Representation Theory of Finite Groups: from Frobenius to Brauer*

Charles W. Curtis

This article is dedicated to the memory of my friend and collaborator, Irving Reiner.

The representation theory of finite groups began with the pioneering research of Frobenius, Burnside, and Schur at the turn of the century. Their work was inspired in part by two largely unrelated developments which occurred earlier in the 19th century. The first was the awareness of characters of finite abelian groups, and their application by some of the great 19th-century number theorists. The second was the emergence of the structure theory of finite groups, beginning with Galois's brief outline of the main ideas in the famous letter written on the eve of his death, and continuing with the work of Sylow and others, including Frobenius himself.

My aim is to give an account of some of the early work, the problems considered, the conjectures made, and then to trace a few threads in the development of the mathematical ideas from their origins to their place in Brauer's theory of modular representations.

Characters of Finite Abelian Groups and 19th-Century Number Theory

A *character* of a finite abelian group A is a homomorphism from A into the multiplicative group of the field $\mathbf{C}$ of complex numbers, in other words, a function $\chi: A \to \mathbf{C}^* = \mathbf{C} - \{0\}$, which satisfies the condition:

$$\chi(ab) = \chi(a)\chi(b) \text{ for all } a,b \text{ in } A.$$

The simplest examples, which occur in elementary number theory, involve the additive and multiplicative groups of the finite field $\mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$ of residue classes $\bar{a} = a + p\mathbf{Z}$, for a prime p. Additive characters of $\mathbf{Z}_p$ are characters of the additive group of $\mathbf{Z}_p$, with the defining property that $\chi(\bar{a} + \bar{b}) = \chi(\bar{a})\chi(\bar{b})$, for all residue classes $\bar{a}$ and $\bar{b}$. These are obtained by taking powers of a pth root of unity, so $\chi(\bar{a}) = \omega^a$, where $\omega^p = 1$ in $\mathbf{C}$. Multiplicative characters of $\mathbf{Z}_p$ are characters of the multiplicative group of $\mathbf{Z}_p$, and include Legendre's quadratic residue symbol $(a/p) = \pm 1$, for a nonzero residue class $\bar{a}$, with $(a/p) = 1$ if $x^2 \equiv a \pmod p$ has a solution and -1 if not.

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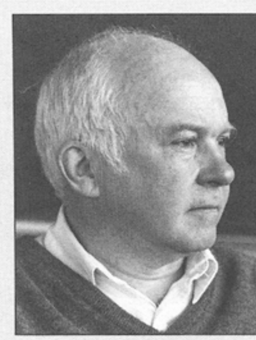

Charles W. Curtis received his Ph.D. from Yale University in 1951. He was a faculty member at the University of Wisconsin-Madison from 1951 to 1963, and has been at the University of Oregon since then. His dissertation and first papers were on noncommutative ring theory. He began to study representation theory soon after his arrival in Madison, and it became his main interest during a fellowship year spent at the Institute for Advanced Study. In that year he met Irving Reiner, and began a collaboration that resulted in the books by Curtis and Reiner, *Representation Theory of Finite Groups and Associative Algebras* (1962), and *Methods of Representation Theory, I* (1981) and *II* (1987). Besides his professional work, Curtis has had a lifelong enthusiasm for playing tennis, and has recently taken up small-scale farming with other members of his family.

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Gauss combined additive and multiplicative characters χ and π, respectively, to form certain sums of roots of unity (today called *Gauss sums*), which have the form

$$g(\chi,\pi) = \Sigma\, \chi(\bar{t})\pi(\bar{t}), \quad \bar{t} \neq 0 \text{ in } \mathbf{Z}_p.$$

In §358 of the *Disquisitiones Arithmeticae* [23], he derived the polynomial equations satisfied by the expressions $g(\chi,\pi)$ in some special cases, using information about the number of solutions of congruences, such as $x^n + y^n \equiv 1 \pmod{p}$. In terms of what we know now, this appears to have been a case of putting the cart before the horse. In fact, Gauss sums have proved to be fundamental for obtaining formulas for the number of solutions of a wide class of polynomial congruences, and for the more general problem of counting the number of solutions of polynomial equations over finite fields. A nice account of these matters, with historical comments, can be found in the first part of Weil's paper on the number of solutions of equations over finite fields [43] (see also [27], §8.3).

Multiplicative characters were used by Dirichlet in his reinterpretation (see [11]) of some of Gauss's work on genera of binary quadratic forms, where the character-theoretic nature of the quadratic residue symbol (a/p) was applied. He also used them in his definition of L-series, and in the proof, using L-series, that certain arithmetic progressions contain infinitely many primes [10].

