Week 3.

Doob's stopping time theorem.

If:

(M_n) - martingale wrt filtration $(\mathcal{F}_n)$

τ - stopping time wrt $(\mathcal{F}_n)$

At least one of two techn. cond.:

(A) $P(\tau=+\infty)=0$ and $\exists m$ such that $|Y_n|<m$ where $Y_n = M_{\min(n,\tau)}$

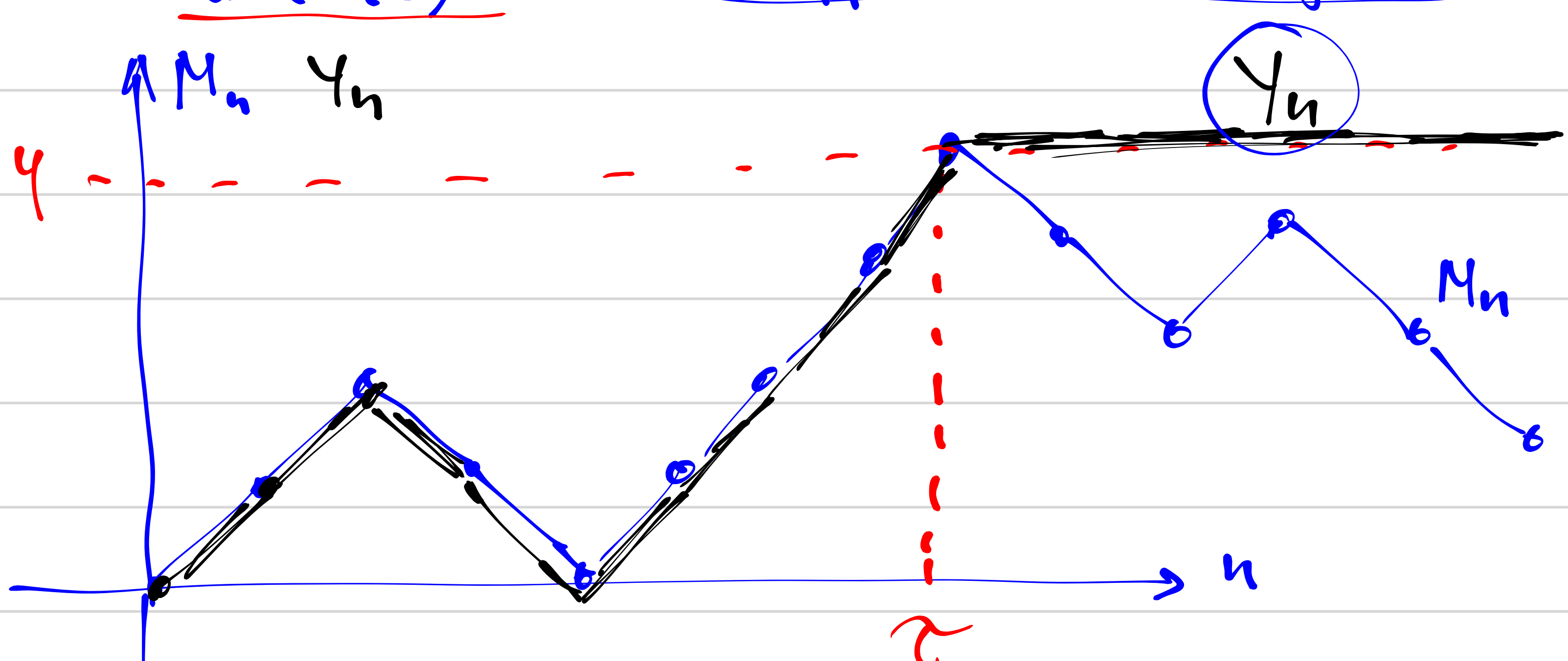
(B) $E(\tau)<\infty$ and $\exists m$ such that $|E(Y_{n+1}-Y_n \mid \mathcal{F}_n)|<m$

then:

$$E(M_\tau) = E(M_0)$$

$Y_n = M_{\min(n,\tau)}$ — stopped martingale.

(M_n) - process
(Y_n) - process
τ - stopp. time
$(\mathcal{F}_n)$ - filtrat.



$\tau = \min\{t \mid M_t = 4\}$ $\qquad Y_n = M_{n\wedge\tau}$

if (M_n) is a mart wrt $(\mathcal{F}_n)$ and τ is a st. time wrt $(\mathcal{F}_n)$ then $Y_n = M_{n\wedge\tau}$ is martingale

$$Y_n = \begin{cases} M_n & n<\tau \\ M_\tau & n \geq \tau \end{cases}$$

useful notation $\quad \min(a,b) = a\wedge b$
$\qquad \max(a,b) = a \vee b$

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

$A\wedge(B\vee C) = (A\wedge B)\vee(A\wedge C)$

$\min(a,\max(b,c)) = \max(\min(a,b),\min(a,c))$

(A) $P(\tau=+\infty)=0$ and stopped (M_n) is bounded

(B) $E(\tau)<\infty$ and stopped (M_n) has bounded predicted increments.

$$|E(\Delta Y_{n+1}|\mathcal{F}_n)|<m$$

$$|E(Y_{n+1}-Y_n|\mathcal{F}_n)|<m$$

Ex. X_n are iid

z	$+1$	-1
$P(X_n=z)$	p	$1-p$

$\{p=0.7\}$ imagine $(p\neq 0.5)$

$$S_n=X_1+X_2+\dots+X_n$$

natural filtration: $\mathcal{F}_n=\sigma(X_1,X_2,X_3\dots X_n)$

a) is (S_n) a martingale wrt $(\mathcal{F}_n)$?

