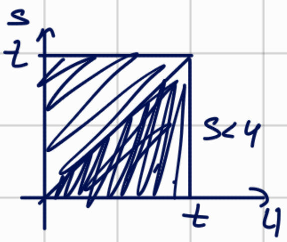
a) $E(J) = \int_0^t u\,du = \frac{t^2}{2}$

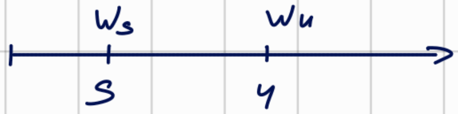
$Var(J) = E(J^2) - (E(J))^2 = E(J^2) - \frac{t^4}{4}$

$E(J^2) = E\left(\int_0^t W_u^2 du \cdot \int_0^t W_s^2 ds\right) = E\left(\int_0^t \int_0^t W_u^2 W_s^2 ds\, du\right) =$

$= \int_0^t \int_0^t E(W_u^2 W_s^2) ds\, du$



$\int_0^t \int_0^u E(W_u^2 W_s^2) ds\, du \cdot 2 = 2 \int_0^t \int_0^u 3s^2 + su - s\, ds\, du =$



$W_u = W_s + \Delta \qquad \Delta \sim N(0, u-s$

$E(W_u^2 W_s^2) = E((W_s + \Delta)^2 W_s^2) = E(W_s^4 + \Delta^2 \cdot W_s^2 + 2 W_s \Delta W_s^2) =$

$= E(W_s^4) + E(\Delta^2) \cdot E(W_s^2) + 2E(\Delta) \cdot E(W_s^3)$

$E(W_s^4) = s^2 \cdot 3 = 3s^2$

$= \int_0^t \int_0^u 4s^2 + 2su\, ds\, du \quad = \quad \int_0^t \frac{4s^3}{3} + u^3 du = \int_0^t \frac{7u^3}{4} du = \frac{7t^4}{16} =$

$= \int_0^t \frac{3u^4}{2} + u^3 - \frac{2u^3}{3} du \quad = \frac{7t^4}{16}$

№4 Page 63.

$dR_t = 4R_t dt + 7dW_t \quad , R_0 = 1$

R_t ? Hints: a) Solve the simple eq-n $dQ_t = 4Q_t dt$

$\frac{dQ_t}{dt} = 4Q_t$

$Q'_t = 4Q_t$

$Q_t = Q_0 \cdot \exp(4t)$

b) variation of const

$R_t = \exp(4t) \cdot B_t$

$dB_t = ?$

$$dR_t = \exp(4t)\cdot dB_t + 4\exp(4t)\cdot B_t\,dt + \frac{1}{2}\cdot 0\,(dB_t)^2$$

$$4R_t\,dt + 7dW_t = \exp(4t)\cdot dB_t + 4\exp(4t)\,B_t\,dt$$

$$4\cancel{(\exp(4t))}\,B_t\,dt + 7dW_t = \exp(4t)\cdot dB_t + 4\cancel{\exp(4t)}\,B_t\,dt$$

$$7dW_t = \exp(4t)\,dB_t$$

$$dB_t = \exp(-4t)\,7dW_t$$

$$\downarrow\ B_t = B_0 + \int_0^t \exp(-4u)\,7du$$

$$R_t = \exp(4t)\cdot\left(B_0 + \int_0^t \exp(-4u)\,7dW_u\right)$$

$$R_0 = \exp(0)\left(B_0 + \int_0^0 \dots dW_u\right)$$

$$1 = 1\cdot(B_0+0)$$

$$B_0 = 1$$

Answer: $R_t = \exp(4t)\cdot\left(1 + \int_0^t \exp(-4u)\,7dW_u\right.$

№5 P63

$X\,Y \sim$ iid Expo, $E(X)=E(Y)=1$

$L=\min(X,Y)$

$R=\max(X,Y)$

$$E(L|R) = \underbrace{E(L|R,Y>X)}\cdot \overset{\frac{1}{2}}{P(Y>X)} + \underbrace{E(L|R,X>Y)}\cdot \overset{\frac{1}{2}}{P(X>Y)} \quad \overset{0}{+P(X=Y)} \quad \checkmark$$

