Week 2.

Two more definitions !!

$\mathcal{F}$ - σ-algebra

$E \to P$, $E \to Var$

$Var(X) = E(X^2) - (E(X))^2 \leftarrow$ is constant

def $\underbrace{Var(X|\mathcal{F})}_{\text{is a random variable}} = E(X^2|\mathcal{F}) - (E(X|\mathcal{F}))^2$

$P(A) = E(I_A)$ $\qquad I_A = \begin{cases} 1, & \text{if } A \\ 0, & \text{if } A^c \end{cases}$

def $P(A|\mathcal{F}) = E(I_A|\mathcal{F})$

Ex. What is bigger $Var(2Y|X)$ or $Var(Y|2X)$?

$Var(2Y|X) = Var(2Y|\sigma(X)) = Var(2Y|\mathcal{F}) = 2^2 Var(Y|\mathcal{F})$

$Var(Y|2X) = Var(Y|\sigma(2X)) = Var(Y|\mathcal{F}) = Var(Y|\mathcal{F})$

$\sigma(X) = \sigma(2X) = \mathcal{F}$

Int: same information

form $\sigma(X)$ - smallest σ-algebra that contains all events of the form $\{X \le t\}$

$\{X \le 5\}$ $\{X \le 20\}$ $\{X \le \sqrt{2}\}$ ....

$\{2X \le 5\}$ $\{2X \le 10\}$ $\{2X \le 20\}$ ...

$\subseteq \Omega$

def. Borel σ-algebra $(\mathcal{B})$ -

- smallest σ-algebra that contains all subsets of real line $\mathbb{R}$ of the form $(-\infty; t]$.

More Properties.

* if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then

$$E\left(E(Y|\mathcal{F}_1)\,|\,\mathcal{F}_2\right) = E\left(E(Y|\mathcal{F}_2)\,|\,\mathcal{F}_1\right) = E(Y|\mathcal{F}_1)$$

Int v

$Var(c) = 0$

* if Y is indep of $\mathcal{F}$ then

$$Var(Y|\mathcal{F}) = Var(Y)$$

* if Y is measurable wrt $\mathcal{F}$ then

$$Var(Y|\mathcal{F}) = 0$$

$$= E(Y^2|\mathcal{F}) - (E(Y|\mathcal{F}))^2 =$$

„take out what is known"

$$= Y^2 - (Y)^2 = 0$$

particular case

* $$E(E(Y|\mathcal{F})) = E(Y)$$

$\mathcal{F}_1 = \{\emptyset, \Omega\}$

„I know nothing"

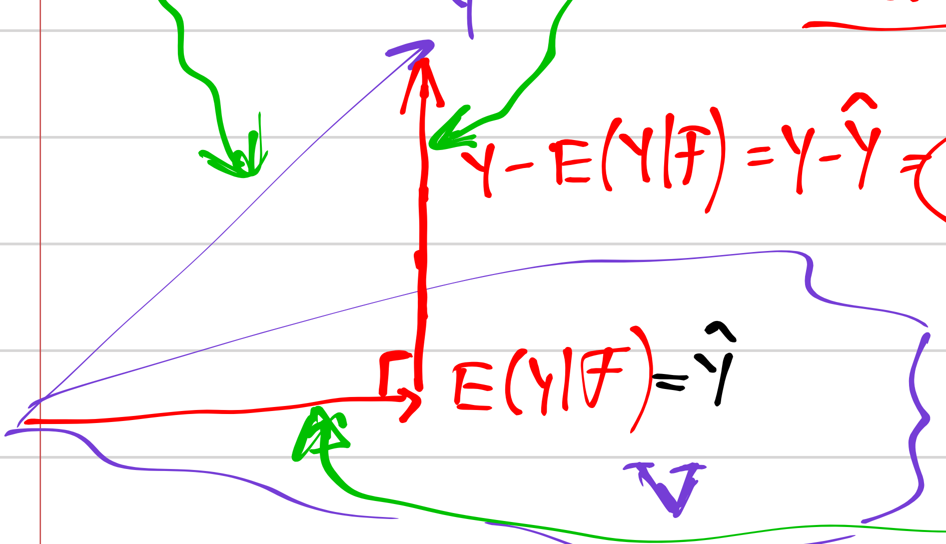
intuition

• Pythagorean theorem.

