Lecture #2

$$Var(R \mid \mathcal{F}) = E(R^2 \mid \mathcal{F}) - (E(R \mid \mathcal{F}))^2$$

RV   σ-algebra

Properties:

① If R is measurable wrt $\mathcal{F}$ then $Var(R/\mathcal{F}) = 0$

② If R and $\mathcal{F}$ are independent then $Var(R/\mathcal{F}) = Var(R)$

③ "Take out known RVs"

if R, S, T - random variables

S, T are known RV (S, T are measurable wrt $\mathcal{F}$) then

$$Var(SR + T / \mathcal{F}) = S^2 \cdot Var(R/\mathcal{F})$$

④ If S is measurable wrt $\mathcal{F}$ then $Cov(R, S \mid \mathcal{F}) = 0$

⑤ $Var(R+S \mid \mathcal{F}) = Var(R \mid \mathcal{F}) + Var(S/\mathcal{F}) + 2\, Cov(R, S \mid \mathcal{F})$

$E(\cdot \mid \mathcal{F})$ → $Corr(\cdot, \cdot \mid \mathcal{F})$, $Var(\cdot \mid \mathcal{F})$, $P(\cdot \mid \mathcal{F})$, $Cov(\cdot, \cdot \mid \mathcal{F})$

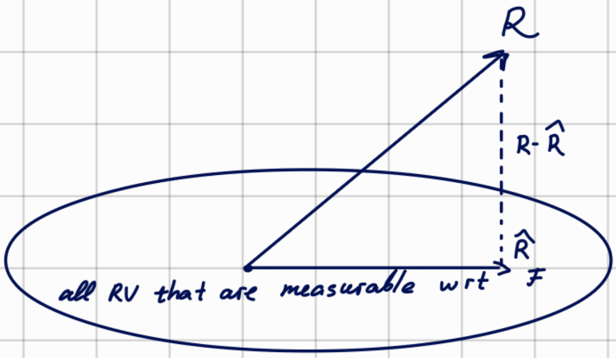
def $Cov(R, S / \mathcal{F}) = E(RS / \mathcal{F}) - E(R / \mathcal{F}) \cdot E(S / \mathcal{F})$

Wow! $Var(R) = E(Var(R \mid F)) + Var(E(R/\mathcal{F}))$

It is Pythagorean theorem
$c^2 = a^2 + b^2$

$$\boxed{\text{If } S \text{ is measurable wrt } \mathcal{F}}$$

$$Var(S \mid \mathcal{F}) \underset{\text{by def}}{=} E(S^2 / \mathcal{F}) - (E(S \mid \mathcal{F}))^2 = S^2 - (S)^2 = 0$$



def $E(R \mid \mathcal{F}) = \hat{R}$

① $\hat{R}$ is meas. wrt. $\mathcal{F}$

② $E(\hat{R}) = E(R)$

③ $Cov(\hat{R} - R, I_A) = 0$

if $A \in \mathcal{F}$

$$Var(R) = \underbrace{Var(\hat{R})}_{Var(E(R \mid \mathcal{F}))} + \underbrace{Var(R - \hat{R})}_{E(Var(R \mid \mathcal{F}))} + \underbrace{2\, cov(R - \hat{R}, \hat{R})}_{0}$$

$$Var(R-\hat{R}) = E((R-\hat{R})^2) - (E(R-\hat{R}))^2$$ by def

$$= \underbrace{E((R-\hat{R})^2)}_{W} = E(E(R-\hat{R})^2 | \mathcal{F}))$$

Tower property of $E(\cdot | \mathcal{F})$

$$E(E(W|\mathcal{F})) = E(W)$$

$Cov(R-\hat{R}, \hat{R}) = 0$

omitted

$$= E(E(R^2 - 2R\hat{R} + \hat{R}^2 | \mathcal{F}))$$

$$= E(E(R^2|\mathcal{F}) - 2\hat{R}\underbrace{E(R|\mathcal{F})}_{\hat{R}} + \hat{R}^2) = E(E(R^2|\mathcal{F}) - 2\hat{R}\hat{R} + \hat{R}^2) = E(E(R^2|\mathcal{F}) - E(R|\mathcal{F}))^2)$$

$$= E(E(R^2|\mathcal{F}) - \hat{R}^2)$$

def the process (X_t) is a martingale.

