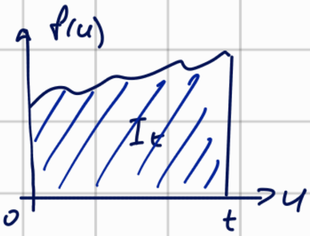
Lecture #4

Wiener process → Stochastic (Ito) integral

$\int_0^t f(u)\,du$ → Area under curve



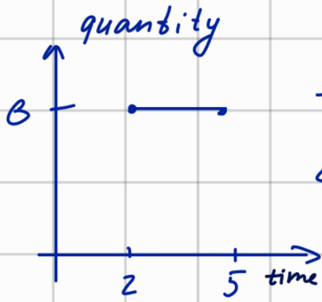
→ $f(u)$ - quantity of shares

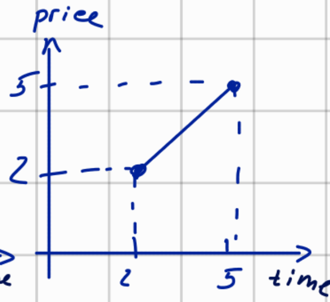
with this interpret. → u - share price at time u

I_t - the net profit

$$\int_2^5 6\,du = 6\cdot 5 - 6\cdot 2$$

quantity price



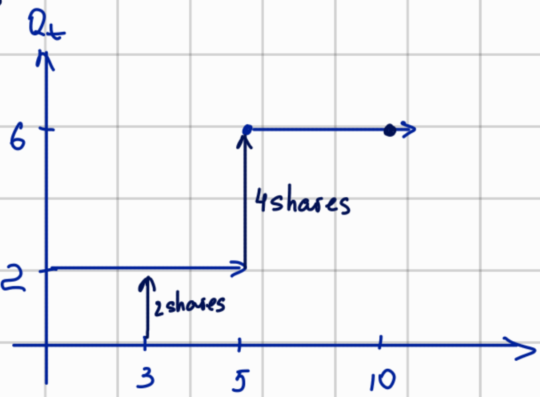


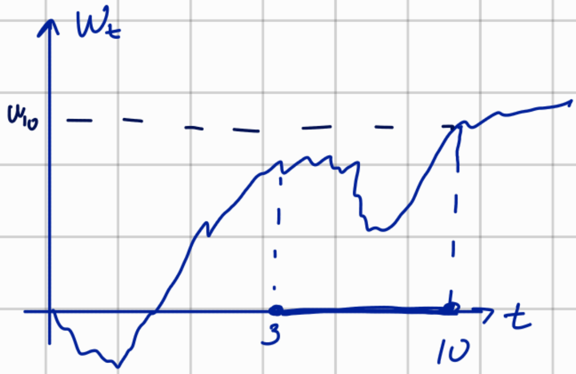
Ex. (W_t) is Wiener Process

$$Q_t = \begin{cases} 2 & t\in[0;5) \\ 6 & t\in[5;\infty) \end{cases}$$

Random process

$$\int_3^{10} Q_t\,dW_t = \dots = \overbrace{6\cdot W_{10}}^{\text{final wealth}} - \underbrace{2\cdot W_3}_{\text{investment at } t=3} - 4W_5$$ – Stochastic integral





$$E\left(\int_3^{10} Q_t\,dW_t\right) \overset{?}{=} E(6W_{10} - 2W_3 - 4W_5) = 0$$

1. $W_0 = 0$
2. $W_t - W_s \sim N(0, t-s)$, $s\le t$
   $W_t \sim N(0,t)$
3. independant increments

