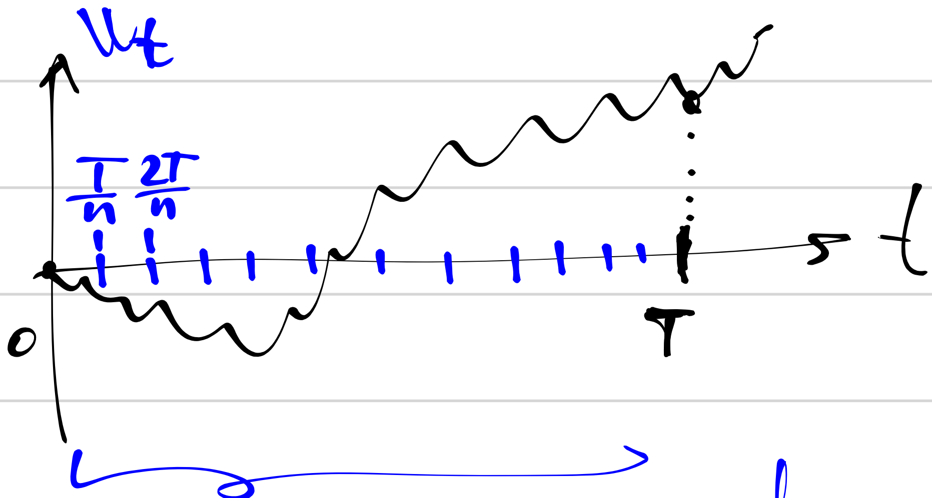
Ito's integral

$R_n \xrightarrow{M.S.} R$ $\boxed{\text{limit in mean square}}$

$$\lim_{n\to\infty} E\left((R_n - R)^2 \right) = 0.$$

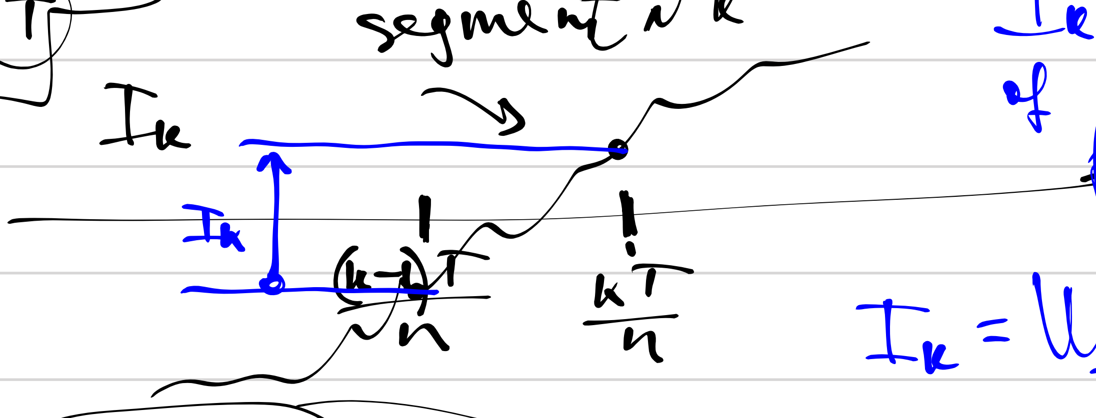
Ex. (!)



$$R_n = I_1^2 + I_2^2 + \dots + I_n^2$$

T = const
$n \uparrow \infty$

$\boxed{\begin{array}{l} \operatorname{Var} R_n \to 0 \\ E R_n = T \end{array}}$



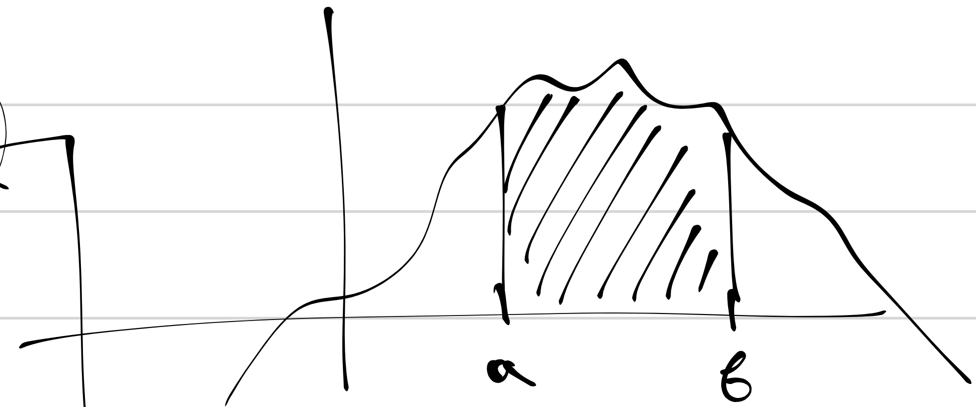
I_k - Increment of Wiener process for segment № k

$$I_k = W_{\frac{kT}{n}} - W_{\frac{(k-1)T}{n}}$$

$$R_n \xrightarrow[n\to\infty]{M.S.} T$$

$\boxed{\begin{array}{l} \text{Law of large Numbers.} \\ \lim \bar{X} = EX_1 \\ \text{if } X_1, X_2, X_3 \dots \text{ are iid.} \end{array}}$

$$\int_a^b f(t)\,dt$$



Interpr. 1. Area under $f(t)$ on the segment $[a;b]$.

Financial interpretation

$$\int_a^b f(t)\,dt$$

quantity function. price function

= net profit if $f(t)$ is the number shares and t is the price of one share.

$$\int_2^5 7\,dt = -7\cdot 2 + 7\cdot 5 = 7\cdot 3 = 21 \quad !!$$

price at t.

7 shares.

$t=2$ buy 7 shares. $t=5$ sell 7 shares.

