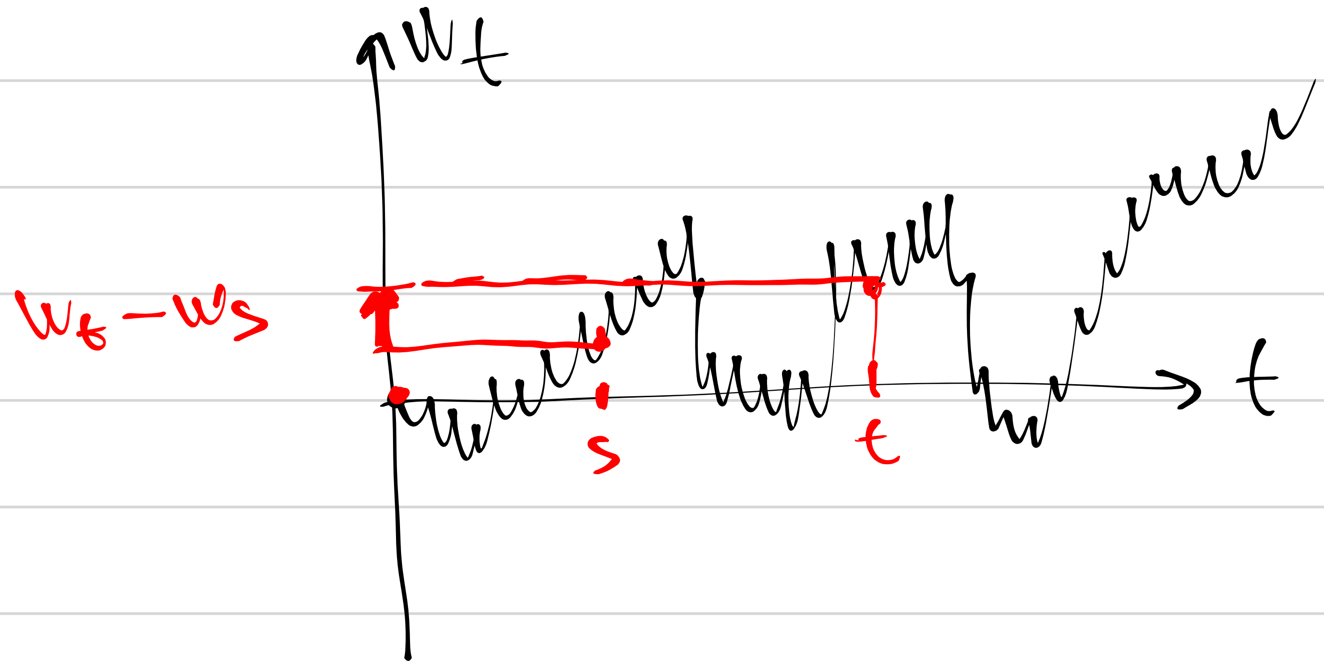
Wiener process.

(like a line in ordinary calculus.

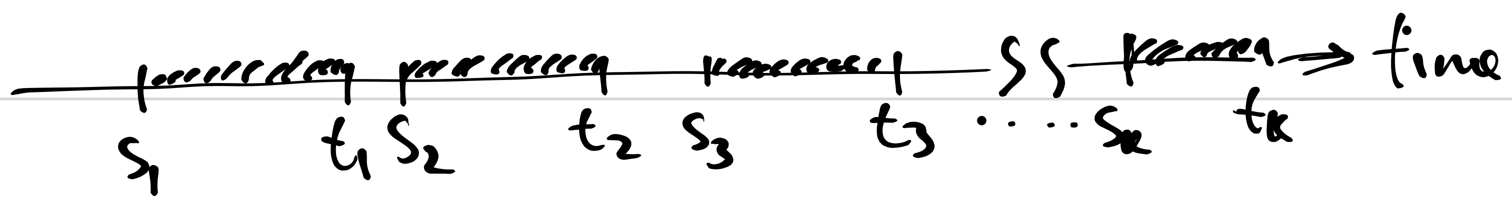


def $(W_t)_{t\in[0;+\infty)}$ is a Wiener process if

① $W_0 = 0$

② $W_t - W_s \sim N(0; t-s)$ for $\forall\, t \geq s \geq 0$

③a



If:

$s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots \leq s_k \leq t_k$

overlap is prohibited

$\Delta_i = W(t_i) - W(s_i)$

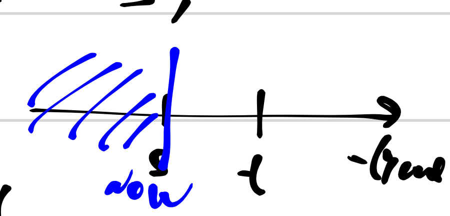
then: $\Delta_1, \Delta_2 \dots \Delta_k$ are independent random variables.

"Increments are independent"

③a ⟺ ③b ← "future increment is independent of the past"

$s \leq t$ $\quad \Delta = W_t - W_s$ is independent of $\mathcal{F}_s$,

where $\mathcal{F}_s = \sigma(W_u, u \in [0; s])$



④ P(trajectory of (W_t) is continuous) $= 1$.

Properties of Wiener Process.

(Ex.)

(W_t) - is a Wiener Process.

a) $E(W_t)$, $Var(W_t)$, $Cov(W_s, W_t)$

b) $E(W_t \mid W_s)$? $t \geq s$

$Var(W_t \mid W_s)$ $t \geq s$

c) Is (W_t) a martingale?

d) $Y_t = W_t^2 - t$, is (Y_t) a martingale?

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a)

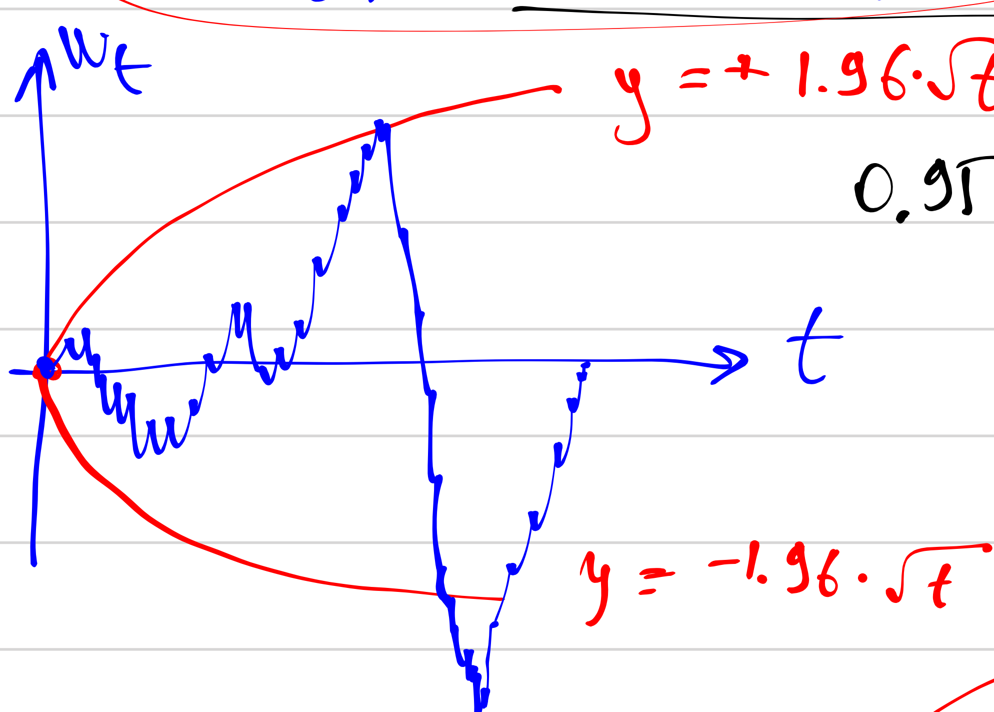
$E(W_t)$?

