Mathematical Matching

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In 1962 a paper by David Gale and Lloyd Shapley appeared at the RAND Corporation, whose title, “College Admissions and the Stability of Marriage,” raised eyebrows.

Actually, the paper dealt with a matter of some urgency.

According to Gale, the paper owes its origin to an article in the *New Yorker*, dated September 10, 1960, in which the writer describes the difficulties of undergraduate admissions at Yale University.

Then as now, students would apply to several universities and admissions officers had no way of telling which applicants were serious about enrolling.

The students, who had every reason to manipulate, would create the impression that each university was their top choice, while the universities would enroll too many students, assuming that many of them would not attend.

The whole process became a guessing game. Above all, there was a feeling that actual enrollments were far from optimal.

Having read the article, Gale and Shapley collaborated.

First, they defined the concept of stable matching, and then proved that stable matching between students and universities always exists.

For simplicity, Gale and Shapley started with unrealistic case in which there are n universities and n applicants and each university has exactly one vacancy.

A more realistic description of this case is a matching between men and women – hence the title of their paper.

The Matching Problem

Consider a community of men and women where the number of men equals the number of women.

Objective: Propose a good matching system for the community.

To be able to propose such a system, we shall need relevant data about the community.

Accordingly, we shall ask every community member to rank members of the opposite sex in accordance with his or her preferences for marriage partner.

We shall assume that no man or woman in the community is indifferent to a choice between two or more members of the opposite sex.

Example

The men are Al, Bob, Cal, Dan.

The women are Ann, Beth, Cher, Dot.

Their list of preferences is

	Ann	Beth	Cher	Dot
Al	3, 1	4, 1	1, 3	2, 2
Bob	2, 2	3, 2	4, 1	1, 3
Cal	1, 3	2, 3	3, 2	4, 1
Dan	3, 4	4, 4	2, 4	1, 4

Given everyone's preferences, can you propose a matching system for the community?

A Possible Proposal:

Al – Dot

Bob – Ann

Cal – Beth

Dan - Cher

This is indeed a possible proposal, but it is not a good one.

Cher is displeased, because she is paired off with her last choice. She can propose to Bob, but she will be turned down because she is his last choice. She will fare no better with Cal, because she is his third choice while he is paired off with his second choice. On the other hand, if Cher propose to Al, he will be very pleased, because she is his first choice.

The proposal is rejected, because Cher and Al prefer each other to their actual mates, and one can reasonably assume that they will reject the matchmaker’s proposal.

Another Possible Proposal:

Let us try to pair off all the men with their first choice.

Al – Cher

Bob – Dot

Cal – Ann

Dan – Beth

Three of the four men are paired off with their first choice.

Do you think this proposal will be accepted or rejected?

Still Another Possible Proposal:

Now we shall try to pair off all the women with their first choice. Is it possible?

Ann’s first choice is Al.

Beth’s first choice is Al.

Cher’s first choice is Bob.

Dot’s first choice is Cal.

The new matching system is:

Ann – Al

Beth – Dan

Cher – Bob

Dot – Cal

Three of the four women are paired off with their first choice. Will they accept or reject this matching System?

Exercise:

Analyze the second proposal above and see whether it can be rejected by any pair of men and women.

The first proposal was rejected, but we can turned the failed effort to our advantage. Indeed, we have learned that a matching system must satisfy the following requirements:

A matching system must be such that under it there cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates.

Explanation: The matching system must be such that under it Ms. X cannot be paired off with Mr. x and Ms. Y cannot be paired off with Mr. y, when Ms. X prefer Mr. y to Mr. x and Mr. y prefers Ms. X to Ms. Y.

If couples X-x and Y-y were paired off according to the matchmaker’s recommendation, then Ms. X could say to Mr. y, “You prefer me to your actual mate and I prefer you to mine. Let’s leave them and pair up.”

Discussion:

Will Ms. X and Mr. y pair themselves off with each other? Not necessarily! “Yes, I prefer you, X, to Y, but I prefer Z to you.”

Definition: A matching system is called *Stable* if under it there cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates.

Example:

For simplicity, we substitute letters for names.

The men: a, b, c, d.

The women: A, B, C, D.

