W04 Optimization

Sufficiency conditions for opt with constraints!

3 approaches

First. (best possible) — Graphical approach
* draw constr. set
* level curves of f

underst.

Second. Weierstrass theorem

Last. Bordered Hessian

blind

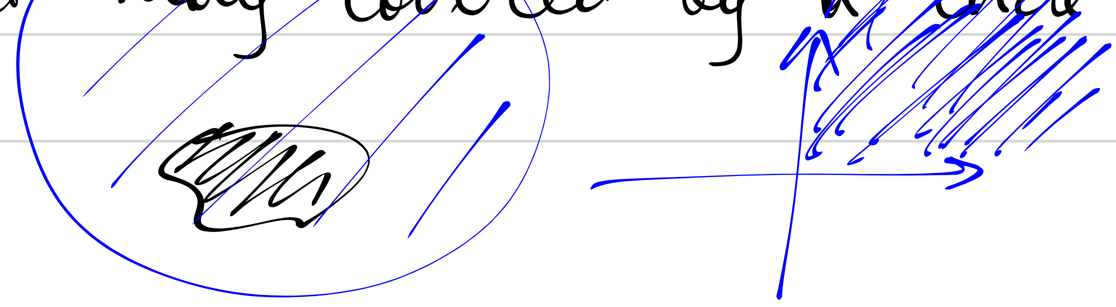
Weierstrass theorem

If the function f is continuous and the constr. set is compact = {closed, bounded}

Then the function f attains both optimal values (minim. value & maximal value) on the constr. set.

closed = the set contains all the boundary points.

bounded = the set may covered by a circle (possibly big)



In practice:

(1) If you search for local extrema and W.T. is applic and you have only two suspicious points then they are local and global extrema.

(2) If you search for global max and W.T is applic then just select the susp-s point with the highest value!

Bordered Hessian approach

(!) tedious (!)

(!) long (!)

the cookbook procedure.

After the F.O.C. we have $A_1, A_2 \dots A_z$ suspicious points.

For every pt A_i:

Size of this matrix depends on the number of active const. at the A_i

Step 1 Create bordered Hessian.

$\partial^2 L/\partial x_1 \partial \lambda_2$

$$
\begin{array}{c|cc|ccc}
 & \lambda_2 & \lambda_5 & x_1 & \dots & x_n \\
\hline
\lambda_2 & & & & & \\
\lambda_5 & & & & & \\
\hline
x_1 & & & \frac{\partial^2 L}{\partial x_1^2} & & \\
\vdots & & & & & \\
x_n & & & & &
\end{array}
$$

only λ_i for active constraints

all variables

the matrix of second order derivatives

not all λ_i!

non-active $h_i > 0$

active constraint $h_i = 0$

Step 2 susp point A

k active constraints.

n original variables $(x_1 \dots x_n)$

let's number them: $\lambda_1, \lambda_2, \dots, \lambda_k$

$$\underbrace{\lambda_1 \dots \lambda_k}_{\text{active}} \quad \underbrace{\lambda_{k+1} \dots \lambda_e}_{\text{non-active}}$$

$$H = \begin{array}{c|cc} & \lambda_1 \dots \lambda_k & x_1 \dots x_n \\ \hline \lambda_1 \vdots \lambda_k & & \\ x_1 \vdots x_n & & \end{array}$$

$H_{[(n+k) \times (n+k)]}$

Calculate $(n-k)$ biggest corner minors.

Examples $n=2$, $k=1$

$$H = \begin{bmatrix} \frac{\partial^2 L}{\partial \lambda_1^2} & \frac{\partial^2 L}{\partial \lambda_1 \partial x_1} & \frac{\partial^2 L}{\partial \lambda \partial x_2} \\ \frac{\partial^2 L}{\partial \lambda \partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial \lambda_1 \partial x_2} \\ \frac{\partial^2 L}{\partial \lambda_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} \end{bmatrix}$$

$H_{[3 \times 3]}$

$n-k = 2-1 = 1$

check only Δ_3

Example $n=3$, $k=1$

$$H = \begin{array}{c|cccc} & \lambda_1 & x_1 & x_2 & x_3 \\ \hline \lambda_1 & & & & \\ x_1 & & & & \\ x_2 & & & & \\ x_3 & & & & \end{array}$$

$n-k = 3-1 = 2$

check only: Δ_3, Δ_4

example $n=3$, $k=2$

$n-k = 3-2 = 1$

check only: Δ_5

$$\begin{array}{c|ccccc} & \lambda_1 & \lambda_2 & x_1 & x_2 & x_3 \\ \hline \lambda_1 & & & & & \\ \lambda_2 & & & & & \\ x_1 & & & & & \\ x_2 & & & & & \\ x_3 & & & & & \end{array}$$

Sufficiency conditions table.

