European option pricing in BS model

$\circledast$ risk neutral - neutral prob-ty measure.

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in intro course

$P(w)$	0.2	0.3	0.5
$X(w)$	2	1	7
$Y(w)$	-1	0	5
$Q(w)$	0	0	1
$p^*(w)$	1/3	1/3	1/3

$E(X) =$

$E(Y) = -0.2 + 2.5 = 2.3$

$E_Q(Y) = 1.5 = 5$

$E_*(Y) = \frac{-1+5}{3} = \frac{4}{3}$

Binomial tree model

$t=0$ $\qquad\qquad$ $t=1$

$$\begin{tikzcd}
 & S_0 \cdot u = 66 = S_1, \quad B_0\cdot(1+r) = \underline{B_1} \\
S_0 = 60, \ \underline{B_0} \arrow[ur, "p_u = 0.1"] \arrow[dr, "p_d = 0.9"'] & \\
 & S_0 \cdot d = 48 = S_1, \quad B_0\cdot(1+r) = \underline{B_1}
\end{tikzcd}$$

S_t - share price

$u = 1.1$

$d = 0.8$

$B_0 \to B_0\cdot(1+r)$

$r = 0.05$

$X_1(u) = 10$

$X_1(d) = 20$

X_0 ?

$\boxed{B_0 = \frac{B_1}{1+r}}$

$S_1(u) = 66$

$S_1(d) = 48$

1. assets are inf-ly div-ble
2. no trans-on costs.
3. no arbitrage opportunity (you can't make money without risk).

Naive idea!

$$\frac{1}{1.05}\left(10\cdot 0.1+20\cdot 0.9\right)=18.09\ldots$$

maybe? $X_0 = E\left(\frac{X_1}{1+r}\right)$ wrong!

let's save this beautiful wrong idea!
let's invent p_u^*, p_d^* that $X_0 = E_*\left(\frac{X_1}{1+r}\right)$

create replication portfolio for X

at $t=0$: α shares + B_0 in bank.

up-state: $\alpha\cdot S_1(u) + B_0\cdot(1+r) = X_1(u)$
down-state: $\alpha\cdot S_1(d) + B_0(1+r) = X_1(d)$

Solve for α and B_0!

$$\alpha = \frac{X_1(u)-X_1(d)}{S_1(u)-S_1(d)} = \frac{10-20}{66-48} = \frac{-10}{18} = \frac{-5}{9}$$
$$= -0.555\ldots$$

from eq1: $B_0 = \frac{X_1(u) - \alpha\cdot S_1(u)}{1+r} =$

$$= \frac{1}{1+r}\left[X_1(u) - (X_1(u)-X_1(d))\cdot\frac{S_1(u)}{S_1(u)-S_1(d)}\right] =$$

$\alpha = \frac{X_1(u)-X_1(d)}{S_1(u)-S_1(d)}$

$$B_0 = \frac{1}{1+r}\left[X_1(u)\cdot\frac{-S_1(d)}{S_1(u)-S_1(d)} + X_1(d)\cdot\frac{S_1(u)}{S_1(u)-S_1(d)}\right]$$

$$= \frac{1}{1.05}\left(10\cdot\frac{-48}{18} + 20\cdot\frac{66}{18}\right) \approx 44.(4)$$

$-0.555\ldots$ shares $+\ 44.44\ldots$ $\to R_1(u) = 10$, $\to R_1(d) = 20$

$$\| $$

$$X_0 = -\frac{5}{9}\cdot 60 + \frac{400}{9} = \frac{100}{9} = 11.1111\ldots$$

$$X_0 = E_*\left(\frac{X_1}{1+r}\right)$$

$$(1+r)\cdot X_0 = X_1(u)\cdot p_u^* + X_1(d)\cdot p_d^*$$

Solve: p_u^*, p_d^* !

