}\mu_{28} = \mu_{48} - 1 \end{array}\right\} \Rightarrow \mu_{48} = 4 \\ \mu_{38} = 2 + \frac{1}{2}\mu_{28} \end{cases}$$

$\mu_{28} = 6$

$\mu_{38} = 5$

$\mu_{18} = 1 + 3 + \frac{5}{2} = 6{,}5$

$E(N) = 6{,}5$

- $Var\,N = E(N^2) - E^2(N) = E(N^2) - \left(\frac{13}{2}\right)^2$

$$\begin{cases} \mu_{18}^2 = \frac{1}{2}(1+\mu_{28})^2 + \frac{1}{2}(1+\mu_{38})^2 \\ \mu_{28}^2 = \frac{1}{2}(1+\mu_{28})^2 + \frac{1}{2}(1+\mu_{48})^2 \\ \mu_{48}^2 = \frac{1}{2}(1+\mu_{28})^2 + \frac{1}{2}\cdot 1 \\ \mu_{38}^2 = \frac{1}{2}(1+\mu_{28})^2 + \frac{1}{2}(\mu_{58}+1)^2 \\ \mu_{58}^2 = \frac{1}{2}(1+\mu_{58})^2 + \frac{1}{2} \end{cases}$$

$$\mu_{58}^2 = \frac{1}{2}(1+2\mu_{58}+\mu_{58}^2) + \frac{1}{2} \rightarrow \text{sq. eq.}$$

$$\frac{1}{2}\mu_{58}^2 = \frac{1}{2} + 2 + \frac{1}{2} \Rightarrow \mu_{58}^2 = 6 \quad \text{compute} \ldots.$$

$$\mu_{18}^2 = E(N^2) = 61{,}5 \Rightarrow VAR(N) = 61{,}5 - 6{,}5^2 = 19{,}25$$

Stationary distributions in Markov chain

states

- recurrant (infinit num. of visits) A, B
- transient (finite num. of visits) C
- absorbing (terminal) D

N_A - num. of visits A $\quad N_A + N_B = N$

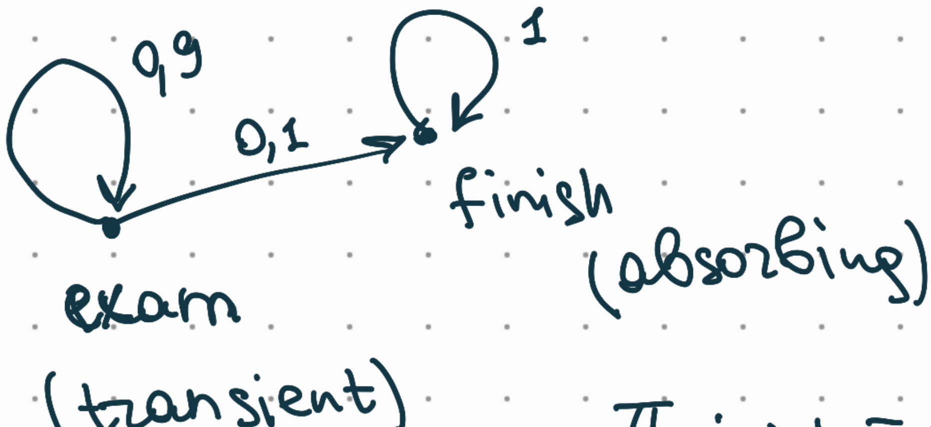
N_B - num of visits B

$\frac{N_A}{N}, \frac{N_B}{N} \rightarrow$ part of time in each state

$N \to \infty \Rightarrow \frac{N_A}{N} \to \pi_A, \ \frac{N_B}{N} \to \pi_B$

(π_A, π_B) - stationary distribution

Task 1 from sem 1



$\left.\begin{array}{l} \pi_{finish} = 0 \\ \pi_{exam} = 0 \end{array}\right\}$ stationary distribut.

Task 2

$p = 97$

70 71 72 73

Is there a stationary distribution?