$$\int_2^5 7\cdot dW_t = \text{net profit} = -7\cdot W_2 + 7\cdot W_5$$

quantity

price at time t

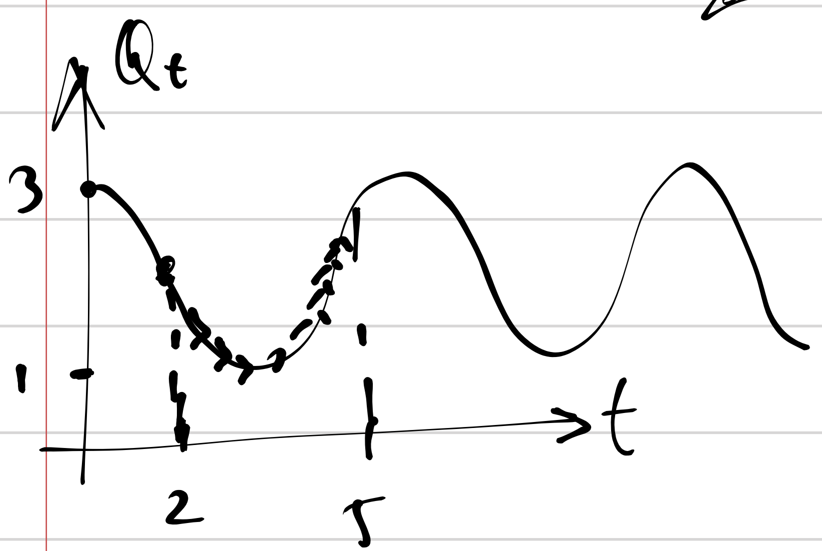
$t=2$ buy 7 shares

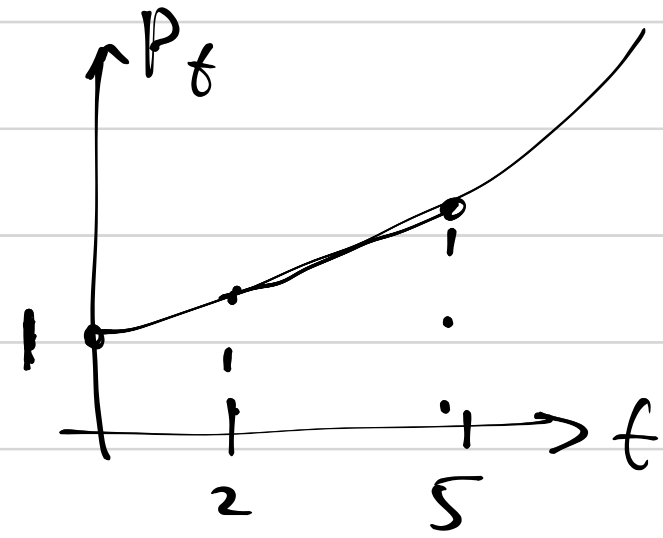
$t=5$ sell 7 shares.

Ex. net profit. $= \int_2^5 \underbrace{(2+\cos t)}_{Q_t} \cdot d\underbrace{\exp t}_{P_t}$

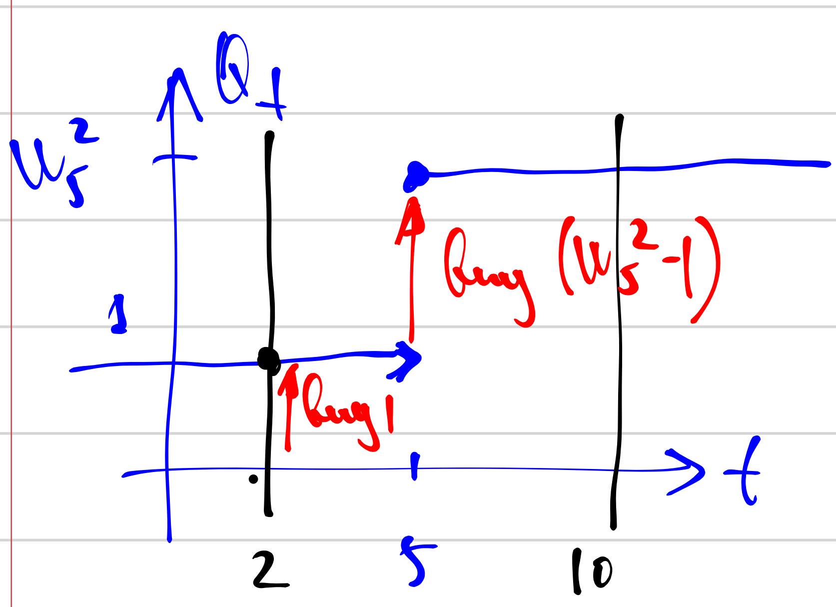
Area??

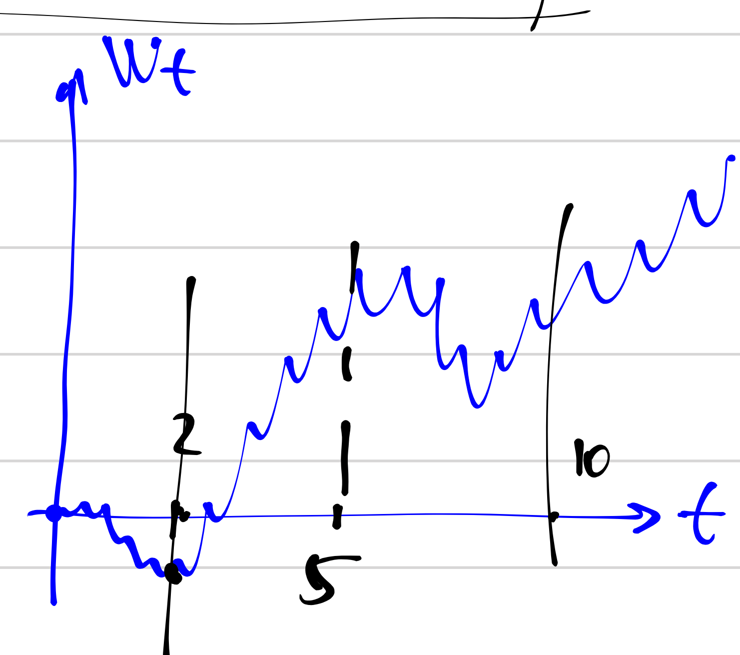
Financial interpret.





Ex. $\int_2^{10} Q_t \, dW_t = ?$ $\qquad Q_t = \begin{cases} 1 & t<5 \\ W_5^2 & t \geq 5 \end{cases}$





$$\overset{?}{=} \underbrace{-1 \cdot W_2}_{\text{at } t=2} \; \underbrace{-(W_5^2-1) \cdot W_5}_{\text{at } t=5} \; \underbrace{+ W_5^2 \cdot W_{10}}_{\text{at } t=10}$$

case A : $W_5^2 < 1$ $\quad (W_5^2 - 1) < 0$ $\quad$ sell shares at $t=5$

case B : $W_5^2 > 1$ $\quad (W_5^2 - 1) > 0$ $\quad$ buy shares at $t=5$

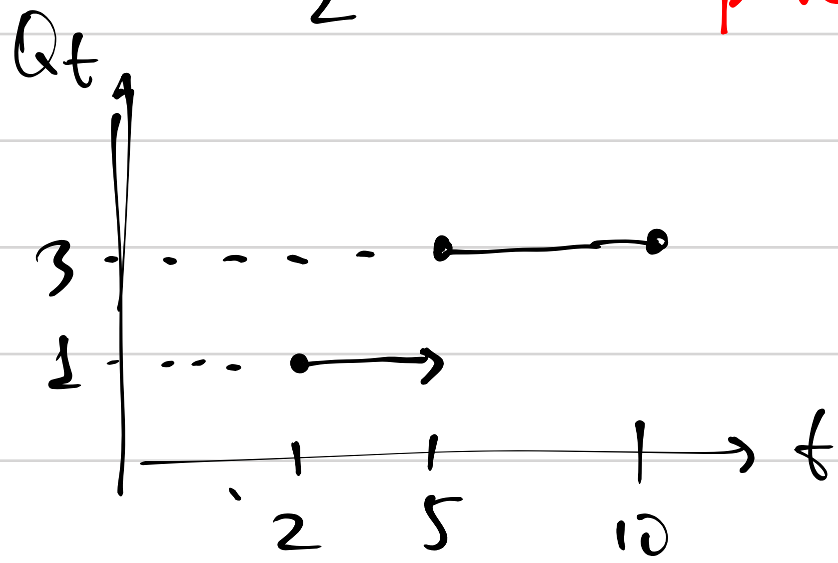
$$\boxed{Q_t = \begin{cases} 1 & t < 5 \\ 3 & t \geq 5 \end{cases}}$$

