Can you hear me? !!
see? !!

1) Short
2) Adv math.

$\boxed{\text{2HA + exam}} \to$ retake $\to$ retake 2

st. calc — st. calc — st. calc

wiki.cs.hse.ru $\to$ "stoch. calc fall 22"
- $\to$ all past exams.
- $\to$ video
- $\to$ tg chat

open access.

st 1, st 2 } main slots } new topics.

st 3 $\to$ no new topics, Q&A.

Information / б-algebras / conditional expected value

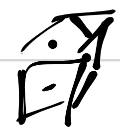
liter.

Zastawniak "Basic stoch. processes"

Shreve "Stoch. calc"

M. Steele "Stoch. calc with financial appl"

Klebaner "Intro to stoch calc with appl"

Ex. faire dice 

X – the result $\quad X \in \{1, 2, 3 \dots 6\}$

Alice : knows 1, 2, many.

Bob : remembers whether X is odd or even

2, 4, 6, ...

1, 3, 5, 7, ...

[inform def] σ-algebra „sigma-algebra"

σ-algebra – collection of all events that a rational ind.-l can distinguish. [always]

– collection of questions that rat. indiv-l can [always] answer.

$\mathcal{A}$ – σ·algebra for Alice

Q. $\{X=1\} \in \mathcal{A}$ True $\qquad \{X<1\} = \emptyset \in \mathcal{A}$

$\{X=3\} \notin \mathcal{A}$ No

$\{X>2\} \in \mathcal{A}$ True

$\{X=2\} \in \mathcal{A}$

$\{X \neq 1\} \in \mathcal{A}$

$$\mathcal{A} = \{ \{X=1\}, \{X=2\}, \{X>2\}, \{X\neq 1\}, \{X \neq 2\}, \{X\in\{1,2\}\}, \emptyset, \{X \in \{1,1,3,4,5,6\}\} \}$$

two trivial events enter any σ-algebra: $\emptyset, \Omega$

$\{X \in \{1,3,5\}\}$ = „X is equal to 1 or 3 or 5"

Bob $\quad \mathcal{F}$ – Bob's σ-algebra $\qquad F$ – event; $\mathcal{F}$ – collection of events

$\{X=1\} \notin \mathcal{F}$

$\{X \in \{1,3,5\}\} \in \mathcal{F}$

$\{X \in \{1,2,3,4\}\} \notin \mathcal{F}$

$\{X \in \{1,2,3,4,5,6\}\}$

$$\mathcal{F} = \{ \{X \in \{1,3,5\}\}, \{X\in\{2,4,6\}\}, \emptyset, \Omega \}$$

formal def: Ω - the set of all outcomes.

$\mathcal{F}$ - σ-algebra if:

(1) $\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$

(2) "rationality"

If events $A_1, A_2, A_3 \ldots\ldots \in \mathcal{F}$ then any [countable] comb-n of $A_1, A_2, \ldots$ using any set operation $(A \cup B,\ A \cap B,\ A^c,\ A \setminus B)$ belongs to $\mathcal{F}$.

$\Updownarrow$

[formal def.] books $\mathcal{F}$ - collection of events:

(1) $\emptyset \in \mathcal{F}$

(2a) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

(2b) If $A_1, A_2 \ldots\ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

event F $\uparrow$ not coll.

$\mathcal{F}$ $\leftarrow$ collection of events.
$\uparrow$ coll.

Ex. Why Ω belons to $\mathcal{F}$?

