

## Submission details

Deadline HA1: 26 November 2023, 23:59

Deadline HA2: 07 December 2023, 23:59

Deadline HA3: 17 December 2023, 23:59

You have one honey-day. The honey-day allows you to postpone one of the three deadlines by 24 hours.

# 1 HA-1

1. Consider the following joint distribution of  $X$  and  $Y$ :

	$X = -1$	$X = 0$	$X = 1$
$Y = 0$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

- Find explicitly  $\sigma(X)$ ,  $\sigma(Y)$ ,  $\sigma(X \cdot Y)$ ,  $\sigma(X^2)$ ,  $\sigma(2X + 3)$ .
  - How many elements are there in  $\sigma(X, Y)$ ,  $\sigma(X + Y)$ ,  $\sigma(X, Y, X + Y)$ ?
2. More  $\sigma$ -algebra questions :)
- You observe the result of 10 independent coin tosses. How many elements the corresponding  $\sigma$ -algebra contains?
  - Prove that a finite  $\sigma$ -algebra can contain only  $2^k$  elements.
  - Is union of two  $\sigma$ -algebras always a  $\sigma$ -algebra? Prove your statement.
  - Is intersection of two  $\sigma$ -algebras always a  $\sigma$ -algebra? Prove your statement.
3. Is it true that for any two  $\sigma$ -algebras  $\mathcal{F}$  and  $\mathcal{H}$  and for any random variable  $Y$

$$E(E(Y | \mathcal{F}) | \mathcal{H}) = E(Y | \mathcal{F} \cap \mathcal{H})?$$

Prove the statement or provide a counter-example.

4. I throw a fair die until the first six appears. Let's denote the total number of throws by  $X$  and the number of odd integers thrown by  $Y$ .
- Find  $\mathbb{P}(Y = y | X)$ ,  $E(Y | X)$ ,  $\text{Var}(Y | X)$ ;
  - Find  $E(X | Y)$ .
5. I throw 100 coins. Let's denote by  $X$  the number of coins that show «heads». I throw these  $X$  coins once again, leaving other coins as they are. Let's denote by  $Y$  the number of coins that show «heads» now. Find  $\mathbb{P}(Y = y | X)$ ,  $E(Y | X)$ ,  $\text{Var}(Y | X)$ ,  $E(Y)$ ,  $\text{Var}(Y)$ .
6. Random variables  $X$  and  $Y$  have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

- Find  $E(Y|X)$ ,  $\text{Var}(Y|X)$ ,  $E(XY|X)$  and  $\text{Var}(XY|X)$ .
- Using standard normal cumulative distribution function find  $\mathbb{P}(YX > 2021|X)$ .

## 2 HA-2

1. The random variables  $Z_1, Z_2, \dots$  are independent and identically distributed with  $\mathbb{P}(Z_n = 1) = 0.7$  and  $\mathbb{P}(Z_n = -1) = 0.3$ . Consider the cumulative sum process,  $S_n = Z_1 + \dots + Z_n$  with  $S_0 = 0$ .

- (a) Find all values of  $a$  such that  $\exp(aS_n)$  is a martingale.
- (b) If possible find the constants  $\alpha$  and  $\beta$  such that  $Y_n = S_n^2 + \alpha S_n + \beta n$  is a martingale.

2. The random variables  $Z_1, Z_2, \dots$  are independent and identically distributed with  $\mathbb{P}(Z_n = +1) = 0.1$ ,  $\mathbb{P}(Z_n = -1) = 0.1$  and  $\mathbb{P}(Z_n = 0) = 0.8$ . Consider the cumulative sum process,  $S_n = Z_1 + \dots + Z_n$  with  $S_0 = 0$ .

Let  $\tau$  be the first moment when  $S_n = 10$  or  $S_n = -20$ .

- (a) Is  $S_n$  a martingale?
- (b) Find  $\mathbb{P}(S_\tau = 10)$ .
- (c) If possible find the non-random function  $a_n$  such that  $Y_n = S_n^2 + a_n$  is a martingale.
- (d) Find  $\mathbb{E}(\tau)$ .

3. Let  $(W_t)$  be a standard Wiener process.

- (a) Find  $\mathbb{E}(\sin(\alpha W_t))$ .
- (b) Find  $\mathbb{E}(\exp(\alpha W_t))$ .
- (c) Find  $\mathbb{E}(\cos(\alpha W_t))$ .