W_t - price at t

number of shares

$$\int_2^{10} Q_t \, dW_t = -1 \cdot W_2 - 2 \cdot W_5 + 3 \cdot W_{10}$$

price of a share.



transactions:

at $t=2$ buy 1 share

at $t=5$ buy 2 more shares

at $t=10$ sell 3 shares

(!) When (W_t) and (Q_t) are random processes old rules of integration are not applicable.

non-random functions:

$$\int_0^t f(u)\, df(u) = [\text{by Newton-Leibnitz}] = \frac{f^2(t)}{2} - \frac{f^2(0)}{2}$$

random processes:

$$\int_0^t W_u \cdot dW_u \neq \frac{W_t^2}{2} - \frac{W_0^2}{2}$$

Some old results are ok!

$$\int_0^t A_u \, dW_u + \int_0^t B_u \, dW_u = \int_0^t (A_u + B_u)\, dW_u$$

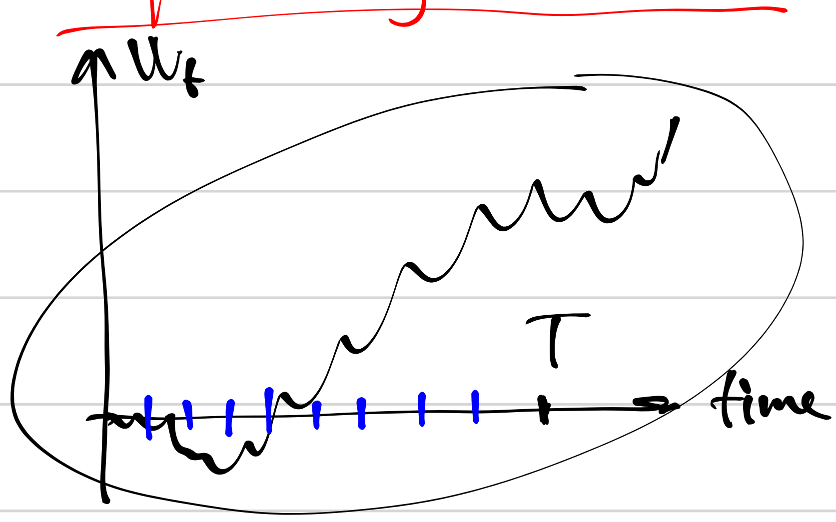
$$\int_0^t \alpha \cdot X_u \, dW_u = \alpha \int_0^t X_u \, dW_u$$

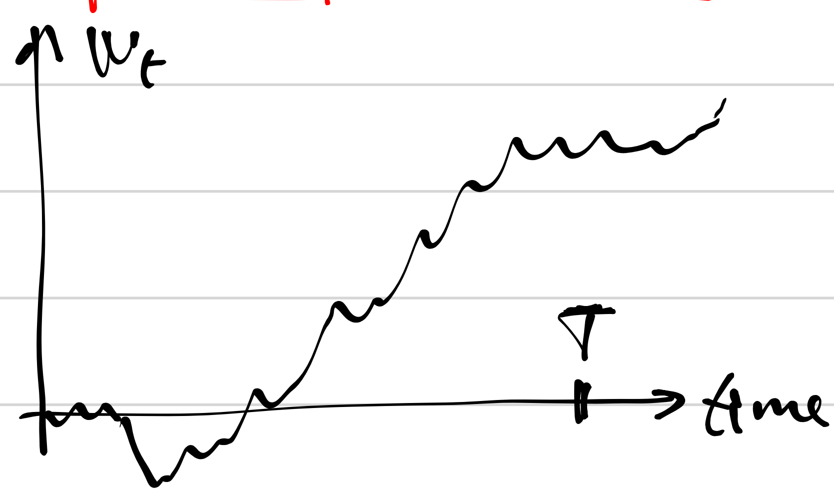
Exercise.

$$\int_0^t W_u \, dW_u$$ using financial interpretation.

