W05 - optimization

① mixed constraints case $\quad \cdots \geq 0$, $\cdots = 0$

② AM - GM inequality

mixed constraints case.

I follow the notation in Sundaram's "First course in optim."

If:

$\max f \qquad f(x) \qquad x \in \mathbb{R}^n$

constraints: $\qquad x = (x_1 \ldots\ldots x_n)$

$$\boxed{\begin{array}{ll} g_1(x) = 0 & h_1(x) \geq 0 \\ g_2(x) = 0 & h_2(x) \geq 0 \\ \vdots & \vdots \\ g_k(x) = 0 & h_\ell(x) \geq 0 \end{array}}$$

constraint set $\qquad U$ - open subset in $\mathbb{R}^n$

$$D = U \cap \{x \mid g_1 = 0 \ldots g_k = 0,\ h_1 \geq 0 \ldots h_\ell \geq 0\}$$

$f, g_1 \ldots g_k, h_1 \ldots h_\ell$ are C^1

$$\boxed{L = f + \lambda_1 \cdot h_1 + \ldots + \lambda_\ell \cdot h_\ell + \mu_1 \cdot g_1 + \ldots + \mu_k \cdot g_k}$$

and x^* - the maximizer

rank of Jacobian matrix of active constraints at the point x^* = number of active constr.

Then: x^* satisfies F.O.C.

$$\boxed{\frac{\partial L}{\partial x_i} = 0} \qquad \boxed{\lambda_j \geq 0 \quad \frac{\partial L}{\partial \lambda_j} = h_j \geq 0 \quad \lambda_j \cdot \frac{\partial L}{\partial \lambda_j} = 0}$$

$$\boxed{\partial L / \partial \mu_s = 0}$$

$i \in \{1, \ldots n\} \qquad j \in \{1 \ldots \ell\} \qquad s \in \{1, \ldots k\}$

Remark: What should I do with strict inequalities?

$x_1 + x_2 \le 5 \longrightarrow \boxed{5 - x_1 - x_2 \ge 0}$ $h_i \ge 0$

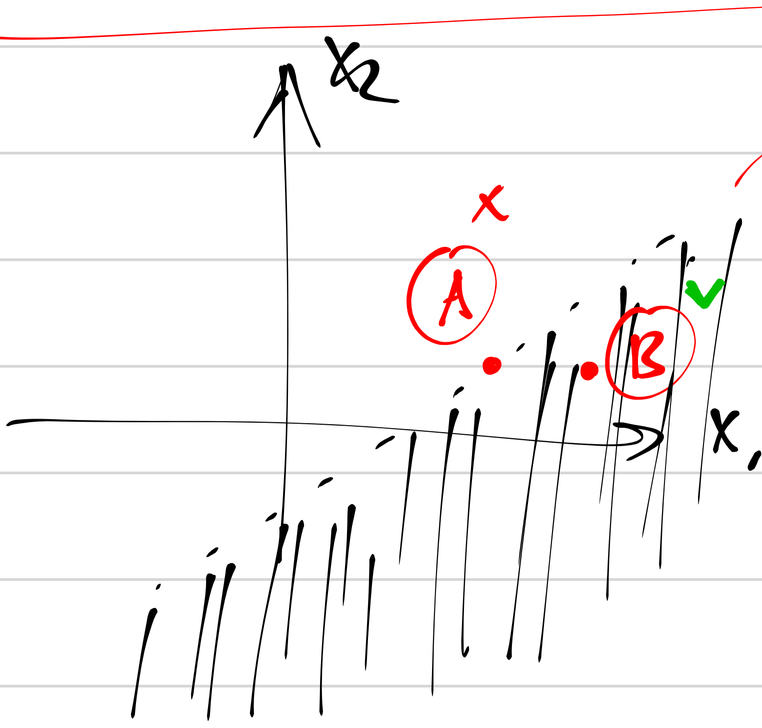
$x_3 \cdot x_2 = 6$ $\qquad \boxed{x_3 \cdot x_2 - 6 = 0}$ $g_i = 0$

$x_1 - x_2 > 3$?

describes an open subset.

Answer: ① do not include it in L

② add this constraint to F.O.C.



$x_1 - x_2 > 3 \sim \boxed{x_1 - x_2 \ge 3 \text{ non-act.}}$

$x_2 < x_1 - 3$

Advice: use simple tools whenever possible.

