W2 R for finance

ETS ARIMA

~ 1 hour clean

~ 10 minutes simple models

simple models for short-term univariate TS forecasts.

Q1. ARIMA model? Y×3

Q2. ETS model?

ETS „exponential smoothing without st. assumptions. "ugly child" ~1970

Statistical model! ~2005

DLT - model ~2010

fpp3 Rob Hyndman „Forecasting principles and practice".

~~fpp3~~

~30 models

ETS = Error + Trend + Seasonality

ETS(AAA) — Additive Error, Additive Trend, Additive Seasonality

ETS(ANM) — Additive Error, No trend, Multiplicative Seasonality

. . . .

ETS(AAdN) — Add Error, Additive dumped trend, No Seasonality

ETS(ANN) ← simplest

ETS(AAA) [monthly obs-s]

Error: u_t $\qquad u_t \sim N(0; \sigma^2)$ indep.

trend component. "level" $\qquad l_t$

[smth that changes slowly]

seasonal component $\qquad s_t$

[smth that changes slowly across seasons]

no randomness

$s_{-11}+s_{-10}+\ldots+s_0=0$

$$y_t = l_{t-1} + b_{t-1} + s_{t-12}$$

$$l_t = l_{t-1} + b_{t-1}$$

$$b_t = b_{t-1}$$

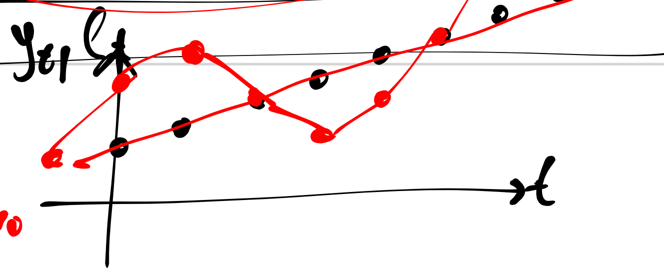
b_t is constant

$$s_t = s_{t-12}$$

?

l_0, b_0

$s_0, s_{-1}, s_{-2} \ldots s_{-10}$



ETS(AAA)

add noise!

$(u_t) \sim N(0; \sigma^2)$ indep

$$y_t = l_{t-1} + b_{t-1} + s_{t-12} + u_t$$

$$l_t = l_{t-1} + b_{t-1} + \alpha \cdot u_t$$

$$b_t = b_{t-1} + \beta \cdot u_t$$

$$s_t = s_{t-12} + \gamma \cdot u_t$$

$$s_{-11} + s_{-10} + \ldots + s_0 = 0$$

$l_0, b_0, s_0, s_{-1}, \ldots s_{-10}, \alpha, \beta, \gamma, \sigma^2$

17 par

method of estimation: $\boxed{\text{maximum likelihood}}$

$\begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} \rightarrow \boxed{\begin{matrix}\text{ETS(AAA)} \\ \text{max lik}\end{matrix}} \rightarrow \hat{\ell}_0, \hat{b}_0, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\sigma}^2, \hat{s}_0, \ldots, \hat{s}_{-10}$

phase I: estimat.

often written without „hats".

all other values are calculated.

explicit decomposition into components! $\rightarrow$ $\boxed{\hat{\ell}_1, \hat{b}_1, \hat{s}_1 \ldots\ldots \hat{\ell}_T, \hat{b}_T, \hat{s}_T}$

phase II: decomposit.

