# Lecture #6

Black and Scholes model

Share price: $S_t$ $\qquad$ random: $dS_t = \mu S_t \cdot dt + \delta \cdot S_t \, dW_t$

Bond price: $B_t$ $\qquad$ $dB_t = r \cdot B_t \cdot dt$ nonrandom

time continuous

Exer $S_t, B_t$ ? $\qquad$ $\mu, \delta, r$ – are constants

$$dB_t = r \cdot B_t \cdot dt$$

$$\frac{dB_t}{dt} = r \cdot B_t$$

$$B_t = B_0 \cdot \exp(r \cdot t)$$

$$dS_t = \mu \cdot S_t \cdot dt + \delta S_t \cdot dW_t \qquad \boxed{S_t > 0}$$

$$\boxed{Y_t = \ln S_t} \quad \boxed{h(s,t) = \ln S}$$

$$dY_t \overset{?}{=}$$

If $Y_t = h(S_t, t)$ $\qquad$ [ . . . . . ] then

$$dY_t = h'_s \cdot dS_t + h'_t \, dt + \frac{1}{2} \cdot h''_{ss} \cdot (dS_t)^2$$

$$\Rightarrow dY_t = \underbrace{\frac{1}{S_t}}_{h'_s} \cdot dS_t + \underbrace{0}_{h'_t} \cdot dt + \frac{1}{2}\left(-\frac{1}{S_t^2}\right) \cdot (dS_t)^2$$

$dW_t \, dW_t = dt$

$dt \cdot dW_t = 0$

$dt \cdot dt = 0$

$$dY_t = \frac{1}{S_t}\left(\mu S_t \, dt + \delta S_t \, dW_t\right) - \frac{1}{2S_t^2}\left(\mu S_t dt + \delta \cdot S_t \, dW_t\right)^2$$

$$dY_t = \mu dt + \delta dW_t - \frac{1}{2S_t^2}\left(\mu^2 S_t \underbrace{(dt)^2}_{0} + \delta^2 S_t^2 \underbrace{(dW_t)^2}_{dt} + 2\mu\delta \cdot S_t^2 \underbrace{dt \, dW_t}_{0}\right)$$

# taking log simplifies $B_t$ and makes it ± linear

by def

$$dY_t = \mu dt + \delta dW_t - \frac{\delta^2}{2} dt$$

$$Y_t = Y_0 + \int_0^t \mu du + \int_0^t \delta dW_u - \int_0^t \frac{\delta^2}{2} du$$

$$\boxed{\ln S_t = \ln S_0 + \mu \cdot t + \delta \cdot (W_t - W_0) - \frac{\delta^2}{2} t}$$

① $W_0 = 0$

② $W_t - W_s \sim N(0, t-s)$

⋮

$$\Downarrow$$

$$S_t = \exp(\ldots\ldots) = S_0 \cdot \exp\left(\mu t + \delta W_t - \frac{\delta^2}{2} t\right)$$

$Y_t = h(X_t, t)$

$dY_t = h_x \cdot dX_t + h'_t \cdot dt + \frac{1}{2} h''_{xx} (dX_t)^2$

Answers

$$\begin{cases} dS_t = \mu S_t dt + \delta \cdot S_t dW_t \\ dB_t = z \cdot B_t dt \end{cases} \longrightarrow \begin{cases} S_t = S_0 \cdot \exp(\mu t - \frac{\delta^2}{2} t + \delta W_t) \\ B_t = B_0 \cdot \exp(zt) \end{cases}$$

Black and Sch. model

discounted price process.

$X_t = \exp(-zt) \cdot B_t$     in discr. time $\frac{P_t}{(1+z)^t}$

$Y_t = \exp(-zt) \cdot S_t$     in cont. time $\frac{P_t}{\exp(zt)}$

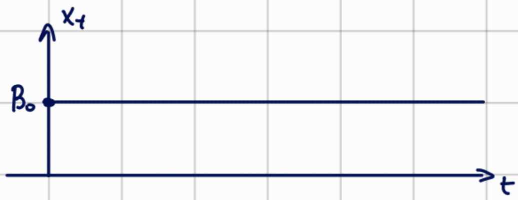
for small $z \approx 0$

$\exp(z) \approx 1+z$

Ex. Is $X_t$ a martingale?

