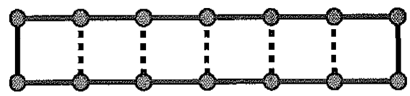

Figure 2. Boundary edges are solid; interior edges are dashed.

win by roughly $n/6$. The essence of the strategy is to avoid any parity switching of who leads the game, which we show is possible, unlike general boards. In Sections 3 and 4, we give preliminary results for the $2 \times n$ game and for the Swedish $1 \times n$ game (where the boundary is initially drawn).

Terminology. See Figure 2. A *boundary* edge is an edge of the bounding rectangle. An *interior* edge is any nonboundary edge.

2. Misère 1 x *n*

In a $1 \times n$ board, there are $n - 1$ interior edges; the remaining $2n + 2$ edges are boundary. We distinguish the leftmost and rightmost boxes as *end* boxes.

Theorem 1. *For all* $n \geq 1$, *misère* $1 \times n$ *Dots-and-Boxes is a first-player win.*

Proof. First we describe Player 1's strategy, which divides the game into two phases. In Phase I, some interior edges remain untaken, and Player 1 always takes such an edge. The initial choice of interior edge is any not incident to an end box, if there is one, and otherwise an arbitrary interior edge. We ignore any boxes that Player 2 takes, and instead focus on the last edge played. If Player 2 takes an interior edge, Player 1 takes another arbitrary interior edge. If Player 2 takes a boundary edge, Player 1 takes one of the two incident interior edges, if one of them is untaken, and otherwise an arbitrary interior edge. This rule may cause Player 1 to take a box, in which case Player 1 takes another, arbitrary interior edge (if any exist). In Phase II, when all interior edges are depleted, Player 1 takes any boundary edge that does not complete a box; we will show that such an edge always exists.

We show that no edge can ever complete two boxes simultaneously. During Phase I, Player 1 goes first and takes only interior edges, so the number of taken interior edges is always at least the number of taken boundary edges. Further we claim that, within each nonboundary box except possibly the one in which Player 2 just played, the number of taken interior edges is always at least the number of taken boundary edges. The claim is trivially true before either player has played in the box. If Player 1 plays in the box, the claim certainly remains true. Whenever Player 2 plays in the box, Player 1's next move will be to play in the box, unless both interior edges have already been taken; in either case, the claim remains true. For boundary boxes, the number of boundary edges can exceed the number of interior edges, but only when all (zero or one) interior edges have

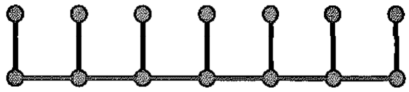

Figure 3. A spanning tree of a $1 \times n$ grid has $2n + 1$ edges.

been taken. Thus, at any time, no box except possibly the one in which Player 2 just played could be completed by an interior edge; all other boxes can be completed only by boundary edges, each of which is incident to only one box. Therefore at no time can any edge complete two boxes simultaneously.

Next we prove that Player 1 completes no boxes during Phase II. By definition, Player 1 will take a box during Phase II only if every box is either completed or one edge from being completed. At such a time, the taken edges must include a spanning tree of the grid, which consists of $2n + 1$ edges (see Figure 3), plus exactly one edge for each completed box (because the cycle formed by each completed box can be broken by a single edge removal). Because we proved that each edge completed at most one box, the number of complete turns must be the number of taken edges minus the number of completed boxes. Thus the number of complete turns must be $2n + 1$, meaning that it is Player 2's turn. Therefore Player 1 completes no boxes during Phase II.

We claim that Player 1 never completes an end box. An end box has at most one interior edge, so there is only one possible move by Player 1 that could complete the box in Phase I. But when Player 2 plays the top or bottom edge of the box, Player 1 will take the interior edge, before Player 2 could have played the opposite (bottom or top) edge of the box. Therefore this move by Player 1 did not complete the box.

If Player 1 plays first in a nonend box of the board, then we claim that Player 1 will not complete this box; refer to Figure 4. If Player 1 also plays second in this box, then the claim is obvious: Player 1 will play in the box only if it does not complete the box. If Player 2 plays second in the box, then by definition Player 1 will immediately take the remaining interior edge of the box. As this is only the third move in the box, this move does not complete the box. Player 1 will not play the final boundary edge of the box because that would complete the box.

Finally we show that Player 1 completes at most $\lfloor (n-1)/3 \rfloor$ boxes. As argued above, for Player 1 to complete a box, it must not be an end box and Player 2 must play in it first. Indeed, Player 2 must play in that box again, taking the other boundary edge, or else we would have already entered Phase II. Thus, every box taken by Player 1 can be charged to two moves by Player 2, as well as the two following interior edges taken by Player 1. Furthermore, the completed box means that Player 1 also takes another interior edge (if there is one). Thus every box completed by Player 1 corresponds to an increase in the number of taken interior by at least 3. Therefore Player 1 completes at most $(n-1)/3$ boxes. $\square$