W06

optimization

today :

① special case : $x \geq 0$, $y \geq 0$ ....
$x \geq 5$

② Cauchy - Schwartz inequality.

General setup.

$\max f$

s.t. $\begin{cases} h_1(x) \geq 0 \\ \vdots \\ h_\ell(x) \geq 0 \end{cases}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

→ Step. 1 . Check NDCQ.

Step 2. Write $L = f + \sum_{j=1}^{\ell} \lambda_j \cdot h_j$

Step 3. F O C.

→ Step 4. Sufficiency conditions.

Special case

$x_1 \geq 0$ $x_2 \geq 0$ ..... $x_n \geq 0$

λ_1 λ_2 λ_n

small trick to dismiss all these λ_j (for $x_j \geq 0$)

Standard KKT:

Consider $\boxed{x_1 \geq 0}$

$$L = f + \boxed{\lambda_1 \cdot x_1} + \sum_{j=2}^{\ell} \lambda_j \cdot h_j = \lambda_1 x_1 + L^*$$

$$L^* = f + \sum_{j=2}^{\ell} \lambda_j \cdot h_j$$

F.O.C.

$$\begin{cases} \dfrac{\partial L}{\partial \lambda_j} \geq 0 \quad \checkmark & \lambda_j \geq 0 \quad \checkmark \qquad \dfrac{\partial L}{\partial \lambda_j} \cdot \lambda_j = 0. \quad \checkmark \\ \dfrac{\partial L}{\partial x_i} = 0 & \end{cases}$$

General 4 cond + λ_1

$$\boxed{\frac{\partial L}{\partial \lambda_1} = x_1 \geq 0} \qquad \lambda_1 \geq 0 \qquad \boxed{\lambda_1 \cdot x_1 = 0.} \quad ''$$

$$\frac{\partial L}{\partial x_1} = \lambda_1 + \frac{\partial L^*}{\partial x_1} = 0$$

$$\lambda_1 = -\frac{\partial L^*}{\partial x_1}$$

3 cond no λ_1

$$x_1 \geq 0 \qquad \boxed{\frac{\partial L^*}{\partial x_1} \leq 0} \qquad \frac{\partial L^*}{\partial x_1} \cdot x_1 = 0$$

Result: If you have non-neg-ty constraints $x_1 \geq 0 \quad x_2 \geq 0 \quad \ldots \quad x_n \geq 0.$

$h_1 \geq 0$
$h_2 \geq 0$
$\vdots$
$h_\ell \geq 0$

We ignore $x_1 \geq 0 \quad x_2 \geq 0 \ldots \quad x_n \geq 0$

$$L^* = f + \sum_{j=1}^{\ell} \lambda_j \cdot h_j$$

(ℓ lambdas)

FOC*

$$x_i \geq 0 \qquad \frac{\partial L^*}{\partial x_i} \leq 0$$

$$x_i \cdot \frac{\partial L^*}{\partial x_i} = 0$$

$$\lambda_j \geq 0 \qquad \frac{\partial L^*}{\partial \lambda_j} \geq 0 \qquad \lambda_j \cdot \frac{\partial L^*}{\partial \lambda_j} = 0$$

Old (General case)

FOC

$$\frac{\partial L}{\partial x_i} = 0.$$

$$\lambda_j \geq 0 \qquad \frac{\partial L}{\partial \lambda_j} \geq 0 \qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0$$

$\ell + n$ lambdas

Example

[In real appl-ns use always the simplest available method]

$$\max \quad x^2 - y$$

s.t. (1) $x \geq 0$

(2) $y \geq 0$

(3) $x + 3y \leq y^2 + 10$

$\to$ $y^2 + 10 - x - 3y \geq 0.$

$\to$ graphical approach (!)

Lagrangean function for non-neg. constraints.

$\boxed{\text{Step 1}}$

$J = \left(\frac{\partial h_i}{\partial x} \;\; \frac{\partial h_i}{\partial y}\right)$

(1) $J = (1 \;\; 0)$ $\Downarrow$ rank $= 1$

(2) $J = (0, 1)$ $\Downarrow$ rank $= 1$

(3) $J = (-1, \; 2y-3)$ $\Downarrow$ rank $= 1$

(1) and (2) $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\Downarrow$ rank $= 2$

(2) and (3) $J = \begin{pmatrix} 0 & 1 \\ -1 & 2y-3 \end{pmatrix}$ rank $= 2$ $\Downarrow$

(1) and (3) $J = \begin{pmatrix} 1 & 0 \\ -1 & 2y-3 \end{pmatrix}$ $\begin{cases} x = 0 \\ y^2 + 10 - x - 3y = 0 \end{cases}$

$\boxed{y^2 + 10 - 3y = 0}$

? $y = \frac{3}{2}$ the (3) is not act.

