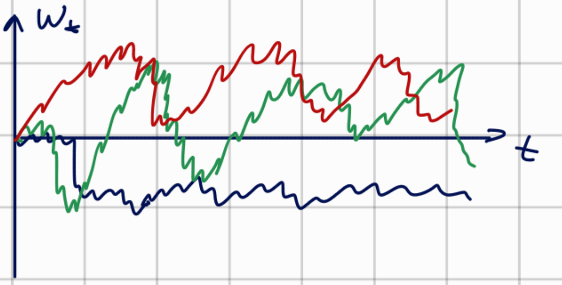
Lecture #3 Time is continuous $t \in [0; +\infty)$

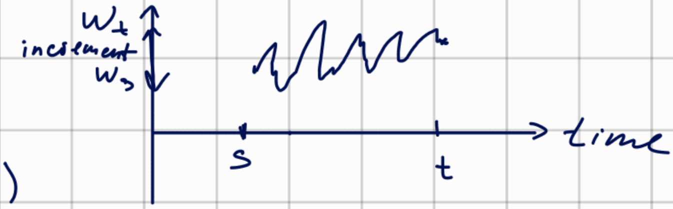
def $(W_t) \to$ Wiener Process / Brownian Motion



① $W_0 = 0$

② for $\forall\ s \le t$

increment $W_t - W_s \sim N(0; t-s)$



③ Increments on non-overlapping intervals are independant

$\forall\ t_1 \le t_2 \le t_3 \le \dots \le t_n$

$\Delta_1 = W(t_2) - W(t_1)$ ← independant increments

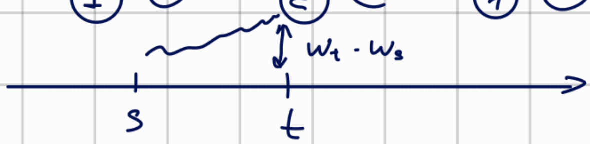
$\Delta_2 = W(t_3) - W(t_2)$,

④ the P(trajectory (W_t) is continuous$) = 1$

def (W_t) is wiener Process wrt filltration $(\mathcal{F}_t)$

if: ① " ② " ④ "   " - mean the same

$\boxed{3'}$



$\forall\ s \le t$

future increments is independant of current information

$(W_t - W_s)$ is indep with $\mathcal{F}_s$

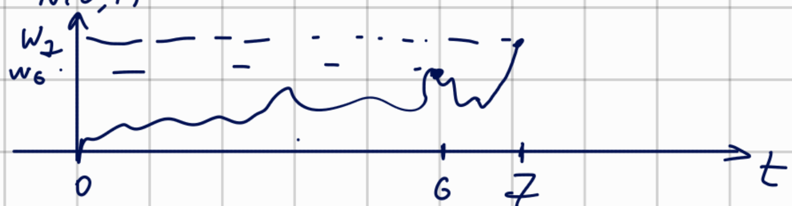
Ex $E(W_7 | W_6)$ $Var(W_{2025} | W_{2020})$

express as a number or using cdf of $N(0;1)$

→ $P(W_7 > 2W_6)$

→ $P(W_7 > 2W_6 \mid W_5 = 3)$

① $W_0 = 0$



$W_7 = (W_7 - W_0) + (W_6 - W_0)$

$E(W_7|W_6) = E(\underbrace{W_7 - W_6}_{\text{future}} + \underbrace{W_6}_{\text{known}}|W_6) = E(W_7 - W_6|W_0) + E(W_6|W_6) =$

$= E(W_7 - W_6) + W_6 = 0 + W_6$

$= W_7 - W_6 \sim N(0; 7-6)$

$\to E(W_7 - W_6)$

$Var(W_{2025}|W_{2020}) = Var(\underbrace{W_{2025} - W_{2020}} + \underbrace{W_{2020} - W_0}|W_{2020}) =$

futur value = past value + increment

$= Var(W_{2025} - W_{2020}|W_{2020}) + \underbrace{Var(W_{2020} - W_0|W_{2020})}_{=0}$