quantity function

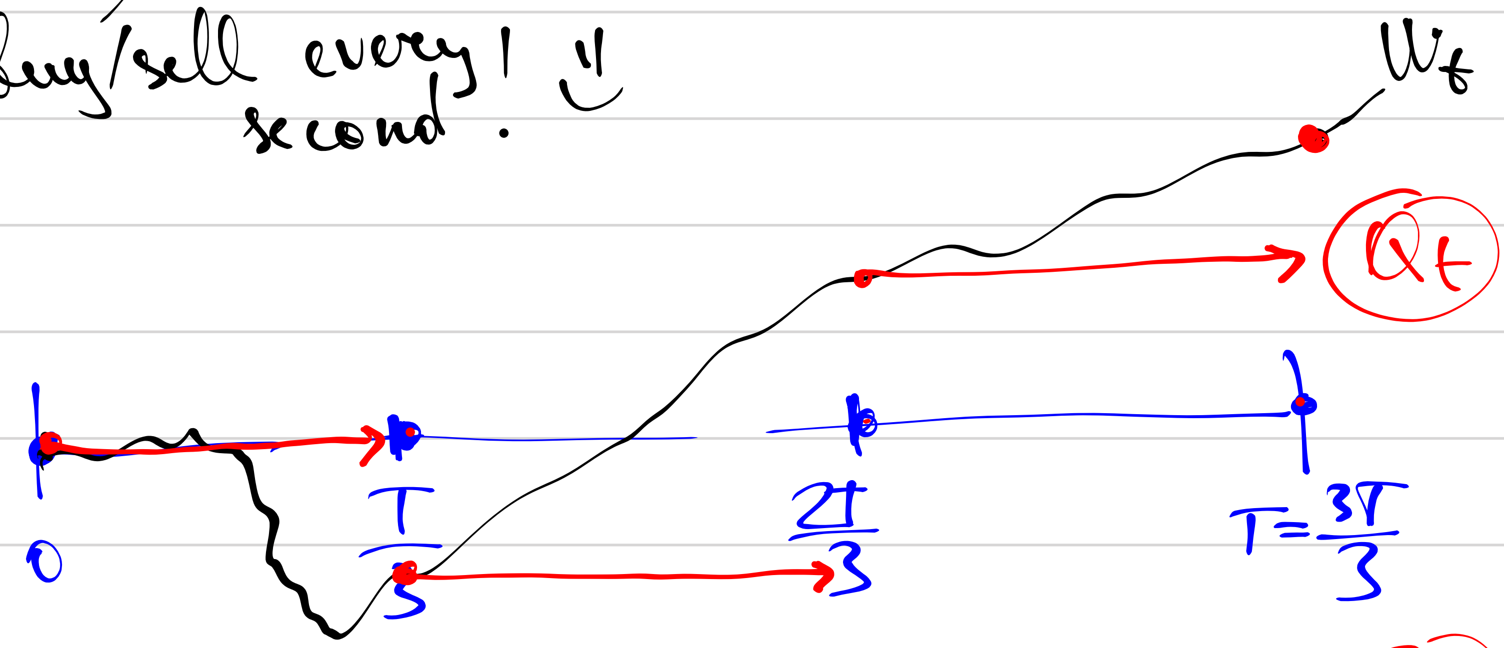
price function





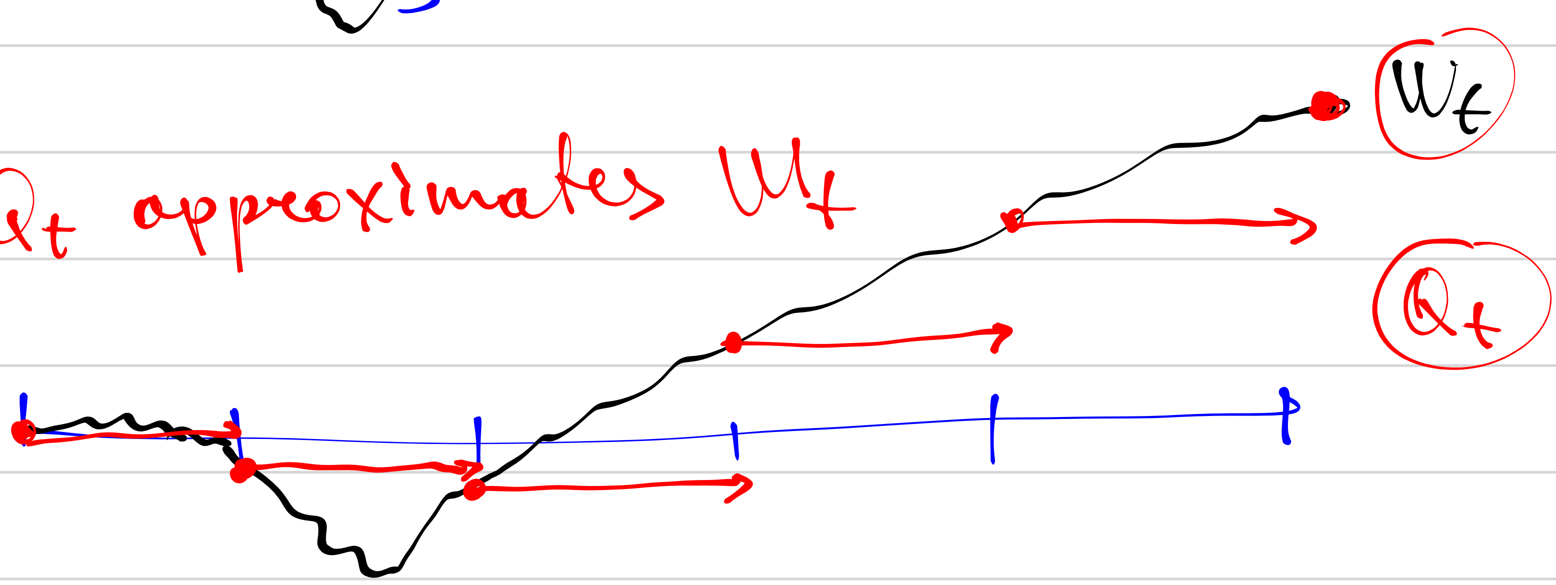
buy/sell every second! ‼

$n=3$



Q_t approximates W_t

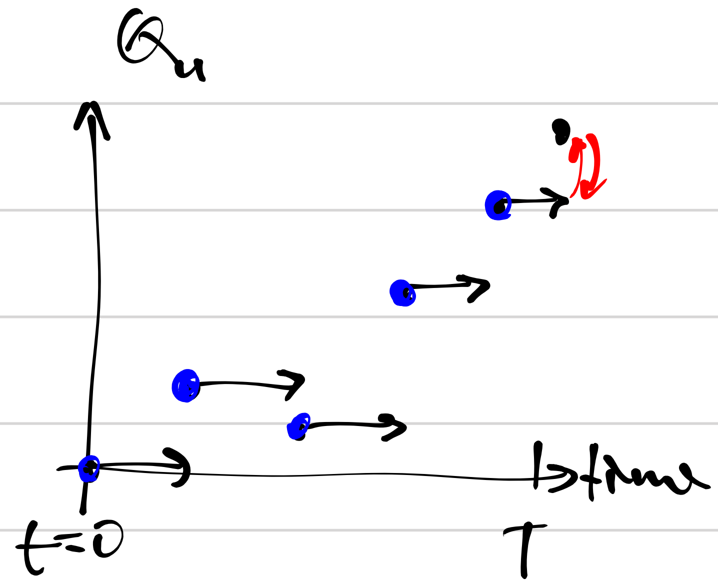
$n=5$

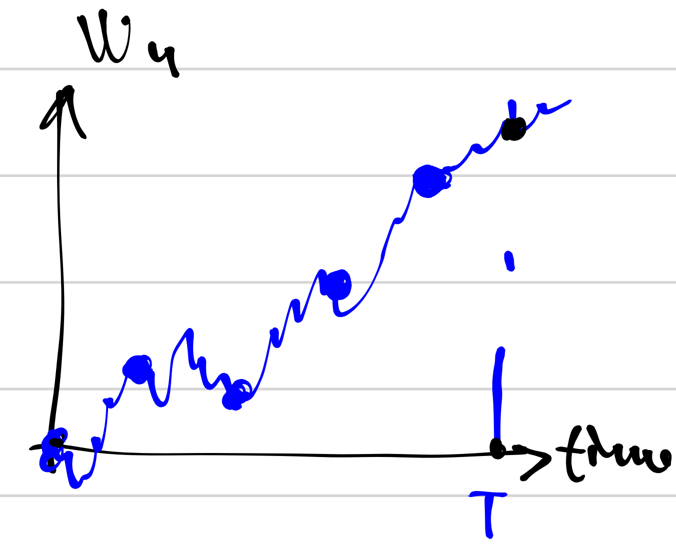


$n \to \infty$ $Q_t \to W_t$ for all t due to continuous traj-ries of W_t

$\boxed{\text{def}}$ $$\int_0^T W_u \, dW_u := \underset{n\to\infty}{\text{limit in M.S.}} \int_0^T Q_u \, dW_u$$

