The table below is for grading. Please, leave it blank :)

Problem	1	2	3	4	5	6	7	8	9	10	Total
Maximum	10	10	10	10	10	10	10	10	10	10	100
Points											

You may write your solutions in Russian or in English.

Good luck!

1. (10%) Evaluate the following limit:

$$\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right).$$

The function may be written in the following way $\sqrt{x+\sqrt{x}}-\sqrt{x}=\sqrt{x}\left(\sqrt{1+x^{-1/2}}-1\right)$. Using substituion $x^{-1/2}=y$ one may find the Taylor expansion of the expression inside brackets. This will give **(7 points)**

$$\sqrt{x}\left(\left(1+\frac{1}{2}\sqrt{\frac{1}{x}}+o\left(\sqrt{\frac{1}{x}}\right)\right)-1\right).$$

The limit is equal to **(3 points)**

$$\lim_{x\to\infty}\sqrt{x}\left(\frac{1}{2}\sqrt{\frac{1}{x}}+o\left(\sqrt{\frac{1}{x}}\right)\right)=\frac{1}{2}.$$

There is also another approach: multiply and divide by conjugate expression $\sqrt{x+\sqrt{x}}+\sqrt{x}$. Idea is worth **(5 points)**. All the rest is worth **(5 points)**.

2. (10%) Let $D(x)$ be so-called Dirichlet function, which equals 1 if its argument is rational and 0 otherwise, and let k be a natural number.

   Prove that the function $x^kD(x)$ is nowhere differentiable if $k=1$ and is differentiable only at $x=0$ if $k=2017$.

For every k the function $x^kD(x)$ is continuous only at $x=0$. To see it one may sketch the graph of the function. So the function is not differentiable for $x\neq 0$. **(3 points)**

Let's consider the derivative at $x=0$ for $k=1$:

$$\lim_{\Delta x\to 0}\frac{\Delta x D(\Delta x)-0\cdot D(0)}{\Delta x}=\lim_{\Delta x\to 0}D(\Delta x).$$

If all Δx are rational then $D(\Delta x)=1$. If all Δx are irrational then $D(\Delta x)=0$. The derivative does not exist at $x=0$. **(3 points)**

Let's consider the derivative at $x=0$ for $k=1$:

$$\lim_{\Delta x\to 0}\frac{\Delta x^{2017}D(\Delta x)-0\cdot D(0)}{\Delta x}=\lim_{\Delta x\to 0}\Delta x^{2016}D(\Delta x).$$

As $D(\Delta x)$ is bounded and Δx^{2016} tends to zero, their product tends to zero. So, the limit exists and the function is differentiable at $x=0$. **(4 points)**

3. The matrices A and B are symmetric 3×3 matrices. Eigenvalues of the matrix A are $\lambda_1^A = 4, \lambda_2^A = 2, \lambda_3^A = 1$, eigenvalues of the matrix B are $\lambda_1^B = 11, \lambda_2^B = 5, \lambda_3^B = 1$.

(a) (2%) Find the trace (the sum of diagonal elements) of the matrix $A + B$.

We note that $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$. Hence,

$$\operatorname{tr}(A+B) = \sum_{i=1}^{3} \lambda_i^A + \sum_{i=1}^{3} \lambda_i^B = 7 + 17 = 24$$

(b) (3%) Let the matrix C be 3×3 matrix with $\det(C) = 1$. Find $\operatorname{tr}(C^{-1}AC)$.

Using the properties of trace, we get: $\operatorname{tr}(C^{-1}AC) = \operatorname{tr}(AC^{-1}C) = \operatorname{tr}(A) = 7$.

(c) (5%) Prove that $\operatorname{tr}(A^k) = \sum_i \lambda_i^k$.

Diagonalize the matrix A: $A = P\Lambda P^{-1}$, where P — the matrix of eigenvectors, Λ — the diagonal matrix with eigenvalues on the diagonal. Hence, $A^k = P\Lambda^k P^{-1}$, where Λ^k — diagonal matrix with k-th powers of eigenvalues on the diagonal. Then $\operatorname{tr}(A^k) = \operatorname{tr}(P\Lambda^k P^{-1}) = \operatorname{tr}(\Lambda^k) = \sum_i \lambda_i^k$.

4. Consider the matrix

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ -18 & 18 & 8 \end{pmatrix}.$$

(a) (4%) Find the eigenvalues and eigenvectors of matrix A.

$det(A-\lambda I) = (1-\lambda)(-1-\lambda)(8-\lambda) = 0$. Therefore, eigenvalues: $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 8$. Eigenvectors are $v_1 = (1,1,0), v_2 = (1,0,2), v_3 = (0,0,1)$ respectively.

(b) (6%) Find the matrix $A^{1/3}$. By definition, $A^{1/3}$ is such a matrix that $(A^{1/3})^3 = A$.