$$\begin{cases} ① & W_0 = 0 \\ ② & W_t - W_s \sim N(0; t-s) \end{cases}$$

$t \geq s \geq 0$

$W_t = W_t - W_0 \sim N(0; t-0)$

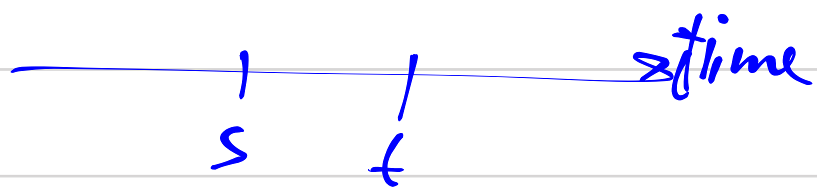
$E(W_t) = 0$ $Var(W_t) = t$ $\forall t$



$0.95 = P(N(0;1) \in [-1.96; 1.96])$

$$\frac{W_t - 0}{\sqrt{t}} \sim N(0;1)$$

$s \leq t$ $Cov(W_s, W_t) = Cov(W_s, W_s + (W_t - W_s)) =$



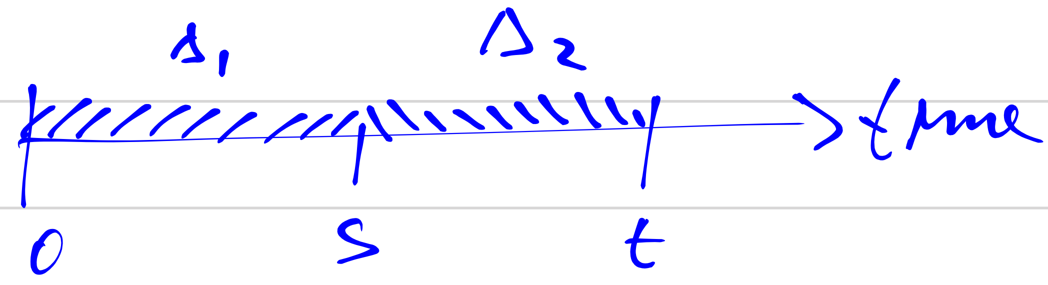
$W_t = W_s + (W_t - W_s)$

W_s is known at $\boxed{s}$; $(W_t - W_s)$ (by assump 3) is unpredictable at $\boxed{s}$.

$= Cov(W_s, W_s) + \boxed{Cov(W_s, W_t - W_s)} = Var(W_s) + 0 = s$

$Cov(W_s, W_t - W_s) = 0$

$\boxed{3a}$



$\Delta_1 = W_s - W_0 = W_s$

$\Delta_2 = W_t - W_s$

Δ_1 and Δ_2 are independent

$Cov(\Delta_1, \Delta_2) = 0$ $\quad$ $Cov(W_s, W_t - W_s) = 0.$

$s \le t$ $\quad$ $Cov(W_s, W_t) = s$

$$cov(W_s, W_t) = \min\{s, t\}$$

$$E(W_t | W_s) = E(W_s + (W_t - W_s) | W_s) =$$

known

Future value = Current value + Increment !!

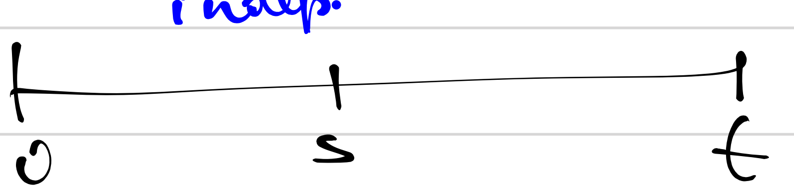
$$W_t = W_s + \underbrace{(W_t - W_s)}$$

$$= W_s + E(\underbrace{W_t - W_s}_{\Delta_2} | \underbrace{W_s}_{\Delta_1}) = W_s + E(W_t - W_s) =$$

indep.

$= W_s$

as $W_t - W_s \sim N(0; t-s)$



by assum 3a Δ_1 and Δ_2 are independent

$$E(W_6 | W_4) = W_4$$

$$\operatorname{Var}(W_t \mid W_s) = \operatorname{Var}(\underbrace{W_s}_{} + W_t - W_s \mid \underbrace{W_s}_{}) =$$

$t \geq s$

fixed

known

$$= \operatorname{Var}(W_t - W_s \mid W_s) =$$

independ.

$\operatorname{Var}(R+42) = \operatorname{Var}(R)$

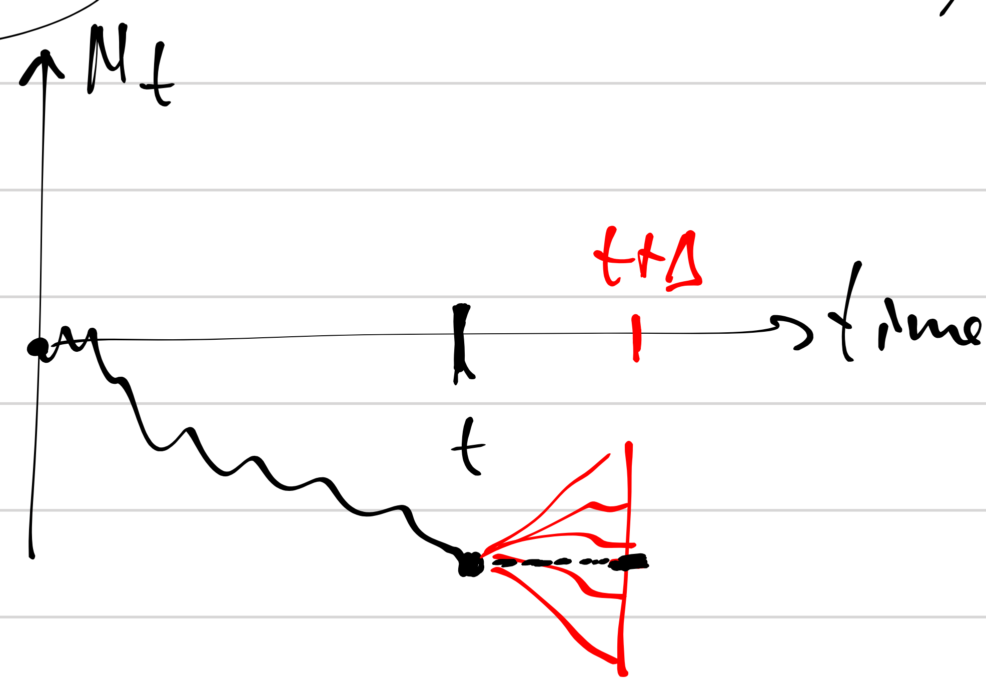
$\operatorname{Var}(R+42X \mid X) = \operatorname{Var}(R \mid X)$

$$= \operatorname{Var}(W_t - W_s) = t - s$$

ex $\quad \operatorname{Var}(W_7 \mid W_4) = 3$

Recap. $(M_t)_{t\in[0;\infty)}$ is a martingale $\forall t$ $\forall \Delta \geq 0$

$$E(M_{t+\Delta} \mid \mathcal{F}_t) = M_t \qquad \mathcal{F}_t = \sigma((M_u)_{u\in[0;t]})$$



$$\to E(W_{t+\Delta} \mid \mathcal{F}_t) = E(W_t + (W_{t+\Delta} - W_t) \mid \mathcal{F}_t) =$$

