MORE PICTURE WRITING

A PREPRINT

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July 16, 2021

ABSTRACT

In this article we give more picture writing examples that motivate and explain generating functions.

1 Picture writing approach

The picture writing approach to motivate generating functions was introduced in an old article Pólya [1956]. The approach is visually appealing and really explains *why* generating functions just count the objects of interest.

Despite its beauty the picture writing approach is almost unknown. So we'll try to provide more examples!

2 Fibonacci sequence

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It's funny to draw bunnies on blackboard, they certainly catch student's attention. To simplify drawing one may just write B for Big bunny and b for baby bunny.

3 Paths to win one dollar

Imagine that I bet one dollar on a fair coin toss. My fortune may go up $\uparrow$ or down $\downarrow$ in every bet. I bet until I will increase my wellfare by one dollar. Let's count all the possible paths!

For example the set of possible possible paths S contains the path $\uparrow$ and the path $\downarrow\uparrow\uparrow$ but not the path $\uparrow\downarrow\downarrow\uparrow\uparrow$. The path $\uparrow\downarrow\downarrow\uparrow\uparrow$ is not included in S because I will stop after the first $\uparrow$.

Hence

$$S = \{\uparrow, \downarrow\uparrow\uparrow, \downarrow\uparrow\downarrow\uparrow\uparrow, \downarrow\downarrow\uparrow\uparrow\uparrow, \ldots\}$$

Instead of writing curly brackets and comma to denote a set we will just use the plus sign $+$.

$$S = \uparrow + \downarrow\uparrow\uparrow + \downarrow\uparrow\downarrow\uparrow\uparrow + \downarrow\downarrow\uparrow\uparrow\uparrow + \ldots$$

We don't make the difference between one sequence, say $\downarrow\uparrow\uparrow$, and a set containing one sequence, $\{\downarrow\uparrow\uparrow\}$. When writing one the wall you may omit curly brackets! It's ok!

We need to invent multiplication of paths. That's easy! Just write two sequences next one to another. By this definition the multiplication is non-commutative:

$$\uparrow\downarrow \cdot \downarrow\downarrow\uparrow = \uparrow\downarrow\downarrow\downarrow\uparrow \neq \downarrow\downarrow\uparrow\uparrow\downarrow = \downarrow\downarrow\uparrow \cdot \uparrow\downarrow$$

The multiplication of two sets or more sets is defined elementwise: For example

$$\uparrow\downarrow \cdot(\downarrow\downarrow\uparrow + \downarrow) = \uparrow\downarrow\downarrow\downarrow\uparrow + \uparrow\downarrow\downarrow$$

Now we are ready to state the main recurrent equation for our set S.

If I loose the first dollar then I need to win one dollars twice!

$$S = \uparrow + \downarrow \cdot S \cdot S$$

This is exactly the equation for generating functions! We just simplify our notation. Now we don't care whether the move is up or down and insted of both $\uparrow$ and $\downarrow$ we just write t.

By definition $g(\uparrow) = t$, $g(\uparrow\uparrow\downarrow) = t^3$ and $g(\uparrow\uparrow + \downarrow\downarrow + \uparrow) = 2t^2 + t$. The polynomial $g(A) = t^3 + 2t^{15}$ means that the set A contains one sequence with three arrows and two sequences with fifteen arrows.

In formal world the function g is a homomorphism between up or down sequencies and polynomials. The function g preserves addition $+$ and multiplication $\cdot$.

Hence we obtain the recurrence relation for the generating function

$$g(S) = t + tg(S)g(S)$$

Now the standard machine of generating function may be used. I reproduce it here for completeness of exposition.

There are two solutions of the quadratic equation

$$g(S) = \frac{1 \pm \sqrt{1-4t^2}}{2t}$$

Instead of boring differentiation in the Taylor expansion formula we use Newton's approach

$$(1+x)^a = \binom{a}{0} + \binom{a}{1}x + \binom{a}{1}x^2 + \dots$$

Here $\binom{a}{0} = 1$ and for $k > 0$:

$$\binom{a}{k} = \frac{a(a-1)(a-2)\cdot\ldots\cdot(a-k+1)}{k!}$$

Hence

$$g(S) = \frac{1 \pm \left(1 + 0.5(-4t^2) + \frac{0.5(0.5-1)}{2!}(-4t^2)^2 + \frac{0.5(0.5-1)(0.5-2)}{3!}(-4t^2)^3 + \dots\right)}{2t}$$

All coefficients in $g(S)$ should be non-negative as they just counts the number of corresponding sequences. Ruling out this case we should leave the $-$ in nominator.

Finally we get

$$g(S) = \sum_{k=0}^{\infty} \binom{0.5}{k}(-1)^k 2^{2k-1} t^{2k-1} = t + t^3 + 2t^5 + \dots$$

So the number of paths to win one dollar exactly in $2k-1$ tosses is

$$c_{2k-1} = \binom{0.5}{k}(-1)^k 2^{2k-1}.$$

It can be simplified to

$$c_{2k-1} = \frac{2^{k-1}\cdot 1\cdot 3\cdot 5\cdot\ldots(2k-3)}{k!}$$

4 Catalan numbers

The representation of Catalan numbers as triangulations of convex n-gon was considered in Pólya [1956]. So I will consider some other representations of Catalan numbers.

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References

George Pólya. On picture-writing. *The American Mathematical Monthly*, 63(10):689–697, 1956.