EDITORIAL COMMENTS ON YOUR MONTHLY PAPER FROM THE EDITOR

Congratulations on your paper reaching Provisional Acceptance with the *Monthly*. The *Monthly* accepts less than 10% of submissions, so we want to help you through the final steps to publication. Please find a markup of your paper with changes the Editor believes are necessary for publication. Before starting in on the changes, please note the following:

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- Also attached is a document containing the *Monthly*'s formatting style for references ("MAA Journal Reference Guide").
- Please include in your final cover letter the following information:
    - The 5 digit primary and up to two 5 digit secondary MSC index codes. A link to the MSC index can be found here: http://www.ams.org/mathscinet/
    - Please note if you or any of your co-authors are 40 years of age or younger. Authors in this category are eligible for the MAA's Merten M. Hasse Prize.
- We have partnered with Taylor & Francis to produce the *Monthly*. After final acceptance, you will receive a copyright transfer form from Taylor & Francis which you should return as soon as possible.
- Please make sure you attach in Editorial Manager both your LaTeX and PDF files. If applicable please upload your eps graphic files as well.

Failure to follow these steps will cause a delay in the final processing of your paper. When all is in order, you will receive an official Acceptance via Editorial Manager for your work. **Your paper is not officially accepted until this occurs.** Questions can be directed to the Editorial Office at monthly@maa.org. We thank you for submitting your work to *The American Mathematical Monthly*.

Best Wishes,

Della Dumbaugh

Della Dumbaugh, Editor-Elect

The American Mathematical Monthly

THE EXPECTED NUMBER OF n-SIDED DICE THROWS TO COLLECT k POINTS IS A GEOMETRIC SERIES FOR k≤n

--Manuscript Draft--

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Full Title:	THE EXPECTED NUMBER OF n-SIDED DICE THROWS TO COLLECT k POINTS IS A GEOMETRIC SERIES FOR k≤n
Article Type:	Filler
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Keywords:	geometric series, dice throws, expected value, recurrence relation
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Order of Authors:	Boris Demeshev, -
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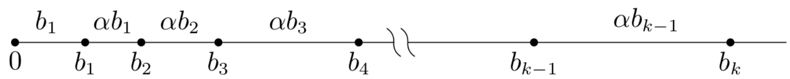
Manuscript Region of Origin:	RUSSIAN FEDERATION
Corresponding Author E-Mail:	boris.demeshev@gmail.com
Manuscript Classifications:	60: Probability Theory and Stochasitic Processes

The expected number of n-sided dice throws to collect k points is a geometric series for $k \le n$

Let's start with a visual proof of a formula for the sum of a finite geometric series. We consider a growing geometric series (b_j) where $b_{j+1} = rb_j$ with ratio $r > 1$. We represent the ratio as $r = 1 + \alpha$, so $b_{j+1} = b_j + \alpha b_j$.



Adding all the ~~segments lengths~~ we see that

$$b_k = b_1 + \alpha(b_1 + b_2 + b_3 + \ldots + b_{k-1})$$

This property of geometric series also has an economic intuition. ~~You~~ final welfare b_k equals your initial welfare b_1 plus all interest payments, where α is the nominal rate of compound interest.

Now let's toss a n-sided dice with faces marked $1, 2, \ldots, n$ until we collect k of these points or more with $k \le n$. We denote the random number of tosses by X_k and the expected number of tosses by $b_k = E(X_k)$.

We collect one point or more with exactly one toss, $X_1 = 1$ and hence $b_1 = 1$.

Now we need to collect k points or more. Let's consider the first toss. With probability $(n-k)/n$ zero more tosses will be required. And with probability $1/n$ we collect $1, 2, \ldots,$ or $k-1$ points.

The recurrence formula is

$$b_k = 1 + \frac{n-k}{n} \cdot 0 + \frac{1}{n}(b_1 + b_2 + \ldots + b_{k-1}).$$

This is exactly the geometric series equation with $b_1 = 1$ and $\alpha = \frac{1}{n}$.

Hence we obtain

$$b_k = E(X_k) = \left(1 + \frac{1}{n}\right)^{k-1}$$

This result appears in the disguised form $E(X_k) = \sum_{i=0}^{k-1} \binom{k-1}{i}/n^i$ in [**?**]. The particular case with $k = n$ is proven in [**?**]. ~~In both these sources the Christmas stockings theorem is invoked in the proof.~~ The approach with recurrence equation is used in [**?**], where the chase for the general k hides the simple structure for $k \le n$. The limiting case $n = k \to \infty$ is equivalent to a quite common textbook example with the answer $E \min\{T : U_1 + \ldots + U_T > 1\} = e$ where U_i are i.i.d. uniform on $[0;1]$ random variables.

REFERENCES

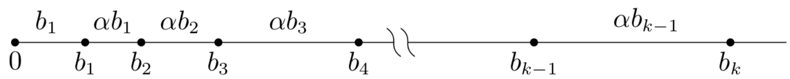
1. Conroy, M (2021). *A collection of dice problems*. `www.madandmoonly.com/doctormatt/mathematics/dice1.pdf`.
2. Treviño, Enrique (2020). Expected Number of Dice Rolls for the Sum to Reach n. *Amer. Math. Monthly*. 127(3):257-257.
3. Usual Suspect (2019). *Expected number of dice rolls require to make a sum greater than or equal to K?* `https://stats.stackexchange.com/q/146114`.

—Submitted by

doi.org/10.XXXX/amer.math.monthly.122.XX.XXX
MSC: Primary 00X00, Secondary 11Y11; 22Z22

The expected number of n-sided dice throws to collect k points is a geometric series for $k \leq n$

Let's start with a visual proof of a formula for the sum of a finite geometric series. We consider a growing geometric series (b_j) where $b_{j+1} = rb_j$ with ratio $r > 1$. We represent the ratio as $r = 1 + \alpha$, so $b_{j+1} = b_j + \alpha b_j$.



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$$b_k = b_1 + \alpha(b_1 + b_2 + b_3 + \ldots + b_{k-1})$$

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Hence we obtain

$$b_k = E(X_k) = \left(1 + \frac{1}{n}\right)^{k-1}$$

This result appears in the disguised form $E(X_k) = \sum_{i=0}^{k-1} \binom{k-1}{i}/n^i$ in [**1**]. The particular case with $k = n$ is proven in [**2**]. In both these sources the Christmas stockings theorem is invoked in the proof. The approach with recurrence equation is used in [**3**], where the chase for the general k hides the simple structure for $k \leq n$. The limiting case $n = k \to \infty$ is equivalent to a quite common textbook example with the answer $E \min\{T : U_1 + \ldots + U_T > 1\} = e$ where U_i are i.i.d. uniform on $[0; 1]$ random variables.

REFERENCES

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A simplier (I hope) proof of a more general result about expected number of dice rolls than in `https://www.tandfonline.com/doi/full/10.1080/00029890.2020.1693213`.

The list of submitted files:

- `filler-nsided.tex`: the main tex file.
- `filler-nsided.pdf`: compiled pdf.
- `geometric_series.png`: the picture. The corresponding tikz-code is commented out in the tex file.

UPDATE! Reply to the referee:

1. Totally agree. Updated the text.
2. Partially agree. However I think that numbering equation in a very short filler is too much. So I replaced the proposed (1) by (the geometric series equation).
3. Totally agree. Updated the text.
4. Not needed as the filler still fits one page limit after modifications.