$= Var(W_{2025} - W_{2020}) = 5$

$W_{2025} - W_{2020} \sim N(0; \underbrace{2025 - 2020}_{5})$

$\leftarrow Var(W_{2025} - W_{2020})$

$Var(R|\mathcal{F})$

if $\boxed{R \text{ is measurable wrt } \mathcal{F}}$

$Var(R|R) = 0$

$W_0 = 0$

$\underbrace{W_7 - 2W_6} = \underbrace{W_7 - W_6}_{N(0;1)} - \underbrace{W_6}_{N(0;6)}$

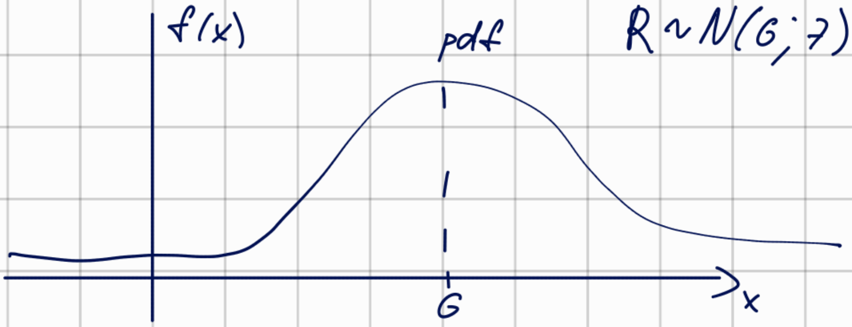
indep.

$Var(L - R) = Var(L) + Var(R) - 2Cov(L, R)$

$E(L - R) = E(L) - E(R) = 0$

$W_7 - 2W_6 \sim N(0, 7)$ unconditional distribution

$P(W_7 - 2W_6 > 0) = \frac{1}{2} \quad \Rightarrow P(W_7 > 2W_6) = \frac{1}{2}$



$R \sim N(6; 7)$

$f(x) = \frac{1}{\sqrt{2\pi \cdot 7}} \exp\left(-\frac{1}{2}\frac{(x-6)^2}{7}\right)$

$P(R > 10) = \int_{10}^{\infty} f(x)dx$

$P(R > 6) = \int_{6}^{\infty} f(x)dx = \frac{1}{2}$

$P(W_7 > 2W_6 \mid W_5 = 3)$?

$E(W_7 - 2W_6 \mid W_5 = 3) = -3$

$W_7 - 2W_6 = (W_7 - W_6) - (W_6 - W_5) -$

$- (W_5 - W_0)$ all 3 are indep.

$E(W_7 - W_6 - (W_6 - W_5) - 3 \mid W_5 = 3) = E(W_7 - W_6 - (W_6 - W_5) - 3) = \underline{-3}$

$Var(W_7 - 2W_6 \mid W_5 = 3) = Var(W_7 - W_6 - (W_6 - W_5) - 3 \mid W_5 = 3) =$

$= Var((W_7 - W_6) - (W_6 - W_5) - 3) = Var(\underbrace{(W_7 - W_6) - (W_6 - W_5)}_{\text{independant}}) =$

$= Var(W_7 - W_6) + Var(W_6 - W_5) = 1 + 1 = 2$

$(W_7 - 2W_6 \mid W_5 = 3) \sim N(\underset{\text{cond } E}{-3}; \underset{\text{cond } Var}{2})$

unconditional distribution: $W_7 - 2W_6 \sim N(0; 7)$

$R \sim N(8; 16) \longrightarrow \underset{\text{standard}}{N(0,1)}$

$\frac{R-8}{\sqrt{16}}$

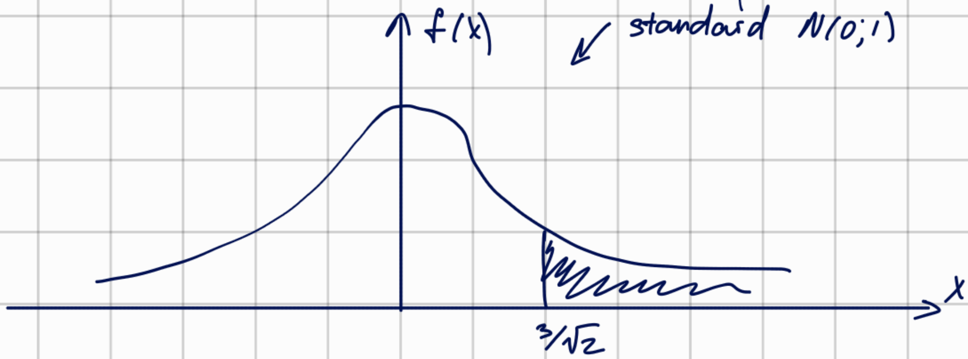
$R \longrightarrow \frac{R - E(R)}{\sqrt{Var(R)}}$

