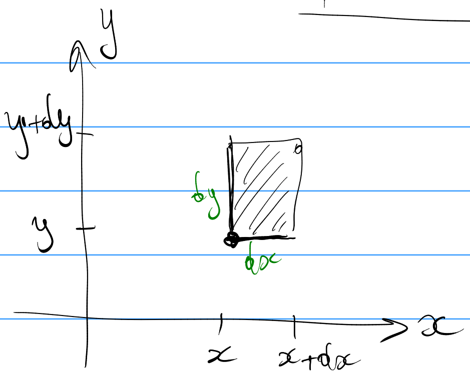
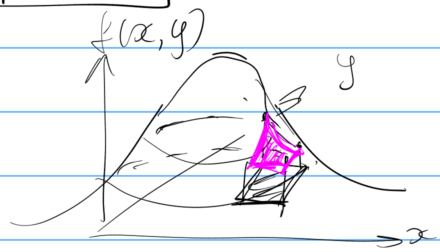
Плотности многомерные.

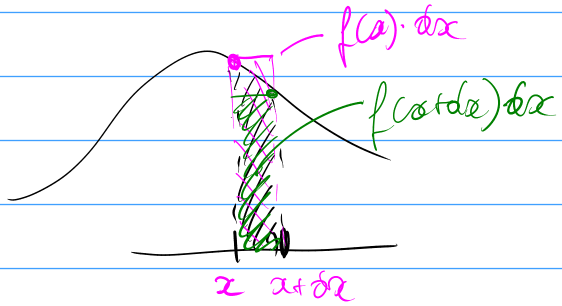


$$\begin{pmatrix} X \\ Y \end{pmatrix}$$

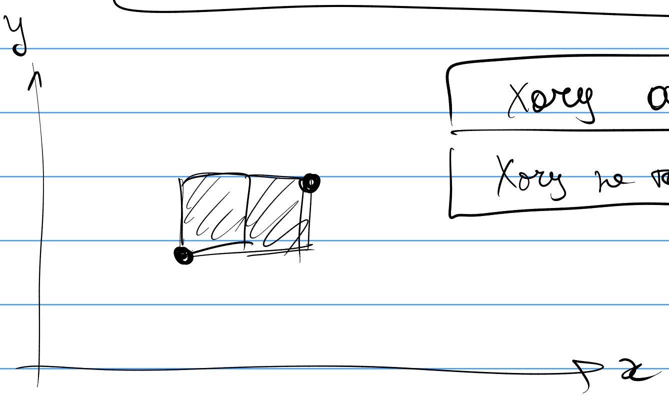


$$f(x,y)\cdot dx\cdot dy \approx P(X\in[x;x+dx],\ Y\in[y;y+dy])$$

$$f(x+dx,\ y+dy)\cdot dx\cdot dy \approx$$

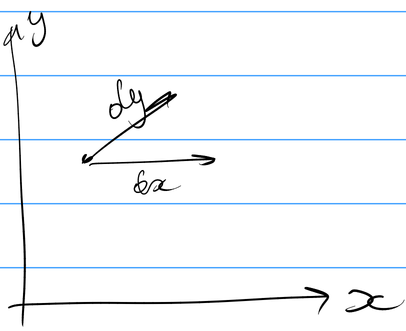


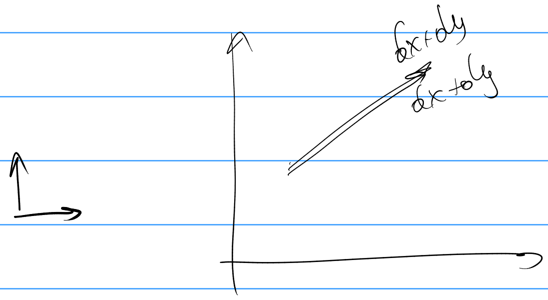
Хотим чтобы было легко складывать наши области.



Хочу аддитивности!

