Hi :)

W4

Wiener process → stochastic integral

Some transformations of W.P.

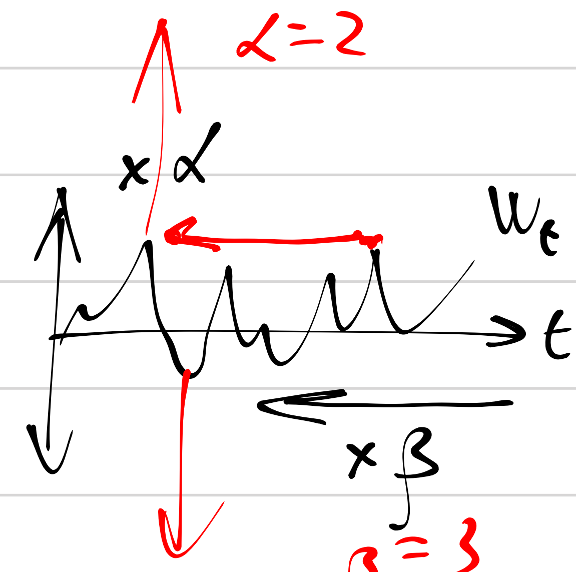
Ex.

(W_t) - Wiener process

scale time

scale units of meas..

$Y_t = \alpha \cdot W_{\beta t}$



How α and β should be related if Y_t is also a Wiener process?

* $W_0 = 0$ $\Rightarrow$ $Y_0 = 0$ ✓

* $W_t - W_s \sim N(0; t-s)$ $\Rightarrow$ $Y_t - Y_s = \alpha W_{\beta t} - \alpha W_{\beta s} =$

$$= \alpha \cdot \underbrace{(W_{\beta t} - W_{\beta s})}_{N(0;\,\beta t - \beta s)} \sim N(0; \alpha^2\beta(t-s))$$

should $N(0; t-s)$

$\alpha^2 \beta = 1$ $\quad \beta = \frac{1}{\alpha^2}$

We know this: We check this:

* $P(\text{traject of } W_t \text{ is cont}) = 1$ $\Rightarrow$ $P(\text{traject of } Y_t \text{ is cont}) = 1$

* $t_0 \; t_1 \; t_2 \; t_3 \; \dots \; t_n$ $\quad\rightarrow\quad$ Δ_i^Y are indep.

$\Delta_1, \Delta_2, \dots\dots \Delta_n$ are indep. t

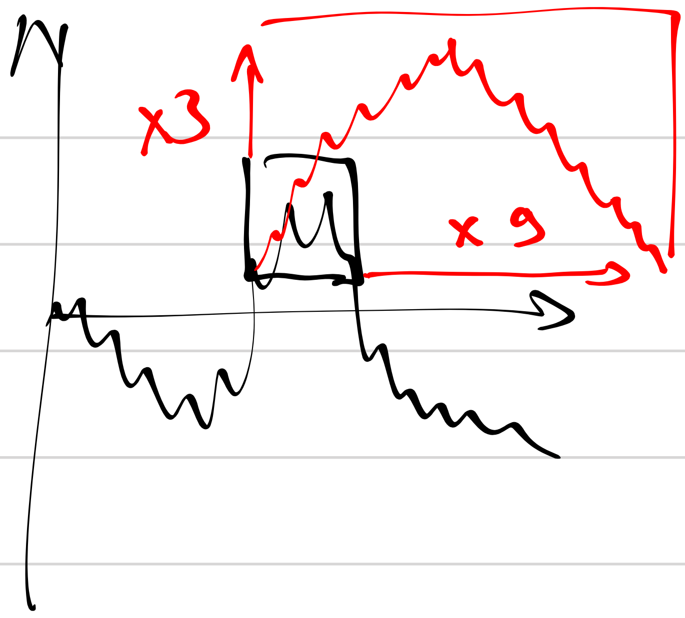
$\Delta_i = W(t_i) - W(t_{i-1})$

$s_0 \; s_1 \; \dots \; s_n$

$\Delta_i^Y = Y(s_i) - Y(s_{i-1}) =$

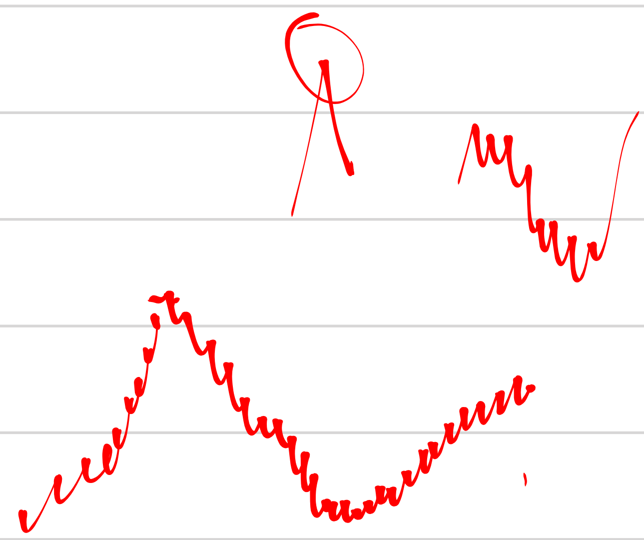
$= \alpha\left(W(\beta s_i) - W(\beta s_{i-1})\right)$

$t_0 = s_0 \cdot \beta \; \dots \; t_n = s_n \cdot \beta.$



scale by 3

scale by $3^2 = 9$



$\alpha^2 \cdot \beta = 1$

$\beta = \frac{1}{\alpha^2}$

$3 f(x)$ x3

$f\left(\frac{1}{9}x\right)$ x9

Inversion.

Ex.

$$Y_t = \begin{cases} f(t) \cdot W_{1/t} & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

a) Find $f(t)$ such that $Y_t - Y_s \sim N(0; t-s)$ $\quad t > s$

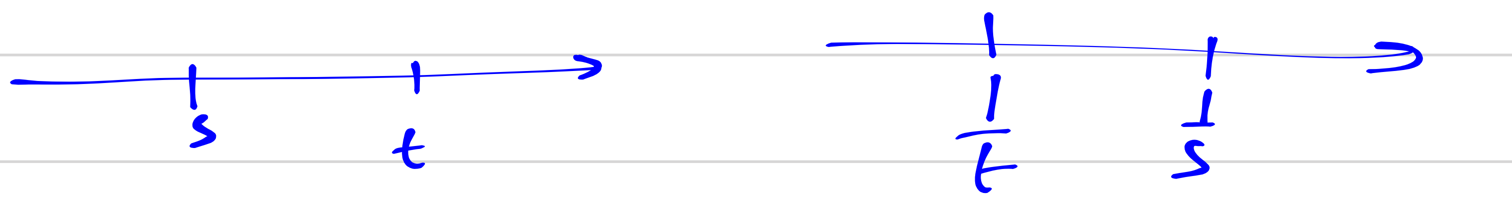
b) Is trajectory of Y_t continuous for $t > 0$? and $f(t)$ cont-s

c) Y_0 ? !!

$$Y_t - Y_s = f(t) \cdot W_{\frac{1}{t}} - f(s) \cdot W_{\frac{1}{s}} =$$

$\boxed{t > s}$ $\quad \frac{1}{t} < \frac{1}{s}$

what is unproven:
* control of $t=0$
* $\Delta_1, \Delta_2 \ldots \Delta_n$ are indep