Preference structure:

	A	B	C	D
a	4, 1	2, 2	1, 4	3, 2
b	2, 2	1, 4	3, 2	4, 1
c	3, 3	1, 1	4, 1	2, 3
d	2, 4	4, 3	1, 3	3, 4

Let’s look at the following matching system:

(A – b, B – c, C – d, D – a)

2x2 1x1 1x3 3x2

The numbers below each couple indicate what rank one member of a couple assigns to the other member. The number on the left indicates what rank the man assigns to the woman, the number on the right, what rank the woman assigns to the man.

Is it a stable matching system?

Verification: Mr. c and Mr. d are paired with their first choice so they need look no further. Mr. b prefers Ms. B to his actual mate, Ms. A, but Ms. B will turn him down because he is her last choice. Mr. a prefers Ms. B and Ms. C to his actual mate, D. If he propose to one of them he will be turn down because they are better off with their actual mates.

Further Examples

Example 1

The preference structure

	A	B
a	1, 1	2, 1
b	1, 2	2, 2

i. (A – b, B – a)

↘ ↙

A – a

This matching is unstable. The system can be undermined by A and a.

ii. (A – a, B – b)

This matching system is stable because A and a are paired off with their first choice and therefore will not deviate.

Example 2

The preference structure is:

	A	B
a	1, 2	2, 1
b	1, 1	2, 2

The possible matching systems are:

i. (A – b, B – a)

This matching system is stable because A and b are paired off with their first choice.

ii. (A – a, B – b)

A – b This matching system is unstable because A and b will pair themselves off with each other.

Example 3

The preference structure is:

	A	B
a	1, 2	2, 1
b	2, 1	1, 2

The possible matching systems are:

i. (A – b, B – a)

This matching system is stable because A and B are paired off with their first choice.

ii. (A – a, B – b)

This matching system is stable, too, because a and b are paired off with their first choice.

A stable matching system is not necessarily a system under which everyone is satisfied. A matching system is stable when no unmatched pair will find it beneficial to deviate from the matching and form their own match.

In other words, a stable matching system serves the interests of the matchmaker, whose recommendation will be honored, but it does not necessarily serve the interest of all community members.

Example 4

The preference structure is:

	A	B	C
a	1, 3	2, 2	3, 1
b	3, 1	1, 3	2, 2
c	2, 2	3, 1	1, 3

Two stable matching systems are straightforward in this example:

i. (A – a, B – b, C – c)

The men obtain their first choice.

ii. (A – b, B – c, C – a)

The women obtain their first choice.

There is also a third stable matching system, where men and women obtain their second choice

(A – c, B – a, C – b)

In fact, all those who try to obtain their first choice will be rejected because they themselves are their favorite's third choice. (Verify it!)

Example 5: The Roommate Problem

We need to divide an even-numbered set of boys into pairs of roommates. Here again, a set of pairs will be called stable if in it there cannot be found two boys who are not roommates but prefer each other to their actual roommates.

We shall see that a stable division does not always exist.

In the following example there are three ways of dividing the boys into pairs. But in each case the division is unstable.

	a	b	c	d
a	-	1	2	3
b	2	-	1	3
c	1	2	-	3
d	1	2	3	-

The arrows indicate the pairs likely to undermine the division.

i. (a – d, b – c)

↘ ↙

ii. (a – c, b – d)

↘ ↙

iii. (a – b, c – d)

↘↙

To sum up, the last example shows that in the case of the roommate problem it is possible that no stable matching exists.

On the other hand, all the examples of the marriage problem provided thus far have had a stable matching.

Whether this is always the case in marriage problems is no evident and requires either a proof or a counter-example.

A Procedure for Finding Stable Matching Systems (The Gale-Shapley Algorithm)

In connection with a procedure for finding stable matching systems, three questions naturally arise:

1. Given a preference structure, is one always certain of finding a stable matching system? Let us recall that a stable division does not always exist for the roommate problem. Does a stable division always exist for the matching problem?

2. How does one find a stable matching system?

3. Does only one stable matching system exist?

(The answer is no; we have already seen examples where more than one stable matching exists.)

The Gale-Shapley algorithm for finding a stable matching system

First Stage: Every man turns to the woman who is first on his list and propose to her. Every woman who receives more than one proposal selects her favorite from among those who propose to her and tells the others that she will never marry them. Every man who is not rejected is put on a “waiting list” of the woman to whom he proposed.

Second Stage: Every man who was rejected turn to the woman who is second on his list and propose to her. Every woman who receives more than one proposal, including any proposals from the previous stage, selects her favorite, and puts him on her waiting list. She informs the others that they are rejected.