Хочу не только квадратики!





$$S(dx+dy,\ dx+dy)=0$$

$$\boxed{(dx+dy)\wedge(dx+dy)=0}$$

$$dx\wedge dx + dy\wedge dx + dx\wedge dy + dy\wedge dy = 0$$

$$\boxed{\begin{aligned} dx\wedge dy &= -dy\wedge dx \\ dx\wedge dx &= 0 \end{aligned}}$$

$$\det\begin{pmatrix}1&1\\1&1\end{pmatrix} = 0$$

Хорошо!

$$\det\begin{pmatrix}1&1\\1&0\end{pmatrix} + \det\begin{pmatrix}1&0\\1&1\end{pmatrix} =$$

$$= \det\begin{pmatrix}1&1\\0&0\end{pmatrix} + \underbrace{\det\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{} + \underbrace{\det\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{} + \det\begin{pmatrix}0&0\\1&1\end{pmatrix} = 0$$

$$\begin{cases} da \wedge da = 0 \\ (da + db)\wedge dc = da\wedge dc + db \wedge dc \end{cases} \Rightarrow dx\wedge dy = -dy\wedge dx$$

Упр. $X, Y \sim$ незав $\mathrm{Exp}(\lambda = 1)$

$$Z = (X+Y)/2, \quad W = \frac{X}{Y}$$

а) $f(x,y)$? $f_{X,Y}(x,y)$

б) $f(z,w)$? $f_{Z,W}(z,w)$

$$\begin{pmatrix} z \\ w\end{pmatrix} = \begin{pmatrix} (X+Y)/2 \\ X/Y\end{pmatrix}$$

$$w+1 = \frac{x+y}{y}$$

$$\frac{2z}{w+1} = \frac{x+y}{\left(\frac{x+y}{y}\right)} = y \qquad x = w\cdot\frac{2z}{w+1}$$

$$\begin{pmatrix} z \\ w\end{pmatrix} = h\begin{pmatrix} X \\ Y\end{pmatrix} \approx \begin{pmatrix} 5+6X+7Y \\ 8-2X+3Y\end{pmatrix}$$

$$f(x) = \begin{cases}\lambda\cdot\exp(-\lambda x) & \text{при } x \geq 0 \\ 0, & \text{иначе}\end{cases}$$

$$f(x) = \exp(-x) \qquad [\text{при } x \geq 0]$$

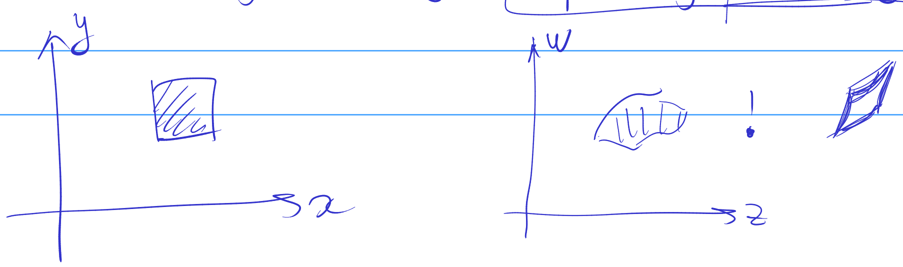
$$\updownarrow$$

$$f_X(x)$$

$$f(y) = f_Y(y) = \exp(-y)$$

$$f(x,y) = f(x)\cdot f(y) = \boxed{\exp(-x-y)}$$

$$f(x,y)\,dx\wedge dy = \boxed{\exp(-x-y)\,dx\wedge dy} \sim P(X\in[x;x+dx], Y\in[y;y+dy])$$



$$\frac{2z}{w+1} = \frac{x+y}{\left(\frac{x+y}{y}\right)} = y \qquad x = w\cdot\frac{2z}{w+1} \qquad \exp(-x-y) = \exp\left(\frac{-2zw}{w+1} - \frac{2z}{w+1}\right) = $$

