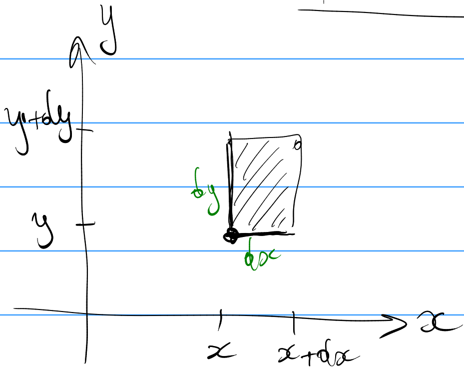
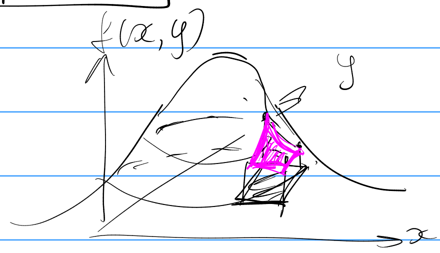
Плотности многомерные

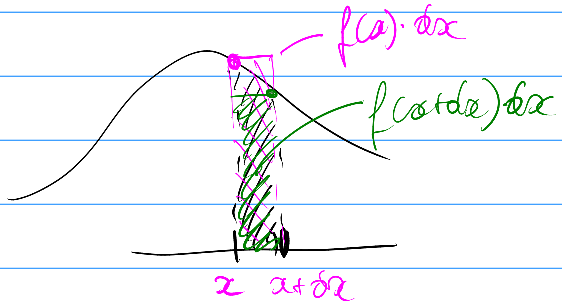


$$\begin{pmatrix}X\\Y\end{pmatrix}$$

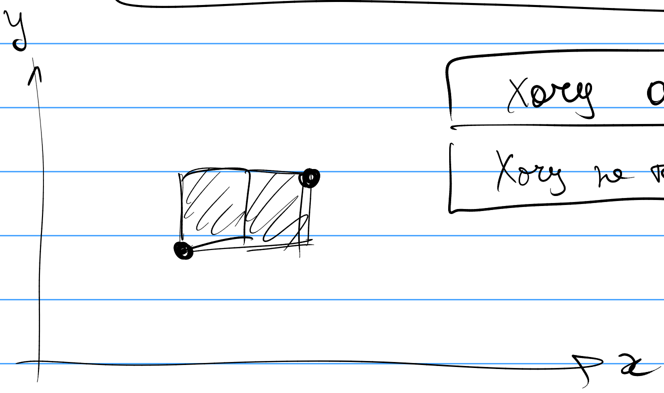


$$f(x,y)\cdot dx\cdot dy \approx P(X\in[x;x+dx],\ Y\in[y;y+dy])$$

$$f(x+dx,y+dy)\cdot dx\cdot dy \approx$$

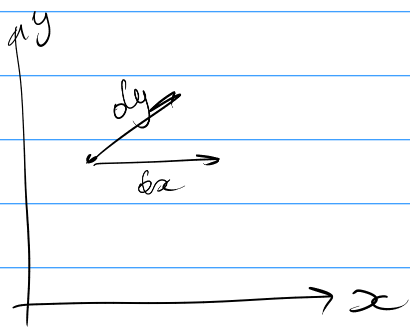


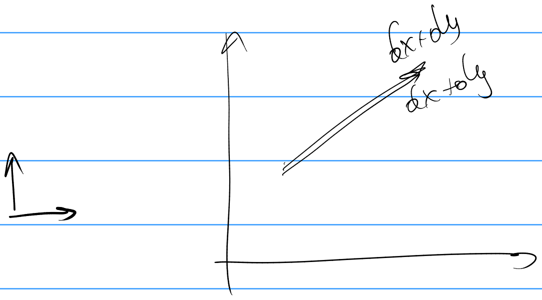
Хотим чтобы было легко складывать наши области.



Хочу аддитивности!

