Привет !! Сем 5

инт. по частям

→ трюк Фейнмана
→ умение видеть экспоненту.
→ корзинка с нахожденками

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\cos px}{1+x^2}\,dx = \dots = \frac{\pi}{2}\cdot\exp(-p) \qquad p\geq 0$$

10*

$$I(p) = \int_0^\infty \left(\frac{\sin(px)}{x}\right)^2 dx$$

$$\left(\frac{\sin(px)}{x}\right)^2 \leq \frac{1}{x^2}$$

$$I'(p) = \int_0^\infty \frac{1}{x^2}\cdot \boxed{2\sin(px)\cdot\cos(px)}\cdot x\cdot dx =$$

$$= \int_0^\infty \frac{\sin(2px)}{x}\,dx = \int_0^\infty \frac{\sin(2px)}{2px}\,d(2px) = \frac{\pi}{2}$$

$$I(p) = \frac{\pi}{2}\cdot p + \underline{\underline{const}}$$

нач. условия:

$$I(0) \overset{?}{=} \int_0^\infty \frac{(\sin 0)^2}{x^2}\,dx = 0$$

$$I(0)=0 \qquad \frac{\pi}{2}\cdot 0 + const = 0$$

$$I(p) = \frac{\pi}{2}p$$

9*

$$\int_0^\infty \frac{1}{(x^2+p)^n}\,dx = J(p)$$

$$J'(p) = \int_0^\infty (-n)\cdot\frac{1}{(x^2+p)^{n+1}}\,dx$$

→ Фейнмана

→ по индукции рабоче-крестьянский

PK.

$x=\sqrt{p}\cdot u \qquad dx=\sqrt{p}\cdot du$

$x^2 = p\cdot u^2$

$$I_n = \int_0^\infty \frac{1}{(u^2p+p)^n}\cdot\sqrt{p}\cdot du = \sqrt{p}\cdot\frac{1}{p^n}\int_0^\infty \frac{1}{(u^2+1)^n}\,du$$

$$\int_0^\infty \frac{1}{(1+u^2)^n}\,du = \int_0^\infty \frac{1+u^2}{(1+u^2)^n}\,du - \int_0^\infty \frac{u^2}{(1+u^2)^n}\,du \qquad n>1$$

$1 = 1+u^2-u^2$ $\qquad I_{n-1}$ $\qquad$ берётся!

$$\int_0^\infty \underset{\downarrow \text{Diff}}{u} \cdot \overset{\uparrow \text{Int}}{\frac{1}{(1+u^2)^n}} \cdot u\,du = \boxed{u\cdot (1+u^2)^{-n+1}\cdot \frac{1}{2(1-n)}\Big|_0^\infty} - \qquad = 0$$

$$\left((1+u^2)^{-n+1}\right)' = (-n+1)\cdot(1+u^2)^{-n}\cdot 2u$$

$$-\int_0^\infty 1\cdot (1+u^2)^{-n+1}\cdot \frac{1}{2(1-n)}\,du =$$

$n=1$ $\qquad I_1 = \int_0^\infty \frac{1}{1+u^2}\,du = \frac{\pi}{2}$

$n\in\mathbb{N}$

$n>1$ $\qquad I_n = I_{n-1} - \left(0 - \left(\frac{1}{2(1-n)}\cdot I_{n-1}\right)\right)$

рекуррентное соотношение !!

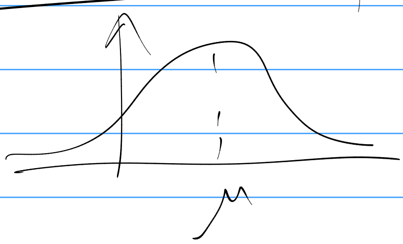
7. $\int_0^\infty x^2 e^{-px}\,dx$

8. $\int_0^\infty x^4 e^{-px^2}\,dx$

$\boxed{Y\sim Exp(\lambda) \quad E(Y^2)\,?}$?