$$E(S_{n+1}|\mathcal{F}_n)=E(\underbrace{S_n}_{\text{is known}}+X_{n+1}|\mathcal{F}_n)=$$

$$=S_n+E(X_{n+1}|\mathcal{F}_n)=S_n+E(X_{n+1})=$$

indep.

$$=S_n+(p\cdot 1-1\cdot(1-p))=$$

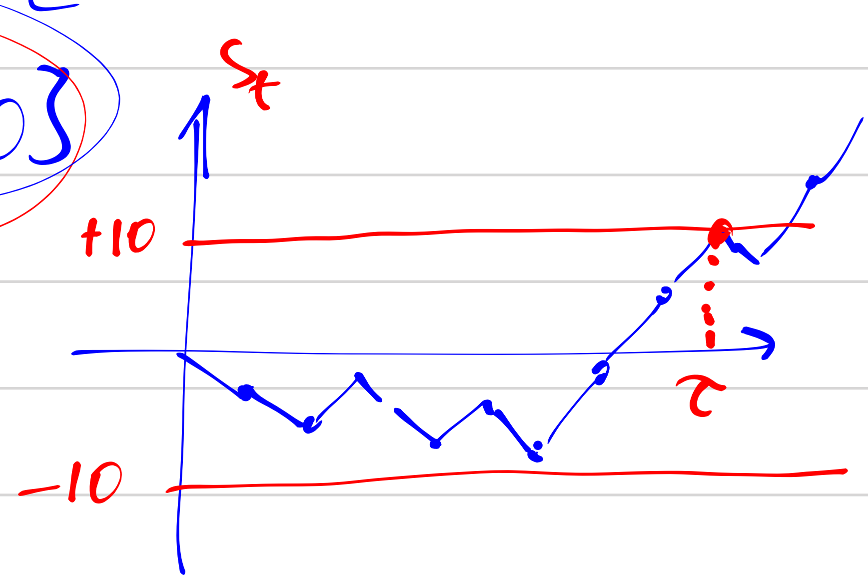
$$=S_n+(2p-1)$$

{mart. only if $p=\frac{1}{2}$}

b) $\tau=\min\{t \mid |S_t|=10\}$

$P(S_\tau=+10)$?

c) $E(\tau)$?



S_n $\xrightarrow{\text{transform?}}$ martingale

$$M_n=S_n-n\cdot(2p-1)$$

$$E(M_{n+1}|\mathcal{F}_n)=E(S_{n+1}-(n+1)(2p-1)|\mathcal{F}_n)=$$

$$=S_n+2p-1-(n+1)(2p-1)=$$

$$=S_n-n(2p-1)=M_n$$

M_n - mart

τ - st. time

A or B is satisfy?

$$Y_n = M_{n \wedge \tau}$$

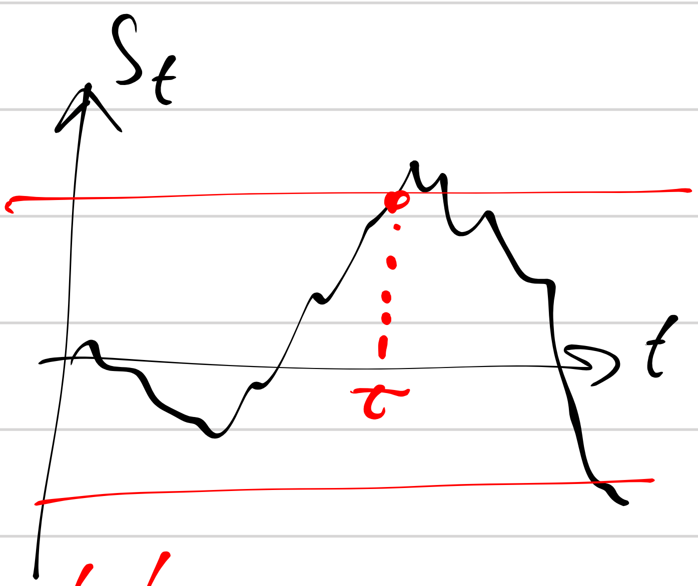
(A) $P(\tau = +\infty) = 0$ and Y_n is bounded

(B) $E(\tau) < \infty$ and predictions (one step ahead) of ΔY_{n+1} are bounded.

$Y_n = M_{n \wedge \tau}$

$$M_n = S_n - n(2p-1)$$

$$\tau = \min\{t \mid |S_t| = 10\}$$



before τ: $|S_t| \le 10$

$n(2p-1)$ is not bounded

$$\Delta Y_{n+1} = \begin{cases} \Delta M_{n+1} & \text{before } \tau \\ 0 & \text{after } \tau \end{cases}$$

$$\Delta M_{n+1} = \Delta S_{n+1} - (2p-1) \qquad |\Delta M_{n+1}| \le 2$$

$\Delta S_{n+1} \in \{-1, 1\}$

(B) is satisfied.