Let it be: $\begin{cases} \pi_0, \pi_1, \pi_2 \dots \pi_n \dots \sim \text{time} \\ \sum \pi_i = 1 \end{cases}$

„in" arrows: $\begin{cases} \pi_0 = (1-p)\pi_0 + (1-p)\pi_1 \\ \pi_1 = p\pi_0 + (1-p)\pi_2 \\ \vdots \\ \pi_n = p\pi_{n-1} + (1-p)\pi_{n+1} \end{cases}$

(we want to know how to get to the state, not to leave it)

$$\Rightarrow \pi_1 = \frac{p}{1-p}\pi_0$$

$$\pi_2 = \frac{\pi_1 - p\pi_0}{1-p} = \frac{p - p(1-p)}{(1-p)^2}\pi_0 = \left(\frac{p}{1-p}\right)^2 \pi_0 \dots$$

$$\dots \pi_n = \left(\frac{p}{1-p}\right)^n \pi_0$$

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 + \frac{p}{1-p}\pi_0 + \dots \left(\frac{p}{1-p}\right)^n \pi_0 + \dots$$

converge if and only if

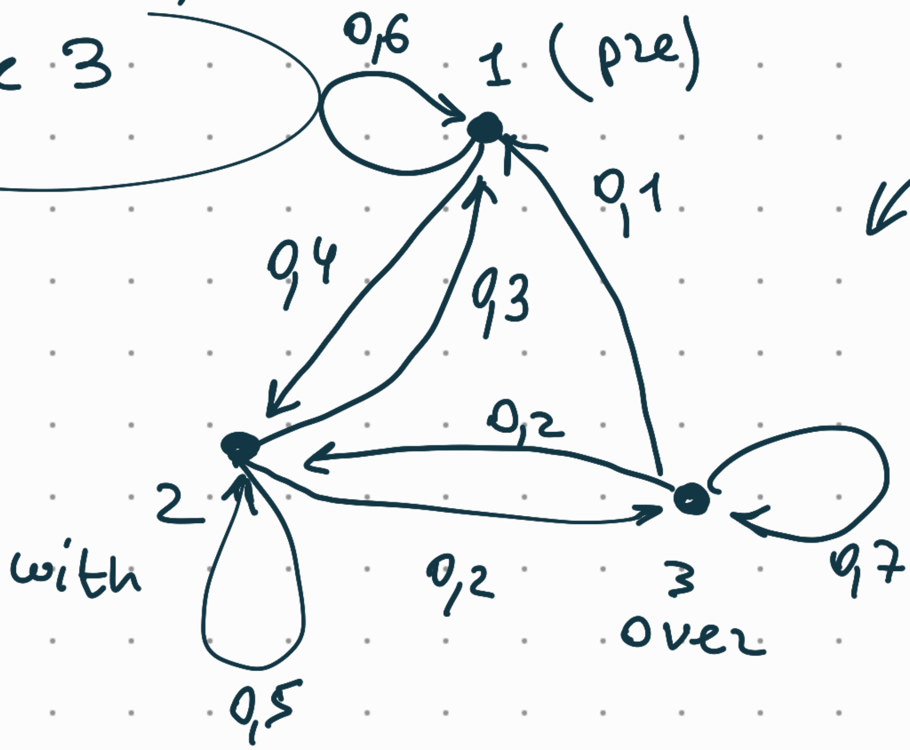
$$\frac{p}{1-p} < 1 \Rightarrow p < 1-p \Rightarrow p < \frac{1}{2}!$$

$0{,}7 > \frac{1}{2} \Rightarrow$ no stationary distrib.

Task 3

all states are recurrent

a, b)



$$P = \begin{bmatrix} 0,6 & 0,4 & 0 \\ 0,3 & 0,5 & 0,2 \\ 0,1 & 0,2 & 0,7 \end{bmatrix}$$

c) 200 breaks (200 steps)

Lemma: if all states of finite markov chain are recurrent, there is a stationary distribution.

so you do not need to calculate P^{200} to find probabilities

$$\begin{cases} \pi_1 = 0,6\,\pi_1 + 0,3\,\pi_2 + 0,1\,\pi_3 \end{cases}$$

$$\begin{cases} \pi_2 = 0{,}4\pi_1 + 0{,}5\pi_2 + 0{,}2\pi_3 \\ \pi_3 = 0{,}2\pi_2 + 0{,}7\pi_3 \quad \text{or} \quad \pi_3 = 1 - \pi_1 - \pi_2 \end{cases}$$

$$\begin{cases} 0{,}4\pi_1 = 0{,}3\pi_2 + 0{,}1 - 0{,}1\pi_1 - 0{,}1\pi_2 \\ 0{,}5\pi_2 = 0{,}4\pi_1 + 0{,}2 - 0{,}2\pi_1 - 0{,}2\pi_2 \end{cases}$$

$$\begin{cases} 0{,}5\pi_1 = 0{,}1 + 0{,}2\pi_2 \\ 0{,}7\pi_2 = 0{,}2 + 0{,}2\pi_1 \end{cases} \qquad \begin{cases} 5\pi_1 = 1 + 2\pi_2 \\ 7\pi_2 = 2 + 2\pi_1 \end{cases}$$