Intuit. The best forecast of the next value is the current value

$$E(X_{t+1} | X_t, X_{t-1}, X_{t-2}, X_{t-3}, \dots, X_1) = X_t$$

ex. 1. $X_1, X_2, X_3, \dots$ iid $P(X_t = 1) = P(X_t = -1) = \frac{1}{2}$

$S_t = X_1 + X_2 + \dots + X_t$ a) $E(X_t)$ $E(S_t)$ $E(M_t)$

$M_t = S_t^2 - t$ b) which one is a martingale?

$E(X_t) = 0$

$E(S_t) = E(X_1 + X_2 + \dots + X_t) = 0 + 0 + \dots + 0 = 0$

$E(M_t) = E(S_t^2) - t = Var(S_t) - t = t \cdot Var(X_1) - t = t(E(X_1^2) - 0) - t = t - t = 0$

smth is const → smth does not depend on time

↓

smth is non-random

$$E(X_{t+1} | X_t, X_{t-1}, \dots, X_1) = E(X_{t+1}) = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$$

indep

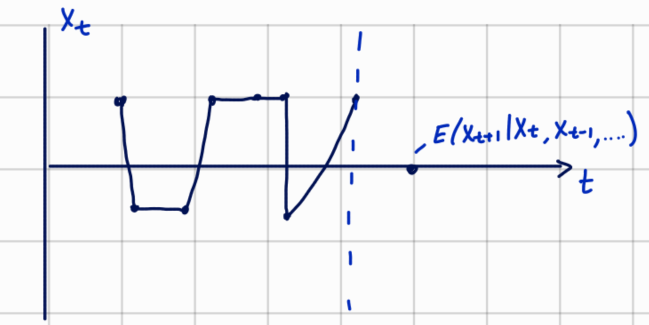
is $E(X_{t+1} | X_t, X_{t-1}, \dots, X_1) \overset{?}{=} X_t$

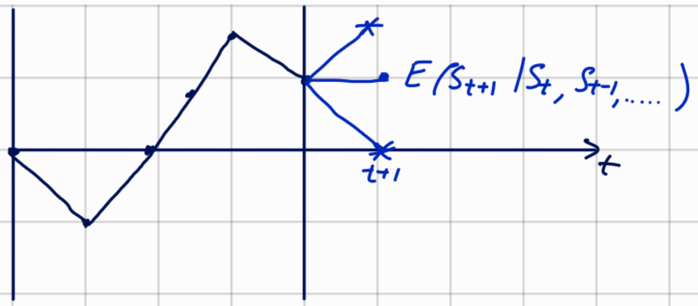
No, X_{t+1} is not a martingale

$$E(S_{t+1} | S_t, S_{t-1}, S_{t-2}, \dots, S_1) = E(S_t + X_{t+1} | S_t, S_{t-1}, \dots, S_1) =$$

known

$$= S_t + E(X_{t+1} | X_t + X_{t-1} + X_{t-2}, \dots, X_1, \dots, X_2 + X_1, \dots X_1) = S_t + E(X_{t+1}) = S_t$$ – is a martingale





Let's shorten the notation:

$\mathcal{F}_t = \sigma(X_1, X_2, \dots, X_t)$

$$E(M_{t+1}|\mathcal{F}_t) = E(S_{t+1}^2 - (t+1) | \mathcal{F}_t) = E((S_t + X_{t+1})^2 | \mathcal{F}_t) - t - 1 =$$

$$= E(S_t^2 + \underbrace{X_{t+1}^2}_{=1} + 2S_t X_{t+1} | \mathcal{F}_t) - t - 1 = E(\underbrace{S_t^2 + 2S_t X_{t+1} | \mathcal{F}_t}_{\text{known}}) + 1 - t - 1 =$$

$$= S_t^2 + 2S_t \cdot E(\underbrace{X_{t+1} | \mathcal{F}_t}_{\text{indep}}) - t = S_t^2 + 2S_t \cdot E(X_{t+1}) - t = S_t^2 + 0 - t = M_t$$ is a martingale