*active

minor size	even number of constraints* $k=0,2,4,\dots$ min	even number of constraints* $k=0,2,4,\dots$ max	odd number of constraints* $k=1,3,5,7,\dots$ min	odd number of constraints* $k=1,3,5,7,\dots$ max
smallest	+	−	−	+
smallest+1	+	+	−	−
smallest+2	+	−	−	+
smallest+4	+	+	−	−
⋮	⋮	⋮	⋮	⋮

from those we need to check

switch.

conditions for unc. opt.

most common cases.

$n=2$ $k=1$

	min	max
Δ_3	−	+

$n=3$ $(k=1)$

	min	max
Δ_3	−	+
Δ_4	−	−

$n=3$ $k=2$

	min	max
Δ_5	+	−

ex.

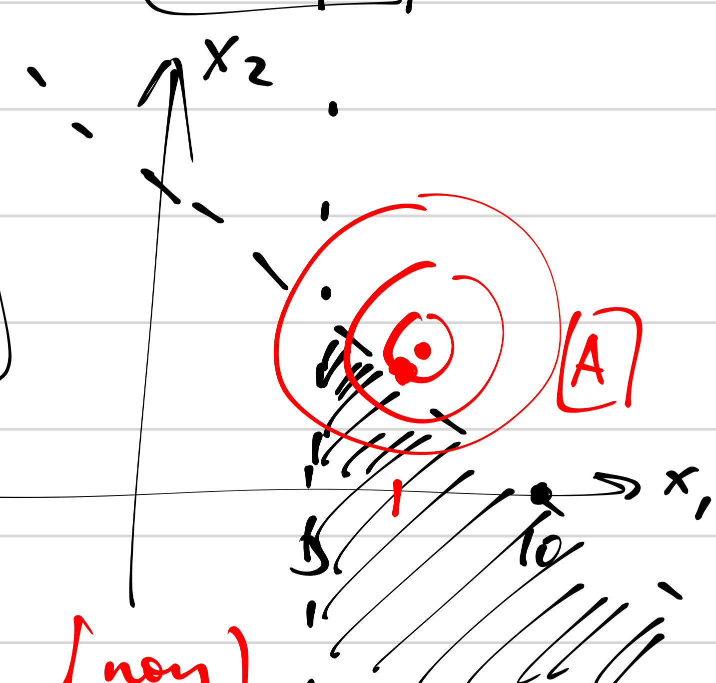
$$\min_{x_1,x_2} \; +(5-x_1)^2+(6-x_2)^2$$

s.t. $\begin{cases} x_1 \geq 3 \\ x_1+x_2 \leq 10 \end{cases}$

$$\boxed{(5-x_1)^2+(6-x_2)^2=c}$$

center $(5,6)$

Advice! $\boxed{\text{Graph.}}$



$x_1 \geq 3$ [non act]

$x_1+x_2=10$ [act]

$$\min_{x_1} \; (5-x_1)^2 + (6-(10-x_1))^2$$

$$25 + x_1^2 - 10x_1 + x_1^2 + 16 - 8x_1$$

$$h(x_1) = 2x_1^2 - 18x_1 + 41$$

$$x_1^* = \boxed{-\frac{b}{2a}} = \frac{18}{4} = 4.5 \qquad x_2^* = 5.5$$

Let's close our eyes.

NDCQ:

$x_1 \geq 3$ act $\qquad J = (1, 0)$ rank $J = 1$ ok.

$x_1 + x_2 \leq 10$ act $\qquad J = (1, 1)$ rank $J = 1$ ok.

$x_1 \geq 3$ act, $x_1 + x_2 \leq 10$ act $\qquad J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ rank $J = 2$ ok.