$$Var(Y) = E(Var(Y|\mathcal{F})) + Var(E(Y|\mathcal{F}))$$

$$Var(Y) = Var(Y - E(Y|\mathcal{F})) + Var(E(Y|\mathcal{F}))$$

is indep. of everything you know



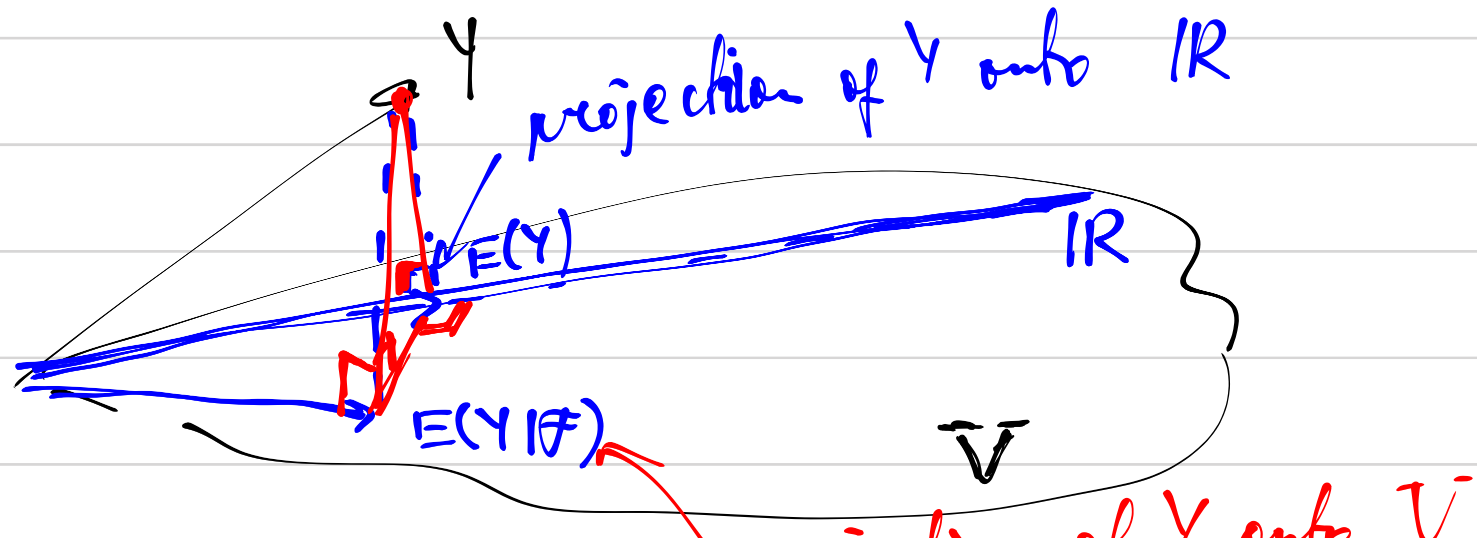
$V = \{R \mid R \text{ is measur. wrt } \mathcal{F}\}$

$MSE(Y-\hat{Y}) = E((Y-\hat{Y})^2)$ is lowest

Idea: $E(Y|\mathcal{F})$ - projection of Y onto the set of all R.V. measurable wrt $\mathcal{F}$.

$$E\big(\underbrace{E(Y|\mathcal{F})}\big)=\underline{E(Y)}$$

„Tower property"
Теор о 3-х перпенд.



$V=\{R \mid R$ is meas. wrt $\mathcal{F}\}$

Ex. Maria picked up 100 mushrooms. [independent]

„L"	„M"	„R"
0,2	0,3	0,5

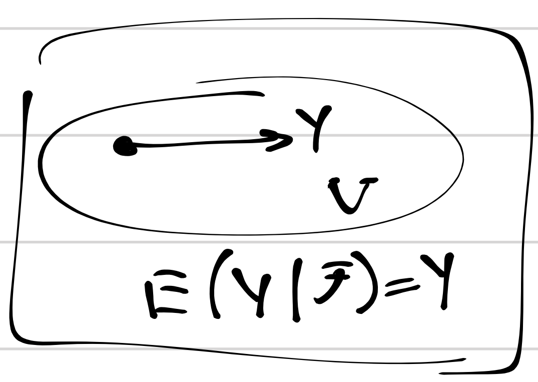
random variables:
- L - number of mushr. of type „L"
- M — // — „M"
- R — // — „R"

a) $E(R+L|M)$? $E(M|R+L)$?
b) $Var(R|L)$? $E(R|L)$?
c) $P(R=0|L)$? $P(E(R|L)=0)$?
d) $Var(R+L|M)$?

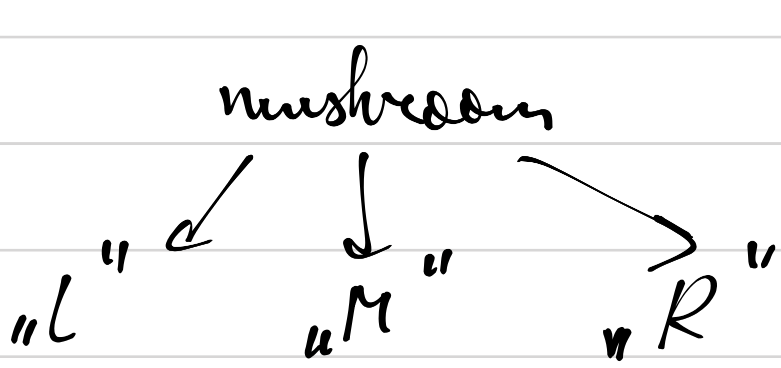
...

