Stochastic Calculus

1. lecture
2. Standart class
3. Catch-up class

new topics

no new topics

1/7

LMS $\longrightarrow$ wiki.cs.hse.ru (past exams collection)

→ Zastavniak, „Basic Stochastic processes"
→ Steele „Stochastic Calculus with Financial applic"
→ Shreve : 2 volumes
„Stochastic Calculus for Finance"

Sigma-algebra

goal: describe information as a math object

2x

	Y=-1	Y=0	Y=1
X=0	0.1 a	0.2 b	0.2 c
X=1	0.3 d	0.1 e	0.1 f

Alice knows X
Bob knows Y
Claude knows XY

Intuitive def
Sigma-algebra – $\boxed{\text{list of events}}$ which states are known to a rational person

$\{X=0\}$ $\{Y=1\}$ $\{Y\neq 0\}$ $\{X\neq 0\}$ ... $\{X=Y\}$ ...

$$\mathcal{F}_A = \{ \{X=0\}, \{X\neq 0\}, \varnothing, \Omega \}$$

$\{X\neq 1\}$

$\{X>7\} \neq \varnothing$ $\{X<1\}$ $\{\sin X=0\}$

$\{X<100\} = \Omega$

list of question the participant can give answers

Notation: F - event
$\mathcal{F}$ - list of events

$$\mathcal{F}_B = \{ \{Y=1\}, \{Y\neq 0\}, \varnothing, \{Y=-1\}, \{Y=0\}, \{Y\neq 1\}, \{Y\neq -1\}, \Omega \} = \{\{c,f\}, \{a,b,e,f\}, \dots \}$$

outcomes : a, b, c, d, e, f $\quad \emptyset = \{\ \} \quad \{\{\ \}\} \neq \emptyset$

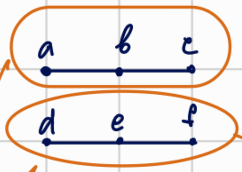
event (set of outcomes): $\Omega = \{a, b, c, d, e, f\}$ $\quad \{X=1\} = \{d, e, f\}$

$\{Y=1\} = \{c, f\}$

σ-algebra : set of events

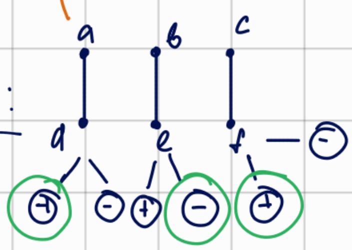
$\mathcal{F}_c = \{ \underbrace{\{X=0\}}_{\{a,b,c\}}, \{Y=0\}, \underbrace{\{X=0, Y=0\}}_{\{b\}}, \dots\dots \}$

Alice



$\{ \{X=0\}, \{X\neq 0\}, \Omega, \emptyset \} \quad 2^2 = 4$

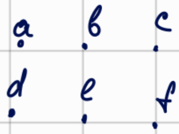
Bob:



2^3

$\{a, d, c, f\} = \{Y \neq 0\}$

Claude



in $\mathcal{F}_c$ we have $2^6 = 64$

Formal definition : $\mathcal{F}$ called sigma-algebra if :

① if $A_1, A_2, A_3, \dots \in \mathcal{F}$ then any event that can be constructed in countably many steps using $(\cup, \cap, \setminus)$ $B \in \mathcal{F}$

② $\emptyset \in \mathcal{F}$, $\Omega \in \mathcal{F}$

St. Formal def:

$\Omega \setminus A = A^c = \bar{A}$

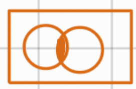
① $\Omega \in \mathcal{F}$

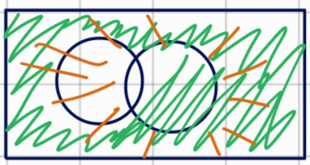
② If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$

③ If $A_1, A_2, A_3, \dots \in \mathcal{F}$ then $A_1 \cup A_2 \cup A_3 \cup \dots \in \mathcal{F}$

