Hi ☺

ETS(Z Z Z) model
- error
- trend
- seasonality

ETS(A A A) model

ARIMA

source of randomness

$u_t \sim N(0; \sigma^2)$ indep.

$y_t = c + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \ldots + u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} \ldots$

google: Forecasting principles and practice by Hydman,

→ ETS models

---

$u_t \sim N(0; \sigma^2)$ independent

$y_t = \boxed{l_{t-1} + b_{t-1}} + \boxed{s_{t-12}} + \boxed{u_t}$

$y_t$ ← observed

$l_t$, $b_t$, $s_t$ are unicorns

seas. effect

$s_t = s_{t-12} + \gamma \cdot u_t$

level / deseasoned $y_t$ / smoothed $y_t$ / trend

$l_t = l_{t-1} + b_{t-1} + \alpha u_t$

+ initial conditions:

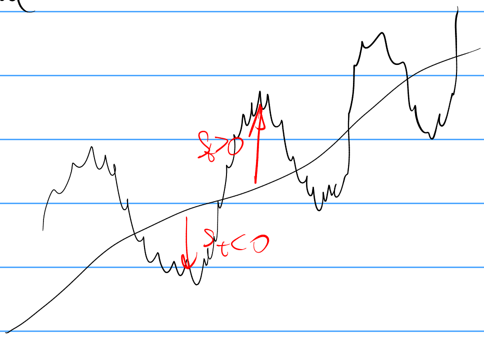
$b_t = b_{t-1} + \beta u_t$

$b_0$, $l_0$, $s_0, s_1, s_2, s_3 \ldots \ldots s_{11}$

local slope of trend

+ 1 constr.

$s_0 + s_1 + s_2 + \ldots + s_{11} = 0$




Q. How many free par-s do we have?

$\sigma^2, \alpha, \beta, \gamma, b_0, l_0, s_0, s_1, s_2 \ldots s_{11}$

s.t.: $s_0 + s_1 + s_2 + \ldots + s_{11} = 0$

A. 17 free param-s (18 par-s − one constr).

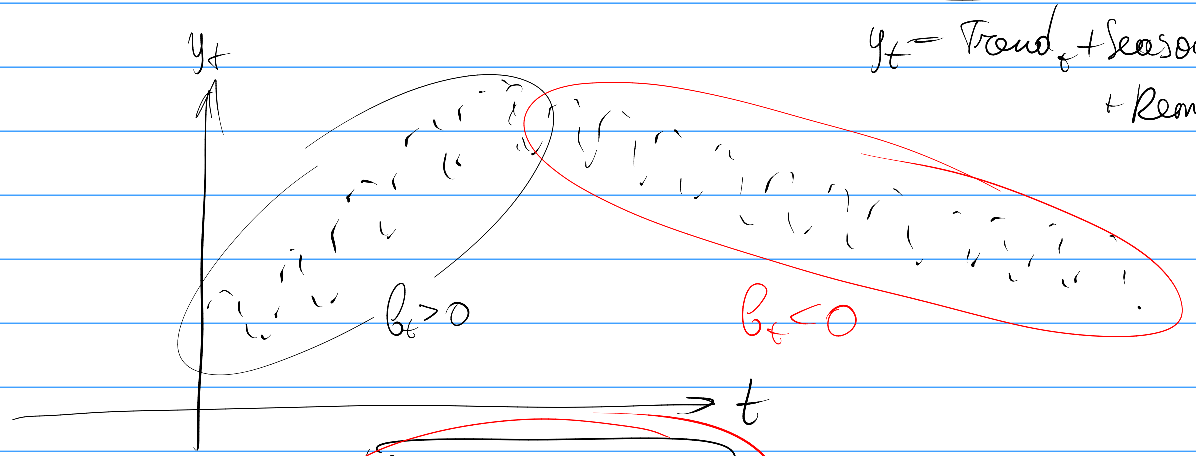
To estimate ETS model we need ≥ 4-5 years.

---

ARIMA model : predictions

ETS model : predictions + decomposition.