$Var(Y) = E(Y^2) - (E(Y))^2$

№3 $\boxed{Z\sim N(0;\sigma^2) \quad E(Z^4)\,?}$ $\rightarrow$ $E(Z^4) = \int_{-\infty}^{\infty} z^4\cdot f(z)\cdot dz =$



$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\cdot \exp\left(-\frac{1}{2\sigma^2}(z-\mu)^2\right)$$

$$= \underbrace{\int_{-\infty}^{\infty} z^4\cdot \frac{1}{\sqrt{2\pi\sigma^2}}\cdot \exp\left(-\frac{z^2}{2\sigma^2}\right)}_{\text{чётная !!}} dz = \frac{2}{\sqrt{2\pi\sigma^2}} \boxed{\int_0^\infty z^4\cdot \exp(-pz^2)\cdot dz}$$

$p = \frac{1}{2\sigma^2}$

$Z \sim \boxed{N(0;1)}$ т Фейнмана $\boxed{\text{(функция произв-щая моментов)}}$

$$E(\exp(+uZ)) = \int_{-\infty}^{\infty} \cancel{z} \, \exp(+uz) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \cdot z^2\right) dz = M_Z(u)$$

$$\exp(uz) = 1 + uz + \frac{(uz)^2}{2!} + \frac{(uz)^3}{3!} + \frac{(uz)^4}{4!} \ldots$$

$\boxed{\text{вот это!}}$

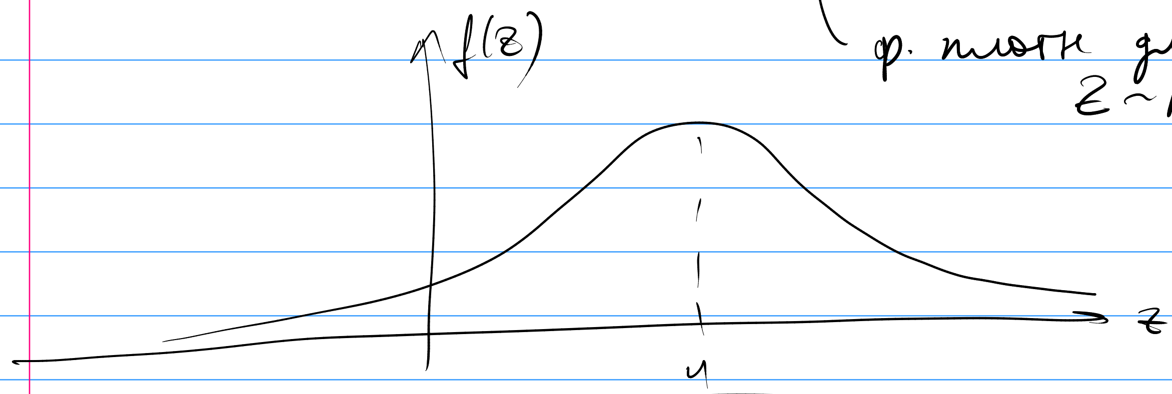
Шаг 1. найдем $M(u)$?

Шаг 2 вычислим $E(Z^4)$ $E(Z^6)$ $E(Z^{2022})$ ....

$$M(u) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z^2 - 2uz)\right) dz =$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z^2 - 2uz + u^2) + \frac{u^2}{2}\right) dz$$

$$= \exp\left(\frac{u^2}{2}\right) \cdot \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z-u)^2\right)}_{\text{ф. плотн для } Z \sim N(0;u)} dz = \exp\left(\frac{u^2}{2}\right) \cdot 1$$



Итог Шага 1: $\boxed{\underline{E(\exp(uZ))} = \exp\left(\frac{u^2}{2}\right)}$ $Z \sim N(0;1)$

$M(u)$

$$M''''(u) \stackrel{?}{=} \boxed{E(Z^4 \exp(uZ)) = \left(\exp\left(\frac{u^2}{2}\right)\right)''''}$$

$$M''''(0) = E(Z^4) = ?$$

$$M^{(2022)}(0) = E(Z^{2022}) = ?$$

$$M(u) = \exp\left(\frac{u^2}{2}\right) = 1 + \frac{u^2}{2} + \frac{(u^2/2)^2}{2!} + \frac{(u^2/2)^3}{3!} + \ldots$$