Third Stage: Every man who is rejected turns to the woman who is next on his list – the second on his list if he was put on the waiting list at a previous stage, or the third on his list if he was rejected twice. Once again, every woman selects her favorite from among those who have proposed to her, including anyone on her waiting list from the previous stage, puts him on the waiting list, and rejects the others.

The procedure continues – until such time as no man is rejected. At that stage every man on a waiting list becomes a mate, and the procedure terminates.

Example 1

The preference structure is:

	A	B	C	D
a	1, 3	2, 3	3, 2	4, 3
b	1, 4	4, 1	3, 3	2, 2
c	2, 2	1, 4	3, 4	4, 1
d	4, 1	2, 2	3, 1	1, 4

Stage 1:

A	B	C	D
a	c		d
b*			

Mr. b, marked with an asterisk (*), is rejected. The others are "waitlisted."

Stage 2:

A	B	C	D
a	c		d*
			b

Mr. d is rejected(despite having been waitlisted at the previous stage).

Stage 3:

A	B	C	D
a	c^*		b
	d		

Mr. c is rejected.

Stage 4:

A	B	C	D
a^*	d		b
c			

Mr. a is rejected.

Stage 5:

A	B	C	D
c	d		b
	a*		

Mr. a is rejected again.

Stage 6:

A	B	C	D
c	d	a	b

The procedure ends and the propose matching system is:
(A – c, B – d, C – a, D – b)

Show that the above matching system is stable

Example 2

The preference structure is:

	A	B	C	D
a	1, 1	2, 3	3, 4	4, 3
b	1, 2	2, 1	4, 2	3, 4
c	1, 3	2, 2	4, 3	3, 1
d	4, 4	1, 4	2, 1	3, 2

Stage 1:

A	B	C	D
a	d		
b*			
c*			

b and c are rejected.

Stage 2:

A	B	C	D
a	d*		
	b		
	c*		

c and d are rejected.

Stage 3:

A	B	C	D
a	b	d	c

The procedure ends and the proposed matching system is: (A – a , B – b , C – d , D – c)

A Stable Matching System Always Exists

Theorem:

The Gale-Shapley algorithm terminates after a finite number of steps.

Proof:

(1) The number of men in the community equals the number of women. Therefore, as long there is a woman in the community with more than one proposal, there is another woman without any proposal.

(2) Once a woman has a proposal, she will always have one, because someone will always be on her waiting list.

(3) When every woman has a proposal, every woman will have exactly one proposal, because the number of men equals the number of women. At this stage the procedure terminates, and there remains only to show that it indeed possible to get to this stage.

(4) It is possible to get to the stage at which every woman has a proposal, because at every stage the men propose to the women who are next on their list; therefore they cannot backtrack and propose again to the women who rejected them. Because there is a finite number of men and women in the community, and no going back, a stage must be reached at which every woman has a proposal. Based on step (3) of this proof, the procedure terminates at this stage.

Theorem:

The Gale-Shapley algorithm terminates in a stable matching system.

Proof:

Let us consider a matching system that is an outcome of the Gale-Shapley algorithm. We shall limit our attention to two couples in this system.

… R … S …

… r … s …

Suppose Mr. s prefers Ms. R to his actual mate, Ms. S. It must be shown that Ms. R does not prefer Mr. s to Mr. r, and therefore will refuse to be paired off with him if he suggests it. Indeed, if Mr. s prefers Ms. R, then he must have proposed to her at one of the previous stages.

But the fact that he is not paired off with her means that she rejected him. Why was he rejected? Because at that stage Ms. R had a proposal from another man whom she preferred to s (not necessarily r). It is possible that at one of the later stages she rejected this man whom she preferred, in favor of another man whom she preferred even more, etc.

Eventually, Mr. r proposed, and Ms. R preferred him to all the men who had proposed to her up to that stage, including Mr. s.

We have now proved that at the end of the Gale-Shapley algorithm there cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates. The conclusion is that the algorithm terminates in a stable matching system.

Generalization

1. *The number of men does not equal the number of women*

   In every matching system obtained in a community where the number of men does not equal the number of women, there will be men or, alternatively, women who are not paired off. The Gale-Shapley algorithm and its conclusions can be generalized to this case.

Example:

Women: A,B,C,D,E **Men:** a , b, c

	A	B	C	D	E
a	5,1	1,2	2,1	3,2	4,1
b	4,3	1,3	2,2	5,1	3,2
c	5,2	4,1	1,3	3,3	2,3

(1) The Men Propose

A	B	C	D	E
	a	c		
	b*			
	a	c*		
		b		
-	a	b	-	c

The procedure terminates in the matching system

(A - , B – a, C – b, D - , E – c)

A and D are single.