Is $Y_t$ a martingale?

$X_t = \frac{B_t}{\exp(zt)} = B_0 = \text{const}$




$E(X_{t+h} | \mathcal{F}_t) = B_0 = X_t$

$\overbrace{(Y_t \text{ is a martingale})}^{?} \iff dY_t = A_t \cdot dW_t + 0 \cdot dt$

$Y_t = \exp(-zt) \cdot S_t = h(s, t)$

$dY_t = \underbrace{\exp(-zt)}_{h'_s} \cdot dS_t + \underbrace{(-z) \cdot \exp(-zt) \cdot S_t}_{h'_t} \cdot dt + \underset{h''_{ss}}{0} (dS_t)^2$

$= \exp(-zt) [\mu S_t dt + \delta S_t dW_t - z S_t dt]$

$= \exp(-zt) S_t \cdot \delta \cdot [dW_t + \frac{\mu - z}{\delta} \cdot dt]$

if $\mu \neq z$ then $(Y_t)$ is not a martingale

$Y_t = \exp(-zt) \cdot S_t$

# different probabilities

| | $X^*=0$ | $X^*=1$ | $X^*=2$ |
|---|---|---|---|
| $X=0$ | $p^*=0{,}1$ <br> $p=0{,}1$ | $p^*=0{,}1$ <br> $p=0{,}1$ | $p^*=0$ <br> $p=0$ |
| $X=1$ | $p^*=0{,}1$ <br> $p=0{,}1$ | $p^*=0{,}7$ <br> $p=0{,}2$ | $p^*=0$ <br> $p=0{,}5$ |

a) $E_p(X) = 0\cdot 0{,}2 + 1\cdot 0{,}8 = 0{,}8$

b) $X \overset{?}{\sim} Bern(0{,}8)$, $Bin(n=1, 0{,}8)$

c) $X \cancel{\overset{?}{\underset{p}{\sim}}} X^*$ $P(X^*=2)=0{,}5$ $P(X=2)=0$ don't have identical dist.

d) $E_p(X^*) = 0{,}3\cdot 1 + 0{,}5\cdot 2 = 1{,}3$

Let's invent a new prob measure $p^*$

① $X \underset{p^*}{\sim} X^*$

② $X \underset{p^*}{\sim} Bern(0.8)$

③ $P(X=0, X^*=0) = p^*(X=0, X^*=0)$

\# $P^*(X^*=0) = 0{,}2$

$P^*(X^*=1) = 0{,}8$

$Var_p(X^*) \neq Var_{p^*}(X^*)$

[Girsanov] theorem

If $(W_t)$ is a Wiener process under $P$ and $W_t^* = W_t + c\cdot t$ then there exists $p^*$ such that $(W_t^*)$ is a Wiener process under $p^*$

Ex $W_t$ - Wiener process under $P$

$W_t^* = W_t + 2t$ - Wiener process under $p^*$

$P(W_2 > W_1)$

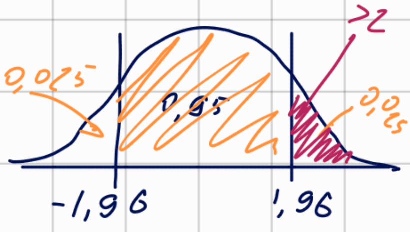
$p^*(W_2 > W_1)$

$E_p(W_7)$ $\quad$ $E_p(W_7^*)$

$E_{p^*}(W_7)$ $\quad$ $E_{p^*}(W_7^*)$

$P(W_2 > W_1) = P(\underbrace{W_2 - W_1}_{N(0,\,2-1)} > 0) = P(N(0;1) > 0) = \frac{1}{2}$

$W_t - W_s \overset{P}{\sim} N(0, t-s)$




$p^*(W_2 > W_1) = p^*(W_2^* - 2\cdot 2 > W_1^* - 2\cdot 1) = P^*(W_2^* - W_1^* > 2) = p^*(N(0,1) > 2) \approx$