Ex. ETS(AAN)

$$y_t = \ell_{t-1} + b_{t-1} + u_t$$
$$\ell_t = \ell_{t-1} + b_{t-1} + \alpha \cdot u_t$$
$$b_t = b_{t-1} + \beta u_t$$

y_1	10	
y_2	11	
y_3	13	
$\vdots$	$\vdots$	

$\ell_0 = 9$

$b_0 = 1$

$\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$ $\sigma^2 = 16$

calculate: b_1, ℓ_1, b_2, ℓ_2

	y_t	ℓ_t	b_t	u_t
$t=0$	—	9	1	—
$t=1$	10	10 ③	1 ②	0 ①
$t=2$	11	11 ⑥	1 ⑤	0 ④
$t=3$	13	12.5 ⑨	1.5 ⑧	1 ⑦

line 1

$y_1 = \ell_0 + b_0 + u_1$

$10 = 9 + 1 + u_1$

$u_1 = 0$

$b_1 = b_0 + \frac{1}{2} \cdot u_1$

$b_1 = 1$

$\ell_1 = \ell_0 + b_0 + \frac{1}{2} \cdot u_1$

$\ell_1 = 9 + 1 + \frac{1}{2} 0 = 10$

line 2

$y_2 = \ell_1 + b_1 + u_2$

$11 = 10 + 1 + u_2$

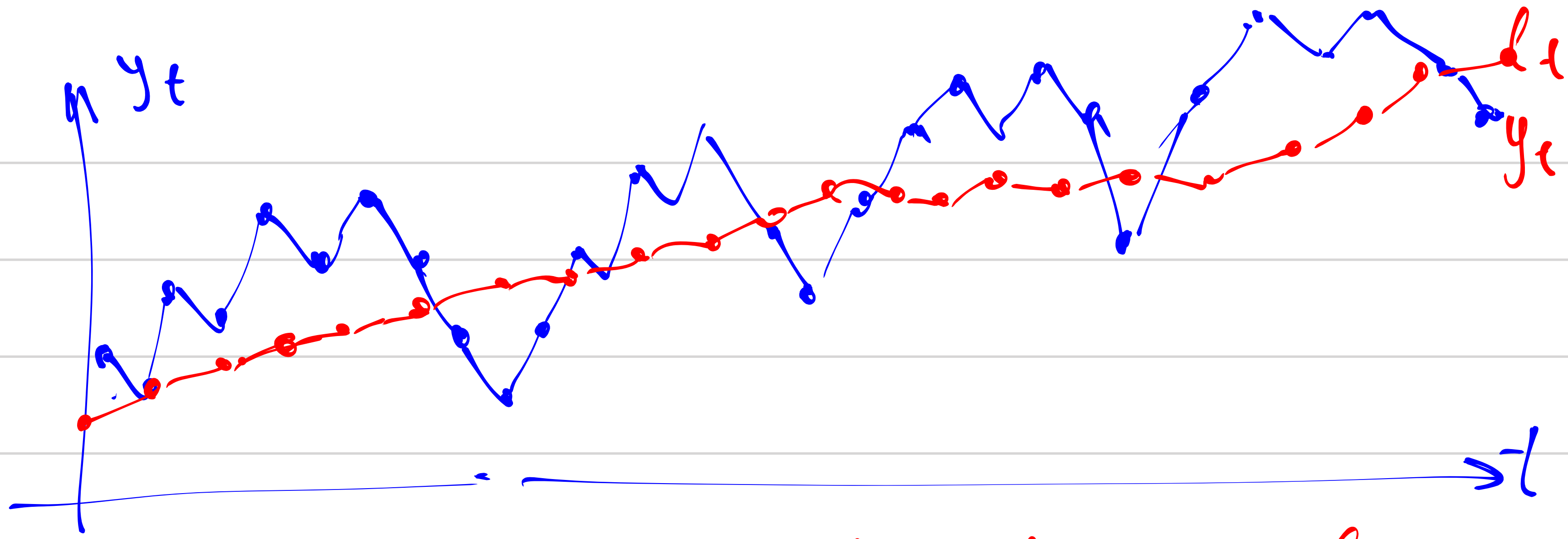
$u_2 = 0$

line 3

$y_3 = \ell_2 + b_2 + u_3$

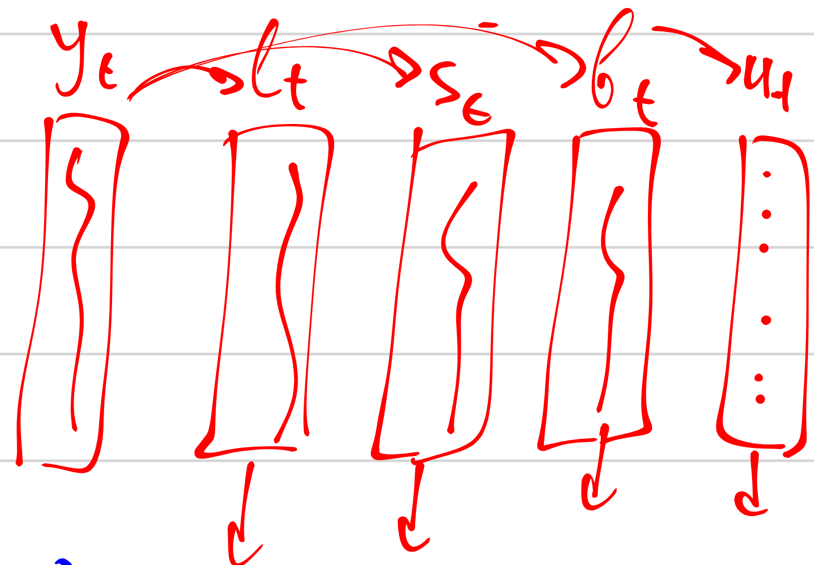
$13 = 11 + 1 + u_3$ $u_3 = 1$

$[\ell_{t-1}, u_t$ are indep$]$



phase I: estimate par-rs

phase II: extract components



phase III: forecasting. ETS(AAA)

