Rules: 120 minutes, one A4 cheat sheet and calculator is ok, (W_t) denotes a Wiener process. You may use the standard normal cumulative distribution function $F()$ in your answers.

1. [10] Let (W_t) be a standard Wiener process.
   (a) [3] What is the distribution of $3W_7 + W_8$?
   (b) [5] What is the conditional distribution of $(3W_7 + W_8 \mid W_1 = 2)$?
   (c) [2] Find the probability $\mathbb{P}(3W_7 + W_8 > 1 \mid W_1 = 2)$.

2. [10] Consider the process $Y_t = t + \int_0^t uW_u du$.
   (a) [2 + 3] Find $\mathbb{E}(Y_t)$ and $\mathbb{V}\mathrm{ar}(Y_t)$.
   (b) [5] Find $\mathbb{C}\mathrm{ov}(Y_t, W_t)$.

   Hint: do not forget about Ito's lemma :)

3. [10] Let $M_t = h(t) \cdot \cos W_t$.
   (a) [6] Find a non-zero function $h(t)$ such that M_t is a martingale.
   (b) [4] Find $\mathbb{E}(\cos W_t)$.

4. [10] Consider the Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

   Measure $\mathbb{P}^*$ denotes the risk-neutral probability that makes discounted share price process a martingale and measure $\mathbb{P}$ denotes the probability under which (W_t) is a Wiener process and $dS_t = \mu S_t dt + \sigma S_t dW_t$.
   (a) [2 + 2] Find $\mathbb{P}^*(S_T > \exp(1))$ and $\mathbb{P}(S_T > \exp(1))$.
   (b) [6] Find the current price X_0 of an option that pays you $X_T = \max\{1, -\ln S_T\}$ at fixed time T.

5. [10] Let $dX_t = X_t dt + X_t dW_t$ with $X_0 = 0$.

   Find at least one solution of this stochastic differential equation.

6. [10] The random variables X_1, X_2, ... are independent and exponentially distributed with rate $\lambda = 1$. Let $Y_n = \max\{X_1, X_2, \ldots, X_n\}$.
   (a) [2] Provide an example of event A such that $A \in \sigma(Y_5)$ but $A \notin \sigma(X_5)$.
   (b) [4 + 4] Find $\mathbb{E}(Y_{n+1} \mid Y_n)$ and $\mathbb{V}\mathrm{ar}(Y_{n+1} \mid Y_n)$.

   Note: here $\sigma(.)$ is a minimal sigma-algebra generated by random variable.