A historical approach to understanding four-dimensional polytopes based on Boole Stott's work

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Abstract

At the end of the 19th century the amateur mathematician Alicia Boole Stott developed a method to calculate the three-dimensional sections of the four-dimensional regular polytopes. Her work on the topic follows a natural algorithmic structure and it is presented in this text making use of her original drawings and models.

Keywords: Regular polytopes, Three-dimensional sections, Boole Stott, Fourth dimension, Coxeter.

1 Introduction

With this text we would like to present the work of the mathematician Alicia Boole Stott, a woman born in the middle of the 19th century who worked on four-dimensional geometry. Although her contribution to this field was important and genuine in many aspects, her work has remained almost unnoticed. As far as we know, no detailed description of her work can be found in modern texts. We intend to recover what we find a very original and inspiring approach to the understanding of four-dimensional objects. In order to do that, the method she used to compute the sections of regular four-dimensional polytopes will be explained. Her results on this

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topic are collected in her publication of 1900 *On certain series of sections of the regular four-dimensional hypersolids* (see [1]). Together with her results she presented a collection of drawings and cardboard models of the three-dimensional sections in order to ease their visualization (see Figure 2). One of the most remarkable facts about Boole Stott's method is the idea to transform a four-dimensional problem into a three-dimensional one, making it possible to use our intuition on the three-dimensional space to solve a four-dimensional problem. We intend to display her work in the simplest possible way so it becomes accessible for a very general public.

We would like to emphasize that, as a woman born in the 19th century, Boole Stott had hardly any educational opportunity and did never receive any formal mathematical training. Her discoveries came from an extraordinary capacity to visualize the fourth dimension. Rigorous mathematical proofs can therefore not be expected in her work, but instead, a waterfall of surprising and original ideas.

In the next section we will briefly describe the most important aspects of Boole Stott's life. A more detailed bibliography can be found in [11].

2 Boole Stott's live

Alicia Boole Stott was born in Castle Road, near Cork (Ireland) on June 8th, 1860 [8]. She was the third daughter to the today famous logician George Boole (1815-1864) and Mary Everest (1832-1916). George Boole died from fever at the age of 49. George's widow Mary and five daughters were left with very little money, so Mrs Boole was forced to move to London, taking Alicia's four sisters with her. Alicia had to stay at Cork with her grandmother Everest and an uncle of her mother [8]. At the age of eleven, she moved to London to live with her mother and sisters for seven years.

In England colleges did not offer degrees to women, and women could only aspire to study some classical literature and other arts, and hardly any science [9]. In Alicia's case, her knowledge of science consisted only of the first two books of Euclid [6].

She was only four years old when her father died, so she could not have received much mathematical influence from him. However, she certainly received a good tuition from her mother. Mary Everest Boole had studied with her husband, George Boole. When George Boole died, Everest Boole moved to England and was offered a job at Queen's College in London as a librarian. Her passion however was teaching, and she liked giving advice to the students [9]. She had innovating ideas about education, believing

Figure 1: Alicia Boole Stott. Undated. (Courtesy of the University of Bristol)

for example that children should manipulate things in order to make the unconscious understanding of mathematical ideas grow [9].

Apart from the education provided by her mother, Alicia was also strongly influenced by the amateur mathematician Howard Hinton, whom she met during her London period. Hinton was a school teacher, and was very interested in four-dimensional geometry.

Inspired by Howard Hinton, Boole Stott undertook a study of four-dimensional geometry between 1880 and 1890. She worked as an amateur, without any scientific education or scientific contacts. Probably unaware of the existence of the six regular polytopes in the fourth dimension, she succeeded in finding them by herself again [8] (see Section 4). Five of these polytopes are the four-dimensional analogues of the five regular polyhedra, namely the hypercube, hyperoctahedron, hypertetrahedron, 120-cell and 600-cell. The extra one is called the 24-cell and has no three-dimensional analogue (see Sections 3 and 5).

Boole Stott also calculated series of sections of all six three-dimensional regular polytopes, building them in beautiful cardboard models. These sections consist of a set of increasing semiregular polyhedra, that vary in shape and colour. Figure 2 shows a picture of a showcase kept at the Groningen University Museum containing her models.

Figure 2: Models of perpendicular sections of the 600-cell. (Courtesy of the University Museum of Groningen.)