$P(W_7 > 2W_6 \mid W_5 = 3) = P(W_7 - 2W_6 > 0 \mid W_5 = 3) =$

$= P(W_7 > 2W_6 \mid W_5 = 3) =$

$= P\left(\frac{W_7 - 2W_6 - (-3)}{\sqrt{2}} > \frac{-(-3)}{\sqrt{2}} \middle| W_5 = 3\right) = P\left(Z > \frac{3}{\sqrt{2}} \middle| W_5 = 3\right)$

standard $N(0;1)$



$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$

$$* \int_{\frac{3}{\sqrt{t}}}^{\infty} f(x)\,dx$$

stop here on the exam

cdf of $N(0,1)$

$$1 - P\left(Z \le \frac{3}{\sqrt{2}} \,\middle|\, W_5 = 3\right) = \boxed{1 - F\left(\frac{3}{\sqrt{2}}\right)} = 1 - \text{norm.cdf}\left(\frac{3}{\text{sqrt}2}\right) =$$

$= 0,0169$

```
from scipy stats import norm
from numpy import sqrt
```

<u>def</u> A process (M_t) is a martingale in continuous time if $E(M_{t+h} \mid \mathcal{F}_t) = M_t \quad \forall t, \forall h > 0$

Ex. $W_t \sim$ Wiener process

$F_t = \sigma((W_s)_{s=0}^{t})$

a) Is (W_t) is a martingale? It is a martingale

b) Is $Q_t = W_t^2 - t$ a martingale? It is a martingale.

a) $E(W_{t+h} \mid \mathcal{F}_t)$

$E(W_1 \mid W_2)$

$E(W_1 \mid \mathcal{F}_2) = W_1$ - know all the values

$$E(W_{t+h} \mid \mathcal{F}_t) = E(W_{t+h} - W_t + W_t - W_0 \mid \mathcal{F}_t)$$

$$= E(W_{t+h} - W_t \mid \mathcal{F}_t) + E(\underbrace{W_t - W_0 \mid \mathcal{F}_t}_{\text{known}})$$

$$= E(W_{t+h} - W_t) + W_t - 0 = W_t$$

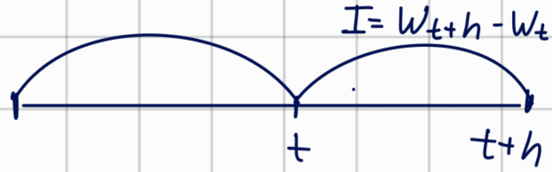
② $W_{t+h} - W_t \sim N(0, h)$

b) $E(Q_{t+h} \mid \mathcal{F}_t) = E(W_{t+h}^2 - (t+h) \mid \mathcal{F}_t)$

$= E((W_t + (W_{t+h} - W_t))^2 \mid \mathcal{F}_t) - t - h)$

Increment

$I = W_{t+h} - W_t$



$= E((W_t + I)^2 \mid \mathcal{F}_t) - t - h = E(W_t^2 + I^2 + 2I \cdot W_t \mid \mathcal{F}_t) - t - h = *$

indep

known

$* = W_t^2 + E(I^2) + E(2W_t I \mid \mathcal{F}_t) - t - h = W_t^2 + h + 2W_t \cdot 0 - t - h$

known

$= W_t^2 + E(I^2) + 2W_t(E(I \mid \mathcal{F}_t) - t - h = W_t^2 + h + 2W_t \cdot 0 - t \cdot h$

