Variant A

1. (10%) Evaluate the following limit:

$$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x}$$

2. (10%) Find and classify the discontinuity points of the following function:

$$f(x)=\frac{\frac{1}{x}-\frac{1}{x+1}}{\frac{1}{x-1}-\frac{1}{x}}.$$

3. Let S be the $n\times n$ «shipbuilding timber» matrix, i.e. the square matrix with all elements equal to 1 and I be the $n\times n$ identity matrix. Let $A=aI+bS$ where a and b are scalar parameters.
   (a) (7%) Find the inverse of A if it is known that it exists and can be represented as a linear combination of I and S
   (b) (3%) Using your result in previous part or otherwise find the inverse of

$$\begin{pmatrix}-1 & 1 & 1 & 1\\ 1 & -1 & 1 & 1\\ 1 & 1 & -1 & 1\\ 1 & 1 & 1 & -1\end{pmatrix}$$

4. Matrices A, B and M are $n\times n$ real matrices, A' denotes the transpose of A, I is $n\times n$ identity matrix.
   (a) (5%) Solve the matrix equation for Y and simplify the answer

$$A'(Y')^{-1}A^2-B=M.$$

   You may assume that all necessary inverse matrices exist.
   (b) (5%) The matrix H is $m\times n$ real matrix of full rank with $m>n$. The matrices X and Y are defined by $X=I-H(H'H)^{-1}H'$ and $Y=I-X$. Prove that $X=X'=X^2$ and $Z=Z'=Z^2$.

5. (10%) For all values of a find and classify the conditional extremum of

$$G(x,y;a)=\frac{-6a^2y+12axy-9ay^2+2a^2x+18xy^2-a^3}{3y+a}$$

   subject to $x+y=1$

6. (10%) Solve the nonhomogeneous differential equation of the fourth order:

$$y''''-3y'''+4y'=4\cos 2x$$

7. (10%) Solve the differential equation

$$y'+xy=2xy^2$$

8. The great wizard Theodore of N-sk knows that aliens spy on him! There are two alien satellites (the red one and the blue one) and one alien battleship flying around the Earth. Aliens can attack Theodore and try to steal his magical power: the red satellite will succeed in stealing with probability 0.1, the blue with probability 0.2, the battleship with probability 0.9. If there is more than one spacecraft, they attack him independently and simultaneously. It is possible that more than one spacecraft succeed in stealing his power. Aliens have one problem: satellites can attack only when they are flying above Theodore (it happens with probabilities 0.7 and 0.4 for red and blue satellites, respectively), and battleship can attack if both satellites are above Theodore and can not attack in other cases.
   (a) (1%) What is the probability that the battleship can attack Theodore?
   (b) (2%) What is the probability that aliens will steal the power of Theodore, if there are two satellites above him?
   (c) (2%) What is the probability that exactly one satellite is flying above Theodore?
   (d) (2%) Theodore knows that there is at least one satellite above him. What is the probability that the battleship can attack him?
   (e) (3%) Someone has stolen the power of Theodore. What is the probability that only the red satellite succeeded?

9. Manager desires to estimate the expected value m of the demand for apples. The firm operates N points of sale. Let's denote the demand in the points of sale by X_1, X_2, ..., X_n, the average demand by $\bar{X}$ and the sample variance by S^2.

   From the previous experience it's known that the distribution of $Z = \sqrt{N}\frac{\overline{X}-m}{\sqrt{S^2}}$ is not normal but is well approximated by the density function:

$$f(z) = \frac{1}{a}\begin{cases} 0, & z \le -a \\ \frac{1}{a}z + 1, & -a < z \le 0 \\ -\frac{1}{a}z + 1, & 0 < z \le a \\ 0, & z > a \end{cases}, \text{ for some } a > 0.$$

   (a) (7%) Find the length of the shortest 90% confidence interval for m in terms of a
   (b) (3%) Describe what happens with the length with the increase of a

10. Random variable Y is uniformly distributed on $[a, b]$.

    You have 5 observations on Y: $y_1 = y_2 = y_3 = y_4 = 4, y_5 = 9$.
    (a) (4%) Calculate first and second raw sample moments and sample estimate of population variance
    (b) (6%) Using sample mean and sample variance calculate method of moments estimates of parameters a and b.

May the Force be with You!