$y_t = \text{Trend}_t + \text{Seasonal}_t + \text{Remainder}_t$




$$\boxed{b_t = b_{t-1} + \beta \cdot u_t}$$

slope of the trend

**Problem 1**

Parameters: (and decomp-n)

par-s: $\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$ $\gamma = \frac{1}{2}$

decomp: $l_{100} = 20$, $b_{100} = 1$, $s_{89} = +2$, $s_{90} = -1$

par: $\sigma^2 = 16$

95% Predictive interval for $y_{101}$, $y_{102}$ ?

$$y_{101} = \underbrace{l_{100} + b_{100} + s_{89}}_{\text{known at time } t=100} + u_{101}$$

$u_{101}$: indep of what we know at time $t=100$

$\mathcal{F}_t = \sigma(y_t, y_{t-1}, y_{t-2}, \dots)$

$$\begin{cases} u_t \sim N(0; \sigma^2) \text{ independent} \\ y_t = l_{t-1} + b_{t-1} + s_{t-12} + u_t \\ s_t = s_{t-12} + \gamma \cdot u_t \\ l_t = l_{t-1} + b_{t-1} + \alpha u_t \\ b_t = b_{t-1} + \beta u_t \end{cases}$$

$$E(y_{101} | \mathcal{F}_{100}) = l_{100} + b_{100} + s_{89} + 0 = 20 + 1 + 2 = 23$$

$$E(u_{101} | \mathcal{F}_{100}) = E(u_{101}) = 0$$

$$Var(y_{101} | \mathcal{F}_{100}) = Var(\underbrace{l_{100} + b_{100} + s_{89}}_{\text{are known}} + u_{101} | \mathcal{F}_{100}) = Var(u_{101} | \mathcal{F}_{100}) = Var(u_{101}) = \sigma^2 = 16$$

$$PI: [23 - 1{,}96\sqrt{16} \; ; 23 + 1{,}96\sqrt{16}]$$

future = past (known) part + random (unpredict.) part

$$y_{102} = l_{101} + b_{101} + s_{90} + u_{102} =$$

$$= (l_{100} + b_{100} + \alpha \cdot u_{101}) + (b_{100} + \beta \cdot u_{101}) + s_{90} + u_{102} =$$

$$= \underbrace{(l_{100} + 2b_{100} + s_{90})}_{\text{predictable at time 100}} + \underbrace{(\alpha+\beta)\cdot u_{101} + u_{102}}_{\text{independent of what we know at time 100}}$$

$$E(y_{102} \mid \mathcal{F}_{100}) = l_{100} + 2b_{100} + s_{90} = 20 + 2\cdot 1 - 1 = 21$$

$$Var(y_{102} \mid \mathcal{F}_{100}) = Var((\alpha+\beta)u_{101} + u_{102}) = (\alpha+\beta)^2 \cdot \sigma^2 + \sigma^2 = \left(\tfrac{1}{2}+\tfrac{1}{2}\right)^2 \cdot 16 + 16 = 32$$

$$(y_{102} \mid \mathcal{F}_{100}) \sim N\left(l_{100} + 2b_{100} + s_{90};\ (\alpha+\beta)^2\sigma^2 + \sigma^2\right)$$

$y_{102}$ PI: $[21 - 1.96\sqrt{32};\ 21 + 1.96\sqrt{32}]$

3 tasks:

- data $\to$ estimate parameters.
- $\to$ data + est. par-s $\to$ decomposition of $y_t$ . !! =
- decomp + estim. of par-s $\to$ forecasts

$\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$ $\gamma = \frac{1}{2}$ $l_0 = 10$ $b_0 = 2$

$s_0 = s_{-1} = s_{-2} = \ldots = s_{-5} = -3$

$s_{-6} = s_{-7} = \ldots\ldots = s_{-11} = +3$

| $t$ | $y_t$ | $l_t$ | $b_t$ | $s_t$ | $u_t$ |
|---|---|---|---|---|---|
| $t=0$ | NA | 10 | 2 | $-3$ | NA |
| $t=1$ | 12 | 10.5 | 0.5 | 1.5 | ? $-3$ |
| $t=2$ | 14 | 11 | 0.5 | 3 | 0 |
| $t=3$ | 10 | | | | |
| $\vdots$ | $\vdots$ | | | | |

$u_t = y_t - l_{t-1} - b_{t-1} - s_{t-12}$

① $u_1 = \underset{12}{y_1} - \underset{10}{l_0} - \underset{2}{b_0} - \underset{3}{s_{-11}}$