$$= \exp(-2z)$$

$$y = \frac{2z}{w+1}$$

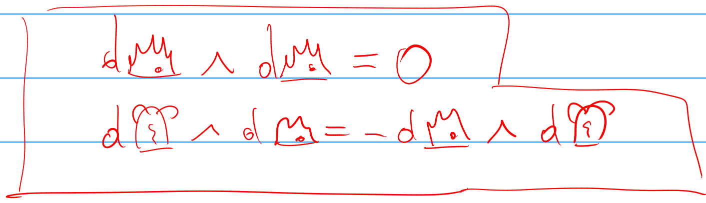
$$x = \frac{2zw}{w+1}$$

$$f(x,y)\,dx\wedge dy = \exp(-x-y)\cdot dx\wedge dy$$

$$dy = d\left(\frac{2z}{w+1}\right) = \frac{2}{w+1}\cdot dz - \frac{2z}{(w+1)^2}\cdot dw$$

$$dx = d\left(\frac{2zw}{w+1}\right) = \frac{2w}{w+1}\cdot dz + \frac{2z\cdot(w+1)-2zw}{(w+1)^2}\,dw = \frac{2w}{w+1}\,dz + \frac{2z}{(w+1)^2}\,dw$$

$$dx\wedge dy = \left(\frac{2}{w+1}\,dz - \frac{2z}{(w+1)^2}\,dw\right)\wedge\left(\frac{2w}{w+1}\,dz + \frac{2z}{(w+1)^2}\,dw\right) =$$



$$= \frac{2}{w+1}\cdot\frac{2z}{(w+1)^2}\,dz\wedge dw + \left(\frac{-2z}{(w+1)^2}\right)\cdot\frac{2w}{w+1}\,dw\wedge dz =$$

$$= \left(\frac{2}{w+1}\cdot\frac{2z}{(w+1)^2} + \frac{2z\cdot 2w}{(w+1)^3}\right)\cdot dz\wedge dw =$$

$$= \frac{4z\cdot(w+1)}{(w+1)^3}\,dz\wedge dw = \frac{4z}{(w+1)^2}\,dz\wedge dw = dx\wedge dy$$

$$P(Z\in[z, z+dz], W\in[w, w+dw]) \sim \underbrace{\exp(-2z)\cdot\frac{4z}{(w+1)^2}}_{\text{высота столбов плотности}}\,dz\wedge dw$$

$$f(z,w) \overset{?}{=} \begin{cases} \exp(-2z)\cdot 4z/(w+1)^2, & \text{если } z>0,\ w\geq 0 \\ 0, & \text{иначе} \end{cases}$$

$$(dx + dy - 2dz) \wedge (dz + 3dy) \wedge (dz + dx) \overset{?}{=}$$

$$d\square \wedge d\square = 0$$

$$= (dx \wedge dz + dy \wedge dz + 3dx \wedge dy - 6dz \wedge dy) \wedge (dz + dx);$$

$$dx \wedge \boxed{dz \wedge dz} = 0$$

$$dx \wedge dz \wedge dx = 0$$

$$= -dx \wedge dx \wedge dz = 0$$

Q. Зачем $f(x,y)$?

$$E(X) \overset{?}{=} \iint_{\mathbb{R}\,\mathbb{R}} \boxed{x} \boxed{f(x,y)\,dx\,dy}$$

A. чтобы найти
$E(\cdot)$, $Var(\cdot)$,
$Cov(\cdot)$, $H(\cdot)$,
$H(\cdot|\cdot)$, $CE(\cdot\|\cdot)$
$P(\cdot)$ $P(\cdot|\cdot)$

$$E(X\cdot Y) = \iint_{\mathbb{R}\,\mathbb{R}} xy \cdot f(x,y)\,dx\,dy$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Упр. $X_1, X_2, X_3 \ldots X_{10} \sim$ незав $Exp(\lambda = 1)$

упорядочим $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(10)}$

$Y_1\ Y_2\ Y_3$
$6, 1, 3$
Y_1
Y_2
Y_3
$Y_{(1)} \leq Y_{(2)} \leq Y_{(3)}$

$P(Y_{(1)} = 1) = 1 - (\frac{5}{6})^3$

$P(Y_1 = 1) = \frac{1}{6}$

$X_2 \quad X_5 \quad X_1 \quad X_4 \quad X_3 \ldots$

$0 \qquad X_{(1)} \quad X_{(2)} \quad X_{(3)} \qquad \infty$

$\quad\ \ \| \qquad\ \ \| \qquad\ \ \|$

$\quad R_1 \quad R_2 \quad R_3$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{10} \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \\ 7 \\ \vdots \end{pmatrix}$$