$$(1+r)\cdot(\alpha\cdot S_0 + B_0) = X_1(u)\,p_u^* + X_1(d)\,p_d^*$$

$$\alpha = \frac{X_1(u)-X_1(d)}{S_1(u)-S_1(d)} \qquad B_0 = \frac{1}{1+r}\left[X_1(u)\cdot\frac{-S_1(d)}{S_1(u)-S_1(d)} + X_1(d)\cdot\frac{S_1(u)}{S_1(u)-S_1(d)}\right]$$

$$\frac{1}{S_1(u)-S_1(d)}\Big[S_0(1+r)\cdot\big(X_1(u)-X_1(d)\big) + X_1(u)\cdot(-S_1(d)) + X_1(d)\cdot S_1(u)\Big] = X_1(u)\cdot p_u^* + X_1(d)\cdot p_d^*$$

$$\frac{1}{S_0 u - S_0 d}\cdot\Big[X_1^u\cdot\big(S_0(1+r) - S_0\cdot d\big) + X_1^d\cdot\big(S_0\cdot u - S_0(1+r)\big)\Big] = X_1^u\,p_u^* + X_1^d\,p_d^*$$

$$X_1^u\cdot\frac{1+r-d}{u-d} + X_1^d\cdot\frac{u-(1+r)}{u-d} \equiv X_1^u\cdot p_u^* + X_1^d\cdot p_d^*$$

$$p_u^* = \frac{1+r-d}{u-d} = \frac{1.05-0.8}{1.1-0.8} = \frac{25}{30} = \frac{5}{6}$$

$$p_d^* = \frac{u-(1+r)}{u-d} = \frac{1.1-1.05}{1.1-0.8} = \frac{5}{30} = \frac{1}{6}$$

$$X_0 = E_*\left(\frac{X_1}{1+r}\right)$$

$S_2 = 726$ $X_2(uu) =$ $X_2 = 100$

0.3

$S_1 = 660$

$S_2 = 528$ $X_2'(ud) = 100$

0.7

$X_1 = \frac{1}{1.05}\left(\frac{5}{6}\cdot 100 + \frac{1}{6}\cdot 100\right) = \frac{100}{1.05}$

0.5

$S_0 = 600$

$S_2 = 528$

$X_2 = 100$

0.2

$S_1 = 480$

$X_1 = \frac{1}{1.05}\cdot\left(\frac{5}{6}\cdot 100 + \frac{1}{6}\cdot 0\right)$

$t = 0$ 0.5

0.8

$S_2 = 384$

$X_2(dd) = 0$

$X_0 = \frac{1}{1.05}\left(\frac{5}{6}\cdot\frac{100}{1.05} + \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{1.05}\cdot 100\right)$

$t = 1$

$t = 2$

$u = 1.1$

$d = 0.8$

$r = 0.05$ (per period)

$p_u^* = \frac{1+r-d}{u-d} = \frac{5}{6}$

$p_d^* = \frac{u-(1+r)}{u-d} = \frac{1}{6}$

f@uest.

$$X_2 = \begin{cases} 100 & \text{if } S_2 > 520 \\ 0 & \text{if } S_2 \le 520 \end{cases}$$

X_0 ?

$$X_0 = E_*\left(\frac{X_2}{(1+r)^2}\right)$$

Th. [Girsanov]

If (W_t) is a Wiener process w.r.t. $P(\,)$ and $W_t^* = W_t + \alpha \cdot t$ then there is a new probability $P^*(\,)$ such that (W_t^*) is a Wiener process w.r.t. $P^*(\,)$.

Example.

(W_t) - Wiener process wrt $P(\,)$

$W_t^* = W_t + 2 \cdot t$

there is P^* such that (W_t^*) is a W.P. wrt P^*.

$P(W_1 > 0)$? $P(W_1^* > 0)$?

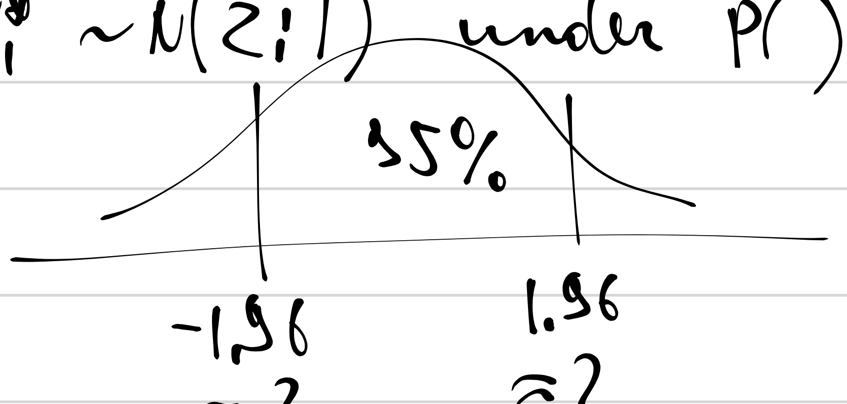
$P^*(W_1^* > 0)$? $P^*(W_1 > 0)$?