Dedekind edited Dirichlet's lectures on number theory for publication, and added supplements containing material of his own. In view of the different ways characters had been applied in Dirichlet's work, he called attention to the general notion of characters of abelian groups in one of the supplements (see [11], page 345, footnote, and pages 611, 612). Weber had also become interested in abelian group characters, had published a paper on them, and gave a full account of them in his *Lehrbuch der Algebra* [41], including their construction using the factorization of abelian groups as direct products of cyclic groups.

The starting point of the representation theory of finite groups was Dedekind's work, apparently unpublished, on the factorization of the group determinant of a finite abelian group, and his suggestion, in a letter to Frobenius in 1896, that perhaps Frobenius might be interested in the same problem for general (not necessarily abelian) groups. Here is a statement of the problem.

Let $\{x_g\} = \{x_{g_1}, \ldots, x_{g_n}\}$ be a set of n indeterminates over the field $\mathbf{C}$ of complex numbers, indexed by the elements $\{g_1, \ldots, g_n\}$ of a finite group G of order n. Form the $n \times n$ matrix whose entry in the ith row and jth column is the indeterminate $x_{g_i g_j^{-1}}$. The group determinant of G is the determinant $\Theta = |x_{gh^{-1}}|$ of this matrix, and is a polynomial in the indeterminates x_{g_i}, with integer coefficients. Dedekind had proved the elegant result that, for a finite abelian group, the group determinant Θ factors over the complex numbers as a product of linear factors, whose coefficients are given by the different characters of χ of the group:

$$\Theta = \prod_{\chi} (\chi(g)x_g + \chi(g')x_{g'} + \ldots).$$

As he communicated to Frobenius, he had also investigated the factorization of Θ for nonabelian groups in some special cases, and had observed that, in the cases he had examined, Θ had irreducible factors of degree greater than one.

The factorization of Θ is not as special a problem as it appears. It is related to the problem of factoring the characteristic polynomial, in the regular representation, of an element of the group algebra $\Sigma x_g g$ with indeterminate coefficients x_g, into its irreducible factors. Exactly the same idea was used, with great success, by Killing and Cartan, and by Cartan and Molien, to obtain the structure of semisimple Lie algebras and associative algebras, over the field of complex numbers [26].

Frobenius's First Papers on Character Theory

With Dedekind's letter as a spur, Ferdinand Georg Frobenius (1849–1917) burst onto the scene with three papers, published in 1896, in which he created the

Ferdinand Georg Frobenius.

theory of characters of finite groups, factored the group determinant for nonabelian groups, and established many of the results that have become standard in the subject. At this point in his career, he had assumed Kronecker's chair in Berlin, and was already widely known for his research on theta functions, determinants and bilinear forms, and the structure of finite groups, all of which contributed ideas he was able to use in his new venture.

His first task in "Über Gruppencharaktere" [16] was to define characters of nonabelian finite groups. The key to his approach was the study of the multiplicative relations satisified by the conjugacy classes $\{C_1, \ldots, C_s\}$ in a finite group G. From his previous work in finite group theory, he was well aware of the importance of counting the numbers of solutions of equations in a group G. His starting point was the consideration of the integers $\{h_{ijk}\}$, denoting the numbers of solutions of the equations $abc = 1$, with $a \in C_i$, $b \in C_j$, and $c \in C_k$. From them, he defined a new set of integers, $a_{ijk} = h_{i'jk}/h_i$, where $C_{i'} = C_i^{-1}$, and h_i is the number of elements in the class C_i. He then made the crucial observation that the a_{ijk} satisfy identities which imply that the bilinear multiplication defined on a vector space E over $\mathbf{C}$ with basis elements $\{e_1, \ldots, e_s\}$ by the formulas

$$e_j e_k = \Sigma\, a_{ijk} e_i \qquad (1)$$

is associative and commutative; that is, we have

$$e_i(e_j e_k) = (e_i e_j)e_k \text{ and } e_i e_j = e_j e_i$$

for all i, j, k.

This was a situation familiar to him, in view of "Über vertauschbare Matrizen" [15]. He summoned into play a theorem on what we now call the irreducible representations of commutative, semisimple algebras. The theorem asserts that, under a condition equivalent to semisimplicity of the algebra, there exist $s = \dim E$ linearly independent numerical solutions $(\rho_1, \ldots, \rho_s)$ of the equations (1), so that $\rho_j \rho_k = \Sigma a_{ijk} \rho_i$. The condition is that $\det (p_{k\ell}) \neq 0$, where $(p_{k\ell})$ is the matrix with entries

$$p_{k\ell} = \sum_{i,j} a_{ijk} a_{ji\ell}\ .$$

He proved it, in this case, by an ingenious direct argument based on properties of the class intersection numbers $\{h_{ijk}\}$. Special cases of the result had been obtained by Dedekind, Weierstrass, and Study ([6], [42], [39]), and the definitive theorem, with a new proof, was given by Frobenius himself, in "Über vertauschbare Matrizen" [15], the first paper in the 1896 series.