$$= W_t + E(W_{t+\Delta} - W_t \mid \mathcal{F}_t) =$$

independent

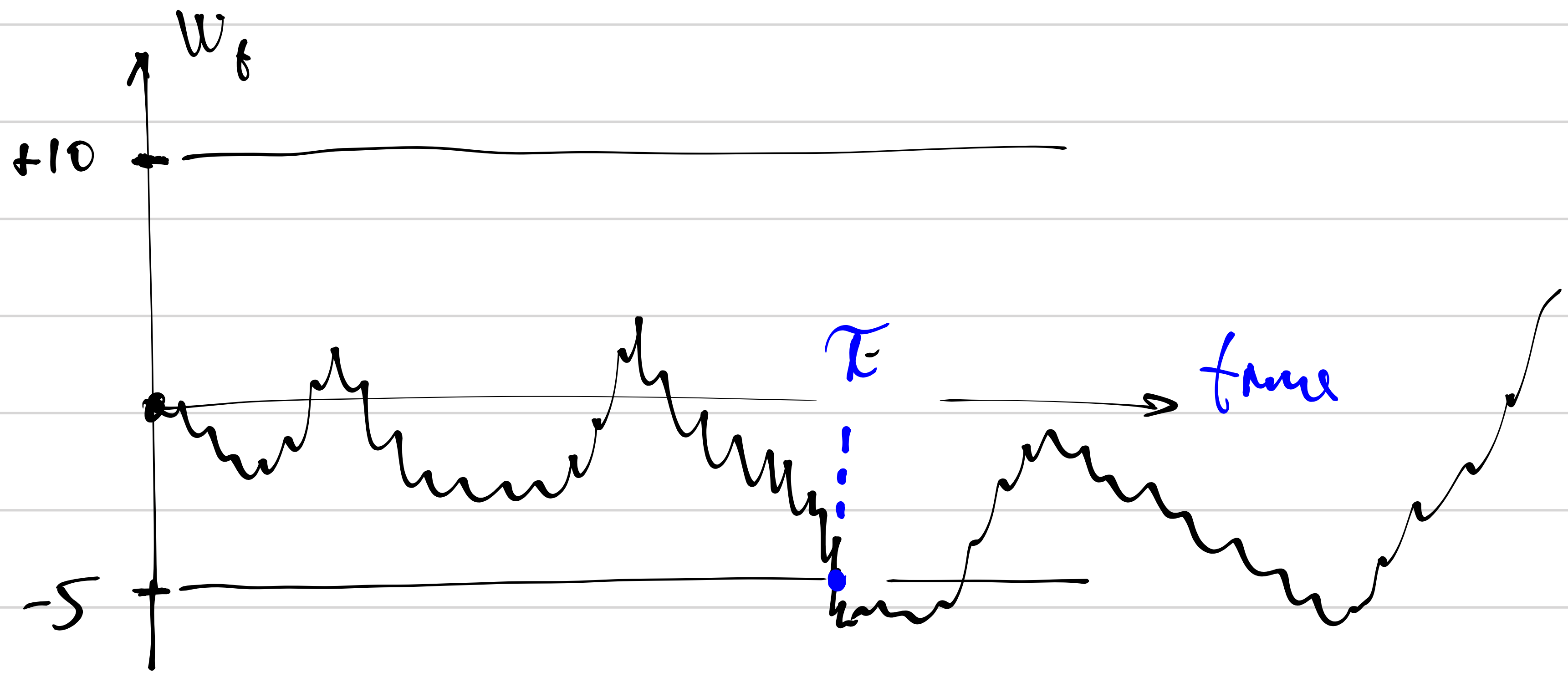
by 3b

$$= W_t + E(W_{t+\Delta} - W_t) = W_t + 0$$

(W_t) is a martingale. $\qquad W_{t+\Delta} - W_t \sim N(0;\Delta)$

Ex.

$$\boxed{\text{Assume that Doob's theorem is valid} \quad E(M_\tau) = M_0.}$$



$$\tau = \min\{t \mid W_t = +10 \text{ or } W_t = -5\}$$

$P(W_\tau = 10)$?

→ (W_t) - martingale

→ (τ) - stopping time.

$$E(W_\tau) = W_0 = 0$$

$$0 = P(W_\tau = 10)\cdot 10 + P(W_\tau = -5)\cdot(-5)$$

$$10p - 5\cdot(1-p) = 0$$

$$p = \frac{5}{15}$$

$$P(W_\tau = 10) = \frac{1}{3}$$

$$P(W_\tau = -5) = \frac{2}{3}$$

$$P(\tau = +\infty) = 0$$

d) $Y_t = W_t^2 - t$. Is (Y_t) a martingale?

$$E(Y_{t+\Delta} \mid \mathcal{F}_t) =$$

$$= E\left(W_{t+\Delta}^2 - (t+\Delta) \mid \mathcal{F}_t\right) =$$

$$= \boxed{\text{Future value} = \text{current value} + \text{Increment}} =$$

$$= E\left((W_t + I)^2 - (t+\Delta) \mid \mathcal{F}_t\right) =$$

$$I = W_{t+\Delta} - W_t$$

$$= E\left(\underbrace{\boxed{W_t^2} + I^2 + 2 \cdot I \cdot W_t \boxed{-t}}_{Y_t} - \Delta \mid \mathcal{F}_t\right) =$$

known at time t

$$= Y_t + E\left(I^2 + 2I \cdot W_t - \Delta \mid \mathcal{F}_t\right) =$$

non-random

by 36: $I = W_{t+\Delta} - W_t$ is independent of $\mathcal{F}_t$

$$= Y_t + E(I^2) - \Delta + 2 \cdot E(I \cdot W_t \mid \mathcal{F}_t) =$$

known

$$I = W_{t+\Delta} - W_t \sim N(0; t+\Delta - t)$$
$$\sim N(0; \Delta)$$

$$E(I) = 0 \qquad Var(I) = \underbrace{E(I^2)}_{= \Delta} - \underbrace{(E(I))^2}_{= 0} = \Delta$$

$$= Y_t + \Delta - \Delta + 2 \cdot W_t \cdot E(I \mid \mathcal{F}_t) =$$

indep by 36

$$= Y_t + 0 + 2 \cdot W_t \cdot E(I) = Y_t$$

(Y_t) is a martingale.

(Y_t) $\qquad Y_t = W_t^2 - t$

$\mathcal{F}_t = \sigma\left((Y_u)_{u\in[0;t]}\right)$

$\{W_1^2 > 5\} \in \mathcal{F}_1$

$\{W_1 > \sqrt{5}\} \notin \mathcal{F}_1$

$\mathcal{F}'_t = \sigma\left((W_u)_{u\in[0;t]}\right)$ $\qquad \mathcal{F}_t \subseteq \mathcal{F}'_t$

$E(W_t \cdot I \mid \mathcal{F}_t)$?

tower property of conditional expected values.

$E(W_t I \mid \mathcal{F}_t) = E\left(E(W_t \cdot I \mid \mathcal{F}'_t) \mid \mathcal{F}_t\right) =$

