Group Theory MC15:
More problems for fun and practice

The commutator subgroup

Jubilation, it commutes again,
Divide out by all commutators!

Definition. Let G be a group. The *commutator subgroup* G' is the subgroup generated by all elements of G of the form $aba^{-1}b^{-1}$, for some $a, b \in G$.

Note that not every element of G' is necessarily of the form $aba^{-1}b^{-1}$; it may be a product of two or more elements in this form.

1. Show that if G is abelian then G' is trivial (i.e. $G' = \{e\}$).
2. Show that G' is normal in G.
3. Show that G/G' is abelian. This kind of make sense intuitively: you're "getting rid" of all the things that prevent G from being abelian, by putting them all into the identity coset. In other words, G "commutes again" (I'm not sure why "again") when you "divide out by all commutators"! The group G/G' is called the *abelianization of* G, denoted G_{ab}.
4. Show that if H is normal in G and G/H is abelian then $G' \subseteq H$. In other words, the only way to "make" G into an abelian group is to mod out by G'. You can mod out by more (a bigger group H), but G' is the minimum required to get something abelian.
5. Let $G = D_8$. Find G' and figure out the multiplication table for G/G'. Now try doing this for D_{2n}. (Warning: the answer depends on whether n is even or odd.)

Products of groups

Definition. Let G and H be groups. We define a new group

$$G \times H = \{(g, h) : g \in G, h \in H\}.$$

In other words, the elements of $G \times H$ are ordered pairs. (This is kind of like $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, whose elements are pairs of real numbers.) The operation on

$G \times H$ is defined to be

$$(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

1. Verify that $G \times H$ is indeed a group. Show that if G and H are both abelian, so is $G \times H$.

2. Show that $G \times \{e\}$ is a normal subgroup of $G \times H$ and the quotient group is isomorphic to H.

3. Let G be a group of order 4 where every element except e has order 2. Show that $G \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$.

4. (Challenge!) Let G be a group in which every element except e has order 2. Show that
$$G \simeq \mathbb{Z}/2 \times \mathbb{Z}/2 \times \cdots \times \mathbb{Z}/2,$$
for some number of copies of $\mathbb{Z}/2$. In particular, $|G| = 2^k$ for some k. (Hint: Induction on the order of G. Let H be a subgroup of G of order 2, define a function
$$\phi : G \to H \times G/H,$$
and show that it's an isomorphism.)

5. Suppose $\gcd(m, n) = 1$. Consider the function
$$\phi : \mathbb{Z}/mn \to \mathbb{Z}/n \times \mathbb{Z}/m,$$
given by $\phi(k) = (k \mod m, k \mod n)$. Show that this map is an isomorphism. (Congratulations: you have just proved a famous theorem from number theory! Which one?)

6. Show that if $\gcd(m, n) \neq 1$, then $\mathbb{Z}/mn$ is NOT isomorphic to $\mathbb{Z}/n \times \mathbb{Z}/m$. (Not just "not necessarily isomorphic" but "definitely not isomorphic"). In particular, show that no two of the groups $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ are isomorphic to each other.

Classifying groups of order 8

So far, we have seen four non-isomorphic groups of order 8:

$$\mathbb{Z}/8, \quad \mathbb{Z}/4 \times \mathbb{Z}/2, \quad \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \text{and} \quad D_8.$$

It turns out that there is one more. The *quaternion group* Q is a group of order 8 with elements

$$\pm 1, \pm i, \pm j, \pm k.$$

The identity of Q is 1, and the group operation (which we will call multiplication) works as follows:

$$\begin{aligned}
&(-1)^2 = 1 \\
&(-1)i = -i, (-1)j = -j, (-1)k = -k, \\
&i^2 = j^2 = k^2 = ijk = -1
\end{aligned}$$

(If this reminds you of the complex numbers and/or of the cross-product in 3 dimensions, you are right! More on the context and history of Q at the end of this section.)

Note: In this context, you should treat the minus sign as just part of the name of the element (and a useful a mnemonic, since it behaves as you might expect). Q is part of a larger algebraic structure called the *quaternion algebra*, where addition is actually meaningful. (See end of this section.) But we are just dealing with Q itself right now, where there is no addition or subtraction, only multiplication.

1. Construct the multiplication table for Q. Is Q abelian?
2. Show that Q is not isomorphic to any of the groups of order 8 that we've seen before.
3. Q has exactly one normal subgroup N of order 2. Construct the multiplication table for Q/N.
4. (Challenge!) Show that there are exactly 5 groups of order 8 up to isomorphism.

A little history: Here is the famous story of the invention of the quaternions by the Irish mathematician William Rowan Hamilton in 1843 (from Wikipedia):

Hamilton knew that the complex numbers could be interpreted as points in a plane, and he was looking for a way to do the same for points in three-dimensional space. Points in space can be represented by their coordinates, which are triples of numbers, and for many years he had known how to add and subtract triples of numbers. However, Hamilton had been stuck on the problem of multiplication and division for a long time. He could not figure out how to calculate the quotient of the coordinates of two points in space.

The great breakthrough in quaternions finally came on Monday 16 October 1843 in Dublin, when Hamilton was on his way to the Royal Irish Academy where he was going to preside at a council meeting. As he walked along the towpath of the Royal Canal with his wife, the concepts behind quaternions were taking shape in *his mind. When the answer dawned on him, Hamilton could not resist the urge to carve the formula for the quaternions,*

$$i^2 = j^2 = k^2 = ijk = -1,$$

into the stone of Brougham Bridge as he paused on it.

What Hamilton was actually thinking about was not just the group Q but the *quaternion algebra*,

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\},$$

which plays an important role in many areas of mathematics and physics. Talk to Steve if you want to know more about $\mathbb{H}$.

And finally...

This problem set started with a song (*Nonabelian*). Let's end it with a fairy tale:

Once upon a time, in Sherwood Forest, there lived 101 people. The most famous of them were Robin Hood, his sweetheart Maid Marion, his sidekicks Little John and Will Scarlet, the bard Alan-a-Dale, and the priest Friar Tuck. Everyone in the forest had a mate except the priest. Each person had written a song. Singing was contagious.

Every song was sung to each person by a singer. (In particular, each person's own song was sung to him or her by some singer.) The priest, to whom everyone sang their own song, had a song that each person sang to himself. The priest sang to each person the song written by that person's mate, and, having no mate, he sang his own song to himself.

Any singer who sang to a first person the song of the singer of a second person's song to a third, was the same singer who sang to the singer of the third person's song to the first person the song of the second person.

If Robin Hood sang Will Scarlet's song to Alan-a-Dale and Marion sang Robin Hood's song to Little John, who sang Ala-a-Dale's song to Will Scarlet? Who sang Marion's song to her?