The characters $\chi = (\chi_1, \ldots, \chi_s)$ of the finite group G were defined in terms of the solutions ρ_j of the equations (1), by the formulas

$$h_j \chi_j / f = \rho_j,$$

where f is a proportionality factor, and $h_j = |C_j|$ as above. This is hardly an intuitively satisfying definition. Things become a little clearer if we realize, as Frobenius did, that the characters can be viewed as complex-valued class functions $\chi : G \to \mathbf{C}$, constant on the conjugacy classes (this is what it means to be a class function), satisfying the relations

$$\chi_j \chi_k = f\, \Sigma\, a_{ijk} \chi_i, \qquad (2)$$

where $\chi_j = \chi(x)$ for $x \in C_j$, and the constant f, called the degree of the character χ, is $\chi(1)$, the value of χ at the identity element 1 of G. The algebra E, as Frobenius realized somewhat later ([18], §6), is isomorphic to the center of the group algebra of G, so that for abelian groups, the constants a_{ijk} describe the multiplication in the group algebra, and the equations (2) are clear generalizations of the definition, given previously, of characters of abelian groups.

The first main results about characters were what are now called the *orthogonality relations*, for two characters χ and ψ, which assert that

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}) = \begin{cases} 1 \text{ if } \chi = \psi \\ 0 \text{ if } \chi \neq \psi \end{cases}.$$

(These involve the choice of the constant f taken above.) By the theorem used to obtain the characters, the number of different characters and the number s of conjugacy classes are the same, so the characters define an $s \times s$ matrix, called the *character table* of G, whose (i,j)th entry is the value of the ith character at an element in the jth conjugacy class. The orthogonality relations express the fact that, in a certain sense, the rows and columns of the character table are orthogonal.

The question arises, what information about a finite group G is contained in its character table. Frobenius took up the problem himself, and it has fascinated group-theorists ever since. His first contribution to it followed easily from his approach to characters ([15], §4). Using the orthogonality relations, he deduced a formula for the class intersection numbers h_{ijk} in terms of the character table, a result which later proved to be fundamental for applications of character theory to finite groups.

Another interpretation of the orthogonality formulas is that the characters form an orthonormal basis for the vector space of class functions on G, with respect to the hermitian inner product defined by

$$(\zeta,\eta) = |G|^{-1} \sum_{g \in G} \zeta(g)\overline{\eta(g)}, \text{ for class functions } \zeta,\eta. \qquad (3)$$

This makes it possible to do a kind of Fourier analysis in the vector space of class functions, in which the "Fourier coefficients" a_χ in the expansion of a class function $\zeta = \Sigma a_\chi \chi$ in terms of the characters, are given by the inner products $a_\chi = (\zeta,\chi)$, for each character χ.

After establishing the foundations of character theory in the second 1896 paper, he turned, in the third, to the solution of the problem raised by Dedekind, about the factorization of the group determinant $\Theta = |x_{gh^{-1}}|$ of a finite group G. He settled the problem with a flourish, proving that $\Theta = \Pi\Phi^f$, with s irreducible factors Φ, whose coefficients are given in terms of the s different characters of G, and the really difficult result, which he called the fundamental theorem in the theory of the group determinant, that the degree of each irreducible factor Φ and the multiplicity with which it occurs in the factorization of Θ coincide, and are both equal to the degree f of the corresponding character. He pointed out the consequence that, if n is the order of the finite group G, then n is the sum of the squares of the degrees of the characters:

$$n = \Sigma f_\chi^2, \quad \text{where } f_\chi = \deg \chi.$$

The Group Determinant of the Symmetric Group S_3

Elements of the Group:

$g_1 = 1, g_2 = (12), g_3 = (23), g_4 = (13), g_5 = (123), g_6 = (132)$

Indeterminates: $x_1, x_2, \ldots, x_6$ with $x_i = x_{g_i}$

The group determinant:

$$\Theta = |x_{g_i g_j^{-1}}| = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_6 & x_5 \\ x_2 & x_1 & x_5 & x_6 & x_4 & x_3 \\ x_3 & x_6 & x_1 & x_5 & x_2 & x_4 \\ x_4 & x_5 & x_6 & x_1 & x_3 & x_2 \\ x_5 & x_4 & x_2 & x_3 & x_1 & x_6 \\ x_6 & x_3 & x_4 & x_2 & x_5 & x_1 \end{vmatrix}$$

The factorization of Θ as predicted by Frobenius [17] (see also [29], Section 4):

$$\Theta = F_1 F_2 (F_3)^2,$$

with

$$F_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$
$$F_2 = x_1 - x_2 - x_3 - x_4 + x_5 + x_6$$

$$F_3 = x_1^2 - x_2^2 + x_2x_3 - x_3^2 + x_2x_4 + x_3x_4 - x_4^2 - x_1x_5 + x_5^2 - x_1x_6 - x_5x_6 + x_6^2.$$