$= E\left(W_t \cdot E(I \mid \mathcal{F}'_t) \mid \mathcal{F}_t\right) =$

$= E(W_t \cdot 0 \mid \mathcal{F}_t) = 0$

$I = W_{t+\Delta} - W_t \sim N(0;\Delta)$

$\boxed{E(I) = 0}$ $\quad Var(I) = \Delta = E(I^2)$

---

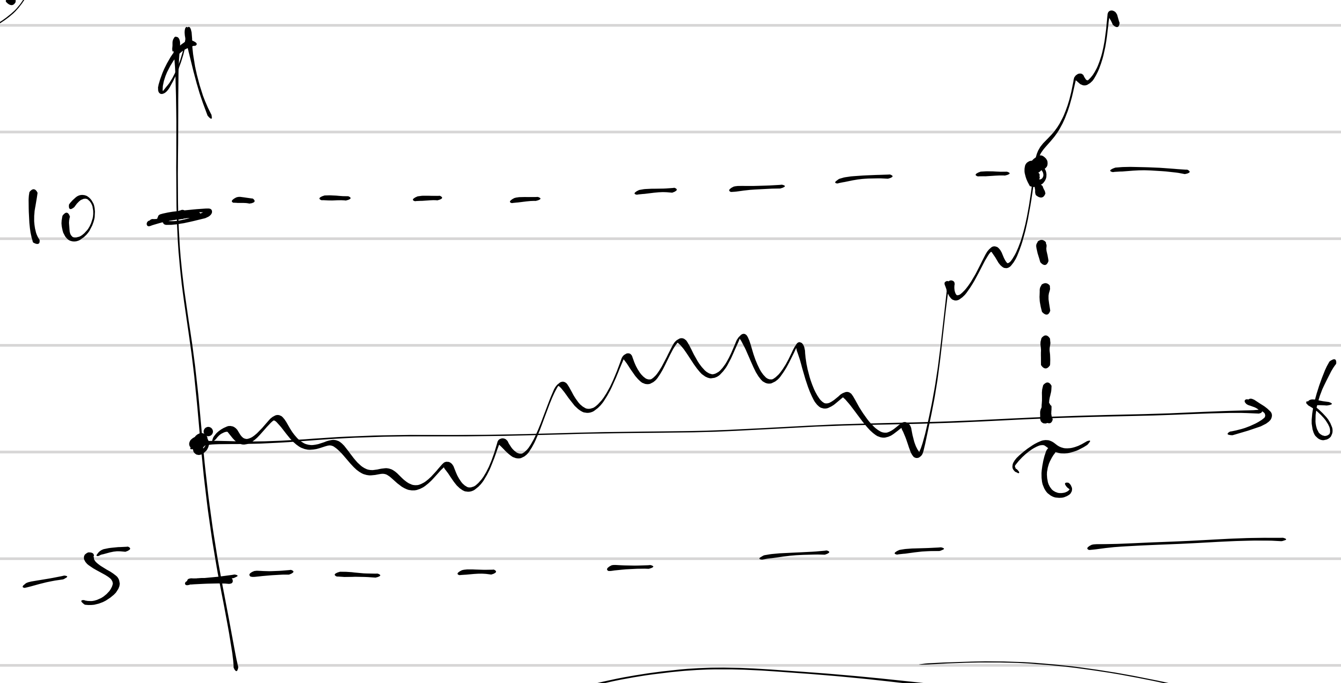
correct penalty

$Y_t = W_t^2 - t$

$E(W_t^2) = Var(W_t) = t$

$V_t = W_t^2$ is not a martingale

(Ex.)



$E(\tau)$? $\boxed{Y_t - \text{martingale!}}$

$\boxed{\text{By Doob's theorem}}$ $E(Y_\tau) = Y_0 = W_0^2 - 0 = 0$

$$E(W_\tau^2 - \tau) = 0$$

$$E(\tau) = E(W_\tau^2) = 10^2 \cdot P(W_\tau = 10) + (-5)^2 \cdot P(W_\tau = -5)$$

$$= 10^2 \cdot \frac{1}{3} + 25 \cdot \frac{2}{3}$$

Wiener process → Wiener Process.

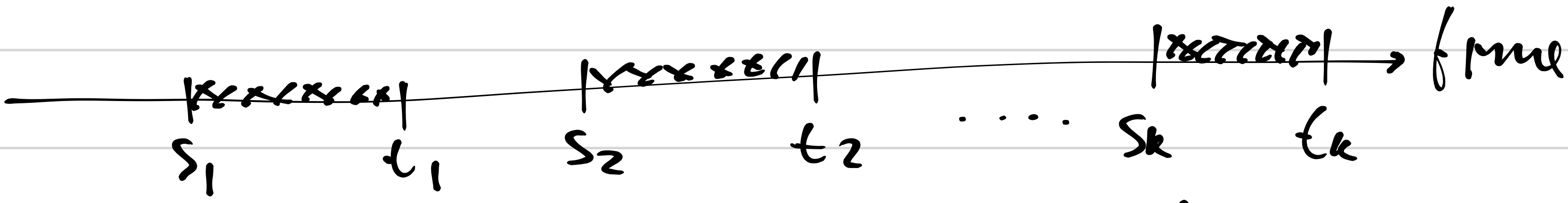
Ex.

$\boxed{\begin{array}{l} (W_t) - \text{Wiener process.} \\ X_t = \alpha \cdot W_{2t} \\ \alpha? \text{ if } (X_t) - \text{Wiener process.} \end{array}}$

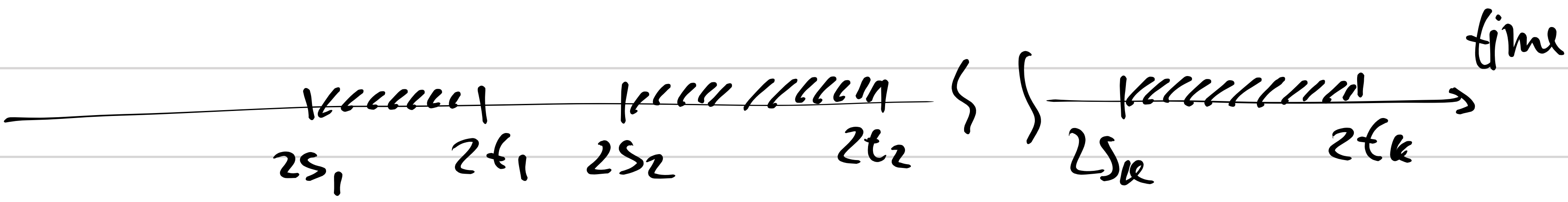
① $W_0 = 0 \Rightarrow X_0 = 0$

④ $P(W_t \text{ has continuous trajectory}) = 1 \Rightarrow P(X_t \text{ has continuous trajectory}) = 1$

(3a)



$$\underbrace{X(t_1)-X(s_1)}_{\Delta_1^X} = \alpha\left(\underbrace{W(2t_1)-W(2s_1)}_{\Delta_1^W}\right)$$



$\Delta_1^W, \Delta_2^W \; \ldots \; \Delta_k^W$ are indep. $\Rightarrow$

$\Rightarrow \Delta_1^X, \Delta_2^X \; \ldots\ldots \; \Delta_k^X$ are indep.

Assump. (2). $\quad X_t - X_s \overset{?}{=} \alpha\cdot(W_{2t}-W_{2s}) \sim N(0;t-s)$ for (X_t)