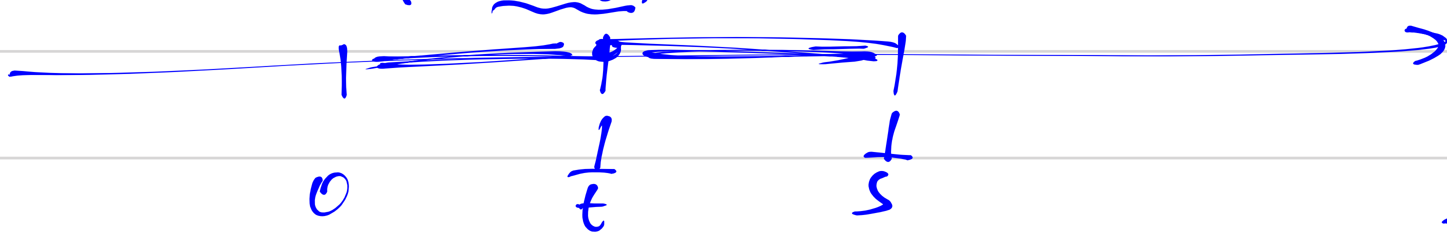
$$Y_t - Y_s = f(t)\cdot W_{\frac{1}{t}} - f(s)\cdot W_{\frac{1}{s}} =$$

$\boxed{t>s}$ $\qquad \frac{1}{t} < \frac{1}{s}$

$$= -f(s)\cdot\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right) - f(s)\cdot W_{\frac{1}{t}} + f(t)\cdot W_{\frac{1}{t}} =$$

$$= -f(s)\cdot\left(W_{\frac{1}{s}} - W_{\frac{1}{t}}\right) + \left(f(t)-f(s)\right)\cdot\left(W_{\frac{1}{t}} - W_0\right)$$

$\sim N\left(0; \frac{1}{s}-\frac{1}{t}\right)$ $\qquad$ indep $\qquad$ $\sim N\left(0; \frac{1}{t}-0\right)$



$$\sim N\left(0; f^2(s)\left(\frac{1}{s}-\frac{1}{t}\right) + \left(f(t)-f(s)\right)^2\cdot\frac{1}{t}\right)$$

$\forall t \geq s$ $\qquad f^2(s)\cdot\left(\frac{1}{s}-\frac{1}{t}\right) + \left(f(t)-f(s)\right)^2\cdot\frac{1}{t} = t-s$ $\qquad \cdot st$

$$f^2(s)\cdot(t-s) + \left(f(t)-f(s)\right)^2\cdot s = st(t-s)$$

$f(t)?$ $\qquad$ guess!! $\qquad f(t) = t.$

$$s^2(t-s) + (t-s)^2\cdot s \overset{?}{=} st(t-s)$$

$$s + t - s = t \quad !!$$

$s=1 \qquad t\geq 1 \qquad f^2(1)\cdot(t-1) + \left(f(t)-f(1)\right)^2 = t(t-1)$

$$\left(f(t)-f(1)\right)^2 = (t-1)\cdot\left(t-f^2(1)\right)$$

$$f(t)-f(1) = +\sqrt{(t-1)\cdot\left(t-f^2(1)\right)}$$

$$f(t)-f(1) = -\sqrt{\ldots\ldots\ldots}$$

$$Y_t = \begin{cases} f(t)\cdot W_{1/t} & \text{if } t>0 \\ 0 & \text{if } t=0 \end{cases}$$

$f(1) \to +1$
$f(1) \to -1$

$$Y_1 = f(1)\cdot W_1$$

$N(0;1) \sim Y_1$ $\qquad W_1 \sim N(0;1)$

two continuous choices: $f(t)=t$ $\quad f(t)=-t$.

Theorem:

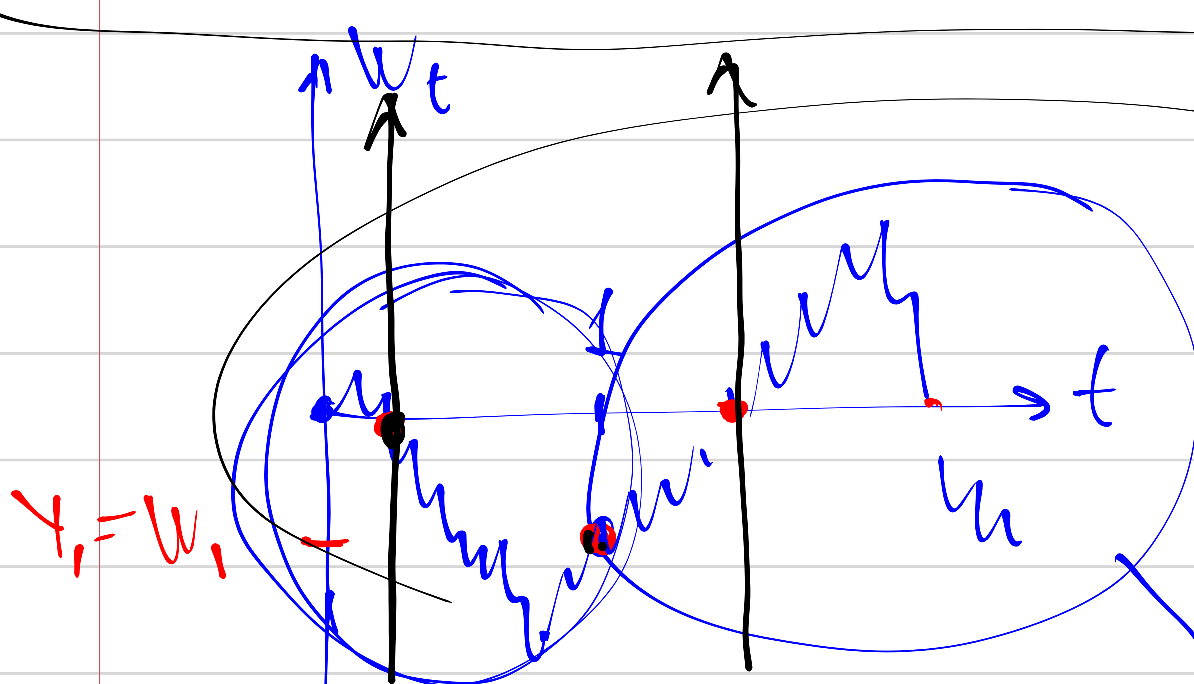
If (W_t) is a Wiener process then

$$Y_t = \begin{cases} t\cdot W_{1/t} & \text{if } t>0 \\ 0 & \text{if } t=0 \end{cases}$$

and

$$R_t = \begin{cases} -t\, W_{1/t} & \text{if } t>0 \\ 0 & \text{if } t=0 \end{cases}$$

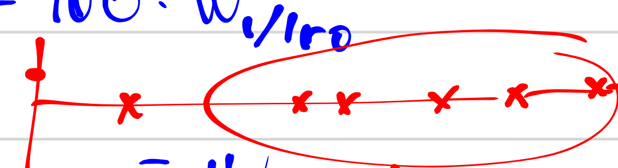
are also Wiener processes.