symmetry

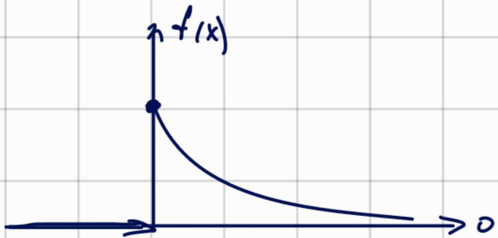
$E(R|L)=$

$$E(X|Y,\ Y|X)$$

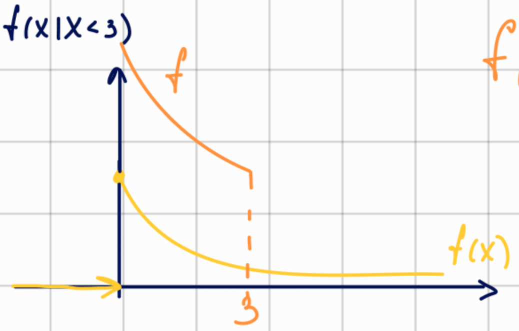
$$f(x,y) = f(x)\cdot f(y) = \begin{cases}\exp(-x)\cdot\exp(-y) & x,y>0\\ 0, & \text{otherwise}\end{cases}$$

$$f(x|y,\ x<y)$$

$$f(x) = \begin{cases}\exp(-x), & x>0\\ 0, & \text{otherwise}\end{cases}$$



$f(x \mid x<3)$



$$f(x \mid X<3) = \begin{cases} \dfrac{\exp(-x)}{P(X<3)} & x \in [0;3] \\ 0 & \text{, otherwise} \end{cases}$$

$$\begin{cases} \dfrac{\exp(-x)}{P(X<y)}, & x \in [0;y] \\ 0, & x \notin [0,y] \end{cases} = f(x \mid y, x<y)$$

$$P(X<y) = \int_0^y \exp(-x)\,dx = 1 - \exp(-y)$$

$$f(x \mid y, X<y) = \begin{cases} \dfrac{\exp(-X)}{1-\exp(-y)} & x \in [0;y] \\ 0, & \text{otherwise} \end{cases}$$

$$E(X \mid Y, X<Y) = \int_0^Y x \cdot \frac{\exp(-x)}{1-\exp(-Y)}\,dx = \frac{1}{1-\exp(-Y)} \int_0^Y x \cdot \exp(-x)\,dx$$

$$= \frac{1}{1-\exp(-Y)} \cdot \Big(\underbrace{x}_{u} \underbrace{(-\exp(-x))}_{v} \Big|_0^Y - \int_0^Y \underbrace{1}_{u'} \underbrace{(-\exp(-x))}\,dx \Big]$$

$$= \frac{1}{1-\exp(-Y)} \cdot \Big(-Y \cdot \exp(-Y) - \exp(-x) \Big|_{x=0}^{x=Y} \Big)$$

$$= \frac{1}{1-\exp(-Y)} \cdot \big(-Y\exp(-Y) + 1 - \exp(-Y)\big)$$

$$E(L \mid R) = \frac{-R\exp(-R) + 1 - \exp(-R)}{1-\exp(-R)}$$

ex 2. P63

Simplify

$Y_t = \int_0^t \exp(-W_u - \frac{u}{2})\, dW_u \qquad \exp(-W$

$X_0 = \exp(0) = 1$

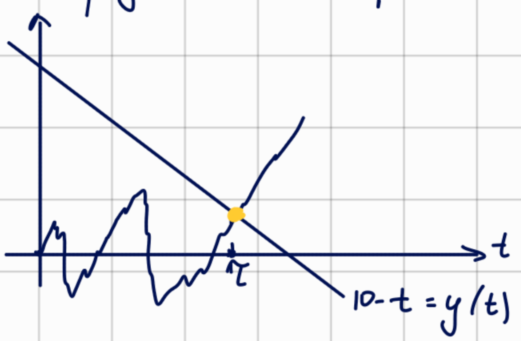
$X_t = \exp(-W_t - \frac{t}{2}) = \exp(-W_t) \cdot \exp(-\frac{t}{2}) = h(t, W_t)$

$dX_t = h'_w \, dW_t + h'_t \, dt + \frac{1}{2} h''_{ww} (dW_t)^2 = -\exp(-W_t) \cdot \exp(-\frac{t}{2}) dW_t -$

$- \frac{1}{2} \exp(-W_t) \cdot \exp(-\frac{t}{2}) dt = -\exp(-W_t) \cdot \exp(-\frac{t}{2}) dW_t$

$Y_t = 1 - \exp(-W_t - \frac{t}{2})$

ex 5. page 62 20 pts.



$E(\tau) = ?$

$Var(\tau) = ?$

Hints: The process $M_t = \exp(\delta W_t - \frac{\delta^2}{2} t)$ a martingale ?

