The table below is for grading. Please, leave it blank :)

Problem	1	2	3	4	5	6	7	8	9	10	Total
Maximum	10	10	10	10	10	10	10	10	10	10	100
Points											

You may write your solutions in Russian or in English.

Good luck!

1. (10%) Evaluate the following limit:

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right).$$

2. (10%) Let $D(x)$ be so-called Dirichlet function, which equals 1 if its argument is rational and 0 otherwise, and let k be a natural number.

   Prove that the function $x^k D(x)$ is nowhere differentiable if $k=1$ and is differentiable only at $x=0$ if $k=2017$.

3. The matrices A and B are symmetric 3×3 matrices. Eigenvalues of the matrix A are $\lambda_1^A=4, \lambda_2^A=2, \lambda_3^A=1$, eigenvalues of the matrix B are $\lambda_1^B=11, \lambda_2^B=5, \lambda_3^B=1$.
   (a) (2%) Find the trace (the sum of diagonal elements) of the matrix $A+B$.
   (b) (3%) Let the matrix C be 3×3 matrix with $\det(C)=1$. Find $\operatorname{tr}(C^{-1}AC)$.
   (c) (5%) Prove that $\operatorname{tr}(A^k)=\sum_i \lambda_i^k$.

4. Consider the matrix

$$A=\begin{pmatrix}-1 & 2 & 0\\ 0 & 1 & 0\\ -18 & 18 & 8\end{pmatrix}.$$

   (a) (4%) Find the eigenvalues and eigenvectors of matrix A.
   (b) (6%) Find the matrix $A^{1/3}$. By definition, $A^{1/3}$ is such a matrix that $(A^{1/3})^3=A$.

5. (10%) Solve the differential equation:

$$(3x^2y^4+2xy)\,dx+(2y^2-3x^2)\,dy=0$$

6. Let $F(x,y)=xy$ and $G(x,y;a,b)=y+bx-a$.
   (a) (3%) For each value of parameters (a,b) find the conditional extremum if it exists, classify it and find the extremal value $F^*(a,b)$.
   (b) (4%) Find all possible values of $F^*(a,b)$ in the region

$$D_1=\begin{cases}a\ge b\\ a\in(0,1)\end{cases}$$

   (c) (3%) Find all possible values of $F^*(a,b)$ in the region

$$D_2=\begin{cases}a\ge b\\ a\in(0,1)\\ b\ge 0.5\end{cases}$$

7. The island is populated with knights and knaves. Each sentence of a knight is true with probability 0.9 independently of other sentences. Each sentence of a knave is true with probability 0.2 independently of other sentences. The proportion of knights on the island is equal to 0.7. You meet one person on the island at random and asked him, whether he is a knight.
   (a) (2%) What is the probability that he will say «I am a knight»?
   (b) (4%) What is the conditional probability that he is a knight given that he said «I am a knight»?
   (c) (4%) What is the conditional probability that he is a knight given that he said «I am a knight», paused and said «I am not a knight»?

8. The joint density of random variables X and Y is given by the function

$$f(x,y) = \begin{cases} x+y, & \text{if } x \in [0;1],\ y \in [0;1] \\ 0, & \text{otherwise} \end{cases}$$

   (a) (2%) Are X and Y independent? Give short argument.
   (b) (4%) Find $\mathbb{P}(Y > 2X)$ and $\mathbb{E}(XY)$
   (c) (4%) Find marginal density $f_X(x)$ and conditional density $f_{Y|X}(y|x)$

9. Boris loves hunting Pokemons. Today he randomly captured three Pokemons.

   Boris has sorted Pokemons by their height in ascending order and obtained their ranks H_i. The lowest Pokemon gets the rank $H_i = 1$, the tallest gets the rank $H_i = 3$. After sorting Pokemons by their combat power Boris obtained the ranks C_i in the same manner. Height and combat power of Pokemons are continuously distributed, so ties are impossible.

   Boris would like to test the hypothesis H_0: height and combat power are independent. He calculates $\hat{\rho}$, sample Pearson correlation coefficient between ranks C_i and H_i.
   (a) (6%) Find the distribution of $\hat{\rho}$ under H_0, that is find all possible values of $\hat{\rho}$ and their probabilities.
   (b) (4%) Find the minimal threshold value ρ^* that will be exceeded by $\hat{\rho}$ with probability less or equal to 0.2 under H_0.

10. (10%) You estimated two models using 47 observations:
   A. $\hat{y}_i = 40 + 0.3x_i + 0.8z_i - 1.8w_i$, $R^2 = 0.82$
   B. $\hat{y}_i = 65 + 0.6x_i + 0.51z_i$, $R^2 = 0.7$

   Test the hypothesis $\beta_w = -1$ against $\beta_w \neq -1$ on 5% significance level. Here β_w is the coefficient before the variable w in the first regression.