1. [10] The random variable Y has exponential distribution with rate $\lambda = 1$. Consider $W = \lfloor Y/n \rfloor \cdot n$.

   (a) [4] Find the entropy of W for $n = 1$.

   (b) [3] Find the entropy of W for general n.

   (c) [3] Find the entropy of Y.

2. Let X be the result of a fair dice throw. Elon Musk make a forecast $\hat{X}$ and his forecast may be $+0.5$ or -0.5 off with equal probabilities independently of the value of X.

   (a) [5] Find entropies $\mathbb{H}(X, \hat{X})$, $\mathbb{H}(X + \hat{X})$.

   (b) [5] Find conditional entropies $\mathbb{H}(\hat{X} \mid X)$ and $\mathbb{H}(X \mid \hat{X})$.

3. [10] Each day you can bet any amount from zero up to your wealth. Your bet is multiplied by 0.5 or by 3 with equal probabilities. You optimize your wealth in the long-run period.

   (a) [8] What is your optimal strategy?

   (b) [2] Which maximal long-term daily interest rate is attainable?

4. [10] Consider 5 observation. Here y is the continuous target variable and a is the predictor.

y	100	50	40	120	120
a	1	2	3	4	5

   Find all the optimal splits in the first node if we try to minimize the total absolute error, $\sum |y_i - \hat{y}_i|$.

5. [10] Elon Musk observes the random variable x and forecasts the random variable y using the formula $\hat{y} = x^2$, but actually $y = 2x + u$. Random variables x and u are independent and uniform on $[0; 1]$.

   (a) [2] Calculate the mean squared error of the forecast, $MSE = \mathbb{E}((y - \hat{y})^2)$.

   (b) [8] Decompose the mean squared error onto three parts: part due to forecast bias, part due to forecast error variance and unpredictable part.

6. [10] Consider the naive bootstrap for a sample y_1, ..., y_n of n observations from normal distribution.

   (a) [4] What is the probability that y_5 will not enter the bootstrap sample? What is the limit of this probability when $n \to \infty$?

   (b) [6] What is the law of distribution of the number of copies of original observation y_5 in the bootstrap sample for $n \to \infty$?