As we noted earlier, the definition (2) of characters of a finite group G does not have the same immediate relation to the structure of G enjoyed by the concept of characters of abelian groups. In the following year, 1897, he clarified the situation by introducing, for the first time, the concept of representation of a finite group. This he defined, as we do today, to be a homomorphism $T : G \to GL_d(\mathbf{C})$, where $GL_d(\mathbf{C})$ is the group of invertible $d \times d$ matrices over $\mathbf{C}$, and d is called the *degree of the representation,* so we have

$$T(gh) = T(g)T(h), \text{ for all } g,h \in G.$$

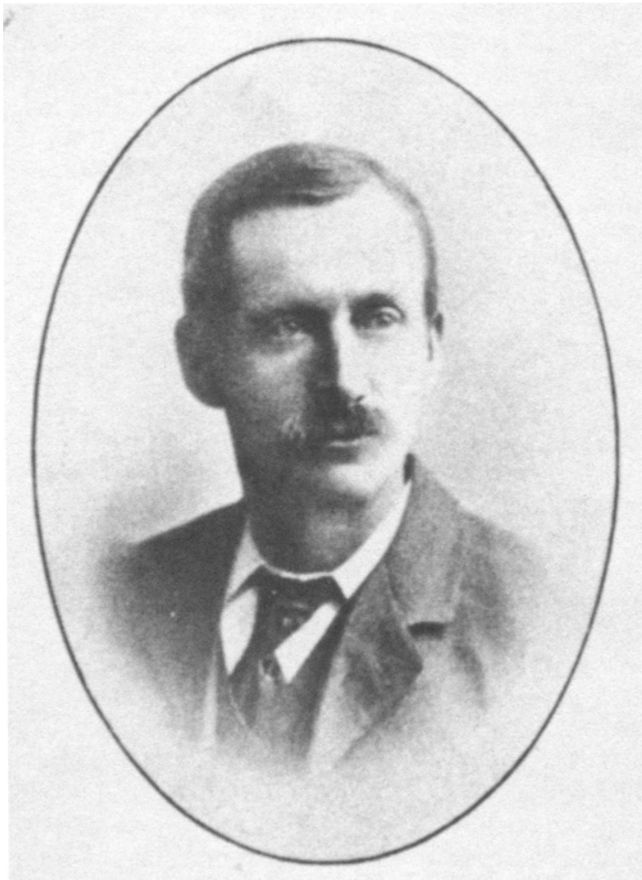
For an abelian group, the characters, defined previously, are representations of degree one. In the general case, he defined two representations T and $T' : G \to GL_{d'}(\mathbf{C})$ to be *equivalent* if they have the same degree, $d = d'$, and the representations T and T' are intertwined by a fixed invertible matrix X, so that $T(g)X = XT'(g)$, or $X^{-1}T(g)X = T'(g)$, for all $g \in G$; in other words, the representation T' is obtained from T by a change of basis in the underlying vector space. In particular, the matrices $T(g)$ and $T'(g)$ are similar, for $g \in G$, and therefore have the same numerical invariants associated with similarity: the same set of eigenvalues, the same characteristic polynomial, trace, and determinant. The important invariant for representation theory is the trace function,

$$\chi(g) = \text{Trace } T(g), \qquad g \in G,$$

which Frobenius called the *character of the representation.* The characters defined earlier by the formulas (2) turned out to be the trace functions of certain representations characterized by the irreducibility of polynomials analogous to the group determinant associated with them.

False modesty was not a weakness of Frobenius. From the beginning of his research on the theory of characters, he was keenly aware of its potential importance for algebra and group theory. He was on to a good thing, and he knew it. Altogether, he published more than 20 papers between 1896 and 1907, extending the theory of characters and representations in various directions, and applying the results to finite group theory.

One of the highlights among the papers published after 1896 was a deep analysis of the relation between characters of a group G and the characters of a subgroup H of G [19]. As he stated in the introduction, an understanding of this relationship is crucial for the practical computation of representations and characters—a statement as true now as it was then! One of the main ideas in the paper was the definition of the *induced class function* ψ^G, for a class function ψ on a subgroup H of G, by the formula



William Burnside.

$$\psi^G(g) = |H|^{-1} \sum_{x\in G} \dot{\psi}(xgx^{-1}), \qquad g \in G,$$

where $\dot{\psi}$ is the function on G defined by

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}.$$

He proved the fundamental result, now called the Frobenius Reciprocity Law, which states that

$$(\psi^G, \zeta)_G = (\psi, \zeta|_H)_H,$$

for class functions ψ on H and ζ on G, respectively, where $(\ , \ )_G$ and $(\ , \ )_H$ are the inner products (3) on the vector spaces of class functions on G and H, and $\zeta|_H$ denotes the restriction of the class function ζ to H. Using the Fourier analysis for expansions of class functions in terms of characters, the Reciprocity Law implies that ψ^G is the character of a representation of G if ψ is the character of a representation of H, and gives the desired information about the relationship between characters of G and H.