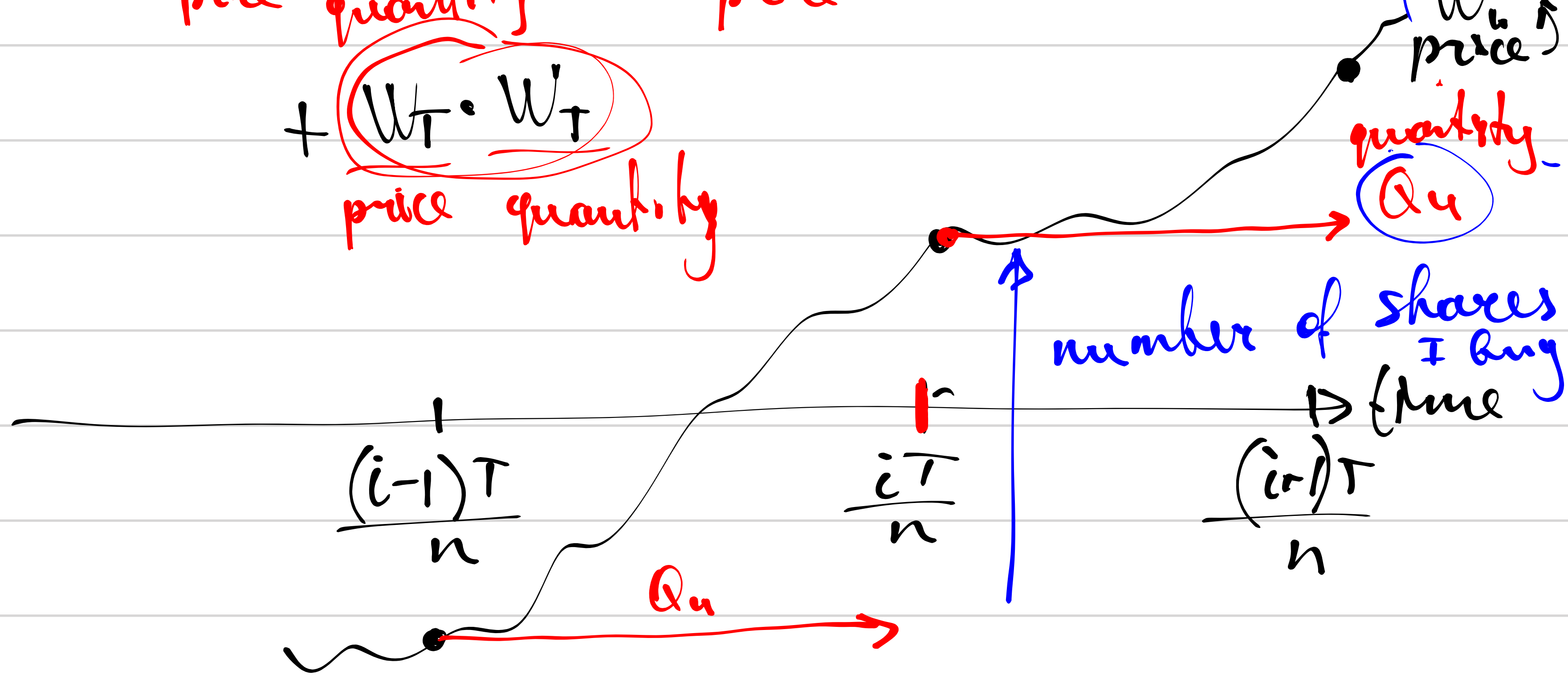
$$X_n = \int_0^T Q_u \, dW_u =$$





$$X_n = \underbrace{\underset{\text{price}}{-W_0} \cdot \underset{\text{quantity}}{W_0}}_{\text{initial transaction}} \underbrace{- \sum_{i=1}^{n} \underset{\text{price}}{W\left(\tfrac{iT}{n}\right)} \cdot \left(W\left(\tfrac{iT}{n}\right) - W\left(\tfrac{(i-1)T}{n}\right)\right)}_{\text{interim transact.}} +$$

$$+ \underset{\text{price}}{W_T} \cdot \underset{\text{quantity}}{W_T}$$



$$X_n = -W_0^2 + W_T^2 - \sum_{i=1}^{n} W\left(\tfrac{iT}{n}\right) \cdot \left(W\left(\tfrac{iT}{n}\right) - W\left(\tfrac{(i-1)T}{n}\right)\right)$$

$$X_n \xrightarrow{M.S.} ?$$

$$R_n = \sum_{i=1}^{n} I_i^2 = \sum_{i=1}^{n} \left(W\left(\tfrac{iT}{n}\right) - W\left(\tfrac{(i-1)T}{n}\right)\right)^2 \xrightarrow{M.S.} T$$

$$X_n = -W_0^2 + W_T^2 - \sum_{i=1}^{n} W\left(\frac{iT}{n}\right)\cdot\left(W\left(\frac{iT}{n}\right) - W\left(\frac{(i-1)T}{n}\right)\right)$$

$$X_n \xrightarrow{M.S.} ?$$

$$R_n = \sum_{i=1}^{n} I_i^2 = \sum_{i=1}^{n}\left(W\left(\frac{iT}{n}\right) - W\left(\frac{(i-1)T}{n}\right)\right)^2 \xrightarrow{M.S.} T$$

let's simplify notation

$$\underbrace{W\left(\frac{iT}{n}\right)} = \boxed{V_i}$$

segment n^i

$\frac{iT}{n}$

$$? \xleftarrow{M.S.} X_n = -V_0^2 + V_n^2 - \sum_{i=1}^{n} V_i\cdot(V_i - V_{i-1})$$

$$R_n = \sum_{i=1}^{n}(V_i - V_{i-1})^2 = \sum_{i=1}^{n}\left(V_i^2 + V_{i-1}^2 - 2V_iV_{i-1}\right) =$$

$$= \sum_{i=1}^{n} V_i^2 + \underbrace{\sum_{i=1}^{n} V_{i-1}^2}_{\sum_{i=1}^{n} V_i^2 - V_n^2 + V_0^2} - 2\sum V_iV_{i-1} =$$

$$= 2\sum_{i=1}^{n} V_i^2 - 2\sum V_iV_{i-1} - V_n^2 + V_0^2 =$$

$$T \xleftarrow{M.S.} R_n = \boxed{2\cdot\sum V_i(V_i - V_{i-1}) - V_n^2 + V_0^2}$$

$$X_n = -V_0^2 + V_n^2 - \frac{R_n}{2} - \frac{V_n^2}{2} + \frac{V_0^2}{2}$$

$$\boxed{X_n = \frac{V_n^2}{2} - \frac{V_0^2}{2} - \frac{R_n}{2}} \xrightarrow{M.S.} \frac{W_T^2}{2} - \frac{T}{2}$$

$$R_n \xrightarrow{M.S.} T$$

$$V_n = W_T \xrightarrow[n\to\infty]{M.S.} W_T \qquad V_0 = W_0 = 0 \text{ (by def of Wiener process)}$$

We are brave !!

$$\boxed{\int_0^T W_u\, dW_u = \frac{W_T^2}{2} - \frac{T}{2}}$$

Newton leibnitz

$$\neq \frac{W_T^2}{2} - \frac{W_0^2}{2}$$

$$\int_0^t \exp(-u^2)\,du = \text{!!}$$ no explicit formula with elementary functions.

our integration table !!

$$\int_0^t 1\,dW_u = -1\cdot W_0 + 1\cdot W_t = W_t$$

$$\boxed{\int_0^t W_u\,dW_u} = \frac{W_t^2}{2} - \frac{t}{2}$$

why $\frac{W_t^2}{2} - \frac{W_0^2}{2}$ is not consisten with our intuition and $\frac{W_t^2}{2} - \frac{t}{2}$ is consistent with our intuition?

the price is W_t (Wiener process)

$s \le t$ — s, t

$W_t - W_s$ is independent of $\mathcal{F}_s$.