Хочу не только квадратики!





$$S(dx+dy,\ dx+dy)=0$$

$$\boxed{(dx+dy)\wedge(dx+dy)=0}$$

$$\underbrace{dx\wedge dx}_{0} + dy\wedge dx + dx\wedge dy + \underbrace{dy\wedge dy}_{0} = 0$$

$$\boxed{\begin{aligned} dx\wedge dy &= -dy\wedge dx \\ dx\wedge dx &= 0 \end{aligned}}$$

$$\det\begin{pmatrix} | & | \\ | & | \end{pmatrix} = 0 \quad \text{Хотим!}$$

$$\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =$$

$$= \underbrace{\det\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{0} + \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\det\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}}_{0} = 0$$

$$\begin{cases} da \wedge da = 0 \\ (da + db) \wedge dc = da \wedge dc + db \wedge dc \end{cases} \Rightarrow dx \wedge dy = -dy \wedge dx$$

Упр. $X, Y \sim$ незав $\mathrm{Exp}(\lambda = 1)$

$Z = (X+Y)/2$

$W = \dfrac{X}{Y}$

а) $f(x,y)$? $f_{X,Y}(x,y)$

б) $f(z,w)$? $f_{Z,W}(z,w)$

$$\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} (X+Y)/2 \\ X/Y \end{pmatrix}$$

$$w+1 = \frac{X+Y}{Y}$$

$$\frac{2Z}{W+1} = \frac{X+Y}{\left(\frac{X+Y}{Y}\right)} = Y \qquad X = W \cdot \frac{2Z}{W+1}$$

$$\begin{pmatrix} Z \\ W \end{pmatrix} = h\begin{pmatrix} X \\ Y \end{pmatrix} \approx \begin{pmatrix} 5 + 6X + 7Y \\ 8 - 2X + 3Y \end{pmatrix}$$

$$f(x) = \begin{cases} \lambda \cdot \exp(-\lambda x) & \text{при } x \geq 0 \\ 0, & \text{иначе} \end{cases}$$

$$f(x) = \exp(-x) \quad [\text{при } x \geq 0]$$

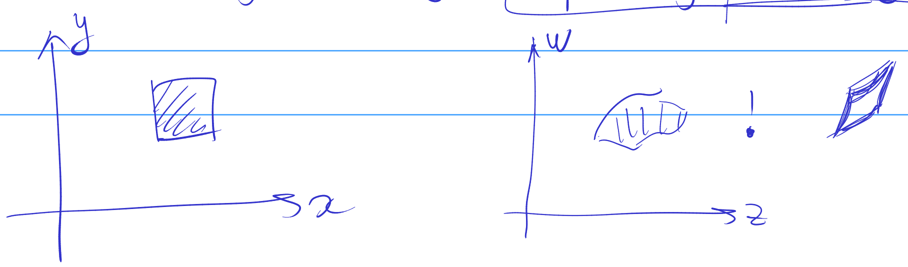
$$\updownarrow$$

$$f_X(x)$$

$$f(y) = f_Y(y) = \exp(-y)$$

$$f(x,y) = f(x) \cdot f(y) = \boxed{\exp(-x-y)}$$

$$f(x,y)\, dx \wedge dy = \boxed{\exp(-x-y)\, dx \wedge dy} \sim P(X \in [x; x+dx],\ Y \in [y; y+dy])$$



$$\frac{2z}{w+1} = \frac{X+Y}{\left(\frac{X+Y}{Y}\right)} = Y \qquad X = W \cdot \frac{2z}{w+1} \qquad \exp(-x-y) = \exp\left(\frac{-2zw}{w+1} - \frac{2z}{w+1}\right) =$$

$$= \exp(-2z)$$

$$Y = \frac{2z}{w+1}$$

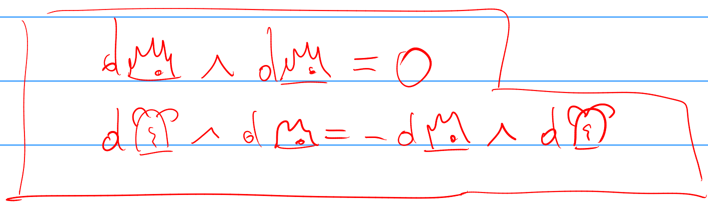
$$X = \frac{2zW}{W+1}$$

$$f(x,y)\, dx \wedge dy = \exp(-x-y) \cdot dx \wedge dy$$

$$dy = d\left(\frac{2z}{w+1}\right) = \frac{2}{w+1} \cdot dz - \frac{2z}{(w+1)^2} \cdot dw$$