$W_1 \sim N(0;1)$ under $P(\,)$

$P(W_1 > 0) = \frac{1}{2}$

$P(W_1^* > 0) = P(W_1 + 2 > 0) = P(W_1 > -2) \approx$

$W_1^* \sim N(2;1)$ under $P(\,)$ ≈ 0.975



$W_1^* \sim N(0;1)$ under P^*

$W_1 = W_1^* - 2$

$W_1^* = W_1 + 2$

$P^*(W_1^* > 0) = \frac{1}{2}$

$P^*(W_1 > 0) = P^*(W_1^* - 2 > 0) = P^*(W_1^* > 2) \approx$

≈ 0.025

in full form $\qquad W_t^* = W_t + \alpha \cdot t$

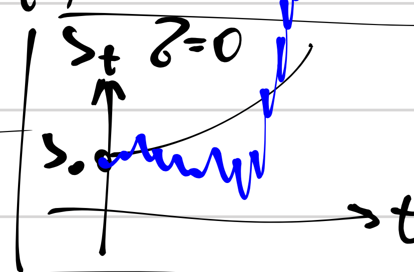
in short form $\qquad dW_t^* = f_t' \cdot dt + f_W' \cdot dW_t + \frac{1}{2} f_{WW}'' (dW_t)^2 =$

(Itô's lemma)

$= \alpha \cdot dt + 1 \cdot dW_t + 0$

(W_t) - Wiener process wrt to P.

Girsanov: if $W_0^* = 0$ and $\boxed{dW_t^* = \alpha \cdot dt + dW_t}$ then $\exists\, P^*$ such that (W_t^*) is a W.P. wrt P^*



$E(S_t) = S_0 \cdot \exp(\mu t)$

BS. $\qquad t \in [0; \infty)$

$$S_t = \boxed{S_0 \cdot \exp(\mu t)} \cdot \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

$$dS_t = \mu S_t \cdot dt + \sigma \cdot S_t \cdot dW_t$$

$$B_t = B_0 \cdot \exp(rt)$$

idea: let's invent such prob-s that for every $\boxed{\text{European type}}$ asset X

$$X_0 = E_*\left(\frac{X_T}{(1+r)^T}\right) \qquad \text{discr.}$$

$$X_0 = E_*\left(\frac{X_T}{\exp(rT)}\right) \qquad \text{cont.}$$

T : payoff time. $\qquad$ contract is signed at $t=0$.

$0 \qquad t \qquad T$

dream $\qquad X_t = E_*\left(\frac{X_T}{\exp(r(T-t))} \middle| \mathcal{F}_t\right)$

$$\underbrace{\exp(-rt) \cdot X_t} = E_*\left(\exp(-rT) \cdot X_T \mid \mathcal{F}_t\right)$$

discounted price! $\qquad M_t = E_*(M_T \mid \mathcal{F}_t) \qquad T \geq t$

this should work for every asset X.
even for our share.

$M_t = \exp(-rt)\cdot S_t$ should be a martingale under P^*.

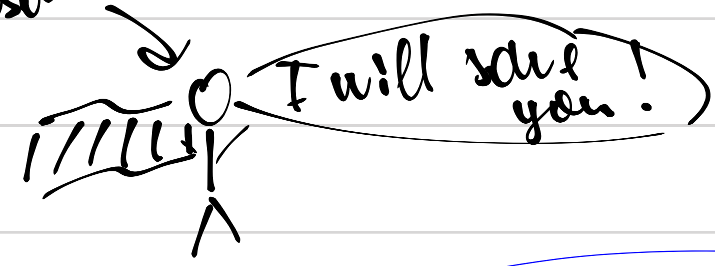
$$dM_t = f'_t\cdot dt + f'_S \cdot dS_t + \frac{1}{2} f''_{SS}(dS_t)^2 =$$

