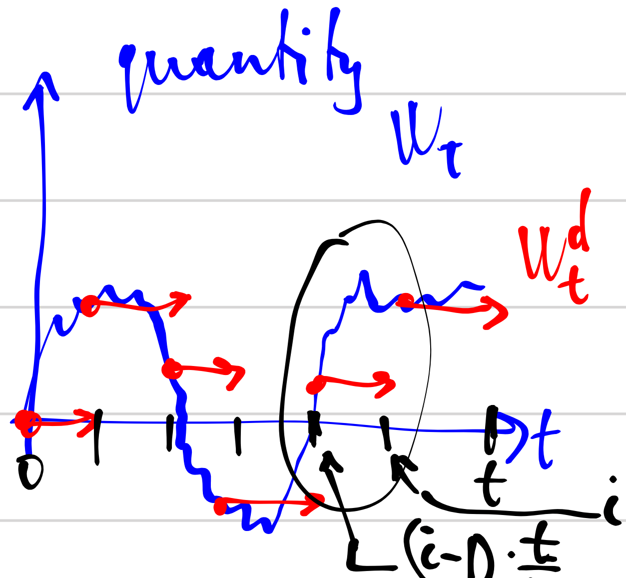
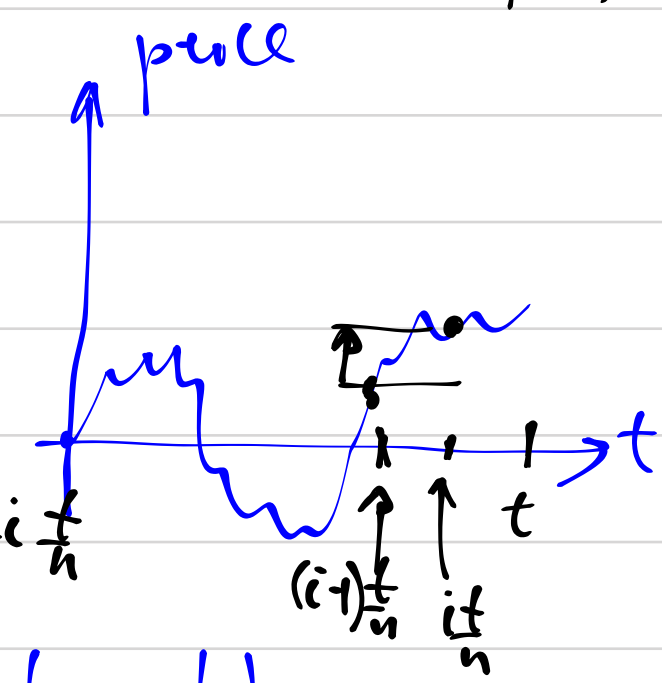
Week 5

goal $I = \int_0^t W_u \, dW_u = ?$

f

$f(x)$





approx $I_n = \int_0^t W_u^d \cdot dW_u =$ (last week) $=$

$$= \sum_{i=1}^{n} \underbrace{W\left(\frac{(i-1)t}{n}\right)}_{\text{quant.}} \cdot \underbrace{\left(W\left(\frac{it}{n}\right) - W\left(\frac{(i-1)t}{n}\right)\right)}_{\text{price change}}$$

$$I_n \xrightarrow[n\to\infty]{L^2} ?$$

fact. (last lecture)

$$J_n = \sum_{i=1}^{n} \left(W\left(\frac{it}{n}\right) - W\left(\frac{(i-1)t}{n}\right)\right)^2 \xrightarrow[n\to\infty]{L^2} t$$

$$\sum_{i=1}^{n} \Delta_i^2 \xrightarrow[n\to\infty]{L^2} t \qquad \Delta^2$$

[idea of the proof: $E\left(\sum \Delta_i^2\right) = t$, $Var\left(\sum \Delta_i^2\right) = v(n) \to 0$]

$$R_n \xrightarrow{L^2} R \qquad \left\{ \lim_{n\to\infty} E\left(|R_n - R|^2\right) = 0 \right.$$

$$V_i = W\left(\frac{it}{n}\right)$$

$W_{(2)}$ → $W(2)$, → $W\left(2\frac{t}{n}\right)$

$$I_n = \sum V_{i-1}\cdot(V_i - V_{i-1}) = \sum V_i V_{i-1} - \sum V_{i-1}^2$$

$$J_n = \sum (V_i - V_{i-1})^2 = \underbrace{\sum V_i^2} + \underbrace{\sum V_{i-1}^2} - 2\sum V_i V_{i-1} =$$