$\approx 0.025$

$E_p(W_7) = 0$

$E_{p^*}(W_7) = E_{p^*}(W_7^* + 2\cdot 7) = 0 - 14 = -14$

$E_p(14) = 14$ $\quad$ $E_p(W_7^*) = E_p(W_7 + 7\cdot 2) = 14$

$E_{p^*}(14) = 14$ $\quad$ $E_{p^*}(W_7^*) = 0$

## Seminar #6.

$W_t$ - is a Wiener process under $p$

$$W_t^* = W_t + \underbrace{\frac{\mu - r}{\sigma} \cdot t}_{h(W_t, t)} \qquad (W_t^*) \text{ is a Wiener process under } p^*$$

$$dW_t^* = \underbrace{1}_{h'_w} \cdot dW_t + \underbrace{\frac{\mu - r}{\sigma}}_{h'_t} dt + \frac{1}{2} \underbrace{0}_{h''_{ww}} \cdot (dW_t)^2$$

$dW_t^*$

$$Y_t = \exp(-rt)\, S_t$$

$$dY_t = \sigma \cdot S_t \cdot \exp(-rt) \cdot \left[ dW_t + \frac{\mu - r}{\sigma} dt \right]$$

$$dY_t = \sigma \cdot S_t \cdot \exp(-rt) \cdot dW_t^*$$

$Y_t$ is not a martingale if you use $p$ probability

$Y_t$ is a martingale if you use $p^*$ probability

$\exp(-rt)\, B_t \leftarrow$ const $B_0$ from $p$-viewpoint

$\exp(-rt)\, B_t \leftarrow$ const $B_0$ from $p^*$ viewpoint

**def** European type asset — time moment $T$ is fixed


1. payoff is received at $T$
2. payoff is based on price history of $S_t$ for $t \in [0; T]$

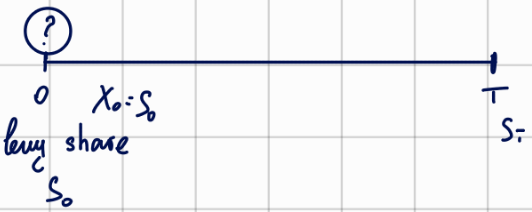
Ex
$$X_T = \begin{cases} 1, & \text{if } S_{T/2} > 100\,\$ \\ 0, & \text{otherwise} \end{cases}$$

$$X_T = S_T$$

$$X_T = \begin{cases} S_T, & \text{if } S_T \geq 100 \\ 0, & \text{otherwise} \end{cases}$$

**Theorem.** In BS model one can replicate every European type claim by using some portfolio of shares and bonds.

$X_T = S_T$



$X_T = 2S_T + 7$

$X_0 = 2S_0 + NB_0 = 2S_0 + 7 \cdot \exp(-2T)$

$B_T \cdot N$

$7 = B_0 \cdot \exp(2t) \cdot N$

$N = \frac{7}{B_0} \exp(-2T)$

number bonds

Bond price

$N \cdot B_T$

$2 S_T + 7$

2 shares

share price

$N^B \cdot B_T = 7$

$N^B \cdot B_0 \cdot \exp(+2T) = 7$

$N^B \cdot B_0 = 7 \cdot \exp(-2T)$

$X_t$ - price of portfolio (claim)

$\# N^S$ - number of shares

$$X_T = X_0 + \int_0^T N_t^S dS_t + \int_0^T N_t^B dB_t$$

$$dX_t = N_t^S \cdot dS_t + N_t^B \cdot dB_t$$

Ex. $d(\underbrace{\exp(-2t) X_t}_{\text{discounted portfolio price}}) = \exp(-2t) dX_t + (-2)\exp(-2t) X_t dt = \exp(-2t)(N_t^S dS_t + N_t^B dB_t - 2X_t dt) = \exp(-2t)(N_t^S \cdot \mu S_t \cdot dt + N_t^S \cdot \sigma \cdot S_t \cdot dW_t + N_t^B \cdot 2 \cdot B_t \cdot dt - 2 X_t dt) =$

$Q_t = \exp(-2t) \cdot X_t$

$X_t = S_t \cdot N_t^S + B_t N_t^B$

$= \exp(-2t) \cdot N_t^S \cdot \sigma \cdot S_t \, dW_t + dt \cdot (N_t^S \mu S_t + \cancel{N_t^B 2 B_t} - 2 S_t N_t^S - \cancel{2 B_t N_t^B}) =$