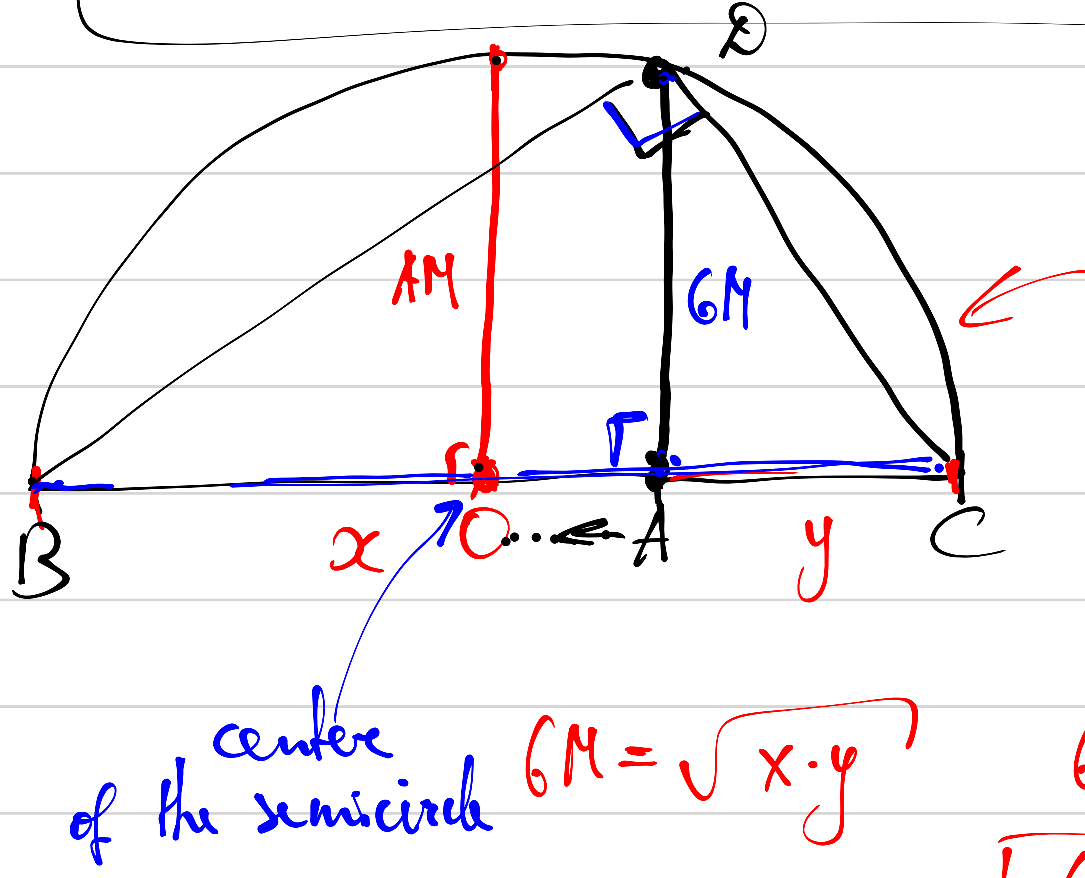
AM - GM inequality
(GM: geometric mean; AM: arithmetic mean)

$AM = \frac{x+y}{2}$ $\qquad GM = \sqrt{x \cdot y}$

$AM = \frac{1+9}{2} = 5$ $\qquad GM = \sqrt{1 \cdot 9} = 3$

theorem if $x>0$ $y>0$ then
$AM \geq GM$
with $AM = GM$ only if $x=y$.

$AM = GM$
$O = A$
$x = y$



$AM = \frac{x+y}{2} = R$

$GM = \sqrt{x \cdot y}$

$GM^2 = x \cdot y$

$$\boxed{\frac{GM}{x} = \frac{y}{GM}}$$

$\Delta BAD \sim \Delta DAC$

$$\frac{DA}{x} = \frac{y}{DA}$$

$$\boxed{AM \geq GM}$$

if $x, y > 0$ then $\frac{x+y}{2} \geq \sqrt{x \cdot y}$

$\frac{x+y}{2} = \sqrt{x \cdot y}$ only if $x = y$

$$\boxed{\begin{array}{ll} \max & 2xy \\ \text{st} & 3x+5y \le 10 \end{array}}$$

simplify.
if $3x+5y<10$ then we can $x\uparrow$ and increase $2xy$ hence in the opt.
$3x+5y=10$