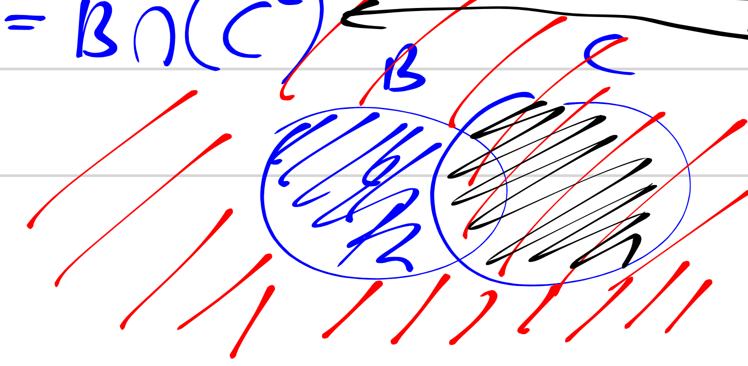
Ans: $\emptyset \in \mathcal{F}$ (by 1)

$(\emptyset)^c \in \mathcal{F}$ (by 2a)

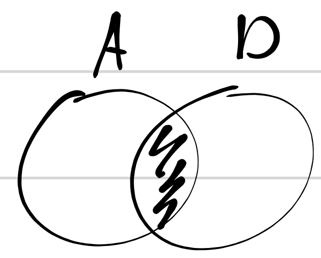
$(\emptyset)^c = \Omega \in \mathcal{F}$

Ex $A \cap (B \setminus C)$ $\leftarrow$ express this event using only unions and complements.

$B \setminus C = B \cap (C^c) \longleftrightarrow (C \cup B^c)^c$



$$A\cap D = (A^c \cup D^c)^c \qquad D = B\setminus C$$



Notation A, B, C - events.

$\sigma(A,B,C)$ - the smallest σ-algebra that contains A, B and C.

Ex.

$X \sim U[0;22]$

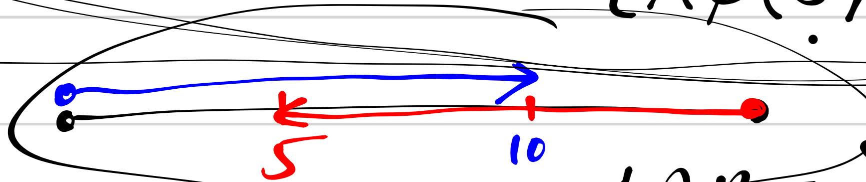
$A = \{X<10\}$
$B = \{X>5\}$ $\Rightarrow$

I know about
$A\cap B$?
$A\cup B$?
$A\setminus B$ .. $B\setminus A$ ..

$\mathcal{F} = \sigma(A,B)$? $\{X\in\mathbb{R}\} = \{X>-1\}$!!

X is not measurable wrt $\mathcal{F}$

$$\sigma(A,B) = \{\varnothing, \Omega, \{X<10\}, \{X>5\}, \{X\in(5,10)\}, \{X\le 5\}, \{X\ge 10\}, \{X\notin(5;10)\}\}$$



$5 \neq V$

$A\cap B = \{X\in(5;10)\}$

"primitive events" - partition of Ω

$\{X\le 5\}$ $\{X\in(5;10)\}$ $\{X\ge 10\}$

$A_1\cup A_2\cup A_3 = \Omega$
$A_i\cap A_j = \varnothing$ $(i\neq j)$

$\{X\le 5\}$	$\{X\in(5;10)\}$	$\{X\ge 10\}$	
+	+	+	$\to \{X\in\mathbb{R}\} = \{X\in[0;22]\}$
+	+	−	$\to X<10$
+	−	+	$\to X\notin(5;10)$
−	+	−	$\to X\in(5;10)$
. . .	. . .	. . .	

$\{X=4\} = \{X=2+2\}$

$\{X>-5\} = \{X>-3\} = \{X\in[0;22]\}$

In total $2^3 = 8$ events in $\mathcal{F}$

Inf def.

Random variable X is measurable wrt σ-algebra $\mathcal{F}$ if $\mathcal{F}$ contains enough info to calculate X.

