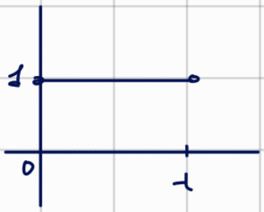
# Lecture #5

$$\int_0^t 1\, dW_t = 1 \times W_t - 1 \times W_0 = W_t$$

quantity ↑ ↖ price





$$\int_0^t W_u\, dW_u =$$

No explicit formula

1. $E\left(\int_0^t A_u\, dW_u\right) = 0$

2. $Var\left(\int_0^t A_u\, dW_u\right) = \int_0^t E\left(A_u^2\right) du$

3. $Cov\left(\int_0^t A_u\, dW_u, \int_0^t B_u\, dW_u\right) = \int_0^t E\left(A_u B_u\right) dw_u$

**def** Ito's process $(Y_t)$

$$Y_t = Y_0 + \int_0^t A_u\, dW_u + \int_0^t B_u\, du$$

↑ const

Short notation: $dY_t = A_t\, dW_t + B_t\, dW_t$

Black Sholes model

$$dB_t = r B_t\, dt$$

$$B_t = B_0 + \int_0^t r B_u\, du$$

$$dS_t = \mu S_t\, dt + \sigma \cdot S_t \cdot dW_t$$

$$S_t = S_0 \int_0^t \mu S_u\, du\, t + \int_0^t \sigma S_u\, dW_u$$

$$Y_t = \int_0^t W_u\, dW_u$$

$$dY_t = W_t\, dW_t$$

$$dY_u = W_u\, dW_u$$

Theorem [under tech. cond]

$(Y_t)$ is a martingale if and only if $dY_t = A_t dW_t$

$(Y_t)$ is a martingale if and only if $Y_t = Y_0 + \int_0^t A_u dW_u$

Ito's lemma

If $h(x,t)$ has continuous $h''_{xx}$, $h'_t$, [some more tech. cond] and $Y_t = h(X_t, t)$ then $dY_t = h'_x \cdot dX_t + h'_t \cdot dt + \frac{1}{2} h''_{xx} \cdot (dX_t)^2$,

where $(dX_t^2)$ is calculated using follow rules: $dt \cdot dW_t = 0$

$$dt \cdot dt = 0$$

$$dW_t \cdot dW_t = dt$$

Chain rule: $h(x,t)$ - non random

$$dh = h_x' \cdot dx + h_t' \cdot dt$$

ex. 1. $Y_t = W_t^2$ $\quad Y_t = h(W_t, t) \quad h(x,t) = x^2$

(a) $dY_t$ - ?

(b) find $\int_0^t W_u dW_u$ - ?

$$dY_t = \underbrace{2W_t}_{h'_x} \cdot dW_t + 0 \cdot dt + \frac{1}{2} \cdot 2 \, (dW_t)^2$$

$$dY_t = 2W_t \cdot dW_t + 1 \cdot dt$$

$$Y_t = Y_0 + \underbrace{\int_0^t 2W_u dW_u}_{\text{express the stat integral}} + \int_0^t 1 \, du$$

$$\Rightarrow \int_0^t 2W_u dW_u = \frac{Y_t - Y_0 - \int_0^t 1 du}{2} = \frac{W_t^2 - 0 - t}{2}$$

$Y_t = \exp(5W_t) \qquad h(x,t) = \exp(5x)$

(a) $dY_t$ ?

(b) Is $(Y_t)$ is a martingale?

(c) $f(t)$ ? $Y_t f(t)$ is a martingale

(d) $E(Y_t)$?

# Not a martingale, because there is a dt

$$dY_t = 5\cdot\exp(5W_t)\cdot dW_t + 0dt + \frac{1}{2}25\cdot\exp(5W)(dW_t)^2$$

short form: $dY_t = 5\exp(5W_t)dW_t + 12{,}5\exp(5W_t)\cdot dt$

$$Y_t = Y_0 + \int_0^t 5\exp(5W_u)dW_u + \int_0^t 12{,}5\cdot\exp(5W_u)\cdot du$$