?

$$V_1^2 + V_2^2 + \dots V_n^2 \qquad V_0^2 + V_1^2 + \dots + V_{n-1}^2$$

$$W\left(\frac{0t}{n}\right) = 0$$

$$V_n = W\left(\frac{n t}{n}\right) = W(t)$$

$$J_n = 2\sum V_i^2 - W^2(t) - 2\sum V_i V_{i-1}$$

$$I_n = -\sum V_i^2 + W^2(t) + \sum V_i V_{i-1}$$

$$I_n = \frac{J_n}{-2} + \frac{W^2(t)}{2}$$

$$I_n \xrightarrow[n\to\infty]{L^2} \frac{t}{-2} + \frac{W^2(t)}{2} \overset{def}{=} \int_0^t W_u \, dW_u$$

!!

$$\int_0^t W_u \, dW_u = \frac{W^2(t) - t}{2}$$

! Old rules of integr-n are not applicable!

$$\int_0^t \sin u \, d\sin(u) = \frac{\sin^2(u)}{2}\Bigg|_{u=0}^{u=t} = \frac{\sin^2(t)}{2}$$

Only a few stoch. integrals can be calculated explicitely.

$\boxed{\text{Integration table}}$

(1) $\int_0^t 1 \, dW_u = W_t$

(2) $\int_0^t W_u \, dW_u = \dfrac{W_t^2 - t}{2}$

Properties of st. integrals.

[technical conditions]

usually : (W_t) is adapted to $(\mathcal{F}_t)$
(A_t) is adapted to $(\mathcal{F}_t)$
$\int_0^t E(A_u^2)\,du$ is finite

Provided (tech. cond)

Linearity: $\int_0^t A_u + B_u \, dW_u = \int_0^t A_u \, dW_u + \int_0^t B_u \, dW_u$

Martingale $I_t = \int_0^t A_u \, dW_u$ — martingale

$$E\left(\int_0^{t+s} A_u \, dW_u \,\middle|\, \mathcal{F}_t \right) = \int_0^t A_u \, dW_u \qquad t, s \geq 0$$

$$E\left(\int_0^{t+s} A_u \, dW_u \right) = E\left(\int_0^t A_u \, dW_u \right).$$

$t=0$

$$E\left(\int_0^s A_u \, dW_u \right) = E(0) = 0$$

Ito's isometry

Step 1. $\int_0^t A_u \, dW_u$ replace by $\int_0^t A_u^d \, dW_u$

Step 2 take lim

$$\mathrm{Var}\left(\int_0^t A_u \, dW_u \right) = \int_0^t E(A_u^2)\, du$$

$$\mathrm{Cov}\left(\int_0^t A_u \, dW_u, \int_0^t B_u \, dW_u \right) = \int_0^t E(A_u B_u)\, du$$

idea: $E((\Delta W)^2) = \Delta t$

Old property

$$E\left(\int_0^t A_u \, du \right) = \int_0^t E(A_u)\, du$$

$$E\left((W_t - W_s)^2 \right) = t - s$$

Ex.

$$I_t = \int_0^t W_u^2 u^2 \, dW_u$$

- a) $E(I_t) = 0$ (property !!)
- b) $Var(I_t) = \int_0^t E(W_u^4 u^4) \, du = \int_0^t 3u^6 \, du = \frac{3t^7}{7}$
- c) $Cov(I_t, W_t) =$

$$E(W_u^4) \overset{?}{=} E(X^4 \cdot u^2) = 3u^2 \qquad X \sim N(0;1)$$

$W_u \sim N(0;u)$

$\frac{W_u}{\sqrt{u}} \sim N(0;1)$

Stein's lemma

If $X \sim N(0;1)$ and h does not grow too fast $\left[\lim_{t \to \pm\infty} h(t)/\exp(t^2/2) = 0\right]$ then

$$E(X \cdot h(X)) = E(h'(X))$$

Ex.