$W_t - W_s \sim \mathcal{N}(0; t-s)$

On average I should make no profit if I buy and sell shares with the price (W_t)

$\boxed{\text{Newton-Leibnitz}}$: $E\left(\frac{W_t^2}{2} - \frac{W_0^2}{t}\right) = \frac{E(W_t^2)}{2} = \frac{t}{2}$

our answer: $E\left(\frac{W_t^2}{2} - \frac{t}{2}\right) = \frac{E(W_t^2)}{2} - \frac{t}{2} = 0$

→ Michael Steele „Stoch. calculus and financial applications"

standard:

Steven Shreve „Stochastic calculus for finance I, II"

„Under some mild technical conditions" !

$\int_0^t \alpha \cdot A_u dW_u = \alpha \cdot \int_0^t A_u dW_u$

$\int_0^t A_u dW_u + \int_t^{t+\Delta} A_u dW_u = \int_0^{t+\Delta} A_u dW_u$

„boring" properties

① $\int_0^t A_u dW_u + \int_0^t B_u dW_u = \int_0^t A_u + B_u \, dW_u$

② $X_t = X_0 + \int_0^t A_u dW_u$ is a martingale
X_0 = constant.

if we invest in a Wiener process we can't make money on average.

$t \geq s$ $\quad E(X_t | X_s) = X_s$

$E(E(X_t | X_s)) = E(X_s) \quad \forall t \geq s$

$E(X_t) = E(X_s) = X_0$

$$\boxed{E\left(\int_0^t A_u dW_u\right) = 0.}$$

„$E(A_t dW_t) = 0$"

$A_t dW_t$ is not a random variable

Popular notation.

In mathematics, (in general)

$A = B$, $B = C \Rightarrow$
$\Rightarrow A = C$

exceptions:

$$\frac{t^3 = o(t)}{t^4 = o(t)} \not\Rightarrow t^3 = t^4$$

we write:

$$\boxed{t^3 = o(t)}$$

we really mean:

$$\boxed{\lim_{t \to 0} \frac{t^3}{t} = 0}$$

In stochastic Calculus

we write	we mean
$dX_t = A_t\,dt + B_t \cdot dW_t$	$X_T = X_0 + \int_0^t A_u\,du + \int_0^t B_u\,dW_u$
$E(B_t\,dW_t) = 0$	$\rightarrow E\left(\int_0^t B_u\,dW_u\right) = 0$
dW_t does not exist! dX_t does not exist!	(W_t) - random process! (X_t) - random process!
$Var(\underbrace{B_t\,dW_t}_{\text{does not exist}}) = E(\underbrace{B_t^2\,dt}_{\text{does not exist}})$	$Var\left(\int_0^t B_u\,dW_u\right) = E\left(\int_0^t B_u^2\,du\right)$

(3) $$E\left(\int_0^t A_u\,du\right) = \int_0^t E(A_u) \cdot du$$

(4) Most important property !!

Ito's lemma

(chain rule for stochastic integrals)

If $X_t = h(W_t, t)$

$\frac{\partial^2 h}{\partial w^2}$ is cont, $\frac{\partial h}{\partial t}$ is cont

[some more technical conditions]

then

$$\boxed{dX_t = h'_w \cdot dW_t + h'_t \cdot dt + \frac{1}{2} \cdot h''_{ww} \cdot dt}$$

$$\boxed{X_t = X_0 + \int_0^t h'_w \cdot dW_u + \int_0^t h'_t \cdot du + \frac{1}{2} \int_0^t h''_{ww} \cdot du}$$

$$\boxed{\int_0^t W_u \, dW_u = \frac{W_t^2}{2} - \frac{t}{2}}$$

[limit in M.S.]

Ex. $X_t = W_t^2$

Ito's lemma

full form

$$\boxed{dX_t \overset{?}{=} 2 W_t \cdot dW_t + 0 \cdot dt + \frac{1}{2} 2 \cdot dt}$$

$$X_t = X_0 + \int_0^t 2 \cdot W_u \cdot dW_u + \int_0^t 1 \cdot du$$

$$W_t^2 = \underset{\substack{\parallel \\ 0}}{W_0^2} + 2 \cdot \int_0^t W_u \, dW_u + t$$

$$\frac{W_t^2 - t}{2} = \int_0^t W_u \, dW_u$$

Ex.

$$\int_0^t W_u^2 \, dW_u \; ?$$

$$\int_0^t f^2(u)\, df(u) \overset{NL}{=} \frac{f^3(t)}{3} - \frac{f^3(0)}{3}$$

non random!

Ito's lemma
$$dX_t = h_w' \cdot dW_t + h_t' \cdot dt + \frac{1}{2} h_{ww}'' \cdot dt$$

$X_t = W_u^3$

$$dX_t \overset{?}{=} 3W_t^2 \, dW_t + 0 \cdot dt + \frac{1}{2} 6 \cdot W_t \cdot dt$$ (short hand notation!)

$$\underset{W_t^3}{X_t} = \underset{\underset{0}{\parallel}}{\underset{W_0^3}{X_0}} + \int_0^t 3W_u^2 \, dW_u + \int_0^t 3W_u \, du$$

$$\boxed{\int_0^t W_u^2 \, dW_u = \frac{W_t^3}{3} - \int_0^t W_u \, du}$$

Ex

$$X_t = \boxed{\exp(3W_t + 2t)}$$

Ito's lemma
$$dX_t = h_w' \cdot dW_t + h_t' \cdot dt + \frac{1}{2} h_{ww}'' \cdot dt$$

$$dX_t = ? \quad 3 \cdot \exp(3W_t + 2t) \cdot dW_t + 2\exp(3W_t + 2t) \cdot dt +$$
$$+ \frac{1}{2} \cdot 9 \cdot \exp(3W_t + 2t) \cdot dt$$

$$\underbrace{X_t}_{\exp(3W_t+2t)} = \underbrace{X_0}_{\overset{\parallel}{1}} + \int_0^t 3\exp(3W_u + 2u) \cdot dW_u + \int_0^t 2\exp(3W_u + 2u) + \frac{9}{2} \exp(3W_u + 2u) \, du$$

Moral: Many stochastic processes can be represented in the form

$$\boxed{X_t = X_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du}$$

def (X_t) is an Ito's process if it can be written as this form.

theorem: [ordinary calculus]
All differentiable functions may be represented as an integral of continuous function

---

Ex. $X_t = 5 + \boxed{\int_0^t W_u^2 \, dW_u}$

$E(X_t)$? $Var(X_t)$?