$((A_1 \cup A_2) \cap A_3) \setminus A_4 \in \mathcal{F}$

ex. $A \cap B = (A^c \cup B^c)^c$





ex. $\mathcal{F}_1$ - σ-algebra

$\mathcal{F}_2$ - σ-algebra

a) Is $\mathcal{A} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a sigma-algebra [on the same Ω] (in general / no)

b) Is $\mathcal{H} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-algebra (yes)

$\{A, B, C\} \cup \{B\, C D\} = \{A, B, C, D\}$

$\mathcal{F}_A = \{\{X=0\}, \{X \neq 0\}, \varnothing, \Omega\}$

$\mathcal{F}_B = \{\{Y=0\}, \{Y=1\}, \{Y=-1\}, \{Y \neq 0\}, \{Y \neq 1\}, \{Y \neq -1\}, 0, \Omega\}$

$H = \mathcal{F}_A \cap \mathcal{F}_B = \{\varnothing, \Omega\}$

① $\Omega \in \mathcal{H}$ +

② If $A \in \mathcal{H}$ then $A^c \in H$ +

③ If $A_1, A_2, A_3 \dots$

$A_1 \cup A_2 \cup A_3 \dots \in \mathcal{F}$ +

$A = F_A \cup F_B = \{\{X=0\}, \{X \neq 0\}, \{Y=0\}, \{Y \neq 0\}\ \{Y=1\}, \dots\}$

assumption №3 $\Rightarrow \{X=0 \text{ or } Y=0\} \notin A$

A is not a sigma-algebra

① $\Omega \in H$?

$\Omega \in \mathcal{F}_1$

$\Omega \in \mathcal{F}_2$ $\Rightarrow \Omega \in H$

② If $A \in \mathcal{H}$ then $A^c \in \mathcal{H}$

$A \in \mathcal{H} \Rightarrow A \in F_1$

$A \in F_2$ $\Rightarrow A^c \in \mathcal{F}_1$, $A^c \in \mathcal{F}_2$ $\Rightarrow A^c \in \mathcal{H} = \mathcal{F}_1 \cap \mathcal{F}_2$

③ If $A_1, A_2, \dots \in \mathcal{H}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{H}$

$A_1, A_2, \dots \in \mathcal{H} \Rightarrow A_1, A_2 \dots \in \mathcal{F}_1 \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_1$

$A_1, A_2, \dots \in \mathcal{F}_2 \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_2$ $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{H}$

Standart class #1 **Conditional expected value**

$\to E(Y|A)$ — Y: RV, A: event

$E(Y|F)$ — Y: RV, F: σ-algebra

$\to E(Y|X)$ — Y: RV, X: RV

informal def

$E(Y|\mathcal{F})$ - is the best forecast of Y given information in F

Idea 1 knowledge in $\mathcal{F}$ is sufficient to make a forecast

A RV (random variable) L can be calculated using σ-algebra $\mathcal{F}$ if I can compare L with any real number $x \in \mathbb{R}$

$$\{L \le x\} \in \mathcal{F}$$

$\boxed{\text{def}}$ RV L is measurable with respect to (WRT) σ-algebra $\mathcal{F}$ if $\forall x \in \mathbb{R}$ $\{L \le x\} \in \mathcal{F}$

	$Y=-1$	$Y=0$	$Y=1$
$X=0$			
$X=1$			

Notation: $\mathcal{F}_A = \sigma(X)$ - the smallest σ-algebra that contains all events $\{X \le x\}$

$\{X \le -2\} = \varnothing$ $\{X \le \frac{1}{2}\} = \{X=0\}$

$\{X \le 0\} = \{X=0\}$ $\{X \le 5\} = \Omega$

$\{\{X=0\}, \varnothing, \Omega\}$ - σ-algebra

$\mathcal{F}_B = \sigma(Y)$

$\mathcal{F}_C = \sigma(XY)$ - the smallest σ-algebra that contains all events from $\sigma(X)$ and all events from $\sigma(Y)$