(2) The women Propose:

a	b	c	
A^*	D	B	
C			
E^*			
C	D^*	B	
	E	A^*	
C	E	B	
D^*	A^*		
C	E	B^*	
		D	A is out.
C^*	E	D	
B			
B	E^*	D	
	C		
B	C	D^*	
		E	
B	C	E	D is out.

Here again, the procedure terminates in the matching system (A - , B – a, C – b, D - , E – c)

Exercise:

Check whether the above matching system is stable. Your answer should also take into account the women who are not paired off.

Exercise:

Prove that the Gale-Shapley algorithm always leads to a stable matching system(even when the number of men does not equal the number of women).

2. Existence of a preference list that does not include all members of the opposite sex

In the following example, there are men who would rather be single than be paired off with certain women. Similarly, there are women who would rather be single than paired off with certain men.

In such cases, the preference to stay single rather than be paired off with a certain person is marked with a zero (0).

We shall now show that the Gale-Shapley algorithm can be generalized to this case too.

Example:

The preference list is:

Women: A, B, C **Men:** a, b, c, d

	A	B	C
a	3,1	1,1	2,0
b	0,2	1,2	2,0
c	1,0	2,3	3,0
d	1,3	0,4	2,1

The Men propose:

A	B	C	
c^*	a		
d	b^*		
d	a		
	c^*	b^*	b and c are out.
d	a	c^*	C is out.

The matching system obtained is (A – d, B – a, C -)

Show that even those who are left out cannot undermine the system.

(3) Possible Indifference

Let us see what happens when indifference is allowed. A community member who is indifferent to a choice between two or more members of the opposite sex and yet obliged to rank them in order of preference might say: “My first choice is Ms. B; as for my second choice, I am indifferent between A and D; as for my third choice, I am indifferent between C and E; and my fourth choice is F.”

It turns out that there can be a stable matching system even when there is indifference.

Definition:

A matching system is called *stable* if under it there cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates

Remark:

It follows from this definition that we assume that a man will not leave his mate for another woman when he is indifferent between the two, and a woman will not leave her mate for another man when she is indifferent between the two.

Up to now we have made the assumption that the preference order of each member contains no indifference.

We can easily dispense with this constraint by assigning an arbitrary strict preference whenever there is indifference.

Example:

Women: A, B, C **Men:** a, b, c

	A	B	C
a	1,3	2,1	2,3
b	1,2	1,2	2,1
c	1,1	2,3	3,2

In this example, Mr. a’s first choice is Ms. A, but he is indifferent to a choice between B and C, who occupy a lower rank in his order of preferences. Mr. b’s first choice is A and B, but he is indifferent to a choice between them; his second choice is Ms. C. There is no indifference in the women’s preference.

Let us try to follow the Gale-Shapley algorithm.

A	B	C
a		
c		

$\nwarrow \; \nearrow$

b

According to the preference structure, a and c propose to A, who is their first choice, but b hesitates, because he prefers A and B equally

To proceed any further, we need to arbitrarily change the given preference structure to a structure in which there is no indifference.

Specifically, we need to let strict preferences stand and replace indifference, wherever it occurs, by a strict preference.

For example:

	A	B	C
a	1,3	3,1	2,3
b	2,2	1,2	3,1
c	1,1	2,3	3,2

Given this preference structure, we can use the Gale-Shapley algorithm to obtain a stable matching.

In the male courtship procedure:

A	B	C
a*	b	
c		
c	b	a

The stable matching system is (A – c, B – b, C – a)

Is this matching system stable in the original structure too?

Does the relation indicated below threaten the system’s stability?

A B C

c b a

Indeed, B prefers a to her actual mate, but a does not prefer B to C; rather, he is indifferent to the choice between them.

An examination of all the possible relations reveals that deviation from the matchmaker's recommendation is impossible.

Thus, the system is stable in the original preference structure too.

We shall now use another preference structure in which there is no indifference.

For example:

	A	B	C
a	1,3	2,1	3,3
b	2,2	1,2	3,1
c	1,1	2,3	3,2

We shall have the men propose according to the Gale-Shapley algorithm.

A	B	C
a*	b	
c		
c	b*	
	a	
c	a	
b*		
c	a	b

The stable matching system is

(A – c, B – a, C – b)

This matching system is stable in the original preference structure too (verify it!)

