The Classification of the Finite Simple Groups

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The classification of the finite simple groups was completed sometime during the summer of 1980. To the extent that I can reconstruct things, the last piece in the puzzle was filled in by Ronald Solomon of Ohio State University. At the other chronological extreme, the theory of finite groups can be traced back to its beginnings in the early nineteenth century in the work of Abel, Cauchy, and Galois. Hence the problem of classifying the finite simple groups has a history of over a century and a half. The proof of the Classification Theorem is made up of thousands of pages in various mathematical journals with at least another thousand pages still left to appear in print. Many mathematicians have contributed to the proof; some have spent their entire mathematical lives working on the problem.

The problem itself is one of the most natural in mathematics: the group is one of the fundamental structures of modern mathematics; the finite groups are a natural subclass of the class of all groups. Moreover, the finite group theorist is quickly led to consider simple groups via the composition series of a group, and if he is optimistic, to the hope that the finite simple groups might be determined explicitly and much of the structure of the arbitrary finite group retrieved from that of its composition factors.

Despite all of this, and despite the fact that most mathematicians learn this much group theory before receiving their Ph.D., the average mathematician does not seem to known much about the classification problem or the mathematics developed to solve it. Within the obvious space limitations of this article, I hope to convey some idea of how the finite simple groups are classified and to relate some of the history of the effort. A more complete description appears in [6], while a very detailed two volume account (by Daniel Gorenstein) is in preparation. A preliminary version of the first quarter of Gorenstein's work appears in [19]. The proceedings of two recent conferences on simple groups containing expository articles on the classification will soon appear in [12] and [13]. Finally an article by Walter Feit on the history of finite group theory through 1961 will appear in [14].

I have included a reasonably lengthy bibliography. Still, many important papers are omitted as they are not directly encountered in the brief outline provided. Other fundamental papers have yet to appear. More complete bibliographies are contained in some of the books mentioned above.

Section 1. The Finite Simple Groups

What exactly is meant by a "classification of the finite simple groups? " To me, such a classification consists of an explicit list of simple groups together with a proof that every finite simple groups is isomorphic to some member of the list. Moreover if the result is to be of use, our knowledge of the groups on this list must be good enough to answer reasonable questions about simple groups.

It seems therefore that the first order of business is to become acquainted with the list of simple groups. Written concisely, the list is a follows:

Groups of prime order
Alternating groups
Finite groups of Lie type
Sporadic groups

The groups of prime order are the abelian simple groups. The alternating group of degree n is the group of all even permutations of a set of order n, and is simple if $n \geqslant 5$.

The finite groups of Lie type are the finite analogues of the semisimple Lie groups. In the mid fifties Chevalley [11] showed that given a pair (X_n, K) consisting of a complex simple Lie algebra X_n and a field K, there exists a *Chevalley group* $X_n(K)$, which is a matrix group over K with various properties. For example, $X_n(C)$ is a semisimple Lie group; if K is algebraically closed then $X_n(K)$ is a semisimple algebraic group; if $K = GF(q)$ is the finite field of order q, then $X_n(K) = X_n(q)$ is a finite group which is essentially simple.

Shortly thereafter Steinberg [29] and Ree [24, 25] showed that the fixed points of certain outer automorphisms of the finite Chevalley groups give rise to still more finite simple groups, which have come to be called *twisted Chevalley groups*. The finite simple Chevalley groups of ordinary and twisted type are the finite simple groups of Lie type. They are listed in Table 1. Some of these groups can be represented as the group of projective transformations of determinant one of a suitable finite dimensional vector space, possibly endowed with a form. In that event, Artin's notation for the group as a linear group is also included.

Each group in the first three classes of groups is a member of one or more infinite families of finite simple groups. There are however 26 finite simple groups which are not members of any infinite family. These groups are therefore termed *sporadic.* The sporadic groups are listed in Table 2.

In the case of the sporadic groups we have available (or will have available in the near future) the group character table and more or less precise information about the subgroup structure, the automorphism group, and the coverings of the group. This information together with information about representations in prime characteristic is sufficient to answer most reasonable group theoretic questions involving the group. The corresponding situation for the infinite families is not so nice, but not so bad either. Questions for the groups of prime order are easy. The determination of the subgroup structure and representation theory of the alternating groups and finite groups of Lie type will probably be the focus of interest in finite group theory in the near future. These fields have been very active for years and already much information is available.

