Stochastic Processes problems

https://github.com/bdemeshev/stochastic_pro

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Contents

1 First step analysis

1.1 Biden and Trump alternately throw a fair dice infinite number of times. Biden throws first. The person who obtains the first 6 wins the game.

a) What is the probability that Biden will win?

b) What is expected number of turns?

c) What is variance of the number of turns?

d) What is expected number of turns given that Biden won?

e) Find the transition matrix of this four state Markov chain.

1.2 Elon throws an unfair coin until “head” appears. The probability of “head” is $p \in (0;1)$. Let N be the total number of throws.

a) Find $\mathbb{E}(N)$, $\text{Var}(N)$, $\mathbb{E}(N^3)$, $\mathbb{E}(\exp(tN))$.

b) What is the probability than N will be even?

1.3 Alice and Bob throw a fair coin until the sequence HTT or THT appears. Alice wins if HTT appears first, Bob wins if THT appears first.

a) Find the probability that Alice wins.

b) Find the expected value and variance of the total number of throws.

c) Using any open source software find the probability that Alice wins for all possible combinations of three coins sequences for Alice and Bob.

d) Now Alice and Bob play the following game. Alice chooses her three coins winning sequence first. Next Bob, knowing the choice of Alice, chooses his three coins winning sequence. Than they throw a fair coin until either of their sequences appears. What is the best strategy for Alice? For Bob? What is the probability that Alice wins this game?

1.4 You throw a dice unbounded number of times. If it shows 1, 2 or 3 then the corresponding amount of dollars is added in the pot. It it shows 4 or 5 the game stops and you get the pot with money. If it shows 6 the game ends and you get nothing. Initially the pot is empty.

a) What is probability that the game will end by 6?

b) What is expected duration of the game?

c) What is your expected payoff?

d) What is your payoff variance?

e) Consider variation-A of the game. Rules are the same, but initially the pot contains 100 dollars. How will the answers to questions (a)-(d) change?

f) Consider variation-B of the game. Initially the pot is empty. One rule is changed. If the dice shows 5 the content of the pot is burned and the game continues. How will the answers to questions (a)-(d) change?

1.5 Boris Johnson throws a fair coin until 1 appears or until he says “quit”. His payoff is the value of the last throw. Boris optimizes his expected payoff. If many strategies gives the same expected payoff he chooses the strategy that minimizes the expected duration of the game.

a) What is the optimal strategy and the corresponding expected payoff?

b) What is the expected duration?

c) How the answers to points (a) and (b) will change if Boris should pay 0.3 dollars for each throw?

1.6 Winnie-the-Pooh starts wandering from the point $x = 1$. Every minute he moves one unit left or one right with equal probabilities.

Let T be the random moment of time when he reaches $x = 0$.

a) Find the generating function $g(u) = \mathbb{E}(u^T)$.

b) Extract all probabilities $\mathbb{P}(T = k)$ from the function $g(u)$.

1.7 Gleb Zheglov catches one criminal every day. With probability 0.2 the catched criminal is replaced by w new criminals. Initially there are n criminals in the town.

What is the expected time to the ultimate crime eradication in the town?

a) (4 points) Solve the problem for $w = 1$ and $n = 1$.

b) (6 points) Solve the problem for arbitrary w and n.

1.8 Consider infinite ladder with steps numbered from 0 to infinity. I start at step 0. Every day with probability u I go one step up. With probability d I go one step down. With probability $1 - u - d$ I stay on the same step.

If I am at step 0 then I stay there with probability $1 - u$ because it's impossible to go down.

Consider the case $d > u$.

What is the probability that I will be at step 0 after 10^{1000} days?

1.9 I throw a fair die until the sequence 626 appears. Let N be the number of throws.

a) What is the expected value $\mathbb{E}(N)$?

b) Write down the system of linear equations for the moment generating function of N. You don't need to solve it!