$X \sim N(0;1)$ $\qquad E(X^2) = \rightarrow 1 = Var(X) + E(X)^2$ Pythagorean th.

Stein's lemma

$$E(X^2) = E(X \cdot X) = E(1) = 1$$

$$E(X^3) = E(X \cdot X^2) = E(2X) = E(X \cdot 2) = E(0) = 0$$

$$E(X^4) = E(X \cdot X^3) = E(3X^2) = 3 \cdot 1$$

$$E(X^6) = E(X \cdot X^5) = E(5 \cdot X^4) = 5 \cdot 3 \cdot 1$$

$$E(X^{2022}) = 2021 \cdot 2019 \cdot 2017 \cdot \ldots \cdot 3 \cdot 1$$

proof

$$E(X \cdot h(X)) = \int_{-\infty}^{\infty} x \cdot \underbrace{h(x)}_{h'(x)} \cdot \overbrace{\frac{1}{\sqrt{2\pi}} \cdot \exp(-x^2/2)}^{\text{pdf}} \, dx = \left\{ \begin{array}{l} \left(-\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\right)' = \\ = x \cdot \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) \end{array} \right.$$

$$= \underbrace{h(x) \cdot (-\text{pdf}(x)) \Big|_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} h'(x) \cdot \text{pdf}(x) \, dx = E(h'(X))$$

$$h(x) \cdot (-\mathrm{pdf}(x)) \Big|_{-\infty}^{\infty} =$$

$$= h(x) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right) \Big|_{x=-\infty}^{x=+\infty} =$$

$$= \frac{h(x) / \sqrt{2\pi}}{\exp(x^2/2)} \Big|_{x=-\infty}^{x=+\infty} = 0$$

c) $\mathrm{Cov}\left(\int_0^t W_u^2 u^2 \, dW_u, \; W_t\right) = (\text{Ito's isometry}) =$

$W_t = \int_0^t 1 \, dW_u$

$$= \int_0^t E\left(W_u^2 u^2 \cdot 1\right) du = \int_0^t u^3 \, du = \frac{t^4}{4}$$

$E(W_u^2) = u$

$$W_u - W_0 = W_u \sim ? \; N(0; u)$$

Ito's lemma

def. X_t is Ito process if

$$X_t = X_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du$$

full form

const

[A_u, B_u satisfy tech. cond]

