1 HA

Deadline: 18 November 2021, 21:00

1. Some questions about σ-algebras.
   (a) You observe the result of 10 independent coin tosses. How many elements the σ-algebra of your information contains?
   (b) Prove that a finite σ-algebra can contain only 2^k elements.
   (c) Is union of two σ-algebras always a σ-algebra? Prove your statement.
   (d) Is intersection of two σ-algebras always a σ-algebra? Prove your statement.
2. Prove the following statement or provide a counter-example. For any two σ-algebras $\mathcal{F}$ and $\mathcal{H}$ and a random variable Y
$$\mathrm{E}(\mathrm{E}(Y|\mathcal{F})|\mathcal{H}) = \mathrm{E}(Y|\mathcal{F}\cap\mathcal{H})$$
3. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y.
   (a) Find $\mathbb{P}(Y = y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$;
   (b) Find $\mathrm{E}(X|Y)$.
4. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now.
   Find $\mathbb{P}(Y = y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$.
5. Random variables X and Y have joint normal distribution with zero means and covariance matrix
$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$
   (a) Find $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(XY|X)$ and $\mathrm{Var}(XY|X)$.
   (b) Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2021|X)$.
6. Starting from the properties of conditional expected value prove the following properties of variance:
   (a) $\mathrm{Var}(h(X)Y + g(X)|X) = h^2(X)\,\mathrm{Var}(Y|X)$;
   (b) Reincarnation of Pythagoras: $\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}(X \mid \mathcal{F})) + \mathrm{E}(\mathrm{Var}(X \mid \mathcal{F}))$;
7. Have a look in the past exams collection. How many pages does it contain?

2 HAHA

Deadline: 29 December 2021, 21:00

1. Assume all expected values are finite, (X_t) and (Y_t) are martingales with respect to the filtration $(\mathcal{F}_t)$. Prove that the following processes are martingales or provide a counter-example. Assume that the expected value $\mathrm{E}(X_t \cdot Y_t)$ is finite for all t.

   (a) $M_t = X_t + 2Y_t$;

   (b) $K_t = X_t \cdot Y_t$.

2. The random variables Z_1, Z_2, ... are independent and identically distributed with $\mathbb{P}(Z_n = 1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.

   (a) For which value of p the process 2^{S_n} will be a martingale?

   (b) Let $p = 0.3$. If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale.

3. Assume that τ_1 and τ_2 are stopping time with respect to the filtration $(\mathcal{F}_t)$ with $t \in \{0, 1, 2, \ldots\}$. Check whether the following random variables are stopping times.

   (a) $\tau_3 = \min\{\tau_1, \tau_2\}$;

   (b) $\tau_4 = \max\{\tau_1, \tau_2\}$;

   (c) $\tau_5 = \tau_1 + 2\tau_2$;

   (d) $\tau_6 = \tau_1 - 2$.

4. Consider a random variable Z with finite expected value and filtration $(\mathcal{F}_t)$. Check whether $M_t = \mathrm{E}(Z \mid \mathcal{F}_t)$ is a martingale.

5. (*) Anna and Boris throw a coin infinite number of times. Anna wins if the sequence HTHH appears first, Boris wins if the sequence TTHH appears first. The coin is biased with 0.4 probability of head.

   (a) What is the expected number of throws to obtain HTHH?

   (b) What is the probability that Anna will win?

   (c) What is the expected number of throws to obtain HTHH or TTHH?

   The way to solve this problem with martingales is described by Shuo-Yen Robert Li in

   `https://projecteuclid.org/euclid.aop/1176994578`.

3 HAHAHA

Deadline: 13 December 2021, 21:00

1. The processes (X_t) and (Y_t) are independent Wiener processes with respect to filtration $(\mathcal{F}_t)$. The process $Z_t = aX_t + bY_t$ is also a Wiener process.
   (a) For which values of constants a and b is it possible?
   (b) Find correlation $\text{Corr}(Z_t, X_t)$.
   (c) Find $\text{E}(Z_3|X_2)$ and $\text{Var}(Z_3|X_2)$.
   (d) Find $\text{E}(Z_3|\mathcal{F}_2)$ and $\text{Var}(Z_3|\mathcal{F}_2)$.

2. The process $C_t = W_t^3 + aW_t^2 + bW_t + c + d \cdot t \cdot W_t$ is a martingale.
   (a) For which values of constants a, b, c and d is it possible?
   (b) Find covariance $\text{Cov}(C_t, \int W_u^2 dW_u)$.

3. Consider the framework of Black and Scholes model: S_t is the share price. Derive the current price of two European type assets, X_0 and Y_0.

   Future payoffs are given by:
   (a) $X_T = (S_T - K)^3$ where T and K are fixed in the contract.
   (b) $Y_T = S_T^{-2}$ where T is fixed in the contract.

4. Let $Y_t = W_t + 4t$. The moment τ is the first moment when Wiener process hits 10.
   (a) Let α be a constant. Find the function $f(t)$ such that $M_t = f(t)\exp(\alpha Y_t)$ is a martingale.
   (b) Using Doob's theorem find $\text{E}(\exp(-s\tau))$ for arbitrary constant s.

5. Let's consider the process $X_t = \int_0^t u^2 dW_u$. Prove that this process can be represented as a time changed Wiener process. That means that there is a deterministic time-scaling $t(s)$ such that $Y_s = X_{t(s)}$ is a Wiener process with respect to some filtration.

6. Solve the stochatic differential equation

   $$dY_t = -Y_t dt + dW_t,\ Y_0 = 1$$

   If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitely. It's ok.

7. Solve the stochatic differential equation

   $$dY_t = Y_t dt + (t^3 + 4Y_t)dW_t,\ Y_0 = 1$$

   If you are have no clues you may try to represent the process as $Y_t = A_t B_t$, where A_t is the solution of the equation $dA_t = A_t dt + 4A_t dW_t$.