$Y_2 = 2\cdot W_{1/2}$

$Y_5 = 5\, W_{1/5}$

$Y_{100} = 100\cdot W_{1/100}$



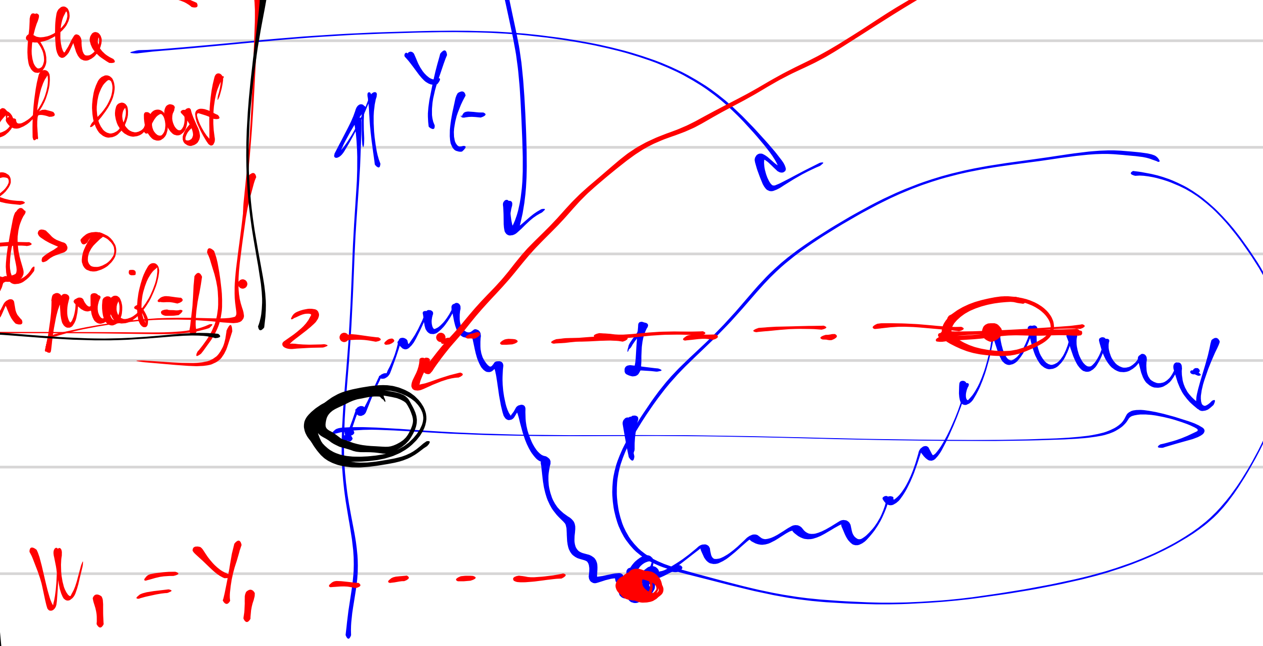
$Y_{0.5} = 0.5\cdot W_2$

$Y_{0.01} = 0.01\cdot W_{100}$

Theorem W_t intersects the line $y=0$ at least once for $t>0$ (with prob $=1$)

$Y_1 = W_1$

restart the timer



theorem W_t intersects the line $y=0$ ∞ number of times with prob. by 1

$\Downarrow$ Inversion

Theorem W_t intersects the line $y=0$ ∞ number of times on $t\in[0;\varepsilon)$ for $\forall\varepsilon>0$

Ex.

a) $E(W_7 \mid W_{10}) \overset{?}{=}$

b) $Var(W_1 \mid W_2) \overset{?}{=}$

c) $E(W_{10} \mid W_7) \overset{?}{=}$ $E(\underbrace{W_7}_{\text{past}} + \underbrace{(W_{10} - W_7)}_{\text{+ increm}} \mid W_7) =$

known past

indep.

$= W_7 + E(W_{10} - W_7 \mid W_7) =$

indep

$= W_7 + E(\underbrace{W_{10} - W_7}_{\sim N(0;10-7)}) = W_7 + 0 = W_7$

a) $E(W_7 \mid W_{10}) =$

$W_7 = 7 \cdot Y_{1/7}$

$W_{10} = 10 \cdot Y_{1/10}$

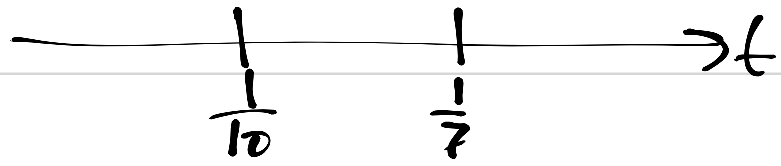
$Y_t \sim$ W.P. !!

$= E(7 \cdot Y_{1/7} \mid 10\, Y_{1/10}) =$

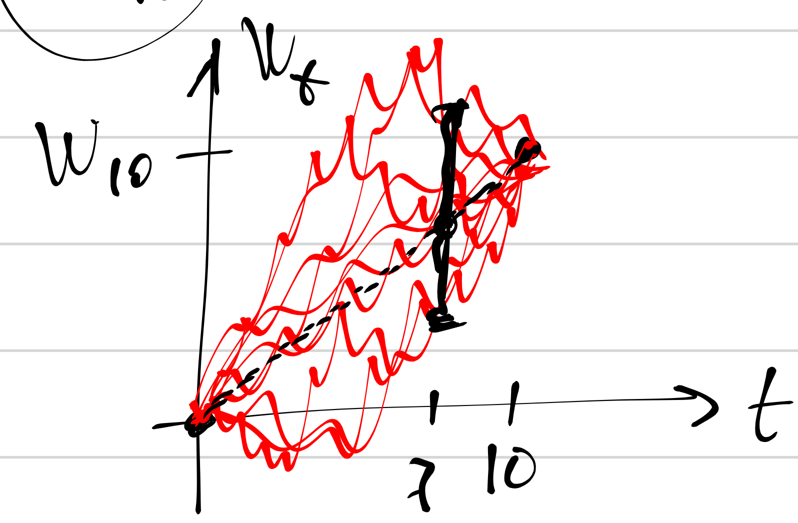
$E(R \mid L) = E(R \mid \sigma(L))$

$\sigma(10\, Y_{1/10}) = \sigma(Y_{1/10})$

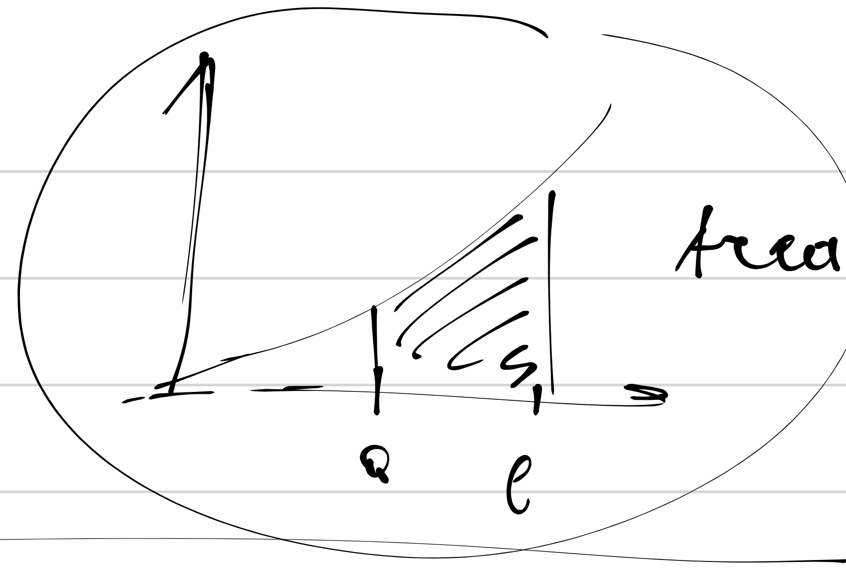
$= 7\, E(Y_{1/7} \mid Y_{1/10}) =$



$= 7 \cdot Y_{1/10} = 7 \cdot \frac{1}{10} \cdot W_{10} = \frac{7}{10} \cdot W_{10}$!!



