MORE PICTURE WRITING

A PREPRINT

Boris Demeshev
Department of Computer Science, Department of Applied Economics
State University Higher School of Economics
Moscow
bdemeshev@hse.ru

July 19, 2021

ABSTRACT

In this article we give more picture writing examples that motivate and explain generating functions.

1 Picture writing approach

The picture writing approach to motivate generating functions was introduced in an old article Pólya [1956]. The approach is visually appealing and really explains *why* generating functions just count the objects of interest.

Despite its beauty the picture writing approach is almost unknown. So we'll try to provide more examples!

2 Fibonacci sequence

The rules to produce the Fibonacci sequence are simple. Start with one young pair of rabbits, $F_1 = 1$. Each young pair will be adult in one month, each adult pair will give birth to a new young pair every month. The number of pairs alive at time n is $F_n = F_{n-1} + F_{n-2}$, where F_{n-2} pairs are alive at $(n-2)$, so they are adult at $(n-1)$, hence they give birth to new pairs at n. Let's draw a picture a little bit like a pedigree tree:

!!!! Here I need a picture

Every living pair of rabbits at every moment of time may be identified with a pedigree string. For example there are three strings of length four: $bBBB$, $bBBb$, $bBbB$.

Let S be the set of all valid pedigrees. A pedigree is valid if it starts with baby bunny b and every baby bunny b is followed by big bunny B.

$$S = \{b, bB, bBB, bBb, \ldots\}$$

Instead of writing a set with curly brackets and comma we will use the plus symbol $+$.

$$S = b + bB + bBB + bBb + \ldots$$

We introduce multiplication of pedigrees. To multiply two pedigrees we just write them one next to another. For example

$$bBB \cdot bB = bBBbB$$

The multiplication of two valid pedigrees may give rise to an unvalid pedigree, as here

$$bBBb \cdot bB = bBBbbB.$$

Let's move on to the list of valid pedigrees. Each valid pedigree with two or more symbols ends with B or with Bb. We take out the corresponding terms:

$$S = b + (b + bB + bBB + bBb + \dots) \cdot B + (b + bB + bBB + bBb + \dots) \cdot Bb$$

Or simply

$$S = b + SB + SBb$$

Now we just simplify our notation. Imagine we don't care whether the bunny is young or adult and we just write t for both types.

By definition $g(B) = t$, $g(bBb) = t^3$ and $g(bB + Bb + B) = 2t^2 + t$. The polynomial $g(A) = t^3 + 2t^{15}$ means that the set A contains one pedigree with three bunnies and two pedigrees with fifteen bunnies.

In formal world the function g is a homomorphism between sets of bunny pedigrees and polynomials. The function g preserves addition $+$ and multiplication $\cdot$.

Hence we obtain the recurrence relation for the generating function

$$g(S) = t + tg(S) + t^2 g(S)$$

Hence

$$g(S) = \frac{t}{1 - t - t^2}$$

It's funny to draw bunnies on blackboard, they certainly catch student's attention. To simplify drawing one may just write B for Big bunny and b for baby bunny.

3 Paths to win one dollar

Imagine that I bet one dollar on a fair coin toss. My fortune may go up $\uparrow$ or down $\downarrow$ in every bet. I bet until I will increase my wellfare by one dollar. Let's count all the possible paths!

For example the set of possible possible paths S contains the path $\uparrow$ and the path $\downarrow\uparrow\uparrow$ but not the path $\uparrow\downarrow\downarrow\uparrow\uparrow$. The path $\uparrow\downarrow\downarrow\uparrow\uparrow$ is not included in S because I will stop after the first $\uparrow$.

Hence

$$S = \{\uparrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow\uparrow\uparrow, \downarrow\downarrow\uparrow\uparrow\uparrow, \dots\}$$

Instead of writing curly brackets and comma to denote a set we will just use the plus sign $+$.

$$S = \uparrow + \downarrow\uparrow\uparrow + \downarrow\uparrow\downarrow\uparrow\uparrow + \downarrow\downarrow\uparrow\uparrow\uparrow + \dots$$

We don't make the difference between one sequence, say $\downarrow\uparrow\uparrow$, and a set containing one sequence, $\{\downarrow\uparrow\uparrow\}$. When writing one the wall you may omit curly brackets! It's ok!