$E(Y|\mathcal{F})$ - is the best forecast of Y given information in $\mathcal{F}$

$E(Y|\mathcal{F}) = \hat{y}$

smallest σ-algebra that contains $\{\hat{Y} \le c\}$ for $c \in \mathbb{R}$

Assumption 1: $\sigma(\hat{y}) \subseteq \mathcal{F}$

„I am able to calculate forecast"

Assumption 2.1.: $E(\hat{Y}) = E(Y)$ „best forecast"

Assumption 2.2.: „I can't predict the errors of the forecast"

$cov(y - \hat{y}, I_A) = 0$ for $\forall A \in \mathcal{F}$

def $\hat{y} = E(Y|\mathcal{F})$

① $\sigma(\hat{y}) \subseteq \mathcal{F}$

$\sigma(\hat{y}) \subseteq \mathcal{F} \Rightarrow \hat{y}_b = \hat{y}_c$

best $\begin{cases} 2.1 & E(\hat{Y}) = E(Y) \\ 2.2. & cov(\hat{Y} - y, I_A) = 0 \quad \forall A \in \mathcal{F} \end{cases}$

$I_{\{b,c\}}$	0	1	1
$I_{\{a\}}$	1	0	0
Ex. y	1	2	3
w	a	b	c
$P(\{w\})$	0,5	0,2	0,3
$\hat{y}$	$\hat{y}_a$	$\hat{y}_b$	$\hat{y}_c$

$\mathcal{F} = \{\{a\}, \{b,c\}, \emptyset, \Omega\}$

a   b — c

$\hat{y}(a)$ $\hat{y}(b) = \hat{y}(c)$

$\hat{y} = \begin{cases} ? & \text{if } a \\ ? & \text{if } b \text{ or } c \end{cases}$

$\hat{Y} = E(Y|\mathcal{F})$?

2.1. $0,5 \cdot \hat{y}_a + 0,2 \cdot \hat{y}_b + 0,3 \cdot \hat{y}_c = 1 \cdot 0,5 + 2 \cdot 0,2 + 3 \cdot 0,3$

$cov(Y, I_A) - cov(\hat{Y}, I_A) = 0$

$cov(Y, I_A) = cov(\hat{Y}, I_A)$

by 2.1

$E(Y \cdot I_A) - E(Y) \cdot E(I_A) =$

$= E(\hat{Y} \cdot I_A) - E(\hat{Y}) \cdot E(I_A)$

in books $E(Y \cdot I_A) = E(\hat{Y} \cdot I_A)$

$A = \{a\}$ $\quad E(Y \cdot I_A) = 0,5 \cdot 1 + 0,2 \cdot 0 + 0,3 \cdot 0$

$E(\hat{Y} \cdot I_A) = 0,5 \cdot \hat{y}_a + 0,2 \cdot 0 + 0,3 \cdot 0$

$0,5 \cdot 1 = 0,5 \cdot \hat{y}_a$

$\{b,c\}$ $\quad E(Y \cdot I_{\{b,c\}}) = 0,5 \cdot 0 + 0,2 \cdot 2 + 0,3 \cdot 3$

$E(\hat{Y} \cdot I_{\{b,c\}}) = 0,5 \cdot 0 + 0,2 \cdot \hat{y}_b + 0,3 \cdot \hat{y}_c$