Apply Doob's theorem:

$$E(M_\tau) = E(M_0) = S_0 - 0 \cdot (2p-1) = 0$$

$$E(S_\tau - \tau(2p-1)) = 0$$

$$\boxed{E(S_\tau) = E(\tau) \cdot (2p-1)}$$

Invent a new martingale (M_t)!

$$M_t = \exp(\beta \cdot S_t) \quad \beta?$$

if $\beta = 0$ then $M_t = \exp(0) = 1$ is a trivial martingale

$$E(M_{t+1} \mid \mathcal{F}_t) = M_t$$

$$E(\exp(\beta S_{t+1}) \mid \mathcal{F}_t) = \exp(\beta S_t)$$

$$E(\underbrace{\exp(\beta S_t)}_{\text{known value}} \cdot \exp(\beta X_{t+1}) \mid \mathcal{F}_t) = \exp(\beta S_t)$$

Idea: future value = current value + increment.

$$E(\exp(\beta X_{t+1}) \mid \mathcal{F}_t) = 1$$

are indep.

$$E(\exp(\beta X_{t+1})) = 1$$

$$p \cdot \exp(\beta \cdot 1) + (1-p) \cdot \exp(\beta \cdot (-1)) = 1$$

$\exp(\beta) = u$

$$p \cdot u + (1-p) \cdot \frac{1}{u} = 1$$

$$p \cdot u^2 - u + (1-p) = 0$$

$\tau = \min\{t \mid |S_t| = 10\}$

before τ: $S_t \in [-10;10]$

$$u_1 = 1 \qquad u_2 = \frac{1-p}{p}$$

$$\beta = \ln u = 0 \qquad \beta = \ln \frac{1-p}{p}$$

martingale

$$\boxed{M_t = \exp\left(\ln\left(\frac{1-p}{p}\right) \cdot S_t\right)} = \left(\frac{1-p}{p}\right)^{S_t} \in \left[0; \left(\frac{1}{p}\right)^{10}\right] \text{ (before } \tau)$$

$Y_t = M_{t \wedge \tau}$

$A_{(T)}$ or $B_{(T)}$?

(A) and (B) are sat-d for $M_t = \left(\frac{1-p}{p}\right)^{S_t}$

By Doob's theorem:

$$\underline{E(M_\tau)} = M_0 = \left(\frac{1-p}{p}\right)^0 = 1$$

$$P(S_\tau = 10) \cdot \left(\frac{1-p}{p}\right)^{10} + \underbrace{P(S_\tau = -10)}_{1 - P(S_\tau = 10)} \cdot \left(\frac{1-p}{p}\right)^{-10} = 1$$

$$P(S_\tau = 10) \cdot \left[\left(\frac{1-p}{p}\right)^{10} - \left(\frac{1-p}{p}\right)^{-10}\right] = 1 - \left(\frac{1-p}{p}\right)^{-10}$$

$$P(S_\tau = 10) = \frac{1 - \left(\frac{1-p}{p}\right)^{-10}}{\left(\frac{1-p}{p}\right)^{10} - \left(\frac{1-p}{p}\right)^{-10}} =$$

$$P(S_\tau = 10) = \frac{\left(\frac{1-p}{p}\right)^{10} - 1}{\left(\frac{1-p}{p}\right)^{20} - 1} = \frac{1}{\left(\frac{1-p}{p}\right)^{10} + 1} = \frac{p^{10}}{(1-p)^{10} + p^{10}}$$

c)

$E(\tau)$

$$\underline{E(S_\tau)} = E(\tau) \cdot (2p-1)$$

$$E(\tau) = \frac{\boxed{10 \cdot \frac{p^{10}}{(1-p)^{10} + p^{10}} + (-10) \cdot \frac{(1-p)^{10}}{(1-p)^{10} + p^{10}}}}{2p-1}$$

ABRACADABRA problem

A monkey can type A–Z (26 letters) $\boxed{\tau \geq 11}$

$L_n \in \{A, B, C, D, \dots, Z\}$ (not numbers!!)

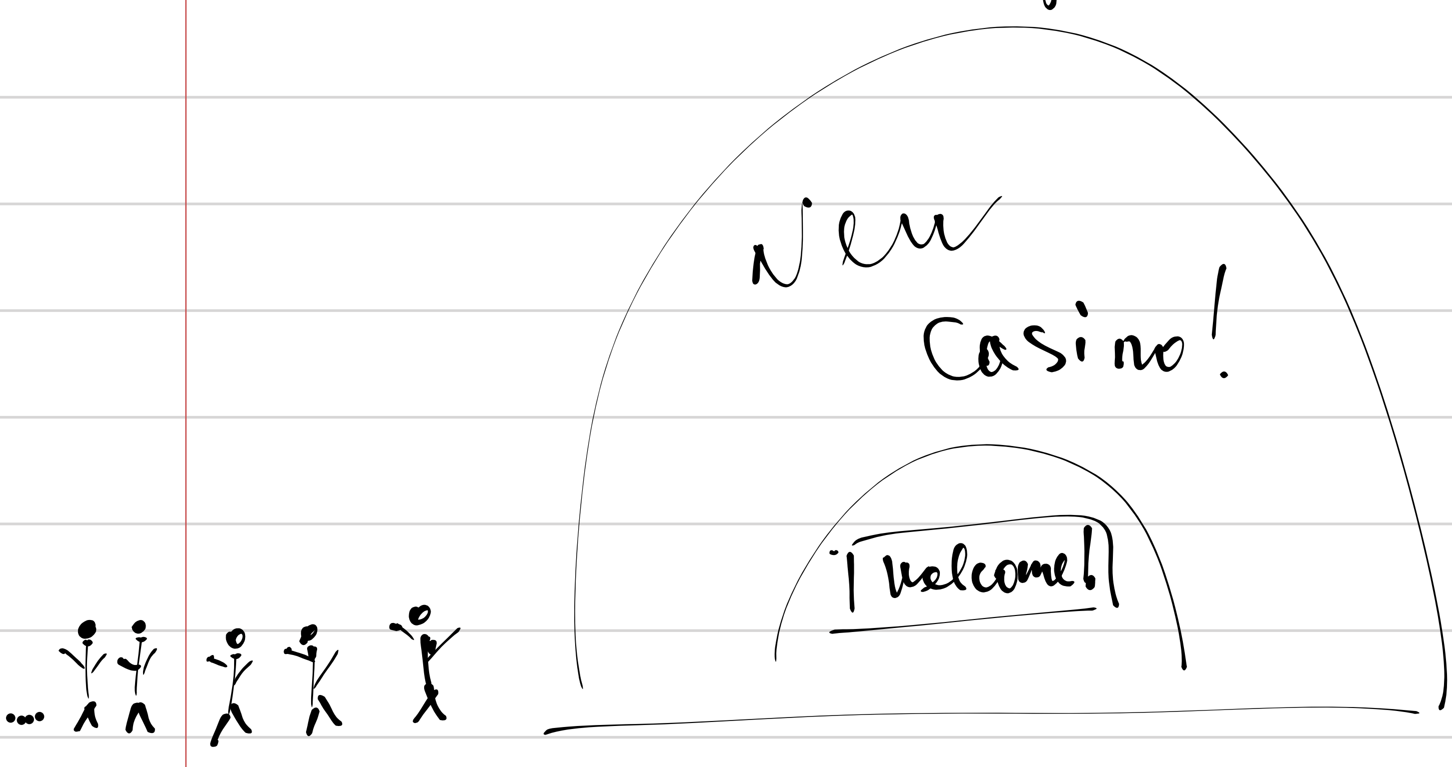
L_n are iid $P(L_n = A) = P(L_n = B) = \dots = \frac{1}{26}$