I can compare X with any real t

formal def. X is meas wrt to $\mathcal{F}$ if

$\forall t \in \mathbb{R}$ $\{X \le t\} \in \mathcal{F}$

wrt = with respect to
LHS = left hand side
RHS = right hand side
wo = without
iff = if and only if
...

break

Can you hear me? !!

informal def $X, Y, Z \ldots$ are random variables

$\sigma(X,Y,Z)$ – the smallest σ-algebra that containts enough info to calculate X, Y, Z.

formal def.

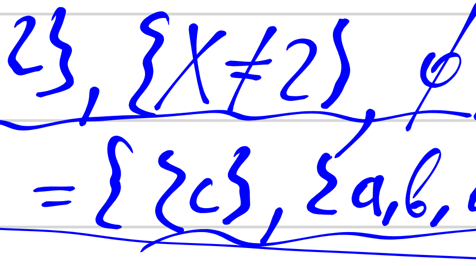
$\sigma(X,Y,Z)$ – the smallest σ-algebra that contains all events of type $\{X \le t\}$, $\{Y \le t\}$, $\{Z \le t\}$ for $\forall t \in \mathbb{R}$.

Ex

$\Omega = \{a, b, c, d\}$

$\left\{\begin{array}{l} a \text{ – outcome} \\ A \text{ – event} \\ \mathcal{A} \text{ – } \sigma\text{-algebra (coll. of events)} \end{array}\right.$

	w	a	b	c.	d.
	$P(\{w\})$	0.1	0.2	0,3	0,4
X	$X(w)$	1	1	2	3
Y	$Y(w)$	2	2	1	1
Z	$Z(w)$	2	1	−1	3



x	1	3	6
y	1	9	36

$y = x^2$

x	1	3	6
$y(x)$	1	9	36

explicitely list all the events in σ-algebras:

a) $\sigma(\{X=2\}) = ?$ $\{\{X=2\}, \{X\ne 2\}, \emptyset, \Omega\} =$
$= \{\{c\}, \{a,b,d\}, \emptyset, \{a,b,c,d\}\}$

b) $\sigma(Y) = ?$

c) $\sigma(X+Y) = ?$

d) „compare" σ-algebras $\sigma(X)$, $\sigma(X+Y)$, $\sigma(Z)$

$\sigma(Y) = \{\{Y=1\}, \{Y=2\}, \emptyset, \Omega\} =$

$= \{\{Y=1\}, \{Y\ne 1\}, \emptyset, \Omega\} =$

$= \{\{c,d\}, \{a,b\}, \{\}, \{a,b,c,d\}\}$

$= \{\{Y>\sqrt{3}\}, \{Y\le\sqrt{3}\}, \emptyset, \Omega\}$

$\{Y>\sqrt{3}\} = \{Y=2\} = \{Y\ne 1\} = \{Y > 1+\sin\frac{\pi}{6}\}$

	w	a	b	c	d
	$P(\{w\})$	0.1	0.2	0.3	0.4
X	$X(w)$	1	1	2	3
Y	$Y(w)$	2	2	1	1
Z	$Z(w)$	2	1	-1	3

$$\{X+Y=4\} = \{X+Y\neq 3\}$$

a b c d $\sigma(Y)$

a b c d $\sigma(X+Y)$

$$\sigma(X+Y)\ ? = \{\{X+Y=3\}, \{X+Y\neq 3\}, \phi, \Omega\} =$$
$$= \{\{a,b,c\}, \{d\}, \phi, \{a,b,c,d\}\}$$

d) $\sigma(X)$, $\sigma(X+Y)$, $\sigma(Z)$

$$\boxed{\sigma(X+Y) \subseteq \sigma(X)}$$

$\sigma(X)$: $\{\{X=1\}, \{X=2\}, \{X=3\},$
$\{X\neq 1\}, \{X\neq 2\}, \{X\neq 3\}, \phi, \Omega\}$

$$\underbrace{\sigma(X+Y)}_{4\text{ events}} \subseteq \underbrace{\sigma(X)}_{8\text{ events}} \subseteq \underbrace{\sigma(Z)}_{?}$$

a b c d

$2^4 = 16$ events

$$\underbrace{\sigma(X+Y)}_{4\text{ events}} \qquad \underbrace{\sigma(Y)}_{4\text{ events}}$$

