Optization. Hi :) (W03)

W02 : KT theorem
W03 cookbook procedure.

We have opt-n problem with inequal-s.

Step 1 Avoid as much as poss. solving opt-n problems with inequalities.

Ex. $\max \; 5 - x_1^2 - x_2^2 + x_1 x_2 + 5x_1$

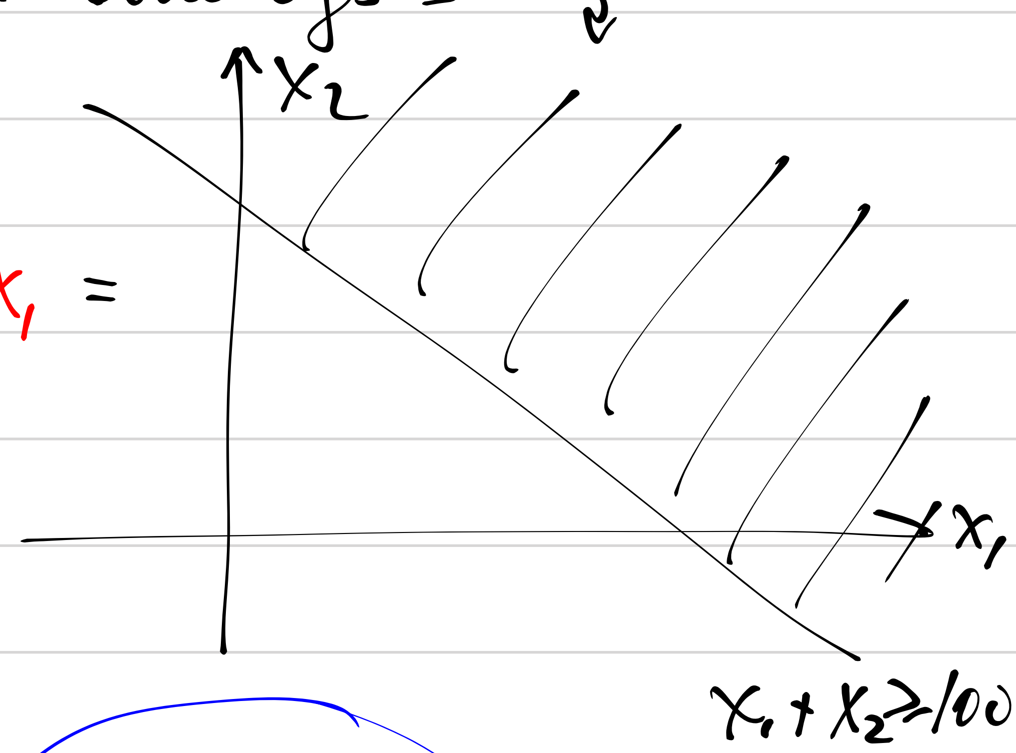
s.t. $x_1 + x_2 \geq 100$

↳ stand way : KT

advice : stop and think whether you can simplify it.

main quest: is it possible to transform inequality into equality?

- monotonicity
- graphical analysis →
- ....




$f(x_1,x_2) = 5 - x_1^2 - x_2^2 + x_1 x_2 + 5x_1 =$

$-x_2^2 + 2 \cdot \frac{1}{2} x_1 x_2 - \frac{1}{4} x_1^2$

$-a^2 + 2ab - b^2$

$= 5 \boxed{- x_2^2 + 2 \cdot \frac{1}{2} x_1 \cdot x_2 - \frac{1}{4} x_1^2} - \frac{3}{4} x_1^2 + 5x_1$

$$= 5 \boxed{-x_2^2 + 2\cdot\frac{1}{2}x_1\cdot x_2 - \frac{1}{4}x_1^2} - \frac{3}{4}x_1^2 + 5x_1 =$$

$$= 5 - \left(x_2 - \frac{1}{2}x_1\right)^2 - \left(\sqrt{\frac{3}{4}}x_1\right)^2 + 2\cdot\frac{1}{2}\cdot\sqrt{\frac{3}{4}}x_1\cdot\sqrt{\frac{4}{3}}\cdot 5 =$$

$$-a^2 + 2\cdot a\, b - b^2$$

$$a = \sqrt{\frac{3}{4}}\cdot x_1 \qquad b = \frac{1}{2}\cdot\sqrt{\frac{4}{3}}\cdot 5 = \frac{5}{\sqrt{3}}$$