2.1 Boole Stott and the Netherlands

In 1894, Schoute ([17], [18] and [19]) described by analytical methods the three-dimensional central sections of the four-dimensional polytopes. According to Coxeter [6], Boole Stott got to know about Schoute's publications from her husband. She realized that Schoute's drawings of the sections were identical to her cardboard models and sent photographs of the models to Schoute. He was very surprised and immediately answered asking to meet her and proposing a collaboration [8]. Schoute came to England during some of his summer holidays to stay with Boole Stott at her maternal cousin's house in Hever [6].

Boole Stott and Schoute worked together for almost twenty years, combining Boole Stott's ability for visualizing four-dimensional geometry with Schoute's analytical method [8]. Schoute persuaded her to publish her results. Her main publications are [2] and [3] published in the journal of the Dutch Academy of Science *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam*. In [3] Boole Stott gives a method to deduce most semiregular polytopes from regular ones (see [12]). Boole Stott's work in also briefly discussed in [14, 15].

2.2 Boole Stott and Coxeter

Boole Stott resumed her mathematical work in 1930, which had stopped since Schoute's death in 1913, when she met H. S. M. Coxeter [6] by her nephew Geoffrey Ingram Taylor (1886-1975). Taylor, aware of Boole Stott's mathematical activities, introduced her aunt to Coxeter[1]. When Boole Stott and Coxeter met, she was then a 70 year old woman whilst Coxeter was only 23. Despite this difference of age, they became friends. They used to meet and work at several topics in mathematics. In a particular occasion [14], Coxeter invited Boole Stott to one of Prof. H.F. Baker's famous "tea parties" for his research students at Cambridge University, where they would deliver a joint lecture. She attended the party bringing along her a set of models which she donated to the department of mathematics[2]. Coxeter's own words [6] describe Boole Stott as:

> *The strength and simplicity of her character combined with the diversity of her interests to make her an inspiring friend.*

In his later work, Coxeter often made reference to her and her work, and called her "Aunt Alice", as Boole Stott's nephew Taylor used to do.

Boole Stott died at 12 Hornsey Lane, Highgate, Middlesex, on December 17, 1940 [21, December 18, 1940].

Since Coxeter and Boole Stott do not have any common publications, it is not always easy to know what precise contributions Boole Stott has made. However, we have some idea about this thanks to several remarks about her work that Coxeter made in his publications.

We now proceed to discuss regular polyhedra and their two-dimensional sections.

3 Platonic solids

The Platonic solids are the three-dimensional convex regular polyhedra. There exist five Platonic solids, namely: the tetrahedron, the cube, the

[1]Taylor became one of the most brilliant and influential mathematical physicist of the 20th century. He was expert on fluid dynamics and wave theory [10]. Both Taylor and Coxeter were fellows at Trinity College (Cambridge) before World War II.

[2]These models of polytope sections are still in safe keeping at the office of Prof. W.B.R. Lickorish at the department of mathematics.

octahedron, the dodecahedron and the icosahedron (see Figure 3).

Figure 3: The five Platonic solids

Looking at the angles that form each vertex of a regular polyhedron it is easy to check that there are not more possibilities. The sum of the angles in each vertex cannot exceed 2π. If the result is precisely 2π one gets a tesselation of a plane, but not a convex regular polyhedron. However, the proof of the existence of the five Platonic solids is more complicate since it requires the construction of them.

Constructing the parallel two-dimensional sections of a Platonic solids (i.e. the sections parallel to one of its faces) is quite elementary. Let us compute, for example, the sections of the cube. In order to do that, we intersect the plane containing a given face of the cube with the cube itself. It is clear that this consists only of the face of the cube, that is, the parallel section is a square. The remain of the sections are obtained by moving the plane towards the center of the cube and intersecting it with the cube. In this simple example all the parallel sections are congruent squares. In a similar way, the parallel sections of the remaining Platonic solids can be obtained leading to: decreasing triangles in the case of the tetrahedron, triangles and hexagons for the octahedron, pentagons and decagons for the dodecahedron and triangles, hexagons and dodecagons for the icosahedron.

A technique to visualize a Platonic solid consists of unfolding it to a plane. Roughly speaking, to unfold means to "cut" certain edges of the polyhedron mapping it to a two-dimensional space. The traditional unfolding of the cube can be seen in Figure 4.