$E(X_t) = 5 + 0$

$\boxed{Var(X_t)} = Var\left(\int_0^t W_u^2 \, dW_u\right) =$

$= E\left(\int_0^t (W_u)^4 \, du\right) =$

$= \int_0^t E(W_u^4) \, du$;

Itô's lemma

$dX_t = h'_w \cdot dW_t + h'_t \cdot dt + \frac{1}{2} h''_{ww} \cdot dt$

$Y_t = W_t^4$ $\quad dY_t = 4 \cdot W_t^3 \cdot dW_t + 0 \cdot dt + \frac{1}{2} \cdot 4 \cdot 3 \cdot W_t^2 \, dt$

$W_t^4 = \int_0^t 4 W_u^3 \, dW_u + 6 \int_0^t W_u^2 \, du$

$E(W_t^4) = E\left(\int_0^t 4 W_u^3 \, dW_u\right) + 6 \cdot \int_0^t E(W_u^2) \, du =$

$= 0 + 6 \cdot \int_0^t u \, du = 6 \cdot \frac{u^2}{2} \Big|_{u=0}^{u=t}$

$= 3t^2$

$Var(X_t) = \ldots = \int_0^t E(W_u^4) \, du = \int_0^t 3u^2 \, du = u^3 \Big|_{u=0}^{u=t}$

$= t^3$

Ex

$$E\left(\exp(W_t)\right) \ ?$$

$$\boxed{E\left(\exp(W_t)\right) = \exp\left(\tfrac{t}{2}\right)}$$

Ito's lemma

$$dX_t = h_w' \cdot dW_t + h_t' \cdot dt + \tfrac{1}{2} h_{ww}'' \cdot dt$$

$$X_t = \exp(W_t)$$

$$dX_t = \exp(W_t) \cdot dW_t + \boxed{0 \cdot dt} + \tfrac{1}{2} \cdot \exp(W_t) \cdot dt$$

$$\rightarrow \exp(W_t) = 1 + \underbrace{\int_0^t \exp(W_u) \cdot dW_u}_{\text{martingale}} + \underbrace{\int_0^t \tfrac{1}{2} \exp(W_u)\, du}_{\text{not a martingale}}$$

$$\boxed{Y_t = \exp(W_t) \cdot f(t)}$$

using $f(t)$ I will remove non-martingale part of X_t.

$$dY_t = \exp(W_t) \cdot f(t) \cdot dW_t + \exp(W_t) \cdot f'(t) \cdot dt + \tfrac{1}{2} \exp(W_t) \cdot f(t)\, dt$$

To remove non-martingale part:

$$\exp(W_t) \cdot f'(t) + \tfrac{1}{2} \exp(W_t) \cdot f(t) \equiv 0$$

$$f'(t) + \tfrac{1}{2} f(t) \equiv 0$$

$$f'(t) = -\tfrac{1}{2} f(t)$$

example: $f(t) = \exp\left(-\tfrac{1}{2} t\right)$

$$Y_t = \exp(W_t) \cdot f(t) = \exp\left(W_t - \tfrac{1}{2} t\right)$$

$\boxed{\text{martingale}}$!

$$E(Y_t) = Y_0 \qquad E\left(\exp\left(W_t - \tfrac{1}{2} t\right)\right) = \exp(0) = 1$$

$$\boxed{E\left(\exp(W_t)\right) = \exp\left(\tfrac{t}{2}\right)} \qquad E\left(\exp(W_t)\right) \cdot \exp\left(-\tfrac{1}{2} t\right) = 1$$

Strategy

old strategy.

1. $\exp(W_t) \longrightarrow$ transform into a martingale
2. use $E\left(\int_0^t A_u \, dW_u\right) = 0.$
3. PROFIT!

$E(\exp(W_t)) = \int_{-\infty}^{\infty} \exp(x) \cdot pdf(x) \cdot dx$

[Under some mild technical conditions]

$X_t = X_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du$ is a martingale if and only if $B_u \equiv 0$.

$$\boxed{X_t = X_0 + \int_0^t A_u \, dW_u}$$

Ex 1 Exam 2022

Ito's lemma

$dX_t = h'_w \cdot dW_t + h'_t \cdot dt + \frac{1}{2} h''_{ww} \cdot dt$

$X_t = \exp(-2W_t - 2t)$

a) dX_t? Is X_t a martingale?

b) $E(X_t)$, $Var(X_t)$?

c) $\boxed{\int_0^t X_u \, dW_u}$

a) $dX_t = -2 \cdot \exp(-2W_t - 2t) \cdot dW_t \cancel{- 2 \cdot \exp(-2W_t - 2t) dt}$
$\cancel{+ \frac{1}{2}(-2)^2 \cdot \exp(-2W_t - 2t) \cdot dt}$

$dX_t = -2 \exp(-2W_t - 2t) \, dW_t$

$$\boxed{X_t = X_0 - 2 \boxed{\int_0^t \exp(-2W_u - 2u) \, dW_u}}$$

$X_0 = \exp(-2 \cdot 0 - 2 \cdot 0) = 1$

X_t - is a martingale

b) $E(X_t) = X_0 = 1$

c) $\int_0^t \exp(-2W_u - 2u) \, dW_u = \frac{X_0 - X_t}{2} = \frac{1 - \exp(-2W_t - 2t)}{2}$

$$\boxed{X_t = X_0 - 2\boxed{\int_0^t \exp(-2W_u - 2u)\,dW_u}}$$

$$\mathrm{Var}(X_t) = 4\cdot \mathrm{Var}\left(\int_0^t \ldots dW_u\right) =$$

$$= 4\cdot \int_0^t E\left(\exp(-4W_u - 4u)\right) du = 4\int_0^t \exp\left(\frac{16u}{2}\right)\cdot \exp(-4u)\,du$$

A4 !

$$\boxed{E(\exp(W_t)) = \exp\left(\frac{t}{2}\right)}$$

$$W_t \sim N(0;t) \qquad \frac{W_t - 0}{\sqrt{t}} \sim N(0;1)$$

$$E\left(\exp(\sqrt{t}\cdot N(0;1))\right) = \exp\left(\frac{t}{2}\right)$$

$$E\left(\exp(N(0;t))\right) = \exp\left(\frac{t}{2}\right)$$

$$-4W_u \sim N(0;16\cdot u)$$

$$E\left(\exp(N(0;16u))\right) = \exp\left(\frac{16u}{2}\right)$$

HA1

4b ← Elchanan Dice Paradox

4a faire dice.