$$dx = d\left(\frac{2zw}{w+1}\right) = \frac{2w}{w+1} \cdot dz + \frac{2z\cdot(w+1) - 2zw}{(w+1)^2} dw = \frac{2w}{w+1} dz + \frac{2z}{(w+1)^2} dw$$

$$dx \wedge dy = \left(\frac{2}{w+1} dz - \frac{2z}{(w+1)^2} dw\right) \wedge \left(\frac{2w}{w+1} dz + \frac{2z}{(w+1)^2} dw\right) =$$



$$= \frac{2}{w+1} \cdot \frac{2z}{(w+1)^2} dz \wedge dW + \left(\frac{-2z}{(w+1)^2}\right) \cdot \frac{2w}{w+1} dw \wedge dz =$$

$$= \left(\frac{2}{w+1} \cdot \frac{2z}{(w+1)^2} + \frac{2z \cdot 2w}{(w+1)^3}\right) \cdot dz \wedge dw =$$

$$= \frac{4z \cdot (w+1)}{(w+1)^3} dz \wedge dw = \frac{4z}{(w+1)^2} dz \wedge dw = dx \wedge dy$$

$$P(Z \in [z, z+dz], W \in [w, w+dw]) \sim \boxed{\underbrace{\exp(-2z)}_{\text{высота}} \cdot \frac{4z}{(w+1)^2}} dz \wedge dw$$

сторона плотности

$$f(z,w) \overset{?}{=} \boxed{\begin{cases} \exp(-2z) \cdot 4z/(w+1)^2, & \text{если } z>0,\ w\geq 0 \\ 0, & \text{иначе} \end{cases}}$$

$$(dx+dy-2dz)\wedge(dz+3dy)\wedge(dz+dx)\overset{?}{=}$$

$$d\square\wedge d\square=0$$

$$=(dx\wedge dz+dy\wedge dz+3dx\wedge dy-6dz\wedge dy)\wedge(dz+dx);$$

$$dx\wedge\boxed{dz\wedge dz}=0$$

$$dx\wedge dz\wedge dx=0$$

$$=-dx\wedge dx\wedge dz=0$$

Q. Зачем $f(x,y)$?

$$E(X)\overset{?}{=}\iint_{\mathbb{R}\,\mathbb{R}}\boxed{x}\,\boxed{f(x,y)\,dx\,dy}$$

A. чтобы найти
$E(\cdot)$, $Var(\cdot)$,
$Cov(\cdot)$, $H(\cdot)$,
$H(\cdot|\cdot)$, $CE(\cdot\|\cdot)$
$P(\cdot)$ $P(\cdot|\cdot)$

$$E(X\cdot Y)=\iint_{\mathbb{R}\,\mathbb{R}}xy\cdot f(x,y)\,dx\,dy$$

$$Cov(X,Y)=E(XY)-E(X)E(Y)$$

$Y_1\ Y_2\ Y_3$
$6,1,3$
Y_1
Y_2
Y_3
$Y_{(1)}\le Y_{(2)}\le Y_{(3)}$
$P(Y_{(1)}=1)=1-\left(\frac{5}{6}\right)^3$
$P(Y_1=1)=\frac{1}{6}$

Упр. $X_1, X_2, X_3 \ldots X_{10}$ ~ незав $Exp(\lambda=1)$

упорядочим $X_{(1)}\le X_{(2)}\le \ldots \le X_{(10)}$

X_2 X_5 X_1 X_4 X_3 ...

0 ∞

$X_{(1)}$ $X_{(2)}$ $X_{(3)}$

$\parallel$ $\parallel$ $\parallel$

R_1 R_2 R_3

$$X=\begin{pmatrix}X_1\\X_2\\\vdots\\X_{10}\end{pmatrix}=\begin{pmatrix}5\\3\\1\\7\\\vdots\end{pmatrix}$$

$$\boxed{X_{(1)}}=\boxed{\min(X_1,\ldots X_{10})}$$

$$X_{(10)}=\max(X_1,\ldots X_{10})$$

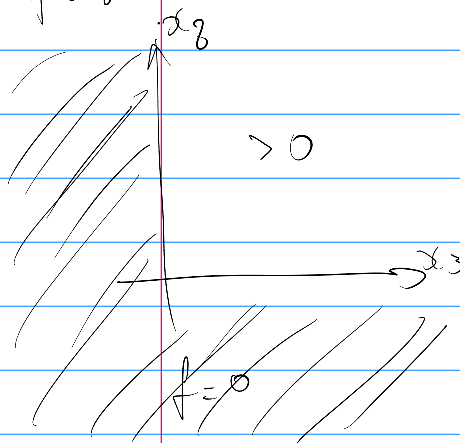
a) $f(x_3,x_8)$? $=\begin{cases}\exp(-x_3-x_8) & \text{при } x_8\ge 0,\ x_3\ge 0\\ 0, & \text{иначе}\end{cases}$

