

Submission details

Deadline HA1: 26 November 2023, 23:59

Deadline HA2: 07 December 2023, 23:59

Deadline HA3: 17 December 2023, 23:59

You have one honey-day. The honey-day allows you to postpone one of the three deadlines by 24 hours.

1 HA-1

1. Consider the following joint distribution of X and Y :

	$X = -1$	$X = 0$	$X = 1$
$Y = 0$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

- Find explicitly $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$, $\sigma(X^2)$, $\sigma(2X + 3)$.
 - How many elements are there in $\sigma(X, Y)$, $\sigma(X + Y)$, $\sigma(X, Y, X + Y)$?
2. More σ -algebra questions :)
- You observe the result of 10 independent coin tosses. How many elements the corresponding σ -algebra contains?
 - Prove that a finite σ -algebra can contain only 2^k elements.
 - Is union of two σ -algebras always a σ -algebra? Prove your statement.
 - Is intersection of two σ -algebras always a σ -algebra? Prove your statement.
3. Is it true that for any two σ -algebras \mathcal{F} and \mathcal{H} and for any random variable Y

$$E(E(Y|\mathcal{F})|\mathcal{H}) = E(Y|\mathcal{F} \cap \mathcal{H})?$$

Prove the statement or provide a counter-example.

4. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y .
- Find $\mathbb{P}(Y = y|X)$, $E(Y|X)$, $\text{Var}(Y|X)$;
 - Find $E(X|Y)$.
5. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now. Find $\mathbb{P}(Y = y|X)$, $E(Y|X)$, $\text{Var}(Y|X)$, $E(Y)$, $\text{Var}(Y)$.
6. Random variables X and Y have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

- Find $E(Y|X)$, $\text{Var}(Y|X)$, $E(XY|X)$ and $\text{Var}(XY|X)$.
- Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2021|X)$.

2 HA-2

1. The random variables Z_1, Z_2, \dots are independent and identically distributed with $\mathbb{P}(Z_n = 1) = 0.7$ and $\mathbb{P}(Z_n = -1) = 0.3$. Consider the cumulative sum process, $S_n = Z_1 + \dots + Z_n$ with $S_0 = 0$.

- (a) Find all values of a such that $\exp(aS_n)$ is a martingale.
- (b) If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale.

2. The random variables Z_1, Z_2, \dots are independent and identically distributed with $\mathbb{P}(Z_n = +1) = 0.1$, $\mathbb{P}(Z_n = -1) = 0.1$ and $\mathbb{P}(Z_n = 0) = 0.8$. Consider the cumulative sum process, $S_n = Z_1 + \dots + Z_n$ with $S_0 = 0$.

Let τ be the first moment when $S_n = 10$ or $S_n = -20$.

- (a) Is S_n a martingale?
- (b) Find $\mathbb{P}(S_\tau = 10)$.
- (c) If possible find the non-random function a_n such that $Y_n = S_n^2 + a_n$ is a martingale.
- (d) Find $\mathbb{E}(\tau)$.

3. Let (W_t) be a standard Wiener process.

- (a) Find $\mathbb{E}(\sin(\alpha W_t))$.
- (b) Find $\mathbb{E}(\exp(\alpha W_t))$.
- (c) Find $\mathbb{E}(\cos(\alpha W_t))$.

Hint: you may solve this with or without Ito's lemma, that's up to you.

4. Let (W_t) be a standard Wiener process and $Y_t = W_t^3 + t^2 W_t^2$.

- (a) Find $\mathbb{E}(Y_t)$ and $\text{Var}(Y_t)$.
- (b) Is Y_t a martingale?
- (c) Find $\mathbb{E}(Y_t \mid W_s)$ for $t \geq s$.

5. The processes (X_t) and (Y_t) are independent Wiener processes adapted to the filtration (\mathcal{F}_t) . The process $Z_t = aX_t + 0.3Y_t$ with $a > 0$ is also a Wiener process.

- (a) Find the value of a .
- (b) Find the correlation $\text{Corr}(Z_t, X_t)$.
- (c) Find $\mathbb{E}(Z_3 \mid X_2)$ and $\text{Var}(Z_3 \mid X_2)$.
- (d) Find $\mathbb{E}(Z_3 \mid \mathcal{F}_2)$ and $\text{Var}(Z_3 \mid \mathcal{F}_2)$.

6. Let $Y_t = W_t + 4t$ and consider the process $M_t = \exp(\alpha W_t - \alpha^2 t/2)$. The moment τ is the first moment when Y_t hits 10.

- (a) Check that M_t is a martingale.
- (b) Find $f(t)$ such that $M_t = f(t) \exp(\alpha Y_t)$.
- (c) Using Doob's theorem for M_t find $\mathbb{E}(\exp(-(4\alpha + \alpha^2/2)\tau))$.
- (d) Find $\mathbb{E}(\exp(-s\tau))$ for $s \geq 0$.
- (e) Find $\mathbb{E}(\tau)$.

Hint: you may believe without penalty that Doob's theorem can be applied in this case.

3 HA-3

1. Consider the process $dX_t = W_t^4 dW_t + W_t^6 dt$ with $X_0 = 2024$.

- (a) Find $E(X_t)$.
- (b) Find dY_t for $Y_t = X_t^2$.
- (c) Find $E(Y_t)$ and $\text{Var}(X_t)$.

2. Consider the process $C_t = W_t^3 + 2W_t^2 - 5W_t^2 \cdot t \cdot W_t$.

- (a) Find dC_t .
- (b) Is C_t a martingale?
- (c) Find the covariance $\text{Cov}(C_t, \int W_u^2 dW_u)$.

3. Solve the stochastic differential equation

$$dY_t = -Y_t dt + dW_t, \quad Y_0 = 1$$

If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitly. It's ok.

4. Solve the stochastic differential equation

$$dY_t = Y_t dt + (t^3 + 4Y_t) dW_t, \quad Y_0 = 1$$

If you are have no clues you may try to represent the process Y_t as $Y_t = A_t B_t$, where A_t is the solution of the equation $dA_t = A_t dt + 4A_t dW_t$.

5. Consider the framework of Black and Scholes model: S_t is the share price. Derive the current price of two European type assets, X_0 and Y_0 .

Future payoffs are given by:

- (a) $X_T = (S_T - K)^3$ where T and K are fixed in the contract.
- (b) $Y_T = S_T^{-2}$ where T is fixed in the contract.

6. Consider the framework of Black and Scholes model with stochastic exchange rate. Now S_t is the share price in dollars with $dS_t = \mu S_t dt + \sigma S_t dW_{1t}$. The exchange rate X_t (price of one dollar in euros) is driven by the equation $dX_t = \alpha X_t dt + \beta X_t dW_{2t}$.

The Wiener processes W_{1t} and W_{2t} are independent, in particular that means $dW_{1t} dW_{2t} = 0$.

The risk free interest rate in euros is r .

The european type option pays you $\ln(S_t X_t)$ in euros at fixed time T .

Find the price of this option in euros at time $t = 0$, write your answer in terms of the share price in euros.