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2 Markov chains

2.1 HSE student lives in two states: "sleep" and "study" and tries to change the state every 1 hour. After the sleep state the student continues sleeping with probability equal to 0.25, otherwise a student starts studying. If the student is studying, the probabilities to continue studying and to start sleeping are equal.

a) Write down the transition matrix of this Markov chain.

b) Draw the graph representation.

c) What is the probability that a Sleeping Student will be a Studying Student after 1 hour? After 2 hours?

d) We know that initially student is sleeping with probability $p = \frac{2}{3}$. Find the probabilities of sleep and study states after 1 and 2 hours.

e) Find the probabilities of sleep and study states after 20 and 100 hours (do it with **matrix** operations and any soft). Is there any difference and why?

2.2 HSE student has three states: pre-coffee, with-coffee and over-coffee. He goes to Jeffrey's each break seeking for a cup of coffee. The line is usually too long, so probability to stay pre-coffee is equal to 60% and to be over-coffee — is zero. Caffeinated students can stay in lines longer, so for with-coffee student the probability to become over-coffee is 20% and to become pre-coffee — 30%. Over-coffee student runs to coffeeshop very fast and able to stay over-coffeed with $p = 0.70$ and can suddenly become pre-coffee with $p = 0.10$.

a) Draw the graph representation of this Markov chain.

b) Write the transition matrix of this Markov chain.

c) What is the probability that morning pre-coffee student will be with-coffee after 1 break? After 3 breaks?

d) What is the probability that morning pre-coffee student will be with-coffee after 1 break? After 3 breaks? After 200 breaks?

2.3 Unteachable students in NOTHSE University try to pass the exams. Students cheat successfully and pass the exams with probability 10%. In the case of a failure students are allowed to infinite number of retakes. All students are unteachable so the amount of knowledge is always the same and doesn't depend of the number of retakes.

a) Draw the graph representation of this Markov chain.

b) What is the probability to graduate using no more than 5 retakes?

c) What is the probability to graduate eventually?

d) Use **first step analysis** to find the average number of retakes per student in this University.

2.4 Every month the real estate Galina agent has two options: to increase her commission and to ask an owner to increase the rent. If the agent has increased the commission, on the next step she increases the commission again with probability $5/8$. If she has asked the owner, she decides to increase the commission with probability equal to $3/4$ on the next step.

a) Write the transition matrix of this Markov chain.

b) Draw the graph representation.

c) Use **first step analysis** to find how many steps the agent does between asking the owner to increase the price.

2.5 Alice and Bob toss a coin, writing down the results. If the last 3 tosses are Head, Head and Tail, Alice wins. If the last 3 tosses are Tail, Head and Head, Bob wins.

a) Is it easy to work with matrix representation in this case?

b) Draw the graph representation. Who is more likely to win the game?

c) Use **first step analysis** to find the probability of Alice's win.

d) Find the probability that the game ends in exactly 4 tosses.

e) Find the expected value and variance of the total number of coin throws in the game.

2.6 HSE student has an unusually caring granny who cooks one pie with probability 0.7 every weekend. Granny's pies are so tasty that HSE student can't resist and he gains 1 kilo for each pie eaten. Without pies the student with more than 70 kilos weight looses 1 kilo per week, yeah, he has a lot of studies! At the beginning of the study year student's weight is $W_0 = 70$ kilos.

Let W_t be the weight of the student t weeks later.

a) Find the probability $\mathbb{P}(W_3 \geqslant 71)$ and expected value $\mathbb{E}(W_3)$.

b) Find the limit weight after infinitely many study weeks $\lim_{t\to\infty} W_t$.

c) Explain whether the chain (W_t) has a stationary distribution.

2.7 The fair price of Sborbank in discrete stock market is somewhere between 100 and 101 rubles. If the price is equal to 100, then the price grows up by 1 ruble with probability $\frac{9}{10}$, otherwise it goes down by 1 ruble. If the price is greater then 100, it grows by 1 ruble with probability $\frac{1}{3}$ or declines by 1 ruble. If the price is lower than 100, it grows by 1 ruble with probability $\frac{2}{3}$ or declines by 1 ruble.

a) Draw the graph representation of the corresponding Markov chain.

b) Do you think this chain has some stationary distribution?

c) Find the average time for the stock price to fall from 102 rubles to 98 rubles.
Hint: you may to decompose the long path into smaller ones and to use the first step analysis.

