Stochastic Processes problems

https://github.com/bdemeshev/stochastic_pro

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Contents

1 First step analysis

1.1 Biden and Trump alternately throw a fair dice infinite number of times. Biden throws first. The person who obtains the first 6 wins the game.

a) What is the probability that Biden will win?

b) What is expected number of turns?

c) What is variance of the number of turns?

d) What is expected number of turns given that Biden won?

e) Find the transition matrix of this four state Markov chain.

1.2 Elon throws an unfair coin until "head" appears. The probability of "head" is $p \in (0; 1)$. Let N be the total number of throws.

a) Find $\mathbb{E}(N)$, $\text{Var}(N)$, $\mathbb{E}(N^3)$, $\mathbb{E}(\exp(tN))$.

b) What is the probability than N will be even?

1.3 Alice and Bob throw a fair coin until the sequence HTT or THT appears. Alice wins if HTT appears first, Bob wins if THT appears first.

a) Find the probability that Alice wins.

b) Find the expected value and variance of the total number of throws.

c) Using any open source software find the probability that Alice wins for all possible combinations of three coins sequences for Alice and Bob.

d) Now Alice and Bob play the following game. Alice chooses her three coins winning sequence first. Next Bob, knowing the choice of Alice, chooses his three coins winning sequence. Than they throw a fair coin until either of their sequences appears. What is the best strategy for Alice? For Bob? What is the probability that Alice wins this game?

1.4 You throw a dice unbounded number of times. If it shows 1, 2 or 3 then the corresponding amount of dollars is added in the pot. It it shows 4 or 5 the game stops and you get the pot. If it shows 6 the game ends and you get nothing. Initially the pot is empty.

a) What is probability that the game will end by 6?

b) What is expected duration of the game?

c) What is your expected payoff?

d) What is your payoff variance?

e) Consider variation-A of the game. Rules are the same, but initially the pot contains 100 dollars. How will the answers to questions (a)-(d) change?

f) Consider variation-B of the game. Initially the pot is empty. One rule is changed. If the dice shows 5 the content of the pot is burned and the game continues. How will the answers to questions (a)-(d) change?

1.5 Boris Johnson throws a fair coin until 1 appears or until he says "quit". His payoff is the value of the last throw. Boris optimizes his expected payoff. If many strategies gives the same expected payoff he chooses the strategy that minimizes the expected duration of the game.

a) What is the optimal strategy and the corresponding expected payoff?

b) What is the expected duration?

c) How the answers to points (a) and (b) will change if Boris should pay 0.3 dollars for each throw?

1.6 Winnie-the-Pooh starts wandering from the point $x = 1$. Every minute he moves one unit left or one right with equal probabilities.

Let T be the random moment of time when he reaches $x = 0$.

a) Find the generating function $g(u) = \mathbb{E}(u^T)$.

b) Extract all probabilities $\mathbb{P}(T = k)$ from the function $g(u)$.

1.7

1.8

2 Generating functions

2.1 The MGF (moment generating function) of the random variable X is give by $M(t) = 0.3\exp(2t) + 0.2\exp(3t) + 0.5\exp(7t)$.

Recover the distribution of the random variable X.

2.2 The random variable Y takes values 1, 2 or 3 with equal probabilities.

Find the MGF of the random variable Y.

2.3 The MGF of the random variable W has a Taylor expansion that starts with $M(t) = 1 + 2t + 7t^2 + 20t^3 + \ldots$.

Find $\mathbb{E}(W)$, $\text{Var}(W)$, $\mathbb{E}(W^3)$.

2.4 The random variable X takes non-negative integer values. The generating function $g(u) = \mathbb{E}(u^X)$ has a Taylor expansion that starts with $g(u) = 0.1 + 0.2u + 0.15u^2 + \ldots$.

Find probabilities $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.

2.5 Random variables X_i are mutually independent and X_i has Gamma distribution $\text{Gamma}(\alpha_i, \beta_i)$.

I sum up the random number N of terms,

$$S = X_1 + X_2 + \ldots + X_N.$$

The number N has Poisson distribution $\text{Pois}(\lambda)$ and is independent of the sequence (X_i).

a) Find the MGF of S. You may the MGF formula for Gamma distribution as known.

b) Find $\mathbb{E}(S)$ and $\text{Var}(S)$.

2.6 The random variable X take non-negative integer values. Its moment generating function is equal to $M(t) = (2 - \exp(t))^{-7}$.

a) Find the probability generating function $g(u) = \mathbb{E}(u^X)$.

b) Find $\mathbb{E}(X)$, $\text{Var}(X)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.

c) Find $\mathbb{P}(X = k)$.

3 Solutions

1.1.

a) $\mathbb{P}(B) = 6/11$, first step equation for $p = \mathbb{P}(B)$ is $p = 1/6 + (5/6)^2 p$ or $p = 1/6 + 5/6 \cdot (1 - p)$.

b) $\mathbb{E}(N) = 6$, first step equation for $m = \mathbb{E}(N)$ is $m = 1/6 + 5/6(m + 1)$.

c) $\mathbb{E}(N^2) = 66$, $\mathrm{Var}(N) = 30$, first step equation is $\mathbb{E}(N^2) = 1/6 + 5/6\mathbb{E}((N + 1)^2)$.

d) $\mathbb{E}(N \mid B) = 61/11$. Start by replacing uncoditional probabilities on the tree by conditional ones. First step equation for $\mu = \mathbb{E}(N \mid B)$ is $\mu = 11/36 + 25/36(\mu + 2)$.

e) $\begin{pmatrix} 0 & 5/6 & 1/6 & 0 \\ 5/6 & 0 & 0 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1.2.

1.3.

1.4.

1.5.

1.6.

1.7.

1.8.

2.1.

2.2. $M(t) = (\exp(t) + \exp(2t) + \exp(3t))/3$.

2.3. $\mathbb{E}(W) = 2$, $\mathrm{Var}(W) = 7 \cdot 2 - 2^2$, $\mathbb{E}(W^3) = 20 \cdot 3!$.

2.4. $\mathbb{P}(X = 0) = 0.1$, $\mathbb{P}(X = 1) = 0.2$, $\mathbb{P}(X = 2) = 0.15$.

2.5.

2.6. $g(u) = g(\exp(t)) = \mathbb{E}(\exp(tX)) = M(t)$

4 Sources of wisdom

[Buz+15] Nazar Buzun et al. "Stochastic Analysis in Problems, part 1 (in Russian)." In: *arXiv preprint arXiv:1508.03461* (2015). URL: https://arxiv.org/abs/1508.03461.