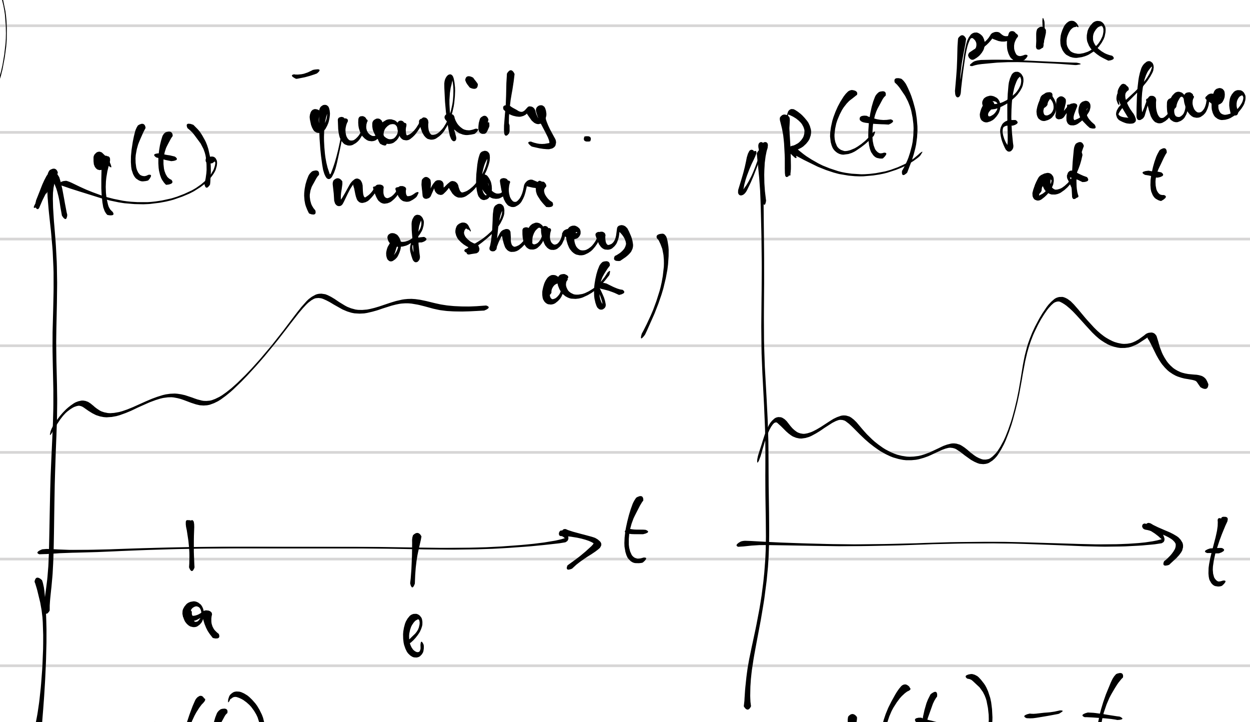
$$\int_a^b t^2\,dt \longleftrightarrow$$



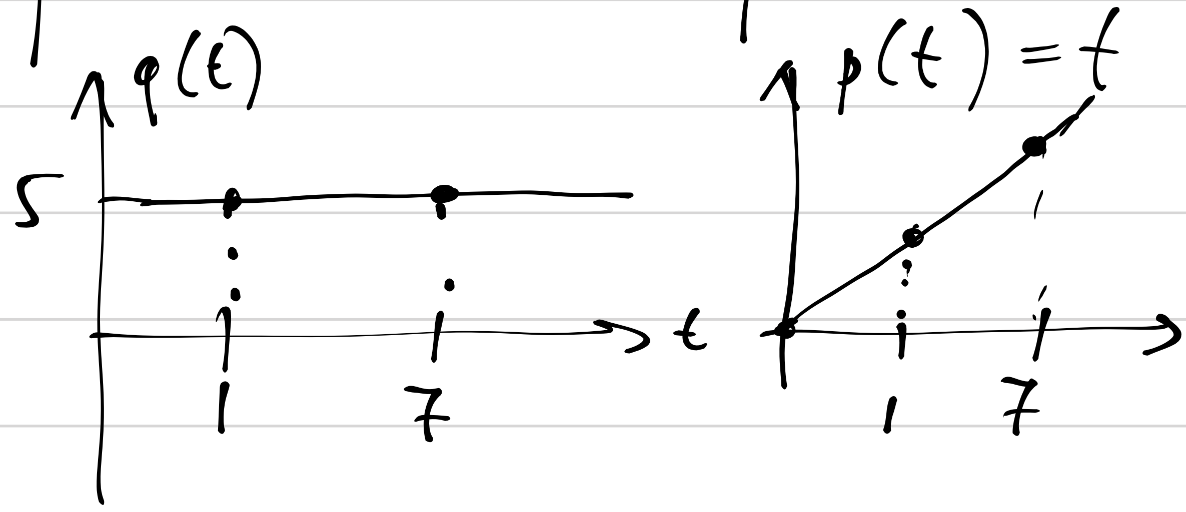
$$\int_a^b t^2 \cdot d(5t\cos t) \longrightarrow ?$$

$$\int_a^b q(t)\,dp(t)$$

↑ profit

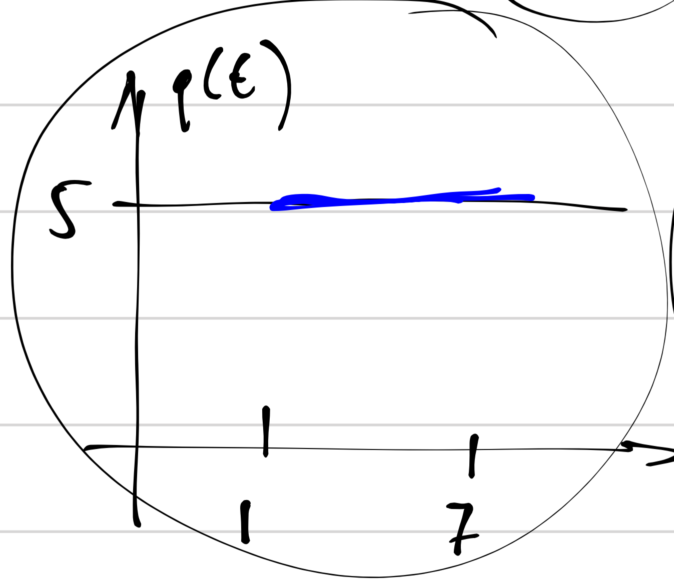


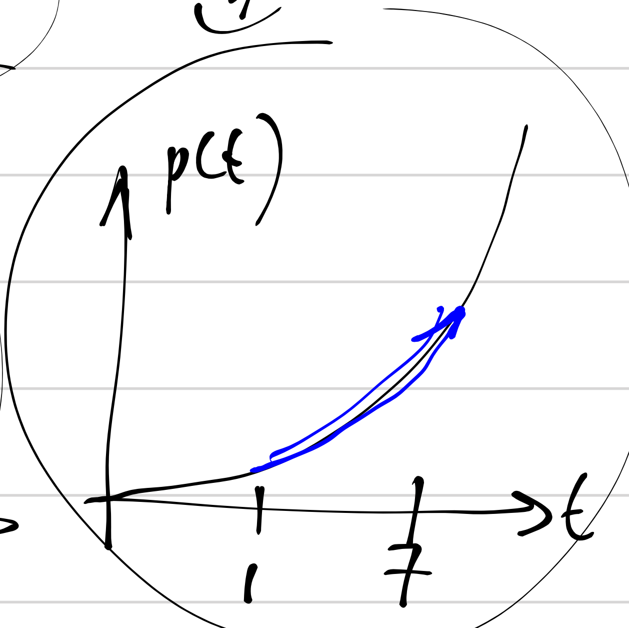
$$\boxed{\int_1^7 5\cdot dt}$$



$5\times 7 - 5\cdot 1 = 30$!!

$$\int_1^7 5\,d(t^2) =$$





= final wealth - initial wealth =

$= 5\times 7^2 - 5\times 1^2 = 5\cdot 48$!!