Table 1. The simple groups of Lie type

Lie notation	Classical notation
$A_n(q), n \geqslant 1$	$L_{n+1}(q)$
$B_n(q), n \geqslant 3$	$P\Omega_{2n+1}(q)$
$C_n(q), n \geqslant 2$	$PSp_{2n}(q)$
$D_n(q), n \geqslant 4$	$P\Omega^+_{2n}(q)$
$E_n(q), 6 \leqslant n \leqslant 8$	
$F_4(q)$	
$G_2(q)$	
${}^2A_n(q), n \geqslant 2$	$U_{n+1}(q)$
${}^2D_n(q), n \geqslant 4$	$P\Omega^-_{2n}(q)$
${}^3D_4(q)$	
${}^2E_6(q)$	
${}^2B_2(2^{2m+1}), m \geqslant 1$	
${}^2F_4(2^{2m+1}), m \geqslant 0$	
${}^2G_2(2^{2m+1}), m \geqslant 1$	

It is now possible to state the Classification Theorem precisely:

Classification Theorem. Every finite simple group is isomorphic to a group of prime order, an alternating group, one of the groups of Lie type listed in Table 1, or one of the sporadic groups listed in Table 2.

For the moment there is some ambiguity involving the sporadic group called the Monster[1]. A finite group G is said to be of *type* F_1 if G has an involution (i.e. an element of order 2) whose centralizer in G is perfect with

[1] That ambiguity has been removed; Simon Norton of Cambridge University has recently completed the proof of the uniqueness of groups of type F_1

Table 2. The sporadic simple groups

Notation	Name	Order
M_{11}	Mathieu	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}		$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}		$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}		$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}		$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1	Janko	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2 = HJ	Hall-Janko	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3 = HJM	Higman-Janko-McKay	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	Janko	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	Higman-Sims	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Mc	McLaughlin	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Sz	Suzuki	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Ly = LyS	Lyons-Sims	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
He = HHM	Held-Higman-McKay	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ru	Rudvalis	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
O'N = O'NS	O'Nan-Sims	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
$Co_3 = \cdot 3$	Conway	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
$Co_2 = \cdot 2$		$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
$Co_1 = \cdot 1$		$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
$M(22) = F_{22}$	Fischer	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
$M(23) = F_{23}$		$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$M(24)' = F_{24}$		$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$F_3 = E$	Thompson	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
$F_5 = D$	Harada	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
$F_2 = B$	Baby Monster	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$F_1 = M$	Monster	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

composition factors F_2 and Z_2, where F_2 denotes the Baby Monster group and Z_2 the cyclic group of order 2. Many facts are known about groups of type F_1; for example such a group has the order given in Table 2 and *if* the group possesses an irreducible complex representation of degree 196, 843, then its isomorphism class is determined and its character table calculated. Recently Robert Griess constructed a group of type F_1 as an automorphism group of a 196, 843 dimensional algebra. However it is not yet known that a group of type F_1 has a 196, 843 dimensional representation, so that it is in theory conceivable that there are several groups of type F_1. On the other hand the uniqueness of groups of type F_1 will follow once the character table of the Baby Monster is completed, and that calculation is nearly done.

A similar uniqueness problem involving the groups ${}^2G_2(3^n)$ was just cleared up during the last year by Bombieri [9], building upon earlier work of Thompson. Similarly the question of the existence and uniqueness of the sporadic group J_4 was settled this year by several mathematicians at Cambridge. Thus for all the simple groups listed, other than possibly the Monster, there exists exactly one group of each type.

Section 2. The Local Theory of Finite Groups

Having stated the Classification Theorem, it is time to examine its proof. That requires an introduction to the local theory of finite groups and a formidable number of terms. To soften the blow, most of this terminology is collected in a local group theoretic dictionary. When you encounter an unfamilar term, look for its definition in the dictionary!