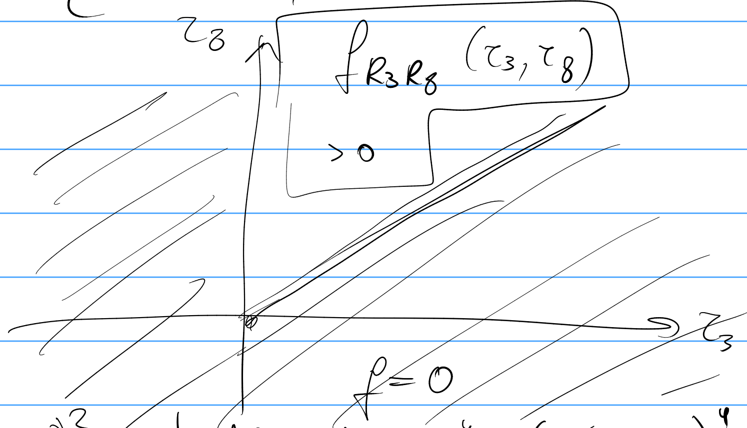
б) $f(r_3,r_8)$?

$$\boxed{f_{R_3, R_8}(\tau_3, \tau_8)\, d\tau_3 \wedge d\tau_8}$$

$$f(\tau_3, \tau_8) = \begin{cases} /////// & \text{при } 0 \le \tau_3 \le \tau_8 \\ 0, & \text{иначе} \end{cases}$$

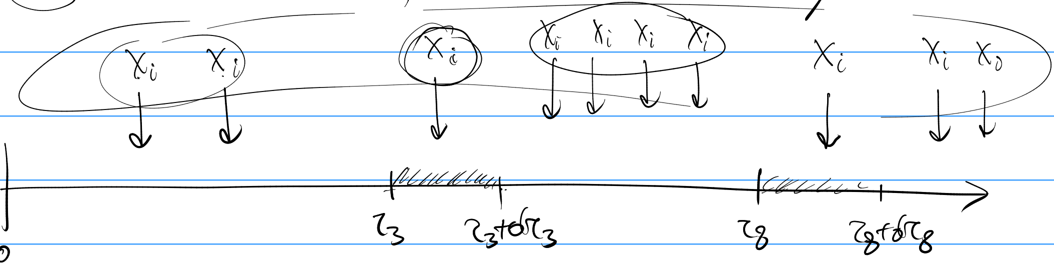
$f_{X_3 X_8}(x_3, x_8)$





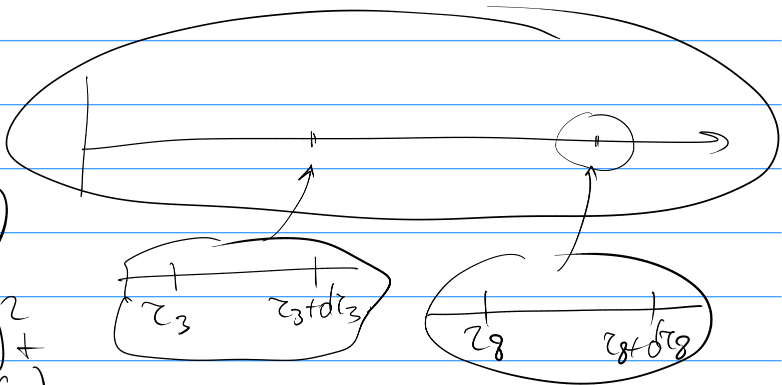
$$= C_{10}^2 \cdot \left(P(X_i \in [0; \tau_3])\right)^2 \cdot C_8^1 \cdot f(\tau_3) \cdot d\tau_3 \cdot C_7^4 \cdot P(X_i \in [\tau_3; \tau_8])^4 \cdot$$
$$\cdot\, C_3^1 \cdot f(\tau_8) \cdot d\tau_8 \cdot C_2^2 \cdot P(X_i \ge \tau_8)^2 + o(\dots)$$