$\varnothing$: $E(y \cdot I_0) = 0$

$E(\hat{y} \cdot I_0) = 0$

Ω: $E(y \cdot I_\Omega) = E(Y)$ 2.1

$E(\hat{y} \cdot I_\Omega) = E(\hat{y})$

$$\begin{cases} 0{,}5 \cdot 1 = 0{,}5 \cdot y_a \\ 0{,}2 \cdot 2 + 0{,}3 \cdot 3 = 0{,}2 \cdot \hat{y}_b + 0{,}3 \cdot \hat{y}_c \\ \hat{y}_b = y_c \\ 0{,}5\hat{y}_a + 0{,}2\hat{y}_b + 0{,}3\hat{y}_c = 1 \cdot 0{,}5 + 2 \cdot 0{,}2 + 3 \cdot 0{,}3 \end{cases}$$

$$\hat{y}_a = 1 \qquad \hat{y}_b = \hat{y}_c = \frac{0{,}2}{0{,}5} \cdot 2 + \frac{0{,}3}{0{,}5} \cdot 3 = 0{,}4 \cdot 2 + 0{,}6 \cdot 3 = 2{,}6$$

$$\hat{y} = \begin{cases} 1, & \text{if } a \\ 2{,}6, & \text{if } b \text{ or } c \end{cases}$$

$$\hat{y} = 1 \cdot I_{\{a\}} + 2{,}6 \cdot I_{\{b,c\}}$$

Theorem: If $\Omega = A_1 \cup A_2 \cup A_3 \cup \dots$ $\quad A_i \cap A_j \neq 0$

$\forall i \neq j$

$\mathcal{F}$ - minimal σ-algebra that contains $A_1, A_2, A_3, \dots$

$\mathcal{F} = \sigma(A_1, A_2, A_3 \dots)$

Then

$$E(Y|F) = \begin{cases} E(Y|A_1) & \text{if } A_1 \text{ happens} \\ E(Y|A_2) & \text{if } A_2 \text{ happens} \\ \vdots \end{cases}$$

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$$X_i = \begin{cases} 1 & \text{card } N_i \text{ is a Queen} \\ 0, & \text{otherwise} \end{cases}$$

52 cards notation:

$$\to E(X_{52}|X_1) = \overbrace{E(X_{52}|\sigma(X_1))}^{RV} = \begin{cases} \underbrace{E(X_{52}|X_1 = 1)}_{\in \mathbb{R} \text{ number}} & \text{if } X_1 = 1 \\ \underbrace{E(X_{52}|X_1 = 0)}_{\in \mathbb{R} \text{ number}} & \text{if } X_1 = 0 \end{cases}$$

$\to E(X_{52}|X_1, X_2)$

$\to E(X_{52}|X_1, X_2, X_3)$

$\to E(X_{52}|X_1, X_2, X_3, \dots, X_4)$

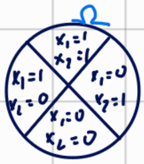
$$E(X_{52} \mid X_1 = 1) = \frac{3}{51} \cdot 1 + \frac{48}{51} \cdot 0 = \frac{3}{51}$$

$$E(X_{52} \mid X_1 = 0) = \frac{4}{51} \cdot 1 + \frac{47}{51} \cdot 0 = \frac{4}{51}$$

$$E(X_{52} \mid X_1] = \begin{cases} \frac{3}{51} & \text{if } X_1 = 1 \\ \frac{4}{51} & \text{if } X_1 = 0 \end{cases}$$

$$E(X_{52} \mid X_1, X_2) = E(X_{52} \mid \mathcal{F}) = \begin{cases} \frac{2}{50} & \text{if } X_1 = 1 \text{ and } X_2 = 1 \\ \frac{3}{50} & \text{if } X_1 = 1 \text{ and } X_2 = 0 \\ \frac{3}{50} & \text{if } X_1 = 0 \text{ and } X_2 = 1 \\ \frac{4}{50} & \text{if } X_1 = 0 \text{ and } X_2 = 0 \end{cases}$$

$\mathcal{F} = \sigma(X_1, X_2)$



$2^4 = 16$ events

$$E(X_{52} \mid X_1 = 1, X_2 = 1) = \frac{2}{50} \cdot 1 + 0 \cdot \frac{48}{50}$$