2.8 The hedgehog Melissa starts at the vertex A of a triangle ΔABC. Each minute she randomly moves to an adjacent vertex with probabilities $\mathbb{P}(A \to B) = 0.7$, $\mathbb{P}(A \to C) = 0.3$, $\mathbb{P}(B \to C) = \mathbb{P}(B \to A) = 0.5$, $\mathbb{P}(C \to B) = \mathbb{P}(C \to A) = 0.5$.

a) What is the probability that she will be in vertex B after 3 steps?

b) Write down the transition matrix of this Markov chain.

c) What proportion of time Melissa will spend in each state in the long run?

2.9 A Hedgehog starts at the point $x = 2$ on the real line. Every minute he moves one step left with probability 0.3 or one step right with probability 0.7. There are two exceptions from this rule: the absorbing point $x = 0$ and the reflecting barrier at $x = 3$.

If the Hedgehog reaches the absorbing point $x = 0$ then he stops moving and stays there. If the Hedgehog reaches the reflecting barrier $x = 3$ then his next move will be one step left with probability 1.

a) [2] Write the transition matrix of this Markov chain.

b) [3] What is the probability that Hedgehog will be at $x = 1$ after exactly 3 steps?

c) [5] What is the expected time to reach the absorbing state?

2.10 Vampire Petr drinks blood of a new victim every day. Unfortunately 20% of the population are vaccinated against vampires. If more than one victim of the last three victims are vaccinated then Petr will be instantaneously cured and will return to the normal life.

For simplicity let's assume that the last three victims were not vaccinated.

a) What is the probability that vampire Petr will be cured in the next three days?

b) How many victims will be bitten by vampire Petr on average?

2.11 A hedgehog moves at random on the vertices A, B, C and D of a regular tetrahedron (тетраэдр). She start at the vertex A and every minute changes her position to one of the adjacent vertices with probability $1/3$ independently of past moves.

a) Write down the transition matrix of this Markov chain.

b) What is the expected time of the first return to the starting vertex A?

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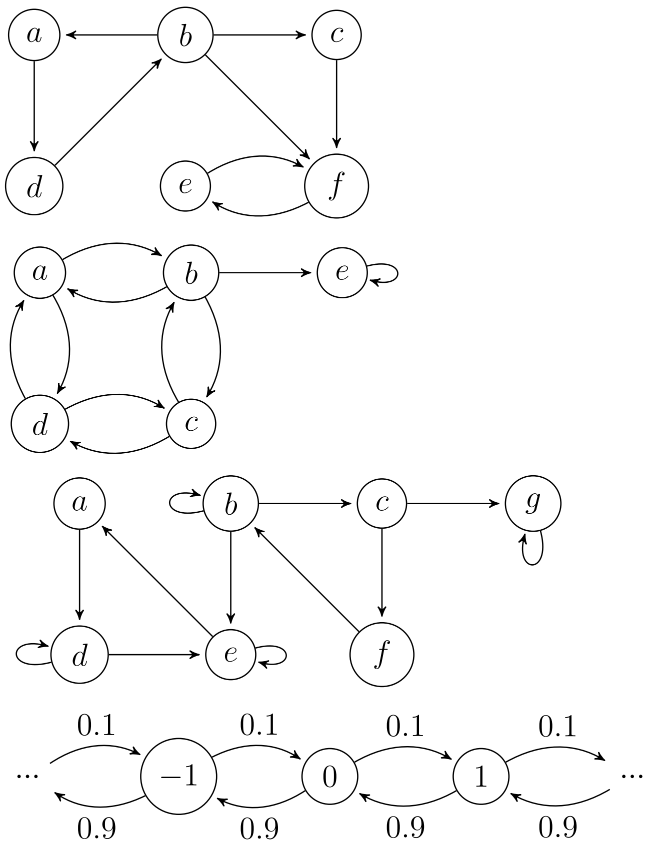
3 Classification of states

3.1 We randomly wander on the graph choosing at each moment of time one of the possible directions. If probabilities are not given we choose equiprobably.



a) Split each Markov chain into communicating classes.

b) Find the period of every state.

c) Classify each state as transient, null-recurrent and positive recurrent.

d) For positive recurrent states find the expected return time.

e) Find all stationary distributions.