$W(t_2)-W(t_1), W(t_3)-W(t_2), \dots, W(t_k)-W(t_{k-1})$

④ $P(\text{trajectory of } (W_t) \text{ is continuous}) = 1$

$$Var\left(\int_0^{10} Q_t \, dW_t\right) = Var\left(6W_{10} - 2W_3 - 4W_5\right) = 36\,Var(W_{10}) + 4\,Var(W_3) +$$
$$+ 16\,Var(W_5) - 2\cdot 2\cdot 6\,Cov(W_{10}, W_3) + 2\cdot 2\cdot 4\,Cov(W_3, W_5) - 2\cdot 6\cdot 4\cdot Cov(W_5, W_{10}) =$$

$$\boxed{Cov(W_5 - W_3,\ W_3 - W_1) = 0}$$

③ property

$$Cov(W_3, W_{10}) = Cov(W_3, W_3 + W_{10} - W_3) =$$
$$= Cov(W_3, W_3) + Cov(W_3, W_{10} - W_3) =$$
$$= 3 + \underbrace{Cov(W_3 - W_0,\ W_{10} - W_3)}_{=0} = 3$$

$$Var(I_t) = 36\cdot 10 + 4\cdot 3 + 16\cdot 5 - 24\cdot 3 + 16\cdot 3 - 48\cdot 5$$

$$I = \int_3^{10} Q_t \, dW_t$$

future value = past value + increments

$$E\left(\int_3^{10} Q_t \, dW_t\right) = 0$$ → We buy and sell Wiener process
It is fair game

→ to continuous time.

<u>def</u> L^2 - convergence

$$R_1, R_2, R_3 \ldots\ldots \xrightarrow{L^2} R$$

if $\lim_{n\to\infty} E\left((R_n - R)^2\right) = 0$

$$E(R_n^2) < \infty, \quad E(R^2) < \infty$$

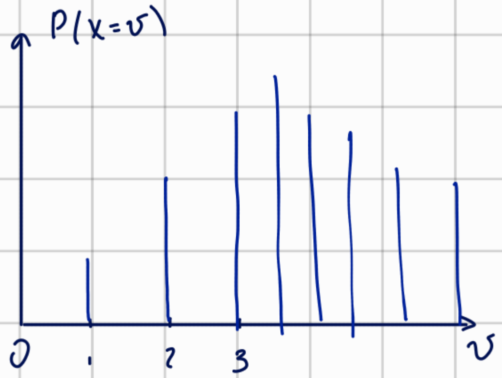
Goal to calculate $\int_0^t W_u \, dW_u$

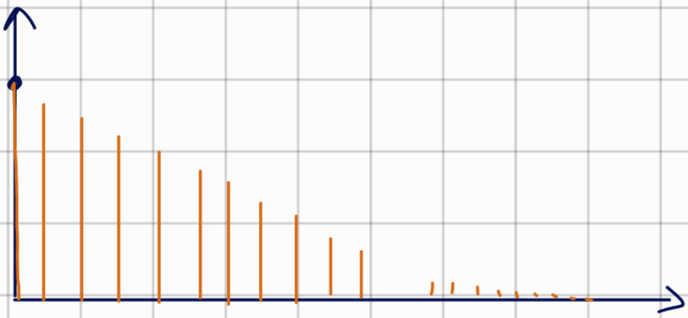
→ old rules of integration are broken $\neq \dfrac{W_t^2}{2} - \dfrac{W_0^2}{2}$

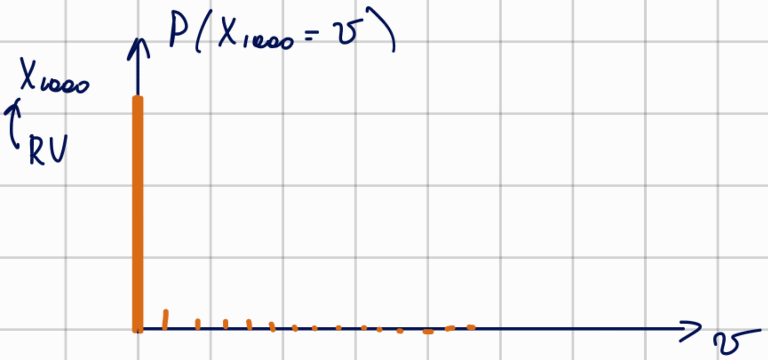
Examples $X_1, X_2, \dots$ are indep $X_k \sim Bin(n=10, p=\frac{1}{k})$