We need to invent multiplication of paths. That's easy! Just write two sequences next one to another. By this definition the multiplication is non-commutative:

$$\uparrow\downarrow \cdot \downarrow\downarrow\uparrow = \uparrow\downarrow\downarrow\downarrow\uparrow \neq \downarrow\downarrow\uparrow\uparrow\downarrow = \downarrow\downarrow\uparrow \cdot \uparrow\downarrow$$

The multiplication of two sets or more sets is defined elementwise: For example

$$\uparrow\downarrow \cdot (\downarrow\downarrow\uparrow + \downarrow) = \uparrow\downarrow\downarrow\downarrow\uparrow + \uparrow\downarrow\downarrow$$

Now we are ready to state the main recurrent equation for our set S.

If I loose the first dollar then I need to win one dollars twice!

$$S = \uparrow + \downarrow \cdot S \cdot S$$

This is exactly the equation for generating functions! We just simplify our notation. Now we don't care whether the move is up or down and insted of both $\uparrow$ and $\downarrow$ we just write t.

By definition $g(\uparrow) = t$, $g(\uparrow\uparrow\downarrow) = t^3$ and $g(\uparrow\uparrow + \downarrow\downarrow + \uparrow) = 2t^2 + t$. The polynomial $g(A) = t^3 + 2t^{15}$ means that the set A contains one sequence with three arrows and two sequences with fifteen arrows.

In formal world the function g is a homomorphism between sets of up or down sequencies and polynomials. The function g preserves addition $+$ and multiplication $\cdot$.

Hence we obtain the recurrence relation for the generating function

$$g(S) = t + tg(S)g(S)$$

Now the standard machine of generating function may be used. I reproduce it here for completeness of exposition.

There are two solutions of the quadratic equation

$$g(S) = \frac{1 \pm \sqrt{1 - 4t^2}}{2t}$$

Instead of boring differentiation in the Taylor expansion formula we use Newton's approach

$$(1 + x)^a = \binom{a}{0} + \binom{a}{1}x + \binom{a}{1}x^2 + \ldots$$

Here $\binom{a}{0} = 1$ and for $k > 0$:

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdot\ldots\cdot(a-k+1)}{k!}$$

Hence

$$g(S) = \frac{1 \pm \left(1 + 0.5(-4t^2) + \frac{0.5(0.5-1)}{2!}(-4t^2)^2 + \frac{0.5(0.5-1)(0.5-2)}{3!}(-4t^2)^3 + \ldots\right)}{2t}$$

All coefficients in $g(S)$ should be non-negative as they just counts the number of corresponding sequences. Ruling out this case we should leave the $-$ in nominator.

Finally we get

$$g(S) = \sum_{k=0}^{\infty} \binom{0.5}{k} (-1)^k 2^{2k-1} t^{2k-1} = t + t^3 + 2t^5 + \ldots$$

So the number of paths to win one dollar exactly in $2k - 1$ tosses is

$$c_{2k-1} = \binom{0.5}{k} (-1)^k 2^{2k-1}.$$

It can be simplified to

$$c_{2k-1} = \frac{2^{k-1} \cdot 1 \cdot 3 \cdot 5 \cdot \ldots (2k-3)}{k!}$$

4 Catalan numbers

The representation of Catalan numbers as triangulations of convex n-gon was considered in Pólya [1956]. So I will consider the representation of Catalan numbers with bracket sequences.

The set of all valid brackets sequences including empty sequence $\square$ is

$$S = \square + () + (()) + ()() + ((())) + (())() + ()(()) + \ldots$$

Multiplication is intuitive and non-commutative

$$() \cdot (()) = ()(())$$

Let's focus on the term that is inside the leftmost opening bracket (and its closing sister).

