Submission details

Deadline HA1: 01 December 2024, 23:59
Deadline HA2: 15 December 2024, 23:59
You have one honey-day. The honey-day allows you to postpone one of these deadlines by 24 hours.

1 HA-1

1. Consider the following joint distribution of X and Y:

	$X=-1$	$X=0$	$X=1$
$Y=0$	0.1	0.1	0.3
$Y=1$	0.3	0.1	0.1

   (a) Find explicitely $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$, $\sigma(X^2)$, $\sigma(2X+3)$.

   (b) Home many elements are there in $\sigma(X, Y)$, $\sigma(X+Y)$, $\sigma(X, Y, X+Y)$?

2. More σ-algebra questions :)

   (a) You observe the result of 10 independent coin tosses. How many elements does the corresponding σ-algebra contain?

   (b) Is union of two σ-algebras always a σ-algebra? Prove your statement.

   (c) Is intersection of two σ-algebras always a σ-algebra? Prove your statement.

3. I throw a fair die until the first six appears. Let's denote the total number of throws by X and the number of odd integers thrown by Y.

   Find $\mathbb{P}(Y = y \mid X)$, $\mathrm{E}(Y \mid X)$, $\mathrm{Var}(Y \mid X)$, $\mathrm{E}(X \mid Y)$.

4. I throw 100 coins. Let's denote by X the number of coins that show «heads». I throw these X coins once again, leaving other coins as they are. Let's denote by Y the number of coins that show «heads» now.

   Find $\mathbb{P}(Y = y \mid X)$, $\mathrm{E}(Y \mid X)$, $\mathrm{Var}(Y \mid X)$, $\mathrm{E}(Y)$, $\mathrm{Var}(Y)$.

5. Random variables X and Y have joint normal distribution with zero means and covariance matrix

$$\begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}.$$

   (a) Find $\mathrm{E}(Y \mid X)$, $\mathrm{Var}(Y \mid X)$, $\mathrm{E}(XY \mid X)$ and $\mathrm{Var}(XY \mid X)$.

   (b) Using standard normal cumulative distribution function find $\mathbb{P}(YX > 2024 \mid X)$.

6. The random variables Z_1, Z_2, ... are independent and identically distributed with $\mathbb{P}(Z_n = 1) = 0.7$ and $\mathbb{P}(Z_n = -1) = 0.3$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.

   (a) Find all values of a such that $\exp(aS_n)$ is a martingale.

   (b) If possible find the constants α and β such that $Y_n = S_n^2 + \alpha S_n + \beta n$ is a martingale.

7. The random variables Z_1, Z_2, ... are independent and identically distributed with $\mathbb{P}(Z_n = +1) = 0.1$, $\mathbb{P}(Z_n = -1) = 0.1$ and $\mathbb{P}(Z_n = 0) = 0.8$. Consider the cumulative sum process, $S_n = Z_1 + \ldots + Z_n$ with $S_0 = 0$.

   Let τ be the first moment when $S_n = 10$ or $S_n = -20$.

   (a) Is S_n a martingale?

   (b) Find $\mathbb{P}(S_\tau = 10)$.

   (c) If possible find the non-random sequence a_n such that $Y_n = S_n^2 + a_n$ is a martingale.

   (d) Find $\mathrm{E}(\tau)$.

2 HA-2

1. Let (W_t) be a standard Wiener process.

   Find $\mathrm{E}(\sin(\alpha W_t))$, $\mathrm{E}(\exp(\alpha W_t))$, $\mathrm{E}(\cos(\alpha W_t))$.

   Hint: you may solve this with or without Ito's lemma, that's up to you.

2. Let (W_t) be a standard Wiener process and $Y_t = W_t^3 + t^2 W_t^2$.

   (a) Find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.

   (b) Is Y_t a martingale?

   (c) Find $\mathrm{E}(Y_t \mid W_s)$ for $t \geq s$.

3. Let $Y_t = W_t + 4t$ and consider the process $M_t = \exp(\alpha W_t - \alpha^2 t/2)$. The moment τ is the first moment when Y_t hits 10.

   (a) Check whether M_t is a martingale.

   (b) Find $f(t)$ such that $M_t = f(t)\exp(\alpha Y_t)$.

   (c) Using Doob's theorem for M_t find $\mathrm{E}(\exp(-(4\alpha + \alpha^2/2)\tau))$.

   (d) Find $\mathrm{E}(\exp(-s\tau))$ for $s \geq 0$.

   (e) Find $\mathrm{E}(\tau)$.

   Hint: you may believe without penalty that Doob's theorem can be applied in this case.

4. Consider the process $dX_t = W_t^4 dW_t + W_t^6 dt$ with $X_0 = 2024$.

   (a) Find $\mathrm{E}(X_t)$.

   (b) Find dY_t for $Y_t = X_t^2$.

   (c) (bonus point) Find $\mathrm{E}(Y_t)$ and $\mathrm{Var}(X_t)$.

5. Consider the process $C_t = W_t^3 + 2W_t^2 - 5W_t^3 \cdot t$.

   (a) Find dC_t.

   (b) Is C_t a martingale?

   (c) Find the covariance $\mathrm{Cov}\left(C_t, \int_0^t W_u^2 dW_u\right)$.

6. Solve the stochastic differential equation $dY_t = -Y_t dt + dW_t,\ Y_0 = 1$.

   If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitely. It's ok.

7. Consider the framework of Black and Scholes model: S_t is the share price. Derive the current price of two European type assets, X_0 and Y_0.

   Future payoffs are given by:

   (a) $X_T = (S_T - K)^3$ where T and K are fixed in the contract.

   (b) $Y_T = S_T^{-2}$ where T is fixed in the contract.