Exam demo-version

1. (10%) Evaluate the following limit:

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}}$$

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}} = \lim_{x\to 0} \left(\cos\sqrt{x}\right)^{1/x} = \lim_{x\to 0} \exp\left(\frac{\ln(\cos\sqrt{x})}{x}\right) = \exp\left(\lim_{x\to 0} \frac{\ln(\cos\sqrt{x})}{x}\right),$$

In the last equality we interchanged limit and continuous function. Now we use Taylor's expansion:

$$\frac{\ln(\cos\sqrt{x})}{x} = \frac{\ln(1-\frac{x}{2}+\frac{x^2}{24}+o(x^2))}{x} = \frac{-\frac{x}{2}+o(x)}{x}$$

It follows that

$$\lim_{x\to 0} \frac{\ln(\cos\sqrt{x})}{x} = -\frac{1}{2},$$

And finally

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}} = \exp\left(-\frac{1}{2}\right) = \frac{1}{\sqrt{e}}.$$

2. (10%) Find and classify the discontinuity points of the following function:

$$f(x) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{x}\right)\right).$$

Точки, в которых данная функция может иметь разрыв: $x = 0$, поскольку в ней равен нулю знаменатель аргумента функции, и точки $x = 1/k, k \in \mathbb{Z}$, поскольку в них $\sin\left(\frac{\pi}{x}\right)$ меняет знак. В точках $x = 1/k, k \in \mathbb{Z}$ функция имеет разрывы первого рода, так как существуют не равные между собой односторонние пределы. Например, рассмотрим $k = 1$. Существует правосторонняя окрестность точки $x = 1$, в которой функция $\sin\left(\frac{\pi}{x}\right)$ положительна. В самом деле, для $x \in (1,2)$ имеет место $\frac{\pi}{2} < \frac{\pi}{x} < \pi$. Для точек из этой окрестности имеем $f(x) = 1$, следовательно, $\lim_{x\to 1+0} f(x) = 1$. С другой стороны, существует левосторонняя окрестность точки $x = 1$, в которой функция $\sin\left(\frac{\pi}{x}\right)$ отрицательна. В самом деле, для $x \in (1/2, 1)$ имеет место $\pi < \frac{\pi}{x} < 2\pi$. Для точек из этой окрестности имеем $f(x) = -1$, следовательно, $\lim_{x\to 1-0} f(x) = -1$. Аналогичные окрестности могут быть найдены для всех рассматриваемых точек.

В точке $x = 0$ функция имеет разрыв второго рода, поскольку не существует односторонних пределов. Действительно, рассмотрим последовательности $a_n = \frac{2}{1+4n}, n \in \mathbb{N}$ и $b_n = \frac{2}{3+4n}, n \in \mathbb{N}$, стремящиеся к нулю справа. Тогда $f(a_n) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{\frac{2}{1+4n}}\right)\right) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{2}+2\pi n\right)\right) = 1$ и $f(b_n) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{\frac{2}{3+4n}}\right)\right) = \operatorname{sgn}\left(\sin\left(\frac{3\pi}{2}+2\pi n\right)\right) = -1$. Тем самым показано, что правостороннего предела $f(x)$ при x стремящемся к нулю не существует. Аналогично можно показать, что не существует левостороннего предела, например, рассмотрев последовательности $-a_n$ и $-b_n$.

3. Matrix A is given by

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

(a) (6%) Find the eigenvalues and eigenvectors of A;

First, we solve the equation $\det(A - \lambda I) = 0$. The roots are $\lambda_1 = 2$, $\lambda_2 = 5$, $\lambda_3 = 10$.
The eigenvectors are

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix};\ v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};\ v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

(b) (4%) Find the eigenvalues of $4A^{-1} + 2I$, where I is identity matrix.

Multiplication by 4 does not change eigenvalues. So we find eigenvalues of $A^{-1} + 0.5I$. They are $a_1 = 0.5 + 0.5 = 1$, $a_2 = 0.2 + 0.5 = 0.7$, $a_3 = 0.1 + 0.5 = 0.6$.