Ex.

$$I = \int_0^{100} f(t)\cdot dW_t$$

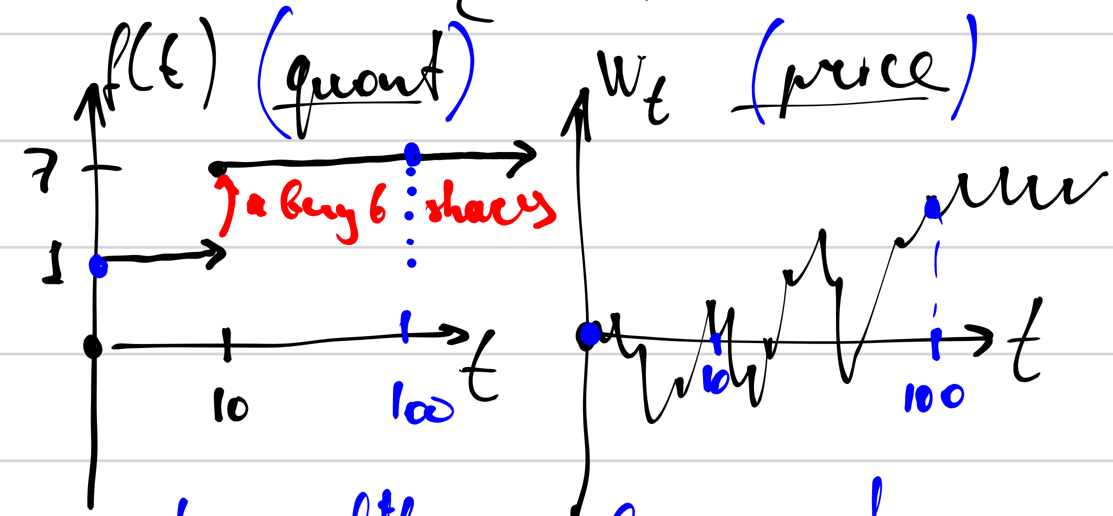
W_t - Wiener process

$$f(t) = \begin{cases} 1 & \text{if } t \in [0;10) \\ 7 & \text{if } t \in [10;+\infty) \end{cases}$$

a) I ?

b) $E(I)$?

c) $Var(I)$?



final wealth $-$ init. wealth $-$ buy. costs

$$= 7\cdot W_{100} - \underbrace{1\cdot W_0}_{0} - (7-1)\cdot W_{10} =$$

$$\int_0^{100} f(t)\, dW_t = 7W_{100} - 6W_{10}$$

b) $E(I) = 0$ (intuit)

c) $Var(7W_{100} - 6W_{10}) =$ (1)

$= Var(7(W_{100} - W_{10} + W_{10}) - 6W_{10}) =$ (2)

$= Var(7\underbrace{(W_{100} - W_{10})} + \underbrace{1\cdot W_{10}}_{W_{10} - W_0}) =$ (3)

indep

(4)

$6\underbrace{(W_{100} - W_{10})} + \underbrace{W_{100}}$

depend

$= 49\cdot(100-10) + 1\cdot(10-0);$

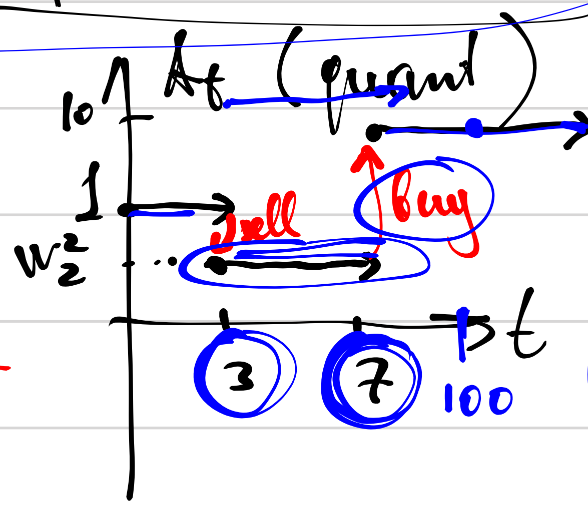
11:20

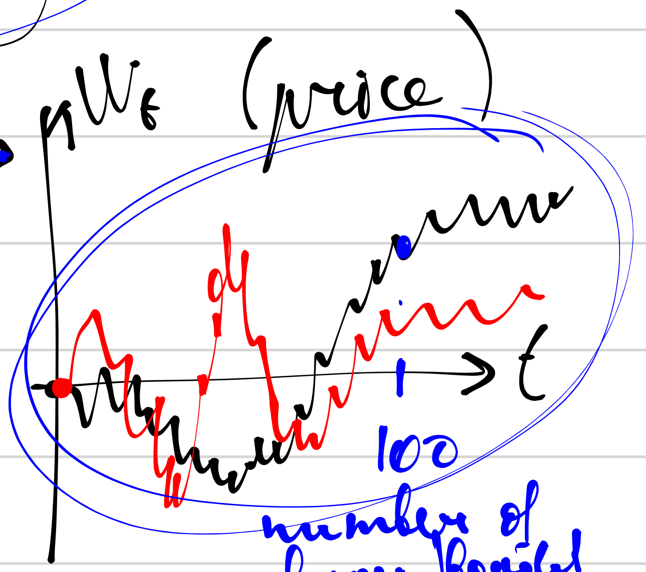
let's continue

Ex. $A_t = \begin{cases} 1 & \text{if } t \in [0;3) \\ W_2^2 & \text{if } t \in [3;7) \\ 10 & \text{if } t \in [7;+\infty) \end{cases}$

a) $I = \int_0^{100} \underbrace{A_t}_{\text{quant}} \, d\underbrace{W_t}_{\text{price}}$?

b) $E(I)$?





number of shares bought at $t=7$

$$I = \underbrace{\text{final wealth}}_{t=100} - \underbrace{\text{init wealth}}_{t=0} - \text{inter. costs} =$$

$$= \underbrace{10}_{\#\text{shares}} \cdot \underbrace{W_{100}}_{\text{price}} - \underbrace{1}_{\#\text{shares}} \cdot W_0 + \underbrace{(1 - W_2^2) \cdot W_3}_{t=3} - \underbrace{(10 - W_2^2) \cdot W_7}_{t=7}$$

$\boxed{t=100}$ $\boxed{t=0}$

$-(W_2^2 - 1) \cdot W_3$

price

$$= 10 \cdot W_{100} + (1 - W_2^2) \cdot W_3 + (W_2^2 - 10) \cdot W_7 \quad !!$$

dep dep

I = sum portfolio price changes

$$\underbrace{\underbrace{1}_{\#\text{shares}} \times (W_3 - W_0)}_{t=0 \to t=3} + \underbrace{\underbrace{W_2^2}_{\#\text{shares}} \times (W_7 - W_3)}_{t=3 \to t=7} + \underbrace{\underbrace{10}_{\#\text{shares}} \times (W_{100} - W_7)}_{t=7 \to t=100}$$

$$E(I) = E(\underbrace{W_3 - W_0}_{\sim N(0;3)}) + E(W_2^2 \cdot (W_7 - W_3)) + 10 \cdot E(\underbrace{W_{100} - W_7}_{N(0;93)})$$

$$= E(\underbrace{W_2^2 \cdot (W_7 - W_3)}_{\text{indep}}) = E((W_2 - W_0)^2 \cdot (W_7 - W_3))$$