$\sigma(X+Y) \not\subseteq \sigma(Y)$

$\sigma(Y) \not\subseteq \sigma(X+Y)$

$$4 = \text{card}(\sigma(X+Y)) = \text{card}(\sigma(Y))$$

$$\sigma(Z) = \{\{Z=1\}, \{Z=-1\} \dots \{Z\in\{1,-1\}\} \dots\}$$
$$= \{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\} \dots$$
$$\dots\}$$

16 events in $\sigma(Z)$

e) T/F

1) Z is measur. wrt $\sigma(X)$ — F

$\sigma(X) \subseteq \sigma(Z)$ 2) X is measur wrt $\sigma(Z)$ — T

$\sigma(X+Y) \subseteq \sigma(X,Y)$ 3) $X+Y$ is meas. wrt $\sigma(X,Y)$ — T

4) X is meas wrt $\sigma(X+Y)$ — F

5) Y is meas wrt $\sigma(X+Y)$ — F

$\sigma(X) \subseteq \sigma(X+Y,Y)$ 6) X is meas wrt $\sigma(X+Y, Y)$ — T

Conditional expected value.

two concepts

$E(X|A) = \frac{1}{P(A)} \cdot E(X \cdot I_A)$ ← constant

X: r.v., A: event

$I_A = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{if } A \text{ does not happen} \end{cases}$

$E(X|\mathcal{F})$ ← random variable

X: r.v., $\mathcal{F}$: σ-algebra.

infor def $E(X|\mathcal{F})$ – best forecast of X given info in $\mathcal{F}$.

lower $E((Y-\hat{Y})^2 | \sigma(X))$

w	a	b	c.	d.
$P(\{w\})$	0.1	0.2	0.3	0.4
X $X(w)$	1	1	2	3
Y $Y(w)$	2	2	1	1
Z $Z(w)$	2	1	-1	3

a) $E(Y|\sigma(X))$?

b) $E(X|\sigma(Y))$?

c) $E(X|\sigma(X+Y))$?

$\{X+Y=4\} = \{d\} = \{X=3\}$

$\{X+Y=3\} = \{a, b, c\}$

a) $E(Y|\sigma(X))$ I have enough info to calc Y!

$E(Y|\sigma(X)) = Y$

$E(Y|\sigma(X)) = \begin{cases} 2 & \text{if } X=1 \\ 1 & \text{if } X=2 \\ 1 & \text{if } X=3 \end{cases}$

b) $E(X|\sigma(Y)) = \begin{cases} 1 & \text{if } Y=2 \\ \frac{18}{7} & \text{if } Y=1 \end{cases}$

$P(X=2|Y=1)$, $P(X=3|Y=1)$

$\frac{0.3}{0.7} \cdot 2 + \frac{0.4}{0.7} \cdot 3 = \cancel{X2}$

$= 2 \cdot \frac{3}{7} + 3 \cdot \frac{4}{7} = \frac{18}{7} =$

≈ 2.57

c) $E(X|\sigma(X+Y)) = \begin{cases} \alpha & \text{if } X+Y=3 \\ \beta & \text{if } X+Y=4 \end{cases}$

$\beta = 3$

$\alpha = \sum_x x \cdot P(X=x|X+Y=3)$

$\frac{0.1}{0.1+0.2+0.3}$, $\frac{0.2}{0.1+0.2+0.3}$, $\frac{0.3}{\ldots}$ ...

$\alpha \overset{?}{=} 1 \cdot \frac{0.1}{0.6} + 1 \cdot \frac{0.2}{0.6} + 2 \cdot \frac{0.3}{0.6} = \frac{1}{6} + \frac{2}{6} + \frac{6}{6} = \frac{9}{6} = 1.5$

$E(X|\sigma(X+Y)) = \begin{cases} 1.5 & \text{if } X+Y=3 \\ 3 & \text{if } X+Y=4 \end{cases}$

inform def → formal def.