4. The characteristic polynomial of a matrix B is given by $f(\lambda) = 6\lambda - 5\lambda^2 - \lambda^3$.

(a) (6%) Find dimensions of B, $\operatorname{rank} B$, $\det B$, sum of diagonal elements of B;

The highest power of λ is 3, so the dimension of B is 3×3. We can solve for λ and find $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -6$. So, $\det B = 0$ and $\operatorname{trace} B = \sum b_{ii} = 0 + 1 - 6 = -5$.

(b) (2%) Suppose additionaly that B is symmetric. Can we find random variables with covariance matrix B?

The quadratic form B is indefinite, so B is not a valid covariance matrix.

(c) (2%) Suppose additionaly that B is symmetric. How many solutions does equation $v^T B v = -2018$ has?

The quadratic form B is indefinite, so the equation has infinitely many solutions.

5. (10%) Solve the following differential equations

(a) (5%) $y'' - 2y' - 8y = 0$,

Let us solve the characteristic equation

$$\lambda^2 - 2\lambda - 8 = 0$$

corresponding to the differential equation (a). It is easy to see that $\lambda_1 = 4$ and $\lambda_2 = -2$ are the solutions of this characteristic equation. Hence, the general solution of the differential equation (a) is

$$y_a(x) = C_1 e^{4x} + C_2 e^{-2x}, \quad \text{where } C_1,\, C_2 \in \mathbb{R}.$$

(b) (5%) $y'' - 2y' - 8y = e^x - 8\cos 2x$.

To solve the differential equation (b) we have to find the particular solutions of the following differential equations

$$y'' - 2y' - 8y = e^x, \tag{1}$$

and

$$y'' - 2y' - 8y = -8\cos 2x. \tag{2}$$

We seek the particular solution of the differential equation (1) in the form

$$y(x) = Ae^x. \tag{3}$$

Substituting expression (3) into equation (1), we obtain $A = -1/9$.

We seek the particular solution of the differential equation (2) in the form

$$y(x) = B_1 \cos 2x + B_2 \sin 2x. \tag{4}$$

Substituting expression (4) into equation (2), we obtain $B_1 = 3/5$ and $B_2 = 1/5$.
The particular solution of the differential equation (b) is a sum of particular solutions of equations (1) and (2). Thus, the particular solution of the differential equation (b) is

$$y(x) = -\frac{1}{9}e^x + \frac{3}{5}\cos 2x + \frac{1}{5}\sin 2x.$$

Therefore, the general solution of equation (b) is

$$y_b(x) = C_1 e^{4x} + C_2 e^{-2x} - \frac{1}{9}e^x + \frac{3}{5}\cos 2x + \frac{1}{5}\sin 2x, \quad \text{where } C_1,\, C_2 \in \mathbb{R}.$$

6. (10%) Find the points of maximum of the function

$$F(u,v) = \sqrt{u}\left(\sqrt{u} - 2\right) - \sqrt{v}\left(\sqrt{v} - 2\right),$$

given that $\sqrt{u} \le 2$, $\sqrt{v} \le 2$

1. We use the change of variables $x = \sqrt{u}$, $y = \sqrt{v}$. Using algebraic manipulations we transform G into: $G(x,y) = (x-1)^2 - (y-1)^2 + 3$. Now constraints have the form $x \in [0,2]$, $y \in [0,2]$
2. First we check for internal extrema:

$$\frac{\partial G(x,y)}{\partial x} = 2(x-1), \ \frac{\partial G(x,y)}{\partial y} = -2(y-1), \ \frac{\partial^2 G(x,y)}{\partial x \partial y} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

   Using Sylvester's criterion we find that the point $(1,1)$ is not a maximum of $G(x,y)$.
3. Now we check for corner solutions.

   Line $\{x = 0,\ y \in [0,2]\}$, $G(0,y) = 1 - (y-1)^2$. At the point (0, 1) we have a **maximum** equal to 1, with values at the borders equal to 0.