$= \exp(-2t) \cdot S_t \cdot N_t^S \cdot (\sigma dW_t + (\mu - 2)dt)$

$Q_t = \exp(-2t) \cdot X_t$

$dQ_t = \exp(-2t) \cdot S_t \cdot N_t^S \cdot \sigma \underbrace{\left[dW_t + \frac{\mu-2}{\sigma} dt\right]}_{[dW_t^* + 0dt]}$

$W_t^* = W_t + \frac{\mu - 2}{\sigma} \cdot t$

① $\boxed{Q_t, \text{ discounted portfolio price is a martingale under } P^*}$

$S_t = S_0 \cdot \exp(\mu t + \sigma W_t - \frac{\sigma^2}{2} t) =$

$S_t = S_0 \cdot \exp(2t + \sigma W_t^* - \frac{\sigma^2}{2} t)$

Ex.

$X_0 - ?$

$$X_T = \begin{cases} 1, & \text{if } S_T > 2 S_0 \\ 0, & \text{otherwise} \end{cases}$$

$$Q_0 = E_{P^*}(Q_t)$$

$$Q_t = \frac{X_t}{\exp(2t)} \longrightarrow \frac{X_0}{\exp(2\cdot 0)} = E_{P^*}\left(\frac{X_T}{\exp(2T)}\right)$$

$$X_0 = \exp(-2T)\cdot E_{P^*}(X_T) =$$

$$= \exp(-2T)\cdot P^*(S_T > 2S_0)$$

$$= \exp(-2T)\cdot P^*\left(S_0 \exp\left(2T + \delta W_t^* - \frac{\delta^2}{2}T\right) > 2 S_0\right) =$$

$$= \exp(-2T)\cdot P^*\left(2T + \delta W_T^* - \frac{\delta^2}{2}T > \ln 2\right) =$$

$$= \exp(-2T)\cdot P^*\left(W_T^* > \frac{\ln 2 + \frac{\delta^2}{2}T - 2T}{\delta}\right) = \exp(-2T)\, P^*\left(N(0,1) > \frac{\ln 2 + \frac{\delta^2}{2}T - 2T}{\delta\sqrt{T}}\right) =$$

$$W_T^* \sim N(0;T) \Rightarrow \frac{W_T^*}{\sqrt{T}} \overset{P^*}{\sim} N(0;1)$$

$$= \exp(-2T)\cdot P^*\left(N(0;1) < \frac{2T - \frac{\delta^2}{2}T - \ln 2}{\delta\sqrt{T}}\right) = \exp(-2T)\cdot F\left(\frac{2T - \frac{\delta^2}{2}T - \ln 2}{\delta\sqrt{T}}\right) = X_0$$

$$S_t = S_0 \cdot \exp\left(2t + \delta\cdot W_t^* - \frac{\delta^2}{2}t\right)$$

$$\boxed{X_0 = \exp(-2T)\cdot E_{P^*}(X_T)}$$

## Check up #6

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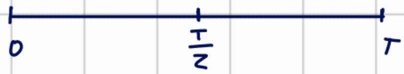
$$X_T = \begin{cases} 1, & \text{if } S_T/S_{T/2} > S_{T/2}/S_0 \\ 0, & \end{cases}$$

[10]



$$X_0 = ? = \exp(-2T)\cdot P^*\left(\frac{S_T}{S_{T/2}} > \frac{S_{T/2}}{S_0}\right)$$

$$\frac{S_0\cdot \exp\left(2T - \frac{\delta^2}{2}T + \delta\cdot W_T^*\right)}{S_0\cdot \exp\left(2\frac{T}{2} - \frac{\delta^2}{2}\frac{T}{2} + \delta\cdot W_{T/2}^*\right)} > \frac{S_0\cdot \exp\left(2\frac{T}{2} - \frac{\delta^2}{2}\frac{T}{2} + \delta W_{T/2}^*\right)}{S_0}$$