для $M''''(0)$

$$M''''(0) = 4! \cdot \frac{(1/2)^2}{2!} = 3$$

$$M^{(2022)}(0) = 2022! \cdot \frac{(1/2)^{1011}}{1011!} = 2021!!$$

$n!! =$
$= n \cdot (n-2)(n-4) \ldots$

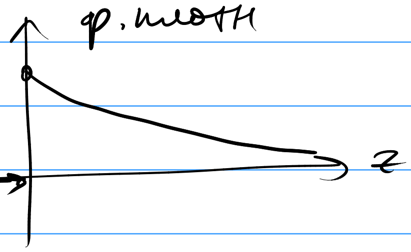
$$\frac{2022}{2} \cdot 2021 \cdot \frac{2020}{2} \cdot \ldots \cdot \frac{1012}{2} = 2021 \cdot 2019 \cdot 2017 \cdot \ldots \cdot 1011 \cdot 1009 \cdot \ldots \cdot 1$$

$$Y \sim N(0; \sigma^2) \qquad E(Y^{40}) =$$

$$= E\left(\left(\frac{Y}{\sigma}\right)^{40} \cdot \sigma^{40}\right) =$$

$$= \sigma^{40} \cdot E(Z^{40}) = \sigma^{40} \cdot 39 \cdot 37 \cdot 35 \cdot \ldots \cdot 1$$

№7 $\quad Z \sim \mathrm{Exp}(\lambda)$



$$f(z) = \begin{cases} \lambda \cdot \exp(-\lambda z) & z \geq 0 \\ 0 & \text{иначе} \end{cases}$$

$$E(Z^2) = \int_0^{\infty} z^2 \cdot \lambda \cdot \exp(-\lambda z)\, dz \qquad (\text{инт № 7})$$

$E(z^4)$ $E(z^{2021})$

$$M(u) = E(\exp(uz)) = \int_0^\infty \exp(uz)\cdot\lambda\cdot\exp(-\lambda z)\,dz$$

$\boxed{\lambda>0}$ $u<\lambda$

$$= \frac{\lambda}{u-\lambda}\exp((u-\lambda)z)\Big|_{z=0}^{z=\infty} = \frac{\lambda}{u-\lambda}\cdot(0-1) =$$

$$= \frac{\lambda}{\lambda-u} = \boxed{E(\exp(uz)) = M(u)}$$

$E(z) = M'(0)$

$E(z\cdot\exp(uz)) = M'(u)$

$E(z\cdot 1) = M'(0)$

$E(z^2) = M''(0)$

$E(z^{2022}) = M^{(2022)}(0)$

$$\frac{\lambda}{\lambda-u} = \frac{1}{1-\left(\frac{u}{\lambda}\right)} = ⊛$$

$$\boxed{\frac{1}{1-q} = 1+q+q^2+q^3+q^4+\ldots}$$

ряд Тейлора для $\frac{1}{1-q}$

$$⊛ = 1+\left(\frac{u}{\lambda}\right)+\left(\frac{u}{\lambda}\right)^2+\left(\frac{u}{\lambda}\right)^3+\left(\frac{u}{\lambda}\right)^4+\ldots$$

$E(z) = M'(0) = \frac{1}{\lambda}$

$E(z^2) = M''(0) = \frac{2}{\lambda^2}$

$Var(z) = E(z^2) - (E(z))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

№7

$$\int_0^\infty z^2\cdot\lambda\cdot e^{-\lambda z}\,dz$$

$z \sim \exp(\lambda)$

$$E(z^{2022}) \overset{?}{=} \frac{2022!}{\lambda^{2022}}$$

№6 $$\int_0^\infty \frac{x\sin(px)}{1+x^2}\,dx \overset{?}{=} \left(\int_0^\infty \frac{-\cos(px)}{1+x^2}\,dx\right)'_p =$$

$$= (-1)\cdot\left(\boxed{\int_0^\infty \frac{\cos(px)}{1+x^2}\,dx}\right)'_p$$ → корзиночный способ

$\boxed{\text{идея}}$

$$I(p)\to I'(p) \quad \begin{cases} \displaystyle\int_0^\infty \frac{\sin(px)}{x}\cdot \; dx = I(p) \\ \displaystyle\int_0^\infty \frac{\exp(-px)}{x}\cdot \; dx = J(p) \end{cases}$$

рассмотрим $I'(p)$, $J'(p)$

идея $$\int_0^\infty x\cdot\sin(px)\cdot \; dx = A'(p)$$

$A'(p)\to A(p)$ $$A(p) = \int_0^\infty (-1)\cdot\cos(px)\cdot \; \cdot dx$$