This term can be any valid bracket sequence S. To the right of them we can also have any valid bracket sequence S.

Hence we obtain the decomposition

$$S = \square + (\cdot S \cdot) \cdot S$$

Each bracket pair is replaced by t. The multiplication of bracket sequences is non-commutative but the multiplication of t-polynomials is commutative. Empty bracket sequence with no brackets is replaced by $t^0 = 1$.

And we get the equation

$$g(S) = 1 + tg(S)g(S)$$

Then the standard generating function technique may be used.

5 Triangular numbers

Imagine a tribe that has only one letter a in the alphabet. The dictionary of all valid words including silence $\square$ is

$$S = \square + a + aa + aaa + aaaa + \dots$$

The dictionary of the tribe satisfies the equation

$$S = \square + a \cdot S$$

Here we do something trivial and replace every a by t.

$$g(S) = 1 + t \cdot g(S)$$

Hence the generating function looks almost like the original dictionary:

$$g(S) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

By the way if there are two letters in the alphabet the equation is

$$S = \square + a \cdot S + b \cdot S.$$

But let's move on to the triangular numbers.

!!!! Here I need a picture

Imagine a group of n persons. We are interested in counting all possible handshakes.

Let's write them as a dictionary!

$$S = \text{😎😎} + \text{☺😎😎} + \text{😎☺😎} + \text{😎😎☺} + \dots$$

We can factor our dictionary as

$$S = (\square + \text{☺} + \text{☺☺} + \dots) \cdot \text{😎} \cdot (\square + \text{☺} + \text{☺☺} + \dots) \cdot \text{😎} \cdot (\square + \text{☺} + \text{☺☺} + \dots),$$

where $\square$ is an empty group of people.

Now replace every symbol by letter t:

$$g(S) = \frac{1}{1-t} \cdot t \cdot \frac{1}{1-t} \cdot t \cdot \frac{1}{1-t}$$

A triangle with n dots on a side corresponds to handshakes of a group with $n+1$ persons. So we need to remove one t to obtain generating function of triangular numbers.

$$g = \frac{t}{(1-t)^3}$$

6 Discussion

This picture writing approach may be used to introduce abstract algebra to college students. We multiply sequences of pictures or just words instead of numbers. We write equations where the solution is a set of picture writings. That's more abstract and more fun!

So far we have focused on counting the number of words of particular lenght. Our counting method is based on the one-to-one correspondance between sets of words and expressions with words combined by plus sign

$$\{aaa, abc, bbb, abcde\} \leftrightarrow aaa + abc + bbb + abcde$$

We have avoided to add the same word to itself. We don't need the same word counted twice when we try to establish the number of valid words with n letters. However our one-to-one corresponding may be extended to allow the addition of a word to itself. All sets of valid words for a given problem is not a ring so we need a bigger object to allow ring structure. We will work not with dictionaries but with bank accounts.

We may think of words as currency names and a bank account of a person may be written as

$$v = 5aaa + 3.5abc - 2bbb + 7abcde.$$

The guy v has a debt of 2 bbb notes, 5 aaa notes, 3.5 abc notes and 7 abcde notes.

The set of all possible bank accounts is an infinite dimensional real vector space. The multiplication of words by concatenation completes the ring structure. This multiplication is correctly defined as you may settle all operations with one-letter currencies, than all operations with two-letters currencies and so on.

The function g that transform every letter into t is a ring homomorphism between all possible bank accounts and polynomials.

In this example

$$g(5aaa + 3.5abc - 2bbb + 7abcde) = 6.5t^3 + 7t^4.$$

Usual sets are bank accounts with zero or one note of every currency.

However the main point of this article is not the formalisation of picture writing approach. My main goal is a call to use directly equations on sets to motivate generating functions.

The equations on sets are more intuitive! Let's work with objects themselves to count them!

References

George Pólya. On picture-writing. *The American Mathematical Monthly*, 63(10):689–697, 1956.