<u>def</u> $(\mathcal{F}_t)$ is a filtration

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \mathcal{F}_4 \subseteq \dots$$

every $\mathcal{F}_t$ is a σ-algebra

<u>def</u> the process (M_t) is a martingale wrt filtration $(\mathcal{F}_t)$ if

$$E(M_{t+1}|\mathcal{F}_t) = M_t$$

Intuition behind the stoping time τ - a stopping time

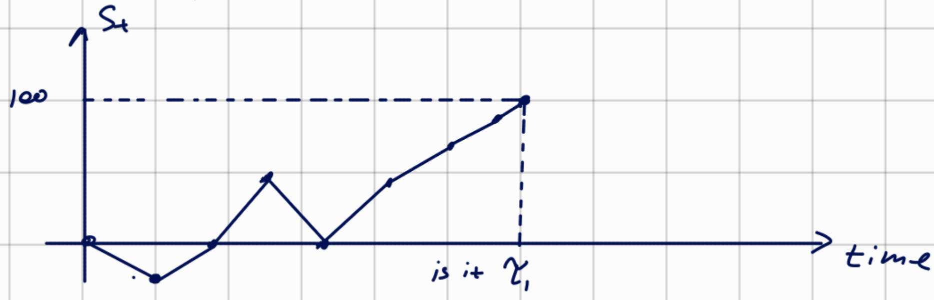
example $(X_t) \sim iid$ $\quad S_t = X_1 + X_2 + \dots + X_t$

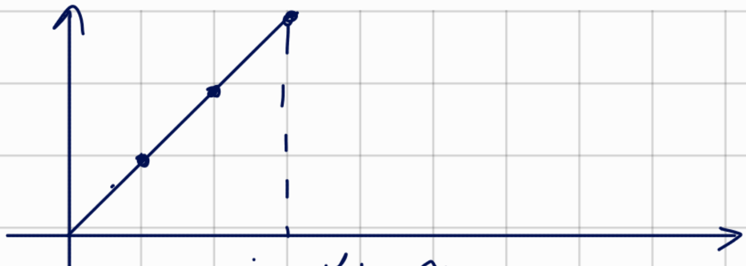
$\mathcal{F}_t = \sigma(X_1, X_2, X_3, \dots, X_t)$ $\quad \tau_1 = \min_t \{t \mid S_t = 100\}$ stopping time

τ_2 = the time of a first local maximum of S_t not a stopping time

X_t → $+1$ $\quad p = \frac{1}{2}$

X_t → -1 $\quad 1 - p = \frac{1}{2}$





is this τ_2

ex. $\tau_3 = \tau_1 + 7$ (+) stopping time

$\tau_4 = \tau_1 - 7$ (−) not

Standart class #2

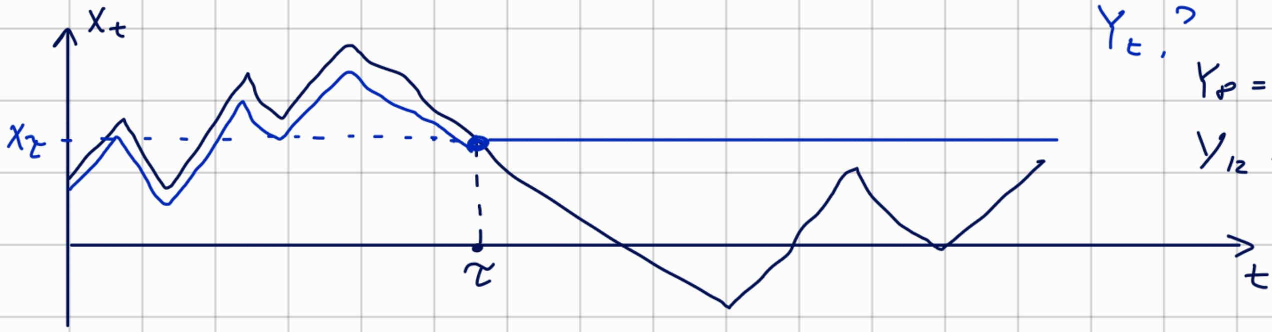
def τ is a stopping time wrt to filtration $(\mathcal{F}_t)$
if $\forall\, t \ \{\tau \le t\} \in \mathcal{F}_t$