„best guess" → „you have no idea how to forecast the forecasting error"

[form. def.]

$$E(X|\mathcal{F}) = \hat{X} \leftarrow \text{r.v.}$$

① $\hat{X}$ is meas-b wrt $\mathcal{F}$

② $E(X-\hat{X})=0$ $\quad \{ E(X)=E(\hat{X})$

③ $Cov(X-\hat{X}, I_A)=0$ if $A \in \mathcal{F}$

[Theorem] If X and Y are discr.

then

$$E(Y|X) = \begin{cases} E(Y|X=x_1) & \text{if } X=x_1 \\ E(Y|X=x_2) & \text{if } X=x_2 \\ \vdots \end{cases}$$

Eq. from def

$$\begin{cases} Cov(X-\hat{X}, I_{Y=1}) = 0 \\ Cov(X-\hat{X}, I_{Y=2}) = 0 \end{cases} \Rightarrow \frac{18}{7}, 1$$

Remark. Notation $\underbrace{E(Y|X)}_{\text{short}} = \underbrace{E(Y|\sigma(X))}_{\text{official}}$

[Theorem] properties of cond. exp. value:

① Addit $\quad E(\alpha X+\beta Y|\mathcal{F}) = \alpha E(X|\mathcal{F})+\beta E(Y|\mathcal{F})$

② Take out what is known! $\quad$ if X is meas. wrt $\mathcal{F}$

$$E(X+Y|\mathcal{F}) = X + E(Y|\mathcal{F})$$

$$E(X\cdot Y|\mathcal{F}) = X\cdot E(Y|\mathcal{F})$$

③ Independence $\quad$ if $\underset{\text{coll.}}{\sigma(X)}$ and $\underset{\text{coll}}{\mathcal{F}}$ are indep. then

$$E(X|\mathcal{F}) = E(X)$$

any A from $\sigma(X)$ and any B from $\mathcal{F}$ are indep.

(...)

Audio? Video? !!

Adv 1 short course !!

1 problem about $E(X|\mathcal{F})$ or σ-alg.

w	a	b	c.	d.
$P(\{w\})$	0.1	0.2	0.3	0.4
$X(w)$	1	1	2	3
$Y(w)$	2	2	1	1
$Z(w)$	2	1	-1	3

b) $E(X|\sigma(Y))$ using formal definition

Answer. [intuit] $E(X|\sigma(Y)) = \begin{cases} 1 & \text{if } Y=2 \\ \frac{18}{7} & \text{if } Y=1 \end{cases}$

$\sigma(Y) = \{\{Y=1\}, \{Y=2\}, \emptyset, \Omega\}$

a b c d

① $\hat{X}$ should be meas wrt $\sigma(Y)$

$\hookrightarrow \hat{X} = \begin{cases} \alpha & \text{if } Y=2 \\ \beta & \text{if } Y=1 \end{cases}$

② $E(X-\hat{X})=0$ $\boxed{E(X) = E(\hat{X})}$

w	a	b	c.	d.
$P(\{w\})$	0.1	0.2	0.3	0.4
$X(w)$	1	1	2	3
$Y(w)$	2	2	1	1
$Z(w)$	2	1	-1	3

$0{,}1\cdot 1 + 0{,}2\cdot 1 + 0{,}3\cdot 2 + 0{,}4\cdot 3 = \alpha\cdot P(Y=2) + \beta P(Y=1)$

$0{,}1 + 0{,}2 + 0{,}6 + 1{,}2 = \alpha\cdot 0{,}3 + \beta\cdot 0{,}7$

$2.1 = \alpha\cdot 0{,}3 + \beta\cdot 0{,}7$

$21 = 3\alpha + 7\beta$

③ $Cov(\underbrace{X-\hat{X}}_{\text{forecast. error}}, \underbrace{I_A}_{\text{I know this RV.}}) = 0$ if $A\in\mathcal{F}$

$A=\emptyset$

$Cov(X-\hat{X}, 0) = 0$!!