$$P(R_3 \in [\tau_3; \tau_3 + d\tau_3],\ R_8 \in [\tau_8; \tau_8 + d\tau_8])$$



$$P(X_7 \in [\tau_8; \tau_8 + d\tau_8]) = f_X(\tau_8) \cdot d\tau_8 + o(d\tau_8)$$

$$P(X_5, X_7 \in [\quad]) = f_X^2(\tau_8)\,(d\tau_8)^2 + \dots = o(d\tau_8)$$



$$P(X_i \in [0; \tau_3]) = \int_0^{\tau_3} \underbrace{\exp(-x)}_{f_X(x)}\, dx = 1 - \exp(-\tau_3)$$

$$P(X_i \in [\tau_3; \tau_8]) = \int_{\tau_3}^{\tau_8} \exp(-x)\, dx = \exp(-\tau_3) - \exp(-\tau_8)$$

$$P(X_i \ge \tau_8) = \int_{\tau_8}^{\infty} \exp(-x)\, dx = \exp(-\tau_8)$$

$\Sigma = 1$

$$C_{10}^2 \cdot P(X_i \in [0; r_3])^2 \cdot C_8^1 \cdot f(r_3)\cdot dr_3 \cdot C_7^4 \cdot P(X_i \in [r_3; r_8])^4 \cdot$$
$$\cdot C_3^1 \cdot f(r_8)\cdot dr_8 \cdot C_2^2 \cdot P(X_i \geq r_8)^2 + o(\dots)$$

$$P(R_3 \in [r_3; r_3 + dr_3],\ R_8 \in [r_8; r_8 + dr_8])$$

если $0 \le r_3 \le r_8$

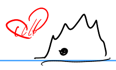
$$f(r_3, r_8) = \begin{cases} \dfrac{10!}{2!\,1!\,4!\,1!\,2!} \cdot (1 - e^{-r_3})^2 \cdot e^{-r_3} \cdot (e^{-r_3} - e^{-r_8})^4 \cdot e^{-r_8} \cdot (e^{-r_8})^2 \\ 0, \text{ иначе} \end{cases}$$

$$C_{10}^2 \cdot C_8^1 \cdot C_7^4 \cdot C_3^1 \cdot C_2^2 = \frac{10!}{2!\,8!} \cdot \frac{8!}{1!\,7!} \cdot \frac{7!}{4!\,3!} \cdot \frac{3!}{1!\,2!} \cdot \frac{2!}{2!}$$

C_{10}^2 – неуд. обозн. $\binom{10}{2}$

$$D(2+8) = \frac{10!}{2!\,8!}$$

$$D(2+1+4+1+2) = \frac{10!}{2!\,1!\,4!\,1!\,2!}$$

13:13 

Q. Как считать совм-ую энтропию?

A. а что это?

$$H(X,Y) = \sum_{i,j} p_{ij} \cdot \log_{\frac{1}{2}} p_{ij}$$

$H(X,Y)$ – кол-во бит информации, (в среднем) чтобы узнать пару (X_i, Y_i)

$\binom{0}{2}$ $\binom{1}{5}$ - - - - - - -

$\binom{X_1}{Y_1}$ $\binom{X_2}{Y_2}$

	$Y=2$	$Y=5$
$X=0$	$1/4$	$1/8$
$X=1$	$1/2$	$1/8$

u	02	05	12	15
$P(Z=u)$	$1/4$	$1/8$	$1/2$	$1/8$

$$H(X,Y) = H(Z)$$

$$H(X,Y) = \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{2}\cdot 1 + \frac{1}{8}\cdot 3$$

$E(X^2) \nearrow \int_{\mathbb{R}} x^2 f(x)\,dx$

$\searrow \iint_{\mathbb{R}\mathbb{R}} x^2 \cdot f(x,y)\,dx\,dy$

11.15 На первом шаге значение X выбирается случайно равномерно на отрезке $[0;1]$. На втором шаге значение Y выбирается случайно и равномерно от 0 до получившегося X.

а) Найдите функции плотности $\boxed{f(y|x), f(x), f(x,y), f(x|y), f(y)}$.

б) Найдите $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(X^2)$, $\mathbb{E}(Y^2)$.