$$= -r\exp(-rt)\cdot S_t\, dt + \exp(-rt)\cdot dS_t + 0 =$$

$$= -r\exp(-rt)\cdot S_t\cdot dt + \exp(-rt)(\mu S_t dt + \sigma S_t dW_t) =$$

$$= \exp(-rt)\cdot S_t\left[(\mu - r)\cdot dt + \sigma\cdot dW_t\right] =$$

Brave Girsanov



$$= \exp(-rt)\cdot S_t\cdot\sigma\left[\frac{\mu-r}{\sigma}\cdot dt + dW_t\right] =$$

is not a martingale!

$$W_t^* = \frac{\mu - r}{\sigma}\cdot t + W_t$$

$$\sigma W_t^* = (\mu - r)\cdot t + \sigma W_t$$

$\exists P^*$ such that W_t^* is W.P. wrt P^*

$$= \exp(-rt)\cdot S_t\cdot\sigma\cdot dW_t^*$$ ← no dt term under P^*

Under P^*

$$X_0 = \exp(-rT)\cdot E_*(X_T)$$

$$X_t = \exp(-r(T-t))\cdot E_*(X_T \mid \mathcal{F}_t)$$

WP under P

$$S_t = \boxed{S_0\cdot\exp(\mu t)\cdot\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)} =$$

$\sigma W_t^* - (\mu - r)t$

$$= S_0\cdot\exp(\mu t)\cdot\exp\left(\sigma W_t^* - \mu t + rt - \frac{\sigma^2}{2}t\right)$$

$$= S_0\cdot\exp(rt)\cdot\exp\left(\sigma W_t^* - \frac{\sigma^2}{2}t\right)$$ WP under P^*

Under P^* $\quad X_0 = \exp(-rT)\cdot E_*(X_T)$

$$X_t = \exp(-r(T-t))\cdot E_*(X_T \mid \mathcal{F}_t)$$

WP under P

$$S_t = \boxed{S_0 \cdot \exp(\mu t)\,\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)} =$$

$$= S_0 \cdot \exp(\mu t)\cdot \exp\left(\sigma W_t^* - \mu t + rt - \frac{\sigma^2}{2}t\right) \qquad \sigma W_t^* - (\mu - r)t$$

$$= \boxed{S_0 \cdot \exp(rt)\cdot \exp\left(\sigma W_t^* - \frac{\sigma^2}{2}t\right)}$$

WP under P^*

$$dS_t = \mu S_t \cdot dt + \sigma S_t \cdot dW_t$$

$$dS_t = r S_t\, dt + \sigma S_t \cdot dW_t^*$$

BS.

$S_0 = 600 \qquad T = 2 \qquad r = 0.05$

$\sigma = 0.1$

$\mu = 0.02$

$$X_T = \begin{cases} 100 & \text{if } S_T \geq 550 \\ 0 & \text{if } S_T < 550 \end{cases}$$

X_0 ?

$$X_0 = \exp(-r\cdot 2)\cdot E_*(X_2) =$$

$$= \exp(-r\cdot 2)\cdot \left[P^*(S_T \geq 550)\cdot 100 + P^*(S_T < 550)\cdot 0\right] =$$

$$= \exp(-r\cdot 2)\cdot 100 \cdot P^*(S_2 \geq 550)$$

$$S_2 = S_0 \cdot \exp(r\cdot 2)\cdot \exp\left(\sigma\cdot W_2^* - \frac{\sigma^2}{2}\cdot 2\right)$$

$$P^*\left(600 \cdot \exp(2r)\cdot \exp(0.1\cdot W_2^* - 0.1^2) \geq 550\right) =$$

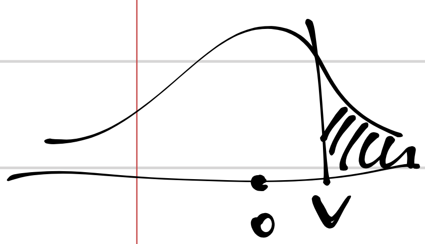
$$= P^*\left(\ln 600 + 2\cdot r + 0.1\cdot W_2^* - 0.01 \geq \ln 550\right) =$$

