Exam demo-version

1. (10%) Evaluate the following limit:

$$\lim_{x\to 0} \sqrt[x]{\cos\sqrt{x}}$$

2. (10%) Find and classify the discontinuity points of the following function:

$$f(x) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{x}\right)\right).$$

3. Matrix A is given by

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

   (a) (6%) Find the eigenvalues and eigenvectors of A;

   (b) (4%) Find the eigenvalues of $4A^{-1} + 2I$, where I is identity matrix.

4. The characteristic polynomial of a matrix B is given by $f(\lambda) = 6\lambda - 5\lambda^2 - \lambda^3$.

   (a) (6%) Find dimensions of B, $\operatorname{rank} B$, $\det B$, sum of diagonal elements of B;

   (b) (2%) Suppose additionaly that B is symmetric. Can we find random variables with covariance matrix B?

   (c) (2%) Suppose additionaly that B is symmetric. How many solutions does equation $v^T Bv = -2018$ has?

5. (10%) Solve the following differential equations

   (a) (5%) $y'' - 2y' - 8y = 0$,

   (b) (5%) $y'' - 2y' - 8y = e^x - 8\cos 2x$.

6. (10%) Find the points of maximum of the function

$$F(u, v) = \sqrt{u}\left(\sqrt{u} - 2\right) - \sqrt{v}\left(\sqrt{v} - 2\right),$$

   given that $\sqrt{u} \le 2$, $\sqrt{v} \le 2$

7. There are three coins in the bag. Two coins are unbiased, and for the third coin the probability of «head» is equal to 0.8. James Bond chooses one coin at random from the bag and tosses it
   (a) (5%) What is the probability that it will show «head»?
   (b) (5%) What is the conditional probability that the coin is unbiased if it shows «head»?

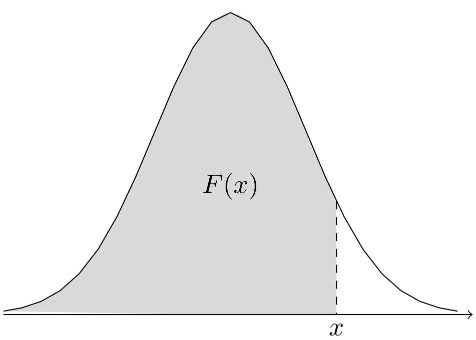
8. The pair of random variables X and Y with $\mathbb{E}(X) = 0$ and $\mathbb{E}(Y) = 1$ has the following covariance matrix
$$\begin{pmatrix} 10 & -2 \\ -2 & 9 \end{pmatrix}.$$
   (a) (5%) Find $\mathbb{V}\text{ar}(X + Y)$, $\mathbb{C}\text{orr}(X, Y)$, $\mathbb{C}\text{ov}(X - 2Y + 1, 7 + X + Y)$
   (b) (5%) Find the value of a if it is known that X is independent of $Y - aX$.

9. You have height measurements of a random sample of 100 persons, $y_1, \ldots, y_{100}$. It is known that $\sum_{i=1}^{100} y_i = 15800$ and $\sum_{i=1}^{100} y_i^2 = 2530060$.
   (a) (3%) Calculate unbiased estimate of population mean and population variance of the height
   (b) (3%) At 4% significance test the null-hypothesis that the population mean is equal to 155 cm, against two-sided alternative.
   (c) (2%) Find the p-value
   (d) (2%) Find the 96% confidence interval for the population mean

10. Let $X = (X_1, \ldots, X_n)$ be a random sample from normal distribution with zero mean and unknown variance $\sigma^2 > 0$.

    If $\xi \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[\xi^4] = 3\sigma^2$.
    (a) (2%) Derive the log-likelihood function of a random sample X.
    (b) (2%) Find the estimator of the parameter σ^2 using maximum likelihood method.
    (c) (2%) Using the realization of a random sample $x = (1, -2, 0, 1)$ find the maximum likelihood estimate of the parameter σ^2 derived in (b).
    (d) (2%) Find the Fisher information $I_n(\sigma^2)$ about the parameter σ^2 contained in n observations of a random sample.
    (e) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ an unbiased estimator of the parameter σ^2?
    (f) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ an efficient estimator of the parameter σ^2?
    (g) (2%) Is the estimator $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ a consistent estimator of the parameter σ^2?
    (h) (2%) Find the following probability limit $\text{plim}_{n\to\infty} \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}$.

Good luck!



x	$F(x)$	x	$F(x)$	x	$F(x)$	x	$F(x)$
0.050	0.520	0.750	0.773	1.450	0.926	2.150	0.984
0.100	0.540	0.800	0.788	1.500	0.933	2.200	0.986
0.150	0.560	0.850	0.802	1.550	0.939	2.250	0.988
0.200	0.579	0.900	0.816	1.600	0.945	2.300	0.989
0.250	0.599	0.950	0.829	1.650	0.951	2.350	0.991
0.300	0.618	1.000	0.841	1.700	0.955	2.400	0.992
0.350	0.637	1.050	0.853	1.750	0.960	2.450	0.993
0.400	0.655	1.100	0.864	1.800	0.964	2.500	0.994
0.450	0.674	1.150	0.875	1.850	0.968	2.550	0.995
0.500	0.691	1.200	0.885	1.900	0.971	2.600	0.995
0.550	0.709	1.250	0.894	1.950	0.974	2.650	0.996
0.600	0.726	1.300	0.903	2.000	0.977	2.700	0.997
0.650	0.742	1.350	0.911	2.050	0.980	2.750	0.997
0.700	0.758	1.400	0.919	2.100	0.982	2.800	0.997

Рис. 1: Distribution function of a standard normal random variable