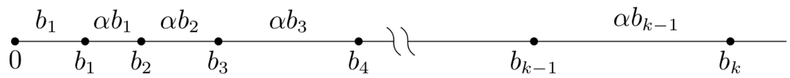
The Expected Number of n-sided Dice Throws to Collect k Points is a Geometric Series for $k \leq n$

Let's start with a visual proof of a formula for the sum of a finite geometric series. We consider a growing geometric series (b_j) where $b_{j+1} = rb_j$ with ratio $r > 1$. We represent the ratio as $r = 1 + \alpha$, so $b_{j+1} = b_j + \alpha b_j$.



Adding all the segment lengths we see that

$$b_k = b_1 + \alpha(b_1 + b_2 + b_3 + \ldots + b_{k-1}).$$

This property of geometric series also has an economic intuition. Your final welfare b_k equals your initial welfare b_1 plus all interest payments, where α is the nominal rate of compound interest.

Now let's toss a n-sided dice until we collect k points or more with $k \leq n$. We denote the random number of tosses by X_k and the expected number of tosses by $b_k = E(X_k)$.

We collect one point or more with exactly one toss, $X_1 = 1$ and hence $b_1 = 1$.

Now we need to collect k points or more. Let's consider the first toss. With probability $(n-k)/n$ zero more tosses will be required. And with probability $1/n$ we collect 1, 2, ..., or $k-1$ points.

The recurrence formula is

$$b_k = 1 + \frac{n-k}{n} \cdot 0 + \frac{1}{n}(b_1 + b_2 + \ldots + b_{k-1}).$$

This equation coupled with the initial conditions defines the geometric series with $b_1 = 1$ and ratio $r = 1 + \frac{1}{n}$.

Hence we obtain

$$b_k = E(X_k) = \left(1 + \frac{1}{n}\right)^{k-1}.$$

This result appears in the disguised form $E(X_k) = \sum_{i=0}^{k-1} \binom{k-1}{i}/n^i$ in [1]. The particular case with $k = n$ is proven in [2]. Both these sources invoke the Christmas stockings theorem in their proofs. The approach with recurrence equation is used in [3], where the chase for the general k hides the simple structure for $k \leq n$.

references

1. Conroy, M (2021). A collection of dice problems. www.madandmoonly.com/doctormatt/mathematics/dice1.pdf.
2. Treviño, Enrique (2020). Expected Number of Dice Rolls for the Sum to Reach n. Amer. Math. Monthly. 127(3):257-257.
3. Usual Suspect (2019). Expected number of dice rolls require to make a sum greater than or equal to K? https://stats.stackexchange.com/q/146114.

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