notation $x \wedge y = \min(x, y)$

$x \vee y = \max(x, y)$

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

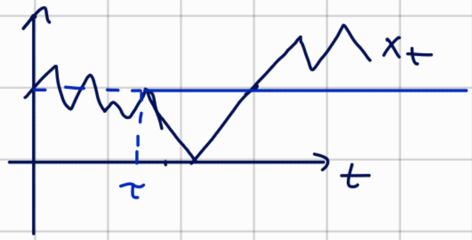
Theorem If (X_t) is a martingale and τ is a stopping time wrt filltration $(\mathcal{F}_t)$ then $Y_t = X_{t \wedge \tau}$ is also a martingale wrt to $(\mathcal{F}_t)$



Y_t ?

$Y_8 = X_{8 \wedge 9} = X_8$

$Y_{12} = X_{12 \wedge 9} = X_9$



Theorem (Optional stopping time / Doob's theorem, теорема Дуба)

If τ is stopping time and (Y_t) is a martingale wrt filtration $(\mathcal{F}_t)$ and at least one of the following conditions are satisfied: then $E(Y_\tau) = E(Y_0)$

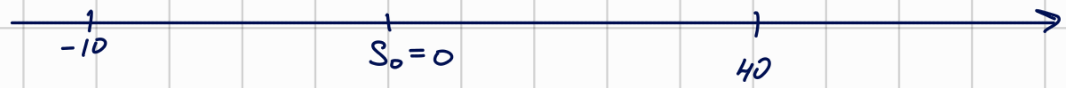
(A) $P(\tau = +\infty) = 0$, $Y_{t \wedge \tau}$ is bounded

$\left[\begin{array}{l} \text{there is a constant } c \\ \text{such that } |Y_{t\wedge\tau}| < c \end{array}\right]$

(B) $E(\tau) < \infty$, $d_t = E(Y_{(t+1)\wedge\tau} - Y_{t\wedge\tau} \mid \mathcal{F}_t)$ is bounded

(C) ....

ex



$S_t = X_1 + X_2 + \dots + X_t$

$X_t \sim iid \begin{cases} +1 & (p = \frac{1}{2}) \\ -1 & (p = \frac{1}{2}) \end{cases}$

$\tau = \min\{t \mid S_t = -10 \text{ OR } S_t = 40\}$

a) $P(X_\tau = 40) = \alpha$ $\quad E(X_\tau)$?

b) $E(\tau) = ?$

S_t is mart

$M_t = S_t^2 - t$ is mart

$\mathcal{F}_t = \delta(X_1, X_2, \dots, X_n)$

$E(S_\tau) = E(S_0) = E(0) = 0$

$S_0 = 0$ $\qquad S_\tau \begin{cases} \nearrow -10 \\ \searrow 40 \end{cases}$

$E(S_\tau) = 40 \cdot \underline{P(S_\tau = 40)} + (-10) \cdot \underline{P(S_\tau = -10)} = 0$