$$\begin{cases} \pi_1 = -1 + \frac{7}{2}\pi_2 \\ -5 + \frac{35}{2}\pi_2 = 1 + 2\pi_2 \end{cases} \qquad \frac{31}{2}\pi_2 = 6$$

$$\pi_2 = \frac{12}{31} < 1 \quad :)$$

$$\pi_1 = -1 + \frac{42}{31} = \frac{11}{31} < 1 \quad :)$$

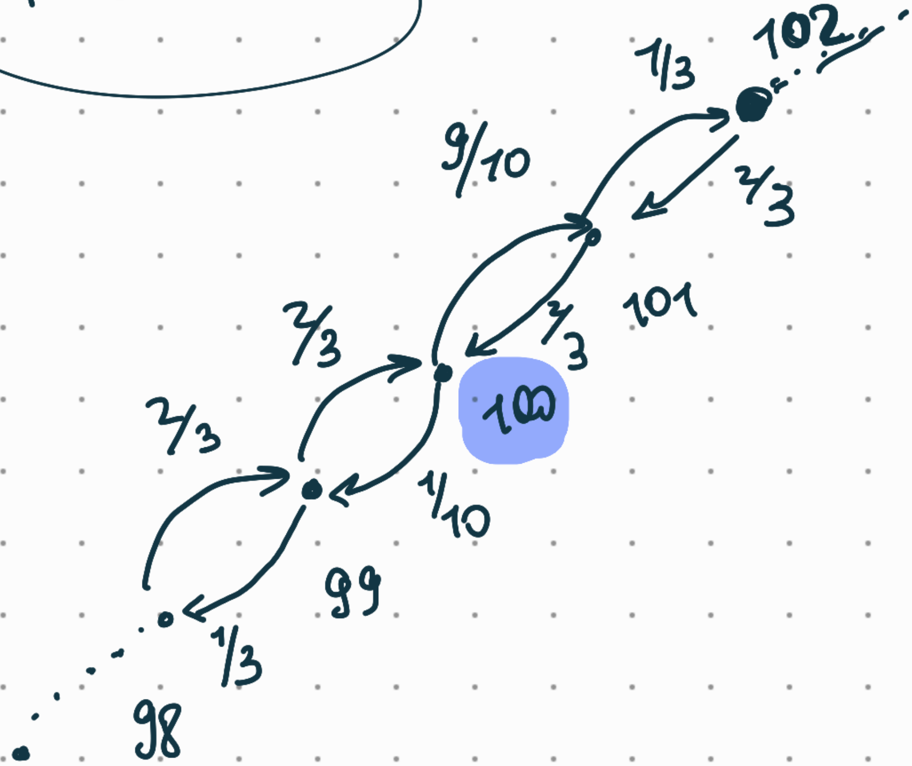
$$\pi_3 = \frac{8}{31}$$

$\Rightarrow$ st. distr. $\left(\frac{11}{31}; \frac{12}{31}; \frac{8}{31}\right)$

* try to calculate P^{200} !

(by soft)

Task 4



1) decomposition : $\mu_{102,98} = \mu_{102,100} + \mu_{100,98}$

different behaviour

2) $\mu_{102,100} = \mu_{102,101} + \mu_{101,100} = 2\mu_{101,100}$

3) first step analysis :

$$\mu_{101,100} = \frac{1}{3}\left(1 + \mu_{102,100}\right) + \frac{2}{3}\left(1 + 0\right) =$$

$$= 1 + \frac{2}{3}\mu_{101,100}$$

$\mu_{101,100} = 3$ $\quad$ $\mu_{102,100} = 6$

4) decomposition $\mu_{100,98} = \mu_{100,99} + \mu_{99,98}$

5) $\mu_{100,99} = \frac{9}{10}\left(1 + \mu_{101,99}\right) + \frac{1}{10}(1+0) =$

$= 1 + \frac{9}{10}\mu_{101,99} = 1 + \frac{9}{10}\Big(\mu_{101,100} +$

$+ \mu_{100,99}\Big) = 1 + \frac{9}{10} \cdot 3 + \frac{9}{10}\mu_{100,99}$

$\mu_{100,99} = 10 + 27 = 37$

6) $\mu_{99,98} = \frac{1}{3}(1+0) + \frac{2}{3}\left(1 + \mu_{100,98}\right) =$

$= 1 + \frac{2}{3}\mu_{100,98} = 1 + \frac{2}{3}\left(\mu_{100,99} + \mu_{99,98}\right) =$

$= 1 + \frac{2}{3} \cdot 37 + \frac{2}{3}\mu_{99,98}$

$\mu_{99,98} = 3 + 37 \cdot 2 = 77$

7) $\mu_{102,98} = 6 + 37 + 77 = 120$