for (W_t) $\quad W_{2t}-W_{2s} \sim N(0;2t-2s)$

$\boxed{W_{\beta}-W_{\alpha} \sim N(0;\beta-\alpha)}$ Assum 2 for W_t

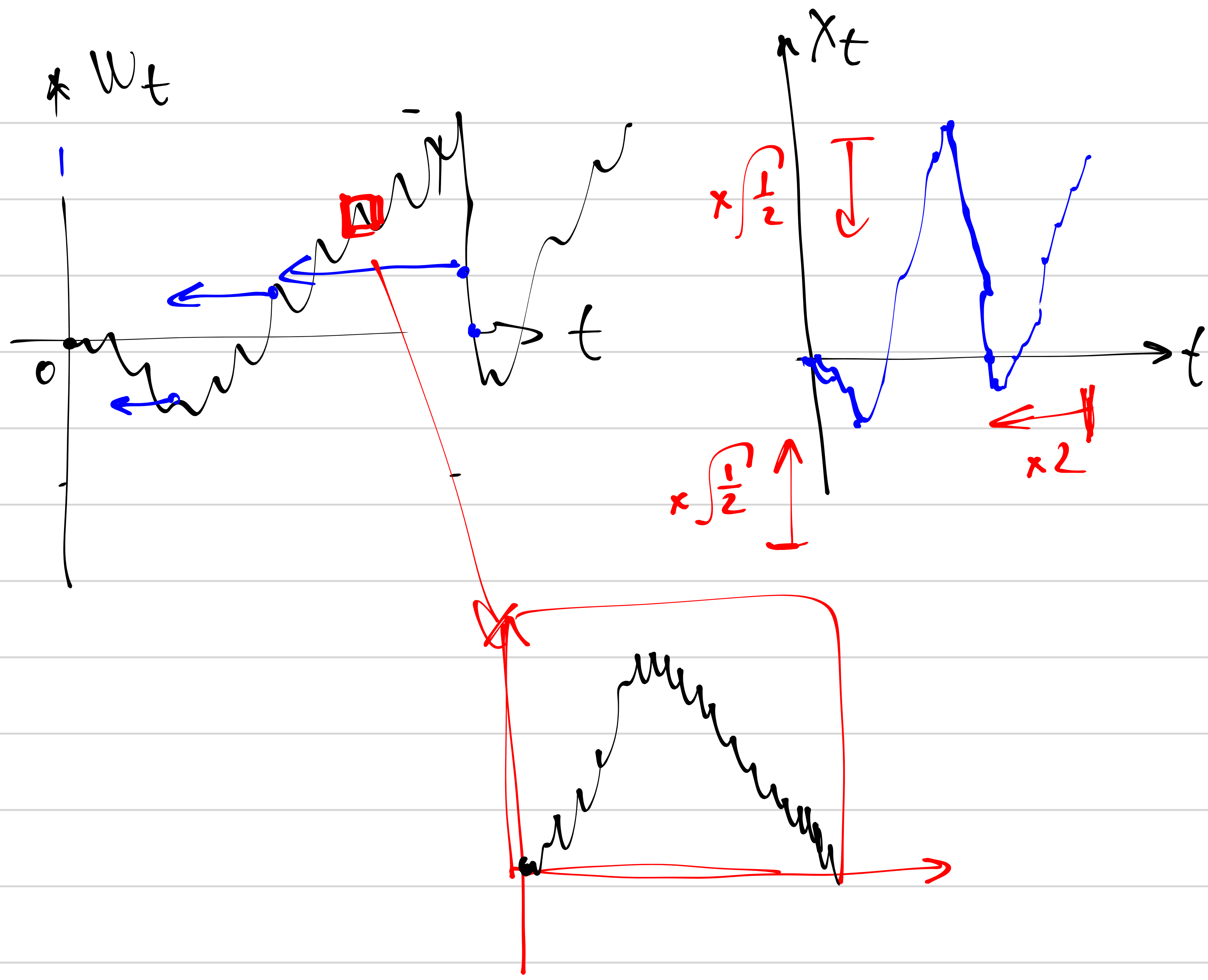
$$\alpha^2\cdot(2t-2s) = t-s \quad \boxed{\forall t \geq s}$$

$\alpha = \sqrt{\frac{1}{2}}$ $\qquad$ $\alpha = -\sqrt{\frac{1}{2}}$

(W_t) – wiener process $\Rightarrow$

$X_t = \sqrt{\frac{1}{2}}\cdot W_{2t}$ – Wiener process

$X_t = -\sqrt{\frac{1}{2}}\cdot W_{2t}$ – Wiener process.



Time Inversion.

Theorem

If: (W_t) – Wiener process. then (X_t) – Wiener process.

$$X_t = \begin{cases} 0, & \text{if } t=0 \\ t\cdot W_{1/t}, & \text{for } t>0. \end{cases}$$

Check ①, ②

① $X_0 = 0$ "

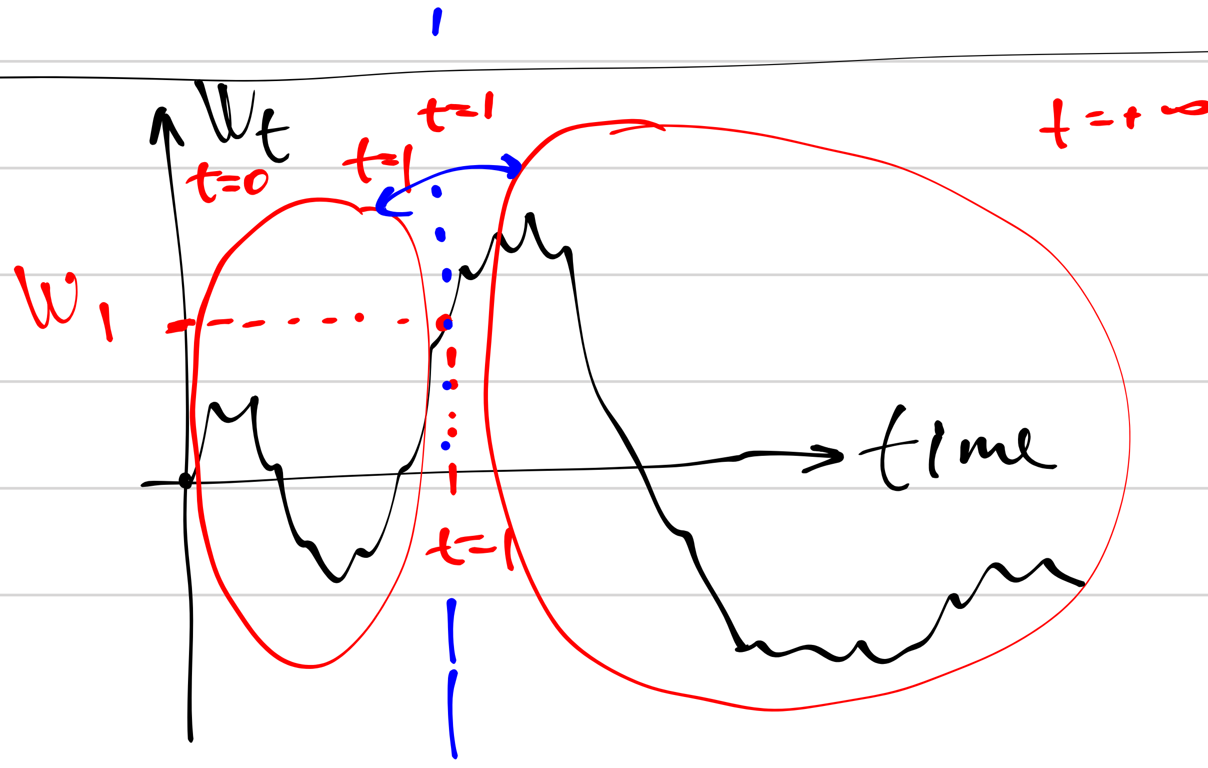
$X_5 = 5\cdot W_{0.2}$

$X_1 = 1\cdot W_1$

$X_{0.2} = 0.2\cdot W_5$

$X_{10} = 10\cdot W_{0.1}$

$X_{0.1} = 0.1\cdot W_{10}$



check assumption 2:

$$X_t - X_s = t \cdot W_{1/t} - s \cdot W_{1/s} \sim N(?;?)$$

$\boxed{t \geq s}$

$\boxed{\frac{1}{t} \leq \frac{1}{s}}$

$$E(X_t - X_s) = t \cdot 0 - s \cdot 0$$

$$Var(X_t - X_s) = t^2 \cdot Var(W_{1/t}) + s^2 \cdot Var(W_{1/s})$$
$$\underline{-2t \cdot s \cdot Cov(W_{1/t}, W_{1/s})} =$$
$$= t^2 \cdot \frac{1}{t} + s^2 \cdot \frac{1}{s} - 2 t \cdot s \cdot \frac{1}{t} =$$
$$= t + s - 2s = t - s \quad \text{☺}$$

$$X_t = \begin{cases} 0 & t=0 \\ t \cdot W_{1/t}, & t>0 \end{cases} \iff W_t = \begin{cases} 0, & t=0 \\ t \cdot X_{1/t}, & t>0 \end{cases}$$

$$Y_t = \begin{cases} 0, & t=0 \\ t \cdot X_{1/t}, & t>0 \end{cases}$$

$$Y_t = t \cdot X_{1/t} = t \cdot \left(\frac{1}{t} \cdot W_{1/(1/t)}\right) = 1 \cdot W_t \quad \text{☺}$$