$2\cdot r$

$$= P^*\left(\ln 600 + 2\cdot r + 0.1\cdot W_2^* - 0.01 \geq \ln 550\right) =$$

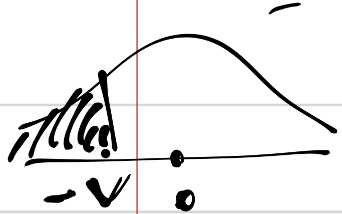
$$= P^*\left(W_2^* \geq \frac{\ln 550 - \ln 600 + 0.01 - 0.1}{0.1}\right) =$$

$$= P^*\left(W_2^* \geq (\ln 55 - \ln 60 - 0.09)\cdot 10\right) =$$



$W_2^* \sim N(0;2)$ under P^*

$$= P^*\left(N(0;1) \geq \frac{(\ln 55 - \ln 60)\cdot 10 - 0.9}{\sqrt{2}}\right) =$$



$$= F\left(\frac{10\cdot \ln 60 - 10 \ln 55 + 0.9}{\sqrt{2}}\right) \approx 0.89$$

$$X_0 \approx 100\cdot \exp(-0.1)\cdot 0.89 = 80.95$$

$$\boxed{\begin{array}{l} W_t^* \text{ is a Wiener Process under } P^* \\ \text{assumpt: } \quad W_t^* - W_s^* \sim N(0; t-s) \\ \qquad\qquad\quad W_t^* \sim N(0; t) \end{array}}$$

$S_0 = 600 \quad r = 0.05 \quad \sigma = 0.1 \quad \mu = 0.02$

$T = 2$

$X_T = S_T^2 \qquad X_0 \, ?$

$0 \qquad\qquad 2$

$$X_0 = \exp(-r \cdot 2) \cdot E_*(S_2^2) =$$

$$= \exp(-r \cdot 2) \cdot E_*\left(S_0^2 \cdot \exp(4r) \cdot \exp(2\sigma W_2^* - 2\sigma^2)\right) =$$

$$S_2 = S_0 \cdot \exp(r \cdot 2) \cdot \exp\left(\sigma \cdot W_2^* - \frac{\sigma^2}{2} \cdot 2\right)$$

$$= S_0^2 \cdot \exp(2r) \cdot E_*\left(\exp(2 \cdot \sigma W_2^* - 2\sigma^2)\right) =$$

4 ♛ − 2 ♛ = 2 ♛

Samurai way

$W_2^* \sim N(0; 2)$ under P^*

$$\int_{-\infty}^{\infty} \exp(2\sigma w - 2\sigma^2) \cdot pdf(w) \cdot dw$$

Ninja way

$M_t = \exp\left(\sigma W_t^* - \frac{\sigma^2}{2} t\right)$ is Mart. under P^*

$$E_*(M_t) = E_*(M_0) = E_*(1) = 1$$

$$E_*\left(\exp(\sigma W_t^*) \cdot \exp\left(-\frac{\sigma^2}{2} t\right)\right) = 1$$

$$\boxed{E_*\left(\exp(\sigma W_t^*)\right) = \exp\left(\frac{\sigma^2}{2} t\right)} \quad \forall t, \sigma$$

$$E_*\left(\exp(2\sigma \cdot W_2^*)\right) = \exp\left(\frac{(2\sigma)^2}{2} \cdot 2\right) = \exp(4\sigma^2)$$