Stand

$$x_1 - 3 \geq 0 \qquad \max_{x_1, x_2} \; -(5-x_1)^2 - (6-x_2)^2$$
$$10 - x_1 - x_2 \geq 0$$

$$L = -(5-x_1)^2 - (6-x_2)^2 + \lambda_1(x_1 - 3) + \lambda_2(10 - x_1 - x_2)$$

F.O.C.

$$\frac{\partial L}{\partial x_1} = +2(5-x_1) + \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = +2(6-x_2) - \lambda_2 = 0$$

$$\lambda_1 \geq 0 \qquad \lambda_2 \geq 0$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 - 3 \geq 0 \qquad \frac{\partial L}{\partial \lambda_2} = 10 - x_1 - x_2 \geq 0$$

$$\lambda_1 \cdot \frac{\partial L}{\partial \lambda_1} = \lambda_1 \cdot (x_1 - 3) = 0 \qquad \lambda_2 \cdot \frac{\partial L}{\partial \lambda_2} = \lambda_2 (10 - x_1 - x_2) = 0$$

$$\begin{cases} \frac{\partial L}{\partial x_1} = +2(5-x_1) + \lambda_1 - \lambda_2 = 0 \\ \frac{\partial L}{\partial x_2} = +2(6-x_2) - \lambda_2 = 0 \\ \lambda_1 \geq 0 \quad \lambda_2 \geq 0 \\ \frac{\partial L}{\partial \lambda_1} = x_1 - 3 \geq 0 \quad \frac{\partial L}{\partial \lambda_2} = 10 - x_1 - x_2 \geq 0 \\ \lambda_1 \cdot \frac{\partial L}{\partial \lambda_1} = \boxed{\lambda_1 \cdot (x_1 - 3) = 0} \quad \lambda_2 \cdot \frac{\partial L}{\partial \lambda_2} = \boxed{\lambda_2 (10 - x_1 - x_2) = 0} \end{cases}$$

Cases:

a) $\lambda_1 > 0 \quad \lambda_2 > 0$

b) $\lambda_1 = 0 \quad \lambda_2 > 0$

c) $\lambda_1 > 0 \quad \lambda_2 = 0$

d) $\lambda_1 = 0 \quad \lambda_2 = 0$.

(A) $x_1 = 3$

$10 - x_1 - x_2 = 0$

$x_2 = 7$

$\lambda_2 = 2 \cdot (6-7) < 0$

no points.

(B) $\lambda_1 = 0 \quad \lambda_2 > 0$

$$\begin{cases} x_1 + x_2 = 10 \\ 5 - x_1 = 6 - x_2 \\ x_2 - x_1 = 1 \end{cases}$$

$x_2 = 5.5 \quad x_1 = 4.5$

$\lambda_2 = 2 \cdot (5 - x_1)$

$\lambda_2 = 2 \cdot (6 - x_2)$

$\lambda_1 = 0$

$\lambda_2 = 2 \cdot 0.5 = 1$

$x_1 > 3$ non-act

$x_1 + x_2 = 10$ act

c) $\lambda_1 > 0 \quad \lambda_2 = 0$

$x_1 = 3 \quad \lambda_1 < 0$ no points.

d) $\lambda_1 = 0 \quad \lambda_2 = 0 \qquad 5 - x_1 = 0$

$6 - x_2 = 0$

$\left.\begin{array}{l} x_1 + x_2 = 11 \\ x_1 + x_2 \leq 10 \end{array}\right\}$ no points

$$L = -(5-x_1)^2 - (6-x_2)^2 + \lambda_1(x_1 - 3) + \lambda_2(10 - x_1 - x_2)$$

S.O.C

$x_1 > 3 \quad \not\to \cancel{\lambda_1}$

$x_1 + x_2 = 10 \to \lambda_2$

$n = 2 \quad \boxed{k = 1}$ (!)

$n - k = 1$

$$H = \begin{array}{c|ccc} & \lambda_2 & x_1 & x_2 \\ \hline \lambda_2 & 0 & -1 & -1 \\ x_1 & -1 & -2 & 0 \\ x_2 & -1 & 0 & -2 \end{array}$$

$\Delta_3 = 0 + 4 + 0 = 4 > 0$ max !!