$40\alpha - 10 \cdot (1 - \alpha) = 0$

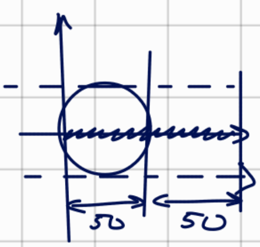
$40\alpha + 10\alpha = 10$

$P(S_\tau = 40) = \alpha = \frac{10}{10 + 40} = \frac{1}{5}$

from A: condition is $Y_{t\wedge\tau}$ is satisfied

$P(\tau = +\infty)$ Probability never quit the game

$P(\tau = +\infty) \le \left(1 - \left(\frac{1}{2}\right)^{50}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{50}\right) \cdot \dots \cdot (\quad) \cdot (\quad) \cdot \dots = 0$



$E(M_\tau) = E(M_0) = E(S_0^2 - 0) = 0$

$E(S_\tau^2 - \tau) = 0$

$E(\tau) = E(S_\tau^2) = -10^2 \cdot \frac{4}{5} + 40^2 \cdot \frac{1}{5} = \frac{400 + 1600}{5} = 400$

$S_\tau^2 \nearrow -10^2 \rightarrow prob = \frac{4}{5}$

$S_\tau^2 \searrow 40^2 \rightarrow prob = \frac{1}{5}$

Example when we cannot use Doob's theorem

(X_t) iid $\nearrow +1$ prob $= \frac{1}{2}$, $\searrow -1$ prob $= \frac{1}{2}$

New rule: I quit the game when I loose first ruble

$S_t = X_1 + X_2 + \ldots + X_t$

$\nu = \min\{t \mid S_t = -1\}$

$(\mathcal{F}_t) = \sigma(X_1, X_2, \ldots, X_t)$

$S_\tau = -1$

$E(S_\tau) = -1$

$\neq$

$E(S_0) = 0$

$\boxed{S_{t \wedge \nu} \text{ is bounded ?}}$ – It is not bounded (no upper-boundary)

$E(\nu)$?

ν - random time necessary to go from 0 to -1

$P(\nu = 1) = \frac{1}{2}$

$P(\nu = 2) = 0$

$P(\nu = 3) = \frac{1}{8}$ . . . .

ν' is a random time necessary to go from 1 to 0

$E(\nu') = E(\nu)$

ν'' is a random time necessary to go from 2 to 1

$E(\nu)$?

$E(\nu) = \frac{1}{2} \cdot 1 + \frac{1}{2} E(\nu + \nu' + 1)$

$E(\nu) = 1 + \frac{1}{2} E(\nu) + \frac{1}{2} E(\nu')$

$E(\nu) = 2 + E(\nu')$

$E(V) = E(V')$

$E(V) = E(V') = +\infty$

$\Rightarrow$ Conditions are violated

Check up class #2

def (X_t) is a martingale

$$E(X_{t+1} \mid X_1, X_2, \dots, X_t) = X_t \qquad (\forall t)$$

def (X_t) is a martingale wrt filltration $(\mathcal{F}_t)$

$$E(X_{t+1} \mid \mathcal{F}_t) = X_t \qquad (\forall t)$$

$$\mathcal{F}_t = \sigma(X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3, \dots, X_t, Y_t, Z_t)$$

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$X_0 = 0,2021$

$\mathcal{F}_t = \sigma(X_0, X_1, X_2, \dots, X_t)$

$P(Y_t = 1 \mid \mathcal{F}_{t-1}) = X_{t-1}$

$X_t = \dfrac{Y_t + X_{t-1}}{2}$

$Y_t \sim \text{Bernoulli}$ $\nearrow 0$, $\searrow 1$

a) Is (X_t) a martingale?

b) distribution of $\lim X_t$?