Ex. bi annual obs-s [2 obs per year]. $m=2$.

$$\begin{cases} y_t = l_{t-1} + b_{t-1} + s_{t-2} + u_t \\ l_t = l_{t-1} + b_{t-1} + \alpha u_t \\ \boxed{b_t = b_{t-1} + \beta u_t} \\ s_t = s_{t-2} + \gamma u_t \\ u_t \sim N(0; \sigma^2) \text{ indep} \\ s_0 + s_{-1} = 0 \end{cases}$$

$b_1 \sim N(b_0; \beta^2 \cdot \sigma^2)$

$b_2 \sim N(b_0; \beta^2 \cdot \sigma^2 \cdot 2)$

$b_t \sim N(b_0; t \cdot \beta^2 \sigma^2)$

$b_{t-1} = l_t - l_{t-1} - \alpha u_t$

$T = 100$

t	y_t	b_t	l_t	s_t
...	...			
98	...	1	10	-2
99	...	2	11	3
100	...	1.5	13	-2.5
		b_t	l_t	s_t

$\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$ $\gamma = \frac{1}{2}$ $\sigma^2 = 16$

Q. 95% predictive interval for y_{101}

$y_{101} = l_{100} + b_{100} + s_{99} + u_{101} = 13 + 1.5 + 3 + u_{101}$
$= 17.5 + u_{101}$

$(y_{101} \mid \text{data}) \sim N(17.5; 16)$

PI for $(y_{101} \mid \text{data})$ $[17.5 - 1.96\sqrt{16}; 17.5 + 1.96\sqrt{16}]$

95% PI for y_{102}?