$$E(Y_{t+h}|\mathcal{F}_t) =$$


$$Y_{t+h} = Y_t + \int_t^{t+h} A_u\, dW_u + \int_t^{t+h} B_u\, du$$

$$\int_t^{t+h} 12{,}5\cdot\exp(5W_u)du \gg 0$$

$$E\left(\int_t^{t+h} A_u\, dW_u\right) = 0$$

$$E(Y_{t+h}) > E(Y_t)$$

(c) $R_t = f(t)\cdot\exp(5W_t)$ $\qquad h(x,t) = f(t)\cdot\exp(5x)$

$f(t)$? s.t. $R_t$ is martingale

$$dR_t = f_t\cdot 5\exp(5W_t)dW_t + f'(t)\cdot\exp(5W_t)dt +$$

$$+\frac{1}{2}f(t)\,25\exp(5W_t)(dW_t)^2$$

$$dR_t = f(t)\cdot 5\cdot\exp(5W_t)dW_t + \underbrace{f'(t)\exp(5W_t) + \frac{1}{2}f(t)\cdot 25\exp(5W_t)}_{0}dt =$$

$$= f'(t)\cdot\cancel{\exp(5W_t)} + \frac{25}{2}f(t)\cdot\cancel{\exp(5W_t)} \equiv 0$$

$$f'(t) = -\frac{25}{2}f(t)$$

$\Downarrow$

$$f(t) = c\cdot e^{-\frac{25}{t}}$$

(d) $E(Y_t)$

$E(\exp(5W_t))$ – it is not a martingale

$E(R_t) = R_0$ # cancel $c$

$E\Big(\underbrace{c \cdot \overbrace{\exp\left(-\frac{25t}{2}\right)}^{1} \overbrace{\exp(5W_t)}^{1}}_{\text{martingale}}\Big) = C = R_0$

$1 = \exp\left(-\frac{25t}{2}\right) E\left(\exp(5W_t)\right) =$

$E(\exp(5W_t)) = \frac{1}{\exp\left(-\frac{25}{2}t\right)} = \exp\left(\frac{25}{2}t\right)$

## Seminar #5

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$dX_t = (2X_t + 1)\,dt + 2\sqrt{X}\,dW_t\, dW_t \quad , X_0 = 1$

$X_t = ?$

$X_t = $ no $X_t$ here

(a) Simplify the eq-n using $Y_t = \sqrt{X_t}$ $\quad Y_t = h(X_t)$

$h(x,t) = \sqrt{x}$

$dY_t = \underbrace{\frac{1}{2\sqrt{X_t}}}_{\frac{1}{2}X_t^{-\frac{1}{2}}} dX_t + 0 \cdot dt + \frac{1}{2}\left(-\frac{1}{4}\right) \cdot X_t^{-\frac{3}{2}} \cdot (dX_t)^2)$

$((2X+1)dt + 2\sqrt{X_t}\,dW_t)^2 =$
$= (2X_t+1)(dt)^2 + 4\sqrt{X_t}(2X_t+1)dt\,dW_t +$
$+ 4X_t(dW_t)^2 = 4X_t dt$

$\Rightarrow dY_t = \left(\frac{2X_t+1}{2\sqrt{X_t}}dt + \frac{2\sqrt{X_t}}{2\sqrt{X_t}}dW_t\right) + \left(-\frac{1}{8}\right)X_t^{-3/2} \cdot 4X_t\, dt$

$dY_t = (dW_t + (\sqrt{X_t} + \frac{1}{2\sqrt{X_t}} - \frac{1}{2}X_t^{-\frac{1}{2}}\quad)dt$

$dY_t = dW_t + Y_t dt$

b) Consider the substitution $Z_t = f(t) \cdot Y_t$ and choose $f(t)$ wisely

$dZ_t = f(t) \cdot dY_t + f'(t) \cdot Y_t \cdot dt + \frac{1}{2} \cdot 0 \cdot dY_t)^2$ $\quad Z_t = h(y,t) = Y_t \cdot f(t)$

$dZ_t = f(t)(dW_t + Y dt) + f'(t) \cdot Y_t dt$ $\quad h_x' dX_t + h_t' dt + \frac{1}{2}h_{xx}''(dX_t)^2$