(1) and (2) and (3). $\begin{cases} x = 0 \\ y = 0 \\ y^2 + 10 - x - 3y = 0 \end{cases}$ not possible

$\to$ $\boxed{\text{NDCQ holds.}}$

$$\max \quad x^2 - y$$

s.t.

(1) $x \geq 0$

(2) $y \geq 0$

(3) $x + 3y \leq y^2 + 10$

Step 2

$$L^* = x^2 - y + \lambda\left(y^2 + 10 - x - 3y\right)$$

Step 3

$$\frac{\partial L^*}{\partial x} \leq 0 \qquad x \geq 0 \qquad \frac{\partial L^*}{\partial x} \cdot x = 0$$

$$(2x - \lambda) \leq 0 \qquad x \geq 0 \qquad x \cdot (2x - \lambda) = 0$$

$$\frac{\partial L^*}{\partial y} \leq 0 \qquad y \geq 0 \qquad \frac{\partial L^*}{\partial y} \cdot y = 0.$$

$$-1 + \lambda(2y - 3) \leq 0 \qquad y \geq 0 \qquad y \cdot (-1 + 2\lambda y - 3\lambda) = 0$$

$$\frac{\partial L^*}{\partial \lambda} \geq 0 \qquad \lambda \geq 0 \qquad \lambda \cdot \frac{\partial L^*}{\partial \lambda} = 0$$

$$y^2 + 10 - x - 3y \geq 0 \qquad \lambda \geq 0 \qquad \lambda \cdot (y^2 + 10 - x - 3y) \geq 0.$$

Case	x	y	λ
1	0	0	0
2	+	+	+
3	+	+	0
4	0	+	+
5	+	0	+
6	0	0	+
7	0	+	0
8	+	0	0

Case 1

$x = 0, \ y = 0, \ \lambda = 0$!!

Case 2

$x > 0, \ y > 0, \ \lambda > 0$

$2x - \lambda = 0 \quad -1 + 2\lambda y - 3\lambda = 0$

$y^2 + 10 - x - 3y = 0.$

$$\lambda = 2x \quad \begin{cases} -1 + 4xy - 6x = 0 \\ y^2 + 10 - x - 3y = 0 \end{cases}$$

....

Cauchy - Schwartz inequality.

Problem

$\mathrm{Corr}(X,Y) \in [-1; 1]$!!

$\max\limits_{X,Y} \mathrm{Corr}(X,Y)$?

$\mathrm{corr}(X,Y) = 1 \iff Y = \alpha + \beta X$

or $X = \alpha + \beta Y$ (a.s)

$\mathrm{Corr}^2(X,Y) \leq 1$

$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}\,X\,\mathrm{Var}\,Y}}$$

goal: $(\mathrm{Cov}(X,Y))^2 \leq \mathrm{Var}\,X \cdot \mathrm{Var}\,Y$

$$(\mathrm{Cov}(X,Y))^2 - \mathrm{Var}(X)\cdot\mathrm{Var}(Y) \overset{?}{\leq} 0$$

$b^2 - ac \leq 0$?

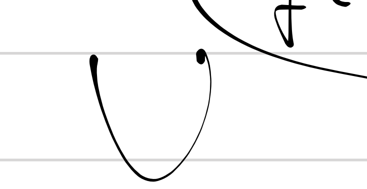
$4b^2 - 4ac \leq 0$?

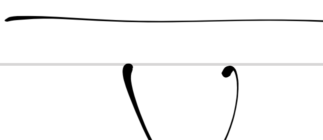
Let's consider

$$f(t) = \boxed{\mathrm{Var}(X)\cdot t^2 + 2\cdot\mathrm{Cov}(X,Y)\cdot t + \mathrm{Var}(Y)}$$

$$D = \boxed{4\cdot\mathrm{Cov}^2(X,Y) - 4\cdot\mathrm{Var}(X)\cdot\mathrm{Var}(Y)}$$

goal: $D \leq 0 \iff f(t) \geq 0 \ (\forall t)$

$D < 0$ 

$D = 0$ 

new goal. Prove that

$$\underline{f(t) = Var(X)\cdot t^2 + 2Cov(X,Y)\cdot t + Var(Y)} \geq 0 \text{ for } \forall t$$

$$f(t) = \underline{Var(tX+Y)} =$$
$$= Var(tX) + Var(Y) + 2Cov(tX, Y) =$$
$$= t^2\cdot Var(X) + Var(Y) + 2t\, Cov(X,Y)$$

$$\boxed{Var(R) = E\left(\underbrace{(R-E(R))^2}\right) \geq 0.} \quad \boxed{\forall t} \quad \forall X, Y$$

$\blacksquare$

Cauchy - Schwartz for numbers !!