Note that some of the edges of the unfolded cube are counted twice. In order to recover the three-dimensional cube from the unfolded version one must identify those edges. Using this idea one can also describe the parallel sections of the cube in a very easy way. An analog of this technic in the fourth dimension was used by Boole Stott in her paper of 1900 and is studied in detail in Section 5. For a historical contextualization of her work, we first present an overview of the results on four-dimensional geometry before and

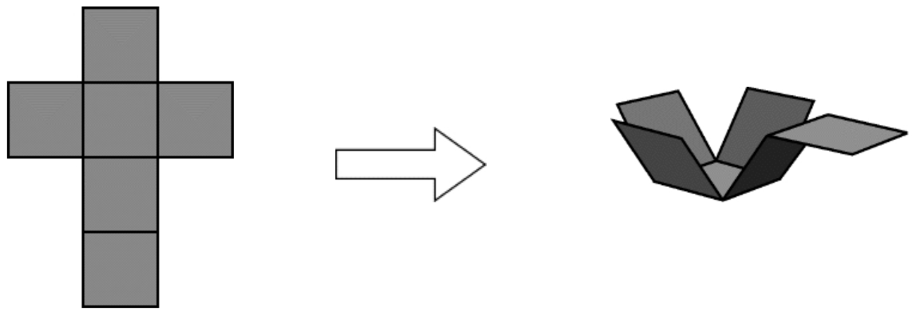

Figure 4: Folding the unfolded cube

during the 19th century .

4 An introduction to four-dimensional polytopes

The most influential paper of the 19th century with respect to four-dimensional geometry was Riemann's *Habilitation* lecture, given on June 10th, 1854 (see [13]). In this lecture, Riemann introduced the notion of an n-dimensional manifold. From that moment on, interest in higher dimensional spaces started increasing. By 1880 several articles on the topic had appeared written by mathematicians such as Arthur Cayley or William Clifford ([4] and [5]).

Four-dimensional polytopes are the four-dimensional analog of polyhedra. They were discovered by the Swiss mathematician Ludwig Schlaefli. Between 1850 and 1852, Schlaefli had developed a theory of geometry in n-dimensions. His work, *Theorie der vielfachen Kontinuität* (see [16]), contains the definition of the n-dimensional sphere and the introduction of the concept of four-dimensional polytopes, which he calls *polychemes*. He proves that there are exactly six regular polytopes in four dimensions and only three in dimensions higher than four. Unfortunately, his work was not accepted for publication, but only some fragments of it were published some years later. The manuscript was not entirely published until 1901. Thus, mathematicians writing about the subject during the second half of the century were unaware of Schlaefli's discoveries.

Between 1880 and 1900 the six regular polytopes were independently rediscovered, among many others, by Stringham [20] in 1880, Gosset [7] in 1900 and Boole Stott [1] in 1900.

5 Boole Stott's sections of polytopes

As mentioned before, Boole Stott's publication *On certain series of sections of the regular four-dimensional hypersolids* [1] describes a rather original method to obtain the three-dimensional sections of the regular polytopes. A *convex regular polytope* in four-dimensional space can be defined as a subset of $\mathbb{R}^4$ bounded by isomorphic three-dimensional regular polyhedra. These polyhedra are called *cells* in the papers of Boole Stott and Schoute. The number of vertices, edges, faces and cells of the six regular polytopes are given in the following table.

Polytope	v	e	f	c	cell
Hypertetrahedron or 5-cell	5	10	10	5	tetrahedron
Hypercube or 8-cell	16	32	24	8	cube
Hyperoctahedron or 16-cell	8	24	32	16	tetrahedron
24-cell	24	96	96	24	octahedron
120-cell	600	1200	720	120	dodecahedron
600-cell	120	720	1200	600	tetrahedron

Figure 5: Polytopes in four dimensions.

In the beginning of her paper, Boole Stott gives an intuitive proof of the uniqueness of the six regular polytopes in four dimensions. Her proof goes roughly as follows. Let P be a regular polytope whose cells are cubes. Let V be one of the vertices of P, and consider the diagonal section of P corresponding to an affine space K, close enough to V so that K intersects all the edges coming from V. The corresponding section must be a regular polyhedron bounded by equilateral triangles. Furthermore, there are only three regular polyhedra bounded by triangles, namely the tetrahedron (bounded by 4 triangles), the octahedron (bounded by 8 triangles) and the icosahedron (bounded by 20 triangles). Then, the polytope can only have 4, 8, or 20 cubes meeting at each vertex. Considering the possible angles in 4 dimensions, Boole Stott shows that the only possibility for P is to have 4 cubes at a vertex (8 and 20 are too many), which gives the 8-cell (also called hypercube). In a similar manner, the remaining five polytopes are found.