3.2 A Knight randomly wanders on the chessboard. At each step he randomly chooses one of the possible Knight-moves with equal probabilities.

a) Find the stationary distribution.

b) Find the expected return time for every square.

c) Find the period of every square.

3.3 Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

a) (3 points) Split the chain in classes and classify them into closed or not closed.

b) (2 points) Classify the states into recurrent or transient.

c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.

What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

3.4 Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) (3 points) Split the chain in classes and classify them into closed or not closed.

b) (2 points) Classify the states into recurrent or transient.

c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.

What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

3.5

3.6

4 Generating functions

4.1 The MGF (moment generating function) of the random variable X is give by $M(t) = 0.3\exp(2t) + 0.2\exp(3t) + 0.5\exp(7t)$.

Recover the distribution of the random variable X.

4.2 The random variable Y takes values 1, 2 or 3 with equal probabilities.

Find the MGF of the random variable Y.

4.3 The MGF of the random variable W has a Taylor expansion that starts with $M(t) = 1 + 2t + 7t^2 + 20t^3 + \ldots$.

Find $\mathbb{E}(W)$, $\text{Var}(W)$, $\mathbb{E}(W^3)$.

4.4 The random variable X takes non-negative integer values. The generating function $g(u) = \mathbb{E}(u^X)$ has a Taylor expansion that starts with $g(u) = 0.1 + 0.2u + 0.15u^2 + \ldots$.

Find probabilities $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.

4.5 Random variables X_i are mutually independent and X_i has Gamma distribution $\text{Gamma}(\alpha_i, \beta_i)$.

I sum up the random number N of terms,

$$S = X_1 + X_2 + \ldots + X_N.$$

The number N has Poisson distribution $\text{Pois}(\lambda)$ and is independent of the sequence (X_i).

a) Find the MGF of S. You may the MGF formula for Gamma distribution as known.
b) Find $\mathbb{E}(S)$ and $\text{Var}(S)$.

4.6 The random variable X take non-negative integer values. Its moment generating function is equal to $M(t) = (2 - \exp(t))^{-7}$.

a) Find the probability generating function $g(u) = \mathbb{E}(u^X)$.
b) Find $\mathbb{E}(X)$, $\text{Var}(X)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.
c) Find $\mathbb{P}(X = k)$.

4.7 The number of players N who will win the lottery is a random variable with probability mass function $\mathbb{P}(N = k) = 7 \cdot 0.3^k/3$ for $k \geqslant 1$. Each player will get a random prize $X_i \sim U[0; 1]$. All random variables are independent. Let S be the sum of all the prizes.

a) Find $\mathbb{E}(S \mid N)$ and conditional moment generating function $M_{S|N}(u)$.
b) Find the unconditional moment generating function $M_S(u)$.
c) What is the probabilistic meaning of $M_S''(0) - (M_S'(0))^2$?

4.8 Prince Myshkin throws a fair coin until two consecutive heads appear. Let N be the number of throws.

Find the moment generating function of N.

4.9 The moment generating function of a random variable X is $1/(1 - 2t)$.

a) Find the moment generating function of $2X$.
b) Find the moment generating function of $X+Y$ where X and Y are independent and identically distributed.
c) Do you remember the sum of geometric progression? Find $\mathbb{E}(X^{2021})$.

4.10 The ultimate goal of this exercise is to prove the good upper bound for tail probability of a normal distribution: if $X \sim \mathcal{N}(0; \sigma^2)$ then $\mathbb{P}(X > c) \leqslant \exp(-c^2/2\sigma^2)$.

Here are the guiding hints (you free to use not use them):

a) State the MGF of X. You may derive it or simply write it if you remember.

b) Consider $Y = \exp(uX)$. Using Markov inequality provide the upper bound for $\mathbb{P}(Y > \exp(uc))$.

c) Prove that $\mathbb{P}(X > c) \leqslant MGF_X(u)\exp(-uc)$ for any u.

d) Find the value of u that makes the upper bound as tight as possible.

4.11 I have an unfair coin with probability of heads equal to $h \in (0; 1)$.

a) Let N be the number of tails before the first head. Find the MGF of N.

b) Let S be the number of tails before k heads (not necessary consecutive). Find the MGF of S.

c) What is the limit of $MGF_S(t)$ when $k \to \infty$ and $k \times h \to 0.5$? What is the name of the corresponding distribution?