$$\boxed{\begin{array}{ll} \max & 2xy \\ \text{st} & 3x+5y = 10 \end{array}}$$

$\downarrow$ AM – GM

$$\frac{10}{2} = \frac{3x+5y}{2} \ge \sqrt{3x\cdot 5y} =$$

$$5 \ge \sqrt{15}\cdot\sqrt{xy}$$

$$\frac{5}{\sqrt{15}} \ge \sqrt{xy}$$

$$xy \le \frac{5^2}{15}$$

$$\boxed{f^* = 2\cdot\frac{5^2}{15}}$$

Optimal p.t.

$$\begin{cases} 3x = 5y \\ 3x+5y=10 \end{cases}$$

$3x = 5$

$5y = 5$

$$\boxed{x^* = \frac{5}{3} \quad y^* = 1}$$

$$\begin{array}{ll} \min & 2x+3y \\ \text{st.} & x\cdot y \ge 50 \\ & x>0 \\ & y>0 \end{array} \longrightarrow \boxed{\begin{array}{ll} \min & 2x+3y \\ \text{s.t} & x\cdot y = 50 \end{array}}$$

$\downarrow$ x2x3

$$\begin{array}{ll} \min & 2x+3y \\ \text{s.t.} & 2x\cdot 3y = 50\cdot 6 \end{array}$$

$$\begin{array}{ll} \min & 2\cdot\dfrac{2x+3y}{2} \\ \text{s.t.} & \sqrt{2x\cdot 3y} = \sqrt{300} \end{array}$$

$=$GM

AM $\ge$ GM

$2x = 3y$

$$\min\ 2\cdot\frac{2x+3y}{2}$$

$$\text{s.t. } \sqrt{2x\cdot 3y} = \boxed{\sqrt{300}}$$

$= GM$

$AM \geq GM$

$2x = 3y$

$2x\cdot 3y = 300$

$9y^2 = 300$

$y > 0$

$$\boxed{y^* = \sqrt{\frac{300}{9}} = \sqrt{\frac{100}{3}} \qquad x^* = \frac{3y}{2} = \frac{3}{2}\sqrt{\frac{100}{3}}}$$

$$\boxed{f^* = 2\cdot\sqrt{300}}$$

AM-GM inequality in higher dim-s.

If $x_1, x_2, \dots, x_n > 0$ then $AM \geq GM$

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} \geq (x_1 \cdot x_2 \cdot \dots \cdot x_n)^{\frac{1}{n}}$$

with exact equality only if $x_1 = x_2 = x_3 = \dots x_n$.

$$\boxed{\begin{array}{ll} \max & x\cdot y\cdot z \\ \text{st} & 2x+3y+4z \leq 100 \end{array}}$$

$$\max\ x\cdot y\cdot z$$

$$\text{s.t. } 2x+3y+4z = 100$$

otherwise we can increase x until the target value.

AM- GM.

$$\frac{2x+3y+4z}{3} = \frac{100}{3} \geq (2x\cdot 3y\cdot 4z)^{\frac{1}{3}}$$

$$\frac{100}{3} \geq (24\cdot x\cdot y\cdot z)^{\frac{1}{3}}$$

$$(100/3)^3 \geq 24\,(x\cdot y\cdot z)$$

$$f^* = \frac{(100/3)^3}{24}$$

$2x = 3y = 4z = \frac{100}{3}$

$x^* = \frac{100}{6}$

$y^* = \frac{100}{9}$ $\quad z^* = \frac{100}{12}$

2:23

$$\boxed{\begin{array}{ll} \max & x^{2}\cdot y^{3} \qquad x,y>0. \\ \text{s.t} & 3x+5y\le 600 \end{array}}$$

$$3x = \frac{3x}{2}+\frac{3x}{2} \qquad 5y = \frac{5y}{3}+\frac{5y}{3}+\frac{5y}{3}$$

$$\max \quad x^2\cdot y^3$$

$$\text{s.t.} \quad \frac{\frac{3x}{2}+\frac{3x}{2}+\frac{5y}{3}+\frac{5y}{3}+\frac{5y}{3}}{5} = \frac{600}{5} \geq$$