Frobenius relished computations, the more challenging the better, and rounded out this great series of papers with computations of the character tables of all the groups in the infinite families consisting of the projective unimodular groups $PSL_2(p)$, for odd primes p (in [16], §10); the symmetric groups S_n (in [20]); and the alternating groups A_n (in [21]). The methods he developed for carrying out these computations involved the full range of his ideas on characters, combined with new techniques from combinatorics and algebra, far ahead of their time, which continue to have a strong influence on research in these areas.

A comprehensive historical analysis of Frobenius's first papers on character theory, his correspondence with Dedekind, and other contemporary work in algebra and representation theory, was provided by T. Hawkins [24], [25], [26].

Character Theory and the Structure of Finite Groups: William Burnside (1852–1927)

At about the same time that Frobenius's first papers on character theory appeared, Burnside published his treatise, *Theory of Groups of Finite Order* (1897). After graduating from Cambridge in 1875, Burnside had followed the Cambridge tradition in applied mathematics, with his research in hydrodynamics, until his appointment as Professor of Mathematics at Greenwich, in 1885. His work in group theory began with a paper on automorphic functions in 1892, and continued with research on discontinuous groups, and then finite groups, leading to his book [5].

At first he was not optimistic about the possible applications of representations to finite group theory. In the preface to the first edition of his book, in reply to the question of why he devoted considerable space to permutation groups, while groups of linear transformations were not referred to, he explained, "My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups [i.e., groups of permutations], it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations."

He was aware of Frobenius's work, however, and developed independently his own approach to representations and characters. It is interesting to speculate on how the work of each one influenced the other. They frequently referred to each other's work in their publications, but as far as I know, they never met, or corresponded extensively with each other.

In the preface to the second edition (1911), he stated, ". . . the reason given in the original preface for omitting any account of it no longer holds good. In fact, it is more true to say that for further advances in the abstract theory one must look largely to the representation of a group by linear substitutions." Later (on p. 269, footnote), he described his indebtedness to the work of Frobenius: "The theory of the representation of a group of finite order as a group of linear substitutions was largely, and the allied theory of group characteristics was entirely, originated by Prof. Frobenius." He then listed the papers of Frobenius discussed in the preceding section, and continues, "In this series of memoirs Prof. Frobenius's methods are, to a considerable extent, indirect; and the same is true of two memoirs, 'On the continuous group that is defined by any given group of finite order,' I and II, Proc. L. M. S. Vol. XXIX (1898) in which the author obtained independently the chief results of Prof. Frobenius's earlier memoirs."

Frobenius expressed himself on the matter, in one of his letters to Dedekind, as follows ([25], page 242; see also [29]): "This is the same Herr Burnside who annoyed me several years ago by quickly rediscovering all the theorems I had published on the theory of groups, in the same order and without exception. . . ."

One of Burnside's best-known achievements in group theory is the theorem, proved using character theory, that every finite group G whose order is divisible only by two primes is solvable: $|G| = p^\alpha q^\beta$, for primes p, q, implies that G is solvable. The $p^\alpha q^\beta$-theorem implies, among other things, that the order of a finite, simple, nonabelian group is divisible by at least three different prime numbers. *Simple* means having no nontrivial normal subgroups. Every finite group has a composition series whose factors are simple groups, so that, in a sense, simple groups are the building blocks of all finite groups. Burnside took a great interest in the classification of finite simple groups, a problem that dominated research in finite group theory until its solution in the 1980s.

It was in this connection that he remarked, in note M of the second edition of his book, "There is in some respects a marked difference between groups of even and those of odd order." He went on to discuss the possible existence of nonabelian simple groups of odd order, remarking that he had shown that the number of possible prime factors of a simple group of composite odd order is at least 7. He continued with the statement: "The contrast that these results shew between groups of odd and even order suggests inevitably that nonabelian simple groups of odd order do not exist."

Further progress on this problem was a long time coming. A breakthrough came with M. Suzuki's proof [40] in 1957 that there are no simple groups of composite odd order having the property that the centralizers of all nonidentity elements are abelian. In his proof, he made heavy use of a subtle extension of Frobenius's work on induced characters, called the theory of exceptional characters. The next step was the theorem of Feit, Hall, and Thompson [13], that the same result held for groups with the property that centralizers of nonidentity elements are nilpotent.