$E(R+L|M)\overset{?}{=}100-M$!!

$E(M|R+L)=100-(R+L)$!!



b)



consider p. case $L=10$

$(R|L=10) \overset{?}{\sim}$ $\sim Bin\left(n=90,\ p=\frac{0.5}{0.8}\right)$

90 unknown mushrooms

$\frac{0.3}{0.8} + \frac{0.5}{0.8} = 1$

M R

$0.2 \quad 0.3 \quad 0.5 = 1$ sum

L M R

$$E(R|L=10) = 90 \cdot \frac{0.5}{0.8}$$
$$Var(R|L=10) = 90 \cdot \frac{0.5}{0.8} \cdot \left(1-\frac{0.5}{0.8}\right)$$

$N \sim Bin(n,p)$
$E(N) = np$
$Var(N) = np(1-p)$

arbitrary case L

$$(R|L) \sim Bin\left(n=100-L,\ p=\frac{0.5}{0.8}\right)$$
$$Var(R|L) = (100-L)\cdot\frac{5}{8}\cdot\left(1-\frac{5}{8}\right)$$
$$E(R|L) = (100-L)\cdot\frac{5}{8}$$

c) $P(R=0|L)$? $P(E(R|L)=0)$?

d) $Var(R+L|M)$?

$$(R|L) \sim Bin\left(n=100-L,\ p=\frac{5}{8}\right)$$

$$P(R=0|L) = \left(\frac{3}{8}\right)^{100-L}$$

$$P(E(R|L)=0) = P\left((100-L)\cdot\frac{5}{8}=0\right) =$$
$$= P(L=100) = (0.2)^{100}$$

cont. 11:10

d) $Var(R+L|M) \overset{?}{=} 0$

$(R+L)$ is measur. wrt. $\sigma(M)$

$\boxed{\text{W2.B}}$ [discrete time] $t \in \{0,1,2,\dots\}$
$t \in \{1,2,3,\dots\}$

def: the sequence of σ-algebras $(\mathcal{F}_t)$ is called filtration if $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ for $\forall t$

int: you know more and more as time goes on

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \mathcal{F}_4 \subseteq \dots$$

def. (Y_t) is a random process adapted to filtration $(\mathcal{F}_t)$ if $\forall t$ Y_t is measurable wrt $\mathcal{F}_t$.

def. (Y_t) is a martingale wrt filtration $(\mathcal{F}_t)$ if

$$E(Y_{t+1} \mid \mathcal{F}_t) = Y_t$$

int: the best forecast of the future value is the current value

def $\boxed{(Y_t) \text{ is a martingale}} \iff$ (Y_t) is a martingale wrt natural filtration $\mathcal{F}_t = \sigma(Y_1, Y_2, \dots Y_t)$.

$$E(Y_{t+1} \mid Y_1, Y_2, \dots Y_t) = Y_t$$

ex.

$X_1, X_2, \dots \sim$ iid

independent

ident. distributed

ξ	-1	$+1$
$P(X_t=\xi)$	$0{,}3$	$0{,}7$

$S_t = X_1 + X_2 + \dots + X_t$

$S_0 = 0$

$\mathcal{F}_t = \sigma(X_1, X_2, \dots X_t)$

a) is (X_t) a mart.le wrt $(\mathcal{F}_t)$?

b) is (S_t) a martingale wrt $(\mathcal{F}_t)$?

c) find α such that $Z_t = S_t + \alpha t$ is a martingale?

d) find β —//— $Y_t = \exp(\beta \cdot S_t)$ is a martingale?

a) $E(X_{t+1} | \mathcal{F}_t) \overset{?}{=} X_t$

$E(X_{t+1} | X_1, X_2, \dots X_t) \overset{?}{=} X_t$

indep.

$\boxed{E(M_{t+h} | \mathcal{F}_t) = M_t}$

$E(X_{t+1}) \overset{?}{=} X_t$

const r.v.

$E(X_{t+1}) = +1 \cdot 0{,}7 - 1 \cdot 0{,}3 = 0{,}4 \neq X_t$

(X_t) is not a martingale

b) (S_t) is a martingale? no

$E(S_{t+1} | \mathcal{F}_t) = E(S_{t+1} | X_1, \dots X_t) =$

$= E(X_1 + X_2 + \dots + X_t + X_{t+1} | X_1, X_2, \dots X_t) =$

known r.v.s

$S_t + 0{,}4$

$= X_1 + X_2 + \dots + X_t + E(X_{t+1} | X_1 \dots X_t) = \boxed{X_1 + X_2 + \dots + X_t + 0{,}4 \neq S_t}$

c) $Z_t = S_t + \alpha \cdot t$ $\quad \alpha?$ $\quad Z_t$ - is martingale.

$$E(Z_{t+1} | \mathcal{F}_t) = Z_t$$

$$E(S_{t+1} + \alpha(t+1) | \mathcal{F}_t) = Z_t$$

$$\boxed{X_1 + \ldots + X_t + 0{,}4} + \alpha(t+1) = S_t + \alpha \cdot t$$

$$S_t + 0{,}4 + \alpha t + \alpha = S_t + \alpha t$$

$$\alpha = -0{,}4.$$

$Z_t = X_1 + X_2 \ldots + X_t - 0{,}4 \cdot t$ is a mart.