$y_{102} = l_{101} + b_{101} + s_{100} + u_{102} = (l_{100} + b_{100} + \frac{1}{2} u_{101}) + (b_{100} + \frac{1}{2} u_{101})$
$+ s_{100} + u_{102} =$

$= 13 + 1.5 + \frac{1}{2} u_{101} + 1.5 + \frac{1}{2} u_{101} + (-2.5) + u_{102} =$
$= 16 + u_{101} + u_{102}$

$$y_1, y_2, y_3 \dots y_T \sim \text{indep } N(7;16)$$

$$\bar{y} = \frac{y_1 + \dots + y_T}{T} \sim N\left(7; \frac{16}{T}\right)$$

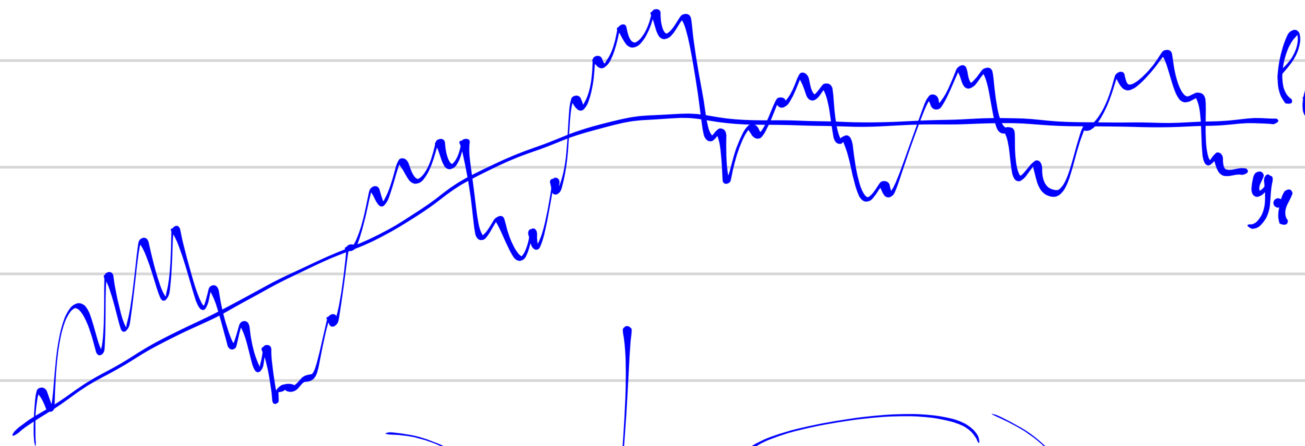
PI for $\bar{y}$ $\quad \left(7 - 1.96\sqrt{\frac{16}{T}}; 7 + 1.96\sqrt{\frac{16}{T}}\right)$

[digression, not ETS]