$\tau = \min\{t \mid \text{when ABRACADABRA appears}\}$

$$\boxed{\tau = \min\{t \mid L_{t-10} = A,\ L_{t-9} = B,\ L_{t-8} = R, \dots, L_t = A\}}$$

$(\mathcal{F}_n) = \sigma(L_1, L_2, L_3 \dots L_n)$ $\boxed{E(\tau)?}$

idea: martingale is a fortune in fair game.



$E\left(\text{bid} \times 26 \cdot \frac{1}{26} + \frac{25}{26} \cdot 0\right) = \text{bid}$

Our Rules: [for every guy that fair!]

1. one person enters every moment
2. you can bring only 1 \$.
3. you bet all you fortune on the next letter of ABR ... RA every moment of time
4. you start by 1.
5. you fortune is $\begin{cases} \times 26 & \text{if you guess right} \\ \times 0 & \text{otherwise} \end{cases}$

Martingales: guy id — M_{it} — fortune of guy i at time t.

guy 1: $M_{10}=1$ (time) $M_{11}, M_{12}, \ldots\ldots$

guy 2: $M_{20}=1$ $M_{21}=1$ M_{22} M_{23} $M_{24}\ldots$ he plays.

player 3: $M_{30}=1$ $M_{31}=1$ $M_{32}=1$ M_{33} $M_{34}\ldots$ she plays

...

$$M_{11} = \begin{cases} +26 & \text{with prob } \frac{1}{26} \\ 0 & \text{with prob } \frac{25}{26} \end{cases}$$

11:25

	$t=1$	$t=2$	$t=3$	$t=4$	...	...
player 1	A $M_{11}=26$	B $M_{12}=0$	R $M_{13}=0$	A $M_{14}=0$	C $M_{15}=0$	...
player 2	-	A $M_{22}=26$	B $M_{23}=26^2$	R $M_{24}=0$	A $M_{25}=0$	...
player 3	-	-	A $M_{33}=0$	B $M_{34}=0$	R $M_{35}=0$	A ...
Monkey	A	A	B	A	R	

11:25

τ = first moment of ABRACADABBA

$E(\tau)$?

Apply Doob's theorem to M_{1t} !!

$$\boxed{E(M_{1\tau}) = M_{10} = 1}$$

$$26^{11}\cdot\frac{1}{26^{11}} + 0\cdot\frac{26^{11}-1}{26^{11}} = 1 \quad !!$$

$1=1$ no τ in the formula

$$X_t = M_{1t} + M_{2t} + M_{3t} + \dots + M_{tt} - t$$

current wealth of all players

initial wealth of t players.

(A) $P(\tau = +\infty) = 0$
$Y_t = X_{t\wedge\tau}$ is bounded

(B) $E(\tau) < +\infty$
and
$E(|\Delta Y_{t+1}| \mid \mathcal{F}_t)$ is bounded
$|\Delta Y_{t+1}| \le 26^1 + 26^4 + 26^{11} + 1$

$$E(X_{t+1} \mid \mathcal{F}_t) = X_t$$

X_t is mart
τ - stopp time
A or B ?

Doob's

$$E(X_\tau) = X_0 = 0$$

$\tau \le 11 \cdot Geom(p)$
$E(\tau) < +\infty$

$$E(M_{1\tau} + M_{2\tau} + \dots + M_{\tau\tau}) - E(\tau) = 0$$

$E(\tau)$?

player $(\tau - 10)$ A B R A C A D A B R A $26^{11} = M_{\tau-10,\tau}$

player $(\tau-3)$ - - - - - - - - - - A B R A $M_{\tau-3,\tau} = 26^4$

player $(\tau-2)$ A B R $M_{\tau-2,\tau} = 0$

player $(\tau-1)$ - - - - - - - - - - - - - - A B $M_{\tau-1,\tau} = 0$

player (τ) - - - - - - - - - - - - - A $M_{\tau\tau} = 26$

monkey: A A B A R A ...... A B R A C A D A B R A

$\tau-2$ $\tau-1$ τ

$$E(26^{11} + 26^4 + 26) = E(\tau)$$

$$E(\tau) = 26^{11} + 26^4 + 26$$

!!

? $E(\tau^2)$?

