Exam demo-version

1. (10%) Evaluate the following limit:

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}}$$

2. (10%) Find and classify the discontinuity points of the following function:

$$f(x) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{x}\right)\right).$$

3. Let A, B and C be square matrix of size $n \times n$. Prove the following statements or provide counterexample:
   (a) (2%) If $B = C^{-1}AC$, then $\det(A) = \det(B)$
   (b) (3%) $\det((A+B)^2) = \det(A^2 + 2AB + B^2)$
   (c) (3%) $\det((A+B)^2) = \det(A^2 + B^2)$
   (d) (2%) If A is invertible, then $(I + A^{-1})^{-1} = A(A+I)^{-1}$

4. Let S be the $n \times n$ «shipbuilding timber» matrix, i.e. the square matrix with all elements equal to 1.
   (a) (2%) Express S^2 in terms of S
   (b) (3%) Find the eigenvalues of S
   (c) (3%) For each eigenvalue of S find at least on eigenvector
   (d) (2%) Find all the eigenvalues of the matrix $A = aI + bS$, where I is the identity matrix.

5. Solve the differential equation:

$$y''' - 4y'' + y' = 2x^2 + 1.$$

6. (10%) Solve the differential equation

$$2xyy' - y' \ln y + y^2 + \ln x = 0$$

The exam continues on the next page

7. (10%) Find the points of maximum of the function

$$F(u,v) = \sqrt{u}\left(\sqrt{u}-2\right) - \sqrt{v}\left(\sqrt{v}-2\right),$$

given that $\sqrt{u} \le 2,\ \sqrt{v} \le 2$

8. Consider a function

$$f(x) = \begin{cases} \frac{1}{x^2},\ \text{if } c_1 < x < c_2 \\ 0,\ \text{otherwise} \end{cases}$$

   (a) (5%) Find all c_1 and c_2 such that the function f is a density function for some random variable X

   (b) (5%) Calculate the expected value and variance of the random variable X for $c_2 = 9$

9. You have height measurements of a random sample of 100 persons, $y_1, \ldots, y_{100}$. It is known that $\sum_{i=1}^{100} y_i = 15800$ and $\sum_{i=1}^{100} y_i^2 = 2530060$.

   (a) (3%) Calculate unbiased estimate of population mean and population variance of the height

   (b) (3%) At 4% significance test the null-hypothesis that the population mean is equal to 155 cm, against two-sided alternative.

   (c) (2%) Find the p-value

   (d) (2%) Find the 96% confidence interval for the population mean

10. Density function of a random variable Y is given by

$$f(y) = \begin{cases} \frac{1}{\theta^2} y e^{-y/\theta},\ \text{if } y > 0 \\ 0,\ \text{otherwise} \end{cases}$$

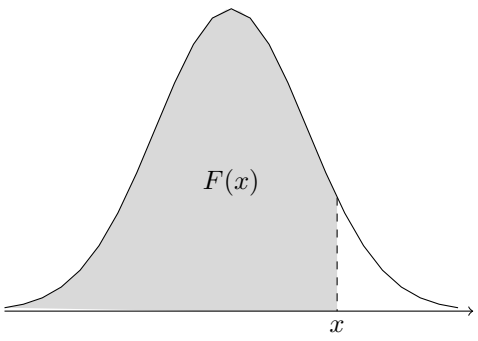
You have 3 observations on Y: $y_1 = 48, y_2 = 50, y_3 = 52$.

   (a) (4%) Using maximum likelihood, find the estimate of θ

   (b) (3%) Is the estimator $\hat{\theta}$ unbiased?

   (c) (3%) Calculate the variance of $\hat{\theta}$

Good luck!



x	$F(x)$	x	$F(x)$	x	$F(x)$	x	$F(x)$
0.050	0.520	0.750	0.773	1.450	0.926	2.150	0.984
0.100	0.540	0.800	0.788	1.500	0.933	2.200	0.986
0.150	0.560	0.850	0.802	1.550	0.939	2.250	0.988
0.200	0.579	0.900	0.816	1.600	0.945	2.300	0.989
0.250	0.599	0.950	0.829	1.650	0.951	2.350	0.991
0.300	0.618	1.000	0.841	1.700	0.955	2.400	0.992
0.350	0.637	1.050	0.853	1.750	0.960	2.450	0.993
0.400	0.655	1.100	0.864	1.800	0.964	2.500	0.994
0.450	0.674	1.150	0.875	1.850	0.968	2.550	0.995
0.500	0.691	1.200	0.885	1.900	0.971	2.600	0.995
0.550	0.709	1.250	0.894	1.950	0.974	2.650	0.996
0.600	0.726	1.300	0.903	2.000	0.977	2.700	0.997
0.650	0.742	1.350	0.911	2.050	0.980	2.750	0.997
0.700	0.758	1.400	0.919	2.100	0.982	2.800	0.997

Рис. 1: Distribution function of a standard normal random variable