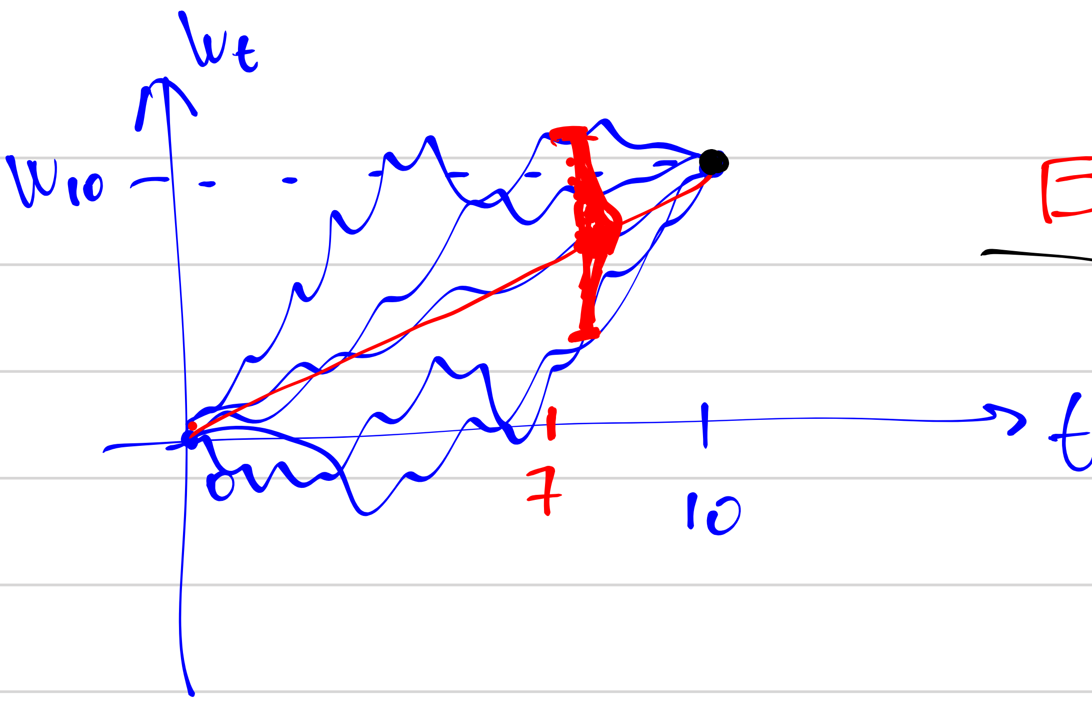
Ex. $E(W_7 | W_{10})$? $Var(W_7 | W_{10})$?

dependent

$$W_7 = 7 \cdot X_{1/7} \qquad W_{10} = 10 \cdot X_{1/10}$$

$$E(W_7 | W_{10}) = E(7 \cdot X_{1/7} | 10 \cdot X_{1/10}) = \quad \underline{(X_t) - \text{Wiener process}}$$

$$= 7 \, E(X_{1/7} | X_{1/10}) = 7 \cdot X_{1/10} = 7 \cdot \frac{1}{10} \cdot W_{10} =$$
$$= \frac{7}{10} \cdot W_{10} \quad \text{☺}$$



$$E(W_7 \mid W_{10}) = \frac{7}{10} \cdot W_{10}$$

$$Var(W_7 \mid W_{10}) = Var(7 \cdot X_{1/7} \mid 10 \cdot X_{1/10}) =$$

$$= 7^2 \cdot Var(X_{1/7} \mid X_{1/10}) = 7^2 \cdot \left(\frac{1}{7} - \frac{1}{10}\right) \quad !!$$

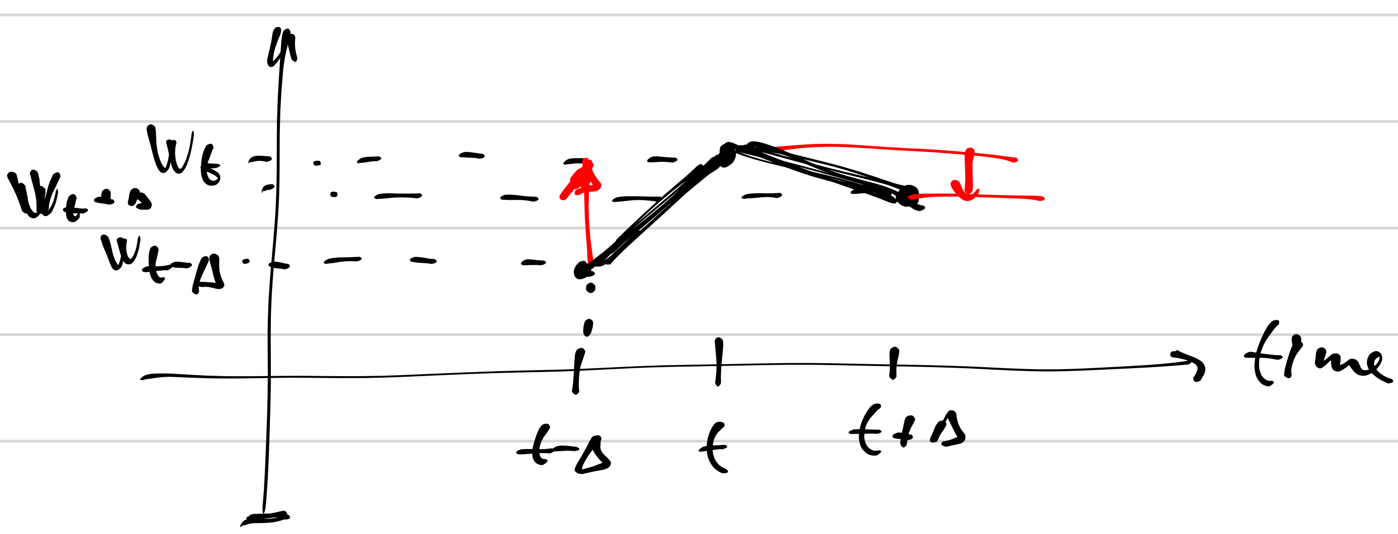
$$\boxed{Var(W_t \mid W_s) = t - s \text{ when } t \geq s}$$

theorem

$P((W_t)$ is nowhere differentiable$) = 1.$

Wow! (4) $P((W_t)$ is continuous$) = 1$

Intuit arg.



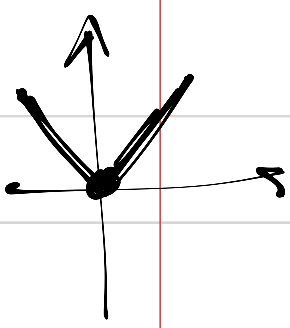
$$W(t) - W(t-\Delta) \sim N(0; \Delta) \quad (\text{by } (2))$$

By Chebyshov inequality: (intuit arg. that cont. by is possible)

$$P(|W_t - W_{t-\Delta}| > a) \leq \frac{\Delta}{a^2} \quad !!$$

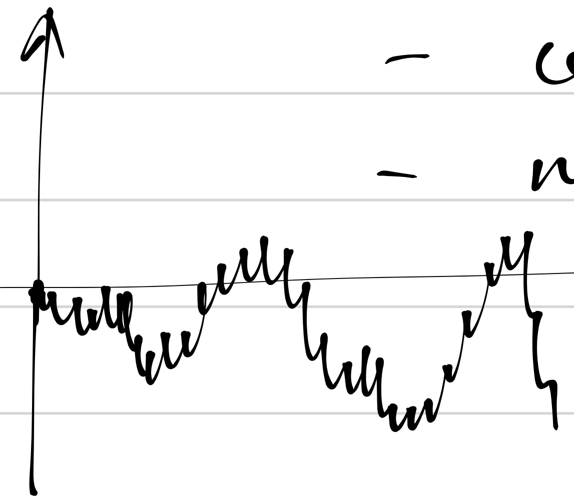
$$R_1 = \frac{W(t) - W(t-\Delta)}{\Delta} \qquad R_2 = \frac{W(t+\Delta) - W(t)}{\Delta}$$