Geom (p) → version 1: the number of failures before first success
→ version 2 — // — trials to the first success.

coin → H 0.7
→ T 0.3

N - number of throws to obtain first „Head"

$E(N) = \frac{1}{p}$

t	1	2	3	4	...
$P(N=t)$	0,7	$0{,}3 \cdot 0{,}7$	$0{,}3^2 \cdot 0{,}7$	$0{,}3^3 \cdot 0{,}7$	...

Continuous time. $t \in [0; +\infty)$

def filtration $(\mathcal{F}_t)_{t=0}^{\infty}$ is a family of σ-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$.

int: in the future you know more than in the past

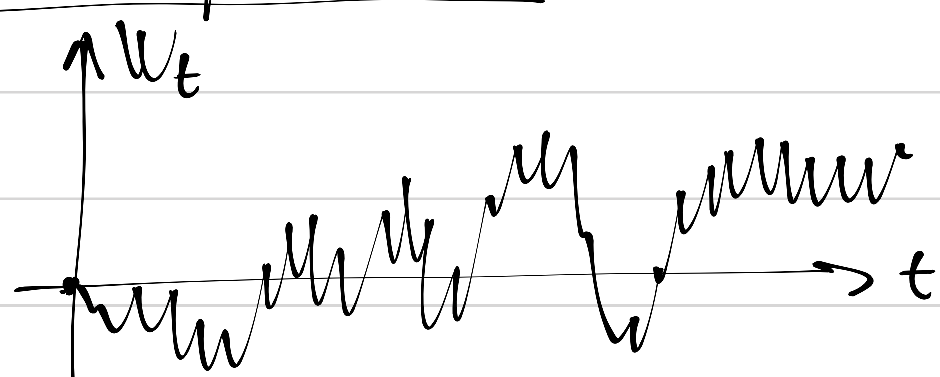
works also for discr

def. Process $(M_t)_{t=0}^{\infty}$ [(M_t) or even M_t] is a martingale wrt to $(\mathcal{F}_t)$

$$E(M_t \mid \mathcal{F}_s) = M_s \quad \text{for } s \le t.$$

$E(M_{s+1} \mid \mathcal{F}_s) = M_s$ (in discrete time)

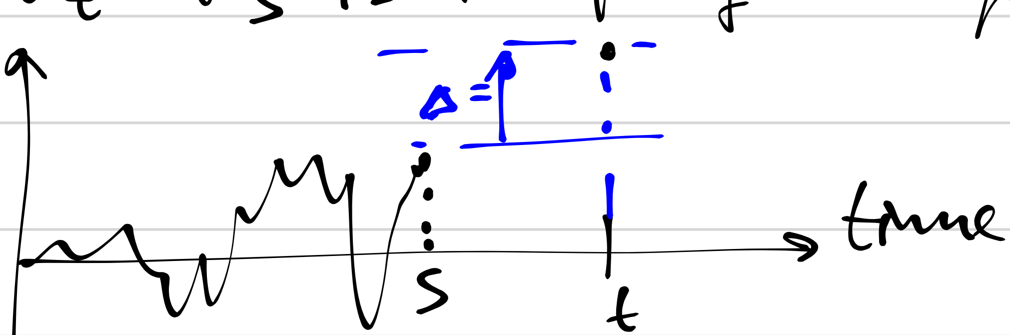
Wiener process (aka Brownian Motion)



def (W_t) is a Wiener Process wrt filtr-n $(\mathcal{F}_t)$

1. $W_0 = 0$
2. (W_t) is adapted to $(\mathcal{F}_t)$ {for $\forall t$ r.v. W_t is measurable wrt $\mathcal{F}_t$}
3. future increment is indep-t of current info

   $\Delta = W_t - W_s$ is indep of $\mathcal{F}_s$ for $s \le t$



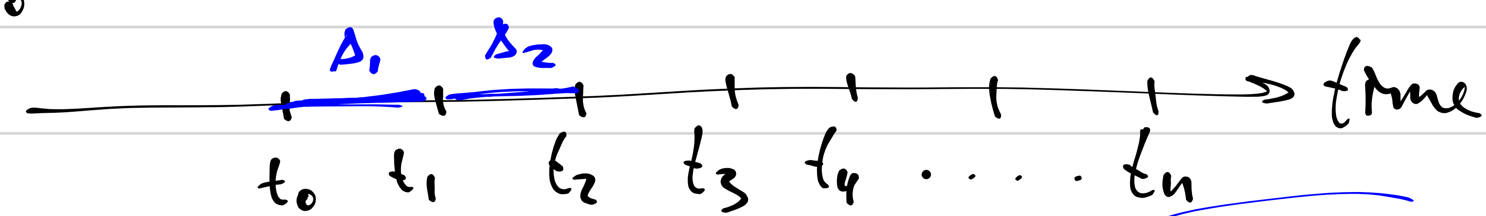
4. $\Delta = W_t - W_s \sim N(0; t-s)$ (for $s \le t$)
5. $P(\text{trajectory of } W_t \text{ is continuous}) = 1$

Alternative def. (without σ-algebra)

old def — (W_t) is a Wiener process

{ (W_t) is a Wiener process wrt to natural filtration }

1. $W_0 = 0$

2. 

$\Delta_i = W(t_i) - W(t_{i-1})$ for $\forall$ $t_0 \le t_1 \le t_2 \le \dots \le t_n$

$\Delta_1, \Delta_2 \dots \Delta_n$ are independent RVs

3. $\Delta = W_t - W_s \sim N(0; t-s)$ for $\forall$ $s \le t$

4. $P(\text{traj-ry of } W_t \text{ is continuous}) = 1.$

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reminder.

natural filtration $\mathcal{F}_t = \sigma\left((W_s)_{s=0}^{t}\right)$

I know current and all past values of the process.