Δ_1 Δ_2

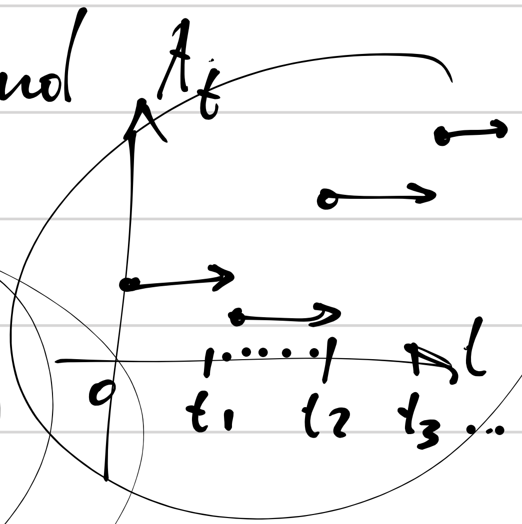
0 2 3 7

$$= E(W_2^2) \cdot E(W_7 - W_3) = 0$$

def

If (W_t) is a Wiener process wrt $(\mathcal{F}_t)$ and (A_t) is adapted to $(\mathcal{F}_t)$ and A_t is piecewise constant:

$$A_t = \begin{cases} a_1 & \text{if } t \in [0; t_1) \\ a_2 & \text{if } t \in [t_1; t_2) \\ a_3 & \text{if } t \in [t_2; t_3) \\ \vdots \end{cases}$$



$\vdots$ a_1 is a $\mathcal{F}_0$-measur. r.v.
a_2 is a $\mathcal{F}_1$-meas r.variable
...

then

$$\int_0^{t_n} A_t \, dW_t \overset{def}{=}$$

$$= \sum_{i=1}^{n} \underbrace{a_i}_{\#\text{ shares}} \cdot \underbrace{(W(t_i) - W(t_{i-1}))}_{\text{price change} \sim N(0; t_i - t_{i-1})} \leftarrow \text{total profit.}$$

Limit in L^2

def.

$R_1, R_2, R_3 \ldots\ldots$

$\forall_n \quad E(R_n^2) < \infty$

$\exists \; R$ such that $\boxed{\lim\limits_{n\to\infty} E\left((R_n - R)^2\right) = 0}$

then R is called L^2 limit of (R_n)

$$R_n \xrightarrow{L^2} R$$

Ex. $X \sim U[0;1]$
$R_n = X^n$
$R_n \xrightarrow{L^2} ?$

guess 0?

$0{,}7;\ 0{,}7^2;\ 0{,}7^3 \ldots\ldots$

$0{,}3;\ 0{,}3^2;\ 0{,}3^3; \ldots\ldots$

$$E((R_n - 0)^2) = E((X^n)^2) = E(X^{2n}) = \int_0^1 x^{2n} \cdot 1 \cdot dx =$$

$$= \frac{x^{2n+1}}{2n+1}\Big|_{x=0}^{x=1} = \frac{1}{2n+1} \xrightarrow{n\to\infty} 0 \quad \left[\begin{matrix}\text{accord}\\ \text{to the def}\end{matrix}\right]$$

Conclusion: $R_n \xrightarrow[n\to\infty]{L^2} 0$

0 ← limit of (R_n)

$$Y = \begin{cases} 0 & \text{if } X<0 \\ 1 & \text{if } X=1 \end{cases}$$

↑ limit of (R_n)

$\underline{P(X=1)=0}$ for $X \sim U\,]0;1]$

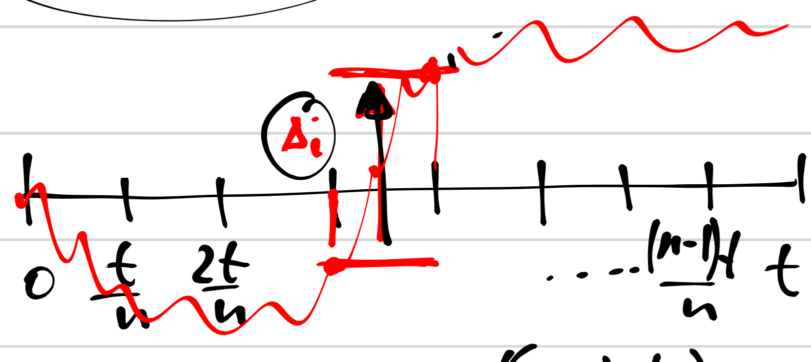
$$E\left((R_n - Y)^2\right) = E\left((R_n-0)^2\right) \xrightarrow[n\to\infty]{} 0$$

theorem: if X_1 and X_2 are two limits of (R_n) in L^2 then $P(X_1 = X_2) = 1$

theorem: If $\lim\limits_{n\to\infty} E(R_n) = \mu$ and $\lim\limits_{n\to\infty} Var(R_n) = 0$

the $R_n \xrightarrow[n\to\infty]{L^2} \mu$ (sufficient condition)

Ex.



n parts

$\Delta t = \frac{t}{n}$

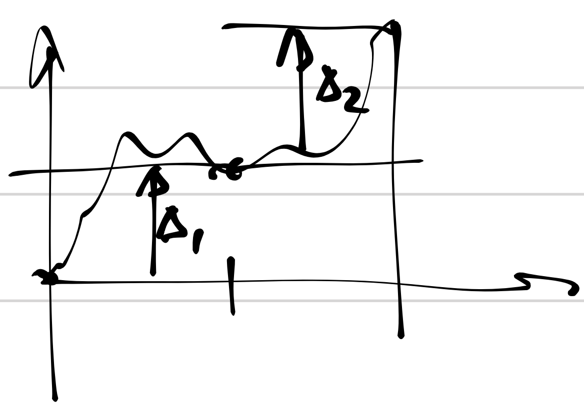
$$\Delta_i = W\left(\frac{it}{n}\right) - W\left(\frac{(i-1)t}{n}\right)$$

a) $\sum\limits_{i=1}^{n} \Delta_i \xrightarrow[n\to\infty]{L^2} W_t$

b) $\sum\limits_{i=1}^{n} \Delta_i^2 \xrightarrow[n\to\infty]{L^2} ?$

$\Delta_1 + \Delta_2 + \dots + \Delta_n = W_t - W_0 = W_t$

$S_n = \sum\limits_{i=1}^{n} \Delta_i^2$

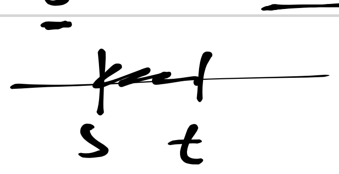


$\Delta_i \sim N\left(0; \frac{t}{n}\right)$

$E(\Delta_i) = 0$

$Var(\Delta_i) =$
$= E(\Delta_i^2) = \frac{t}{n}$

$W_t - W_s \sim N(0; t-s)$



$$S_n=\sum_{i=1}^{n}\Delta_i^2$$

$$\boxed{\Delta_i\sim N\left(0;\frac{t}{n}\right)}\ \text{indep.}$$

$$E(\Delta_i)=0$$

$$Var(\Delta_i)=E(\Delta_i^2)=\frac{t}{n}$$

$$E(S_n)=$$

$$=\sum_{i=1}^{n}E(\Delta_i^2)=$$

$$=\sum_{i=1}^{n}\frac{t}{n}=n\cdot\frac{t}{n}=t$$

$$Var(S_n)=Var\left(\sum\Delta_i^2\right)=\overset{\boxed{\text{no cov!}}}{\sum_{i=1}^{n}}Var(\Delta_i^2)=n\cdot Var(\Delta_1^2)=$$