$$\boxed{X_{(1)}} = \boxed{\min(X_1, \ldots X_{10})}$$

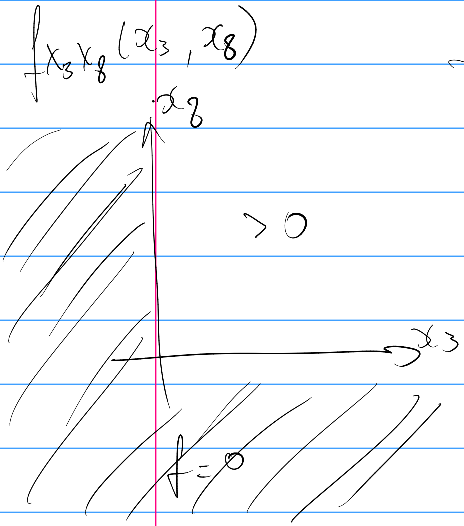
$$X_{(10)} = \max(X_1, \ldots X_{10})$$

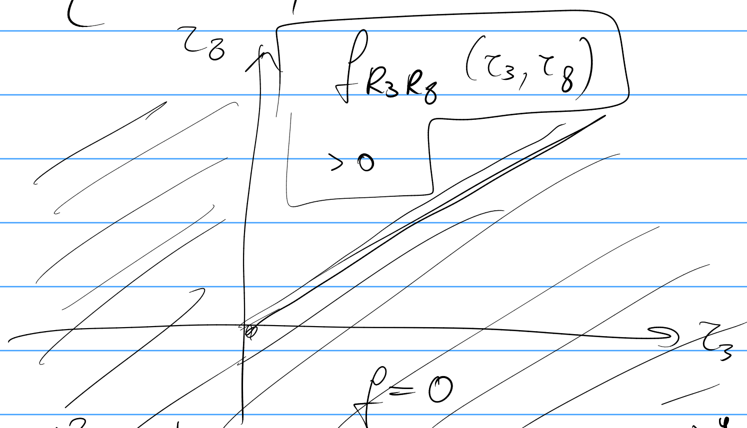
а) $f(x_3, x_8)$? $= \begin{cases} \exp(-x_3 - x_8) & \text{при } x_8 \geq 0,\ x_3 \geq 0 \\ 0, & \text{иначе} \end{cases}$

б) $f(r_3, r_8)$?

$$\boxed{f_{R_3,R_8}(\tau_3,\tau_8)\, d\tau_3 \wedge d\tau_8}$$

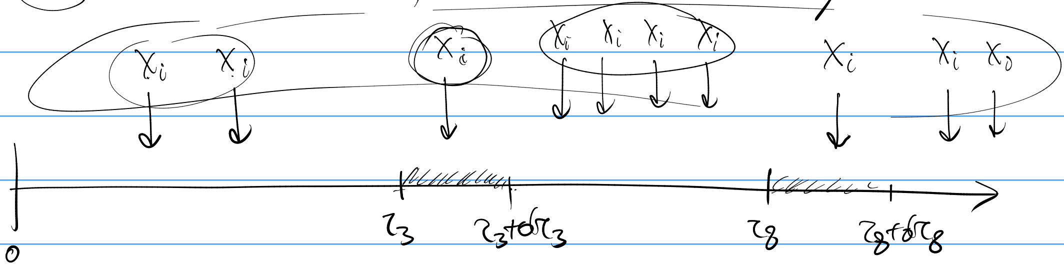
$$f(\tau_3,\tau_8) = \begin{cases} \text{/////} & \text{при } 0 \le \tau_3 \le \tau_8 \\ 0, & \text{иначе} \end{cases}$$