After her proof, Boole Stott proceeds to study three-dimensional perpendicular sections of these polytopes, which can be defined as follows. Let O be the center of a given polytope P, and C be the center of one of its cells. Let H be some affine 3-dimensional subspace, perpendicular to OC. A *perpendicular section* is $H \cap P$.

Although Boole Stott's publication [1] treats only perpendicular sections, Boole Stott also made models of *diagonal sections* of polytopes. These sections are defined as $K \cap P$, where O is the center of the polytope P, V one of its vertices, and K some affine three-dimensional subspace perpendicular to the segment OV.

The idea of Boole Stott's method uses the unfolding of the four-dimensional body in a three-dimensional space, as it was done in Section 2 for one dimension lower. This unfolding operation can be described as "cutting" some of the two-dimensional edges between the three-dimensional faces an mapping the polytope to the third dimension. As an example of this, the unfolded hypercube is illustrated in Figure 6.

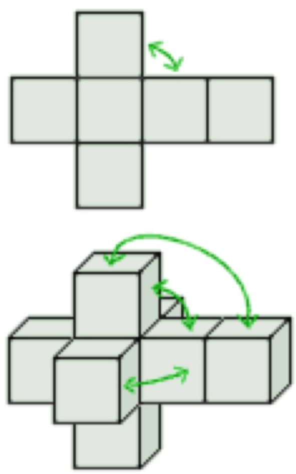

Figure 6: Unfolded cube and unfolded hypercube

Note that some of the two-dimensional faces (i.e., squares) must be identified in order to recover the hypercube (this identification, of course, is only possible in four dimensions).

We now sketch the algorithm behind Boole Stott's work on the computation of the three-dimensional perpendicular sections of the four-dimensional regular polytopes. In order to illustrate her results, the algorithm will be applied afterwards to some of these polytopes. We note that a similar algorithm can be described for the case of diagonal sections.

6 Sketch of the Algorithm

Let P be a four-dimensional regular polytope. The algorithm for the computation of its three-dimensional sections, following Boole Stott's reasoning, is presented in the following steps.

Step 1 Unfold the polytope P into the three-dimensional space (to ease visualization, only a representation of part of the unfolded polytope will be normally used. The complete set of three-dimensional sections may be recovered by using the symmetry of the polytope).

Step 2 Let Γ be the graph whose nodes are the vertices and the midpoints of the edges of the unfolded P, two nodes are connected if one of the nodes is the midpoint of an edge and the other a vertex meeting the edge.

Step 3 Fix a cell C of the polytope P on the unfolded figure.

Step 4 Consider the three-dimensional space H_1 containing C. The first three-dimensional section S_1 of P will be $H_1 \cap P = C$.

Step 5 Let $\mathcal{V}_2$ be the elements of Γ at distance 1 of C. The set $\mathcal{V}_2$ is contained in a hyperplane H_2 parallel to H_1. The elements in Γ at distance 1 to C are just the midpoints of the edges of $P \backslash C$ meeting C. These points will be the set of vertices of a polyhedron S_2 that will be the second section of P. One can easily compute the faces of S_2 as follows: for any cell D of the unfolded polytope that intersects $\mathcal{V}_2$, the polygon, segment or point given by the convex hull of $D \cap \mathcal{V}_2$ will be a face, edge or vertex of S_2 respectively. The natural folding of P gives the identification of every face of the polyhedron S_2.

Step 6 Let $\mathcal{V}_3$ by the elements of Γ at distance 1 of S_2 that are not contained in S_1. $\mathcal{V}_3$ is contained in a hyperplane H_3 parallel to C. $S_3 = P \cap H_3$ is the third section of P. One can easily compute the faces of S_3 as follows: for any cell D of the unfolded hypercube that intersects $\mathcal{V}_3$, the polygon, segment or point given by the convex hull of $D \cap \mathcal{V}_3$ will be a face, edge or vertex of S_3 respectively. The natural folding of P gives the identification of every face of the polyhedron S_3.