What is the local theory of groups? The *p-local-subgroups* of a group G are the normalizers $N_G(X)$ of nontrivial p-subgroups X of G. The *local subgroups* of a group are its various p-local subgroups as p varies over the prime divisors of its order. The *local theory* of finite groups is the study of finite groups from the point of view of the local subgroups. The first theorem of local group theory is Sylow's Theorem.

Certain local subgroups merit more attention than others. For example the normalizers of subgroups of prime order and the 2-local subgroups are of particular importance. When we intersect these collections we obtain the centralizers of involutions, which receive the most attention.

In very general terms, the program for classifying the finite simple groups consists of first demonstrating that each finite simple group possesses some subgroup of prime order whose normalizer has one of several rather specific kinds of structure, and then proving that any finite simple

A Dictionary of Local Group Theoretic Terms

normalizer in G of X: The largest subgroup $N_G(X)$ of a group G in which the subgroup X is normal.

centralizer in G of X: The subgroup $C_G(X)$ of all elements of a group G which commute with the subset X of G.

quasisimple group: A group which is its own commutator subgroup and is simple modulo its center.

subnormal subgroup: A subgroup H of a group G for which there exists a series $G_0, \ldots G_n$ of subgroups of G with $H = G_0$, $G = G_n$, and G_i normal in G_{i+1} for each i.

component: A subnormal quasisimple subgroup of a group.

p-standard subgroup: A quasisimple subgroup L of a group G which is a component of the normalizer of some subgroup of G of prime order p, such that $C_G(L)$ is "small" in some suitable sense. For example, usually we require that $C_G(L)$ has cyclic Sylow p-subgroups, although when $p = 2$ a somewhat weaker hypothesis seems more appropriate.

$O_\pi(G)$: The largest normal subgroup of G whose order is divisible only by primes in the set π of primes. If p is prime, $O_p(G) = 0_{\{p\}}(G)$ and p' denotes the set of primes distinct from p.

component type: A group G is of component type if it possesses an involution t such that $C_G(t)/O_{2'}(C_G(t))$ possesses a component.

characteristic 2-type: A group G is of characteristic 2-type if $C_H(O_2(H)) \leqslant O_2(H)$ for each 2-local H of G.

elementary abelian p-group: A group isomorphic to the direct product of groups of order p.

p-rank: The p-rank of an elementary abelian p-group is its dimension regarded as a vector space over $GF(p)$. The p-rank $m_p(G)$ of a general group G is the maximum rank of an elementary abelian p-subgroup of G.

$e(G)$: The maximum of $m_p(M)$ as p varies over all odd primes and M varies over all 2-locals of G.

$\Gamma(G, k, p)$: The undirected graph whose vertices are the elementary abelian p-subgroups of G of rank at least k, with vertices joined when they commute elementwise.

disconnected group: G is disconnected when $\Gamma(G, 2, 2)$ is a disconnected graph.

strongly p-embedded subgroup: H is a strongly p-embedded subgroup of G if $\Gamma(G, 1, p)$ is disconnected and H is a proper subgroup of G containing the normalizer of a connected component of $\Gamma(G, 1, p)$.

group with such a normalizer is isomorphic to a group on our list. In the most important of these structures, the normalizer contains a so called p-standard subgroup of the simple group, and that standard subgroup is quasisimple. Given a quasisimple group L, the *p-standard form problem for L* is to determine, up to isomorphism, all groups G with a p-standard subgroup isomorphic to L. The existence of a standard subgroup forces G to be nearly simple, and that tells us we are on the right track.

As a first attempt at a proof of the Classification Theorem, we might try to show that every finite simple group possesses a standard subgroup. If we were careful, we know what this subgroup looks like. For let $\mathscr{K}$ denote the class of groups on our list and define a $\mathscr{K}$-group to be a group all of whose composition factors are in $\mathscr{K}$. Proceeding by induction we need only consider simple groups all of whose proper subgroups are $\mathscr{K}$-groups. In particular its standard subgroups are $\mathscr{K}$-groups and to complete the proof of the Classification Theorem we need only solve the standard form problems for all quasisimple $\mathscr{K}$-groups.