$dZ_t = f(t)\,dW_t + (f(t) \cdot Y + f'(t)\, Y_t)\,dt$ $\quad f(t) \equiv 0$

$f'(t) = -\exp(-t)$

$Z_t = \exp(-t) \cdot Y_t$ $\quad f + f' \equiv 0$

$dZ_t = \exp(-t) \cdot dW_t$

$$Z_t = Z_0 + \int_0^t \exp(-u)\,dW_u$$
$$Z_0 = \exp(-0)\, Y_0 = 1$$
$$Z_t = 1 + \int_0^t \exp(-u)\,dW_u$$
$$Y_t = \exp(t)\cdot\left(1 + \int_0^t \exp(-u)\,dW_u\right)$$
$$X_t = \exp(2t)\cdot\left(1 + \int_0^t \exp(-u)\,dW_u\right)^2$$

ex 4.

$$Y_t = \int_0^t \text{sign}(\cos u)\,dW_u$$

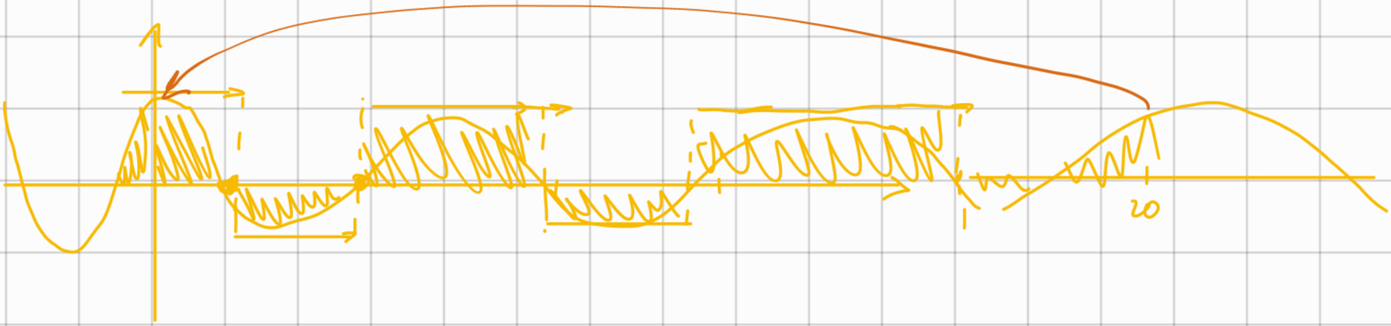
$\text{sign}(5) = +1$

$\text{sign}(-6) = -1$

(a) $\text{Cov}(Y_{2024\pi}, W_{2024\pi})$

$$\text{cov}\left(\int_0^t \text{sign}(\cos(u))\,dW_u,\ W_t = \int_0^t 1\,dW_u\right) = \int_0^t E(\text{sign}(\cos u)\cdot 1)\,du = \int_0^t \text{sign}(\cos u)\,du = 0$$

# nothing is random




(b) Is $(Y_t)$ a Wiener process.

$Y_0 = 0$

$(Y_t - Y_s) \sim N(0, t-s)$

$$Y_\pi - Y_{\frac{\pi}{4}} = \left(Y_\pi - Y_{\frac{\pi}{2}}\right) + \left(Y_{\frac{\pi}{2}} - Y_{\frac{\pi}{4}}\right) = \int_{\frac{\pi}{2}}^{\pi} -1\,dW_u + \int_{\frac{\pi}{4}}^{\pi} 1\,dW_u \sim$$

# distribution is the same

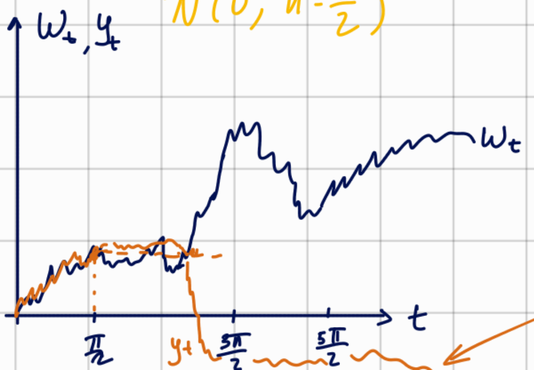
$$\sim -\cancel{1}\left(W_\pi - W_{\frac{\pi}{2}}\right) + 1\left(W_{\frac{\pi}{2}} - W_{\frac{\pi}{4}}\right) = W_\pi - W_{\frac{\pi}{4}} \sim N\left(0; \pi - \frac{\pi}{4}\right)$$

can just erase

$N\left(0; \frac{\pi}{2} - \frac{\pi}{4}\right)$

# they are independant

$N\left(0; \pi - \frac{\pi}{2}\right)$

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