X is the total number of throws

Y is the number of $\{1, 3, 5\}$

4a

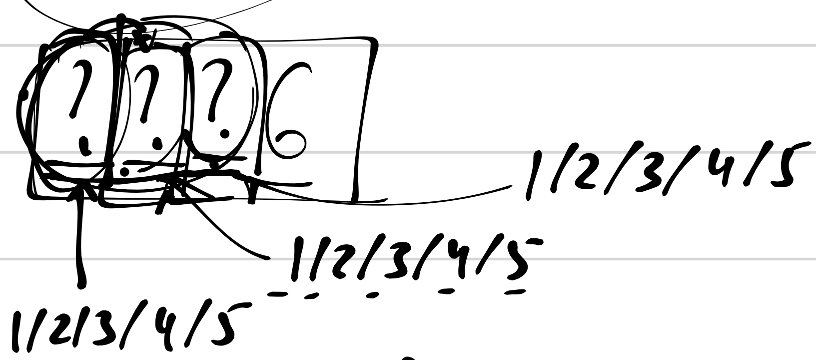
$P(Y=y|X)$?

$E(Y|X)$?

$Var(Y|X)$?

Strategy! Fix X.

$X=4$



$$(Y|X=4) \sim Bin\left(n=3,\ p=\frac{3}{5}\right)$$

$$P(Y=y|X=4) = C_3^y \cdot \left(\frac{3}{5}\right)^y \cdot \left(1-\frac{3}{5}\right)^{4-y}$$

$$(Y|X) \sim Bin\left(n=X-1,\ p=\frac{3}{5}\right)$$

$$P(Y=y|X) = C_{X-1}^{y} \cdot \left(\frac{3}{5}\right)^y \cdot \left(1-\frac{3}{5}\right)^{X-1-y} \quad !!$$

$$E(Y|X) = \text{„}np\text{"} = (X-1)\cdot\frac{3}{5}$$

$$Var(Y|X) = \text{„}np(1-p)\text{"} = (X-1)\cdot\frac{3}{5}\cdot\frac{2}{5} \quad !!$$

$0!=1$

$X=1 \Rightarrow Y=0$

$$P(Y=0|X=1) = C_0^0 \cdot \left(\frac{3}{5}\right)^0 \cdot \frac{2}{5}^{1-1-0}$$

$$\frac{(0+0)!}{0!\,0!} \cdot 1 \cdot 1 = 1$$

(!) Study extreme cases!

4b) $\boxed{E(X|Y)}$

$\boxed{E(X|Y=0)}$

$\boxed{\text{wrong solution!}}$

$Y=0 \Rightarrow$ ~~X X X~~

$1, 2, 4, 6$

$\Rightarrow \text{Geom}\left(p=\frac{1}{3}\right)$

$\Rightarrow E(\text{Geom}) = \frac{1}{p} = \frac{1}{(1/3)} = 3$

wrong intuition $\quad E(X|Y=0)=3$

right solution.

$E(X|Y=0) = E(X \mid$ after the initial sequence of 2 and 4's 6 has app-d $)$

$P(\cancel{X}) = \frac{2}{6}$!!

$Y \geq 1$

$\boxed{2442244}\,5 \;.\;.\;.\;.$

$\boxed{242422}\,1 \;.\;.\;.\;.\; 6$ $\qquad Y=1$

$Y \geq 1$

$\boxed{22244}\,6$

$Y=0.$

$\frac{2}{6} \qquad \frac{4}{6}$

$\boxed{N \text{ - the length of initial sequence of 2 and 4's } +1}$

$= E(N \mid$ after the init sequence of 2s and 4s 6 has app-d $)$

$= E(N \mid$ // has ap-d $)$

$= E(N)$ $\qquad N \sim \text{Geom}\left(p=\frac{4}{6}\right)$

$E(N) = 6/4 = 3/2 = 1.5$!!

$$\boxed{242424}\ \boxed{2224422243}\ \boxed{2422246}$$

$\frac{3}{2}$ $\frac{3}{2}$ $\frac{3}{2}$

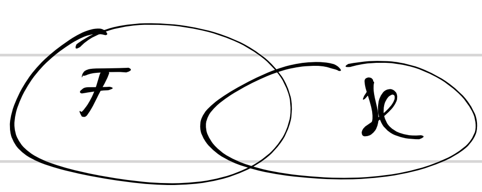
$$\frac{3}{2}\times(4+1)$$

№3 $$E\big(E(Y\mid\mathcal{F})\mid\mathcal{H}\big) \overset{?}{=} E(Y\mid\mathcal{F}\cap\mathcal{H}) \qquad (A)$$

! $\mathcal{F}\subseteq\mathcal{H}$ or $\mathcal{H}\subseteq\mathcal{F}$ $\Rightarrow$ A is TRUE

Tower Property

!



y	1	2	3
$P(Y=y)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
	a	b	c
$E(Y\mid\mathcal{F})$	1	2.5	2.5
$E(E(Y\mid\mathcal{F})\mid\mathcal{H})$	3.5/2	3.5/2	2.5

$\mathcal{F}=\{a, bc, \phi, \Omega\}$

$\mathcal{H}=\{c, ab, \phi, \Omega\}$

$\mathcal{F}\cap\mathcal{H}=\{\phi, \Omega\}$

$E(Y\mid\mathcal{F}\cap\mathcal{H})=E(Y)=2$

$\phi\in\mathcal{A}$ — never happens

$\Omega\in\mathcal{A}$ — always happens

„никогда так не было, чтобы никогда не было" $\approx$ $\boxed{\Omega\in\mathcal{A}}$

Use Risk neutral probability measure

$$\rightarrow\ X_0 = \exp(-rt)\cdot E^*\big((S_t^5-K)\cdot I(S_t\geq K)\big)$$

strat 2

$Y_t = S_t^5$

$Y_t = Y_0 \cdot \exp(5\sigma \cdot W_t \dots)$

$\mu_Y = \dots$

$\sigma_Y = 5\sigma$

$$dS_t = \mu S_t \, dt + \sigma \cdot S_t \cdot dW_t$$

$$S_t = S_0 \cdot \exp\left(\mu t + \sigma \cdot W_t - \frac{\sigma^2}{2} t\right)$$

$$S_t = S_0 \cdot \exp\left(rt + \sigma W_t^* - \frac{\sigma^2}{2} t\right)$$

W_t^* is a Wiener process wrt P^*

$$X_0 = \exp(-rt) \cdot E^*\left((S_t^5 - k) \cdot I(S_t \geq k)\right) =$$

$$= \exp(-rt) \cdot E^*\left(S_t^5 \cdot I(S_t \geq k) - k \cdot I(S_t \geq k)\right)$$

$$E^*\left(k \cdot I(S_t \geq k)\right) =$$

$$= k \cdot P^*(S_t \geq k) =$$

$$= k P^*(W_t^* \geq \dots) =$$

$W_t^* \sim N(0; t)$

$$= k \cdot cdf_{N(0;1)}(\dots)$$

$$E^*\left(S_t^5 \cdot I(S_t \geq k)\right) =$$

$$= E^*\left(S_0^5 \exp(? W_t^* + ? t) \cdot I(W_t^* > ?)\right) =$$

$$= const \cdot E^*\left(\exp(? W_t^*) I(W_t^* > \xi)\right) =$$

$$= const \cdot \int_{\xi}^{+\infty} \exp(? w) \cdot pdf(w) \, dw$$