Theorem 2

If (X, Y) has joint density $f(x, y)$

then $E(Y \mid X) = \int_{-\infty}^{\infty} y \cdot f(y \mid X)\, dy$

$E(Y)$ - constant

$E(Y \mid A)$ - constant

event

$E(Y \mid \mathcal{F})$ - RV

$Var(Y)$ - constant

$Var(Y) = E(Y^2) - (E(Y))^2$

$\boxed{\text{def}}$ $Var(Y \mid \mathcal{F}) = E(Y^2 \mid \mathcal{F}) - (E(Y \mid \mathcal{F}))^2$

$P(A) = E(I_A)$

def $P(A \mid \mathcal{F}) = E(I_A \mid \mathcal{F})$ ← RV

Properties:

1. $E(X + Y \mid \mathcal{F}) = E(X \mid \mathcal{F}) + E(Y \mid \mathcal{F})$
2. If X is measurable WRT $\mathcal{F}$ (we can calculate X provided info in $\mathcal{F}$)

   $E(X \cdot Y \mid \mathcal{F}) = X \cdot E(Y \mid \mathcal{F})$

   „Take out what is known"

③ $E(E(Y|\mathcal{F})) = E(Y)$ (by def)

"on average we are eight"

④ If $\mathcal{F} \leq \mathcal{H}$ then $E(E(Y|\mathcal{H})|\mathcal{F}) = E(Y|\mathcal{F})$

⑤ If Y and $\mathcal{F}$ are independant $E(Y|\mathcal{F}) = E(Y)$

RV: $\Omega \to \mathbb{R}$ collection of events

$$P(A \cap B) = P(A) \cdot P(B) \text{ for all } A \in \sigma(Y) \text{ and } B \in \mathcal{F}$$

def y and $\mathcal{F}$ are indep.

Catch up classes #1

	$Y=-1$	$Y=0$	$Y=1$
$X=0$	a	b	c
$X=1$	d	e	f

$\{a,b,c\}$ $\{a,b,c,d,e,f\} = \Omega$

$$\sigma(X) = \{ \{X \leq 0\}, \{X \leq 52\}, \emptyset, \{X > 0\} = \{def\}, \{X \leq 1\}, \{X \leq -7\}$$

$\{X \leq t\}$ $\forall t \in \mathbb{R}$ $W = g$ $W = 10X + Y$

$\sigma(W) = 2^{\Omega}$

$t = 0$ $t = -1$

$t = 2$ $t = 6.7$ $t = \sqrt{5}$

$t = 123456$

$\sigma(X)$: ① $B(X)$ contains $\{X \leq t\}$ all events

② minimal σ-algebra

def $\hat{y} = E(Y|\mathcal{F})$

① $\sigma(\hat{y}) \subseteq \mathcal{F}$

② $E(\hat{y}) = E(Y)$

$\text{cov}(\hat{y}, I_A) = \text{cov}(Y, I_A)$ for $\forall A \in \mathcal{F}$

$\mathcal{F} = \{\{a\}, \{bcdef\}, \emptyset, \Omega\}$, $\sigma(X) \neq \mathcal{F}$

Retake 15.02.2016 Page 46 ex 1

$$Y_i = \begin{cases} 1 & \text{if toss } N^{\circ}i \text{ is head} \\ 0 & \text{if toss } N^{\circ}i \text{ is tail} \end{cases}$$

N - number of the first head

$E(Y_2 | N) = ?$

$E(N | Y_2) = ?$

$Var(Y_2 | N) = ?$

$$E(Y_2 | N) = \begin{cases} E(Y_2 | N=1) & \text{if } N=1 \\ E(Y_2 | N=2) & \text{if } N=2 \\ E(Y_2 | N=3) & \text{if } N=3 \end{cases}$$

$E(Y_2 | N=1) = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}$

$E(Y_2 | N=2) = 1 \cdot 1$

TH

$P(Y_2 = 1 | Y_1 = 0, Y_2 = 1) = 1$

$E(Y_2 | N=3) = 0$

$Y_1 = 0, Y_2 = 0, Y_3 = 1)$

$$E(Y_2 | N) = \begin{cases} 1/2, & \text{if } N=1 \\ 1, & \text{if } N=2 \\ 0, & \text{if } N=3 \end{cases}$$