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5 Limits

5.1 Polina loves sweet chestnuts. She has infinite sequence of baskets before her. In the basket number n there are n chestnuts in total. Unfortunately only one chestnut in every basket is a sweet one.

She picks chestnuts one by one at random from all the buskets sequentially. First she picks the unique chestnut from the basket number one, than she picks in a random order two chestnuts from the basket number two and so on.

The random variable S_t indicates whether the chestnut number t was a sweet one.

a) Find $\lim S_t$ or prove that the limit does not exist.

b) Find $\operatorname{plim} S_t$ or prove that the limit does not exist.

5.2 Let (X_n) be independent, each variable X_n has exponential distribution with rate $\lambda_n = n$.

a) Find the probability limit $\operatorname{plim} X_n$ or prove that it does not exist.

Let (Y_n) be independent, each variable Y_n has exponential distribution with rate $\lambda_n = n/(n+1)$.

b) Find the probability limit $\operatorname{plim} Y_n$ or prove that it does not exist.

5.3 Let (X_n) be independent normally distributed $\mathcal{N}(5; 10)$.

a) Find the probability limit

$$\operatorname{plim} \frac{X_1 + X_2 + \cdots + X_n}{7n}.$$

b) Find the probability limit

$$\operatorname{plim} \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{7n}.$$

c) Find the probability limit

$$\operatorname{plim} \ln\left(X_1^2 + X_2^2 + \cdots + X_n^2\right) - \ln n.$$

5.4 Let (X_n) be independent uniform on $[0;1]$. Let $Y_n = X_n^2 + X_n^3$.

a) Find the probability limit $\operatorname{plim} V_n$ for
$$V_n = \max\{Y_1, Y_2, \ldots, Y_n\}.$$

b) Find the probability limit $\operatorname{plim} W_n$ for
$$W_n = \max\{X_1 + Y_1, X_1 + Y_2, \ldots, X_1 + Y_n\}.$$

5.5 Consider the random variable X and the sequence of random variables Y_n with $\mathbb{E}(Y_n) = \frac{1}{n}$ and $\operatorname{Var}(Y_n) = \frac{\sigma^2}{n}$. Let $W_n = X + Y_n$.

a) Find the probability limit $\operatorname{plim} Y_n$;

b) Find the probability limit $\operatorname{plim} W_n$.

5.6 The random variables X_i are independent and uniformly distributed on $[0;1]$. Let $Y_n = \min X_1, \ldots X_n$.

a) Find the almost sure limit of Y_n;

b) Find the probability limit of Y_n;

c) Find the limiting distribution of Y_n.

5.7 Let X and Y be independent and uniformly distributed on $[0;1]$. Let $V_n = n^2 Y \cdot I(X \leqslant 1/n)$ and $W_n = Y \cdot I(X > 1/n)$.

a) Find $\operatorname{plim} V_n$ and $\operatorname{plim} W_n$.

b) Does (V_n) converge in mean squared?

c) Does (W_n) converge in mean squared?

5.8 a) As a warm-up find the limit
$$\lim_{n\to\infty} \frac{2n^2 + 3n + 6}{5n^2 + 2n + 9}.$$

Now consider the sequence with parameters:
$$X_n = \frac{Ln^2 + 3n + 6}{Rn^2 + 2n + 9}$$

b) For each value of parameters L and R find the limit $\lim X_n$.

c) Find the almost surely limit of X_n if L and R are independent and Unif$[0;1]$. Does the pointwise limit exist?

d) Random variables L and Q be independent and take values 0 or 1 with equal probabilities. Let $R = L + Q$. Find the almost surely limit of X_n in terms of L and R. Does the pointwise limit exist?

5.9 Consider the sequence $Y_n = U^n$ with parameter U.

a) Find the ordinary limit of Y_n for all values of U for which the sequence converges.

b) Find the almost surely limit of Y_n if $U \sim$ Unif$[0;1]$.

c) What is the probability that Y_n converges if $U \sim \text{Unif}[0;2]$?

d) What is the probability that Y_n converges if U takes values $+1$ or -1 with equal probabilities?

e) Does Y_n converges in distribution if U takes values $+1$ or -1 with equal probabilities?