$= E((W_t + (W_{t+h} - W_t))^2 \mid \mathcal{F}_t) - t - h =$

$I = W_{t+h} - W_t \sim N(0; h)$

$E(I) = 0$

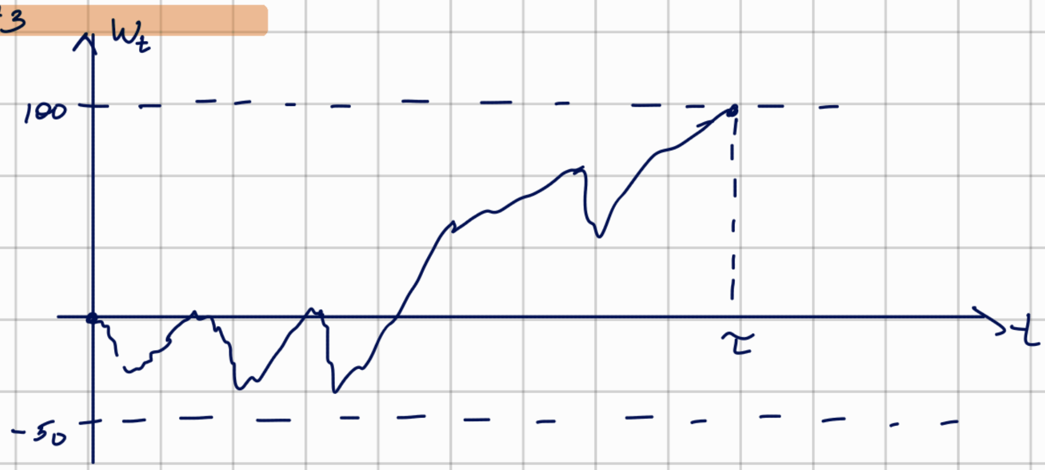
$\underbrace{Var(I)}_{\text{TRUE}} = h$

$\rightarrow E(I^2) - (\cancel{E(I)})^2 \quad \rightarrow 0$

$E(Q_{t+h} \mid \mathcal{F}_t) = \ldots\ldots \quad Q_t$

Seminar #3

Ex.



$\tau = \min_t \{ W_t = 100 \text{ OR } W_t = -50 \}$

a) $P\{W_\tau = 100\}? = \frac{1}{3}$

b) $E(\tau)$?

Hint Assume we can use Doob's theorem

If M_t is a martingale, τ - stopping time wrt filtration (F_t) and $\boxed{\text{technical condition}}$ Then $E(M_\tau)=E(M_0)$

then $E(M_\tau)=E(M_0)$

(W_t) - mart.

(Q_t) - mart.

$F_t := \sigma((W_s)_{s=0}^{t})$

$E(W_t)=E(W_0)=0$

$$100 \cdot \underbrace{P(W_\tau=100)}_{=A} + (-50)\cdot P(W_\tau=-50)=0$$

$$100\cdot A + (-50)\cdot(1-A)=0$$

$$100\cdot A + (-50) + 50A = 0$$

$$150A=50$$

$$A=\frac{1}{3}$$

$E(Q_\tau)=E(Q_0)$ $\qquad Q=W_t^2-t$

$E(W_\tau^2-\tau)=E(0^2-0)=0$

$E(W_\tau^2)-E(\tau)=0$

$E(\tau)=E(W_\tau^2)=100^2\cdot\frac{1}{3}+(-50)^2\cdot\frac{2}{3}=5000$

ABRACADABRA problem

26 letters A-Z

τ = number of keys pressed

$E(\tau)$

To use Doob's theorem open the casino

Rules:
- → enter with 1 $
- → player should bet everything on the next key
- → ORDER of bets: A, B, R, A, C, ...
- → if you are right then your bet multiply by 26
- → if you are wrong you get 0

W_{it} - wellfare of person N_i at time t

$W_{10} = 1$

$W_{11} \nearrow 0 \quad \frac{25}{26}$

$\quad\ \ \searrow 26 \quad \frac{1}{26}$

$W_{12} \nearrow 0 \quad \left(1-\left(\frac{1}{26}\right)^2\right)$

$\quad\ \ \searrow 26^2 \quad \left(\frac{1}{26}\right)^2$

What is a martingale?