Step 7 Repeat step 6 until $\mathcal{V}_i = \emptyset$.

As a preliminar example, let us now compute the three-dimensional parallel sections of the hypercube.

6.1 The 8-cell or hypercube

Let P denote the hypercube. P is the analog of the cube in four dimensions and it is bounded by 8 cubes, 4 of them meeting at every vertex.

Step 1 Unfold the hypercube P into the three-dimensional space. Figure 7 is Boole Stott's representation of part of the unfolding hypercube (only four 4 are drawn). We recall that since it is an unfolding of a four-dimensional object, vertices, edges and faces occur more than once.

Step 2 Let Γ be the graph as explained in the sketch of the algorithm.

Step 3 Fix a cube cell C of the polytope P on the unfolded figure.

Step 4 Consider the three-dimensional space H_1 containing C. The first three-dimensional section S_1 of P will be $H_1 \cap P = C$, that is, the cube itself.

Step 5 Let $\mathcal{V}_2$ be the elements of Γ at distance 1 of C. These are the midpoints of the edges of $P \setminus C$ meeting C that form a cube S_2 that will be the second section of P. Three of the faces of S_2 are the squares $abcd$, $abfg$ and $adef$ (see Figure 7). The natural folding of P gives the identification of every face of the polyhedron S_2.

Step 6 Let $\mathcal{V}_3$ by the elements of Γ at distance 1 of S_2 that are not contained in S_1. Three of the faces of the third sections S_3 will be $KLMN$, $KLOP$ and $KMSO$, which form again, by symmetry and after the necessary identifications, a cube isomorphic to the previous ones.

Step 7 $\mathcal{V}_4 = \emptyset$ and we are done.

This simple example gives the idea of Boole Stott's method. Let us now consider the more interesting case of the 24-cell, which we find still not too difficult to compute.

6.2 The 24-cell

Let P denote now the 24-cell. We note that this polytope is the only one without an analog in three dimensions. Its 24 cells are octahedra, and 6 of them meet at every vertex. Step 1 of the algorithm gives the unfolding of P. In Boole Stott's representation (see Figure 8) only 4 octahedra are drawn. Note that the figure is again an unfolding, and the two A' should be identified and similarly, for the segments AE and AC.

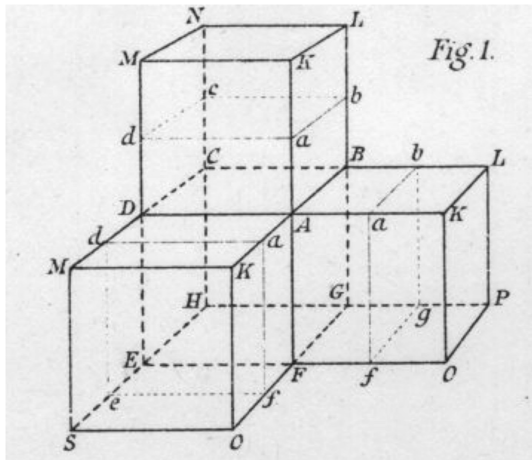

Figure 7: Hypercube: four cubes meeting at a vertex A [1].

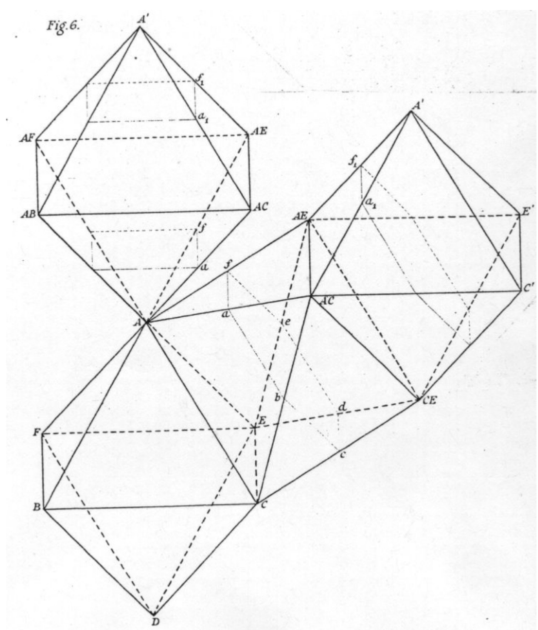

Figure 8: Four octahedra of the 24-cell [1].

