1. Consider $X_t = \exp(-2W_t - 2t)$.
   (a) Find dX_t. Write the corresponding full form. Is X_t a martingale?
   (b) Find $\mathbb{E}(X_t)$ and $\mathbb{V}\text{ar}(X_t)$.
   (c) Find $\int_0^t X_u dW_u$.
2. Let (W_t) be a Wiener process.
   (a) Find $\mathbb{E}(W_5 W_4 \mid W_4)$, $\mathbb{V}\text{ar}(W_5 W_4 \mid W_4)$.
   (b) Find $\mathbb{E}(W_5 W_4 W_3 \mid W_4)$, $\mathbb{V}\text{ar}(W_5 W_4 W_3 \mid W_4)$.
3. Winnie-the-Pooh starts at zero of the real line. Every minute he moves left by one with probability 0.2, right by one with probability 0.2 or does not move and eats a honey-pot with probability 0.6.

   Let X_t be the coordinate of Winnie at time t. Rabbit's hole has x-coordinate 6 and Owl lives at (-4), let τ be the first moment when Winnie visits Owl or Rabbit, $\tau = \min\{t \mid X_t = 6 \text{ or } X_t = -4\}$.
   (a) Is X_t a martingale?
   (b) Find a constant a such that $Y_t = X_t^2 - at$ is a martingale.
   (c) Find $\mathbb{P}(X_\tau = 6)$ and $\mathbb{E}(\tau)$.

   Hint: you don't need to check technical conditions of Doob's optional stopping theorem in point (c).
4. Consider the Vasicek interest rate model
   $$dR_t = (0.05 - R_t)dt + dW_t, \quad R_0 = 0.07.$$
   (a) Write the same models in full form with integrals.
   (b) Let $b_t = \mathbb{E}(R_t)$. Find b_t and sketch it.

   Hint: to find b_t take expected value of both sides of the full form and solve the ordinary differential equation.
5. Consider two-period binomial model with initial share price $S_0 = 600$, Up and down multipliers are $u = 1.2$, $d = 0.8$, risk-free interest rate is $r = 0.05$ per period.

   Consider an option that pays you $X_2 = 100$ at $T = 2$ if $S_2 > S_1$ and nothing otherwise.
   (a) Find the risk neutral probabilities.
   (b) Find the current price X_0 of the asset.
   (c) How much shares should I have at $t = 1$ in the «down» state of the world to replicate the option?
6. Consider Black and Scholes model in continuous time with risk-free interest rate r, volatility σ, initial share price S_0 and exponential growth rate of expected share price μ.
   (a) Let's denote by $\mathbb{P}$ the real probability and by $\mathbb{P}^*$ the risk-neutral probability. Find $\mathbb{P}(S_2 > S_1)$ and $\mathbb{P}^*(S_2 > S_1)$.
   (b) An option pays you the sum $X_2 = \sqrt{S_1}$ at $T = 2$. Find the non-arbitrage price X_0 of the option.

   Hint: the answers may contain the standard normal cumulative distribution function $F()$.