$$E(W_{1t+1} \mid F_t) = \frac{1}{26} \cdot 26 W_t + \frac{25}{26} \cdot 0 = W_{1t}$$

$$X_t = W_{1t} + W_{2t} + W_{3t} + \dots + W_{tt}$$

not a martingale

$$E(X_t) = t$$

$M_t = X_t - t$ is mart

$$E(M_\tau) = E(M_0) = 0$$

$$E(X_\tau - \tau) = 0$$

$$E(X_\tau) = E(\tau)$$

Monkey	C	D	R	N	P	Q	A	D	...	ABRACAD	ABR	A
Person 1	A	.	.	.	.	.	.	. .	.			
Person 2	.	A	.	. .	.	.	.	.				
Person 3												
⋮												
Person$(\tau-10)$												26^{11}
$(\tau-3)$												26^4
τ												26^0

$$X_\tau = 26^{11} + 26^4 + 26^0$$

$$E(\tau) = E(X_\tau) = 26^{11} + 26^4 + 26^0$$

Ex (W_t) - Wiener process

$$X_t = \begin{cases} 0, & t = 0 \\ t \cdot W_{1/t}, & t \neq 0 \end{cases}$$

a) X_0 -? $= 0$

b) $X_1 = ?$ $= 1 \cdot W_{1/t} = W_1$

c) $X_t - X_s$ ·? $\sim N(0, t-s)$ N2

d) Are increments of (X_t) dependant independant N3

$X_t - X_s = t \cdot W_t - S W_{1/s}$

$E(X_t - X_s) = t \cdot E(W_{\frac{1}{t}}) - s \cdot E(W_{\frac{1}{s}})$

$W_{1/t} \sim ? N(0; \frac{1}{t})$

$= t \cdot 0 - S \cdot 0 = 0$

$Var(X_t - X_s) = Var(X_t) + Var(X_s) - 2 cov(X_t, X_s) = t + s - 2s = t - s$

$Var(X_t) = Var(t \cdot W_{1/t}) = t^2 \cdot Var(W_{1/t}) = t^2 \cdot \frac{1}{t} = t$

$Var(X_s) = \ldots\ldots = S$

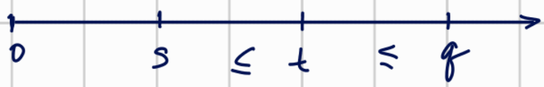
$cov(X_t, X_s) = Cov(t \cdot W_{1/t}, s \cdot W_{1/s}) = t \cdot s \cdot cov(W_{1/t}, W_{1/s})$

$= t \cdot s \cdot Cov(\underbrace{W_{1/t}, W_{1/t}}_{var} + (W_{1/s} - W_{1/t})) = ts \cdot Var(W_{1/t}) = ts \cdot \frac{1}{t} = s$

Ex $cov(W_2, W_8) = Cov(W_2, W_2 + (W_8 - W_2)) = Var(W_2) = 2$

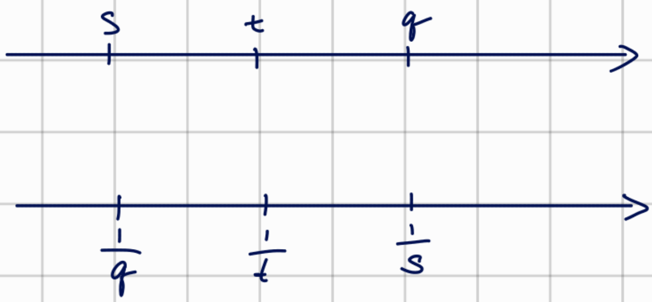
$Cov(W_7, W_{10}) = 7$

$\boxed{Cov(W_t, W_s) = \min(t, s)}$



$(X_q - X_t), (X_t - X_s)$

$$\operatorname{cov}(X_q - X_t, X_t - X_s) = \operatorname{cov}(X_q, X_t) - \operatorname{cov}(X_t, X_t) - \operatorname{cov}(X_q, X_s) + \operatorname{cov}(X_t, X_s) =$$



