Submission details

Deadline HA1: 26 November 2023, 23:59
Deadline HA2: 07 December 2023, 23:59
Deadline HA3: 17 December 2023, 23:59
You have one honey-day. The honey-day allows you to postpone one of the three deadlines by 24 hours.

1 HA-1

1. Consider the following joint distribution of X and Y:

	$X=-1$	$X=0$	$X=1$
$Y=0$	0.1	0.2	0.3
$Y=1$	0.2	0.1	0.1

   (a) Find explicitely $\sigma(X)$, $\sigma(Y)$, $\sigma(X\cdot Y)$, $\sigma(X^2)$, $\sigma(2X+3)$.

   (b) Home many elements are there in $\sigma(X,Y)$, $\sigma(X+Y)$, $\sigma(X,Y,X+Y)$?

2. More σ-algebra questions :)

   (a) You observe the result of 10 independent coin tosses. How many elements the corresponding σ-algebra contains?

   (b) Prove that a finite σ-algebra can contain only 2^k elements.

   (c) Is union of two σ-algebras always a σ-algebra? Prove your statement.

   (d) Is intersection of two σ-algebras always a σ-algebra? Prove your statement.

3. Is it true that for any two σ-algebras $\mathcal{F}$ and $\mathcal{H}$ and for any random variable Y

$$\mathrm{E}(\mathrm{E}(Y|\mathcal{F})|\mathcal{H}) = \mathrm{E}(Y|\mathcal{F}\cap\mathcal{H})?$$

   Prove the statement or provide a counter-example.

4. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y.

   (a) Find $\mathbb{P}(Y=y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$;

   (b) Find $\mathrm{E}(X|Y)$.

5. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now.

   Find $\mathbb{P}(Y=y|X)$, $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$.

6. Random variables X and Y have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

   (a) Find $\mathrm{E}(Y|X)$, $\mathrm{Var}(Y|X)$, $\mathrm{E}(XY|X)$ and $\mathrm{Var}(XY|X)$.

   (b) Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2021|X)$.

2 HA-2

1. The random variables Z_1, Z_2, ... are independent and identically distributed with $\mathbb{P}(Z_n = 1) = 0.7$ and $\mathbb{P}(Z_n = -1) = 0.3$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.
   (a) Find all values of a such that $\exp(aS_n)$ is a martingale.
   (b) If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale.

2. The random variables Z_1, Z_2, ... are independent and identically distributed with $\mathbb{P}(Z_n = +1) = 0.1$, $\mathbb{P}(Z_n = -1) = 0.1$ and $\mathbb{P}(Z_n = 0) = 0.8$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.
   Let τ be the first moment when $S_n = 10$ or $S_n = -20$.
   (a) Is S_n a martingale?
   (b) Find $\mathbb{P}(S_\tau = 10)$.
   (c) If possible find the non-random function a_n such that $Y_n = S_n^2 + a_n$ is a martingale.
   (d) Find $\mathrm{E}(\tau)$.

3. Let (W_t) be a standard Wiener process.
   (a) Find $\mathrm{E}(\sin(\alpha W_t))$.
   (b) Find $\mathrm{E}(\exp(\alpha W_t))$.
   (c) Find $\mathrm{E}(\cos(\alpha W_t))$.

   Hint: you may solve this with or without Ito's lemma, that's up to you.

4. Let (W_t) be a standard Wiener process and $Y_t = W_t^3 + t^2 W_t^2$.
   (a) Find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.
   (b) Is Y_t a martingale?
   (c) Find $\mathrm{E}(Y_t \mid W_s)$ for $t \geq s$.

5. The processes (X_t) and (Y_t) are independent Wiener processes adapted to the filtration $(\mathcal{F}_t)$. The process $Z_t = aX_t + 0.3Y_t$ with $a > 0$ is also a Wiener process.
   (a) Find the value of a.
   (b) Find the correlation $\mathrm{Corr}(Z_t, X_t)$.
   (c) Find $\mathrm{E}(Z_3 \mid X_2)$ and $\mathrm{Var}(Z_3 \mid X_2)$.
   (d) Find $\mathrm{E}(Z_3 \mid \mathcal{F}_2)$ and $\mathrm{Var}(Z_3 \mid \mathcal{F}_2)$.

6. Let $Y_t = W_t + 4t$ and consider the process $M_t = \exp(\alpha W_t - \alpha^2 t/2)$. The moment τ is the first moment when Y_t hits 10.
   (a) Check that M_t is a martingale.
   (b) Find $f(t)$ such that $M_t = f(t)\exp(\alpha Y_t)$.
   (c) Using Doob's theorem for M_t find $\mathrm{E}(\exp(-(4\alpha + \alpha^2/2)\tau))$.
   (d) Find $\mathrm{E}(\exp(-s\tau))$ for $s \geq 0$.
   (e) Find $\mathrm{E}(\tau)$.

   Hint: you may believe without penalty that Doob's theorem can be applied in this case.

3 HA-3

1. Consider the process $dX_t = W_t^4 dW_t + W_t^6 dt$ with $X_0 = 2024$.
   (a) Find $\mathrm{E}(X_t)$.
   (b) Find dY_t for $Y_t = X_t^2$.
   (c) (bonus point) Find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(X_t)$.
2. Consider the process $C_t = W_t^3 + 2W_t^2 - 5W_t^2 \cdot t \cdot W_t$.
   (a) Find dC_t.
   (b) Is C_t a martingale?
   (c) Find the covariance $\mathrm{Cov}(C_t, \int W_u^2 dW_u)$.
3. Solve the stochatic differential equation
$$dY_t = -Y_t dt + dW_t,\ Y_0 = 1$$
   If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitely. It's ok.
4. Solve the stochatic differential equation
$$dY_t = Y_t dt + (t^3 + 4Y_t)dW_t,\ Y_0 = 1$$
   If you are have no clues you may try to represent the process Y_t as $Y_t = A_t B_t$, where A_t is the solution of the equation $dA_t = A_t dt + 4A_t dW_t$.
5. Consider the framework of Black and Scholes model: S_t is the share price. Derive the current price of two European type assets, X_0 and Y_0.

   Future payoffs are given by:
   (a) $X_T = (S_T - K)^3$ where T and K are fixed in the contract.
   (b) $Y_T = S_T^{-2}$ where T is fixed in the contract.
6. Consider the framework of Black and Scholes model with stochastic exchange rate. Now S_t is the share price in dollars with $dS_t = \mu S_t dt + \sigma S_t dW_{1t}$. The exchange rate X_t (price of one dollar in euros) is driven by the equation $dX_t = \alpha X_t dt + \beta X_t dW_{2t}$.

   The Wiener processes W_{1t} and W_{2t} are independent, in particular that means $dW_{1t}dW_{2t} = 0$.

   The risk free interest rate in euros is r.

   The european type option pays you $\ln(S_t X_t)$ in euros at fixed time T.

   Find the price of this option in euros at time $t = 0$, write your answer in terms of the share price in euros.