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

scalar product.

$$\langle v, w\rangle \overset{?}{=} v_1\cdot w_1 + v_2\cdot w_2 + \dots + v_n\cdot w_n$$

$$\|v\| = \sqrt{v_1^2 + \dots + v_n^2} = \sqrt{\langle v, v\rangle}$$

length of v

$$CS: \quad (\langle v, w\rangle)^2 \leq \langle v, v\rangle \cdot \langle w, w\rangle$$

$$(Cov(X,Y))^2 \leq Var(V)\cdot Var(W)$$

$\boxed{\text{goal}}$: $\underbrace{(\langle v, w\rangle)^2}_{b} \overset{?}{\leq} \underbrace{\|v\|^2}_{a} \cdot \underbrace{\|w\|^2}_{c}$

$$b^2 - a\cdot c \overset{?}{\leq} 0 \qquad 4b^2 - 4ac \overset{?}{\leq} 0$$

$$f(t) = \boxed{\|v\|^2\cdot t^2 + 2\langle v, w\rangle\cdot t + \|w\|^2} \; ?$$

$$\mathcal{D} = 4(\langle v, w\rangle)^2 - 4\cdot\|v\|^2\cdot\|w\|^2 \leq 0$$

t_0 root of $f(t)$

$$f(t) = \|v\|^2 \cdot t^2 + 2\langle v, w\rangle t + \|w\|^2 =$$

$$= (\boxed{v_1^2} + v_2^2 + \dots\ v_n^2)\boxed{t^2} + 2(v_1 w_1 + v_2 w_2 + \dots)t +$$
$$+ (w_1^2 + w_2^2 + \dots + w_n^2)$$

$$= \underbrace{(v_1^2 \cdot t^2 + 2 v_1 w_1 \cdot t + w_1^2)} + (\quad \dots \quad)^2 + \dots\dots (\quad n \quad)$$

$$= (v_1 t + w_1)^2 + (v_2 t + w_2)^2 + \dots + (v_n t + w_n)^2 = 0$$

$$\geq 0$$

if $f(t) \geq 0$ for all t then

$$\boxed{D \leq 0}$$

$$w_1 = -v_1 \cdot t_0$$
$$w_2 = -v_2 \cdot t_0$$
$$\vdots$$
$$w_n = -v_n \cdot t_0$$

CS:

$$(\langle v, w\rangle)^2 \leq \|v\|^2 \cdot \|w\|^2$$

$$(v_1 w_1 + \dots + v_n w_n)^2 \leq (v_1^2 + v_2^2 + \dots + v_n^2) \cdot$$
$$\cdot (w_1^2 + \dots + w_n^2)$$

$\|w\|^2$

min $x^2 + 9y^2 + 16z^2$

s.t. $6x + 9y + 20z \geq 70.$

sol-n

$$w = \begin{pmatrix} x \\ 3y \\ 4z \end{pmatrix} \qquad \|w\|^2 = x^2 + 9y^2 + 16z^2$$

$$v = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}$$

$$\langle v, w\rangle = 6x + 9y + 20z = \langle \underbrace{\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}}_{v}, \underbrace{\begin{pmatrix} x \\ 3y \\ 4z \end{pmatrix}}_{w} \rangle$$

$$\left(\left\langle \begin{pmatrix}6\\3\\5\end{pmatrix}, \begin{pmatrix}x\\3y\\4z\end{pmatrix}\right\rangle\right)^2 \le \left\|\begin{pmatrix}6\\3\\5\end{pmatrix}\right\|^2 \cdot \left\|\begin{pmatrix}x\\3y\\4z\end{pmatrix}\right\|^2$$

$$70^2 \le (6x+9y+20z)^2 \le (6^2+3^2+5^2)\cdot\underbrace{(x^2+9y^2+16z^2)}_{\min x^2+9y^2+16z^2}$$

inequality

$$\text{minimal } f = \frac{70^2}{6^2+3^2+5^2}$$

!!

$\boxed{\text{optimal point?}}$ $\boxed{w \parallel v}$

$$\begin{cases} \dfrac{x}{6} = \dfrac{3y}{3} = \dfrac{4z}{5} \\ 6x+9y+20z = 70 \end{cases} \qquad y = \frac{x}{6} \qquad z = \frac{5x}{6\cdot 4}$$

$$6x + \frac{9x}{6} + 20\cdot\frac{5x}{6\cdot 4} = 70$$

$$x\left(6 + \frac{9}{6} + \frac{100}{6\cdot 4}\right) = 70$$

$$x = \frac{70}{6+\frac{9}{6}+\frac{100}{6\cdot 4}} \qquad y = \frac{x}{6} \qquad z = \frac{5x}{6\cdot 4}$$

ny!!