$E(Z_{t+1} | \mathcal{F}_t) = Z_t$

Y_t is a r. process

(Y_t) is a r. process

d) $Y_t = \exp(\beta \cdot S_t)$ $\quad$ /for $\forall t$/

β? $\qquad E(Y_{t+1} | \mathcal{F}_t) = Y_t$ $\qquad \beta_1, \beta_2$

$\beta_1 = 0$ $\qquad Y_t = \exp(0\, S_t) = 1$ $\qquad (Y_t)_{t=1}^{\infty}$ is a r. proc..

$$E(1 | \mathcal{F}_t) = 1$$

$$E(\exp(\beta S_{t+1}) | \mathcal{F}_t) = \exp(\beta S_t)$$

$S_{t+1} = S_t + X_{t+1}$ $\qquad X_1, X_2 \ldots X_t$

$$E(\exp(\beta S_t + \beta X_{t+1}) | \mathcal{F}_t) =$$

$$= E(\exp(\beta S_t) \cdot \exp(\beta X_{t+1}) | \mathcal{F}_t) =$$

take out of E what is known

$$= \exp(\beta S_t) \cdot E[\exp(\beta X_{t+1}) | X_1, X_2 \ldots X_t] =$$

independency: omit irr. info

$$= \exp(\beta S_t) \cdot E[\exp(\beta X_{t+1})] \qquad \stackrel{RHS}{=} \exp(\beta S_t)$$

это „мю"

$$E\left(\exp(\beta X_{t+1})\right) = 1$$

β	-1	$+1$
$P(X_{t+1}=\beta)$	$0{,}3$	$0{,}7$

$$0{,}3\cdot\exp(-\beta)+0{,}7\exp(+\beta)=1$$

$$\exp(\beta)=t$$

$$0{,}3\cdot\frac{1}{t}+0{,}7\cdot t=1$$

$$0{,}7t^2-t+0{,}3=0$$

$$t_1=1 \qquad t_2=\frac{3}{7}$$

$$\beta_1=\ln 1 \qquad \beta_2=\ln\frac{3}{7}$$

$$\beta_1=0 \qquad \beta_2=\ln 3-\ln 7\approx -0{,}85$$

$Y_t=\exp(\beta S_t)$ is a martingale

def τ – r.v. is called a stopping time (момент ост-ки) wrt filtration $(\mathcal{F}_t)$

if (intuit) we have enough info to understand whether τ has occured. (for all t)

[you always can check whether τ is now]

(formally)

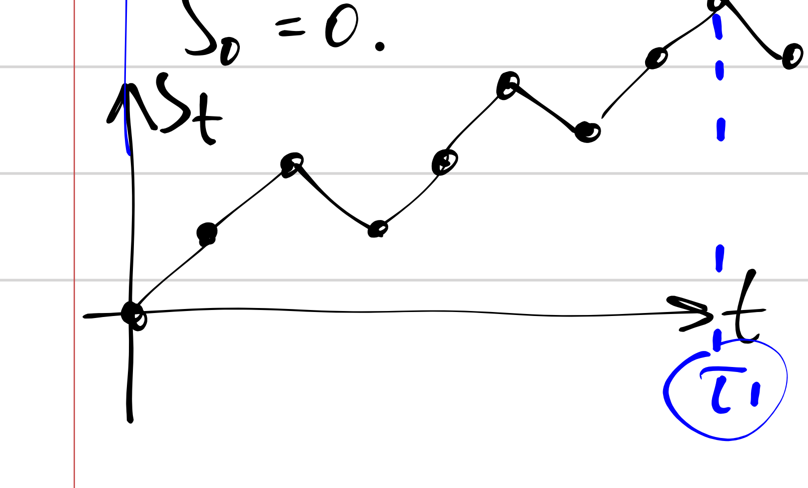
$$(\tau\le t)\in\mathcal{F}_t \text{ for } \forall t$$