Diagonalize matrix A: $A = P\Lambda P^{-1}$, where P — matrix of eigenvectors, Λ — diagonal matrix with eigenvalues on the diagonal. $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 2 & 1 \end{pmatrix}$, $A^{1/3} = P\Lambda^{1/3}P^{-1}$, where $\Lambda^{1/3}$ is a diagonal matrix with $\lambda^{1/3}$ on the diagonal. Hence,

$$A^{1/3} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ -6 & 6 & 2 \end{pmatrix}.$$

5. (10%) Solve the differential equation:

$$(3x^2y^4 + 2xy)\,dx + (2y^2 - 3x^2)\,dy = 0$$

Заметим, что $y = 0$ является решением уравнения. Запомнив это, поделим левую и правую часть исходного уравнения на y^4. Это интегрирующий множитель. Полученное уравнение имеет вид:

$$\left(3x^2 + \frac{2x}{y^3}\right) dx + \left(\frac{2}{y^2} - \frac{3x^2}{y^4}\right) dy = 0$$

Тогда, если $P(x,y) = \left(3x^2 + \frac{2x}{y^3}\right)$ и $Q(x,y) = \left(\frac{2}{y^2} - \frac{3x^2}{y^4}\right)$, то $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -6\frac{x}{y^4}$. Имеем уравнение в полных дифференциалах.

Составляем и решаем систему уравнений для нахождения потенциала $u(x, y)$:

$$\begin{cases} \frac{\partial u}{\partial x} = 3x^2 + \frac{2x}{y^3} \\ \frac{\partial u}{\partial x} = \frac{2}{y^2} - \frac{3x^2}{y^4}. \end{cases}$$

Из первого уравнения находим, что $u\,(x, y) = x^3 + \frac{x^2}{y^3} + h(y)$, где $h(y)$ — произвольная непрерывно дифференцируемая функция от y. Подставляем найденное во второе уравнение:

$$u'_y\,(x, y) = \frac{-3x^2}{y^4} + h'(y)$$

$$\frac{-3x^2}{y^4} + h'\,(y) = \frac{2}{y^2} - \frac{3x^2}{y^4}$$

$$h\,(y) = -\frac{2}{y} - C$$

C — произвольная константа.

Тогда решение исходного уравнения

$$x^3 + \frac{x^2}{y^3} - \frac{2}{y} = C$$

6. Let $F(x, y) = xy$ and $G(x, y; a, b) = y + bx - a$.

(a) (3%) For each value of parameters (a, b) find the conditional extremum if it exists, classify it and find the extremal value $F^*(a, b)$.

One may use the Lagrange multiplier method or direct substitution or graphical analysis. There is no extremum for $b = 0$. For $b > 0$ there is one conditional maximum $(x^*, y^*) = (a/2b, a/2)$. For $b < 0$ the same point is the conditional minimum of F. The optimal value is $F^*(a, b) = a^2/4b$.

(b) (4%) Find all possible values of $F^*(a, b)$ in the region

$$D_1 = \begin{cases} a \geq b \\ a \in (0, 1) \end{cases}$$

We need to find the range of the fraction $a^2/4b$ in the region D_1.

Using negative values of b we may set F^* to arbitrary negative value.

As $a \neq 0$ we note that F^* can't be equal to zero.

Let's consider positive b. If it exists, the minimal value of F^* should be when $a = b$. In this case $F^* = b^2/4b = b/4$. We can take arbitrary small value of b, so on the line $a = b$ we may reach $F^* \in (0, 1/4)$. If we fix the value of a and consider all $b \leq a$ then $F^* \in [1/4, +\infty)$.

Finally, $F^* \in (-\infty; 0) \cap (0; +\infty)$.

(c) (3%) Find all possible values of $F^*(a, b)$ in the region

$$D_2 = \begin{cases} a \geq b \\ a \in (0, 1) \\ b \geq 0.5 \end{cases}$$

We need to find the range of the fraction $a^2/4b$ in the region D_2.
The minimum will be achieved on the line $a = b$, so $F^*_{min} = b^2/4b = b/4$. As $b \geq 0.5$ we see that $F^*_{min} = 1/8$.
The supremum will be achieved for $a = 1$ and $b = 0.5$, so $F^*_{sup} = a^2/4b = 1/2$.
Finally, $F^* \in [1/8; 1/2)$.

7. The island is populated with knights and knaves. Each sentence of a knight is true with probability 0.9 independently of other sentences. Each sentence of a knave is true with probability 0.2 independently of other sentences. The proportion of knights on the island is equal to 0.7. You meet one person on the island at random and asked him, whether he is a knight.

   (a) (2%) What is the probability that he will say «I am a knight»?

   $$\mathbb{P}(B) = 0.7 \cdot 0.9 + 0.3 \cdot 0.8 = 0.87$$

   (b) (4%) What is the conditional probability that he is a knight given that he said «I am a knight»?

   $$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0.7 \cdot 0.9}{0.87} = \frac{63}{87} \approx 0.72$$

   (c) (4%) What is the conditional probability that he is a knight given that he said «I am a knight», paused and said «I am not a knight»?

   $$\mathbb{P}(A|C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{0.7 \cdot 0.9 \cdot 0.1}{0.7 \cdot 0.9 \cdot 0.1 + 0.3 \cdot 0.2 \cdot 0.8} = \frac{63}{111} \approx 0.57$$

8. The joint density of random variables X and Y is given by the function

   $$f(x,y) = \begin{cases} x+y, \text{ if } x \in [0;1], \; y \in [0;1] \\ 0, \text{ otherwise} \end{cases}$$

   (a) (2%) Are X and Y independent? Give short argument.