short notation: $dX_t = A_t \, dW_t + B_t \, dt$

short form

(!) dX_t is not a meaningfull object

~~$E(dX_t)$~~ ~~$\mathrm{Var}(dX_t)$~~

Ex. $X_t = 5 + \int_0^t W_u^3 \, dW_u$

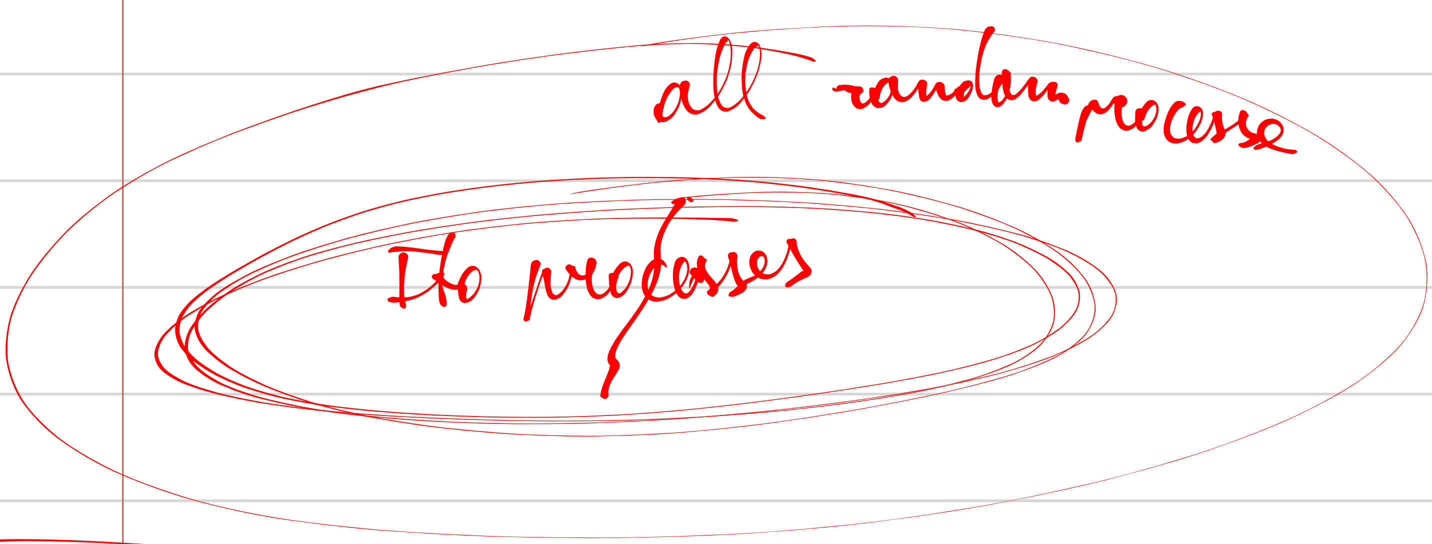
$$dX_t \overset{?}{=} W_t^3 \, dW_t$$

Ex. $dY_t = W_t^4 \, dW_t + \sin W_t \, dt$

$$Y_t = Y_0 + \int_0^t W_u^4 \, dW_u + \int_0^t \sin W_u \, du$$

Theorem Under [techn. conditions] Ito process $X_t = X_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du$ is a martingale if and only if $B_u \equiv 0$ (as).

[only if $dX_t = A_t \, dW_t$]



Ito's lemma

[the hardest point!]

If X_t is an Ito process $\left[\begin{array}{l} X_t = X_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du \\ dX_t = A_t \, dW_t + B_t \cdot dt \end{array} \right]$

and $Y_t = f(t, W_t, X_t)$ [techn. cond: f is twice differentiable]

then Y_t is an Ito process and

$$dY_t = f'_t \cdot dt + f'_w \cdot dW_t + f'_x \cdot dX_t + \frac{1}{2}\left[f''_{ww} (dW_t)^2 + 2 \cdot f''_{wx} \, dW_t \cdot dX_t + f''_{xx} \cdot (dX_t)^2 \right],$$

where the RHS is simplified according to mnemonic rule $dW_t \cdot dW_t = dt$, $dW_t \cdot dt = 0$, $dt \cdot dt = 0$.

ex 1

$$Y_t = W_t^2 = f(t, W_t, X_t)$$

$$dY_t = ?\ \underbrace{f'_t}_{=0} \cdot dt + \underbrace{f'_w}_{=2\cdot W_t} \cdot dW_t + \underbrace{f'_x}_{=0} \cdot dX_t +$$

$$+ \frac{1}{2}\left[\underbrace{f''_{ww}}_{2} \cdot (dW_t)^2 + 2\underbrace{f''_{xw}}_{=0} \cdot (dX_t)(dW_t) + \underbrace{f''_{xx}}_{=0}(dX_t)^2\right] =$$

$$= 2W_t \cdot dW_t + 1 \cdot (dW_t)^2 = 2W_t \cdot dW_t + dt$$

$$dY_t = 2W_t \cdot dW_t + dt \qquad \text{[short form]}$$

$$Y_t = Y_0 + \int_0^t 2W_u dW_u + \int_0^t 1 \cdot du \qquad \text{[full form]}$$