$X_k \xrightarrow[k\to\infty]{L^2} =$ guess

$X_2 \sim Bin(n=10, \frac{1}{2})$ $\qquad$ $X_{100} \sim Bin(n=10, \frac{1}{100})$







$$E((X_k-0)^2) = E(X_k^2) = Var(X_k) + (E(X_k))^2 = 10\cdot\underbrace{\frac{1}{k}\cdot\left(1-\frac{1}{k}\right)}_{} + \left(10\cdot\frac{1}{k}\right)^2 \xrightarrow{k\to\infty} 0$$

„npq" $\qquad$ „np"

The guess was correct

$X_n \xrightarrow{L^2} 0$

$(X_k + Y_n) \xrightarrow{L^2}$

Examples $X_n \sim Bin(n=10, p=\frac{1}{k})$

$Y_n = k\cdot X_n \qquad E(Y_k) = E(k\cdot X_k) = 10\cdot\frac{1}{k}\cdot k = 10$

$Var(Y_n) = Var(k\cdot X_k) = k^2\cdot Var(X_k) = k^2\cdot 10\cdot\frac{1}{k}\left(1-\frac{1}{k}\right) = 10k\left(1-\frac{1}{k}\right)$

$\lim_{k\to\infty} Var(Y_n) = +\infty$

$\lim_{k\to\infty} E((Y_k-0)^2) = \lim_{k\to\infty}(E(Y_k))^2 + Var(Y_k) = \lim_{k\to\infty} 100 + 10k\left(1-\frac{1}{k}\right) =$

$= \infty$

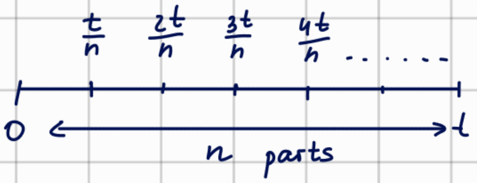
$X_n \xrightarrow{L^2} 0$

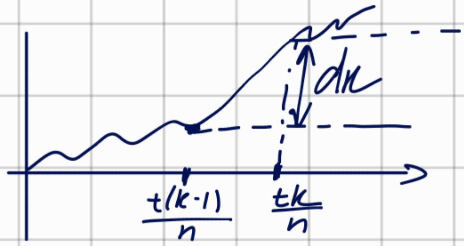
$Y_n \not\xrightarrow{L^2} 0$

Theorem : if $E(X_n) \to \mu$ and $Var(X_n) \to 0$ then

$X_n \xrightarrow{L^2} \mu$

Ex Good exercise :)





d_k = the increment $N_k = W\left(\frac{tk}{n}\right) - W\left(\frac{t(k-1)}{n}\right) \sim N\left(0; \frac{t}{n}\right)$

$\frac{tk}{n} - \frac{t(k-1)}{n} = \frac{t}{n}$

a) $E(d_k)$, $Var(d_k)$

b) $X_n = \sum_{k=1}^{n} d_k^2$ $\quad E(X_n)$ - ? $\quad Var(X_n)$?

c) $X_n \xrightarrow{L^2}$?

(a) $E(d_k) = 0 \qquad Var(d_k) = \frac{t}{n} \Rightarrow E(d_k^2) = \frac{t}{n}$

(b) $E(X_n) = d_1^2 + d_2^2 + d_3^2 + \ldots + d_n^2$

$$E(X_n) = \frac{t}{n} + \frac{t}{n} + \ldots + \frac{t}{n} = \frac{t}{n} \cdot n = t$$

$$Var(X_n) = Var(d_1^2) + Var(d_2^2) + \ldots + Var(d_n^2) = n \cdot Var(d_1^2) = n \cdot Var\left(\frac{t}{n} \cdot Z^2\right)$$

$Z \sim N(0;1)$

$d_1 \sim N\left(0; \frac{t}{n}\right)$

$$= n \cdot \frac{t^2}{n^2} \cdot Var(Z^2) = \frac{t^2}{n} \cdot Var(Z^2)$$

$\frac{d_1}{\sqrt{t/n}} \sim N(0;1)$

$Z = \sqrt{\frac{n}{t}} \cdot d \sim N(0;1)$

$d_1 = \sqrt{\frac{t}{n}} \cdot Z$, where $Z \sim N(0;1)$

$E(X_n) = t \xrightarrow{n\to\infty} t$

$Var(X_n) = \frac{t^2}{n} \cdot \underbrace{Var(Z^2)}_{const} \xrightarrow{n\to\infty} 0$