$$\Delta_i\sim N\left(0;\frac{t}{n}\right)$$

$$z=\frac{\Delta_i-0}{\sqrt{\frac{t}{n}}}\sim N(0;1)\qquad \Delta_i=z\cdot\sqrt{\frac{t}{n}}\sim N\left(0;\frac{t}{n}\right)$$

$$\Delta_i^2=z^2\cdot\left(\frac{t}{n}\right)$$

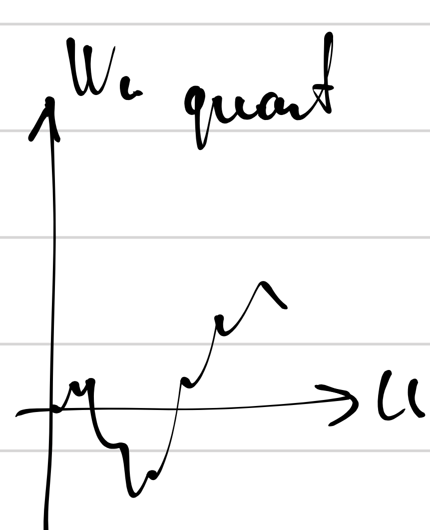
$$=n\,Var\left(z^2\cdot\frac{t}{n}\right)=n\cdot\frac{t^2}{n^2}\cdot Var(z^2)=\frac{t^2}{n}\cdot\underbrace{Var(z^2)}_{\text{const.}}$$

$$\lim_{n\to\infty}\frac{t^2\cdot Var(z^2)}{n}\overset{?}{=}0$$

$$E\left(\sum_{i=1}^{n}\Delta_i^2\right)=t,\qquad Var\left(\sum_{i=1}^{n}\Delta_i^2\right)\to 0\quad\Longrightarrow\quad\sum_{i=1}^{n}\Delta_i^2\xrightarrow{L^2}t$$

Ex.

$$I=\int_0^t W_u\,dW_u=$$



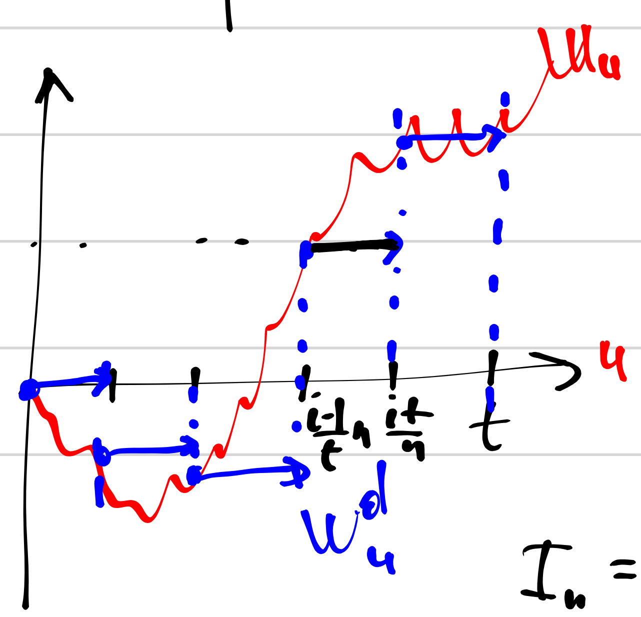
strategy

Step 1 ⇓

$$I_n=\int_0^t W_u^d\cdot dW_u$$

Step 2

$$I_n\xrightarrow{L^2}I$$



$$W^d\left(\frac{it}{n}\right)=W\left(\frac{it}{n}\right)$$

$W^d(u)$ is const if $u\in\left[\frac{it}{n};\frac{(i+1)t}{n}\right)$

shares x pr. chang

$$I_n=\sum_{i=1}^{n}W\left(\frac{i-1}{n}t\right)\cdot\Delta_i$$

goal: $I = \int_0^t W_u \, dW_u$?

<u>approximation</u> $I_n = \sum_{i=1}^{n} W\left(\frac{(i-1)t}{n}\right) \cdot \Delta_i =$

$$= \sum_{i=1}^{n} W\left(\frac{(i-1)t}{n}\right) \cdot \left(W\left(\frac{it}{n}\right) - W\left(\frac{(i-1)t}{n}\right)\right)$$

$$I_n \xrightarrow[n\to\infty]{L^2} I \text{ ?}$$

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let's continue !!

$\boxed{\text{Q + A?}}$

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$\boxed{\begin{array}{l}\text{ideas: 1) future = current + change} \\ \text{2) time inversion}\end{array}}$

a) $E(W_{2019}^3 \mid W_{2018}) \overset{?}{=} a$

b) $E(W_{2018}^3 \mid W_{2019}) \overset{?}{=}$

$$a = E\left((\underbrace{W_{2019} - W_{2018}}_{\Delta} + W_{2018})^3 / W_{2018} \right) =$$

$$= E\left((W_{2018} + \Delta)^3 / W_{2018} \right) =$$

$$= E\left(\Delta^3 + 3\Delta^2 \cdot W_{2018} + 3\Delta \cdot W_{2018}^2 + W_{2018}^3 \middle| W_{2018} \right)$$

Δ is indep of W_{2018}

W_{2018} is completely known.

$\boxed{\begin{array}{l}\text{Answer:} \\ = 3 \cdot 1 \cdot W_{2018} + W_{2018}^3\end{array}}$

$$= E(\Delta^3) + 3E(\Delta^2) \cdot W_{2018} + 3E(\Delta) \cdot W_{2018}^2 + W_{2018}^3$$

$\Delta \sim N(0; 2019-2018) \sim N(0;1)$ $\quad E(\Delta) = 0$

$E(\Delta^3) = \int_{-\infty}^{\infty} u^3 \cdot f(u) \, du = 0$ $\quad Var(\Delta) = 1 = E(\Delta^2)$

$$b = E\left(W_{2018}^3 \mid W_{2019}\right) =$$

$$W_t = t \cdot Y_{1/t} \qquad t > 0$$

$$\frac{1}{t} \cdot W_t = Y_{1/t} \qquad \frac{1}{t} = u$$

funny note!

$$u \cdot W_{1/u} = Y_u$$

$$Y_t = t \cdot W_{1/t}$$

$$= E\left(2018^3 \cdot Y_{1/2018}^3 \mid \underbrace{2019}_{x} \cdot Y_{1/2019}\right) =$$

$$= 2018^3 \cdot E\left(Y_{1/2018}^3 \mid Y_{1/2019}\right) =$$

$$= 2018^3 \cdot E\left(\left(Y_{1/2019} + \Delta\right)^3 \mid Y_{1/2019}\right) =$$

$$\Delta = Y_{1/2018} - Y_{1/2019}$$

$$\Delta \sim N\left(0; \frac{1}{2018} - \frac{1}{2019}\right)$$

$$E(\Delta) = 0$$

$$E(\Delta^2) = \frac{1}{2018} - \frac{1}{2019} = \frac{1}{2018 \cdot 2019}$$

$$E(\Delta^3) = 0$$

$$= 2018^3 \left(Y_{1/2019}^3 + 3 \cdot E(\Delta^2) \cdot Y_{1/2019}\right) =$$

$$= 2018^3 \left(\frac{1}{2019^3} \cdot W_{2019}^3 + 3 \frac{1}{2018 \cdot 2019} \cdot \frac{1}{2019} \cdot W_{2019}\right)$$

$$= \left(\frac{2018}{2019}\right)^3 \cdot W_{2019}^3 + 3 \cdot \left(\frac{2018}{2019}\right)^2 \cdot W_{2019}$$

!!