$$\boxed{W_t^2 = 0 + \int_0^t 2W_u\, dW_u + \underbrace{\int_0^t 1 \cdot du}_{t}}$$

$$\int_0^t 2W_u\, dW_u = W_t^2 - t$$

$$\int_0^t W_u\, dW_u = \frac{W_t^2 - t}{2}$$

Ex BS-model S_t - share price

$$dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dW_t$$

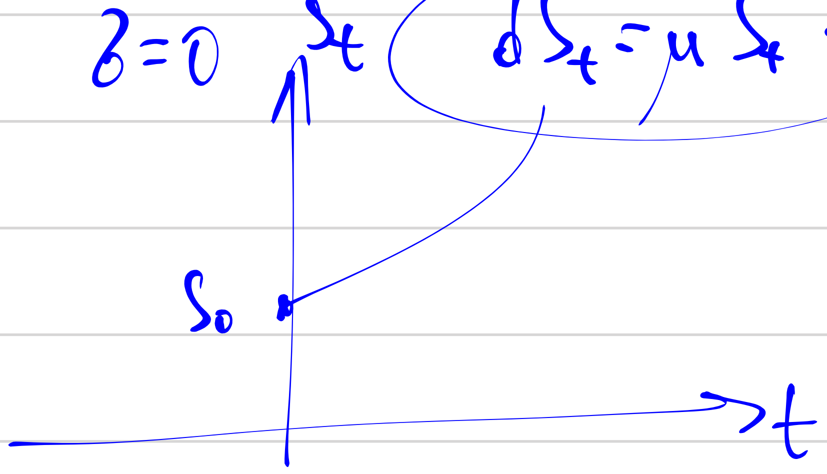
a) what is S_t if $\sigma = 0$? σ (volatility)

b) $Y_t = \ln S_t$ dY_t?

c) write the full of Y_t

d) find explicit expression for S_t

$\sigma = 0$



$$dS_t = \mu S_t \cdot dt$$

$$\frac{dS_t}{dt} = \mu \cdot S_t$$

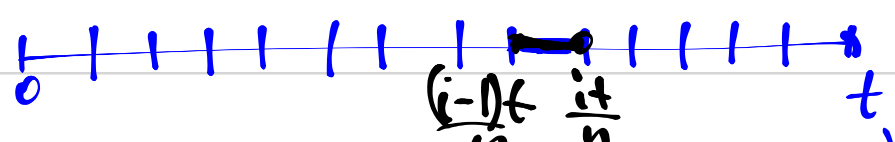
$$S_t = S_0 \cdot \exp(\mu t)$$

$\sigma > 0$

$$dS_t = \mu S_t dt + \sigma \cdot S_t \cdot dW_t$$

simulate?

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u$$



$$S\left(\frac{it}{n}\right) - S\left(\frac{(i-1)t}{n}\right) \approx \mu \cdot S\left(\frac{(i-1)t}{n}\right) \cdot \left(\frac{t}{n}\right) + \sigma \cdot S\left(\frac{(i-1)t}{n}\right) \cdot \left(W\left(\frac{it}{n}\right) - W\left(\frac{(i-1)t}{n}\right)\right)$$

b Y_t

$$Y_t = \ln(S_t) = f(\underbrace{t, W_t}_{\text{no dep}}, S_t)$$

$$dY_t = \underbrace{f'_t}_{=0} \cdot dt + \underbrace{f'_W}_{=0} \cdot dW_t + \underbrace{f'_S}_{=\frac{1}{S_t}} \cdot dS_t +$$

$$+ \frac{1}{2}\left[\underbrace{f''_{WW}}_{=0} \cdot (dW_t)^2 + 2\underbrace{f''_{WS}}_{=0} \cdot (dW_t)\cdot(dS_t) + \underbrace{f''_{SS}}_{=-\frac{1}{S_t^2}} (dS_t)^2\right] =$$

