

1 HA

Deadline: 18 November 2021, 21:00

1. Some questions about σ -algebras.

- (a) You observe the result of 10 independent coin tosses. How many elements the σ -algebra of your information contains?
- (b) Prove that a finite σ -algebra can contain only 2^k elements.
- (c) Is union of two σ -algebras always a σ -algebra? Prove your statement.
- (d) Is intersection of two σ -algebras always a σ -algebra? Prove your statement.

2. Prove the following statement or provide a counter-example. For any two σ -algebras \mathcal{F} and \mathcal{H} and a random variable Y

$$\mathbb{E}(\mathbb{E}(Y|\mathcal{F})|\mathcal{H}) = \mathbb{E}(Y|\mathcal{F} \cap \mathcal{H})$$

3. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y .

- (a) Find $\mathbb{P}(Y = y|X)$, $\mathbb{E}(Y|X)$, $\text{Var}(Y|X)$;
- (b) Find $\mathbb{E}(X|Y)$.

4. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now.

Find $\mathbb{P}(Y = y|X)$, $\mathbb{E}(Y|X)$, $\text{Var}(Y|X)$, $\mathbb{E}(Y)$, $\text{Var}(Y)$.

5. Random variables X and Y have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

- (a) Find $\mathbb{E}(Y|X)$, $\text{Var}(Y|X)$, $\mathbb{E}(XY|X)$ and $\text{Var}(XY|X)$.
- (b) Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2019|X)$.

6. Starting from the properties of conditional expected value prove the following properties

- (a) $\text{Var}(h(X)Y + g(X)|X) = h^2(X) \text{Var}(Y|X)$;
- (b) Reincarnation of Pythagoras: $\text{Var}(X) = \text{Var}(\mathbb{E}(X | \mathcal{F})) + \mathbb{E}(\text{Var}(X | \mathcal{F}))$;

7. Have a look in the past exams collection. How many pages does it contain?

2 HAHHA

Deadline: 29 December 2020, 21:00

1. Assume all expected values are finite, (X_t) and (Y_t) are martingales with respect to the filtration (\mathcal{F}_t) . Prove that the following processes are martingales or provide a counter-example.
 - (a) $M_t = X_t + Y_t$;
 - (b) $K_t = X_t \cdot Y_t$.
2. The random variables Z_1, Z_2, \dots are independent and identically distributed with $\mathbb{P}(Z_n = 1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$. Consider the cumulative sum process, $S_n = Z_1 + \dots + Z_n$ with $S_0 = 0$.
 - (a) For which value of p the process 2^{S_n} will be a martingale?
 - (b) Let $p = 0.4$. If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale.
3. Assume that τ_1 and τ_2 are stopping time with respect to the filtration (\mathcal{F}_t) with $t \in \{0, 1, 2, \dots\}$. Check whether the following random variables are stopping times.
 - (a) $\tau_3 = \min\{\tau_1, \tau_2\}$;
 - (b) $\tau_4 = \max\{\tau_1, \tau_2\}$;
 - (c) $\tau_5 = \tau_1 + \tau_2$;
 - (d) $\tau_6 = \tau_1 - 1$.
4. Consider any random variable Z and filtration (\mathcal{F}_t) . Check whether $M_t = E(Z \mid \mathcal{F}_t)$ is a martingale.
5. (*) Anna and Boris throw a coin infinite number of times. Anna wins if the sequence HTHH appears first, Boris wins if the sequence TTHH appears first. The coin is biased with 0.4 probability of head.
 - (a) What is the expected number of throws to obtain HTHH?
 - (b) What is the probability that Anna will win?
 - (c) What is the expected number of throws to obtain HTHH or TTHH?

The way to solve this problem with martingales is described by Shuo-Yen Robert Li in

<https://projecteuclid.org/euclid.aop/1176994578>.

3 HAHAAHA

Deadline: 13 December 2021, 21:00

- The processes (X_t) and (Y_t) are independent Wiener processes with respect to filtration (\mathcal{F}_t) . The process $Z_t = aX_t + bY_t$ is also a Wiener process.
 - For which values of constants a and b is it possible?
 - Find correlation $\text{Corr}(Z_t, X_t)$.
 - Find $E(Z_3|X_2)$ and $\text{Var}(Z_3|X_2)$.
 - Find $E(Z_3|\mathcal{F}_2)$ and $\text{Var}(Z_3|\mathcal{F}_2)$.
- The process $C_t = W_t^3 + aW_t^2 + bW_t + c + d \cdot t \cdot W_t$ is a martingale.
 - For which values of constants a, b, c and d is it possible?
 - Find covariance $\text{Cov}(C_t, \int W_u^2 dW_u)$.
- Consider the framework of Black and Scholes model: S_t is the share price. Derive the current price of two European type assets, X_0 and Y_0 .
Future payoffs are given by:
 - $X_T = (S_T - K)^2$ where T and K are fixed in the contract.
 - $Y_T = S_T^{-1}$ where T is fixed in the contract.
- Let $Y_t = W_t + 3t$. The moment τ is the first moment when Wiener process hits 10.
 - Let α be a constant. Find the function $f(t)$ such that $M_t = f(t) \exp(\alpha Y_t)$ is a martingale.
 - Using Doob's theorem find $E(\exp(-s\tau))$ for arbitrary constant s .
- Let's consider the process $X_t = \int_0^t u^2 dW_u$. Prove that this process can be represented as a time changed Wiener process. That means that there is a deterministic time-scaling $t(s)$ such that $Y_s = X_{t(s)}$ is a Wiener process with respect to some filtration.
- Solve the stochastic differential equation

$$dY_t = -Y_t dt + dW_t, Y_0 = 1$$

If you have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitly. It's ok.

- Solve the stochastic differential equation

$$dY_t = Y_t dt + (t^3 + 4Y_t) dW_t, Y_0 = 1$$

If you have no clues you may try to represent the process as $Y_t = A_t B_t$, where A_t is the solution of the equation $dA_t = A_t dt + 4A_t dW_t$.