Theorem $E(X_n) \to t$

$Var(X_n) \to 0$

$X_n \xrightarrow{L^2} t$

Stein lemma

If $Z \sim N(0;1)$

$h(Z)$ is a function $\lim\limits_{z\to\infty} \frac{h(z)}{\exp\left(\frac{z^2}{2}\right)} \to 0$

[$R(\cdot)$ does not grow too fast]

Then $Cov(Z, h(Z)) = E(h'(Z))$

$\to E(Z \cdot h(z))$

Seminar #4

Ex $Z \sim N(0;1)$

$Cov(Z, z^3) = E(3Z^2) = 3\underbrace{E(Z^2)}_{Var(Z)} = 3 \cdot E(Z \cdot \underbrace{Z}_{h}) = 3E(1) = 3$

$Cov(Z, h(Z)) = E(Z \cdot h(z))$

$E(h'(Z))$

$E(Z^{2024}) = E(Z \cdot \underbrace{Z^{2023}}_{h}) = E(2023\, Z^{2022}) = 2023 \cdot E(Z \cdot \underbrace{Z^{2021}}_{h(z)}) =$

$= 2023 \cdot E(2021\, E^{2020}) = 2023 \cdot 2021 \cdot E(Z \cdot Z^{2019}) = \ldots =$

$= 2023 \cdot 2021 \cdot 2019 \cdot \ldots \cdot 3 \cdot 1$

$E(Z \cdot h(Z)) = \int_{-\infty}^{\infty} z h(z) \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)}_{pdf} dz = \int_{-\infty}^{\infty} -h(z) \cdot \underbrace{(-z) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{z}{2}\right)^2}_{u'} dz = -h(z) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{+\infty} h'(z) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$

0

$= E(h'(z))$

$v' = -h'(z)$ $\qquad u = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$

$u \cdot v - \int v' u \, dz$

$$Var(Z^2) = E((Z^2)^2) - (E(Z^2))^2 = E(Z^4) - (E(Z^2))^2$$

$$E(Z^2) = E(Z \cdot Z) = E(1) = 1$$

$$E(Z \cdot h(Z)) = E(h'(Z))$$

$$E(Z^4) = E(Z \cdot Z^3) = E(\underbrace{3 \cdot Z^2}_{h'}) = 3$$

goal

$$I = \int_0^t W_u \, dW_u \overset{?}{=}$$

Informal: Approximate I by $I_1, I_2, I_3, I_4 \dots$

Idea: 1. This integral is a Random variable

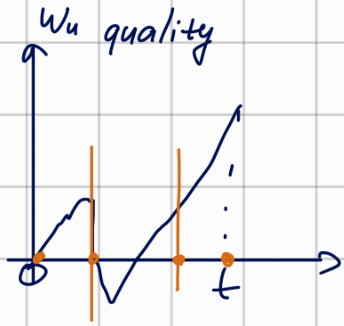
I_n

We will adjust our portfolio at this $\left(\frac{kt}{n}\right)$ moments.

Step 1 $\quad I_n = \int_0^t Q_u \, dW_t$

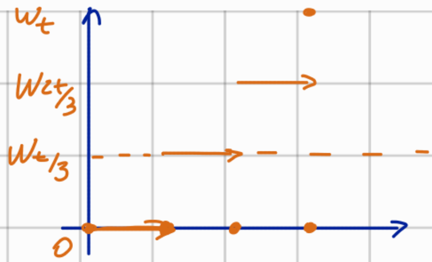
Step 2 $\quad I_n \xrightarrow{L^2} ?$

We need to calculate



Consider approximate strategy

time	0	$t/3$	$2t/3$	$3t/3=t$
target quantity	$W_0=0$	$W_{t/3}$	$W_{\frac{2t}{3}}$	W_t
I buy	0	$W_{t/3}$	$(W_{2t/3}-W_{t/3})$	$W_t - W_{2t/3}$



$$\boxed{I_3 = -0 - W_{t/3}\cdot W_{t/3} - (W_{2t/3} - W_{t/3})\cdot W_{2t/3} - (W_t - W_{2t/3})\cdot W_t + W_t \cdot W_t}$$