$$(y_{102} \mid \text{data}) \sim N(16; 32)$$

$$Var(u_{101} + u_{102}) = 16 + 16 = 32$$

PI for y_{102} $= [16 - 1.96\sqrt{32}; 16 + 1.96\sqrt{32}]$

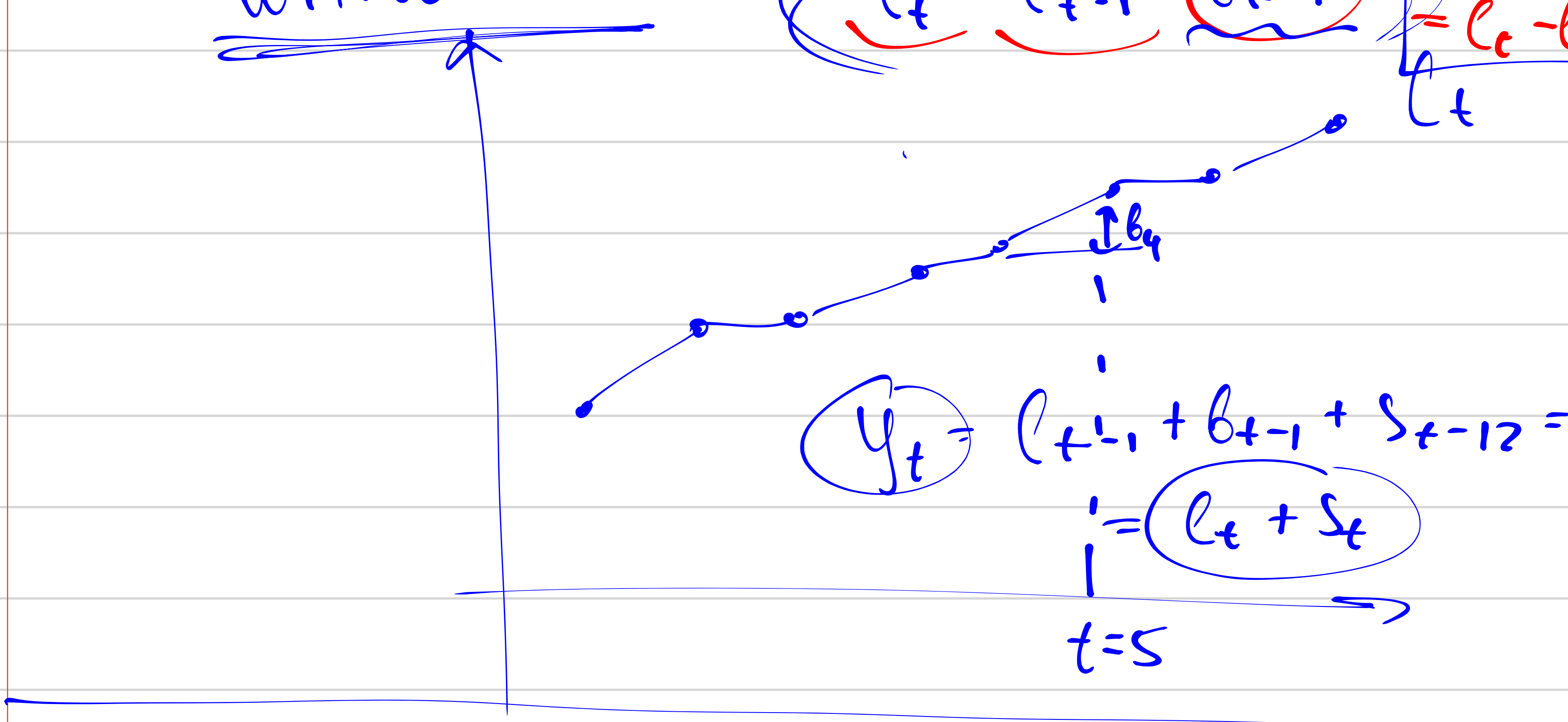


$b_t > 0$ | $b_t \approx 0$

[Break !!]

Q: diff between b_t and l_t?

without noise: $l_t = l_{t-1} + b_{t-1}$ $\quad$ $b_{t-1} = l_t - l_{t-1}$



$$y_t = l_{t-1} + b_{t-1} + s_{t-12} = l_t + s_t$$

ARIMA · model.

Q. I have never heard of ARMA ⊕ true
⊖ false

ARMA-model

def $(y_t) \sim ARMA(p,q)$ model wrt white noise (u_t) if

① $\boxed{y_t \text{ is } MA(\infty) \text{ process wrt } (u_t)}$

② $P_{AR}(L)\cdot(y_t-\mu) = P_{MA}(L)\cdot u_t$

$$P_{AR}(L) = 1 - \alpha_1 L - \alpha_2 L^2 - \dots - \alpha_p L^p$$
$$P_{MA}(L) = 1 + \beta_1 L + \beta_2 L^2 + \dots + \beta_q \cdot L^q$$

③ polynom $P_{AR}(L)$ and $P_{MA}(L)$ have no common roots.

theorem : property ① implies stationarity of ARMA(p,q)

$P_{AR}(L) = (1-0{,}3L)$ $P_{MA}(L) = (1-0{,}3L)\cdot(1+0{,}2L)$

$$(1-0{,}3L)\cdot(y_t-5) = (1-0{,}3L)(1+0{,}2L)\cdot u_t$$

$$y_t - 0{,}3y_{t-1} - 3.5 = u_t - 0{,}1u_{t-1} - 0.06u_{t-2}$$

(y_t) is not ARMA(1,2)