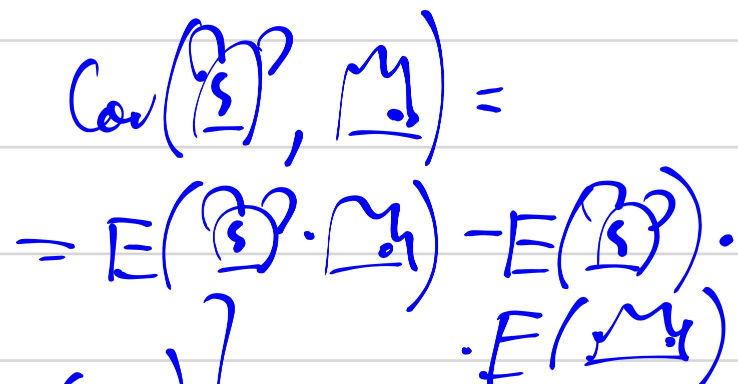
$A = \{Y=1\}$

$$\mathrm{Cov}(X-\hat{X}, I_A) = 0$$

$$\mathrm{Cov}(X, I_A) - \mathrm{Cov}(\hat{X}, I_A) = 0$$

$$E(X\cdot I_A) - E(X)\cdot E(I_A) = E(\hat{X}\cdot I_A) - E(\hat{X})\cdot E(I_A)$$

Assump. №2 $E(X) = E(\hat{X})$



$$\mathrm{Cov}(L,R) = E\left[(L-E(L))\cdot(R-E(R))\right] =$$
$$= E(L\cdot R) - E(L)\cdot E(R) =$$
$$= E\left[(L-E(L))\cdot R\right] =$$
$$= E\left[L\cdot(R-E(R))\right]$$

restated

③ $E(X\cdot I_A) = E(\hat{X}\cdot I_A)$ for any $A \in \mathcal{F}$.

	w	a	b	c	d
	$P(\{w\})$	0,1	0,2	0,3	0,4
X	$X(w)$	1	1	2	3
Y	$Y(w)$	2	2	1	1
Z	$Z(w)$	2	1	-1	3
I_A		0	0	1	1
$X\cdot I_A$		0	0	2	3
$\hat{X}$		α	α	β	β
$\hat{X}\cdot I_A$		0	0	β	β

$A = \{Y=1\}$

$$I_A = \begin{cases} 1 & \text{if } Y=1 \\ 0 & \text{if } Y\neq 1 \end{cases}$$

$E(X\cdot I_A)$

$$0{,}1\cdot 0 + 0{,}2\cdot 0 + 0{,}3\cdot 2 + 0{,}4\cdot 3 = E(\hat{X}\cdot I_A)$$

$$= 0{,}1\cdot 0 + 0{,}2\cdot 0 + 0{,}3\cdot\beta + 0{,}4\cdot\beta$$

$$\beta = \frac{0{,}3}{0{,}7}\cdot 2 + \frac{0{,}4}{0{,}7}\cdot 3 = \frac{18}{7}$$

	w	a	b	c.	d.
	$P(\{w\})$	0.1	0.2	0,3	0,4
X	$X(w)$	1	1	2	3
Y	$Y(w)$	2	2	1	1
Z	$Z(w)$	2	1	-1	3
I_A		1	1	0	0
$X\cdot I_A$		1	1	0	0
$\hat{X}$		α	α	β	β
$\hat{X}\cdot I_A$		α	α	0	0

$$\boxed{A=\{Y=2\}}$$

„best forecast"