Ex 1. $\boxed{\text{Is } (W_t) \text{ a mart? } \quad s \le t}$

"boring proof"

known value (p2)

$$E(\underbrace{W_t}_{\text{future value}} \mid \mathcal{F}_s) = E(\underbrace{W_s}_{\text{current value}} + \underbrace{W_t - W_s}_{\text{increment}} \mid \mathcal{F}_s) =$$

$$= W_s + E(W_t - W_s \mid \mathcal{F}_s) = W_s + E(W_t - W_s) =$$

independ (p3)   (p4) $N(0; t-s)$

$= W_s + 0 = W_s$. yes, (W_t) is a martingale.

Ex

a) $E(W_t) = E(W_t - 0) = E(\overbrace{W_t - W_0}^{p3 \; \sim N(0;\, t-s)}) = \boxed{0}$

b) $E(W_t^2) = Var(W_t)$ $\{$as $E(W_t)=0\}$ $= Var(W_t - W_0) = t-0$

c) $Var(W_t) = \;\; \| \;\; = t-0$

d) $Cov(W_5, W_7) = Cov(W_5, W_5 + W_7 - W_5) = Var(W_5) +$

e) $P(W_6 > 11 W_5) =$

f) $P(|W_{100}| < 20) =$

$+ Cov(W_5, W_7 - W_5) =$

$= \underbrace{Var(W_5)}_{=5} + \underbrace{Cov(W_5 - W_0, W_7 - W_5)}_{\text{by } p3 \;=\; 0}$

$0 \quad 5 \quad 7 \quad t$

$\Delta_1 = W_5 - W_0$

$\Delta_2 = W_7 - W_5$

e) $P(W_6 > 11 W_5) =$

$= P(W_6 - 11 W_5 > 0) =$

$= P(\underbrace{W_6 - W_5}_{N(0;6-5)} - 10 \underbrace{W_5}_{N(0;5)} > 0) = \frac{1}{2}$

indep.

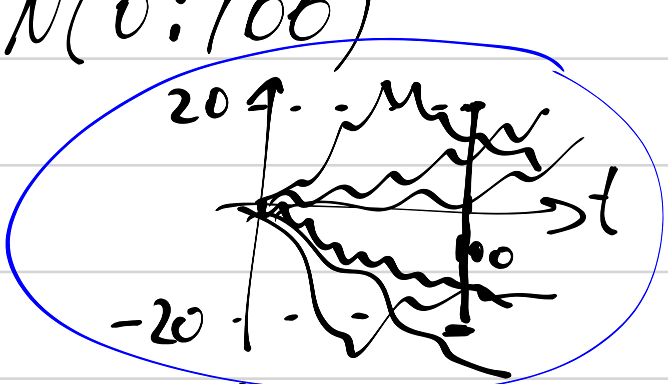
$W_6 - 11 W_5 \sim N(0;\, 1 + 100 \cdot 5) \sim N(0;\, 501)$

$\boxed{5W_1 + 7W_2 + 6W_{15}} = \underline{6(W_{15} - W_2)} + \underline{13(W_2 - W_1)} +$

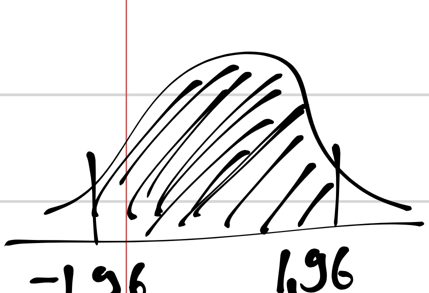
$\underline{+18(W_1 - W_0)}$

$W_{100} \sim N(0 : 100)$

$P(|W_{100}| < 20) \overset{?}{=}$



$= P(-20 < W_{100} < 20) =$



$= P\left(\frac{-20-0}{\sqrt{100}} < \frac{W_{100} - 0}{\sqrt{100}} < \frac{20-0}{\sqrt{100}}\right) =$

$= P(-2 < N(0;1) < 2) \simeq 0{,}95$

Questions !!

Q1

X, R - random variables

$\mathcal{F}$ - σ-algebra

A - event

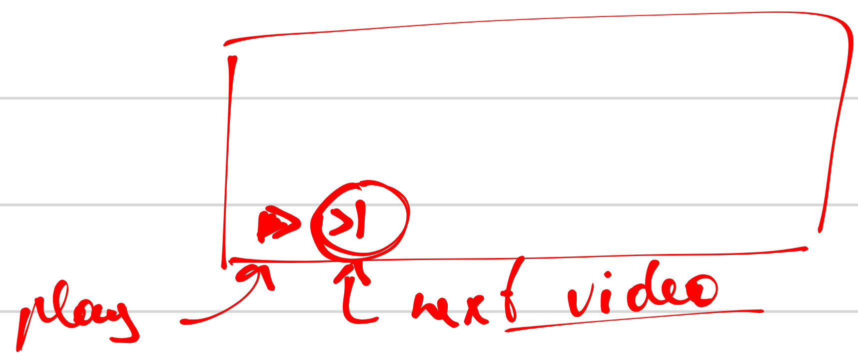
number → $E(X)$

R.V. → $E(X \mid \mathcal{F})$

number → $E(X \mid A)$

R.V. → $E(X \mid R) = E(X \mid \sigma(R))$

Q2



Q3

Q4

$\tau \le \tau_1 \sim 11 \cdot \mathrm{Geom}\left(\frac{1}{26^{11}}\right)$

$\tau = \min \{ t \mid L_{t-10} = A, L_{t-9} = B \dots L_t = A \}$

$\tau_1 = \min \{ t \mid L_{t-10} = A \dots L_t = A,\ t = 11 \cdot k \}$

A, B, R, A, C, A, D, A, B, R, A | A, B, R, A, C, ... A | fine

$t=1$ $t=11$ $t=12$ .... $t=22$

100 ... 110 accept.