$$2T - \frac{\delta^2}{2}T + \delta W_T^* - \left(2\frac{T}{2} - \frac{\delta^2}{2}\frac{T}{2} + \delta W_{T/2}^*\right) > 2\frac{T}{2} - \frac{\delta^2}{2} + \delta\cdot W_{T/2}^*$$

$$\delta W_T^* - \delta W_{T/2}^* > \delta W_{T/2}^*$$

$$\boxed{W_T^* - 2W_{T/2}^* > 0}$$

$$E_*\left(W_T^* - 2W_{T/2}^*\right) = 0 - 2\cdot 0 = 0$$

$Var_*(W_T^* - 2W_{T/2}^*) = Var^*(W_T^*) + 4Var^*(W_{\frac{T}{2}}^*) - 2\cdot 2\cdot Cov^*(W_T^*, W_{T/2}^*) = T$

$Var^*(W_T^*) = T$, $4Var^*(W_{\frac{T}{2}}^*) = \cancel{4\frac{T}{2}}$

$Cov(W_t, W_s) = \min(t,s)$

$\frac{T}{2} < T \Rightarrow \cancel{-4\frac{T}{2}}$

$W_t - W_s \overset{P}{\sim} N(0; t-s)$

$W_t^* - W_s^* \overset{P^*}{\sim} N(0; t-s)$

$x_0 = ? = \exp(-2T)\cdot P^*\left(\frac{S_T}{S_{T/2}} > \frac{S_{T/2}}{S_0}\right) = \exp(-2T)\cdot P^*(N(0;T)>0) = \frac{1}{2}\exp(-2T)$

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BS model

$p^*$ - risk neutral probability

$p$ - real probability

a) $P(S_2 > S_1)$ $\quad$ $p^*(S_2 > S_1)$

b) $X_0$ ? $\quad$ $X_2 = \sqrt{S_1}$

a) $P\left(S_0\cdot\exp(\mu 2 - \frac{\delta^2}{2}2 + \delta W_2) > S_0\cdot\exp(\mu\cdot 1 - \frac{\delta^2}{2}\cdot 1 + \delta\cdot W_1)\right) =$

$= P\left(\mu - \frac{\delta^2}{2} + \delta\underbrace{(W_2 - W_1)}_{N(0;1)} > 0\right) = P\left(W_2 - W_1 > \frac{\frac{\delta^2}{2} - \mu}{\delta}\right) = P\left(N(0,1) > \frac{\delta^2 - 2\mu}{2\delta}\right) =$

$= P\left(N(0,1) < \frac{2\mu - \delta^2}{2\delta}\right) = F\left(\frac{2\mu - \delta^2}{2\delta}\right) = 1 - F\left(\frac{\delta^2 - 2\mu}{2\delta}\right)$

cdf of $N(0;1)$

$p^*(S_2 > S_1) = \ldots = F\left(\frac{2r - \delta^2}{2\delta}\right)$

$X_0 = \exp(-2r)\cdot E_{p^*}\left(\sqrt{S_0\cdot\exp(r_1 - \frac{\delta^2}{2}\cdot 1 + \delta\cdot W_1^*)}\right) =$

$= \exp(-2r)\cdot E\left(\sqrt{S_0}\cdot\exp\left(\frac{r}{2} - \frac{\delta^2}{4}\right)\cdot\exp\left(\frac{\delta}{2}W_1^*\right)\right) =$

$= \exp\left(-2r + \frac{r}{2} - \frac{\delta^2}{4}\right)\sqrt{S_0}\cdot E_{p^*}\left(\exp\left(\frac{\delta}{2}W_1^*\right)\right) = \exp\left(-1{,}5r - \frac{\delta^2}{4}\right)\sqrt{S_0}\cdot\exp\left(\frac{(\delta/2)^2}{2}1\right) =$

