Hi

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Two logins $\rightarrow$ video, $\rightarrow$ audio

→ video recordings + handwritten notes

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Optimization ← general topic

Homogeneous functions. ← today.

$$f(x_1, x_2, x_3, \ldots, x_n) \qquad f: \mathbb{R}^n \to \mathbb{R}$$

def f is homogeneous [однородная] of degree [степени] k if

$$f(tx_1, tx_2, tx_3 \ldots tx_n) = t^k \cdot f(x_1, x_2, x_3 \ldots x_n)$$

[for all points in the domain]

Examples.

? degree of homogeneity of these fun.s

a) $f(x_1, x_2) = x_1 + 7x_2$

b) $f(x_1, x_2) = \min\left(\frac{x_1}{5}, \frac{x_2}{3}, \frac{x_3}{7}\right)$

c) $f(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_3^2$

d) $f(x_1, x_2) = \frac{x_1}{x_2 + 3}$

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a) $f(tx_1, tx_2) = tx_1 + 7tx_2 = t^1 \cdot (x_1 + 7x_2) =$

$= t^1 \cdot f(x_1, x_2)$

degree = 1

b) $(x_1, x_2, x_3) \in \mathbb{R}^3$

$\min(-1, -2, -3) = -3 \neq -1 \cdot \min(1, 2, 3)$

no!

$x_1 > 0 \quad x_2 > 0 \quad x_3 > 0 \quad (t > 0)$

$\min\left(t\frac{x_1}{5}, t\frac{x_2}{3}, t\frac{x_3}{7}\right) = t^1 \cdot \min\left(\frac{x_1}{5}, \frac{x_2}{3}, \frac{x_3}{7}\right)$

homog. degree = 1

c) $f = x_1^2 + x_1 x_2 + x_3^2$

$f(tx_1, tx_2, tx_3) = t^2 \cdot f(x_1, x_2, x_3)$

homog. degree = 2

d) $f(x_1, x_2) = \frac{x_1}{x_2 + 3}$ not homog.

$f(tx_1, tx_2) = \frac{tx_1}{tx_2 + 3} \neq t^k \cdot \frac{x_1}{x_2 + 3}$

$x_1 = 1$
$x_2 = 1$

$LHS = \frac{t}{t + 3}$ $\quad$ $RHS = t^k \cdot \frac{1}{4}$

theorem.

$\boxed{\text{If :}}$ $f_1(x_1, x_2 \dots x_n)$ is hom. of degree k_1
$f_2(x_1, x_2, \dots x_n)$ is hom. of degree k_2

then: $f = f_1 \cdot f_2$ is hom. of degree $k_1 + k_2$

Proof:

$$f(tx_1, tx_2, \dots tx_n) = f_1(tx_1 \dots tx_n) \cdot f_2(tx_1 \dots tx_n) =$$

$$= t^{k_1} \cdot f_1(x_1 \dots x_n) \cdot t^{k_2} f_2(x_1 \dots x_n) =$$

$$= t^{k_1 + k_2} f_1(x_1 \dots x_n) \cdot f_2(x_1 \dots x_n) =$$

$$= t^{k_1 + k_2} \cdot f(x_1 \dots x_n)$$

Theorem.

If: f is homogeneous of degree k
[and differentiable] [f is not const]

then: $\frac{\partial f}{\partial x_i}$ is homogeneous of degree $(k-1)$.

Ex.

$$f(x_1, x_2, x_3) = \frac{x_1^2 \cdot x_2}{x_3^5} \qquad \text{degree} = -2 :$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad f(t \cdot x) = \frac{t^2 \cdot t \cdot x_1^2 x_2}{t^5 x_3^5} = t^{-2} \cdot \frac{x_1^2 x_2}{x_3^5}$$

$$\frac{\partial f}{\partial x_3} = \frac{x_1^2 \cdot x_2}{x_3^6} \cdot (-5) \qquad \frac{\partial f}{\partial x_3}(tx) = \frac{t^2 \cdot t \cdot x_1^2 \cdot x_2}{t^6 \cdot x_3^6} \cdot (-5) =$$

$$= t^{-3} \cdot \frac{\partial f}{\partial x_3}(x)$$

Proof

(0) $$f(tx_1, tx_2 \ldots tx_n) = t^k \cdot f(x_1, x_2 \ldots x_n)$$ identity

(1) $$\boxed{\frac{\partial LHS}{\partial x_1} = \frac{\partial RHS}{\partial x_1}}$$

$$\frac{\partial (tx_1)}{\partial x_1} = t$$

(2) $$\boxed{\frac{\partial f}{\partial x_1}(tx_1, tx_2 \ldots tx_n) \cdot t} = \boxed{t^k \frac{\partial f}{\partial x_1}(x_1, x_2 \ldots x_n)}$$

(3) $$\frac{\partial f}{\partial x_1}(tx_1, tx_2 \ldots tx_n) = t^{k-1} \frac{\partial f}{\partial x_1}(x_1, x_2 \ldots x_n)$$

Euler's theorem on homogeneous fun-s.

th) the function f is homogeneous [of degree k] iff [if and only if]

f satisfies the differential equation:

$$\boxed{x_1 \cdot \frac{\partial f}{\partial x_1} + x_2 \cdot \frac{\partial f}{\partial x_2} + \ldots + x_n \cdot \frac{\partial f}{\partial x_n} = k \cdot f}$$