R_1 and R_2 are indep by (3)



$y = |x|$ — continuous for $\mathbb{R}$
— non-differentiable at $x = 0$

(W_t)



— continuous for $t \geq 0$
— non-differentiable for $t \geq 0$

$\boxed{\text{recap}}$

in mean square

$$\boxed{R_n \xrightarrow{M.S.} R}$$

$$\boxed{R_n \xrightarrow{L^2} R}$$

$E(R^2) < \infty$

$E(R_n^2) < \infty$

$$\lim_{n \to \infty} E(|R_n - R|^2) = 0$$

Ex. 1

$P(R_n = 0) = 1 - \frac{1}{n}$

$P(R_n = n) = \frac{1}{n}$

$\boxed{\text{plim } R_n = 0}$

$R_n \xrightarrow{p} 0$

$\lim E(|R_n - 0|^2) =$

$= \lim \left(n^2 \cdot \frac{1}{n} + 0^2 \left(1 - \frac{1}{n}\right)\right) = \lim n = +\infty$

$R_n \xrightarrow{p} 0$

$R_n \overset{M.S.}{\not\to} 0$

Ex 2.

$P(R_n = 0) = 1 - \frac{1}{n}$

$P(R_n = 1) = \frac{1}{n}$

$\text{plim } R_n = 0$

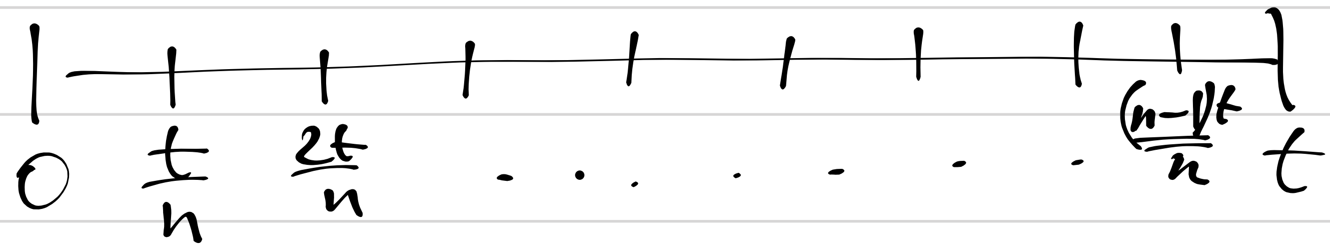
$\lim E(|R_n - 0|^2) =$

$= \lim \left(1^2 \cdot \frac{1}{n} + 0^2 \cdot \left(1 - \frac{1}{n}\right)\right) = 0$

$R_n \xrightarrow{p} 0$

$R_n \xrightarrow{M.S.} 0$

Exercise



$$R_n = \underbrace{\left(W\left(\tfrac{t}{n}\right) - W(0)\right)}_{I_1}^2 + \underbrace{\left(W\left(\tfrac{2t}{n}\right) - W\left(\tfrac{1t}{n}\right)\right)}_{I_2}^2 + \dots + \underbrace{\left(W(t) - W\left(\tfrac{(n-1)t}{n}\right)\right)}_{I_n}^2$$

$$R_n \xrightarrow{M.S.} ?$$

$$R_n = I_1^2 + I_2^2 + \dots + I_n^2$$

$$E(R_n) = n \cdot E(I_1^2)$$

$$I_1 \sim N\left(0; \tfrac{t}{n} - 0\right)$$

$$E(I_1) = 0 \qquad Var(I_1) = E(I_1^2) = \tfrac{t}{n}$$

$$E(R_n) = n \cdot \frac{t}{n} = t \quad !!$$

does not depend on n.

NO COV !!

$$Var(R_n) = Var(I_1^2) + Var(I_2^2) + \dots + Var(I_n^2) \dots$$

$I_1, I_2, I_3, I_4 \dots I_n$ are independent by 3a.

$$I_1 \sim I_2 \dots \sim N\left(0; \tfrac{t}{n}\right)$$

$$Var(R_n) = n \cdot Var(I_1^2) = \qquad \boxed{Z = \frac{I_1 - 0}{\sqrt{\frac{t}{n}}} \sim N(0;1)}$$

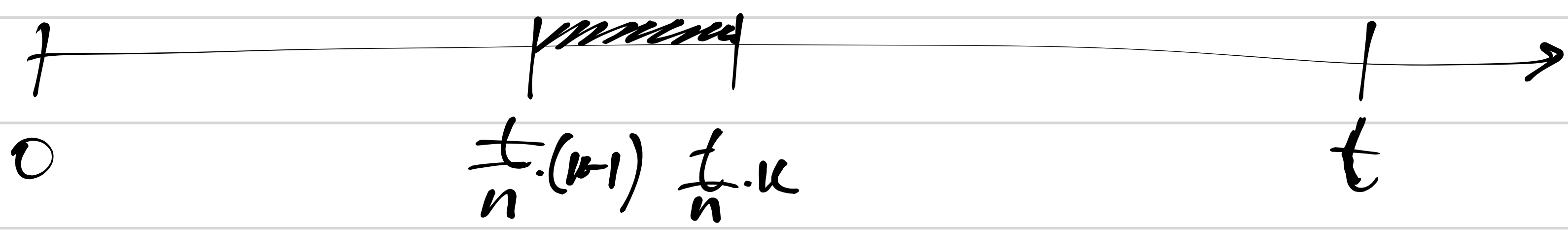
$$I_1 = Z \cdot \sqrt{\tfrac{t}{n}}$$

$$= n \cdot Var\left(\left(Z \cdot \sqrt{\tfrac{t}{n}}\right)^2\right) = n \cdot Var\left(Z^2 \cdot \tfrac{t}{n}\right) =$$

$$= \frac{t^2 \cdot n}{n^2} \cdot \underbrace{Var(Z^2)}_{\text{const.}} \qquad Z \sim N(0;1) \qquad Z^2 \sim \chi_1^2 \quad Var(Z^2) = 2$$

$$R_n = I_1^2 + I_2^2 + I_3^3 + \dots + I_n^2$$

$$I_k = W\left(\tfrac{t}{n}\cdot k\right) - W\left(\tfrac{t}{n}\cdot(k-1)\right)$$



$E(R_n) = t$ [does not depend on n!]