   Line $\{y = 0,\ x \in [0,2]\}$, $G(x,0) = (x-1)^2 - 1$. At the point (1, 0) we have a minimum equal to (-1), with values at the borders equal to 0.

   Line $\{x = 2,\ y \in [0,2]\}$, $G(0,y) = 1 - (y-1)^2$. At the point (2, 1) we have a **maximum** equal to 1, with values at the borders equal to 0.

   Line $\{y = 2,\ x \in [0,2]\}$, $G(x,2) = (x-1)^2 - 1$. At the point (1, 2) we have a minimum equal to (-1), with values at the borders equal to 0.
4. Now we do inverse substitution.

Answer: Maxumum points: (0, 1) и (4, 1)

7. There are three coins in the bag. Two coins are unbiased, and for the third coin the probability of «head» is equal to 0.8. James Bond chooses one coin at random from the bag and tosses it

   (a) (5%) What is the probability that it will show «head»?

$$\mathbb{P}(B) = \mathbb{P}(\text{head}) = \frac{2}{3} \cdot 0.5 + \frac{1}{3} \cdot 0.8 = \frac{18}{30} = \frac{3}{5} = 0.6$$

   (b) (5%) What is the conditional probability that the coin is unbiased if it shows «head»?

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\frac{2}{3} \cdot 0.5}{0.6} = \frac{10}{18} = \frac{5}{9}$$

8. The pair of random variables X and Y with $\mathbb{E}(X) = 0$ and $\mathbb{E}(Y) = 1$ has the following covariance matrix

$$\begin{pmatrix} 10 & -2 \\ -2 & 9 \end{pmatrix}.$$

   (a) (5%) Find $\mathbb{V}\mathrm{ar}(X + Y)$, $\mathbb{C}\mathrm{orr}(X, Y)$, $\mathbb{C}\mathrm{ov}(X - 2Y + 1, 7 + X + Y)$

$$\mathbb{V}\mathrm{ar}(X + Y) = 10 + 9 - 2 \cdot 2 = 15$$

$$\mathbb{C}\mathrm{orr}(X, Y) = \frac{-2}{\sqrt{10 \cdot 9}}$$

$$\mathbb{C}\mathrm{ov}(X - 2Y + 1, 7 + X + Y) = \mathbb{V}\mathrm{ar}(X) - 2\,\mathbb{C}\mathrm{ov}(Y, X) + \mathbb{C}\mathrm{ov}(X, Y) - 2\,\mathbb{V}\mathrm{ar}(Y) = 10 - (-2) - 18 = -6$$

   (b) (5%) Find the value of a if it is known that X is independent of $Y - aX$.

For independent variables the covariance is equal to zero: $\mathbb{C}\mathrm{ov}(X, Y - aX) = 0$. Hence, $a = \frac{\mathbb{C}\mathrm{ov}(X,Y)}{\mathbb{V}\mathrm{ar}(X)} = -0.2$.

9. You have height measurements of a random sample of 100 persons, $y_1, \ldots, y_{100}$. It is known that $\sum_{i=1}^{100} y_i = 15800$ and $\sum_{i=1}^{100} y_i^2 = 2530060$.

   (a) (3%) Calculate unbiased estimate of population mean and population variance of the height

Sample mean: $\bar{y} = 15800/100 = 158$.
Unbiased estimate of the variance:

$$\hat{\sigma}^2 = \frac{\sum (y_i - \bar{y})^2}{n - 1} = \frac{\sum y_i^2 - n\bar{y}^2}{n - 1} = 340$$

   (b) (3%) At 4% significance test the null-hypothesis that the population mean is equal to 155 cm, against two-sided alternative.

Observed value of Z-statistics

$$Z_{obs} = \frac{158 - 155}{\sqrt{340}/\sqrt{100}} = 1.63$$

Critical value of Z-statistics $Z_{crit} = 2.05$.
Conclusion: hypothesis H_0 is not rejected.