   No, the density function can not be represented as a product $f(x,y) \neq f_X(x) f_Y(y)$.

   (b) (4%) Find $\mathbb{P}(Y > 2X)$ and $\mathbb{E}(XY)$

   $$\mathbb{P}(Y > 2X) = \int_0^{1/2} \int_{2x}^1 x + y \, dy \, dx = \int_0^{1/2} x + 0.5 - 4x^2 \, dx = \frac{5}{24}$$

   $$\mathbb{E}(XY) = \int_0^1 \int_0^1 xy(x+y) \, dx \, dy = \frac{1}{3}$$

   (c) (4%) Find marginal density $f_X(x)$ and conditional density $f_{Y|X}(y|x)$

   $$f_X(x) = \int_0^1 f(x,y) \, dy = \begin{cases} x + 0.5, \text{ if } x \in [0;1] \\ 0, \text{ otherwise} \end{cases}$$

   $$f(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{x+y}{x+0.5}, \text{ if } x \in [0;1], \; y \in [0;1] \\ 0, \text{ otherwise} \end{cases}$$

9. Boris loves hunting Pokemons. Today he randomly captured three Pokemons.

   Boris has sorted Pokemons by their height in ascending order and obtained their ranks H_i. The lowest Pokemon gets the rank $H_i = 1$, the tallest gets the rank $H_i = 3$. After sorting Pokemons by their combat power Boris obtained the ranks C_i in the same manner. Height and combat power of Pokemons are continuously distributed, so ties are impossible.

Boris would like to test the hypothesis H_0: height and combat power are independent. He calculates $\hat{\rho}$, sample Pearson correlation coefficient between ranks C_i and H_i.

(a) (6%) Find the distribution of $\hat{\rho}$ under H_0, that is find all possible values of $\hat{\rho}$ and their probabilities.

Let's sort Pokemons by H_i and consider possible orderings of C_i:

Ordering of C_i	Value of $\hat{\rho}$	Probability
1, 2, 3	1	1/6
1, 3, 2	0.5	1/6
2, 1, 3	0.5	1/6
3, 2, 1	-1	1/6
2, 3, 1	-0.5	1/6
3, 1, 2	-0.5	1/6

So the distribution of $\hat{\rho}$ is

$\hat{\rho}$	-1	-0.5	0.5	1
Probability	1/6	2/6	2/6	1/6

(b) (4%) Find the minimal threshold value ρ^* that will be exceeded by $\hat{\rho}$ with probability less or equal to 0.2 under H_0.

From the distribution table we find that $\rho^* = 0.5$. Indeed $\mathbb{P}(\hat{\rho} > 0.5) = 1/6 \leq 0.2$. The value of ρ^* can't be decreased further as probability would jump to 3/6.

10. (10%) You estimated two models using 47 observations:

A. $\hat{y}_i = 40 + 0.3x_i + 0.8z_i - 1.8w_i$, $R^2 = 0.82$

B. $\hat{y}_i = 65 + 0.6x_i + 0.51z_i$, $R^2 = 0.7$

Test the hypothesis $\beta_w = -1$ against $\beta_w \neq -1$ on 5% significance level. Here β_w is the coefficient before the variable w in the first regression.

Поскольку необходимо проверить гипотезу, что $\beta_w = -1$, то необходимо использовать t-тест (F-тест для проверки одной гипотезы эквивалентен t-статистике в квадрате). Необходимая t-статистика рассчитывается по формуле $\frac{\hat{\beta}_w + 1}{se(\hat{\beta}_w)}$, где нам не известен знаменатель.

Теперь обратимся к условию. Вторая модель является ограниченной (restricted) версией первой модели при ограничении $\beta_w = 0$. Поэтому на основе данных об R^2 можно проверить гипотезу $\beta_w = 0$ с помощью соответствующего F-теста. $F = \frac{(R^2_{UR} - R^2_R)/q}{(1-R^2_{UR})/(n-k)} = \frac{(0.82-0.7)/1}{(1-0.82)/(47-4)} = \frac{0.12}{0.18/43} = 29.27$.

Как было сказано выше для тестирования одного ограничения $F(1, n-k) = t^2_{n-k}$, следовательно $t_{43} = -5.41$. Мы берём отрицательное значение, так как сама оценка коэффициента отрицательная. Отсюда можно получить $se(\hat{\beta}_w)$, так как $\frac{\hat{\beta}_w}{se(\hat{\beta}_w)} = -5.41$, следовательно

$$se(\hat{\beta}_w) = \frac{\hat{\beta}_w}{-5.41} = \frac{-1.8}{-5.41} = 0.333.$$

Исходя из полученных значений, необходимая t-статистика равна $\frac{-1.8+1}{0.333} = -2.403$. Критическое значение равно -2.018, соответственно нулевая гипотеза отвергается на 5% уровне значимости.