Summary: To obtain a stable matching system, given a preference structure in which there is indifference, one constructs an alternative preference structure in which there is no indifference and finds a stable matching system using the Gale-Shapley algorithm.

This matching system will be stable in the original preference structure too.

Note, that the Gale-Shapley algorithm, which leads to one stable matching system when there is no indifference, may lead to several stable matching systems when there is indifference.

Claim:

Every stable matching system in a "revised" preference structure in which there is no indifference is also stable in the original preference structure in which there is indifference.

Proof:

Let us assume, on the contrary, that the stable matching system in the revised preference structure is unstable in the original preference structure in which there is indifference.

Now, according to the original preference structure, there exist Ms. X and Mr. y who are not paired off with each other but prefer each other to their actual mates.

Because preference relations(in contrast to indifference relations) do not change in the conversion to the revised preference structure in which there is indifference, X and y prefer each other to their actual mates in this preference structure .

Therefore, this matching system is unstable in the revised preference structure, which contradict our assumption.

The contradiction proves that the assumption made at the beginning of the proof is incorrect; therefore, the stable matching system in the revised preference structure is also stable in the original preference structure where indifference occurs.

OPTIMALITY

It has been seen that some preference structures yield more than one stable matching system. This raises a few questions.

(1) Is there one stable matching system that is everyone's favorite?
Assuming that there is no indifference, the answer is no, because if there are two stable matching systems, then at least one man is paired off with a different woman in the second system, and necessarily prefers one system to the other.

(2) Is there one stable matching system that is the men’s favorite?

Surprisingly, the answer is yes.

The same goes to the women: there is one stable matching system that is the women’s favorite.

We shall illustrate each of these points by way of example.

Example:

Given the following preference structure:

Women: A, B, C, D **Men:** a, b, c, d

	A	B	C	D
a	2,3	1,4	4,1	3,1
b	3,2	2,2	1,3	4,4
c	2,4	4,1	3,2	1,3
d	4,1	2,3	1,4	3,2

When the men propose, the following matching system is obtained by the Gale-Shapley algorithm(verify it!):

System 1:

(A – a, B – d, C – b, D – c)

When the women propose, the following matching system is obtained by the Gale-Shapley algorithm(verify it!):

System 2:

(A – d, B – b, C – c, D – d)

The following matching system, which is not obtained by the Gale-Shapley algorithm, is stable too(verify it):

System 3:

(A – a, B – b, C – c, D – d)

There are no other stable matching systems for this preference structure(verify it!).

In the following table, we include what preference ranking is assigned to each man and woman in the above three stable systems.

	Male Courtship	Female Courtship	Other System
A B C D:	3 3 3 3	1 2 2 1	3 2 2 2
a b c d:	2 1 1 2	3 2 3 4	2 2 3 3
	System 1	**System 2**	**System 3**

System 1 is best for all the men.

System 2 is best for all the women.

To sum up: the matching system that is obtained when the men propose is *optimal for every man* and the matching system that is obtained when the women propose is *optimal for every woman,* when “optimal” is defined as follows:

Definition:

A stable matching system is called *optimal for a given man* if he is at least as well off under it as under any other stable matching system.

Similarly, a stable matching system is called *optimal for a given woman* if she is at least as well off under it as under any other stable matching system.

Note:

We are comparing only stable matching systems. The system in question must be stable, and bears comparison only to other stable matching systems.

The satisfaction of a single individual with an unstable matching system is irrelevant, because an unstable matching system will not last and will be undermined by internal deviations.

Optimality Theorem:

For every preference structure, the matching system obtained by the Gale-Shapley algorithm, when the men propose, is optimal for the men.

The matching system obtained by the Gale-Shapley algorithm, when the women propose, is optimal for the women.

Condition for the Existence of a Unique Stable Matching System

Theorem:

Assuming that there is no indifference, if the male courtship procedure and the female courtship procedure lead to the same matching system, then there is a unique stable matching system for the given preference structure.

Discussion

Gale and Shapley were the first to ask whether their algorithm for matching men and women was applicable to the college admissions problem.

What Gale and Shapley did not know at the time was that the Association of American Medical Colleges had already for ten years been applying the Gale-Shapley algorithm to the task of assigning interns to hospital in the United States.

By a process of trial and error that spanned over half a century, the Association in 1951 adopted the procedure, later rediscovered by Gale and Shapley, that was hospital optimal.