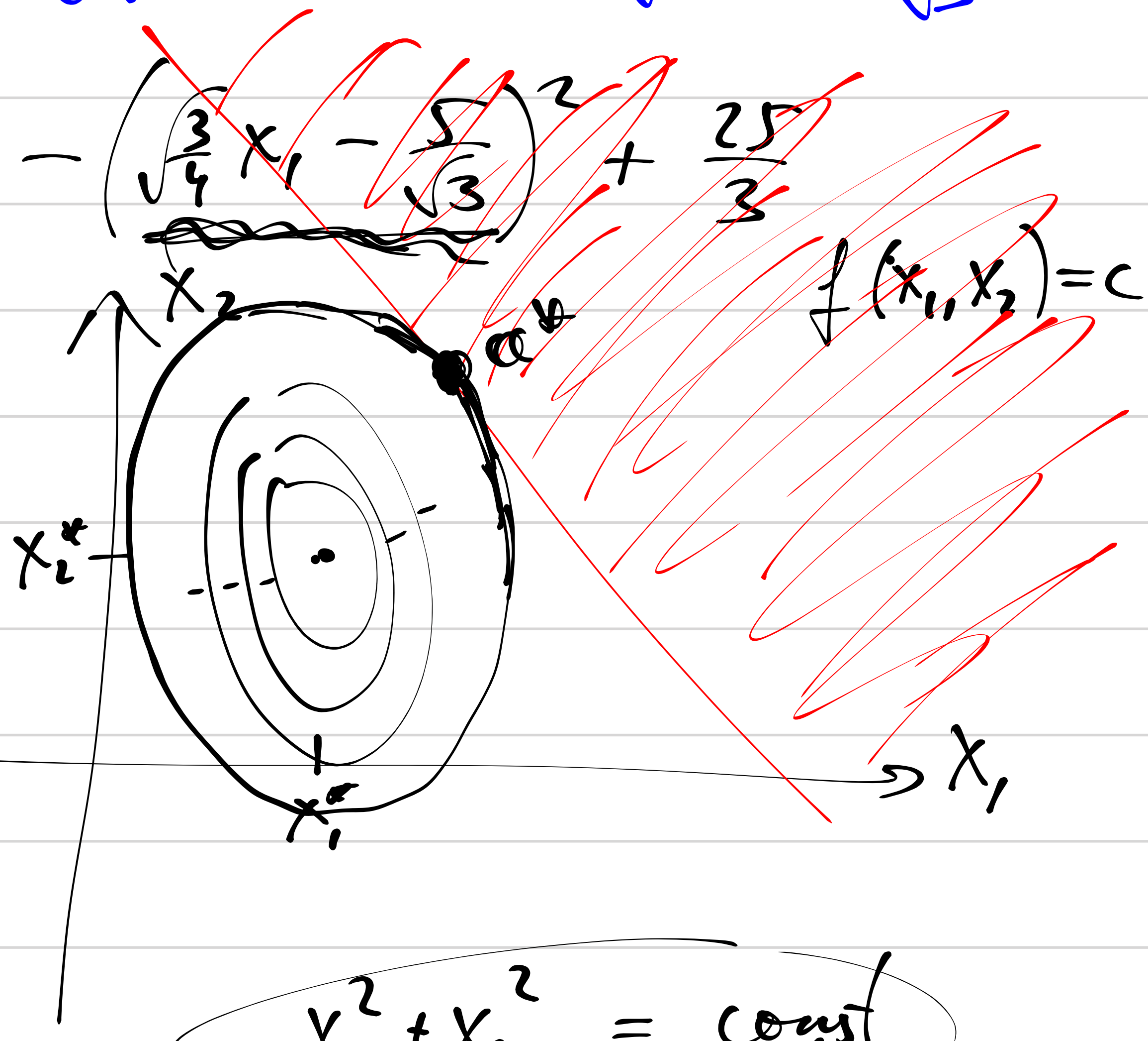
$$= 5 - \left(x_2 - \frac{1}{2}x_1\right)^2 - \left(\sqrt{\frac{3}{4}}x_1 - \frac{5}{\sqrt{3}}\right)^2 + \frac{25}{3}$$