example: $(X_t)\sim$ iid $\quad P(X_t=1)=0{,}7$
$\quad P(X_t=-1)=0{,}3.$

10

$S_t=X_1+\ldots+X_t$

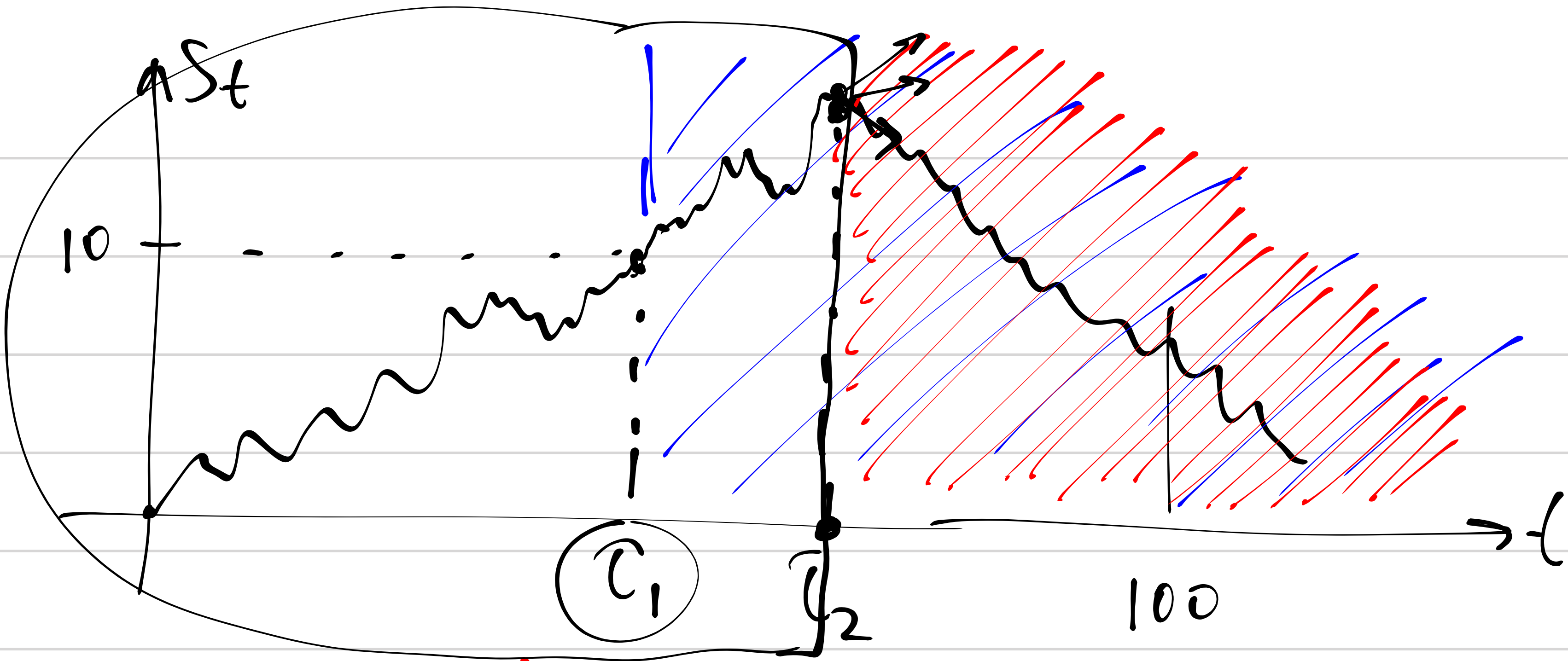
$S_0=0.$



$$\mathcal{F}_t=\sigma(X_1,\ldots,X_t)$$

$$\tau_1=\min\{t \mid S_t=10\}$$

$$\tau_2=\min\{t \mid S_t=\max\{S_1\ldots S_{100}\}\}$$



τ_1 - stopping time
τ_2 - is not a stopping time

$(\tau \le t) \in \underline{\mathcal{F}_t}$ for $\forall t$ def

$$\{\tau = 3\} = \underset{\mathcal{F}_3}{\underset{\cap}{\{\tau \le 3\}}} \setminus \underset{\mathcal{F}_2 \subseteq \mathcal{F}_3}{\underset{\cap}{\{\tau \le 2\}}} \in \mathcal{F}_3$$

int

$\{\tau = 3\} \in \mathcal{F}_3 \longleftrightarrow$ At time $t=3$ we know whether $\tau = 3$

Doob's stopping time theorem

If $\left\{\begin{array}{l}(M_t) \text{ is a martingale wrt } (\mathcal{F}_t) \\ \tau \text{ is stopping time wrt } (\mathcal{F}_t)\end{array}\right\}$ int.

and at least one of the condition (A) and (B) holds $\qquad Y_t = M_{\min(t,\tau)}$

tech. $\left\{\begin{array}{l}\text{(A)} \quad P(\tau = +\infty) = 0 \text{ and } \exists c : \quad |Y_t| < c \\ \text{(B)} \quad E(\tau) < \infty \text{ and } \exists c : \\ \qquad |E(Y_{t+1} - Y_t \mid \mathcal{F}_t)| < c\end{array}\right.$

then

$$E(M_\tau) = E(M_1)$$

Mart Stol.