$$= \frac{1}{S_t} \cdot dS_t - \frac{1}{2S_t^2}(dS_t)^2 = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) -$$

$$- \frac{1}{2S_t^2}(\mu S_t dt + \sigma S_t dW_t)^2 =$$

$$= \mu dt + \sigma dW_t - \frac{1}{2}(\mu dt + \sigma dW_t)^2 =$$

$$= \mu dt + \sigma dW_t - \frac{1}{2}\left(\mu^2 \underbrace{(dt)^2}_{\to 0} + \sigma^2 \underbrace{(dW_t)^2}_{\to dt} + 2\mu \cdot \sigma \cdot \underbrace{dt\, dW_t}_{\to 0}\right)$$

$$\boxed{dY_t = \left(\mu - \frac{1}{2}\sigma^2\right) \cdot dt + \sigma \cdot dW_t}$$

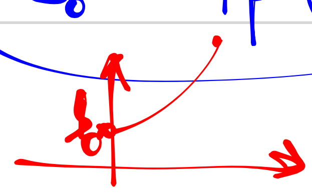
$$Y_t = Y_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) du + \int_0^t \sigma\, dW_u$$

$$Y_t = Y_0 + \left(\mu - \frac{\sigma^2}{2}\right) \cdot t + \sigma \cdot W_t$$

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

$$\boxed{S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$$

$$S_t = \underbrace{S_0 \cdot \exp(\mu t)} \cdot \underbrace{\exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right)}_{\text{martingale}}$$



ex

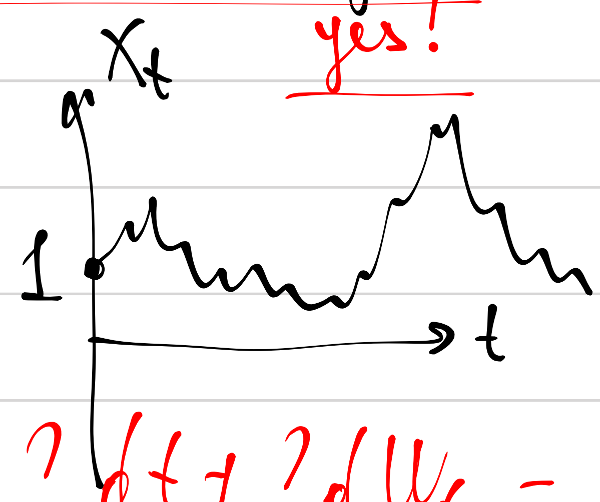
$$X_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

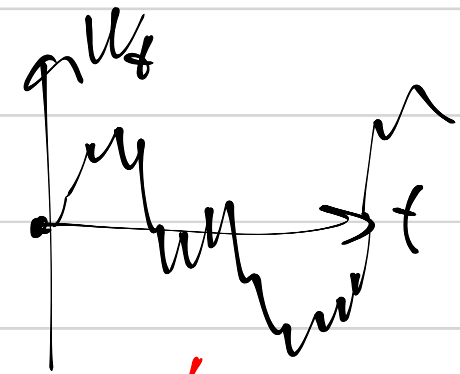
is X_t a martingale? yes!

Wiener process? no!

$X_0 = 1$

$E(X_0) = 1$





$dX_t =$ $?\,dt + ?\,dW_t$ – not mart

$\quad\quad\;\to$ $?\,dW_t$ – a mart

$$X_t = f(t, W_t) = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

Ito's lemma: $dX_t = f'_t \cdot dt + f'_w \cdot dW_t + \frac{1}{2}\left[f''_{ww} \cdot (dW_t)^2 \right]$

$$= -\frac{\sigma^2}{2} \cdot \exp(\dots) \cdot dt + \sigma \cdot \exp(\dots) \cdot dW_t +$$

$$+ \frac{1}{2}\sigma^2 \cdot \exp(\dots) \cdot \underbrace{(dW_t)^2}_{dt} = \sigma \cdot \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right) \cdot dW_t$$

$$X_t = X_0 + \int_0^t \sigma \exp\left(\sigma W_u - \frac{\sigma^2}{2} u\right) dW_u$$

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$$I_t = \int_0^t W_u^2 \, dW_u \qquad J_t = \int_0^t W_u^2 \, du$$

a) $E(I_t)$, $E(J_t)$ b) $Var(I_t)$, $Var(J_t)$

2 pts!! 2 pts!!