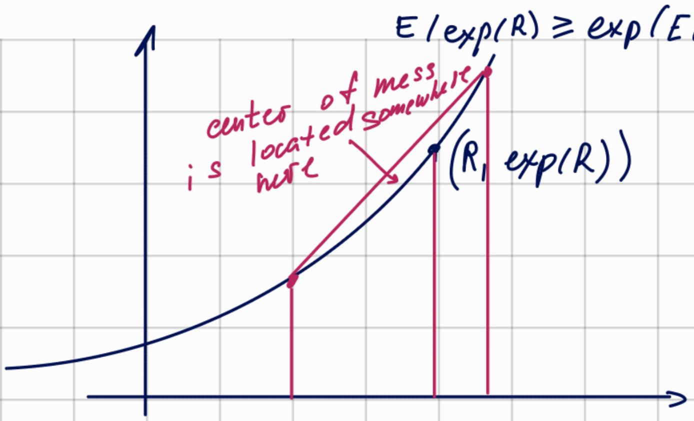
$= \sqrt{S_0}\cdot\exp\left(-1{,}5r - \frac{\delta^2}{8}\right)$

$$\boxed{\begin{aligned} &E(\exp(\alpha\cdot W_t)) = \exp\left(\frac{\alpha^2}{2}\cdot t\right) \\ &E_*(\exp(\alpha\cdot W_t^*)) = \exp\left(\frac{\alpha^2}{2}t\right)\end{aligned}}$$

$E(5 + 6R) = 5 + 6E(R)$

$E(R^2) \geq (E(R))^2$

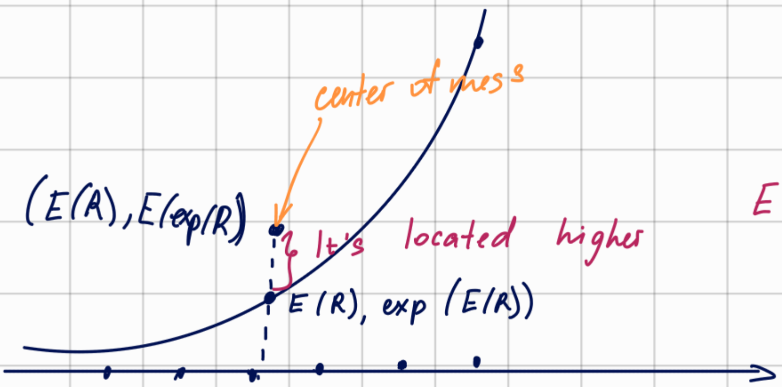
$E(\exp(R)) \geq \exp(E(R)) =$



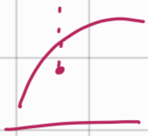

center of mess

$(E(R), E(\exp(R)))$

$(E(R), \exp(E(R)))$




$E(\exp(R)) \geq \exp(E(R))$



case : $E(\sqrt{R}) \leq \sqrt{E(R)}$

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$dS_t = S_t\, dt + \sqrt{S_t} \cdot dW_t$ , $S_0 = 1$

Find all $h(\cdot)$ such that $Y_t = h(t)$ is a martingale

$dY_t = A_t\, dt + B_t\, dW_t$

$A_t = 0$

$dY_t = h_s' \cdot dS_t + \underbrace{h_t'}_{0} \cdot dt + \frac{1}{2} h_{ss}'' \cdot (dS_t)^2 =$

$= h_s' \cdot (S_t dt + \sqrt{S_t} \cdot dW_t) + \frac{1}{2} h_{ss}'' (S_t dt + \sqrt{S_t}\, dW_t)^2 = h_s' S_t \cdot dt + h' \sqrt{S_t}\, dW_t +$

$+ \frac{1}{2} h'' \underbrace{(S_t^2 \cancel{(dt)^2}_{\to 0} + 2 \cdot S_t \cdot \sqrt{S_t}\, \cancel{dt\, dW_t}_{\to 0} + S_t \cdot \cancel{(dW_t)^2}_{\to dt})}_{S_t dt}$

$dW_t\, dW_t = dt$

$dt \cdot dW_t = 0$

$dt \cdot dt = 0$

$h' + \frac{1}{2} h'' = 0$

$h' = g$ $\qquad g + \frac{1}{2} g' = 0$

$g' = -2g$

$g(a) = g_0 \cdot \exp(-2a)$

$h(a) = \frac{g_0}{-2} \cdot \exp(-2a) + c$

$h(a) = c + d \cdot \exp(-2a)$

$h(S_t) = c + d \cdot \exp(-2S_t)$ is a martingale $\forall\, c, d \in \mathbb{R}$