Short rules: 120 minutes, A4 cheat sheet and calculator are allowed.
Copy and sign this short pledge:

«I pledge on my honor that I will not give or receive any unauthorized assistance on this exam».

And now you may solve the problems!

1. (10 points) Ded Moroz would like to receive S_2 roubles at time $T = 2$ if $S_2 < 0.5S_1$ and zero otherwise.

   Assume the framework of Black and Scholes model, S_t is the share price, r is the risk free rate, σ is the volatility.

   How much Ded Moroz should pay now at $t = 0$?

2. (10 points) Find the integral
$$\int_0^t (W_u^2 - u) dW_u.$$

3. (10 points) The process Y_t is defined by
$$dY_t = W_t dt + W_t dW_t,\ Y_0 = 0.$$
   Find $\mathbb{E}(Y_t)$ and $\mathbb{V}\mathrm{ar}(Y_t)$.

4. (10 points) Consider the stochastic differential equation
$$dR_t = h(R_t) dW_t, \quad R_0 = 1.$$
   Let's denote by $f(t)$ the integral $f(t) = \int_0^t 1/h(u) du$.

   Find the stochastic differential equation for $f(R_t)$ in terms of $h'(R_t)$.

5. (10 points) The variables X, Y and Z are independent and exponentially distributed with mean 1. Let $L = \min\{X, Y, Z\}$ and $R = \max\{X, Y, Z\}$.

   Find $\mathbb{E}(L \mid R)$ and $\mathbb{E}(R \mid L)$.

Be brave!! One more problems is waiting to kill you!! The final boss of this game!!

6. (20 points) Consider a process $S_t = Z_1 + Z_2 + \ldots + Z_t$ with $S_0 = 0$. Increments Z_t are identically distributed and take values (-1), 0 and 1 with equal probabilities.

   Let τ be the first moment of time when $|S_t| = 100$.

   Find $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$.

   You are free to use or not to use the following guiding steps:

   (a) Find $f = f(\lambda)$ such that $M_t = f^t \exp(\lambda S_t)$ is a martingale.

   (b) Using Doob's optional stopping time theorem obtain an equation for $\mathbb{E}(f^\tau)$.

   (c) Find $\mathbb{E}(f^\tau)$.

   (d) Find $h(\alpha) = \mathbb{E}(\exp(\alpha\tau))$ for arbitrary α.

   (e) Find $h'(0)$, $h''(0)$ and hence $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$.

   Hint: you may not check technical conditions of Doob's optional stopping time theorem.