As some simple groups possess no standard subgroups, some modification of the approach just described is necessary. Also, in the process of establishing the existence of standard subgroups, extra information is generated which makes it unnecessary to solve *all* standard form problems. But modulo such reservations, the approach above is essentially the one which has been utilized.

The prime 2 plays a special role in simple group theory. Therefore we are led to a division of the class of simple groups into those which possess 2-standard subgroups and those which do not. More precisely we divide the class of simple groups into the groups of component type and the groups of characteristic 2-type. Groups in the first class usually contain a 2-standard subgroup while those in the second do not.

There is another major subdivision: division according to size. The best measure of the size of a simple group is not its order but the p-rank of various subgroups of G for various primes p. The small groups of component type are essentially those of small 2-rank. More precisely, a group of component type is small if it is disconnected. In particular, simple groups of 2-rank at most 2 are disconnected. For small groups of characteristic 2-type a different definition is more appropriate. A group G of characteristic 2-type is small if $e(G) \leqslant 2$. Most of the groups of characteristic 2-type are of Lie type and even characteristic. In such a group, $e(G)$ is a good approximation of the Lie rank of G.

We now have a subdivision of the finite simple groups into four subclasses: the large and small groups of component type and the large and small groups of characteristic 2-type. This leads to five major problems: the classification of the groups in each of the four subclasses plus the problem of showing that the classes are exhaustive. For example the small groups of component type are the disconnected groups of component type. To classify such groups it is sufficient to solve

Problem I. Determine all disconnected groups.

Then to complete the classification of the groups of component type we are left with

Problem II. Determine all connected simple groups of component type.

The small groups G of characteristic 2-type are those with $e(G) \leqslant 2$. Thus we have

Problem III. Determine all groups G of characteristic 2-type with $e(G) \leqslant 2$.

Problem IV. Determine all groups G of characteristic 2-type with $e(G) \geqslant 3$.

Finally it remains to solve

Problem V. Prove that each connected simple group is either of component type or of characteristic 2-type.

This is the basic outline of the classification. In the next two sections we put a little flesh on the bones.

Section 4. The Proof and History of the Classification Theorem; The Fifties and Sixties

I will begin my discussion of the history of the classification in the mid fifties; Feit's article [14] contains a nice treatment of earlier work on finite groups. Most of the work in the fifties and sixties was directed at the small groups, particularly the disconnected groups, although those working in the area presumably did not think in those terms at that time.

In 1954 at the International Congress of Mathematicians in Amsterdam, Richard Brauer proposed that finite simple groups of even order might be classified in terms of the centralizers of their involutions. Earlier in [10], Brauer and Fowler had shown that given a group H there exist at most a finite number of finite simple groups possessing an involution whose centralizer is isomorphic to H. Their argument depends on the fact that the subgroup generated by a pair of involutions is a dihedral group. That fact is also one of the principal reasons why the prime 2 plays a special role in simple group theory.

In the late nineteenth century Burnside had conjectured that all simple groups of odd order are abelian. If Burnside's conjecture were true, then all nonabelian simple groups would contain involutions, and hence be covered by Brauer's program. In the early sixties, building on earlier work of Suzuki, Feit and Thompson established Burnside's conjecture in their famous Odd Order Paper [15]. Aside from the significance of the theorem itself, the Odd Order Paper introduced many important techniques to local group theory. Later Alperin, Brauer, Gorenstein, Walter, and others determined all simple groups of 2-rank at most 2 in series of papers including [1] and [21].

In the late sixties, Helmut Bender classified all groups with a strongly 2-embedded subgroup [8]. His result depends upon earlier work of Suzuki and an ingenious counting argument of Feit. Feit's argument depends once again on the fact that each pair of involutions generates a dihedral group, so there seems to be no hope of extending Bender's proof to odd primes.

A section of a group G is a factor group H/K where $K \lhd H$ and H is a subgroup of G. The *sectional 2-rank* of G is the maximum 2-rank of its sections. In 1970 in [23], Ann McWilliams showed that disconnected 2-groups are of sectional 2-rank at most 4. A year or two later Gorenstein and Koichiro Harada determined all groups of sectional 2-rank at most 4 [20], and hence all groups with a disconnected Sylow 2-group. This work depends upon the earlier classification of simple groups of 2-rank 2. Finally, building upon Bender's theorem, in 1972 Aschbacher [2] determined all disconnected groups with a connected Sylow 2-group, completing the solution of Problem I.