$dM_t = \delta \exp(\delta W_t - \frac{\delta^2}{2} t) dW_t - \cancel{\frac{\delta^2}{2} \exp(\,) dt} + \cancel{\frac{1}{2} \delta^2 \exp(\,) dt} =$

$(dW_t)^2$

$= \delta \exp(\delta W_t - \frac{\delta^2}{2} t) dW_t \Rightarrow$ yes, it is a martingale

Doob's theorem

if M_t is a martingale, τ - stopping time [....] then $E(M_\tau) = M_0$

$m(u) = E(\exp(-u\tau))$

(b) Find $m(u) = E(\exp(-u\tau))$? by apply Doob's theorem

$W_\tau = 10 - \tau$ by def.

$E(M_\tau) = 1$

$E(\exp(\delta \cdot W_\tau - \frac{\delta^2}{2} \tau)) = 1$

$E(\exp(10\delta - \delta \cdot \tau - \frac{\delta^2}{2} \tau)) = 1$

$E\left(\exp(10\delta)\cdot \exp\left(-\tau\left(\delta+\frac{\delta^2}{2}\right)\right)\right)=1$

$E\left(\exp\left(-\tau\left(\underbrace{\delta+\frac{\delta^2}{2}}_{u}\right)\right)\right)=\exp(-10\delta)$

$\delta+\frac{\delta^2}{2}=u$

$\delta-?$ $\quad \delta>0$

$1+2\delta+\delta^2=2u+1$

$\Rightarrow (\delta+1)^2=2u+1$

$\delta=-1\pm\sqrt{2u+1}$

Solution 1: $E(\exp(-\tau u))=\exp(10-10\cdot\sqrt{2u+1})$

Solution 2: $E(\exp(-\tau u))=\exp(10+10\cdot\sqrt{2u+1})$

$m(u)=E(\exp(-\tau u))=\exp(10-10\sqrt{2u+1})$ - answer

$E(\tau)-?$

$(1+x)^a=1+C_a^1x+C_a^2x^2+\ldots$

$\exp(x)=1+x+\frac{x^2}{2!}+\ldots$

c) $Var(\tau)$?

$m(u)=E(\exp(-\tau u))=\exp(10-10\sqrt{2u+1})=m(0)+m'(0)\cdot u+\frac{m''(0)}{2!}u^2+\ldots$

$m'(u)=E(-\tau\cdot\exp(-\tau u))$

$m'(0)=E(-\tau)=\exp(10-10\sqrt{2u+1})'_u$

$E(\tau)=-m'(0)=$

$m''(u)=E((-\tau)(-\tau)\cdot\exp(-\tau u))$

$m''(0)=E(\tau^2)$

$Var(\tau)=E(\tau^2)-E((\tau))^2=m''(0)-(m'(0))^2$

$\sqrt{1+2u}=(1+2u)^{\frac12}=1+C_{\frac12}^1 2u+C_{\frac12}^2(2u)^2+\ldots=1+\frac12\cdot 2u+\frac{\frac12(\frac12-1)}{2}\cdot 4u^2=$

$=1+u-\frac12u^2+o(u^2)$

$m(u)=\exp(10-10-10u+5u^2+o(u^2))=1-10u+55\cdot u^2+\ldots$ $\quad m'(0)$, $\frac{m''(0)}{2!}$

$\exp(-10u+5u^2+\ldots)+\frac{(-10u+5u^2+\ldots)^2}{2}+\ldots=1-10u+(5+50)u^2+\ldots=$

$E(\tau)=-m'(0)=10$

$Var(\tau)=m''(0)-(m'(0))^2=110-10^2=10$

ex. 1. Page 62.

$M_t=\exp\left(W_t^2/(1+2t)\right)g(t)$

$dM_t=\frac{2W_t}{1+2t}\left(\exp\left(\frac{W_t^2}{1+2t}\right)\right)\cdot g(t)\,dW_t+\exp\left(\frac{W_t^2}{1+2t}\right)'_t g(t)+\exp\left(\frac{W_t^2}{1+2t}\right)\cdot g'(t)dt+$

$+\frac{1\cdot g(t)}{2}\left(\left(\frac{2W_t}{1+2t}\right)'_{W_t}\exp\left(\frac{W_t^2}{1+2t}\right)+\frac{2W_t}{1+2t}\cdot\exp\left(\frac{W_t^2}{1+2t}\right)'_{W_t}\right)\underbrace{(dW_t)^2}_{=dt}$