   (c) (2%) Find the p-value

Using tables we find that the area under the curve to the right of 1.63 is approximately 5%. Hense p-value is equal to 10%.

(d) (2%) Find the 96% confidence interval for the population mean

The confidence interval has the form

$$[158 - 2.05 \cdot \sqrt{340/100}; 158 + 2.05 \cdot \sqrt{340/100}]$$

Finally: $[154.2; 161.8]$

10. Let $X = (X_1, \ldots, X_n)$ be a random sample from normal distribution with zero mean and unknown variance $\sigma^2 > 0$.

If $\xi \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[\xi^4] = 3\sigma^2$.

(a) (2%) Derive the log-likelihood function of a random sample X.

The likelihood function of a random sample X is

$$L(x_1, \ldots, x_n; \sigma^2) = f_{X_1, \ldots, X_n}(x_1, \ldots, x_n; \sigma^2) =$$

$$= \prod_{i=1}^{n} f_{X_i}(x_i; \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}} = (2\pi)^{-n/2}(\sigma^2)^{-n/2} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}}.$$

Hence, the log-likelihood function of a random sample X is

$$l(x_1, \ldots, x_n; \sigma^2) := \ln \mathcal{L}(x_1, \ldots, x_n; \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}.$$

(b) (2%) Find the estimator of the parameter σ^2 using maximum likelihood method.

Let us write the likelihood equation:

$$\frac{\partial l(x_1, \ldots, x_n; \sigma^2)}{\partial(\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^4} = 0.$$

Solving this equation with respect to σ^2, we find $\sigma^2 = \sum_{i=1}^{n} x_i^2$. Hence, the maximum likelihood estimator of the parameter σ^2 is $\widehat{\sigma^2}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

(c) (2%) Using the realization of a random sample $x = (1, -2, 0, 1)$ find the maximum likelihood estimate of the parameter σ^2 derived in (b).

The maximum likelihood estimate of the parameter σ^2 is

$$\widehat{\sigma^2}_{ML} = \tfrac{1}{4}\left(1^2 + (-2)^2 + 0^2 + 1^2\right) = \tfrac{3}{2}.$$

(d) (2%) Find the Fisher information $I_n(\sigma^2)$ about the parameter σ^2 contained in n observations of a random sample.

Considering that $l(x_1; \sigma^2) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma^2) - \frac{x_1^2}{2\sigma^2}$, we arrive at

$$\frac{\partial l(x_1; \sigma^2)}{\partial(\sigma^2)} = -\frac{1}{2\sigma^2} + \frac{x_1^2}{2\sigma^4} = \frac{x_1^2 - \sigma^2}{2\sigma^4}.$$

Therefore, the Fisher information $I_1(\sigma^2)$ about the parameter σ^2 contained in a single observation of a random sample is

$$I_1(\sigma^2) = \mathbb{E}\left[\left(\frac{\partial l(X_1;\sigma^2)}{\partial(\sigma^2)}\right)^2\right] = \mathbb{E}\left[\frac{\left(X_1^2-\sigma^2\right)^2}{4\sigma^8}\right] = \frac{\mathbb{E}\left[\left(X_1^2-\sigma^2\right)^2\right]}{4\sigma^8} \underset{=}{\overset{\mathbb{E}[X_1^2]=\sigma^2}{=}}$$

$$= \frac{\mathrm{D}(X_1^2)}{4\sigma^8} = \frac{\mathbb{E}[X_1^4]-\left(\mathbb{E}[X_1^2]\right)^2}{4\sigma^8} = \frac{3\sigma^4-\left(\sigma^2\right)^2}{4\sigma^8} = \frac{2\sigma^4}{4\sigma^8} = \frac{1}{2\sigma^4}.$$

Thus, $I_n(\sigma^2) = n \cdot I_1(\sigma^2) = \frac{n}{2\sigma^4}$.

(e) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ an unbiased estimator of the parameter σ^2?

The estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ is unbiased, since

$$\mathbb{E}[\widehat{\sigma^2}] = \mathbb{E}\left[\tfrac{1}{n}\sum_{i=1}^n X_i^2\right] = \tfrac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i^2] = \tfrac{1}{n}\sum_{i=1}^n \sigma^2 = \tfrac{1}{n}n\sigma^2 = \sigma^2.$$

(f) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ an efficient estimator of the parameter σ^2?

The estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ is efficient, as

$$\mathrm{D}(\widehat{\sigma^2}) = \mathrm{D}\left(\tfrac{1}{n}\sum_{i=1}^n X_i^2\right) = \tfrac{1}{n^2}\sum_{i=1}^n \mathrm{D}(X_i^2) = \tfrac{1}{n^2}\sum_{i=1}^n 2\sigma^4 = \tfrac{1}{n^2}n2\sigma^4 = \frac{2\sigma^4}{n},$$

and

$$I_n^{-1}(\sigma^2) = \left(\frac{n}{2\sigma^2}\right)^{-1} = \frac{2\sigma^2}{n} = \mathrm{D}(\widehat{\sigma^2}).$$

(g) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$ a consistent estimator of the parameter σ^2?

As random variables $X_1^2, \ldots, X_n^2, \ldots$ are independent, have the same distribution, and have finite means, then the we can apply the law of large numbers, according to which we have

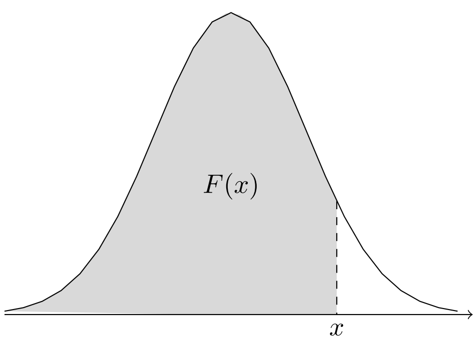
$$\tfrac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_i^2] = \sigma^2 \qquad \text{as } n \to \infty.$$

(h) (2%) Find the following probability limit $\operatorname{plim}_{n\to\infty}\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2}$.

As $\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{\mathbb{P}} \mathbb{E}[X_i^2] = \sigma^2$, and the function $g(x) = \sqrt{x}$ is continuous, by Slutsky's theorem we derive

$$\sqrt{\tfrac{1}{n}\sum_{i=1}^n X_i^2} = g\left(\tfrac{1}{n}\sum_{i=1}^n X_i^2\right) \xrightarrow{\mathbb{P}} g(\sigma^2) = \sqrt{\sigma^2} = \sigma \qquad \text{as } n \to \infty.$$

Good luck!



x	$F(x)$	x	$F(x)$	x	$F(x)$	x	$F(x)$
0.050	0.520	0.750	0.773	1.450	0.926	2.150	0.984
0.100	0.540	0.800	0.788	1.500	0.933	2.200	0.986
0.150	0.560	0.850	0.802	1.550	0.939	2.250	0.988
0.200	0.579	0.900	0.816	1.600	0.945	2.300	0.989
0.250	0.599	0.950	0.829	1.650	0.951	2.350	0.991
0.300	0.618	1.000	0.841	1.700	0.955	2.400	0.992
0.350	0.637	1.050	0.853	1.750	0.960	2.450	0.993
0.400	0.655	1.100	0.864	1.800	0.964	2.500	0.994
0.450	0.674	1.150	0.875	1.850	0.968	2.550	0.995
0.500	0.691	1.200	0.885	1.900	0.971	2.600	0.995
0.550	0.709	1.250	0.894	1.950	0.974	2.650	0.996
0.600	0.726	1.300	0.903	2.000	0.977	2.700	0.997
0.650	0.742	1.350	0.911	2.050	0.980	2.750	0.997
0.700	0.758	1.400	0.919	2.100	0.982	2.800	0.997

Рис. 1: Distribution function of a standard normal random variable