$\mathrm{Var}(R_n) = \frac{t^2}{n}\cdot \text{const.} \xrightarrow[n\to\infty]{} 0$ (less and less random $n\uparrow$)

$\boxed{\text{Guess: } R_n \xrightarrow{M.S.} t}$

$$E\left(|R_n - t|^2\right) \xrightarrow{?} 0$$

$$= E\left((R_n - t)^2\right) = \mathrm{Var}(R_n - t) = \mathrm{Var}(R_n) \to 0$$

$$\mathrm{Var}(R_n - t) = E\left((R_n - t)^2\right) - \Big(\underbrace{E(R_n - t)}_{=0}\Big)^2$$

the fact:

$$\boxed{\begin{gathered} R_n = I_1^2 + I_2^2 + I_3^3 + \dots + I_n^2 \xrightarrow{M.S.} t \\ \text{where } I_k = W\left(k\cdot\tfrac{t}{n}\right) - W\left((k-1)\cdot\tfrac{t}{n}\right) \end{gathered}}$$

Q. Intuitive meaning of convergence in M.S ?

A₁. $d_n = (R_n - R)^2$ how "far" R_n is from R.

$$E(d_n) \to 0 \qquad R_n \xrightarrow{M.S.} R$$

A₂. $$\underbrace{E((R_n - R)^2)}_{\geq 0} = \underbrace{Var(R_n - R)}_{\geq 0} + \underbrace{(E(R_n - R))^2}_{\geq 0}$$

$$R_n \xrightarrow{M.S.} R \iff \begin{cases} Var(R_n - R) \to 0 \\ E(R_n - R) \to 0 \end{cases}$$

Q2 $\mathcal{B}(X)$, $\mathcal{B}(X, X+X)$

A2. $\mathcal{B}(X) = \mathcal{B}(X, 2X)$

Q3. N - number of trials (to get Tail)

$X_2 =$ "tail" $\Rightarrow N \leq 2$

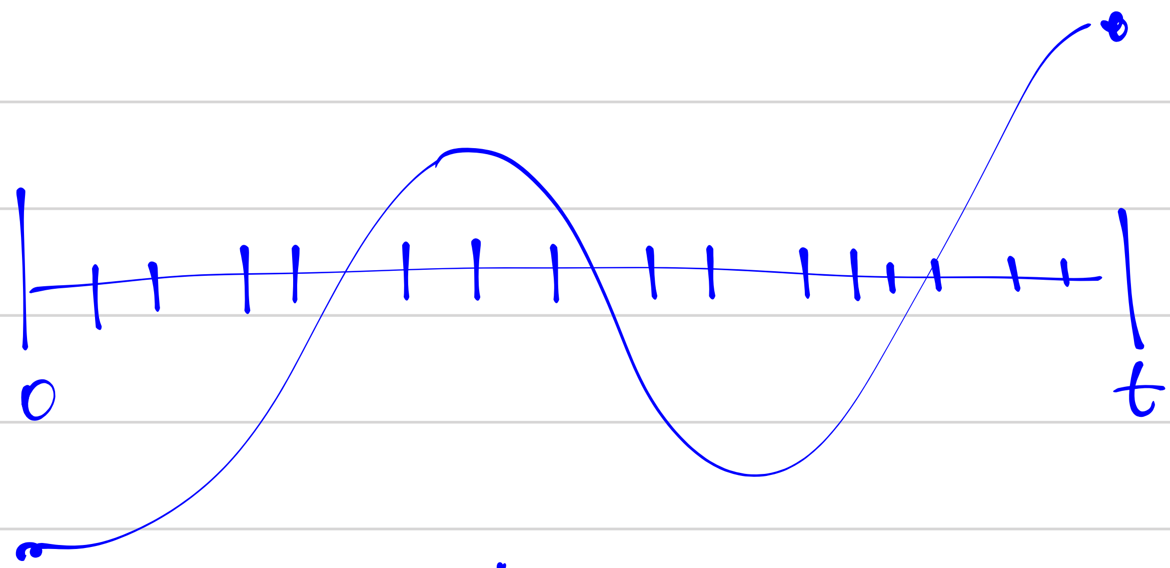
$\boxed{N \text{ and } X_i \text{ are dependent.}}$

$X_2 =$ "head" $\Rightarrow N \neq 2$

X_i and N are dependent.

$N = \min\{t \mid X_t = \text{"tail"}\}$

Q4.



$h(x)$ — cont. differentiable

$\delta_1 = h\left(\frac{t}{n}\right) - h(0)$ $\quad$ $\delta_k = h\left(k\cdot\frac{t}{n}\right) - h\left((k-1)\cdot\frac{t}{n}\right)$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\delta_i^2 \overset{?}{=} \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\delta_i^2}{(t/n)^2}\cdot(t/n)^2 = \lim_{n\to\infty}\int_0^t \left(h'(x)\right)^2\cdot\left(\frac{t}{n}\right)dx = 0$$

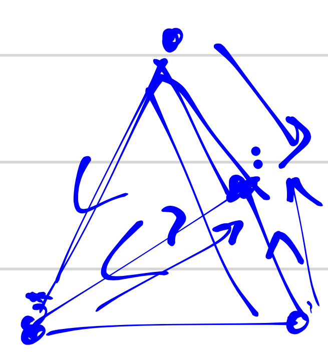
"4"

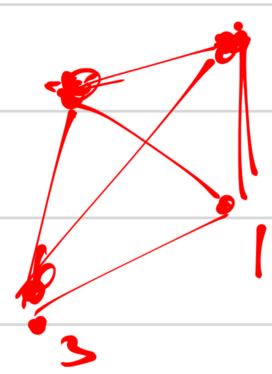
1-1-1-1

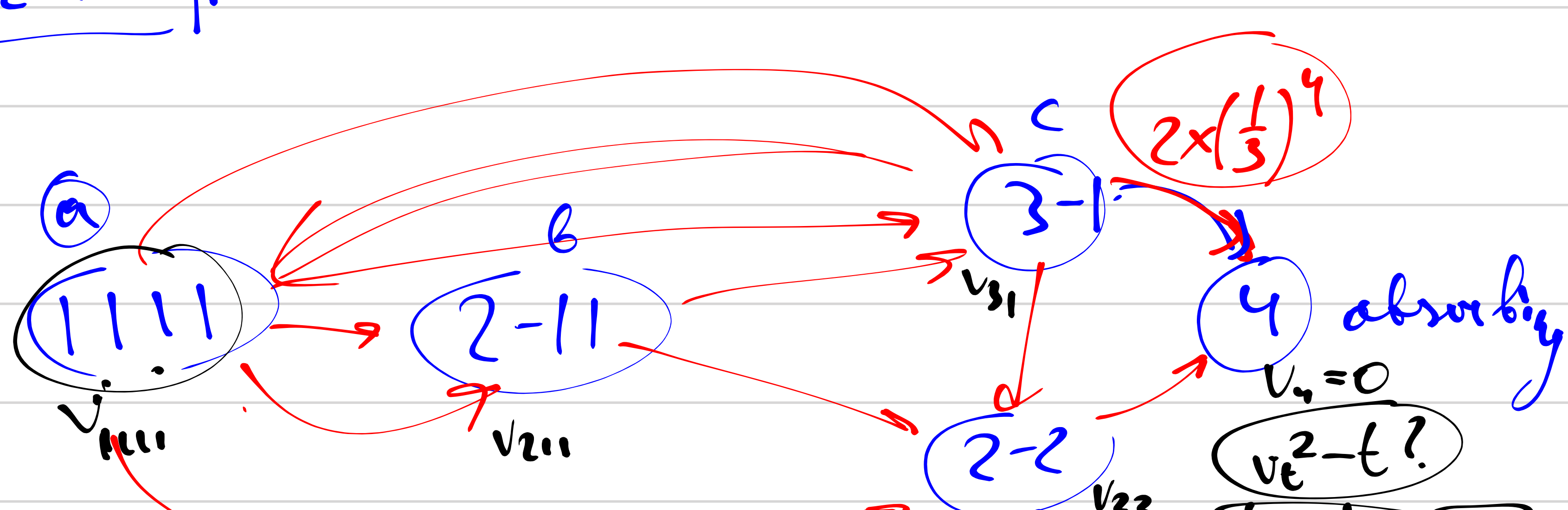
"3-1"

"2-2

2-1-1 ?







$v_t^2 - t$?

strategy 2

V_t

a ?

First step Analysis

strategy 1

b = $\quad$ c =

eq 1 $\quad a = p_{1111\to 211}\cdot(1+b) + p_{1111\to 31}\cdot(1+c) + p_{1111\to 22}\cdot(1+d)$

eq 2 $\quad c = 2\cdot\left(\frac{1}{3}\right)^4\cdot 1 + p_{31\to 22}\cdot(1+d) + \ldots\ldots$