105 ... 115 do not accept

$$E(\tau) \le 11 \cdot E\left(\mathrm{Geom}\left(\tfrac{1}{26^{11}}\right)\right) = 11 \cdot 26^{11}$$

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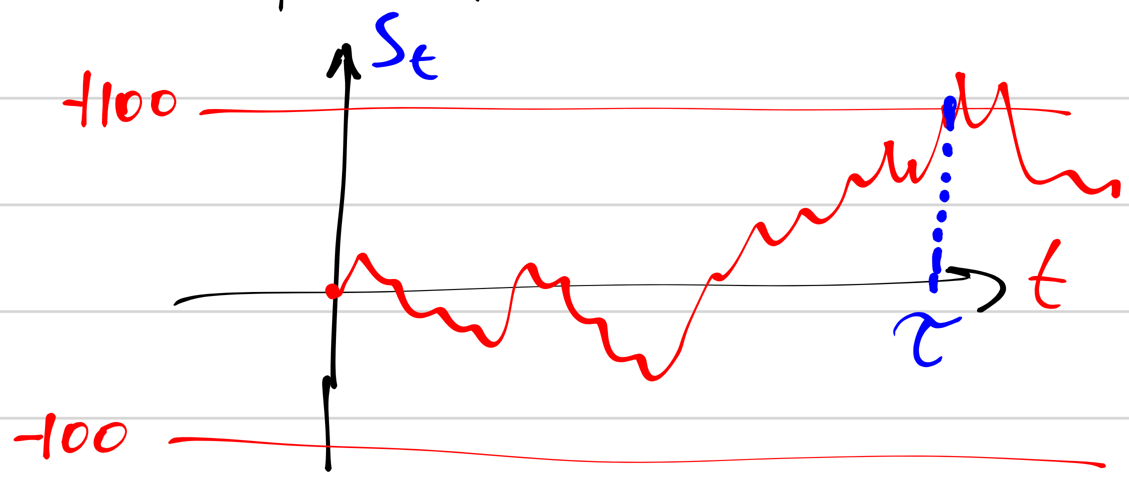
problem №6. [20 pts] ← double points!

$S_t = Z_1 + Z_2 + \ldots + Z_t$ $\quad S_0 = 0$

$Z_t \sim$ iid

z	$+1$	-1
$P(Z_t = z)$	$\frac{1}{2}$	$\frac{1}{2}$

$\tau = \min \{ t \mid |S_t| = 100 \}$



a) $f(\lambda)$? such that $M_t = (f(\lambda))^t \cdot \exp(\lambda S_t)$ is martingale.

$f = f(\lambda)$

$$E(M_{t+1} \mid \mathcal{F}_t) = M_t$$

$$E(f^{t+1} \cdot \exp(\lambda \cdot S_{t+1}) \mid \mathcal{F}_t) = f^t \cdot \exp(\lambda S_t)$$

$$f^{t+1} E(\exp(\lambda S_t) \cdot \exp(\lambda \cdot Z_{t+1}) \mid \mathcal{F}_t) = f^t \cdot \exp(\lambda S_t)$$

is known

$$f \cdot E(\exp(\lambda Z_{t+1}) \mid \mathcal{F}_t) = 1$$

indep.

$$f \cdot E(\exp(\lambda Z_{t+1})) = 1$$

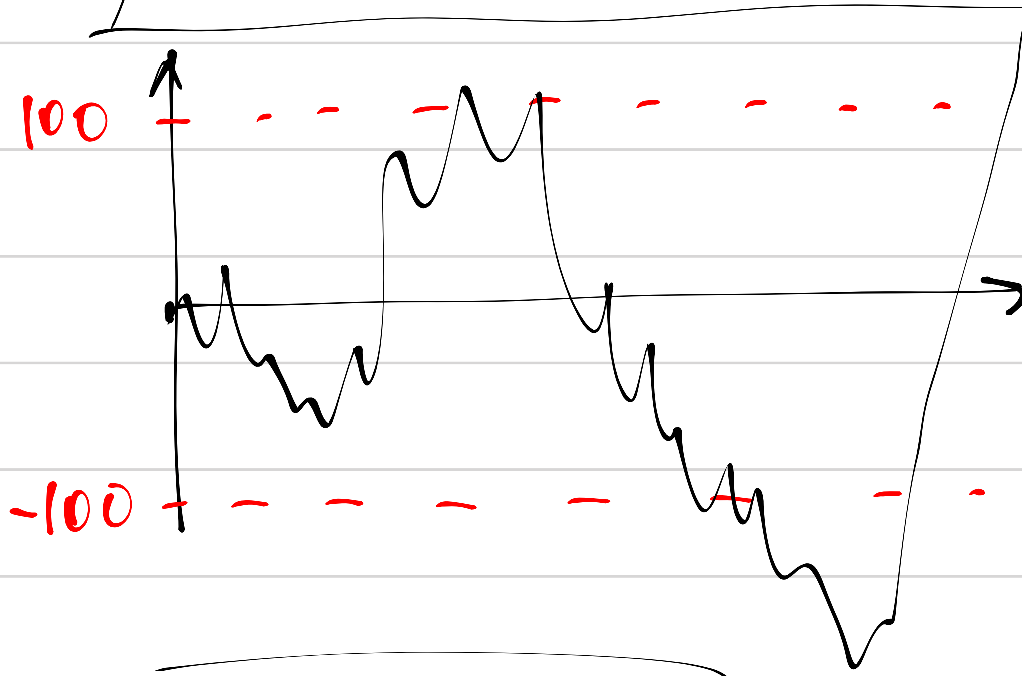
$$f \cdot \left(\frac{1}{2} \cdot \exp(\lambda) + \frac{1}{2} \cdot \exp(\lambda \cdot (-1))\right) = 1$$

$$f = \frac{1}{\frac{1}{2}(\exp(\lambda) + \exp(-\lambda))} = \frac{2}{\exp(\lambda) + \exp(-\lambda)}$$

a) $f(\lambda)$ such that $M_t = (f(\lambda))^t \cdot \exp(\lambda S_t)$ is a mart.

$$f(\lambda) = \frac{2}{\exp(\lambda) + \exp(-\lambda)}$$

b) Are S_τ and τ indep-t?