In the late sixties, inspired by ideas in the Odd Order Paper, Gorenstein introduced the concept of the signalizer functor. Later David Goldschmidt improved the concept and proved a special case of the signalizer functor theorem [18]. Other special cases were established by George Glauberman [17] and Patrick McBride. I have omitted the definition of signalizer functors and the statement of the signalizer functor theorem; suffice it to say that the concept is a fundamental tool for studying $O_{p'}(H)$ for p-locals H of a group.

Around 1970, Bender, Gorenstein, and Walter introduced the concept of the component and established various results about components. In particular Problem V is an easy consequence of the elementary theory of components and the signalizer functor theorem, as Gorenstein and Walter show in [22].

Perhaps the most important single paper in finite simple group theory is John Thompson's N-group Paper. That paper appeared in six parts from 1968–1974 [30]. In it, Thompson classifies the groups in which all local subgroups are solvable: the N-groups. While the N-group paper is not part of the classification program per se, it introduced many important techniques to local group theory, and to a large extent supplied a model for the classification of the groups of characteristic 2-type. For example the parameter $e(G)$ and the subdivison of groups G of characteristic 2-type into those with $e(G) \leqslant 2$ and $e(G) \leqslant 3$ first appear there.

Section 5. The Classification in the Seventies

By 1972 most of the tools and concepts needed to classify the simple groups were available. In that year in a series of talks at the University of Chicago, Gorenstein advanced an outline of a program for classifying the finite simple groups. While that outline has not been followed in all details, it is a very good first approximation of the final shape of the proof. To my knowledge these talks constitute the first attempt to think seriously about systematically classifying the finite simple groups. Gorenstein's talks are reproduced at the end of [19].

In 1973 in [3], Aschbacher proved the Component Theorem. Supplemented by a result of Foote [16], the Component Theorem shows that, with known exceptions, finite simple groups satisfying the B_2-conjecture contain 2-standard subgroups. (See the box below for a discussion of the B_p-conjecture and balanced groups.) The paper also introduced the concept of tightly embedded subgroups and a notion of "small size" for the centralizer of 2-standard subgroups. A subgroup K of G is *tightly* embedded in G if K is of even order but intersects its distinct conjugates in subgroups of odd order. Then a quasisimple subgroup L is 2-standard in G if $C_G(L)$ is tightly embedded in G. Later Aschbacher and Seitz [7] determined the groups with a 2-standard subgroup whose centralizer does not have cyclic Sylow 2-groups, hence reducing to the usual notion of standard.

The B_p-Conjecture and Balanced Groups

The following conjecture describes a property of finite groups which is of fundamental importance to finite simple group theory:

B_p-conjecture. Let p be a prime and G a group with $O_{p'}(G) = 1$. Then for each element x of order p in G and each component $K/O_{p'}(C_G(x))$ of $C_G(x)/O_{p'}(C_G(x))$, there exists a component L of $C_G(x)$ with $K = LO_{p'}(C_G(x))$.

While the B_p-conjecture follows as a corollary to the classification, the majority of the proof of the Classification Theorem is devoted to establishing weak versions of the conjecture.

The concept of balance is related to the B_2-conjecture. A group G is *balanced* if $O_{2'}(C_G(t)) \leqslant O_{2'}(G)$ for each involution t of G. Indeed notice that balanced groups satisfy the B_2-conjecture.