$= t - t - s + s = 0$

for jointly normal RV

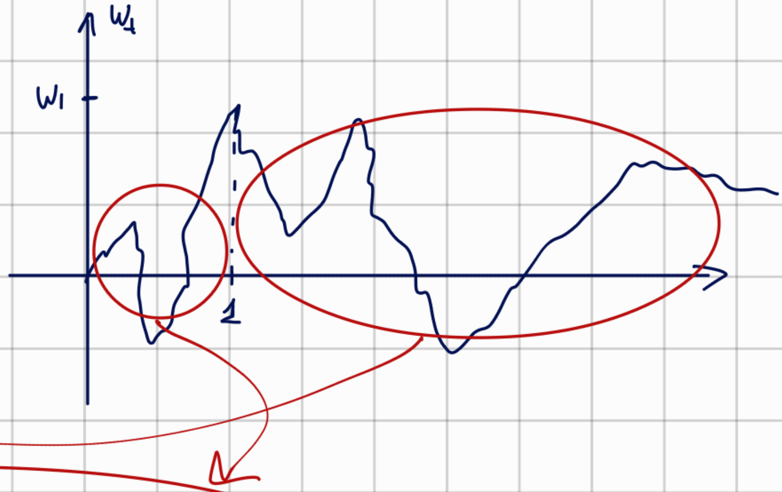
$\operatorname{cov}(R_1, R_2) = 0 \iff$ indep of R_1 and R_2

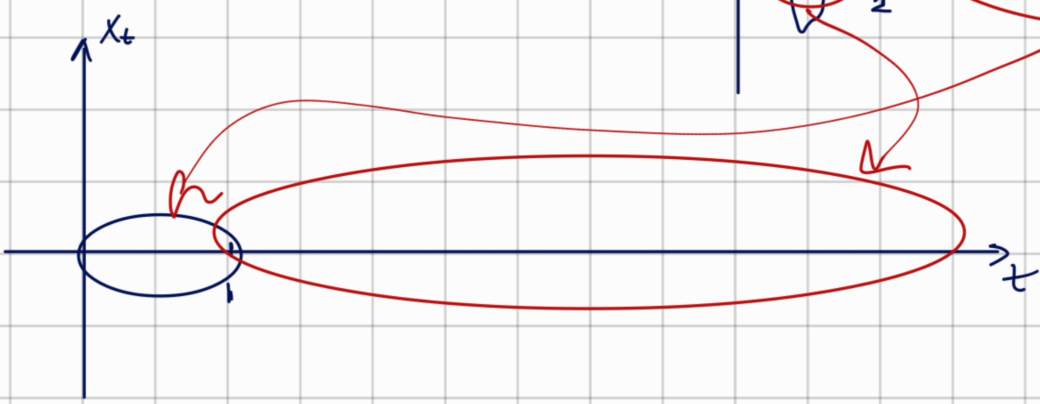
Theorem If W_t is Wiener Process (WP) then

$$X_t = \begin{cases} 0, & t = 0 \\ t W_{1/t}, & t \neq 0 \end{cases}$$ is also a Wiener Process

Ex. $E(W_5 \mid W_t) = E\left(\frac{1}{5} X_{\frac{1}{5}} \,\middle|\, \frac{1}{6} X_{\frac{1}{6}}\right)$

$\operatorname{Var}(W_5 \mid W_6) =$





$X_{\frac{1}{5}} = \frac{1}{5} W_5$

$E(W_5 \mid W_t) = E\left(5 X_{\frac{1}{5}} \,\middle|\, 6 X_{\frac{1}{6}}\right) = 5 \cdot E\left(X_{\frac{1}{5}} \,\middle|\, X_{\frac{1}{6}}\right) = 5 \cdot X_{\frac{1}{6}} = 5 \cdot \frac{1}{6} \cdot W_6 = \frac{5}{6} \cdot W_6$