ex. X_t - are indep. $P(X_t=+1)=P(X_t=-1)=\frac{1}{2}$

$S_t = X_1 + \dots + X_t$

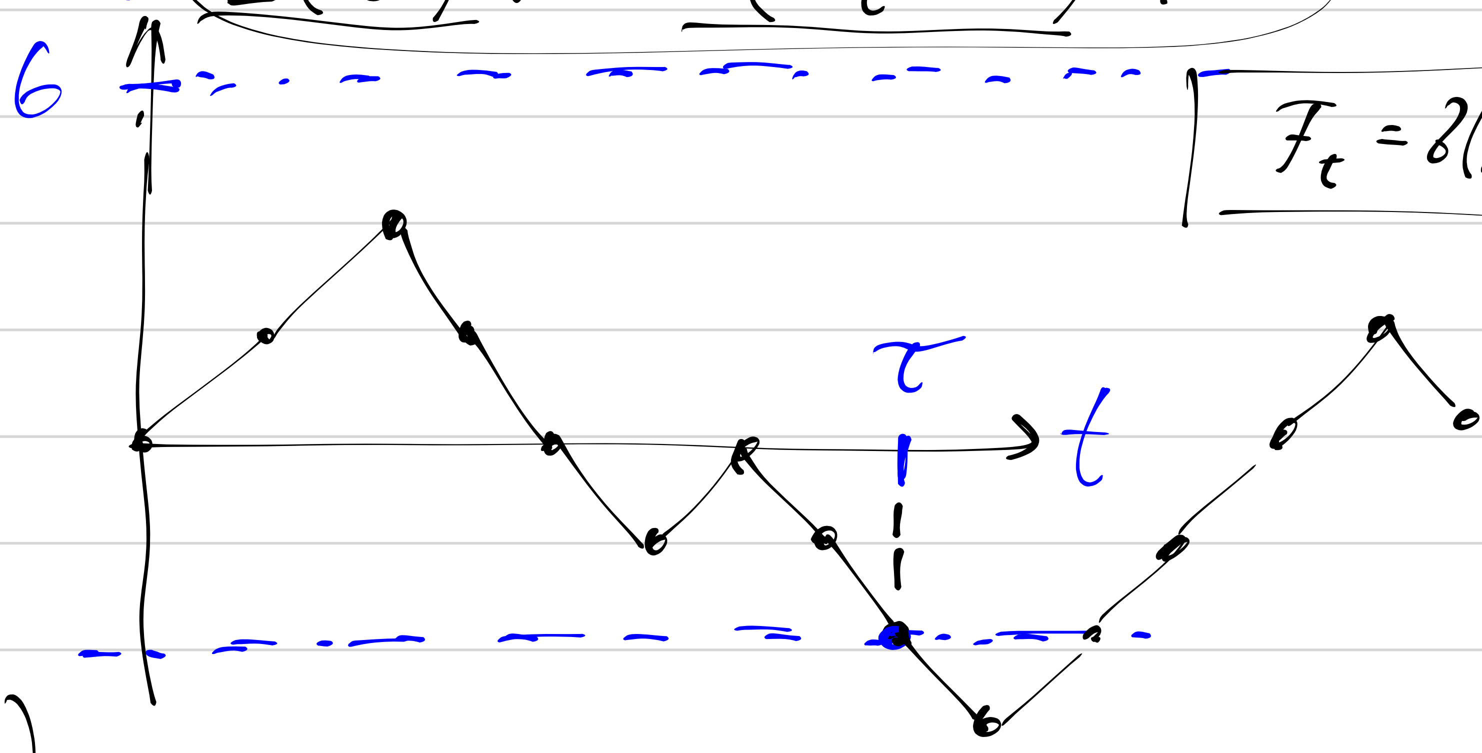
a) S_t - mart $E(S_{t+1} \mid \mathcal{F}_t) = S_t$

S_t - is not stat. $Var(S_t) = t \cdot Var(X_1) \neq$ $\neq$ const

b) is $R_t = S_t^2 - t$ a mart?

c) $\tau = \min\{t \mid S_t = -2 \text{ or } S_t = 6\}$

$E(\tau)$? $P(S_\tau = 6)$?



$\mathcal{F}_t = \sigma(X_1 \dots X_t)$

b)

$$E(R_{t+1} \mid \mathcal{F}_t) \overset{?}{=} E(S_{t+1}^2 - (t+1) \mid \mathcal{F}_t) =$$

$$= E((S_t + X_{t+1})^2 - (t+1) \mid \mathcal{F}_t) =$$

$$= E(S_t^2 + X_{t+1}^2 + 2S_t X_{t+1} \mid \mathcal{F}_t) - t - 1 =$$

$$= S_t^2 + 2S_t \cdot E(X_{t+1} \mid \mathcal{F}_t) + E(\underbrace{X_{t+1}^2}_{1} \mid \mathcal{F}_t) - t - 1 =$$