$E(Y_2 | N) = \frac{1}{2} \cdot I(N=1) + 1 \cdot I(N=2) =$

$$E(N | Y_2) = \begin{cases} E(N / Y_2 = 0) & \text{if } Y_2 = 0 \\ E(N | Y_2 = 1) & \text{if } Y_2 = 1 \end{cases}$$

$E(N | Y_2 = 0) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \ldots \ldots = a$

$y_1 = 0$ $\quad y_1 = 1$ $\quad y_1 = 0$ $\quad y_3 = 0$ $\quad y_4 = 1$

gen. function

deriv

dif

$1 \cdot \frac{1}{2} \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \ldots \ldots = \frac{1}{2} a$

$\frac{1}{2} + \frac{3}{4} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \ldots = \frac{1}{2} a$

$\frac{1}{4} + \frac{3}{8} - \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots = \frac{1}{2} \cdot \frac{1}{2} a$

$\frac{1}{2} + \frac{3}{4} - \frac{1}{4} - \frac{1}{4} - \frac{3}{8} + \frac{1}{8} = \frac{1}{4} a$

$a = 2+3-1-1-1 = 2$

$E(N \mid Y_2 = 1) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = 1{,}5$

$E(N \mid Y_2) = \begin{cases} 2 & \text{if } Y_2 = 0 \\ 1{,}5 & \text{if } Y_2 = 1 \end{cases} = 2 \cdot I(Y_2 = 0) + 1{,}5\, I(Y_2 = 1)$

$= 2 - Y_2/2$

$Var(Y_2 \mid N) = E(Y_2^2 \mid N) - (E(Y_2 \mid N))^2 = \begin{cases} \frac{1}{2} - \left(\frac{1}{2}\right)^2 & N=1 \\ 0 & N=2 \\ 0 & N \geq 3 \end{cases} = \frac{1}{4} \cdot I(N=1)$

Approach

$N=1 \quad Y_2 \to 1 \quad \frac{1}{2}$

$\quad\quad\quad\;\; Y_2 \to 0 \quad \frac{1}{2}$

$E(Y_2 \mid N) = \begin{cases} \frac{1}{2} & N=1 \\ 1 & N=2 \\ 0 & N \geq 3 \end{cases}$

$(Y_2 \mid N=1) \sim Bin\left(n=1, p=\frac{1}{2}\right)$

Bernoulli $\left(p = \frac{1}{2}\right)$

„npq" $= 1 \cdot \frac{1}{2} \cdot \frac{1}{2}$

$E(Y_2^2 \mid N) = \begin{cases} \frac{1}{2} & N=1 \\ 1 & N=2 \\ 0 & N \geq 3 \end{cases}$

$Y_2 = 0 \Rightarrow Y_2^2 = 0$

$Y_2 = 1 \Rightarrow Y_2^2 = 1$

$(Y_2 \mid N=2) \quad Y_2 = 1$

$Var(Y_2 \mid N=2) = 0$

$(Y_2 \mid N=3) \quad Y_2 = 0$

$Var(Y_2 \mid N=3) = 0$

08.02.2013 Retake P. 20 №1

$X, Y \sim$ joinly N

$E(X) = E(Y) = 0$

$Var(X) = Var(Y) = 1$

$Cov(X,Y) = \rho$

$E(Y \mid X) = \int_{-\infty}^{\infty} y\, f(y \mid X)\, dy$

$E(Y^2 \mid X) = \int_{-\infty}^{\infty} y^2\, f(y \mid X)\, dy$

$$f(y|X) = \frac{f(X,y)}{f(X)}$$

$$\longrightarrow \frac{1}{\sqrt{\det(2\pi C)}} \cdot \exp\left(-(xy)(C^{-1})\cdot\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$$\longrightarrow \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad X \sim N(0;1)$$

covariance matrix

$$C = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$