$Y_1 \nearrow 1$ with $p = X_0 = 0,2021$

$Y_1 \searrow 0$ with $1 - p = 1 - X_0 = 0,7979$

$X = \dfrac{Y_1 + X_0}{2}$

$(Y_2 \mid \mathcal{F}_1) \nearrow 1$ with $p = X_1$

$(Y_2 \mid \mathcal{F}_1) \searrow 0$ with $1 - p = 1 - X_1$

$P(Y_1 = 1, Y_2 = 1, Y_3 = 1)$? $= P(Y_1 = 1) \cdot P(Y_2 = 1, Y_3 = 1 \mid Y_1 = 1)$

$$P(Y_1 = 1, Y_2 = 0, Y_3 = 1) \overset{?}{=} \underset{\downarrow \atop X_0}{0{,}2021} \cdot \underbrace{P(Y_2 = 1 / Y_1 = 1)}_{\substack{0{,}60105 \\ \downarrow \\ X_1}} \cdot \underbrace{P(Y_3 = 1 / Y_2 = 2, Y_1 = 1)}_{\substack{\frac{1{,}60105}{2} \\ \downarrow \\ X_2}}$$

$$X_1 = \frac{1 + 0{,}2021}{2} = 0{,}60105$$

$$E(X_{t+1} / \mathcal{F}_t) = E\left(\frac{Y_{t+1} + X_t}{2} \middle| \mathcal{F}_t\right) =$$

← know $X_1, X_2, \ldots X_t$

$$= \frac{X_t}{2} + \frac{1}{2}\left(1 \cdot X_t + 0 \cdot (1 - X_t)\right) = X_t \quad \text{is a martingale}$$

$$X_1 \nearrow \frac{0{,}2021}{2} \qquad 1 - 0{,}2021$$

$$X_1 \searrow \frac{1 + 0{,}2021}{2} \qquad \text{with } 0{,}2021$$

$$X_2 \nearrow \frac{0{,}2021}{2} / 2$$

$$X_2 \to \left(1 + \frac{0{,}2021}{2}\right) / 2$$

$$X_2 \searrow \left(\frac{1 + 0{,}2021}{2}\right) / 2$$

$$X_2 \downarrow \frac{1 + \frac{1 + 0{,}2021}{2}}{2}$$

If $X_t \in [0;1] \Rightarrow X_{t+1} \in [0;1]$

$X_0 \in [0;1]$

$$X_1 = \frac{Y_1^{\le 1} + X_0^{\le 1}}{2} \begin{matrix} \le 1 \\ \ge 0 \end{matrix}$$

$$X_2 = \frac{{}_{\ge} Y_2^{\le 1} + Y_1^{\le 1}{}_{\ge 0}}{2} \in [0;1]$$

Guess $\lim X_t \sim \text{Uniform } [0;1]$

$P(X_t \le a)$

for uniform distribution

$W \sim U[0;1] \quad P(W \le 0{,}7) = 0{,}7$

$P(W \le 0{,}3) = 0{,}3$

$\lim P(X_t \le a) = a$

$P(X_t \le a) = E(I(X_t \le a)) =$

$= E(E(I(X_t \le a) \mid \mathcal{F}_{t-1}))$;

$E(E(\cdot \mid \mathcal{F})) = E(\cdot)$

$E(\cdot \mid \mathcal{F})$ → easy to calculate

$E(\cdot)$ → hard to calculate

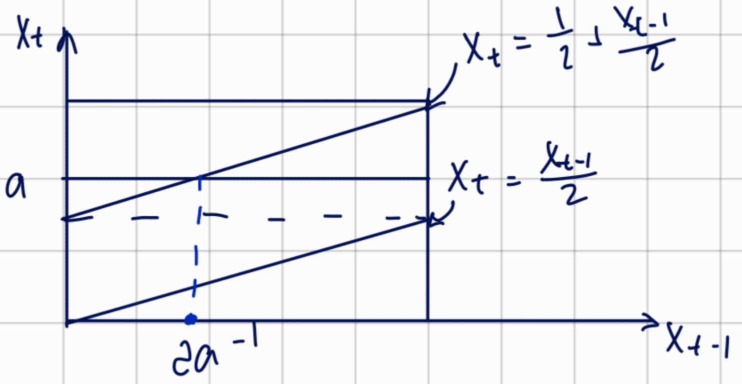
$$E(I(X_t \le a) \mid \mathcal{F}_{t-1}) = P(X_t \le a \mid X_{t-1}) = P\left(\frac{Y_t + X_{t-1}}{2} \le a \mid X_{t-1}\right)$$