в) Найдите $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, $\mathrm{Cov}(X,Y)$, $\mathrm{Corr}(X,Y)$.

г) Найдите $\mathbb{E}(X|Y)$, $\mathbb{E}(Y|X)$, $\mathbb{E}(X^2|Y)$, $\mathbb{E}(Y^2|X)$.

д) Найдите $\mathrm{Var}(Y|X)$, $\mathrm{Var}(X|Y)$.

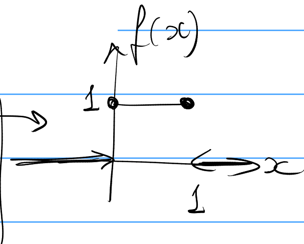
е) Найдите $\mathbb{P}(Y > 0.2|X = 0.5)$, $\mathbb{P}(Y > 0.2|X < 0.5)$.

$E(X^2|Y) = \int_{\mathbb{R}} x^2 \cdot f(x|Y)\,dx$

$\boxed{f(y|x)\ ?}$

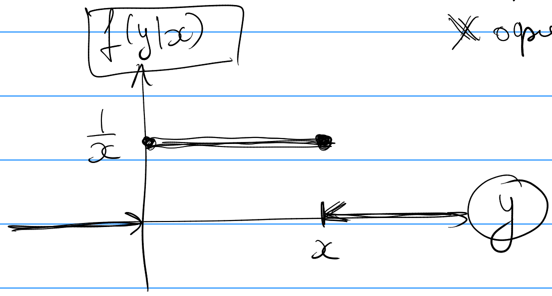
$\boxed{X \sim U[0;1]}$

$\boxed{(Y|X) \sim U[0;X]}$



расп-но

x зафиксировали

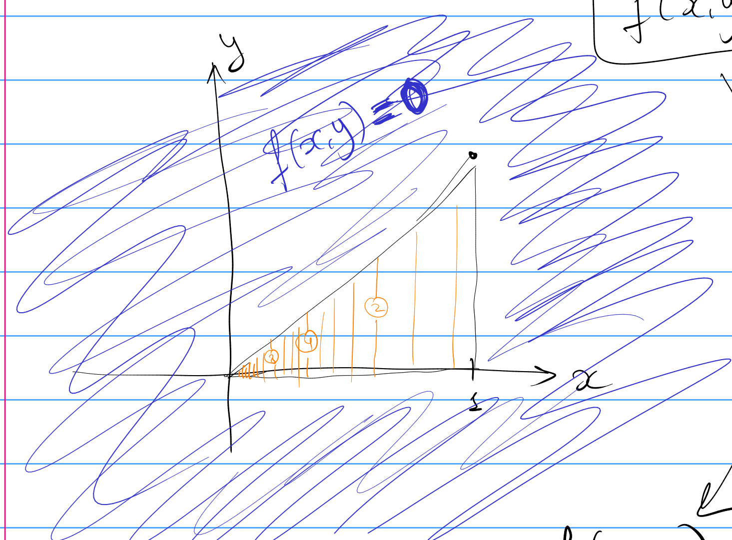


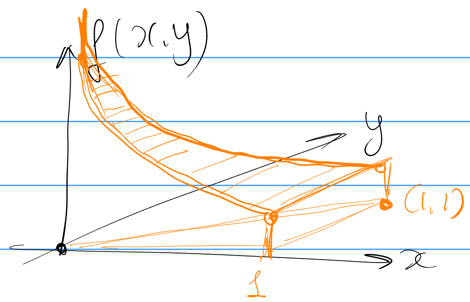
x – параметр (!)