$\to 1$

$X_{t+1} = \pm 1$

$X_{t+1}^2 = 1$

$$= S_t^2 + 2S_t \cdot E(X_{t+1}) + 1 - t - 1 =$$

$$= S_t^2 + 2S_t \cdot 0 + 1 - t - 1 = S_t^2 - t = \boxed{R_t}$$

R_t - mart

S_t - mart.

$$\begin{cases} Y_t = S_{\min(t,\tau)} & \text{cond } A \text{ is satisf.} \\ Y_t = R_{\min(t,\tau)} & \text{cond } B \text{ is satisf.} \end{cases}$$

If S_t is mart, τ stopp time + [tech]
then $E(S_\tau) = E(S_0) = 0$

If R_t is mart, τ stopp time + [tech]
then $E(R_\tau) = E(R_0) = 0$

$$\tau = \min\{t \mid S_t = -2 \text{ or } S_t = 6\}$$

$$S_\tau = -2 \text{ or } S_\tau = 6$$

u	-2	6
$P(S_\tau = u)$	$1-p$	p

$$6 \cdot p + (-2) \cdot (1-p) = 0 \qquad E(S_\tau) = 0.$$

$$p = \frac{2}{8} \quad !!$$

$$P(S_\tau = 6) = \frac{2}{8} \qquad P(S_\tau = -2) = \frac{6}{8}$$

$$E(S_\tau^2) = E(\tau) \qquad E(R_\tau) = 0$$

$$E(S_t^2 - \tau) = 0$$

$$36 \cdot \frac{2}{8} + 4 \cdot \frac{6}{8} = E(\tau)$$

$$E(\tau) = 12$$

Free slot 3

Any Q?

Stopping time. [(Intuit)]

You look outside ☺

not a st. time → $\tau_1 = 10$ min. before the kitten will run away.

stopping time $\tau_2 = 10$ min after the kitten will run away.

int „ you can say „Stop" exactly at the moment τ"

form $\{\tau \le t\} \in \mathcal{F}_t$ for $\forall t$

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let's pass !! M

№2
$$X_0 = 0.2021$$
$$\begin{cases} Y_t = \begin{cases} 1 \text{ with } P(Y_t = 1 \mid \mathcal{F}_{t-1}) = X_{t-1} \\ 0 \text{ with } P(Y_t = 0 \mid \mathcal{F}_{t-1}) = 1 - X_{t-1} \end{cases} \\ X_t = \dfrac{Y_t + X_{t-1}}{2} \end{cases}$$

$$\mathcal{F}_t = \sigma(X_1, X_2, \dots X_t).$$

a) (X_t) is a martingale?

б) $X_t \xrightarrow{as} X_\infty$ distribution of X?

$$E(X_{t+1} \mid \mathcal{F}_t) = E\left(\frac{Y_{t+1} + X_t}{2} \,\Big|\, \mathcal{F}_t\right) =$$

known

$$\boxed{\mathcal{F}_t = \sigma(X_1, \dots, X_t)}$$

$$= \frac{X_t}{2} + \frac{1}{2} E(Y_{t+1} \mid \mathcal{F}_t) = \frac{X_t}{2} + \frac{1}{2}\big(1 \cdot X_t + 0 \cdot (1 - X_t)\big) = X_t$$

Y_{t+1} Bernoulli

X_t → 1, $1 - X_t$ → 0

6) $\boxed{X_t \xrightarrow{as} X_\infty}$

$$P(\lim X_t = X_\infty) = 1$$

(5) $\boxed{X_t} = \boxed{\frac{Y_t + X_{t-1}}{2}}$ $\forall t$ $Y_t = 2X_t - X_{t-1}$

(6) $P(\lim LHS = \lim RHS) = 1$

(7) $$X_\infty = \frac{Y_\infty + X_\infty}{2}$$

Bernoulli (Y_t)

(1) $X_\infty = Y_\infty \Rightarrow X_\infty \in \{0, 1\}$

(2) $E(X_{t+1} \mid \mathcal{F}_t) = X_t$ def (mart)

(3) $E(E(X_{t+1} \mid \mathcal{F}_t)) = E(X_t)$

(4) $(\forall t)$ $E(X_{t+1}) = E(X_t)$ necess $X_t \in [0; 1]$

(5) $E(X_\infty) = E(X_0) = 0.2021$

(6) $$X_\infty = \begin{cases} 1 & \text{with } 0.2021 \\ 0 & \text{wit } 0.7979 \end{cases}$$

$X_0 = 0.2021$

$X_1 = \frac{Y_1 + X_0}{2}$

0.2021 $\swarrow$ $\searrow$ 0.7979

$Y_1 = 1$

$X_1 = \frac{1 + 0.2021}{2} = 0.60105$

0.60 $\swarrow$ $\searrow$ 0.40

$Y_2 = 1$ $\quad$ $Y_2 = 0$

$Y_1 = 0$

$X_1 = \frac{0.2021}{2} = 0.10105$

0.1 $\swarrow$ $\searrow$ 0.9

$Y_2 = 1$ $\quad$ $Y_2 = 0$

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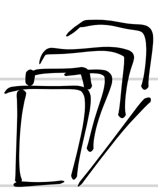
4)