M_t is a martingale: $dt\left(\exp\left(\frac{W_t^2}{1+2t}\right)\cdot g'(t) + \exp\left(\frac{W_t^2}{1+2t}\right) g'(t) + \frac{g(t)}{2}\left(\left(\frac{2W_t}{1+2t}\right)'_{W_t} \cdot \exp\frac{W_t^2}{1+2t}\ldots\right.\right.$
$+ \frac{2W_t}{1+2t}\cdot \exp\left(\frac{W_t z}{1+2t}\right)'_{W_t}\Big) = 0$

$$h = \exp(dt, W_t)\cdot g(t)$$

$$h'_t = \exp(a)\cdot g'_t + \exp(a)\cdot a'_t\, g$$

$$h'_w = \exp(a)\cdot a'_w \cdot g$$

$$h''_{ww} = \exp(a)\cdot a''_{ww}\cdot g + \exp(a)\, a'_w \cdot a'_w \cdot g.$$

$$0 = g'_t + a'_t \cdot g + \frac{1}{2} a''_{ww}\cdot g + \frac{1}{2} a'_w \cdot a'_w\, g$$

$$a = \frac{W_t^2}{1+2t}$$

$$a'_w = \frac{2W_t}{1+2t}$$

$$a''_{ww} = \frac{2}{1+2t}$$

$$a'_t = \frac{W_t^2}{(1+2t)^2}\cdot(-2)$$

$$\Rightarrow g' + \cancel{\frac{W_t^2(-2)}{(1+2t)^2}\, g} + \frac{1}{2}\cdot\frac{1}{1+2t}\, g + \cancel{\frac{4W_t^2}{(1+2t)^2}\frac{1}{2}\, g}$$

$$\frac{2dg}{dt} = -\frac{g}{1+2t}$$

$$\frac{2dg}{g} = -\frac{dt}{1+2t}$$

$$2\ln g = -\frac{1}{2}\ln(1+2t) + C$$

$$g = (1+2t)^{-\frac{1}{4}}\cdot C$$

№4 p 57

Under P^*: $\frac{X_t}{\exp(rt)}$ is a martingale

W_t^* is a Wiener Pr.

BS model: $X_T = (\ln S_T)\cdot(\ln S_{T/2}) =$

$X_0\ ? = E_{P^*}\left(\frac{X_T}{\exp(rt)}\right) =$

$\ln S_T = \ln\left(S_0\cdot\exp\left(rT + \delta\cdot W_T^* - \frac{\delta^2}{2}T\right)\right)$

$S_1 = S_0\cdot\exp\left(\mu T - \frac{\delta^2}{2}T + \delta W_T^*\right)$

S_t - share price

r - risk-free rate

δ - volatility

$= \ln S_0 + \mu T + \sigma W_T^* - \frac{\sigma^2}{2}T)\cdot(\ln(X_n S_0 + \frac{2T}{2} + \sigma W_{\frac{T}{2}}^* - \frac{\sigma^2}{4}T)$

$= E\left(\frac{(\ln S_0 + 2T + \sigma W_T^* - \frac{\sigma^2}{2}T)(\ln S_0 + \frac{2T}{2} + \sigma W_{\frac{T}{2}}^* - \frac{\sigma^2}{4}T)}{\exp(2T)}\right) =$

$$= \frac{(\ln S_0)^2 + \frac{1}{2}(2T)^2 + \frac{\sigma^4}{8}T^2 + \ln S_0\left(\frac{2T}{2} - \frac{\sigma^2 T}{4}\right) + 2T\left(\ln S_0 + \frac{2T}{2} - \frac{\sigma^2}{4}T\right)}{\exp(2T)}$$

$$= \exp(-2T)\cdot\Big((\ln S_0 + 2T - \frac{\sigma^2}{2}T)(\ln S_0 + \frac{2T}{2} - \frac{\sigma^2}{4}T) + \cancel{E(\sigma W_T^*\cdot const)} \to 0 +$$
$$+ \cancel{E_*(\sigma W_{\frac{T}{2}}^*\cdot const)} \to 0 + E_*(\sigma W_T^*\cdot\sigma\cdot W_{\frac{T}{2}}^*)\Big)$$

$\sigma^2\frac{T}{2}$

$E(W_7\cdot W_5) = Cov(W_7, W_5) = Cov(W_5+\Delta, W_5) = 5 + 0 = 5$

ex1. page 47

(W_t) - Wiener. pr.
(V_t) - Wiener. pr
} indep.

1. $W_0 = 0$
2. $W_t - W_s \sim N(0; t-s)$
3. indep-t increments
4. trajectories are cont.