$\cancel{(1-0{,}3L)}$: $y_t - 5 = u_t + 0{,}2u_{t-1}$

$(y_t) \sim ARMA(0,1) = MA(1)$

ARMA(1,1) is stat-ry by def-n.

$$\boxed{(y_t) \sim ARIMA(p, \underline{\underline{d}}, q)} \quad \Leftarrow \quad \left\{ \begin{array}{l} \Delta^1 y_t \text{ is not ARMA} \\ \vdots \\ \Delta^{d-1} y_t \text{ is not ARMA} \\ (\Delta^d y_t) \sim ARMA(p,q) \end{array} \right.$$

$\Downarrow$

equation for y_t

$$P_{AR}(L) \cdot \underline{(1-L)^d \cdot y_t} = P_{MA}(L) \cdot u_t \qquad (\text{without } \mu)$$

$$(1-L)^d y_t \sim MA(\infty) \text{ wrt } (u_t)$$

P_{AR} and P_{MA} have no comm. roots.

$(y_t) \sim$ SARIMA = seasonality + ARIMA.

$$SARIMA \underbrace{(p,d,q)}_{\text{non-seas}} \underbrace{(P,D,Q)}_{\text{seasonal}}$$

1 $$P^S_{AR}(L^{12}) \cdot P_{AR}(L) \cdot (1-L^{12})^D \cdot (1-L)^d \cdot y_t =$$

$$= P^S_{MA}(L^{12}) \cdot P_{MA}(L) \cdot u_t$$

2 $$(1-L^{12})^D \cdot (1-L)^d \cdot y_t \sim MA(\infty) \text{ wrt } (u_t)$$

3 $P^S_{AR}(L^{12}) \cdot P_{AR}(L)$ and $P^S_{MA}(L^{12}) \cdot P_{MA}(L)$ have no comm. roots.

$$P^S_{AR}(L^{12}) = 1 - \alpha^S_1 \cdot L^{12} - \alpha^S_2 \cdot (L^{12})^2 \ldots\ldots - \alpha^S_P \cdot (L^{12})^P$$

$$P^S_{MA}(L^{12}) = 1 + \beta^S_1 \cdot L^{12} + \ldots\ldots + \beta^S_Q \cdot (L^{12})^Q$$

~30 models: ETS → - estimation
- [decomposition into comp-ts.]
- forecasting.

∞ models: ARIMA → estimation
forecasting

How to select the best model?

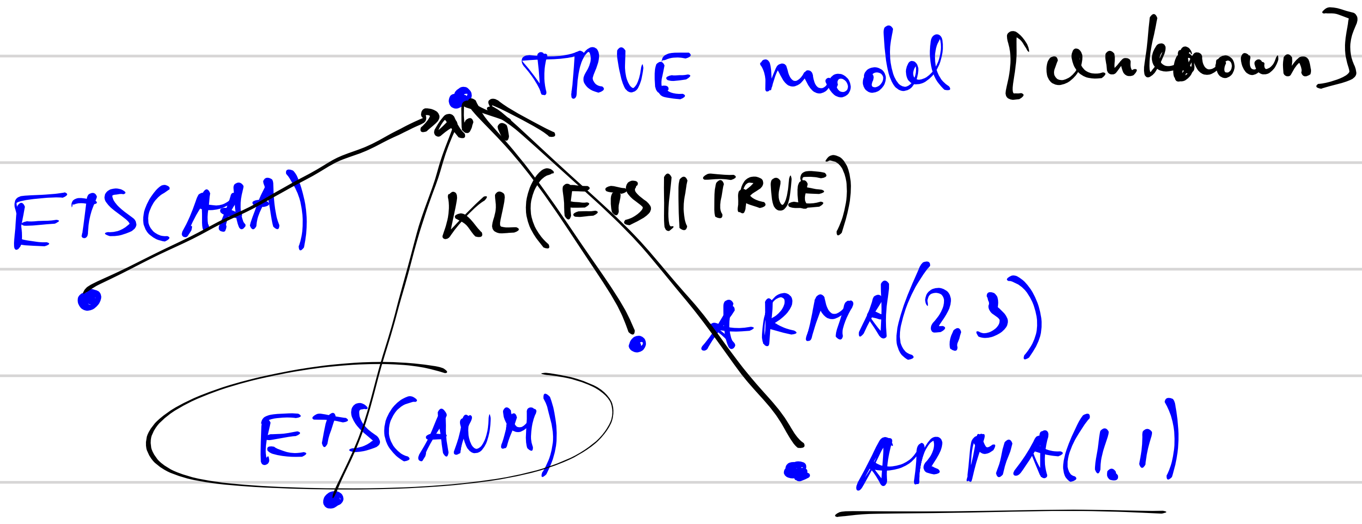
"All models are wrong but some are useful"