Thompson then formulated a program for verifying the B_2-conjecture, which had as one step the classification of groups possessing a tightly embedded subgroup with quaternion Sylow 2-groups. Almost all groups of Lie type and odd characteristic have such a subgroup. In 1974 Aschbacher completed this step of the program; the proof eventually appeared in [4]. Although Thompson's program was never implemented, it led to a discussion among Aschbacher, Thompson and Walter at a meeting in Sapporo in 1974 from which there emerged an approach for determining all unbalanced groups, based on [4] and earlier work of Gorenstein and Walter. Balanced groups satisfy the B_2-conjecture, so if the unbalanced groups could be classified and shown to satisfy the B_2-conjecture, that conjecture would be established. The approach depended in turn on the solution of various 2-standard form problems. Moreover, given the B_2-conjecture and the Component Theorem, the only obstacle to the solution of Problem II was the solution of the 2-standard form problems for $\mathscr{K}$-groups. Walter and Morton Harris have each produced proofs of the B_2-conjecture based on this approach. Numerous group theorists have contributed to the solution of the 2-standard form problems, including R. Solomon [28] and Seitz [26]. The last such problem was completed by Solomon in the summer of 1980; to my knowledge the solution to this standard form problem constitutes the last step in the classification of the simple groups.

The discussion above outlines the solution to Problem II, the major steps being the verification of the B_2-conjecture, the proof of the Component Theorem, and the solution of the 2-standard form problems. The general shape of this solution was clear in 1975, although the details took five more years to complete. However in 1975 the shape of the solutions to Problems III and IV was much less clear.

As indicated earlier, most of the basic outline of the classification of the groups of characteristic 2-type can be retrieved from the N-group Paper. There is however one critical omission: Standard subgroups never appear in the N-group paper. For if L is a p-standard subgroup of G and X a subgroup of order p centralizing L, then since L is not solvable, neither is $N_G(X)$. Hence N-groups do not possess standard subgroups, and we will never be led to consider such objects by classifying the N-groups. Rather it was Gorenstein in his 1972 lectures who suggested that in studying a group G of characteristic 2-type, the methods of the N-group paper could be used first to establish the B_p-conjecture for primes p for which the p-rank of some 2-local is large, and then further to produce p-standard subgroups.

We are led to the following theorem:

Theorem. Let G be a simple $\mathscr{K}$-group of characteristic 2-type with $e(G) \geqslant 3$. Then (essentially) one of the following holds:

1) There is a maximal 2-local subgroup which has no noncyclic normal abelian subgroups.
2) There exists a p-standard subgroup of G which is of Lie type and even characteristic.
3) For each odd prime p such that the p-rank of some 2-local is at least min $\{4, e(G)\}$, G has a strongly p-embedded 2-local subgroup.

The theorem is established by Aschbacher, Gorenstein, and Lyons in a series of papers which have not yet appeared. The standard form problems are solved by Aschbacher, Finkelstein, Frohardt, Gilman, and Griess, again in papers yet to appear. The groups in the first case have been classified in work of various authors, most notably Timmesfeld [31] and S. Smith [27]. Many of the sporadic groups arise in this case.

The third case of the Theorem is called the Uniqueness Case. The solution of the Uniqueness Case and of Problem III can be thought of as the analogue of Problem I for groups of characteristic 2-type. Their solution follows the general outline established in the N-group Paper. The next test case for the method was [5], in which many of the techniques for solving the problem first appeared, and which classifies the groups G with $e(G) = 1$. Aschbacher showed that the Uniqueness Case leads to a contradiction and Geoffrey Mason solved Problem III; neither paper has yet appeared. The mathematics used to solve these last two problems is as difficult as any in finite group theory.

Section 6. Concluding Remarks

This completes the outline of the proof of the Classification Theorem. Much of the mathematics involved has been produced only recently and will no doubt be improved once there is time for the techniques to sink in. Still it is hard to imagine a short proof of the theorem which follows the outline just described. I personally am skeptical that a short proof of any kind will ever appear.

Long proofs disturb many mathematicians. For one thing, as the length of a proof increases, so does the possibility for error. The probability of an error in the proof of the Classification Theorem is virtually 1. On the other hand the probability that any single error cannot be easily corrected is virtually zero, and as the proof is finite, the probability that the theorem is incorrect is close to zero. As time passes and we have an opportunity to assimilate the proof, that confidence level can only increase.

It is also perhaps time to consider the possibility that there are some natural, fundamental theorems which can be stated concisely, but do not admit a short, simple proof. I suspect that the Classification Theorem is such a result. As our mathematics becomes more sophisticated, we may encounter such theorems more frequently.

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