Appoximation #3

Step 1. $\boxed{I_n = -\sum_{k=1}^{n}(W_{kt/n} - W_{(k-1)t/n})\cdot W_{\frac{kt}{n}} + W_t^2}$

$$I_n \xrightarrow[n\to\infty]{L^2}$$

$$X_n = \sum_{i=1}^{n} d_i^2 \xrightarrow[n\to\infty]{L^2} t$$

$$d_i = W_{it/n} - W_{(i-1)t/n}$$

$$V_k = W_{\frac{kt}{n}}$$

$$I_n = -\sum_{k=1}^{n}(W_{kt/n} - W_{(k-1)t/n})\cdot W_{\frac{kt}{n}} + W_t^2$$

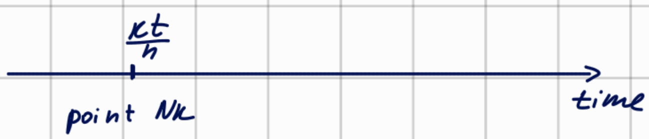
$$I_n = -\sum (V_k - V_{k-1})\cdot V_k + V_k^2$$

$$I_n \xrightarrow{L^2} ?$$

$$\sum_{i=1}^{n}(V_i - V_{i-1})^2 \xrightarrow{L^2} t$$

$$X_n = \sum_{i=1}^{n} V_i^2 + \sum_{i=1}^{n} V_{i-1}^2 - 2\sum_{i=1}^{n} V_i V_{i-1}$$

$V_1^2 + V_2^2 + \dots + V_n^2$ $\qquad \cancel{V_0^2} + V_1^2 + \dots + V_n^2$



$$I_n = -\sum_{i=1}^{n} V_i^2 + \sum_{i=1}^{n} V_i V_{i-1} + V_n^2$$

$$I_n = \frac{V_n^2 - X_n}{2}$$

$$V_n^2 = W_{\frac{nt}{n}}^2 - W_t^2$$

$$I_n = \frac{W_t^2 - X_n}{2} \xrightarrow[n\to\infty]{L^2} \frac{W_t^2 - t}{2}$$

$$\int_0^t W_u \, dW_u = \frac{W_t^2 - t}{2}$$

Intuition: $E\left(\frac{W_t^2}{2}\right)=\frac{1}{2}E(W_t^2)=\frac{1}{2}Var(W_t)=\frac{1}{2}t$

$E\left(\frac{W_t^2-t}{2}\right)=0$

Under [some technical conditions]:

(1) $E\left(\int_0^t Q_u\,dW_u\right)=0$

(2) $\int_0^t (A_u+B_u)\,dW_u=\int_0^t A_u\,dW_u+\int_0^t B_u\,dW_u$

(3) $Y_t=\int_0^t Q_u\,dW_u$ then (Y_t) - a martingale

(4) $Var\left(\int_0^t Q_u\,dW_u\right)=\int_0^t E(Q_u^2)\,du$

(4+) $Cov\left(\int_0^t A_u\,dW_u, \int_0^t B_u\,dW_u\right)=\int_0^t E(A_u B_u)\,du$

Meaning

(1) $\int_0^{10} W_u\,dW_u=\frac{W_{10}^2-10}{2}$

(2) $\int_0^{10}(6W_u+7)\,dW_u=\int_0^{10}6W_u\,dW_u+\int_0^{10}7\,dW_u=6\,\frac{W_{10}^2-10}{2}$

(3) $Var\left(\int_0^t W_u^4\,dW_u\right)=\int_0^t E(W_u^8)\,du=\int_0^t u^4\cdot 5\cdot 7\cdot 3\cdot 1\,du=u^5\Big|_0^t\cdot 7\cdot 3=21t^5$