opt. without constraints is located at: $x_2^* = \frac{1}{2}x_1^*$

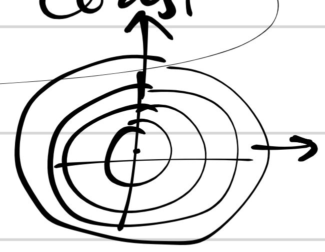
$\sqrt{\frac{3}{4}}x_1^* = \frac{5}{\sqrt{3}}$

$x_1^* = \frac{5}{3}\cdot 2$

$x_2^* = \frac{5}{3}$




$$x_1^2 + x_2^2 = \text{const}$$



$a^*$ is the constr. extremum.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_2 - \frac{1}{2}x_1 \\ \sqrt{\frac{3}{4}}x_1 - \frac{5}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \sqrt{\frac{3}{4}} & 0 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{5}{\sqrt{3}} \end{pmatrix}}$$

Now: max $\quad 5 - x_1^2 - x_2^2 + x_1 x_2 + 5x_1$

s.t. $\quad x_1 + x_2 = 100 \qquad x_2 = 100 - x_1$

$$\max_{x_1} \quad 5 - x_1^2 + x_1(100 - x_1) - (100 - x_1)^2 + \sqrt{7} x_1$$

hope you can finish from here !!

ex 2. Monotonicity:

global $\max \quad x_1^2 \cdot x_2^3 \, x_3^5 \cdot x_4^6 \qquad f(x_1, x_2, x_3, x_4)$

s.t. $\quad x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_4 \geq 0$

$\quad x_1 + x_2 + x_3 + x_4 \leq 600$

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Idea 1: $x_1 = 0$ or $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ are not optimal!

$$\boxed{\text{conclusion: } x_1 > 0 \quad x_2 > 0 \quad x_3 > 0 \quad x_4 > 0}$$

Idea 2. a point $(x_1, x_2, x_3, x_4)$ with $x_1 + x_2 + x_3 + x_4 < 600$ can't be optimal.

in such a point one can increase $x_4$ (or $x_2$ or $x_1$ or $x_3$) a little bit and $f$ will increase

$$\boxed{\text{conclusion: } x_1 + x_2 + x_3 + x_4 = 600}$$

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$$\max \; 5 - x_1^2 - x_2^2 + x_1 x_2 + 5x_1$$

$$\text{s.t.} \quad x_1 + x_2 \geq 100$$

cookbook procedure

Step 1 Check NDCQ

Step 2 write Lagrangean function

Step 3. state (and solve) F.O.C

Step 4. check sufficiency conditions

Step 1.

we need to check NDCQ for all possible combinations of active constraints.

active constr LHS $\rightarrow$ = RHS

$x_1 + x_2 \geq 100$

act: $x_1 + x_2 = 100$

non act: $x_1 + x_2 > 100$

$J = \left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}\right) = (1, 1)$

rank $(1,1) = 1$

number of act $= 1$

$\Downarrow$ ☺

$\emptyset$

rank $= 0$

number of act $= 0$

!! ☺

NDCQ:

rank of the Jacobian matrix of active constrts is equal to the number of active constraints.

$\begin{cases} h_1 \geq 0 \\ h_2 \geq 0 \\ h_3 \geq 0 \end{cases}$

case 1 $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ . . .

case 2 $h_1 > 0$, $h_2 = 0$, $h_3 = 0$ . . . .

case 8

boring !!

