MATH 10
ASSIGNMENT 19: SIGN OF A PERMUTATION
APRIL 17, 2015

Definition. Let f be a permutation of $\{1, \dots, n\}$. An **disorder for** f is a pair i, j such that $i < j$ but $f(i) > f(j)$. For example, for permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

there are 4 disorders: $(1,2)$, $(1,3)$, $(1,4)$, $(3,4)$.

A *sign* of a permutation is defined by

$$\operatorname{sgn}(f) = (-1)^{\#\text{ of disorders}}$$

thus, $\operatorname{sgn}(f) = +1$ if the number of disorders is even (such permutations are called *even*), and $\operatorname{sgn}(f) = -1$ if the number of disorders is odd (such permutations are called *odd*).

1. A *transposition* is a permutation that exchanges exactly two elements and leaves other unchanged, i.e. a cycle of length 2.
   (a) Show that any permutation can be written as a product of transpositions of the form $(i\ i+1)$, so that at each step, we are always exchanging two elements that are next to each other. Write the permutation
   $$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$
   in such a form.
   (b) Write a transposition $(i\ j)$ $(i < j)$ is such a form.

2. Prove that for any transposition $\tau = (ij)$, we have $\operatorname{sgn}(ij) = -1$.

3. Find the sign of a cycle of length n.

4. Let s be a permutation and let $\tau = (i\ i+1)$ be a transposition that exchanges two adjacent elements. Show that then $\operatorname{sgn}(\tau s) = -\operatorname{sgn}(s)$. [Hint: τ changes the order of exactly one pair.]

5. Show that if $s = \tau_1 \dots \tau_k$, where each τ is a transposition of the form $(i\ i+1)$ (compare with problem 1), then $\operatorname{sgn}(s) = (-1)^k$.

6. Show that for any permutations $s, t \in S_n$, we have $\operatorname{sgn}(st) = \operatorname{sgn}(s)\operatorname{sgn}(t)$.

7. For any permutation $s \in S_n$ and a polynomial p in variables $x_1, \dots, x_n$, we can define new polynomial $s(p)$ by permuting $x_1, \dots, x_n$ using s. For example, if $p = x_1^2 + 2x_2 + x_1x_3$, and $s = (1\,2)$, then $s(p) = x_2^2 + 2x_1 + x_2x_3$.
   (a) Show that for the polynomial in 3 variables $p = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$, and any permutation s, we have $s(p) = \operatorname{sgn}(s) \cdot p$.
   (b) Can you construct a polynomial p in n variables such that $s(p) = \operatorname{sgn}(s) \cdot p$ for any permutation $s \in S_n$?

***8.** Explain while the game of 15: *https://en.wikipedia.org/wiki/15_puzzle* is unsolvable