The culmination of this line of research came in 1963, with the publication by Walter Feit and John Thompson of what has become known as the odd-order paper [14], containing one theorem: All finite groups of odd order are solvable. Although a purely group-theoretic proof (not using characters) has been found for Burnside's $p^\alpha q^\beta$-theorem, the proof of the odd-order theorem contains an apparently essential component based on character theory. Feit and Thompson's proof of it takes about 250 pages of close reasoning which to this day resists significant simplification, so perhaps the 50-year wait following Burnside's statement of the problem is not so surprising.

New Foundations of Character Theory: Issai Schur (1875–1941)

Issai Schur entered the University of Berlin in 1894, to study mathematics and physics. Among his instructors, he expressed special thanks, in a brief autobiographical note at the end of his dissertation, to Professors Frobenius, Fuchs, Hensel, and Schwarz. The dissertation itself, on the classification of the polynomial representations of the general linear group, was a work of such distinction as to place him at once on an equal footing with his illustrious predecessors in representation theory.

The difficulty of the proofs of the main theorems in Frobenius's approach to character theory has already been mentioned. If all persons wishing to enter the field had to master the intricacies of group determinants, the representation theory of finite groups might well have remained a closed book to all but a few.

Burnside's account of the foundations of the theory made important strides towards greater accessibility. In particular, he was apparently the first to take irreducible representations and complete reducibility as concepts of central importance. A representation T of a finite group G is called *reducible* if it is equivalent to a representation T' of the form

$$T'(g) = \begin{pmatrix} T_1(g) & A(g) \\ 0 & T_2(g) \end{pmatrix} \qquad \text{for all } g \in G,$$

for representations T_1 and T_2 of lower degree. If this does not occur, the representation is said to be irreducible. Using results of Loewy [31] and E. H. Moore [33] on the existence of G-invariant hermitian forms, Maschke [32] had proved that every representation T is completely reducible, that is, T is either irreducible or equivalent to a direct sum of irreducible representations.

It remained to Schur, however, to give a wholly elementary and self-contained exposition of the main facts about representations and characters [36]. His starting point was the result, now called Schur's Lemma, which as he pointed out had also played an important role in Burnside's account of the theory. He stated the result, in two parts, as follows:

I. Let T and T' be irreducible representations of a finite group G, of degrees d and d', respectively. Let P be a constant $d \times d'$ matrix, such that

$$T(g)P = PT'(g), \qquad \text{for all } g \text{ in } G.$$

Then either $P = 0$, or T and T' are equivalent, and P is an invertible $d \times d$ matrix.

II. The only matrices P which commute with all the matrices $T(g)$, $g \in G$, for an irreducible representation T, are scalar multiples of the identity matrix.

As a consequence, he gave short, understandable proofs of the orthogonality properties of the matrix coefficient functions $\{a_{ij}(g)\}$, and for the characters, of irreducible representations $T(g) = (a_{ij}(g))$, $g \in G$. He also gave a new proof of Maschke's theorem on complete reducibility, replacing an appeal to the existence of invariant bilinear forms by a simple, direct argument, in much the same spirit as the standard proof used today. This work, along with what were by all accounts clear and beautifully presented courses of lectures, put the subject within reach, for students and professional mathematicians, without requiring a specialized background.

One of his students, Walter Ledermann, remarking on the popularity of his lectures, recalls attending his algebra course in a lecture theater filled with about 400 students, and sometimes having to use opera glasses to follow the speaker when he was unlucky enough to get a seat in the back [30].

Schur's research opened up two more important lines of investigation. In the first [38], he introduced what are called *projective representations* of a finite group G, that is, homomorphisms τ from G into the projective general linear group $PGL_n(\mathbf{C}) = GL_n(\mathbf{C})/\{\text{scalars}\}$. He analyzed precisely when such a representation τ could be lifted to an ordinary representation T of a suitably defined covering group $\tilde{G}$, so that the diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{T} & GL_d(\mathbf{C}) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\tau} & PGL_d(\mathbf{C}) \end{array}$$

is commutative, and the kernel of the homomorphism from $\tilde{G}$ to G is contained in the center of $\tilde{G}$. He constructed a universal covering group $\tilde{G}$, which can be put in the diagram above (for some choice of T) for all projective representations τ. The methods used to construct $\tilde{G}$ and the kernel of the homomorphism from $\tilde{G}$ to G are the beginnings of a major chapter in group theory, known as the cohomology of groups.

