EXPECTED NUMBER OF n-SIDED DIE THROWS TO COLLECT k POINTS IS A GEOMETRIC SERIES FOR $k \leq n$

A PREPRINT

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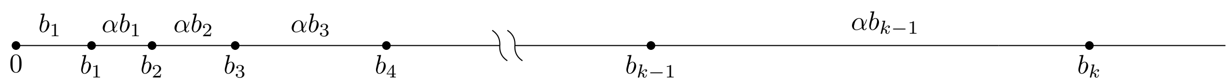
1 Link between die and geometric series

Let's start with a visual proof of a formula for the finite geometric series sum. We consider a growing geometric series (b_j) where $b_{j+1} = rb_j$ with ratio $r > 1$. We represent the ratio as $r = 1 + \alpha$, so $b_{j+1} = b_j + \alpha b_j$.



Adding all segments lengths we see that

$$b_k = b_1 + \alpha(b_1 + b_2 + b_3 + \ldots + b_{k-1})$$

This property of geometric series also has an economic intuition. You final wellfare b_k equals your initial wellfare b_1 plus all interest payments, where α is the nominal rate in the compound interest scheme.

Now let's toss a n-sided die until we collect k points or more with $k \leq n$. We denote the random number of tosses by X_k and the expected number of tosses by $b_k = \mathbb{E}(X_k)$.

For sure we'll collect one point or more with exactly one toss, $X_1 = 1$ and hence $b_1 = 1$.

Now we need to collect k points or more. Let's consider the first toss. With probability $(n - k)/n$ zero more tosses will be required. And with probability $1/n$ we will need to collect 1, 2, ..., $k - 1$ points.

The recurrence formula is

$$b_k = 1 + \frac{n-k}{n} \cdot 0 + \frac{1}{n}(b_1 + b_2 + \ldots + b_{k-1}).$$

This equation coupled with initial condition defines the geometric series with $b_1 = 1$ and ratio $r = 1 + \frac{1}{n}$.

Hence we obtain

$$b_k = \mathbb{E}(X_k) = \left(1 + \frac{1}{n}\right)^{k-1}$$

This result appears in disguised form $\mathbb{E}(X_k) = \sum_{i=0}^{k-1} \binom{k-1}{i}/n^i$ in Conroy [2021]. The particular case with $k = n$ is proven in Treviño [2020]. In both these sources the Christmas stockings theorem is invoked in the proof. The approach with recurrence equation is used in Suspect [2019], where the chase for the general k hides the simple structure for $k \leq n$.

References

M Conroy. A collection of dice problems, 2021. URL www.madandmoonly.com/doctormatt/mathematics/dice1.pdf.

Enrique Treviño. Expected number of dice rolls for the sum to reach n. *The American Mathematical Monthly*, 127(3): 257–257, 2020.

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