Check up #3

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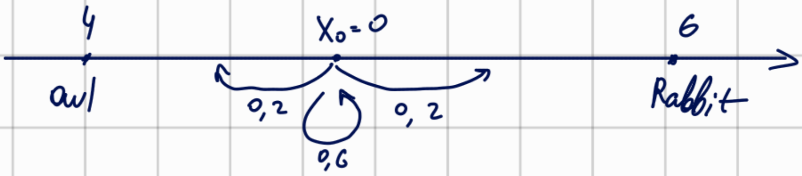
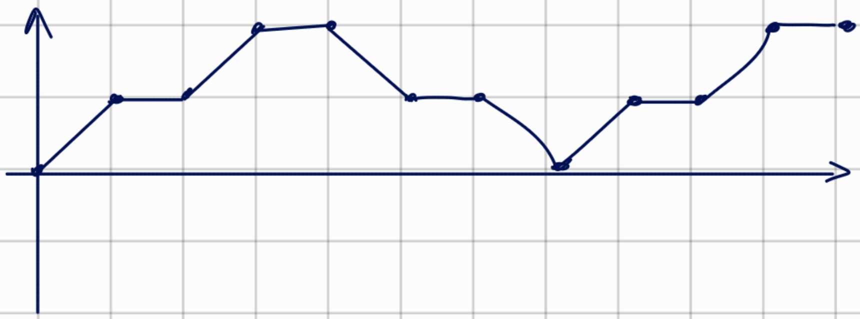
$X_0 = 0$



$\tau = \min\{t \mid X_t = 6 \quad \text{OR} \quad X_t = -4\}$

① Is X_t a martingale

$E(X_{t+1} \mid \mathcal{F}_t) = 0{,}2(X_t+1) + 0{,}2(X_t-1) + 0{,}6X_t = X_t$



б) $Y_t = X_t^2 - at$ - is mart

$E(Y_{t+1} \mid \mathcal{F}_t) = Y_t$

$E(X_{t+1}^2 - at + 1 \mid \mathcal{F}_t) = X_t^2 - at$

$\underbrace{X_{t+1}}_{\text{future}} = \underbrace{X_t}_{\text{current}} + \underbrace{(X_{t+1} - X_t)}_{\text{increment}}$

$E((X_t + I)^2 \mid \mathcal{F}_t) - at - a = X_t^2 - at$

$E(X_t^2 + I^2 + 2X_tI \mid \mathcal{F}_t) - a = X_t^2$

$X_t^2 + E(I^2 \mid \mathcal{F}_t) + E(2X_tI \mid \mathcal{F}_t) - a = X_t^2$

$I \to +1 \quad 0{,}2$
$I \to 0 \quad 0{,}6$
$I \to -1 \quad 0{,}2$

$E(I^2) = 0{,}2 \cdot 1^2 + 0{,}2(-1)^2 = 0{,}4$

$E(I) = 0$

$E(X_{t+1}^2 \mid \mathcal{F}_t) = 0{,}2 \cdot (X_t+1)^2 + 0{,}2(X_t-1)^2 + 0{,}6X_t^2$

$0{,}2 \cdot (X_t+1)^2 + 0{,}2(X_t-1)^2 + 0{,}6X_t^2 - at - a = X_t^2 - at$

c) ? $P(X_\tau = 6)$

$E(\tau)$

(X_t) - martingale

$X_t = X_t^2 - 0,4t$ - martingale

$E(X_\tau) = E(X_0) =$

$E(X_\tau) = 0$

$6 \cdot P(X_\tau = 6) + (-4) \cdot P(X_\tau = -4) = 0$

$6\boxed{} + (-4)\cdot(1 - \boxed{}) = 0$

$6\boxed{} - 4 + 4\boxed{} = 0$

$\boxed{} = 0,4$

$E(Y_\tau) = E(X_\tau^2 - 0,4\tau) = 6^2 \cdot 0,4 + (-4)^2 \cdot 0,6 - 0,4 \cdot E(\tau) = 0$

$E(\tau) = 60..$ Solve it by yourselves

$X_\tau \nearrow 6$, $X_\tau \searrow -4$

№2 (W_t) - WP

a) $E(W_5 \cdot W_4 \mid W_4) = W_4 \cdot E(W_5 \mid W_4) = W_4 \cdot W_4$

$Var(W_5 \cdot W_4 \mid W_4 = W_4^2 \cdot Var(W_5 \mid W_4)$

$= W_4^2 \cdot Var(W_4 + I \mid W_4) = W_4^2 \cdot Var(I \mid W_4)$ indep

$I = W_5 - W_4$

4 → 5 time

$= W_4 \cdot Var(I) = W_4^2 \cdot 1$

$W_5 - W_4 \sim N(0;1)$

$I = W_5 - W_4$

b) $E(W_5 \cdot W_4 \cdot W_3 \mid W_4) = E((W_4 + I) \cdot W_4 \cdot W_3 \mid W_4) =$

