Short rules: 120 minutes, A4 cheat sheet and calculator are allowed.
Copy and sign this short pledge:

«I pledge on my honor that I will not give or receive any unauthorized assistance on this exam».

And now you may solve the problems!

1. (10 points) Ded Moroz would like to receive K roubles at time $T = 2$ if $S_2 < 0.5S_1$ and zero otherwise.

   Assume the framework of Black and Scholes model, S_t is the share price, r is the risk free rate, σ is the volatility.

   How much Ded Moroz should pay now at $t = 0$?

2. (10 points) Simplify as much as possible the integral

$$\int_0^t \exp(-W_u - u/2) dW_u.$$

3. (10 points) The process Y_t is defined by

$$dY_t = W_t^2 dt + W_t dW_t,\ Y_0 = 0.$$

   (a) (6 points) Find $\mathbb{E}(Y_t)$, $\mathbb{E}(Y_t W_t)$, $\mathbb{E}(Y_t W_t^2)$.

   (b) (4 points) Find $\mathbb{V}\text{ar}(Y_t)$.

4. (10 points) Find at least one solution of the stochastic differential equation

$$dR_t = 4R_t dt + 7dW_t, \quad R_0 = 1.$$

   You are free to use the following steps:

   (a) Solve deterministic equation $dQ_t = 4Q_t dt$.

   (b) Now you need to «remove» this deterministic solution Q_t from R_t. To accomplish this goal represent R_t as $R_t = Q_t B_t$ and find a very simple equation for B_t.

   (c) Solve the equation for B_t.

   (d) Finalize your solution.

5. (10 points) The variables X and Y are independent and are exponentially distributed with mean 1. Let $L = \min\{X, Y\}$ and $R = \max\{X, Y\}$.

   Find $\mathbb{E}(L \mid R)$ and $\mathbb{E}(R \mid L)$.

Be brave!! One more problems is waiting to kill you!! The final boss of this game!!

6. (20 points) Consider a process $S_t = Z_1 + Z_2 + \ldots + Z_t$ with $S_0 = 0$. Increments Z_t are identically distributed and take values (-1) and 1 with equal probabilities.

   Let τ be the first moment of time when $|S_t| = 100$. Let's denote by Z the indicator that $S_\tau = 100$.

   (a) Find $f(\lambda)$ such that $M_t = f(\lambda)^t \exp(\lambda S_t)$ is a martingale.

   (b) Using Doob's optional stopping time theorem obtain an equation for $\mathbb{E}(Zf(\lambda)^\tau)$ and $\mathbb{E}((1-Z)f(\lambda)^\tau)$.

   (c) Solve for these two expected values.

   (d) Find $\mathbb{E}(f(\lambda)^\tau)$.

   (e) Find $h(\alpha) = \mathbb{E}(\exp(\alpha\tau))$ for arbitrary α.

   (f) Find $h'(0)$, $h''(0)$ and hence $\mathbb{E}(\tau^2)$.

   Hint: you may not check technical conditions of Doob's optional stopping time theorem.