$E(W_u^8)=E((\sqrt{u}\cdot Z)^8)=E(u^4\cdot Z^8)=u^4\cdot E(Z^8)$

$W_u\sim N(0;u)$

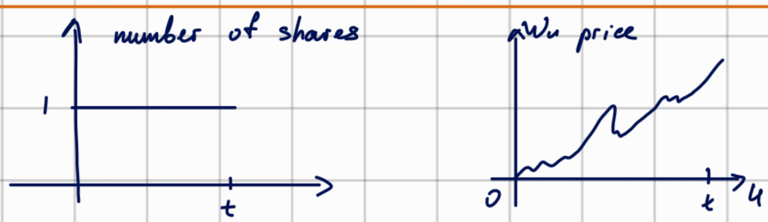
$\frac{W_u}{\sqrt{u}}\sim N(0,1)$

$W_u=\sqrt{u}\,Z \qquad Z\sim N(0;1)$

$*=u^4\cdot E(Z\cdot Z^7)=u^4\cdot E(7\cdot Z^6)=u^4\cdot 7\cdot 5\cdot 3\cdot 1$

$E\left(\int_0^t W_u^4\,dW_u\right)=0$

$Cov\left(W_t,\int_0^t W_u^2\,dW_u\right)\overset{?}{=}Cov\left(\int_0^t 1\,dW_u,\int_0^t W_u^2\,dW_u\right)=\int_0^t E(1\cdot W_u^2)\,du;$ *



$1\cdot W_t-1\cdot W_0$

$\int_0^t 1\,dW_u$

$E(W_u^2)=Var(W_u)=u$ *

$=\int_0^t u\,du=\frac{t^2}{2}$

Check up #4

$\mathcal{E}x$

	$X=0$	$X=2$
$Y=0$	$0{,}2$ $R=0$	$0{,}3$ $R=0$
$Y=1$	$0{,}4$ $R=0$	0.1 $R=2$

$X \cdot Y = R$

$$\sigma(X \cdot Y) = \sigma(R) = \{\{R=0\}, \{R \neq 0\}, \emptyset, \Omega\}$$

Past exam: P 10 ex 2

$$X_t = \int_0^t s\, W_s \cdot dW_s$$

$E(X_t) = 0$

$Var(X_t) = Var\left(\int_0^t s W_s \cdot dW_s\right) = \int_0^t E(sW_s)^2 ds$

$Cov(X_t, W_t) =$

$W_t \sim N(0,t)$

$\frac{W_t}{\sqrt{t}} \sim N(0,1)$

$\left(\frac{W_t}{\sqrt{t}}\right)^2 \Rightarrow \frac{W_t}{t}$

Page 13 $\sqrt{2}$

$Y_t = W_t^3 - t \cdot W_t^4$

$E(Y_t) = E(W_t^3 - t W_t^4) = E(W_t^3) - t\, E(W_t^4) = E((\sqrt{t} \cdot z)^3) - t E((\sqrt{t} \cdot z)^4) =$

$= E(\sqrt{t}^3 \cdot z^3) - t E(\sqrt{t} \cdot z^4) =$

$= t^{\frac{3}{2}} \cdot E(z^3) - t^3 E(z^4) =$

<Apply Stein lemma> $\Rightarrow t^{\frac{3}{2}} \cdot \underbrace{E(2z^2)}_{= 2E(z) = 0} - t^3 \cdot 3\underbrace{E(z^2)}_{= E(zz)=1} = -3t^3$

$Var(Y_t) = Var(W_t^3 - t W_t^4) = Var(W_t^3) + t^2 Var(W_t^4) - 2t\, Cov(W_t^3, W_t^4) =$

$- E($

$Var(Y_t) = E(Y_t^2) - (E(Y_t))^2 = E(Y_t^2) - 9t^6$

$E(Y_t^2) = E((W_t^3 - t W_t^4)^2) = E(W_t^6 - \underbrace{2W_t^3 W_t^4}_{2W_t^7} + t^2 W_t^8) =$