Another theme was Schur's search for arithmetical properties of representations, which brought out connections with algebraic number theory. The central idea is the concept of a splitting field K of a finite group G. This is a subfield K of the complex field $\mathbf{C}$ with the property that each irreducible representation $T : G \rightarrow GL_d(\mathbf{C})$ is equivalent to a K-representation $T' : G \rightarrow GL_d(K)$. A splitting field is minimal if no proper subfield is a splitting field. From the work of Frobenius, it was known that a given finite group G has a splitting field K which is an algebraic number field, that is, a finite extension of the rational field. The splitting field problem was to determine, for a finite group G, the algebraic number fields K which are minimal splitting fields. Splitting fields reflect, in some mysterious way, the structure of the group. For example, splitting fields for cyclic groups require the addition of roots of unity to the rational field, while the field of rational numbers is a splitting field for the symmetric groups S_n.

Both Burnside and Schur were interested in the splitting field problem, and had evidence to support the conjecture that the cyclotomic field of mth roots of unity, where m is the least common multiple of the orders of the elements of G, is always a splitting field. Using a subtle device, known as the Schur index, Schur was able to prove the conjecture for all solvable groups [37].

The Dawn of the Modern Age of Representation Theory: Emmy Noether (1882–1935)

By finding simple algebraic ideas to express the essential structure of a mathematical theory, Emmy Noether reshaped many different parts of 20th-century mathematics. Representation theory of finite groups was no exception: it has never been the same since the publication of her article "Hyperkomplexe Grössen und Darstellungstheorie" (1929) [34]. She presented a basic set of ideas underlying the representation theory of a finite-dimensional algebra over an arbitrary field. In the case of representations of finite groups, the algebra involved was the group algebra KG of the group G over a field K. This is the associative ring whose additive group is the vector space over K with a distinguished basis indexed by the elements of the group G. In order to define multiplication in KG, it is enough to define it for a pair of basis elements, corresponding to elements g and h in G; their product is defined to be the basis element corresponding to the product $g \cdot h$ in G.

Representations of G over the field K may be viewed as homomorphisms $T : G \rightarrow GL(V)$, where V is a finite-dimensional vector space over K and $GL(V)$ is the group of invertible linear transformations on V. The

Emmy Noether.

definition given earlier amounts to choosing a basis in V, and taking note of the resulting isomorphism: $GL(V) \cong GL_d(K)$, where d is the dimension of V over K.

Noether's critical observation was that each representation $T : G \to GL(V)$ defines the structure of a left KG-module on V, with the module operation $a \cdot v$ defined by setting

$$a\cdot v = \sum_{g\in G} \alpha_g T(g)v, \text{ for } v \in V \text{ and } a = \sum_{g\in G} \alpha_g g \in KG.$$

(Here we have identified the element $g \in G$ with the basis element of KG corresponding to it.) Conversely, each finitely-generated left KG-module V defines a representation $T : G \to GL(V)$ by reversing the procedure given above.

It is easily checked that two representations are equivalent if and only if the KG-modules corresponding to them are isomorphic. Thus the main problem of representation theory, which is the classification of the representations of a finite group G up to equivalence, becomes the problem of construction and classification of modules over the group algebra. The problem makes sense for arbitrary fields K. For a field K of characteristic zero, or of prime characteristic p not dividing the order of the group, the left KG-modules are semisimple, that is, direct sums of simple modules, by a version of Maschke's Theorem. This implies that the group algebra KG is semisimple, and the main facts about representations, such as that the number of equivalence classes of irreducible representations in a splitting field is the same as the number of conjugacy classes, become straightforward applications of the Wedderburn structure theorems for semisimple algebras.

For more detailed analysis of the contents of her paper [37], see the article by Jacobson [28].

Richard Brauer (1901–1977) and Modular Representation Theory

The next great surge of activity in the representation theory of finite groups, and one that ties up some of the threads started earlier in this article, centered around the work of Richard Brauer, and its continuation by his students and successors, on modular representation theory. Brauer had been a student in Berlin, and completed his dissertation under Schur's supervision in 1926. His early work on representation theory and the theory of simple algebras, including the invention of what has become known, over his objections, as the Brauer group, firmly established his position in the European mathematical community. When Hitler came to power in 1933, Brauer, Emmy Noether, and many other Jewish university teachers in Germany were dismissed from their positions, and came to the United States. Brauer's major publications on modular representations began to appear soon after his arrival in the United States, and the subject remained a focus of his research throughout the rest of his life.

My interest in representation theory was kindled by lectures given by Brauer that I heard as a graduate student, including all four of his 1948 AMS Colloquium Lectures at the summer meetings in Madison, and a lecture on his solution of Artin's conjecture on L-series with general group characters, at another meeting in New York, when he was awarded the Cole Prize of the American Mathematical Society for this work. I can't say I understood the lectures very well at the time, but they made a strong impression.

A few years later, in 1954, Irving Reiner and I spent a year at the Institute for Advanced Study in Princeton. Neither of us knew much about representation theory, but it seemed to us to be a subject in which exciting things were happening, especially those connected with Brauer's work. We organized an informal seminar devoted to Brauer's work on modular representations and other topics in character theory. This led to our book-writing projects, as a way of learning the subject.