$$E(X\cdot I_A) = E(\hat{X}\cdot I_A)$$

$$+0{,}3\cdot 0+0{,}4\cdot 0$$

$$0{,}1\cdot 1+0{,}2\cdot 1 = 0{,}1\cdot\alpha+0{,}2\cdot\alpha \quad +0{,}3\cdot 0+ +0{,}4\cdot 0$$

$$0{,}3\cdot 1 = 0{,}3\cdot\alpha$$

$$1=\alpha$$

$$\sigma(Y) = \{\{Y=1\},\boxed{\{Y=2\}},\emptyset,\Omega\}$$

$$I_A = \begin{cases}1 & \text{if } A\\ 0 & \text{if } A^c\end{cases} \qquad I_A = \begin{cases}1 & \text{if } Y=2\\ 0 & \text{if } Y\neq 2\end{cases}$$

Ex. X_1, X_2, X_3 are indep. ident. distributed

$$P(X_i=+1) = P(X_i=-1) = \tfrac{1}{2}$$

$$X_i \in \{-1,1\}$$

a) exp-ly list the σ-algebra $\sigma(X_1+X_2)$, $\sigma(X_1\cdot X_2)$

b) how many events are in $\sigma(X_1,X_2,X_3)$

c) $E(X_1\,|\,\sigma(X_1,X_2))$

$E(X_3\,|\,\sigma(X_1,X_2))$

$E(X_2\,|\,\sigma(X_1+X_2))$

$E(X_1+X_2\,|\,\sigma(X_1))$

$E(X_1\cdot X_2\,|\,X_2,X_3) = E(X_1\cdot X_2\,|\,\sigma(X_2,X_3))$

a) $\sigma(X_1+X_2) =$

$\{\{X_1+X_2=-2\}, \{X_1+X_2=0\}, \{X_1+X_2=2\},$

$\{X_1+X_2\neq -2\}, \{X_1+X_2\neq 0\}, \{X_1+X_2\neq 2\},$

$\emptyset, \Omega\}$

$\{X_1+X_2=\pm 2\} = \{(X_1+X_2)^2=4\}$

a) $\sigma(X_1 \cdot X_2) \overset{?}{=} \{\{X_1 \cdot X_2 = 1\}, \{X_1 \cdot X_2 = -1\}, \phi, \Omega\}$

$= \{\{X_1 = X_2\}, \{X_1 \neq X_2\}, \phi, \Omega\}$

b) how many events are in $\sigma(X_1, X_2, X_3)$?

$\{X_1 = 1, X_2 = 1, X_3 = 1\} = A_1$

$\Omega \to \{X_1 = 1, X_2 = 1, X_3 = -1\} = A_2$

$\{X_1 = 1, X_2 = -1, X_3 = 1\} = A_3$

$\vdots$

combinations.

$A_1 \cup A_2 =$

$= \{X_1 = 1, X_2 = 1\}$

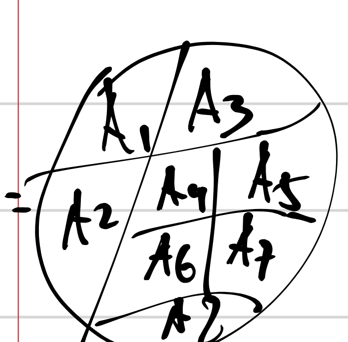
partion: 8 bits

a lot of comb

$\Omega = A_1 \cup A_2 \cup A_3 \ldots \cup A_8$

$A_i \cap A_j = \phi \quad (i \neq j)$

$\text{card } \sigma(X_1, X_2, X_3) = 2^8 = 256$

$\Omega =$ 

A_1	+	+
A_2	+	−
A_3	−	+
$\vdots$	−	−
	−	−
	−	−
A_8	−	−
	$\uparrow$ $A_1 \cup A_2$	$\uparrow$ $A_1 \cup A_3$

