Rules: 120 minutes, one A4 cheat sheet and calculator is ok, (W_t) denotes a Wiener process. You may use the standard normal cumulative distribution function $F()$ in your answers.

1. [10] Let (W_t) be a standard Wiener process.
   (a) [3] What is the distribution of $W_7 + 3W_8$?
   (b) [5] What is the conditional distribution of $(W_7 + 3W_8 \mid W_1 = 2)$?
   (c) [2] Find the probability $\mathbb{P}(W_7 + 3W_8 > 1 \mid W_1 = 2)$.
2. [10] Consider the processes $X_t = t + \int_0^t (W_u^3 + W_u)\, dW_u$.
   (a) [2 + 3] Find $\mathbb{E}(X_t)$ and $\mathbb{V}\mathrm{ar}(X_t)$.
   (b) [5] Find $\mathbb{C}\mathrm{ov}(X_t, W_t)$.
3. [10] Let $X_t = (-W_t + g(t)) \exp(W_t - t/2)$ with $X_0 = 0$.
   (a) [4 + 2] Find dX_t and write X_t as a sum of two integrals.
   (b) [4] Find at least one function $g(t)$ such that X_t is a martingale.
4. [10] Consider the Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

   Measure $\mathbb{P}^*$ denotes the risk-neutral probability that makes discounted share price process a martingale and measure $\mathbb{P}$ denotes the probability under which (W_t) is a Wiener process and $dS_t = \mu S_t dt + \sigma S_t dW_t$.
   (a) [2 + 2] Find $\mathbb{P}^*(S_T > 1)$ and $\mathbb{P}(S_T > 1)$.
   (b) [6] Find the current price X_0 of an option that pays you $X_T = \max\{0, \log S_T\}$ at fixed time T.
5. [10] It's the final exam. Five students are sitting in the last row. The student in the middle of the last row is the only one who brought a calculator. Let's denote the coordinate of this student by 0.

   Every minute his calculator moves one seat to the right (+1) or one seat to the left (-1) with equal probabilities.

   Let's X_n be the coordinate of the calculator at time n with $X_0 = 0$.
   (a) [4] Check whether $M_n = X_n^2 - n$ is a martingale.
   (b) [6] Find the average time for the calculator to reach the end of the row where $|X_n| = 2$.
6. [10] The random variables X_1, X_2, ... are independent and exponentially distributed with rate $\lambda = 1$. Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$.
   (a) [2] Provide an example of event A such that $A \in \sigma(Y_5)$ but $A \notin \sigma(X_5)$.
   (b) [4 + 4] Find $\mathbb{E}(Y_{n+1} \mid Y_n)$ and $\mathbb{V}\mathrm{ar}(Y_{n+1} \mid Y_n)$.

   Note: here $\sigma(.)$ is a minimal sigma-algebra generated by random variable.