Modular representation theory is the classification of kG-modules, where kG is the group algebra of a finite group G over a field k of characteristic $p > 0$. In case p

divides the order of the group G, the group algebra kG is not semisimple, and the kG-modules are not necessarily direct sums of simple modules, so their classification is much more difficult. Modular representations were first considered by Leonard Eugene Dickson [7], [8], [9]; among other things, he was the first to point out the different nature of the theory in case the order of the group is divisible by the characteristic of the field.

One of Brauer's first results in this subject was the theorem ([1], 1935), that the number of equivalence classes of irreducible representations of a finite group G, in a splitting field of characteristic $p > 0$, is equal to the number of conjugacy classes in G containing elements of order prime to p. If p does not divide the order of G, every conjugacy class has this property, and the result agrees with known properties of representations in the complex field $\mathbf{C}$.

Brauer maintained a steady interest in the relation between properties of the irreducible complex-valued characters and the structure of finite groups. One of his objectives was to use modular representation theory to obtain new information about the values of the irreducible characters in $\mathbf{C}$, and to apply it to problems on the structure of groups. Many of his lectures at meetings and research conferences contained lists of unsolved problems, often involving finite simple groups and properties of their characters.

In order to develop the connection between modular representations and complex-valued characters, he introduced what is now called a p-modular system, consisting of an algebraic number field K which is a splitting field for G, a discrete valuation ring R with quotient field K, maximal ideal P, and residue field $k = R/P$ of characteristic p, for a fixed prime number p. As an application of the character theory he had developed in connection with his prize-winning proof [3] of Artin's conjecture, he had also succeeded in proving the splitting field conjecture of Burnside and Schur [2]. For the cyclotomic field containing the nth roots of unity, where n is the order of G, it follows that K, and the residue field $k = R/P$, are both splitting fields.

The next step explains how modular representations are related to representations in the field K. Each KG-module V defines a representation $T : G \to GL_d(K)$. Since R is a principal ideal domain, it follows that there exists a representation $T' : G \to GL_d(R)$ which is equivalent to T: $T'(g) = XT(g)X^{-1}$, for all $g \in G$. The homomorphism $R \to R/P = k$ can be applied to the entries of the matrices $T'(g), g \in G$; this procedure yields a representation $\bar{T}' : G \to GL_d(k)$ and a kG-module $M = \bar{V}$. The representation $\bar{T}'$ and the module $M = \bar{V}$ are obtained from T and V by what is called *reduction mod P*, so $\bar{T}'$ is a modular representation of G. But there is a difficulty connected with this process. The kG-module $M = \bar{V}$ is not determined up to isomorphism by the isomorphism class of the KG-module V. Nevertheless, in a fundamental joint paper [4], Brauer and his Toronto Ph.D. student Cecil Nesbitt proved that the composition factors of M are uniquely determined.

Using the process of reduction mod P, they defined the *decomposition matrix* D as follows. The rows of D are indexed by the isomorphism classes of simple KG-modules (or, what amounts to the same thing, by the equivalence classes of irreducible representations $T : G \to GL_d(K)$), the columns by the isomorphism classes of simple kG-modules, and an entry d_{ij} of D gives the number of times the jth simple kG-module appears as a composition factor in the module obtained by reduction mod P from the ith simple KG-module. They also introduced the Cartan matrix C, whose entry c_{ij} counts the number of times the jth simple kG-module occurs as composition factor in the ith indecomposable left ideal occurring in a suitably indexed list of indecomposable direct summands of kG. In 1937 they proved the remarkable fact that the Cartan matrix and the decomposition matrix satisfy the relation $C = {}^tDD$, where ${}^tD$ is the transpose of D. This establishes a deep connection between the representation theory of G in the field K of characteristic zero and the representation theory of G in the field k of characteristic p. Its proof used a result of Frobenius [22], which was a refinement of his previous work on the factorization of the group determinant.

The preceding result is only the beginning of Brauer's theory. The refinements of character theory he was seeking came from his theory of p-blocks. These describe a partition of the set of irreducible characters in subsets, called p-blocks, corresponding to the decomposition of the group algebra RG as a direct sum of indecomposable two-sided ideals. To each p-block of irreducible characters, he associated a certain p-subgroup of G, called the defect group of the block. He obtained precise information about the values of the irreducible characters in a given p-block, using the modular representation theory of the defect group and its normalizer. This work, in turn, led to applications of the theory of p-blocks of characters by Brauer, Suzuki, and others to important early steps in the classification of finite simple groups (see [12], Chapters VIII and XII).

The belief that representation theory of finite groups had a bright future was shared by Frobenius, Burnside, Schur, Noether, and Brauer. The high level of current research activity in the subject and its connections with other parts of mathematics seem to support their judgment.

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