10 pts + !!

$$I_t = \int_0^t A_u \, dW_u \qquad E(I_t) = 0 \quad !!$$

quant price

p61 Q2 b)

$$X_t = \exp(-\alpha t)\cdot\left(1 + \int_0^t \exp(\alpha u)\, dW_u\right)$$

$$E(X_t) \overset{?}{=} \exp(-\alpha t)\cdot E\Big(1 + \underbrace{\int_0^t \dots dW_u}\Big) =$$

$$= \exp(-\alpha t) \quad !!$$

2pts/10pts

p53 Q1 n times

X = # „fives" in the first $(n-1)$ throws.
Y = # „fives" in the last $(n-1)$ throws.

$$R_i = \begin{cases} 1 & \text{if throw } Ni \text{ is „five"} \\ 0 & \text{otherwise.} \end{cases}$$

$$X = R_1 + R_2 + \dots + R_{n-1} \qquad Y = R_2 + R_3 + R_4 + \dots + R_n$$

$n \geq 3$ a) $E(Y|X)$?
b) $E(X|Y)$?
c) $Var(Y|X)$?

$$E(R_2 + R_3 + \dots + R_n \mid R_1 + R_2 + \dots R_{n-1})?$$

$$= E(R_n \mid R_1 + \dots + R_{n-1}) + E(R_2 \mid R_1 + \dots + R_{n-1}) + E(R_3 \mid R_1 + \dots + R_{n-1}) + \dots + E(R_{n-1} \mid R_1 + \dots + R_{n-1}) =$$

indep

$$= E(R_n) + (n-2)\cdot E(R_2|X)$$

$$P(R_2 = k, X = \ell) = P(R_3 = k, X = \ell)$$

$$E(R_n) = \frac{1}{6}\cdot 1 + \frac{5}{6}\cdot 0 = \frac{1}{6}$$

$$E(R_2|X)\ ?$$

Advice: consider specific case if you are stuck

$n=10$

$E(R_2|X=5)$ $\qquad\qquad$ $R_2 \in \{0,1\}$

$$E(R_2 \mid R_1+R_2+\dots+R_9=5) =$$

$$= 1\cdot P(R_2=1 \mid R_1+\dots+R_9=5) + 0\cdot P(R_2=0|\dots)$$

$$= \frac{5}{9} \xrightarrow{\text{gen.}} E(R_2|X) = \frac{X}{n-1}$$

a) $E(Y|X) = \frac{1}{6} + (n-2)\cdot\frac{X}{n-1}$ $\qquad$!!

b) $E(X|Y) = \frac{1}{6} + (n-2)\cdot\frac{Y}{n-1}$ $\qquad$ (by symmetry)

c) $Var(Y|X) =$

$$= Var(R_2+R_3+\dots+R_n \mid R_1+R_2+\dots+R_{n-1}) \quad ?$$

alone and indep. of.

$$= Var(R_n) + Var(R_2+R_3+\dots R_{n-1} \mid R_1+R_2+\dots+R_{n-1})$$

$R_n \sim Binomial(n=1, p=\frac{1}{6})$

$R_n \sim Bernoulli(p=\frac{1}{6})$

$E(R_n) = \frac{1}{6}$

$Var(R_n) = \frac{1}{6}\cdot(1-\frac{1}{6}) = \frac{5}{36}$

$R_1+R_2+\dots+R_{n-1}=7$

$R_2+\dots+R_{n-1}=7$ $\qquad$ $R_2+\dots R_{n-1}=6$

$Var(R_2+\dots+R_{n-1} \mid R_1, R_2, \dots, R_{n-1}) = 0$

general: $Var(R_2+\dots+R_{n-1}|X) = \frac{X}{n-1}\cdot\frac{n-1-X}{n-1}$

part case $\qquad n=10$

$$Var(R_2+\dots+R_9 \mid R_1+R_2+\dots+R_9=7) = \frac{7}{9}\cdot\frac{2}{9}$$

y	6	7
$P(Y=y\mid X=7)$	7/9	2/9

~6 + Bernoulli

$$Var(Y|X) = \frac{5}{36} + \frac{X}{n-1} \cdot \frac{n-1-X}{n-1}$$

p 48 Q1

$Y_1, Y_2, Y_3 \ldots$ iid

y	0	1
$P(Y_i = y)$	$\frac{1}{2}$	$\frac{1}{2}$

$$N = \min_n \{ n \mid Y_n = 1 \}$$

a) $E(Y_2|N)$

b) $Var(Y_2|N)$

c) $E(N|Y_2)$

$$E(Y_2 / N = 10) = 0$$

$$E(Y_2 / N = 2) = 1$$

$$E(Y_2 / N = 1) = E(Y_2 | Y_1 = 1) = E(Y_2) = \frac{1}{2}$$

$$E(Y_2/N) = \begin{cases} 0 & \text{if } N > 2 \\ 1 & \text{if } N = 2 \\ \frac{1}{2} & \text{if } N = 1 \end{cases}$$

$$Var(Y_2/N) = \begin{cases} 0 & \text{if } N > 2 \\ 0 & \text{if } N = 2 \\ \frac{1}{4} & \text{if } N = 1 \end{cases} = \begin{cases} 0 & \text{if } N \geq 2 \\ \frac{1}{4} & \text{if } N = 1 \end{cases}$$

c) $E(N|Y_2 = 1) = 1 \cdot P(N = 1 | Y_2 = 1) + 2 \cdot P(N = 2 | Y_2 = 1)$

$= 1 \cdot P(Y_1 = 1 | Y_2 = 1) +$

$+ 2 \cdot P(Y_1 = 0 | Y_2 = 1) =$

? 1 ? ?

N

$= 1 \cdot P(Y_1 = 1) + 2 \cdot P(Y_1 = 0) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \boxed{1.5}$

$P(Y_1 = 1 | Y_2 = 0) = \frac{1}{2}$

$E(N | Y_2 = 0) = 1 \cdot P(N = 1 | Y_2 = 0) +$

$+ 3 \cdot P(N = 3 | Y_2 = 0) + 4 \cdot P(N = 4 | Y_2 = 0) + \ldots =$

$= 1 \cdot \frac{1}{2} +$
$+ 3 \cdot \left(\frac{1}{2}\right)^2 + 4 \cdot \left(\frac{1}{2}\right)^3 +$
$+ 5 \cdot \left(\frac{1}{2}\right)^4 + \ldots$

$$P(N = 4 | Y_2 = 0) = \frac{P(N = 4, Y_2 = 0)}{P(Y_2 = 0)} =$$

$$= \frac{P(0001)}{1/2} = \left(\frac{1}{2}\right)^4 \cdot 2$$

$$\alpha = E(N \mid Y_2 = 0) = \frac{1}{2} + 3\cdot\left(\frac{1}{2}\right)^2 + 4\cdot\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \dots$$

$$- \quad \frac{1}{2}\alpha = \frac{1}{4} + 3\cdot\left(\frac{1}{2}\right)^3 + 4\cdot\left(\frac{1}{2}\right)^4 + 5\cdot\left(\frac{1}{2}\right)^5 + \dots$$

$$\frac{1}{2}\alpha = \frac{1}{4} + 3\cdot\left(\frac{1}{2}\right)^2 + \underbrace{\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \dots}_{\text{geom sum}}$$

$$\frac{1}{2}\alpha = \underbrace{\frac{1}{4} + 3\cdot\frac{1}{4}} + \frac{(1/2)^3}{1-\frac{1}{2}} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$\alpha = \frac{5}{2} \quad !!$$

$$E(N \mid Y_2) = \begin{cases} 1.5 & \text{if } Y_2 = 1 \\ \frac{5}{2} & \text{if } Y_2 = 0. \end{cases}$$