$\boxed{\max}$ (1) $5 - x_1^2 - x_2^2 + x_1 x_2 + 5x_1$

s.t. $x_1 + x_2 \geq 100 \longrightarrow x_1 + x_2 - 100 \geq 0$ (2)

Step 2

$$\boxed{\begin{array}{l} \max f(x_1 \dots x_n) \\ h_i(x_1 \dots x_n) \geq 0 \end{array}}$$

min $5 - x_1^2 - x_2^2 \dots$

max $-5 + x_1^2 + x_2^2 \dots$

$L(x_1, x_2, \lambda) = f \; \pm \; \lambda \cdot h$ (3)

$L = 5 - x_1^2 - x_2^2 + x_1 x_2 + 5x_1 + \lambda(x_1 + x_2 - 100)$

Step 3 F.O.C.

$\frac{\partial L}{\partial x_1} = 0$ $\quad \lambda \geq 0$ (4)

$\frac{\partial L}{\partial x_2} = 0$ $\quad \boxed{\frac{\partial L}{\partial \lambda} = x_1 + x_2 - 100 \geq 0}$

compl. slack: $\lambda \cdot \frac{\partial L}{\partial \lambda} = 0$

start solving from here

case 1 $\quad \lambda = 0$

$\partial L / \partial x_1 = -2x_1 + x_2 + 5 + \lambda = 0$

$\partial L / \partial x_2 = -2x_2 + x_1 + \lambda = 0$

$$\begin{cases} \lambda = 0 \\ x_1 = 2x_2 \\ -2(2x_2) + x_2 + 5 = 0 \end{cases}$$

$x_2 = \frac{5}{3}$

$x_1 = \frac{10}{3}$

$\lambda = 0.$

$\frac{5}{3} + \frac{10}{3} - 100 \geq 0$ [FALSE]

no points in case 1

Case 2 $\quad \lambda > 0 \quad \frac{\partial L}{\partial \lambda} = 0$

$\partial L / \partial x_1 = -2x_1 + x_2 + 5 + \lambda = 0$

$\partial L / \partial x_2 = -2x_2 + x_1 + \lambda = 0$

$\partial L / \partial \lambda = x_1 + x_2 - 100 = 0$

$$\begin{cases} -3x_1 + 3x_2 + 5 = 0 \\ x_1 + x_2 = 100 \quad (\times 3) \end{cases}$$

$6x_2 + 5 = 300$

$x_2 = 295/6$

$x_1 = 100 - x_2 = \frac{600}{6} - \frac{295}{6} = \frac{305}{6}$

$\lambda = 2x_2 - x_1 = \frac{590 - 305}{6} = \frac{285}{6}$

one sssp point: $x_1 = \frac{295}{6}$ $x_2 = \frac{305}{6}$

$\lambda = \frac{285}{6}$

Step 4

* [graphical analysis]
* [ soon ] bordered hessian
* Weierstrass theorem
* .....

we have already done it

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Example Step 1 only check NDCQ

$$\begin{cases} x_1^2 + 2x_2^2 \le 100 \\ x_1^3 \le 1000 \end{cases}$$

case ①

$x_1^2 + 2x_2^2 < 100$

$x_1^3 < 1000$

0 constr

NDCQ holds

case ②

$$\begin{cases} x_1^2 + 2x_2^2 = 100 \\ x_1^3 < 1000 \end{cases}$$

→ if $x_1 = 0$ then $x_2 \neq 0$

$J = (2x_1 ; 4x_2)$ rank $J = 1$

number of act-ve const = 1

NDCQ holds.

case 3

$$\begin{cases} x_1^2 + 2x_2^2 < 100 \\ x_1^3 = 1000 \end{cases}$$

$x_1 = 10$

$J = (3x_1^2 ; 0) = (300 ; 0)$ rank $J = 1$

number of active constr = 1

NDCQ holds.

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case 4.

number of act-ve const = 2

$$\begin{cases} x_1^2 + 2x_2^2 = 100 \\ x_1^3 = 1000 \end{cases}$$

→ $x_2 = 0$

↳ $x_1 = 10$

$J = \begin{pmatrix} 2x_1 & 4x_2 \\ 3x_1^2 & 0 \end{pmatrix}$

$J = \begin{pmatrix} 20 & 0 \\ 300 & 0 \end{pmatrix}$ rank = 1

NDCQ fails at $(x_1 = 10, x_2 = 0)$ → we need to check it!