② $l_1 = l_0 + b_0 + \alpha \cdot u_1 = 12 - \frac{3}{2}$

$b_1 = b_0 + \beta \cdot u_1 = 2 - \frac{3}{2} = \frac{1}{2}$

$s_1 = s_{-11} + \gamma \cdot u_1 = 3 + \frac{1}{2}\cdot(-3) = 1.5$

observ-ns  parameters

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} \qquad \theta = \begin{pmatrix} \sigma^2 \\ \alpha \\ \beta \\ \gamma \\ \ell_0 \\ b_0 \\ s_0 \\ s_{-1} \\ \vdots \\ s_{-10} \end{pmatrix}$$

$\boxed{\text{Max. Likelihood!}}$

$$\ln L = \ln f(y_1, y_2, \ldots\ldots y_T) = \ln L(y, \theta)$$

$$= \ln f(y_1) + \ln f(y_2 | y_1) + \ln f(y_3 | y_2, y_1) + \ldots\ldots + \ln f(y_T | y_{T-1}, y_{T-2}, \ldots\ldots) \to \max_\theta$$

$$\boxed{s_{-11} = 0 - \sum_{t=-10}^{0} s_t}$$

$$\boxed{(y_t | y_{t-1}, y_{t-2}, \ldots) \sim N(\ell_{t-1} + b_{t-1} + s_{t-12}; \sigma^2)}$$

$N(\mu; \sigma^2)$

$$\boxed{\begin{aligned} \ln f &= \ln \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{1}{2} \exp\left(-\frac{1}{2\sigma^2}(y_t - \mu)^2\right) = \\ &= -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\exp\left(-\frac{1}{2\sigma^2}(y_t - \mu)^2\right) \end{aligned}}$$

RHS: par-s

$$\ln f(y_1) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\exp\left(-\frac{1}{2\sigma^2}(y_1 - (\ell_0 + b_0 + s_{-11}))^2\right)$$

RHS: param-s and $y_1$

$$\ln f(y_2 | y_1) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\exp\left(-\frac{1}{2\sigma^2}(y_2 - (\ell_1 + b_1 + s_{-10}))^2\right)$$

where $\ell_1 = \ell_0 + b_0 + \alpha \cdot u_1 =$

$= \ell_0 + b_0 + \alpha(y_1 - \ell_0 - b_0 - s_{-11})$

$b_1 = b_0 + \beta \cdot u_1 = b_0 + \beta(y_1 - \ell_0 - b_0 - s_{-11})$

$$\ln f(y_3 | y_2, y_1) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\exp\left(-\frac{1}{2\sigma^2}(y_3 - (\ell_2 + b_2 + s_{-9}))^2\right)$$

$\ell_2 = \ell_1 + b_1 + \alpha \cdot u_2 =$

$= \ell_1 + b_1 + \alpha(y_2 - \ell_1 - b_1 - s_{-10}) =$

$= \ldots = \ldots\ldots =$ $\boxed{\text{only } \ell_0, b_0, s_0, s_{-1}, s_{-2} \ldots y_1, y_2}$

One may use AKAIKE inf crit.

$$AIC = -2\underbrace{\ln L}_{\text{max of log likelihood}} + 2 \cdot k$$

$k$ ← number of free params. ETS(AAA)? $k=17$

ETS(AAN) — No season. Add trend. Add error.

ETS(ANA) — No trend. Additive error. Additive season-ty

Ex

How many free par-s?

$s_t = 0$    AAN: $\boxed{\sigma^2, \alpha, \beta, \ell_0, b_0}$    $k = ?$ 5

$\boxed{b_t = 0}$    ANA    $\cancel{\beta}$ $\cancel{b_0}$    $k = ?$ 15

$$\begin{cases} u_t \sim N(0; \sigma^2) \text{ independent} \\ y_t = \boxed{\ell_{t-1} + b_{t-1}} + \boxed{s_{t-12}} + \boxed{u_t} \\ s_t = s_{t-12} + \gamma \cdot u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ b_t = b_{t-1} + \beta u_t \end{cases}$$

$y_t$ ← observed

$\ell_t, b_t, s_t$ are unicorns

+ initial conditions:

$b_0, \ell_0, s_0, s_{-1}, s_{-2}, s_{-3}, \ldots s_{-11}$

+ 1 constr.