AM - GM

$$\geq \left(\frac{3x}{2}\cdot\frac{3x}{2}\cdot\frac{5y}{3}\cdot\frac{5y}{3}\cdot\frac{5y}{3}\right)^{\frac{1}{5}}$$

$$\boxed{f^* = \frac{120^5\cdot 12}{125}}$$

$$\left(\frac{600}{5}\right)^5 \geq \frac{3}{2}\cdot\frac{3}{2}\cdot\frac{5}{3}\cdot\frac{5}{3}\cdot\frac{5}{3}\left(x^2\cdot y^3\right)$$

$$\frac{3x}{2} = \frac{5y}{3} = \frac{600}{5} = 120$$

$$\frac{600}{5} = 120$$

$$\boxed{x^2\cdot y^3 \le \frac{120^5\cdot 4\cdot 3}{5^3}}$$

$$\boxed{x^* = \frac{2}{3}\cdot 120 \qquad y^* = \frac{3}{5}\cdot 120}$$

$$\boxed{\begin{array}{ll} \min & 20xy+5xz+3yz \\ \text{s.t.} & x\cdot yz \geq 600 \end{array}} \qquad x,y,z>0$$

$$\boxed{\begin{array}{ll} \min & 20xy+5xz+3yz \\ \text{s.t} & x\cdot y\cdot z = 600 \end{array}}$$

$$\boxed{\begin{array}{ll} \min & 3\cdot\dfrac{20xy+5xz+3yz}{3} \\ \text{s.t} & xyz=600 \end{array}}$$

AM - GM

$$\frac{20xy+5xz+3yz}{3} \geq$$

$$\geq \left(20xy\cdot 5xz\cdot 3yz\right)^{\frac{1}{3}} =$$

$$= \left(20\cdot 5\cdot 3\cdot 600\cdot 600\right)^{\frac{1}{3}}$$

$$\boxed{f^* = 3\cdot\left(300\cdot 600^2\right)^{\frac{1}{3}}}$$

$$\min\ 3\cdot\left(\frac{20xy+5xz+3yz}{3}\right)$$
$$\text{s.t. } xyz = 600$$

opt-l point: $\boxed{20xy = 5xz = 3yz}$

$$\frac{xyz}{20xy} = \frac{xyz}{5xz} = \frac{xyz}{3yz}$$

$$\boxed{\frac{z}{20} = \frac{y}{5} = \frac{x}{3}}$$

$$\boxed{\frac{z\cdot y\cdot x}{20\cdot 5\cdot 3} = \frac{600}{20\cdot 5\cdot 3}}$$

$$\frac{z}{20} = \left(\frac{600}{20\cdot 5\cdot 3}\right)^{\frac{1}{3}} \qquad \frac{y}{5} = \left(\frac{600}{20\cdot 5\cdot 3}\right)^{\frac{1}{3}}$$
$$\frac{x}{3} = \left(\frac{600}{20\cdot 5\cdot 3}\right)^{\frac{1}{3}}$$

$$z^* = 20\cdot\left(\frac{600}{20\cdot 5\cdot 3}\right)^{\frac{1}{3}} \qquad y^* = 5\cdot\left(\frac{600}{20\cdot 5\cdot 3}\right)^{\frac{1}{3}}$$
$$x^* = 3\cdot\left(\frac{600}{20\cdot 5\cdot 3}\right)^{\frac{1}{3}}$$

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* mixed constr. $h \geq 0$ $\quad g = 0$

$$L = f + \sum \lambda_i \cdot h_i + \sum \mu_j \cdot g_j$$

$$\frac{\partial L}{\partial x_i} = 0 \quad \left\{ \begin{array}{l} \lambda_i \geq 0 \\ \frac{\partial L}{\partial \lambda_i} = h_i \geq 0 \\ \lambda_i \cdot h_i = 0 \end{array} \right. \quad \Bigg| \quad \frac{\partial L}{\partial \mu_j} = 0$$

* AM - GM ineq. $\quad \frac{x+y}{2} \geq \sqrt{xy} \quad (x>0,\ y>0)$

AM ≥ GM

$$\boxed{AM = GM \quad x = y,\ x_1 = x_2 = \ldots = x_n}$$

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq (x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}}$$