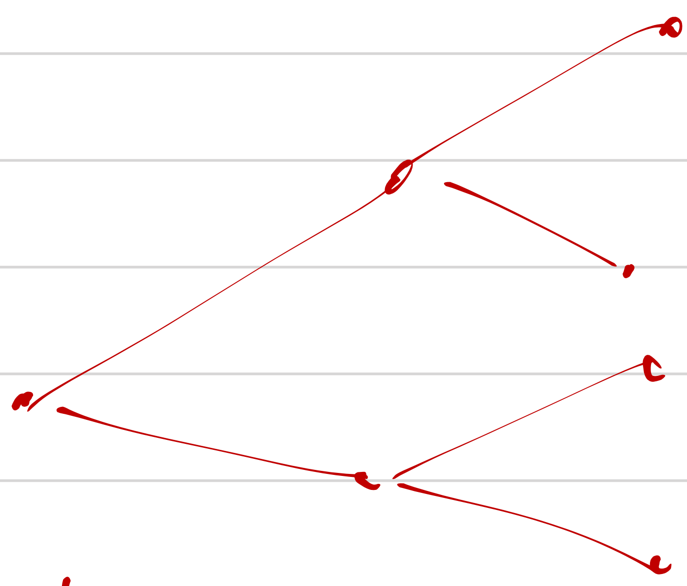
$$= S_0^2 \cdot \exp(2r) \cdot \underbrace{E_*\left(\exp\left(\underline{2 \cdot \sigma W_2^*} - 2\sigma^2\right)\right)} =$$

$$= S_0^2 \exp(2r) \cdot E_*\left(\exp(2\sigma W_2^*)\right) \cdot \exp(-2\sigma^2) =$$

$$= S_0^2 \exp(2r) \cdot \exp(4\sigma^2) \cdot \exp(-2\sigma^2) =$$

$$= S_0^2 \cdot \exp(2r) \cdot \exp(2\sigma^2)$$

X_0



$$B_t = (1+r)^t \cdot B_0$$

$$B_1 = (1+r) \cdot B_0$$

discr

$$B_2 = (1+r)^2 \cdot B_0$$

cont.

$$B_t = \exp(rt) \cdot B_0$$

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BS model.

$$X_T = \begin{cases} 1 \text{ share} & \text{if } S_T > S_{T/2} \\ 0 & \text{otherwise.} \end{cases}$$

X_0 ?

$$X_0 = \exp(-rT) \cdot E_*(X_T) \qquad !!$$

$$X_T = S_T \cdot I(S_T > S_{T/2})$$

$$E_*(S_T \cdot I(S_T > S_{T/2}))$$

$$S_T = S_0 \cdot \exp(rT) \cdot \exp\left(\sigma W_T^* - \frac{\sigma^2}{2} T\right)$$

$$S_{T/2} = S_0 \exp(rT/2) \cdot \exp\left(\sigma W_{T/2}^* - \frac{\sigma^2}{2} T/2\right)$$

$$A = \{S_T > S_{T/2}\} =$$

$$= \left\{ \exp(rT) \cdot \exp\left(\sigma W_T^* - \frac{\sigma^2}{2} T\right) > \exp\left(\frac{rT}{2}\right) \cdot \exp(\ldots) \right\}$$

$$= \left\{ rt + \sigma W_T^* - \frac{\sigma^2}{2} T > \frac{rT}{2} + \sigma W_{T/2}^* - \frac{\sigma^2}{2} T/2 \right\} =$$

$$= \left\{ W_T^* - W_{T/2}^* > \frac{\frac{\sigma^2 T}{4} - \frac{rt}{2}}{\sigma} \right\} = \left\{ W_T^* - W_{T/2}^* > \frac{\sigma^2 T - 2rT}{4\sigma} \right\}$$

$$E_*(S_T \cdot I_A) = E_*\left(S_0 \cdot \exp(rT) \cdot \exp\left(\sigma W_T^* - \frac{\sigma^2}{2} T\right) \cdot I_A\right)$$

$$= S_0 \cdot \exp(rT) \cdot \exp\left(-\frac{\sigma^2}{2} T\right) \cdot E_*\left(\exp\left(\sigma(W_T^* - W_{T/2}^*) + \sigma W_{T/2}^*\right) \cdot I_A\right);$$

future = past value + increment

$$E_*\left(\exp\left(\sigma(W_T^* - W_{T/2}^*) + \sigma W_{T/2}^*\right)\cdot I_A\right) =$$