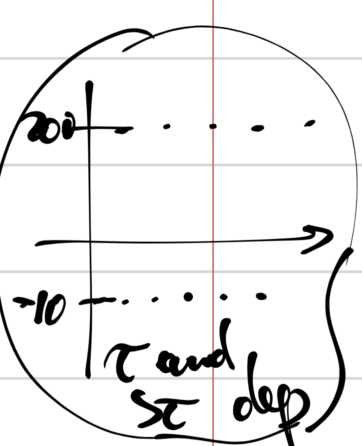


$S_\tau = 100$

τ ?

$\tau = 256$

$S_\tau \nearrow 100$?
$\searrow -100$?



yes, independent.

$P(S_\tau = 100 \mid \tau = t) = P(S_\tau = 100) = \frac{1}{2}$

(due to the symmetry)

c) Using Doob's theorem find $E(f(\lambda)^\tau) = \varphi(\lambda)$

We are not req-d to check:

$\tau \leq \tau_1$ $\quad$ 200 $\quad$ (+1 +1 +1 +1 .....)

$E(\tau) \leq 200 \cdot E(\tau_1) = 200 \cdot \frac{1}{(\frac{1}{2})^{200}}$

$$E\left(f_\lambda^\tau \cdot \exp(\lambda \cdot S_\tau)\right) = f_\lambda^0 \cdot \exp(\lambda \cdot S_0)$$

$= E(M_0)$

$$E\big(\underbrace{f_\lambda^{\tau} \cdot \exp(\lambda S_\tau)}_{\text{independent}}\big) = 1$$

$$\underbrace{E(f_\lambda^{\tau})}_{?} \cdot E(\exp(\lambda S_\tau)) = 1$$

$$E(\exp(\lambda S_\tau)) = \frac{1}{2} \cdot \exp(100\lambda) + \frac{1}{2}\exp(-100\lambda)$$

c) $$E(f_\lambda^{\tau}) = \frac{1}{\frac{1}{2}\exp(100\lambda) + \frac{1}{2}\exp(-100\lambda)} =$$

$f = \frac{2}{\exp(\lambda)+\exp(-\lambda)}$

$$\boxed{E(f_\lambda^{\tau}) = \frac{2}{\exp(100\lambda) + \exp(-100\lambda)}}$$ [in terms of λ]

reverse (e), (d)

f - constant.

d) $\boxed{\text{find } G(f) = E(f^{\tau}) \text{ explicitly or just find } G'(1)}$

e) how can we use $G(f)$ to find $E(\tau)$?

$$G(f) = E(f^{\tau})$$

$$G'(f) = E(\tau \cdot f^{\tau-1}) \leftarrow \text{derivative [without full proof]}$$

$$\boxed{G'(1) = E(\tau)}$$

in terms of λ

$$E\left[\left(\frac{2}{\exp(\lambda)+\exp(-\lambda)}\right)^{\tau}\right] = \frac{2}{\exp(100\lambda) + \exp(-100\lambda)}$$

$$G(f) = E(f^{\tau}) = \quad .$$

in terms of f.

$$E(\tau) = \boxed{G'(1)}$$ directly.

$$H(\lambda) = G\left(\frac{2}{\exp(\lambda)+\exp(-\lambda)}\right) = \frac{2}{\exp(100\lambda)+\exp(-100\lambda)}$$

calculer !

$$= 2\cdot(\ldots)^{-1}$$

$G'(1)$?

$$H(\lambda) = G(f(\lambda))$$

$$\frac{dH}{d\lambda} = G'(f(\lambda))\cdot f'(\lambda)$$

$\lambda \to 0 \quad f(\lambda) \to 1$

$$G'(f(\lambda)) = \frac{H'(\lambda)}{f'(\lambda)}$$

$$\lim_{\lambda\to 0} G'(f(\lambda)) = \lim_{\lambda\to 0} \frac{H'(\lambda)}{f'(\lambda)} =$$

$$= \lim_{\lambda\to 0} \frac{-1\cdot(\exp(100\lambda)+\exp(-100\lambda))^{-2}\cdot(100\exp(100\lambda) - 100\exp(-100\lambda))}{-1\cdot(\exp(\lambda)+\exp(-\lambda))^{-2}\cdot(\exp(\lambda)-\exp(-\lambda))} =$$

$$= \lim_{\lambda\to 0} \frac{100\exp(100\lambda) - 100\exp(-100\lambda)}{\exp(\lambda)-\exp(-\lambda)} =$$

L'Hopital

$\exp(u) \sim 1+u$ (taylor)

$$= \lim \frac{100(1+100\lambda) - 100\cdot(1-100\lambda) + o(\lambda)}{1+\lambda-(1-\lambda) + o(\lambda)} =$$

$$= \lim_{\lambda\to 0} \frac{20000\lambda + o(\lambda)}{2\lambda + o(\lambda)} = 10000 \ !!$$

$$G'(1) = 10000$$

$E(\tau)$!!