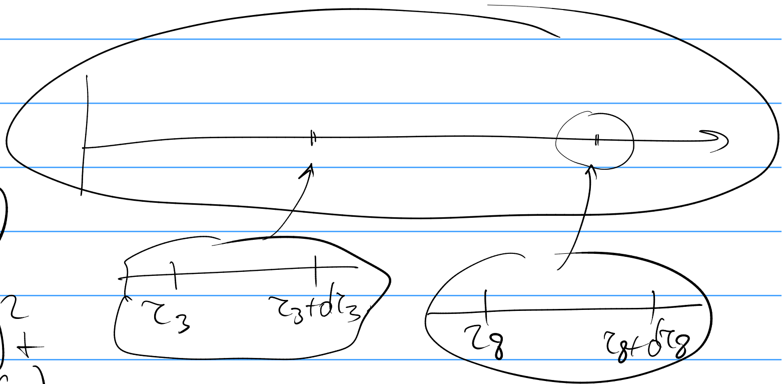


$$= C_{10}^2 \cdot \left(P(X_i \in [0;\tau_3])\right)^2 \cdot C_8^1 \cdot f(\tau_3) \cdot d\tau_3 \cdot C_7^4 \cdot P(X_i \in [\tau_3;\tau_8])^4 \cdot$$
$$\cdot\, C_3^1 \cdot f(\tau_8) \cdot d\tau_8 \cdot C_2^2 \cdot P(X_0 \ge \tau_8)^2 + o(\ldots)$$

$$P(R_3 \in [\tau_3; \tau_3 + d\tau_3],\ R_8 \in [\tau_8; \tau_8 + d\tau_8])$$



$$P(X_7 \in [\tau_8; \tau_8 + d\tau_8]) = f_X(\tau_8) \cdot d\tau_8 + o(d\tau_8)$$



$$P(X_5, X_7 \in [\quad]) = f_X^2(\tau_8)\,(d\tau_8)^2 + \ldots = o(d\tau_8)$$

$$P(X_i \in [0;\tau_3]) = \int_0^{\tau_3} \underbrace{\exp(-x)}_{f_X(x)}\, dx = 1 - \exp(-\tau_3)$$

$$P(X_i \in [\tau_3;\tau_8]) = \int_{\tau_3}^{\tau_8} \exp(-x)\, dx = \exp(-\tau_3) - \exp(-\tau_8)$$

$$P(X_0 \ge \tau_8) = \int_{\tau_8}^{\infty} \exp(-x)\, dx = \exp(-\tau_8)$$

$\Sigma = 1$

$$C_{10}^2 \cdot \left(P(X_i \in [0;\tau_3])\right)^2 \cdot C_8^1 \cdot f(\tau_3)\cdot d\tau_3 \cdot C_7^4 \cdot P(X_i \in [\tau_3;\tau_8])^4 \cdot C_3^1 \cdot f(\tau_8)\cdot d\tau_8 \cdot C_2^2 \cdot P(X_i \geq \tau_8)^2 + o(\dots)$$

$$P(R_3 \in [\tau_3; \tau_3 + d\tau_3], R_8 \in [\tau_8; \tau_8 + d\tau_8])$$

если $0 \leq \tau_3 \leq \tau_8$

$$f(\tau_3, \tau_8) = \begin{cases} \dfrac{10!}{2!\,1!\,4!\,1!\,2!} \cdot (1-e^{-\tau_3})^2 \cdot e^{-\tau_3} \cdot (e^{-\tau_3} - e^{-\tau_8})^4 \cdot e^{-\tau_8} \cdot (e^{-\tau_8})^2 \\ 0, \text{ иначе} \end{cases}$$

$$C_{10}^2 \cdot C_8^1 \cdot C_7^4 \cdot C_3^1 \cdot C_2^2 = \frac{10!}{2!\,\cancel{8!}} \cdot \frac{\cancel{8!}}{1!\,\cancel{7!}} \cdot \frac{\cancel{7!}}{4!\,\cancel{3!}} \cdot \frac{\cancel{3!}}{1!\,\cancel{2!}} \cdot \frac{\cancel{2!}}{2!}$$

$\boxed{C_{10}^2}$ = неуд. обозн. $\boxed{\binom{10}{2}}$ $\boxed{D(2+8)} = \dfrac{10!}{2!\,8!}$

$$D(2+1+4+1+2) = \frac{10!}{2!\,1!\cdot 4!\cdot 1!\,2!}$$

$\boxed{13:13}$