1. [10] The random variable Y takes values 0 or 1 with $\mathbb{P}(Y = 1) = p$.
   (a) [3] Find the entropy of Y and plot it as a function of p.
   (b) [3] Find the Gini impurity index of Y and plot it as a function of p.
   (c) [3] Find the second order Taylor expansion of the entropy as a function of p.
   (d) [1] Is the second order entropy Taylor expansion exactly equal to the Gini impurity index?

2. [10] Random variables X and Y are discrete and independent.
   (a) [3] Is it possible that $\mathbb{H}(X+Y) = \mathbb{H}(X) + \mathbb{H}(Y)$? Provide an example or prove that it is not possible.
   (b) [3] Is it possible that $\mathbb{H}(X+Y) < \mathbb{H}(X) + \mathbb{H}(Y)$? Provide an example or prove that it is not possible.
   (c) [4] Is it possible that $\mathbb{H}(X+Y) > \mathbb{H}(X) + \mathbb{H}(Y)$? Provide an example or prove that it is not possible.

3. [10] Each day you can bet any amount from zero up to your wealth. Your bet is multiplied by 0.5 or by 4 with equal probabilities. You optimize your wealth in the long-run period.
   (a) [8] What is your optimal strategy?
   (b) [2] Which maximal long-term daily interest rate is attainable?

4. [10] Consider the following dataset $y = (1, 1, 1, 1, 0, 0)$, $a = (0, 0, 0, 1, 0, 1)$ and $b = (1, 1, 0, 0, 0, 1)$. Here y is the target variable and a and b are predictors.
   (a) [8] Construct a classification tree. Use Gini impurity index as splitting criterion. Grow the tree until it is not possible to split further.
   (b) [2] Calculate the total impurity drop due to each predictor.

5. [10] Elon Musk observes the random variable x and forecasts the random variable y using the formula $\hat{y} = 2x$, but actually $y = 2x^2 + u$. Random variables x and u are independent and uniform on $[0; 1]$.
   (a) [2] Calculate the mean squared error of the forecast, $MSE = \mathbb{E}((y - \hat{y})^2)$.
   (b) [8] Decompose the mean squared error onto three parts: part due to forecast bias, part due to forecast variance and unpredictable part.

6. [10] Random variables y_1, ..., y_n are independent identically distributed with $\mathbb{P}(y_i = 1) = p$, $\mathbb{P}(y_i = 0) = 1 - p$. Consider a naive bootstrap sample y_1^*, ..., y_n^*.
   (a) [4] Find $\mathbb{P}(y_1^* = y_1)$ and $\mathbb{P}(y_1^* = y_2^*)$.
   (b) [4] Find $\mathbb{E}(y_i^*)$ and $\mathbb{V}\text{ar}(y_i^*)$.
   (c) [2] Find $\mathbb{C}\text{ov}(y_1^*, y_2^*)$.