$\Delta = W_T^* - W_{T/2}^*$

$$A = \left\{ W_T^* - W_{T/2}^* > \frac{\sigma^2 T - 2rT}{4\sigma} \right\}$$

$\Delta \overset{P^*}{\sim} N(0; T - \frac{T}{2})$

$$= \underline{E_*\left(\exp(\sigma W_{T/2}^*)\right)} \underline{E_*\left(\exp(\sigma\Delta)\cdot I\left(\Delta > \frac{\sigma^2 T - 2rT}{4\sigma}\right)\right)}$$

if X and Y are uncorrelated then

$$E(X\cdot Y) = E(X)\cdot E(Y)$$

$$\to \exp\left(\frac{\sigma^2}{2}\cdot\frac{T}{2}\right)$$

if $X \sim N(0;1)$ $\qquad E(X^{2n}) = E(X \cdot X^{2n-1}) = E((2n-1)X^{2n-2}) =$

$$= \ldots = \underline{(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot\ldots\cdot 3\cdot 1}$$

$$E(X^8) = E(X\cdot X^7) = E(7X^6) = 7E(X^6) =$$

$$= 7\cdot E(X\cdot X^5) = 7\cdot E(5X^4) =$$

$$= 7\cdot 5\cdot E(X^4) = \ldots$$

$\frac{d}{dx}$

if $W_t \sim WP$.

$\triangle$ $\qquad E(W_t^{2n}) = E\left(\left(\sqrt{t}\cdot\frac{W_t}{\sqrt{t}}\right)^{2n}\right) = t^n\cdot E\left(\left(\frac{W_t}{\sqrt{t}}\right)^{2n}\right) =$

$$= t^n\cdot(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot\ldots\cdot 3\cdot 1$$

$$E(\exp(\sigma W_t)) = \exp\left(\frac{\sigma^2 t}{2}\right) \quad \triangle$$

$$E_*\left(\exp(\sigma\cdot\Delta)\cdot I\left(\Delta > \frac{\sigma^2 T - 2rT}{4\sigma}\right)\right) =$$

$$\boxed{\exp\left(t\frac{\sigma^2}{2}\right)\cdot F\left(\sigma\sqrt{t} - \frac{\alpha}{\sqrt{t}}\right)}$$

$$\Delta \sim N(0; \tfrac{T}{2}) \sim W_{T/2}^*$$

$$= \exp\left(\frac{T}{2}\cdot\frac{\sigma^2}{2}\right)\cdot F\left(\sigma\cdot\sqrt{T/2} - \frac{\sigma^2 T - 2rT}{4\sigma\sqrt{T/2}}\right) \quad !!$$

$$\boxed{\begin{aligned} &E\left(\exp(z\cdot W_t)\cdot I(W_t>\alpha)\right) \overset{?}{=} \\ &E_*\left(\exp(z\, W_t^*)\cdot I(W_t^*>\alpha)\right) \overset{?}{=} \exp\left(t\tfrac{z^2}{2}\right)\cdot F\left(z\sqrt{t}-\tfrac{\alpha}{\sqrt{t}}\right)\end{aligned}}$$

Samurai way!

$$E(h(X))=\int_{-\infty}^{\infty} h(x)\,\mathrm{pdf}(x)\,dx$$

$X\sim N(0;1)$ $\qquad \frac{W_t^*}{\sqrt{t}} \overset{P^*}{\sim} X \qquad W_t^* \overset{P^*}{\sim} \sqrt{t}\cdot X$

$$= E_*\left(\exp(X\cdot z\sqrt{t})\cdot I\left(X>\frac{\alpha}{\sqrt{t}}\right)\right)=$$

$$=\int_{\alpha/\sqrt{t}}^{\infty}\exp(x\, z\sqrt{t})\cdot\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{x^2}{2}\right)dx=$$

$$=\int_{\alpha/\sqrt{t}}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(x^2-2x\cdot z\sqrt{t}+tz^2-tz^2\right)\right)dx=$$