* Idea [takes a lot of time]
→ check the forecast quality
MAE, MSE

$$y_1 \dots y_T \mid y_{T+1} \dots \dots y_{T+h}$$

$$MAE = \frac{\sum_{t=T+1}^{T+h} |y_t - \hat{y}_{t|t-1}|}{h}$$

* AIC [a little bit of theory.]
fast



$$\frac{AIC_{mod\,A} - AIC_{mod\,B}}{2} \simeq KL(mod\,A \,||\, TRUE) - KL(mod\,B \,||\, TRUE)$$

(!) AIC can compare only the models with the same number of obs-s.

joint prob-ty density for

$y_t \sim \underline{ARMA(1,1)}$ ← model for $y_1 \ldots y_T$

$y_t \sim ARIMA(1,1,0)$ ← model for $\Delta y_2, \Delta y_3, \ldots \Delta y_T$

$\Delta y_t \sim ARMA(1,0)$

joint prob. density

! you can't compare ARMA(1,1) and ARIMA(1,1,0) with AIC.

$\underline{ARMA(3,1)}$
$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{pmatrix} ; \begin{pmatrix} \gamma_0 & \gamma_1 & & \\ \gamma_1 & \gamma_0 & \gamma_1 & \\ & \ddots & \ddots & \gamma_1 \\ & & \gamma_1 & \gamma_0 \end{pmatrix} \right)$$

$y_t \sim$ wnoise normal
$$\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \begin{pmatrix} \sigma^2 & 0 & 0 & 0 \\ 0 & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2 & 0 \\ 0 & 0 & 0 & \sigma^2 \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{pmatrix} \right)$$

normal

$y_t \sim AR(1)$
$\boxed{\mu = 0}$
$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; \begin{pmatrix} \frac{\sigma^2}{1-\beta^2} & \cdots \\ & \ddots \\ & & \ddots \end{pmatrix} \right)$$

$y_t = \beta \cdot y_{t-1} + u_t$

$Var(y_t) = \beta^2 \cdot Var(y_{t-1}) + Var(u_t) + 0$

$$\boxed{\max_{\sigma^2, \beta} f(y_1 \ldots y_T)}$$

$$Var(y_t) = \frac{\sigma^2}{1-\beta^2}$$

fast approximation.

"y" "x"

$$\begin{bmatrix} y_2 \\ \vdots \\ y_T \end{bmatrix} \quad \begin{bmatrix} y_1 \\ \vdots \\ y_{T-1} \end{bmatrix}$$

„auto ETS" Step 1. Estimate ≈ 30 ETS models
Step 2 chooses the best one by AIC

„auto ARIMA" = Hyndman-Khandakar

Step 1 Choose d { KPSS test is used (or ADF)
Step 2 Choose D { STL decomposition + „seasonality intensity"

Step 3 Estimate many models with fixed d, fixed D
Step 4. Choose the best by AIC.