$a = 0,7$

$X_{t-1} = 0,5$

$$\frac{Y_t + X_{t-1}}{2} \nearrow \frac{1+0,5}{2} = 0,75 \quad \text{prob } 0,5 \ (X_{t-1})$$

$$\searrow \frac{0,5}{2} = 0,25 \quad \text{prob } 1-0,5 = 1-X_{t-1}$$



$a > \frac{1}{2}$

$1 - X_{t-1} + X_{t-1} \cdot I(X_{t-1} \le 2a-1)$

$2a = 1 + X_{t-1}$

$$\boxed{P(X_t \le a) = E(1 - X_{t-1} + X_{t-1} \cdot I(X_{t-1} \le 2a-1))}$$

try at home

$\lim X_t \overset{\text{guess}}{\sim} U[0;1]$

$\lim P(Y_t = 1)$

$X_{1000000} \approx U[0,1]$

$P(Y_t = 1) = E(E(Y_t \mid X_{t-1})) = E(E(Y_t \mid X_{t-1})) =$

$= E(X_{t-1})$

exam 2017

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standard dice in times

$w_i \nearrow 1$ if „five" appears

$\searrow 0$, otherwise

$Y = W_2 + W_3 + \ldots + W_n$

$X = W_1 + W_2 + \ldots + W_{n-1}$

$E(X \mid Y)$ $\quad$ $Var(Y \mid X)$

$E(Y \mid X)$

depends on information

$$E(X|Y) = E(\underbrace{W_1}_{ind} + W_2 + \dots + W_{n-1} | W_2 + W_3 + \dots W_n) = \underbrace{E(W_1)}_{\frac{1}{6}} +$$

$$+ E(W_2 + \dots + W_{n-1} | W_2 + W_3 + \dots + W_n)$$

$$E(W_2 + W_3 + \dots + W_n | W_2 + W_3 + W_4 + \dots + W_n) = W_2 + W_3 + \dots + W_n = (n-1)R$$

$$E(W_2 | W_2 + \dots + W_n) = R =$$

$$E(W_3 | W_2 + \dots + W_n) = R =$$

$$E(W_2 | W_2 + W_3) \qquad E(W_3 / W_2 + W_3 = 1) = \frac{1}{2}$$

$$R = \frac{W_2 + W_3 + \dots + W_n}{n-1}$$

$$W_2 + W_3 = 0 \qquad E(W_2 / W_2 + W_3 = 0) = 0$$

$$W_2 + W_3 = 1 \qquad E(W_2 | W_2 + W_3 = 1) = \frac{1}{2}$$

$$E(W_2 | W_2 + W_3) = \frac{W_2 + W_3}{2}$$

$$E(W_3 | W_2 + W_3) = \frac{W_2 + W_3}{2}$$

$$E(W_2) = \frac{1}{6}$$

$$E(W_1 + W_2 + \dots + W_{n-1} | W_2 + \dots + W_n) = \frac{1}{6} + (n-2) \cdot \frac{W_2 + W_3 + \dots + W_n}{n-1}$$

$$E(X|Y) = \frac{1}{6} + \frac{n-2}{n-1} \cdot Y$$

$$E(Y|X) = \text{the same} = \frac{1}{6} + n-2 \; \frac{X}{n-1}$$

$$E(W_2 + W_3 + \dots + W_n | W_1 + W_2 + \dots + W_{n-1})$$

unknown

$$E(\underbrace{W_2} + \underbrace{W_3} + \underbrace{W_4} | W_2 + W_3 + W_4 + W_5 + W_6 = 4) = 3 \cdot E(W_2 / W_2 + \dots + W_6 = 4) =$$

$$= 3 \cdot \frac{4}{5}$$

$$E(W_2 | W_2 + \dots + W_6) = \frac{W_2 + W_3 + \dots + W_6}{5}$$