Fix an octahedron cell of P (step 3). Let H_1 be the three-dimensional space containing the octahedron $ABCDEF$. The first section $H_1 \cap P_2$ is clearly the octahedron $ABCDEF$ itself (step 4).

Let H_2 be the space parallel to H_1 and passing through the point a (the mid-point between A and AC). The second section $H_2 \cap P_2$ is a three-dimensional solid whose faces are either parallel to the faces of the octahedron $ABCDEF$ or to the rectangle $BCEF$. In Figure 9 two of these faces are shadowed. Since the drawing of the octahedra meeting at A is not complete (3 octahedra are missing), we only see part of the final section. The remaining part can be deduced by symmetry (step 5).

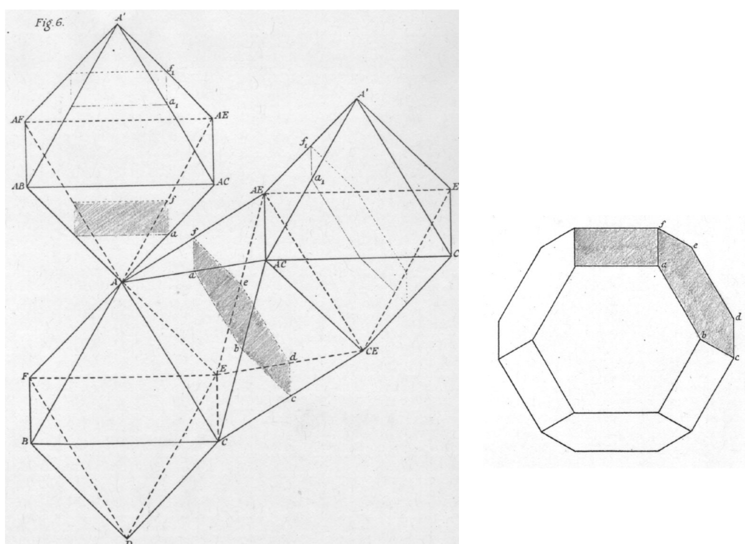

Figure 9: Section $H_2 \cap P$ of the 24-cell [1].

Following step 6, let H_3 be the space parallel to H_1 and passing through the vertex AC. The section $H_3 \cap P$ contains a rectangle $ABACAEAF$ parallel to the rectangle $BCEF$ and a triangle $AEACCE$ parallel to the face ACE (the shadowed faces of Figure 10).

By symmetry, the fourth section passing through a_1 (the mid-point between AC and A') is isomorphic to the second section (step 7). Again by symmetry, the last section through A' is an octahedron (step 8).

We briefly discuss the results concerning the remaining polytopes.

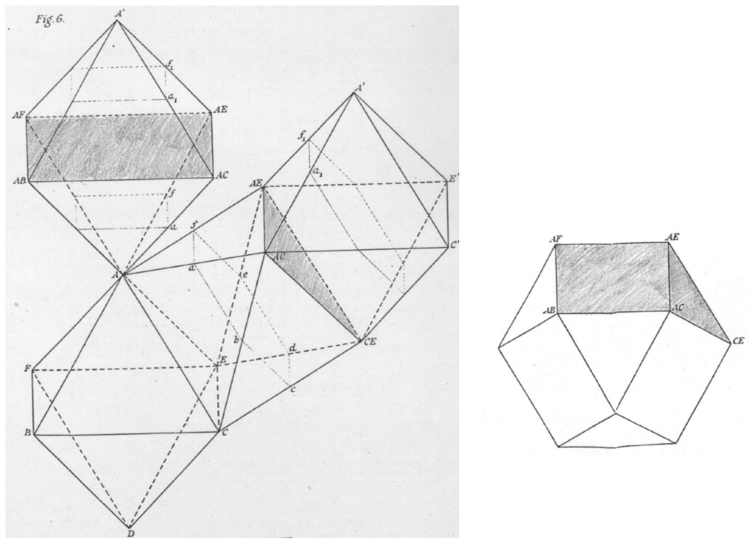

Figure 10: Section $H_3 \cap P$ of the 24-cell [1].