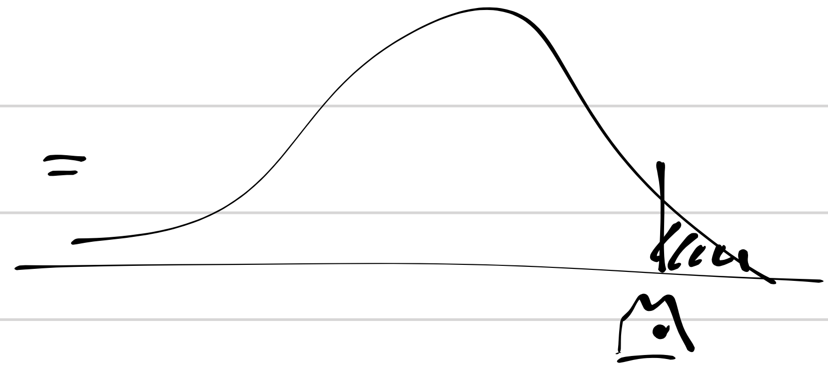
$$=\int_{\alpha/\sqrt{t}}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{1}{2}(x-z\sqrt{t})^2\right)\cdot\exp\left(\frac{tz^2}{2}\right)dx$$

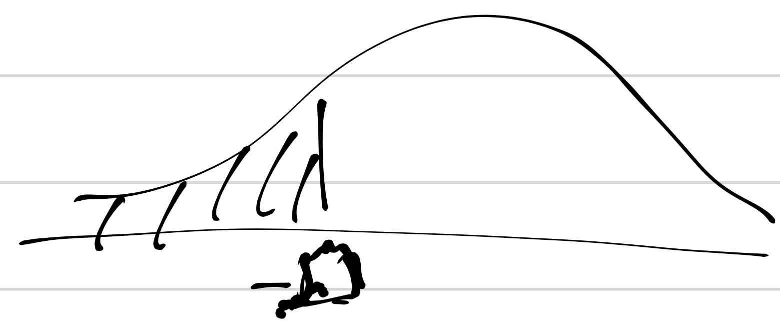
$$=\exp\left(\frac{tz^2}{2}\right)\cdot\int_{\alpha/\sqrt{t}}^{\infty}\underbrace{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(x-z\sqrt{t})^2\right)}_{\mathrm{pdf}\ N(z\sqrt{t};1)}dx=$$

$F(t)=P(N<t)$

$$=\exp\left(\frac{tz^2}{2}\right)\cdot P\left(N(z\sqrt{t};1)>\frac{\alpha}{\sqrt{t}}\right)=$$

$$=\exp\left(\frac{tz^2}{2}\right)\cdot\underbrace{P\left(N(0;1)>\frac{\alpha}{\sqrt{t}}-z\sqrt{t}\right)}=\boxed{\exp\left(\frac{tz^2}{2}\right)\cdot F\left(z\sqrt{t}-\frac{\alpha}{\sqrt{t}}\right)}$$

$P(N(0;1) > M) =$ 

$= P(N(0;1) < -M) =$ 

$= F(-M)$

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page 59 exp 4   BS model

$X_T = \ln S_T \cdot \ln S_{T/2}$

X_0 ?

$X_0 = \exp(-rT)\cdot E_*(\ln S_T \cdot \ln S_{T/2}) =$

$X_0 \neq \exp(-rT)\cdot E(\ln S_T \cdot \ln S_{T/2})$

DO 30min 10%

SC 120min 90%

dec 29 9:30 R201

$\ln S_T = \ln S_0 + rT + \sigma W_T^* - \frac{\sigma^2}{2}T$

$\ln S_{T/2} = \ln S_0 + rT/2 + \sigma W_{T/2}^* - \frac{\sigma^2}{2}T/2$

$= \exp(-rT)\cdot E_*\left[\left(\ln S_0 + rT + \sigma W_T^* - \frac{\sigma^2}{2}T\right)\cdot\left(\ln S_0 + r\frac{T}{2} - \frac{\sigma^2}{2}T/2 + \sigma W_{T/2}^*\right)\right] =$

$E_*(\text{const}\cdot W_T^*) = 0$ $E_*(\text{const}\cdot W_{T/2}^*) = 0$

$\underbrace{E_*(W_T^* \cdot W_{T/2}^*) = T/2}$

$Cov(W_T^*, W_{T/2}^*) = T/2$

$= \exp(-rT)\cdot\left(\left(\ln S_0 + rT - \frac{\sigma^2}{2}T\right)\cdot\left(\ln S_0 + \frac{rT}{2} - \frac{\sigma^2 T}{4}\right) + \sigma^2\cdot\frac{T}{2}\right)$