Examples:

$$f = \frac{x_1^2 \cdot x_2}{x_3^5}$$

$$(-2) \cdot \frac{x_1^2 \cdot x_2}{x_3^5} =$$

$$x_1 \cdot \frac{\partial f}{\partial x_1} + x_2 \cdot \frac{\partial f}{\partial x_2} + x_3 \cdot \frac{\partial f}{\partial x_3} =$$

$$= x_1 \cdot \frac{2x_1 x_2}{x_3^5} + x_2 \cdot \frac{x_1^2}{x_3^5} + x_3 \cdot \frac{x_1^2 \cdot x_2}{x_3^6}(-5) =$$

$2 + 1 - 5 =$ $k = -2$

def $\quad f(t\cdot x) = t^k \cdot f(x)$ $\quad\Downarrow\Uparrow$

E.C. $\quad \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = k \cdot f(x)$

def $\rightarrow$ E.C. $\quad$ [2 variables]

$$f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$$

$\downarrow \quad \frac{\partial LHS}{\partial t} = \frac{\partial RHS}{\partial t}$

$\frac{\partial (tx_1)}{\partial t} = x_1$

$$\frac{\partial f}{\partial x_1}(tx_1, tx_2) \cdot x_1 + \frac{\partial f}{\partial x_2}(tx_1, tx_2) \cdot x_2 = k \cdot t^{k-1} \cdot f(x_1, x_2)$$

$\frac{\partial (tx_2)}{\partial t} = x_2$

plug-in $t = 1$.

$$\boxed{\frac{\partial f}{\partial x_1}(x_1, x_2) \cdot x_1 + \frac{\partial f}{\partial x_2}(x_1, x_2) \cdot x_2 = k \cdot f(x_1, x_2)}$$

ex. $f = \cos\left(\frac{x_2}{x_3}\right) \cdot x_1 \qquad \frac{\partial f}{\partial x_2} = -\sin\left(\frac{x_2}{x_3}\right) \cdot x_1 \cdot \frac{1}{x_3}$

$f = x_1^a \cdot x_2^b \cdot x_3^c \cdot x_4^d$

E.C. $\rightarrow$ def.

$$\boxed{x_1 \cdot \frac{\partial f}{\partial x_1} + x_2 \cdot \frac{\partial f}{\partial x_2} = k \cdot f} \quad (\forall x)$$

[idea is not obvious]

$$h(t) = f(t\cdot x_1, t\cdot x_2)$$

$$h'(t) \overset{?}{=} \frac{\partial f}{\partial x_1}(tx_1, tx_2) \cdot x_1 + \frac{\partial f}{\partial x_2}(tx_1, tx_2) \cdot x_2$$

$$t \cdot h'(t) \overset{?}{=} \underbrace{\frac{\partial f}{\partial x_1}(tx_1, tx_2) \cdot tx_1 + \frac{\partial f}{\partial x_2}(tx_1, tx_2) \cdot tx_2}_{} \overset{(E.C.)}{=}$$

$$= k \cdot f(tx_1, tx_2) = k \cdot h(t)$$

We've got a diff. equat.

$$\boxed{t \cdot h'(t) = k \cdot h(t)}$$

$$t \cdot \frac{dh}{dt} = k \cdot h$$

$$\frac{dh}{h} = k \cdot \frac{dt}{t} \qquad k = \text{const}$$

$$\int \frac{dh}{h} = k \int \frac{dt}{t} \qquad h > 0,\ t > 0$$

$$\ln h = k \cdot \ln t + c$$

$$h = t^k \cdot d \qquad \text{does not depend on } t$$

$$f(tx_1, tx_2, \dots tx_n) = t^k \cdot d$$

$t = 1$

$$f(x_1, x_2, \dots x_n) = d$$

Voila!

$$f(tx_1, tx_2 \dots tx_n) = t^k \cdot f(x_1, x_2, \dots x_n)$$

[def]

M

Q. f - homogeneous of degree 5

$$\underbrace{f(x_1, \dots x_n)} \cdot \underbrace{\frac{\partial f}{\partial x_1}(x_1, \dots x_n)} \cdot \underbrace{\frac{\partial^2 f}{\partial x_2 \partial x_3}(x_1, \dots x_n)}$$

is homog? what is the degree?

A? $\boxed{\text{d. of h.}} = 5 + 4 + 3 = 12$!!

Homogeneous differential equation

$$\boxed{\frac{dy}{dx} = f(x,y)} \quad f \text{ is homogeneous.}$$

Strategy to solve: → divide y by x

→ replace $y(x)$ by

$y(x) = x \cdot b(x)$

$\frac{y(x)}{x} = b(x)$

It will help! (probably !!)

Ex $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ ← RHS is homog. $\boxed{k=0}$

$$\frac{x^2t^2 + y^2 \cdot t^2}{2xy \cdot t \cdot t} = t^0 \cdot \frac{x^2+y^2}{2xy}$$

Follow the advice ... !! $y(x) = x \cdot b(x)$

LHS : $\frac{dy}{dx} = b(x) + x \cdot b'(x)$

RHS : $\frac{x^2+y^2}{2xy} = \frac{x^2 + x^2 \cdot b^2(x)}{2x \cdot x \cdot b(x)}$

$$b(x) + x \cdot b'(x) = \frac{1+b^2}{2b}$$

$$b + x \cdot \frac{db}{dx} = \frac{1+b^2}{2b}$$

$$x \cdot \frac{db}{dx} = \frac{1+b^2}{2b} - b$$

$$\frac{1+b^2-2b^2}{2b} = \frac{1-b^2}{2b}$$

$$x \cdot \frac{db}{dx} = \frac{1-b^2}{2b} \longrightarrow \boxed{\frac{db}{1-b^2} \cdot 2b = \frac{dx}{x}}$$