$f(y|x)$ $\quad f(x)$ $\quad f(x,y)$

$$f(y|x) = \frac{f(x,y)}{f(x)} \qquad f(x,y) = f(x)\cdot f(y|x)$$

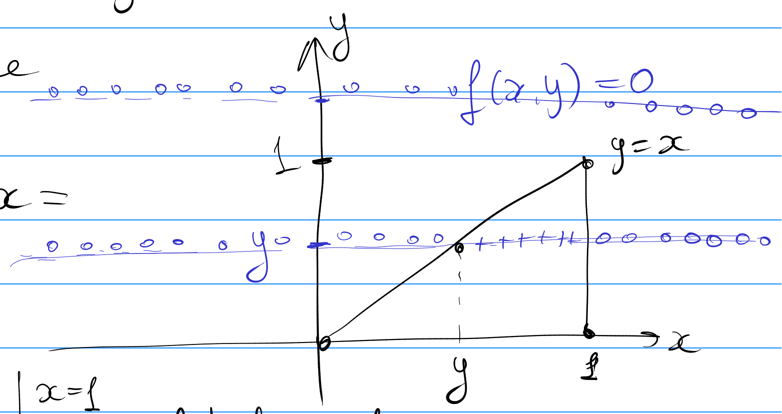
$$\boxed{f(x,y)} = \begin{cases} 1\cdot\frac{1}{x}, & \text{если } 0 \le y \le x \le 1 \\ 0, & \text{иначе} \end{cases}$$





$$f(x|y) = \frac{f(x,y)}{f(y)} \qquad f(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

$$f(x,y)=\begin{cases}\frac{1}{x} & \text{если } 0\le y\le x\le 1\\ 0, & \text{иначе}\end{cases}$$



$$f(y)=\int_{\mathbb{R}} f(x,y)\,dx=$$

$$=\int_y^1 \frac{1}{x}\,dx=\ln x\Big|_{x=y}^{x=1}=\ln 1-\ln y=-\ln y$$

$$\boxed{\boxed{f(y)}=\begin{cases}-\ln y, & \text{если } 0\le y\le 1\\ 0, & \text{иначе}\end{cases}}$$

если бы!

$$P(X\in[0;0{,}5],\ Y\in[0,\ 0.3])=\int_0^{0,5}\int_0^{0,3} f(x,y)\,dy\,dx$$

$f(y)$?

$$P(Y\in[y;y+dy])\approx f(y)\cdot dy$$

$$P(Y\in[y,y+dy],\ X\in[x;x+dx])=f(x,y)\cdot dx\,dy$$

$$f(y)=\frac{1}{dy}\int_x P(Y\in\ \_\ \_\ \_\ \_\ )=\int f(x,y)\cancel{\cdot dy}\,dx$$

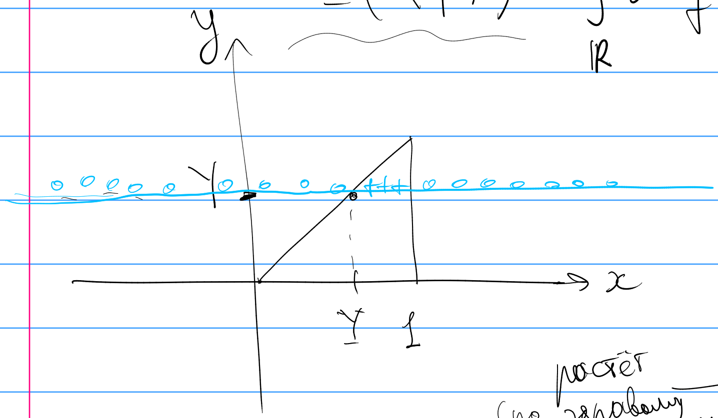
	$Y=2$	$Y=3$	$Y=5$
$X=0$			
$X=1$			

$$P(Y=y)=\sum_{x} P(X=x,\ Y=y)$$

$$f(y)=\int_{\mathbb{R}} f(x,y)\,dx$$

$$\boxed{f(x|y)=\frac{1/x}{-\ln y}\quad 0\le y\le x\le 1}$$

$$E(X^2|Y) = \int_{\mathbb{R}} x^2 \cdot f(x|Y)\,dx = \int_Y^1 x^2 \cdot \frac{1/x}{-\ln Y}\,dx =$$



$$= \int_Y^1 \frac{x\,dx}{-\ln Y} = \frac{x^2/2}{-\ln Y}\Bigg|_{x=Y}^{x=1} =$$

растёт (по здравому смыслу)

$$= \frac{1-Y^2}{-2\ln Y} \geq 0$$ [illegible]

среднее значение X^2 при изв-м Y

$$\lim_{y\to 1} \frac{1-y^2}{-2\ln y} = 1$$ (по здр. смыслу)

$$\mathrm{Var}(X|Y) = \underbrace{E(X^2|Y)}_{\int_{\mathbb{R}} x^2 \cdot f(x|Y)\,dx} - \big(\underbrace{E(X|Y)}_{\int_{\mathbb{R}} x \cdot f(x|Y)\,dx}\big)^2$$

листок 11