$E(I_t) = 0$ $\quad E\left(\int_0^t W_u^2 \, du\right) = \int_0^t E(W_u^2) \, du =$

$$= \int_0^t u \, du = \frac{t^2}{2}$$

$$E\left(M_{t+\Delta} \mid \mathcal{F}_t\right) = M_t$$ (def of a martingale)

$$E\left(E\left(M_{t+\Delta} \mid \mathcal{F}_t\right)\right) = E(M_t)$$

$$E(M_{t+\Delta}) = E(M_t)$$ [necess. cond. for a mart.]

$$\mathrm{Var}\left(\int_0^t W_u^2 \, dW_u\right) \underset{\text{Ito's isometry}}{=} \int_0^t E\left(\left(W_u^2\right)^2\right) du =$$

$$= \int_0^t \underbrace{E(W_u^4)}_{3u^2} du = \int_0^t 3u^2 du = t^3$$

$$\mathrm{Var}\left(\int_0^t W_u^2 \, du\right) \quad ????$$

strat: $\int \ldots du \to \ldots \int \ldots dW_u$ (Ito's lemma)

$$dY_t = W_t^2 \, dt \qquad Y_t = \int_0^t W_u^2 \, du$$

$$\boxed{\text{Ito's lemma } df = f_t' \cdot dt + f_w' \cdot dW_t + \frac{1}{2} f_{ww}'' \cdot \underbrace{(dW_t)^2}_{dt}}$$

guess: $X_t = t \cdot W_t^2 \qquad X_0 = 0 \cdot W_0^2$

$$dX_t = W_t^2 \cdot dt + 2 W_t \cdot t \cdot dW_t + \frac{1}{2} \cdot 2 \cdot t \cdot (dW_t)^2$$

$$dX_t = W_t^2 \, dt + 2t W_t \, dW_t + t \cdot dt$$

$$X_t = X_0 + \int_0^t 2u W_u \, dW_u + \int_0^t W_u^2 \, du + \int_0^t u \, du$$

$$X_t = X_0 + \int_0^t 2u W_u\,dW_u + \int_0^t W_u^2\,du + \int_0^t u\,du$$

import. step.

$$\boxed{\int_0^t W_u^2\,du = t\cdot W_t^2 - 0 - 2\int_0^t u W_u\,dW_u - \frac{t^2}{2}}$$

$$\mathrm{Var}\left(\int_0^t W_u^2\,du\right) = \mathrm{Var}(t\cdot W_t^2) + 4\,\mathrm{Var}\left(\int_0^t u W_u\,dW_u\right) - 4\,\mathrm{Cov}\left(tW_t^2, \int_0^t u W_u\,dW_u\right);$$

$$\mathrm{Var}(t\cdot W_t^2) = t^2\,\mathrm{Var}(W_t^2) = t^2\left(E(W_t^4) - \left(E(W_t^2)\right)^2\right) =$$

$$= t^2\cdot(3t^2 - t^2) = 2\cdot t^4$$

$$\mathrm{Var}\left(\int_0^t u W_u\,dW_u\right) \overset{\text{Ito's isometry}}{=} \int_0^t E(u^2 W_u^2)\,du =$$

$$= \int_0^t u^2\cdot u\,du = \frac{t^4}{4}$$

$$\mathrm{Cov}\left(t\cdot W_t^2, \int_0^t u W_u\,dW_u\right) = t\,\mathrm{Cov}\left(W_t^2, \int_0^t \dots dW_u\right) =$$

$$t + \int_0^t 2W_u\,dW_u = W_t^2$$

$$= t\cdot \mathrm{Cov}\left(t + \int_0^t 2W_u\,dW_u, \int_0^t u W_u\,dW_u\right) =$$

$$= 2t\,\mathrm{Cov}\left(\int_0^t W_u\,dW_u, \int_0^t u W_u\,dW_u\right) = (\text{Ito's isometry})$$

$$= 2t\cdot\int_0^t E(u\cdot W_u^2)\,du = 2t\cdot\int_0^t u^2\,du = 2t\cdot\frac{t^3}{3}$$

