Submission details

Submit you home assignments at `https://www.gradescope.com` with coursecode: NXE75N.
Deadline HA1: 13 December 2022, 23:59
Deadline HA2: 18 December 2022, 23:59
You have one honey-day. The honey-day allows you to postpone one of the two deadlines by 24 hours.

1 HA

1. Some questions about σ-algebras.

   (a) You observe the result of 10 independent coin tosses. How many elements the σ-algebra of your information contains?

   (b) Prove that a finite σ-algebra can contain only 2^k elements.

   (c) Is union of two σ-algebras always a σ-algebra? Prove your statement.

   (d) Is intersection of two σ-algebras always a σ-algebra? Prove your statement.

2. Prove the following statement or provide a counter-example. For any two σ-algebras $\mathcal{F}$ and $\mathcal{H}$ and a random variable Y

$$\mathrm{E}(\mathrm{E}(Y|\mathcal{F})|\mathcal{H}) = \mathrm{E}(Y|\mathcal{F} \cap \mathcal{H})$$

3. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y.

   (a) Find $\mathbb{P}(Y = y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$;

   (b) Find $\mathrm{E}(X|Y)$.

4. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now.

   Find $\mathbb{P}(Y = y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$.

5. Random variables X and Y have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

   (a) Find $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(XY|X)$ and $\mathrm{Var}(XY|X)$.

   (b) Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2021|X)$.

6. The random variables $Z_1, Z_2, \ldots$ are independent and identically distributed with $\mathbb{P}(Z_n = 1) = p$ and $\mathbb{P}(Z_n = -1) = 1 - p$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.

   (a) For which value of p the process 2^{S_n} will be a martingale?

   (b) Let $p = 0.3$. If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale.

2 HAHA

1. The processes (X_t) and (Y_t) are independent Wiener processes with respect to filtration $(\mathcal{F}_t)$. The process $Z_t = aX_t + bY_t$ is also a Wiener process.

   (a) For which values of constants a and b is it possible?

   (b) Find correlation $\mathrm{Corr}(Z_t, X_t)$.

   (c) Find $\mathrm{E}(Z_3|X_2)$ and $\mathrm{Var}(Z_3|X_2)$.

   (d) Find $\mathrm{E}(Z_3|\mathcal{F}_2)$ and $\mathrm{Var}(Z_3|\mathcal{F}_2)$.

2. The process $C_t = W_t^3 + aW_t^2 + bW_t + c + d \cdot t \cdot W_t$ is a martingale.

   (a) For which values of constants a, b, c and d is it possible?

   (b) Find covariance $\mathrm{Cov}(C_t, \int W_u^2 dW_u)$.

3. Consider the framework of Black and Scholes model: S_t is the share price. Derive the current price of two European type assets, X_0 and Y_0.

   Future payoffs are given by:

   (a) $X_T = (S_T - K)^3$ where T and K are fixed in the contract.

   (b) $Y_T = S_T^{-2}$ where T is fixed in the contract.

4. Let $Y_t = W_t + 4t$. The moment τ is the first moment when Wiener process hits 10.

   (a) Let α be a constant. Find the function $f(t)$ such that $M_t = f(t)\exp(\alpha Y_t)$ is a martingale.

   (b) Using Doob's theorem find $\mathrm{E}(\exp(-s\tau))$ for arbitrary constant s.

5. Solve the stochatic differential equation

$$dY_t = -Y_t dt + dW_t,\ Y_0 = 1$$

   If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitely. It's ok.

6. Solve the stochatic differential equation

$$dY_t = Y_t dt + (t^3 + 4Y_t)dW_t,\ Y_0 = 1$$

   If you are have no clues you may try to represent the process as $Y_t = A_t B_t$, where A_t is the solution of the equation $dA_t = A_t dt + 4A_t dW_t$.