$A_1 \cup A_2 = \{X_1 = 1, X_2 = 1\}$

$A_1 \cup A_3 = \{X_1 = 1, X_3 = 1\}$

$\{X_1 \geq \max\{X_2, X_3\}\}$

$\{X_1 \leq X_2 \leq X_3\}$

c) $E(X_1 \mid \sigma(X_1, X_2))$ ‼

$E(X_3 \mid \sigma(X_1, X_2))$ ‼

$E(X_2 \mid \sigma(X_1+X_2))$

$E(X_1+X_2 \mid \sigma(X_1))$ ‼

$E(X_1 \cdot X_2 \mid X_2, X_3)$ ‼

1. Linearity.
2. Take out what is known
3. Remove irrelevant information
...

→ X_3 is ind of X_1 and X_2

$$E(X_3 \mid X_1, X_2) = E(X_3) = 0$$

→ X_1 is measurable wrt $\sigma(X_1, X_2)$

$$E(X_1 \mid X_1, X_2) = X_1$$

indep.

$$E(X_1+X_2 \mid X_1) = \underbrace{E(X_1 \mid X_1)}_{\text{Take out what is known}} + E(X_2 \mid X_1) =$$

$$= X_1 + E(X_2) = X_1 + 0 = X_1$$

$$E(X_1 \cdot X_2 \mid X_2, X_3) = X_2 \cdot \underbrace{E(X_1 \mid X_2, X_3)}_{\text{remove irrel. info}} = X_2 \cdot E(X_1) =$$

$$= X_2 \cdot 0 = 0$$

$$E(X_2 \mid X_1+X_2) = \begin{cases} \alpha & \text{if } X_1+X_2 = -2 \\ \beta & \text{if } X_1+X_2 = 0 \\ \gamma & \text{if } X_1+X_2 = 2 \end{cases}$$

$\alpha \stackrel{?}{=} -1$ ‼

$\gamma \stackrel{?}{=} +1$ ‼

$$\beta = E(X_2 \mid X_1+X_2 = 0) = \sum_x x \cdot P(X_2 = x \mid X_1+X_2 = 0)$$

$$= 1 \cdot P(X_2 = 1 \mid X_1+X_2 = 0) + (-1) \cdot P(X_2 = -1 \mid X_1+X_2 = 0)$$

way 1 : symmetry : $\frac{1}{2}$

way 2:

$$P(X_2=1 \mid X_1+X_2=0) = \frac{P((X_2=1)\cap(X_1+X_2=0))}{P(X_1+X_2=0)} =$$

$$\boxed{P(A \mid B) = \frac{P(A\cap B)}{P(B)}}$$

$$= \frac{P(X_2=1 \text{ and } X_1=-1)}{P(X_1+X_2=0)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$\beta = 0$

$$E(X_2 \mid X_1+X_2) = \begin{cases} 1 & \text{if } X_1+X_2=2 \\ 0 & \text{if } X_1+X_2=0 \\ -1 & \text{if } X_1+X_2=-2 \end{cases} =$$

$$= \frac{X_1+X_2}{2}$$

another way to solve !!

$$E(X_1+X_2 \mid X_1+X_2) = X_1+X_2$$ „take out what is known"

$$E(X_1 \mid X_1+X_2) + E(X_2 \mid X_1+X_2) = X_1+X_2$$ Linearity

$$2E(X_1 \mid X_1+X_2) = X_1+X_2$$ symmetry.

$$E(X_1 \mid X_1+X_2) = \frac{X_1+X_2}{2}$$

---

d) $E(X_1+X_2 \mid X_1 \cdot X_2) \overset{?}{=} 0$

$$= \begin{cases} \alpha & \text{if } X_1\cdot X_2 = 1 \\ \beta & \text{if } X_1 \cdot X_2 = -1 \end{cases}$$

$\alpha = 0$?

$\beta = 0$

$$\alpha = 2 \cdot P(X_1+X_2=2 \mid X_1 \cdot X_2 = 1) + (-2)\cdot P(X_1+X_2=-2 \mid X_1\cdot X_2=1) = 0$$