Hint: you may solve this with or without Ito's lemma, that's up to you.

4. Let  $(W_t)$  be a standard Wiener process and  $Y_t = W_t^3 + t^2 W_t^2$ .

- (a) Find  $\mathbb{E}(Y_t)$  and  $\text{Var}(Y_t)$ .
- (b) Is  $Y_t$  a martingale?
- (c) Find  $\mathbb{E}(Y_t \mid W_s)$  for  $t \geq s$ .

5. The processes  $(X_t)$  and  $(Y_t)$  are independent Wiener processes adapted to the filtration  $(\mathcal{F}_t)$ . The process  $Z_t = aX_t + 0.3Y_t$  with  $a > 0$  is also a Wiener process.

- (a) Find the value of  $a$ .
- (b) Find the correlation  $\text{Corr}(Z_t, X_t)$ .
- (c) Find  $\mathbb{E}(Z_3 \mid X_2)$  and  $\text{Var}(Z_3 \mid X_2)$ .
- (d) Find  $\mathbb{E}(Z_3 \mid \mathcal{F}_2)$  and  $\text{Var}(Z_3 \mid \mathcal{F}_2)$ .

6. Let  $Y_t = W_t + 4t$  and consider the process  $M_t = \exp(\alpha W_t - \alpha^2 t/2)$ . The moment  $\tau$  is the first moment when  $Y_t$  hits 10.

- (a) Check that  $M_t$  is a martingale.
- (b) Find  $f(t)$  such that  $M_t = f(t) \exp(\alpha Y_t)$ .
- (c) Using Doob's theorem for  $M_t$  find  $\mathbb{E}(\exp(-(4\alpha + \alpha^2/2)\tau))$ .
- (d) Find  $\mathbb{E}(\exp(-s\tau))$  for  $s \geq 0$ .
- (e) Find  $\mathbb{E}(\tau)$ .

Hint: you may believe without penalty that Doob's theorem can be applied in this case.

### 3 HA-3

1. Consider the process  $dX_t = W_t^4 dW_t + W_t^6 dt$  with  $X_0 = 2024$ .

- (a) Find  $E(X_t)$ .
- (b) Find  $dY_t$  for  $Y_t = X_t^2$ .
- (c) (bonus point) Find  $E(Y_t)$  and  $\text{Var}(X_t)$ .

2. Consider the process  $C_t = W_t^3 + 2W_t^2 - 5W_t^3 \cdot t$ .

- (a) Find  $dC_t$ .
- (b) Is  $C_t$  a martingale?
- (c) Find the covariance  $\text{Cov}\left(C_t, \int_0^t W_u^2 dW_u\right)$ .

3. Solve the stochastic differential equation

$$dY_t = -Y_t dt + dW_t, \quad Y_0 = 1$$

If you have no clues you may try a substitution  $Z_t = f(t)Y_t$ . Do not forget that the final answer may contain integrals that can't be calculated explicitly. It's ok.

4. Solve the stochastic differential equation

$$dY_t = Y_t dt + (t^3 + 4Y_t) dW_t, \quad Y_0 = 1$$

If you have no clues you may try to represent the process  $Y_t$  as  $Y_t = A_t B_t$ , where  $A_t$  is the solution of the equation  $dA_t = A_t dt + 4A_t dW_t$ .

5. Consider the framework of Black and Scholes model:  $S_t$  is the share price. Derive the current price of two European type assets,  $X_0$  and  $Y_0$ .

Future payoffs are given by:

- (a)  $X_T = (S_T - K)^3$  where  $T$  and  $K$  are fixed in the contract.
- (b)  $Y_T = S_T^{-2}$  where  $T$  is fixed in the contract.

6. Consider the framework of Black and Scholes model with stochastic exchange rate. Now  $S_t$  is the share price in dollars with  $dS_t = \mu S_t dt + \sigma S_t dW_{1t}$ . The exchange rate  $X_t$  (price of one dollar in euros) is driven by the equation  $dX_t = \alpha X_t dt + \beta X_t dW_{2t}$ .

The Wiener processes  $W_{1t}$  and  $W_{2t}$  are independent, in particular that means  $dW_{1t} dW_{2t} = 0$ .

The risk free interest rate in euros is  $r$ .

The European type option pays you  $\ln(S_t X_t)$  in euros at fixed time  $T$ .

Find the price of this option in euros at time  $t = 0$ , write your answer in terms of the share price in euros.