$Z_t = \frac{1}{2}W_t + \frac{1}{2}V_t$

$Q_t = \frac{1}{\sqrt{2}}W_t + \frac{1}{\sqrt{2}}V_t$

check whether (Z_t) and (Q_t) are Wiener processes

	1	2	3	4
Z_t	☆	X	☆	☆
Q_t	☆	☆	☆	☆

$(Z_b - Z_a)$, $(Z_c - Z_b)$

$\frac{1}{2}(W_b - W_a) + \frac{1}{2}(V_b - V_a)$, $\frac{1}{2}(W_c - W_b) + \frac{1}{2}(V_c - V_b)$

indep

indep

by assumption they are indep.

indep

$E(Z_t) = 0$, $E(Q_t) = 0$

$Var(Z_t) = \left(\frac{1}{2}\right)^2 \cdot t + \left(\frac{1}{2}\right)^2 \cdot t = \frac{1}{2} t$

$Var(Q_t) = \left(\frac{1}{\sqrt{2}}\right)^2 t + \left(\frac{1}{\sqrt{2}}\right)^2 t = t$

$Q_t - Q_s = \frac{1}{\sqrt{2}} \cdot (W_t - W_s) + \frac{1}{\sqrt{2}} (V_t - V_s) \sim N(0; \frac{1}{2}(t-s) + ^2 \frac{1}{2} \cdot (t-s))$

Z_t is not a Wiener process

Q_t is a Wiener process

ex. 3. p 47

X_t → $+1$ 0,5 ; $+0$ 0,1 ; -1 0,1

$\tau = \min_t \{ t \mid X_t = 0 \text{ OR } Y_t = 300 \}$

$X_t = \underset{10}{X_0} + Y_1 + Y_2 + \ldots + Y_t$

a) $M_t = a^{X_t}$ is a martingale?

b) $P(X_\tau = 300)$?

$E(M_{t+1} | \mathcal{F}_t) = M_t$

$M_{t+1} = a^{X_t + Y_{t+1}} = a^{X_t} \cdot a^{Y_{t+1}} = M_t \cdot a^{Y_{t+1}}$

$E(\underbrace{M_t \cdot a^{Y_{t+1}}}_{\text{known}} | \mathcal{F}_t) = M_t$

$M_t \cdot E(a^{Y_{t+1}} | \mathcal{F}_t) = M_t$

$E(a^{Y_{t+1}} \overset{\text{indep}}{|} \mathcal{F}_t) = 1$

$E(a^{Y_{t+1}}) = 1$

$0{,}5 \cdot a^1 + 0{,}1 \cdot a^0 + 0{,}4 \cdot a^{-1} = 1 \quad \Big| \cdot 10a$

$5a^2 + 1a + 4 = 10a$

$5a^2 - 9a + 4 = 0$

$a_1 = 1 \qquad a_1 \cdot a_2 = \frac{4}{5} = 0{,}8$

$a_2 = 0{,}8$

$1^{X_t} = 1$ ← mart ⟶ always 1

$0{,}8^{X_t}$ ← mart ⟶ can go up and down depends on X

b) $p = P(X_\tau = 300)$?

Doob's theorem

$E(M_\tau) = M_0 = 0{,}8^{10}$

$X_0 = 10$

$$p \cdot 0{,}8^{300} + (1-p)\cdot 0{,}8^{0} = 0{,}8^{10}$$

$$p = P(X_\tau = 300) \overset{?}{=} \frac{0{,}8^{10} - 0{,}8^{0}}{0{,}8^{300} - 0{,}8^{0}}$$ answer.

№2 Page 47

$$Y_t = \int_0^t (W_u + u)^2 \, dW_u$$

$E(Y_t) = 0$

$$Var(Y_t) = \int_0^t E(W_u + u)^4 \, du$$

cubed symmetric around 0

$$E((W_u + u)^4) = E(W_u^4 + \underbrace{4W_u^3 \cdot u}_{0} + 6W_u^2 \cdot u^2 + \underbrace{4 \cdot W_u \cdot u^3}_{0} + u^4) = 3u^2 + 6u^3 + u^4$$

$E(W_u^2) = u$

$E(W_u^4) = 3u^2$

$$\Rightarrow \int_0^t (3u^2 + 6u^3 + u^4)\,du = u^3 + \frac{6u^4}{4} + \frac{u^5}{5}\Big|_0^t =$$

$$= t^3 + \frac{3t^4}{2} + \frac{t^5}{5}$$