Q5. page 67 exam 2021

$$dS_t = S_t\,dt + \sqrt{S_t}\cdot dW_t \qquad S_0 = 1$$

Find all transformations $Y_t = h(S_t)$ such that Y_t is a martingale?

$$dY_t = h'_s \cdot dS_t + \frac{1}{2}\left(h''_{ss} \cdot (dS_t)^2\right) =$$

$$= h'_s \cdot \left(S_t \cdot dt + \sqrt{S_t}\cdot dW_t\right) + \frac{h''}{2}\left(S_t\cdot dt + \sqrt{S_t}\cdot dW_t\right)^2 =$$

$(dt)^2 = 0 \quad dt\cdot dW_t = 0$

$(dW_t)^2 = dt$

$$= h'\left(S_t \cdot dt + \sqrt{S_t}\cdot dW_t\right) +$$

$$+ \frac{h''}{2}\cdot S_t \cdot dt =$$

$$\boxed{\begin{aligned} dt\cdot d\,\text{any} &= 0 \\ (dW_t)^2 &= dt \end{aligned}}$$

$$= \underbrace{\left(h'\cdot S_t + \frac{h''}{2}\cdot S_t\right)}dt + h'\cdot\sqrt{S_t}\,dW_t$$

Y_t is a martingale

$$h' + \frac{h''}{2} = 0$$

$$h'' = -2\cdot h'$$

$$h'(t) = c\cdot \exp(-2t)$$

$$h(t) = d\cdot \exp(-2t) + f$$

$$Y_t = 11\cdot \exp(-2S_t) + 3 \qquad Y_t \text{ is martingale!}$$

$$E(Y_t) \overset{?}{=} E(Y_0) = 11\cdot \exp(-2\cdot S_0) + 3$$

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$$dR_t = 4R_t\,dt + 7\,dW_t\ ,\quad R_0 = 1$$

Find R_t?

a) solve $dQ_t = 4Q_t \cdot dt$

b) $B_t = \frac{R_t}{Q_t}$ find B_t

c) Find R_t

a) $Q_t = Q_0 \cdot \exp(4t)$

$Q_t = \exp(4t)$ (example)

$$B_t = \frac{R_t}{\exp(4t)} = \exp(-4t)\cdot R_t = f(t, R_t)$$

$$dB_t = f_t' \cdot dt + f_R' \cdot dR_t + \frac{1}{2}\cdot \underbrace{f_{RR}''}_{0} \cdot \underbrace{(dR_t)^2}_{49\,dt} =$$

$$= -4\cdot\exp(-4t)R_t\,dt + \exp(-4t)\cdot 1 \cdot dR_t + 0 =$$

$$= -4\exp(-4t)R_t\,dt + \exp(-4t)\left(4R_t\,dt + 7\,dW_t\right) =$$

$$= \exp(-4t)\cdot 7 \cdot dW_t$$

$$dB_t = \exp(-4t)\cdot 7\,dW_t \qquad B_t = B_0 + \int_0^t \exp(-4u)\cdot 7\,dW_u$$

(By the way, B_t is a martingale)

$$R_t = Q_t \cdot B_t = \exp(-4t)\cdot\left(B_0 + 7\int_0^t \exp(-4u)\,dW_u\right)$$

$R_0 = 1 \Rightarrow B_0 = 1$

$$R_t = \exp(-4t)\cdot\left(1 + 7\int_0^t \exp(-4u)\,dW_u\right)$$

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