MATH 10
ASSIGNMENT 18: PERMUTATIONS

APRIL 10, 2016

A **permutation** of some set S is a function $f\colon S \to S$ which is a bijection (one-to-one and onto, or invertible). We will only be discussing permutations of finite sets, usually the set $S = \{1, \dots, n\}$. In this case one can also think of a permutation as a way of permuting n items placed in boxes labeled $1, \dots, n$: namely, move item from box 1 to box $f(1)$, item from box 2 to $f(2)$, etc. The set of all permutations of $\{1, \dots, n\}$ is denoted by S_n.

Permutations can be composed in the usual way: $f \circ g(x) = f(g(x))$.

Notation: the permutation f which sends 1 to a_1, 2 to a_2, etc, is usually written as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

An alternative way of writing permutations is using cycles. A **cycle** $(a_1 a_2 \dots a_k)$ is a permutation which sends a_1 to a_2, a_2 to a_3, …, a_n to a_1 (and leaves all other elements unchanged). For example, (123) is the permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(a) = a$ for all other a. The same cycle can also be written as (231).

We can also consider products (i.e. compositions) of several cycles. For example, (123)(45) is a permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f(4) = 5$, $f(5) = 4$. It is also customary not to write cycles of length one: instead of writing (123)(4), we write just (123).

1. How many permutations of the set $\{1, \dots, n\}$ are there?

2. Compute the following compositions (a) $(12) \circ (13)$ (b) $(12) \circ (23)$ (c) $(23) \circ (12)$ (d) $(12) \circ (13) \circ (12)$ (e) $(123) \circ (132)$ (f) $(38) \circ (123456) \circ (38)$

3. Find the inverse of permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 6 & 4 & 2 & 5 \end{pmatrix}$$

Write this permutation as a product of cycles.

4. Show that any permutation can be written as a product of non-intersecting cycles.

5. Fifteen students are meeting in a classroom which has 15 chairs numbered 1 through 15. The teacher requires that every minute they change seats following this rule:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	5	10	8	11	14	15	6	13	1	4	9	7	2	12

(e.g., the student who was sitting in the chair number 1 would move to chair number 3).

(a) Write this permutation as product of cycles.
(b) In how many minutes will the students return to their original seats?

6. An order of a permutation f is the smallest number d such that $f^d = id$, where id is the identity permutation: $id(a) = a$.

(a) Find the order of a cycle of length n
(b) Find the order of a permutation (12)(34795)(6 10 11 12 13 14 15)
(c) Let a permutation f be a product of non-intersecting cycles of lengths $n_1, n_2, \dots, n_l$ (in this case, we will say that it has the **type** $\langle n_1, n_2, \dots n_l \rangle$). What is the order of f?
(d) Find permutations of the set $\{1, \dots, 9\}$ which have orders 7, 10, 12, 11 (if they exist).

7. Show that any permutation can be written as a product of transpositions (i.e., a permutation that interchanges two elements leaving all other unchanged — same as a cycle of length 2). Do that for the permutation in problem5.