6.3 The 16-cell or hyperoctahedron

This polytope is the analog of the octahedron in four dimensions. It is bounded by 16 tetrahedra, 8 of them meeting at every vertex (see Figure 11). The first and last sections are clearly tetrahedra (since the 16-cell is bounded by tetrahedra). The remaining sections are shown in Figure 12.

6.4 The 120-cell

The 120-cell is bounded by 120 dodecahedra. Due to the complexity of this solid, Boole Stott also made drawings of its sections (see Figure 13) and cardboard models.

6.5 The 600-cell

This is the most complicated polytope. It is bounded by 600 tetrahedra. Like in the case of 120-cell, Boole Stott made drawings and cardboard models of its sections. The models are displayed in Figure 2.

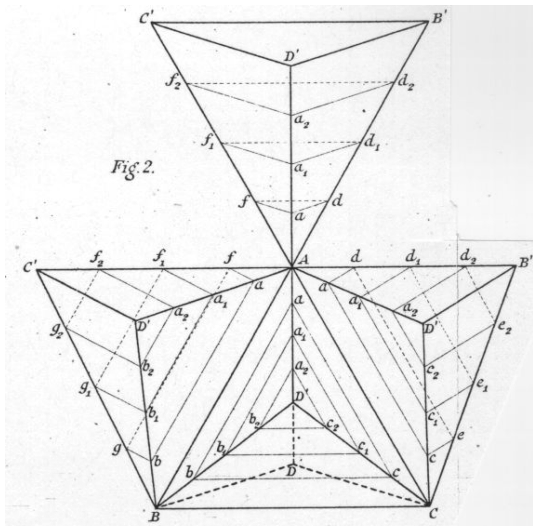

Figure 11: Five tetrahedra of the 16-cell. [1].

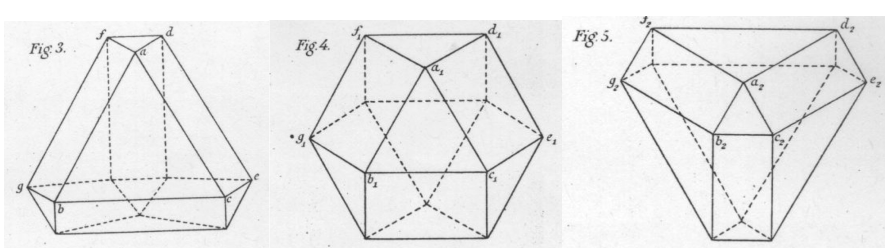

Figure 12: 2nd, 3rd and 4th sections of the 16-cell. [1].

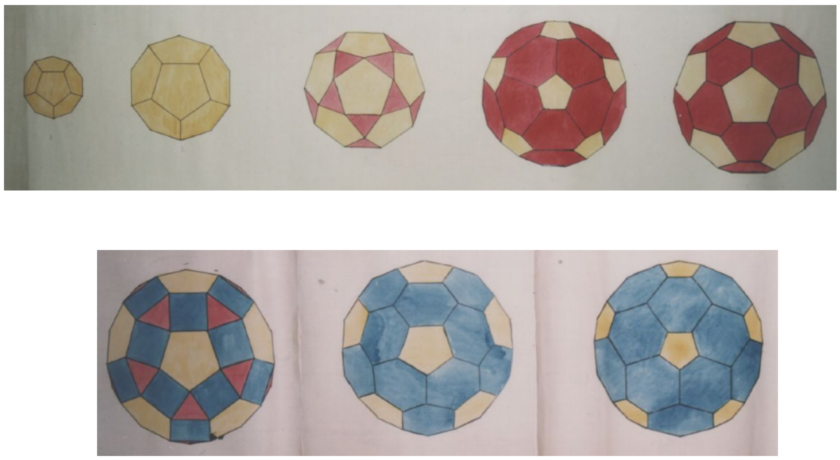

Figure 13: Sections of the 120-cell. (Courtesy of the University Museum of Groningen.)

7 Conclusions

The work of the amateur mathematician Boole Stott on the computation of three-dimensional sections of the regular polytopes has been presented. After an introduction to the theory of three- and four-dimensional polytopes and their sections, an algorithm based on Boole Stott's method is given in Section 6. This algorithm provides an efficient and easy to visualize way of dealing with the computation of the sections of these four-dimensional objects. As an illustration of this method, the algorithm has been applied to some of the regular polytopes, resulting in series of polyhedra represented by her original drawings and cardboard models.

Acknowledgments

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