52 cards

$$X_i = \begin{cases} 1 & \text{if card n } i \text{ is } Q \\ 0 & \text{otherwise.} \end{cases}$$

$Y_n = E(X_{52} | X_1, \dots X_n)$ $(\forall n)$ $\quad n \in \{1, \dots 52\}$

a) is Y_n a martingale?

b) $Cov(Y_{50}, Y_{51})$

$$Y_1 = E(X_{52} | X_1)$$
$$Y_2 = E(X_{52} | X_1, X_2)$$

$Y_{51} \in \{0; 1\}$

$$Y_{51} = E(X_{52} | X_1, X_2, \dots X_{51}) =$$
$$= X_{52} = 4 - X_1 - X_2 - \dots - X_{51}$$

$Cov(Y_{51}, Y_{50}) =$

$= E(Y_{51} \cdot Y_{50}) - E(Y_{50}) \cdot E(Y_{51})$ $\quad \frac{4}{52}$ $\quad \frac{4}{52}$

$$X_{52} = 4 - X_1 - X_2 - \dots X_{51}$$

$$Y_{50} = E(X_{52} | X_1, X_2 \dots\dots X_{50}).$$

$$Y_{50} = \begin{cases} 0 & \text{if } X_1 + \dots + X_{50} = 4 \\ 1 & \text{if } X_1 + \dots + X_{50} = 2 \\ \frac{1}{2} & \text{if } X_1 + \dots + X_{50} = 3 \end{cases}$$

	$Y_{51}=0$	$Y_{51}=1$
$Y_{50}=0$	a $\frac{48\cdot 47}{52\cdot 51}$	b 0
$Y_{50}=1/2$	c $\frac{4\cdot 48}{52\cdot 51}$	d $\frac{4\cdot 48}{52\cdot 51}$
$Y_{50}=1$	f 0	g $\frac{4\cdot 3}{52\cdot 51}$

$f=0$

$g = [\ \ |Q|Q] = \frac{4}{52}\cdot\frac{3}{51}$

$a = P(Y_{50}=0,\ Y_{51}=0) = P($ last two cards are not $Q) =$ correct $= \frac{48}{52}\cdot\frac{47}{51}$

$48\cdot 47 + 2\cdot 4\cdot 48 + 4\cdot 3 = 52\cdot 51$

$b=0$

$c = \frac{48}{52}\cdot\frac{4}{51}$

$d = \frac{4}{52}\cdot\frac{48}{51}$

$E(Y_{51}) = ?$ $E(Y_{50}) = ?$ ...

$Cov(Y_{50}, Y_{51}) = ?$

$Y_0 = \frac{4}{52}$

a) check Y_t is a martingale?

$S_t = X_1 + \ldots + X_t$ ← number of open Q

$4 - S_t$ ← number of closed Q

$E(Y_{t+1} \mid \mathcal{F}_t) =$

$Y_t = \frac{4-S_t}{52-t}$

$= E\left(\frac{4-S_{t+1}}{52-(t+1)} \,\Big|\, \mathcal{F}_t\right) = \frac{1}{51-t}\cdot\left(4 - E(S_{t+1} \mid \mathcal{F}_t)\right) =$

$$= \frac{1}{51-t}\left(4 - E(S_{t+1} \mid \mathcal{F}_t)\right) =$$

S_{t+1} = number of open Q

$$S_{t+1} = S_t + X_{t+1}$$

$$E(S_{t+1} \mid \mathcal{F}_t) = E(S_t + X_{t+1} \mid \mathcal{F}_t) =$$

$$= S_t + E(X_{t+1} \mid \mathcal{F}_t) = S_t + \frac{4-S_t}{52-t}$$

$$= \frac{1}{51-t}\left[4 - S_t - \frac{4-S_t}{52-t}\right] =$$

$$= \frac{1}{51-t}\left[(4-S_t)\cdot\left(1 - \frac{1}{52-t}\right)\right] =$$

$$= \frac{1}{51-t}(4-S_t)\cdot\frac{52-t-1}{52-t} = \frac{4-S_t}{52-t} = Y_t$$

$$E(Y_{t+1} \mid \mathcal{F}_t) = Y_t \quad \text{(a martingale).}$$

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faster sol.

$$Y_t = E(X_{52} \mid \mathcal{F}_t)$$

$$Y_{t+1} = E(X_{52} \mid \mathcal{F}_{t+1})$$

$$E(Y_{t+1} \mid \mathcal{F}_t) = E\left(E(X_{52} \mid \mathcal{F}_{t+1}) \mid \mathcal{F}_t\right) =$$

(by property)

$$= E(X_{52} \mid \mathcal{F}_t) = Y_t \quad \text{(mart)}$$