$= W_4 [E(W_4 \cdot W_3 \mid W_4) + E[I \cdot W_3 \mid W_4)] = *$

$I \sim N(0, 5-4)$

$E(I \cdot W_3 \mid W_4) = E(I \mid W_4) \cdot E(W_3 \mid W_4)$

$E(I) = 0$

(W_t) - Wp => time inversion $X_t = \begin{cases} 0, & t = 0 \\ t \cdot W_t, & t > 0 \end{cases}$

$s = \frac{1}{t}$ $\quad X_{\frac{1}{s}} = \frac{1}{s} \cdot W_s$

$W_s = s \cdot X_{1/s}$

$X_t = t \cdot W_{1/t}$

$X_t = t \cdot X_{1/t} \sim W_t$

$E(\cdot|R) = E(\cdot|4R)$

$W_3 = 3 X_{\frac{1}{3}}$

$W_4 = 4 \cdot X_{\frac{1}{4}}$

$* = W_4 \cdot W_4 \cdot E(W_3|W_4) = W_4^2 \cdot E(3 \cdot X_{\frac{1}{3}} \,|\, 4 \cdot X_{\frac{1}{4}}) = 3 \cdot W_4^2 \cdot E(X_{\frac{1}{3}}|X_{\frac{1}{4}}) = *$

Ex. $E(Y|R) = \begin{cases} 5 & \text{if } R=0 \\ 7 & \text{if } R=1 \end{cases}$

$E(Y|4R) = \begin{cases} 5 & \text{if } 4R=0 \\ 7 & \text{if } 4R=1 \end{cases}$

R	0	1
4R	0	4
Y	5	7

$\sigma(R) = \sigma(4R)$

$* = 3 \cdot W_4^2 (X_{\frac{1}{4}}) = 3W_4^2 \cdot \frac{1}{4} \cdot W_4$

b) $Var(W_5 \cdot W_4 \cdot W_3 | W_4) = W_4^2 \cdot \underbrace{Var(W_5 W_3 | W_4)}$

$(W_4 + I)$

$I = W_5 - W_4$

$Var(W_5 \cdot W_3 | W_4) = Var((W_4 + I) \cdot W_3 | W_4) =$

$= Var(W_4 \cdot W_3 + I \cdot W_3 | W_4) \begin{matrix} \nearrow Var/cov = \\ \searrow E \end{matrix}$

$= E(W_4^2 \cdot W_3^2 + I^2 \cdot W_3^2 + 2 W_4 W_3 I W_3 | W_4) - (\frac{3}{4} W_4^2)^3$

$E(2W_4 W_3 W_3 \cdot I | W_4) = E(I) \cdot E(2W_4 W_3 \cdot W_3 | W_4) =$

$= W_5 - W_4 = I \sim N(0;1)$

$E(W_4^2 W_3^2 | W_4) = W_4^2 \cdot E(W_3^2|W_4) = W_4^2 \cdot E(3X_{\frac{1}{3}} | 4 \cdot X_{\frac{1}{4}}) = 9W_4^2 \cdot E(X_{\frac{1}{3}}^2 / X_{\frac{1}{4}})$

$W_t^2 - t$ - martingale

$= 9W_4^2 \cdot (X_{\frac{1}{4}}^2 + \frac{1}{3} - \frac{1}{4})$

1 $E(W_5^2 - 5 | W_4) = W_4^2 - 4$

$X_{\frac{1}{4}} = \frac{1}{4} \cdot W_4$

8 $E(W_5^2 / W_4) = W_4^2 + 1$

1. $E(W_7^2 - 7 | W_2) = W_2^2 - 2$

2. $E(W_7^2 | W_2) = W_2^2 + 7 - 2$

$E(W_{0,3}^2 | W_{0,1}) = W_{0,3}^2 + 0,3 - 0,1$