5.10 The random variables X_i are independend and uniformly distributed on $[0;2]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^{10} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^3}{n+1}\right\}.$$

5.11 The random variables X_i are independend and uniformly distributed on $[0;1]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^{n} X_i}{n}, \frac{2\sum_{i=1}^{n} X_i^2}{n}\right\}.$$

5.12 The random variables X_i are independend and uniformly distributed on $[0;2]$. Find

$$\operatorname*{plim}_{n\to\infty} \frac{(X_1-\bar{X})^3 + (X_2-\bar{X})^3 + \ldots + (X_n-\bar{X})^3}{n+2022}.$$

5.13 Consider the stochastic process (X_n), where X_0 is uniform on $[0;2]$ and $X_n = (1+X_{n-1})/2$.

a) Find $\mathbb{E}(X_n)$ and $\text{Var}(X_n)$.

b) Find the probability limit $\text{plim}\, X_n$.

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6 Conditional expected value without sigma-algebras

6.1 We randomly uniformly select a point inside triangle $A=(6,0)$, $B=(0,2)$ and $O=(0,0)$. Let (X,Y) be coordinates of this random point.

a) Find conditional expected values $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

b) Find conditional variances $\text{Var}(Y \mid X)$ and $\text{Var}(X \mid Y)$.

6.2 The pair of random variables X and Y has joint probability density

$$f(x,y) = \begin{cases} x+y, & \text{if } x \in [0;1], y \in [0;1]; \\ 0, & \text{otherwise.} \end{cases}$$

a) Find the marginal densities $f_X(x)$ and $f_Y(y)$.

b) Find the conditional densities $f(x \mid y)$ and $f(y \mid x)$.

c) Find the conditional expected values $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

d) Find the conditional variances $\text{Var}(Y \mid X)$ and $\text{Var}(X \mid Y)$.

6.3 The random variables X and Y are independend with Poisson distribution with rate $\lambda = 1$. Let $S = X + Y$.

a) Find conditional probabilities $\mathbb{P}(X = x \mid S = s)$ and $\mathbb{P}(Y = y \mid S = s)$.
b) Find conditional expected values $\mathbb{E}(X \mid S)$ and $\mathbb{E}(Y \mid S)$.
c) Find conditional variances $\text{Var}(X \mid S)$ and $\text{Var}(Y \mid S)$.
d) How the answers will change if $X \sim \text{Pois}(\lambda_x)$ and $Y \sim \text{Pois}(\lambda_y)$?

6.4 Let X and Y be independent and exponentially distributed with rate $\lambda = 1$ and $S = X + Y$.

a) Find conditional densities $f(x \mid s)$ and $f(y \mid s)$.
b) Find conditional expected values $\mathbb{E}(X \mid S)$ and $\mathbb{E}(Y \mid S)$.
c) Find conditional variances $\text{Var}(X \mid S)$ and $\text{Var}(Y \mid S)$.
d) Find $\text{Cov}(X, Y \mid S)$ and $\text{Corr}(X, Y \mid S)$.
e) How the answers will change if $X \sim \text{Expo}(\lambda_x)$ and $Y \sim \text{Expo}(\lambda_y)$?

6.5 The random variable X has Poisson distribution with rate $\lambda = 1$. The random variable Y has uniform distribution on $[1; 2]$. Random variables X and Y are independent.

Find $\mathbb{E}(XY \mid X)$, $\text{Var}(XY + X^3 \mid X)$, $\text{Cov}(X, Y \mid X)$, $\text{Cov}(XY, X^2Y \mid X)$.

6.6 The random variables X_1 and X_2 are independent and normally distributed, $X_1 \sim \mathcal{N}(1; 1)$, $X_2 \sim \mathcal{N}(2; 2)$. I choose X_1 with probability 0.3 and X_2 with probability 0.7 without knowing their values.

Casino pays me the value Y that is equal to the chosen random variable.

Let the indicator I be equal to 1 if I choose X_1 and 0 otherwise.

a) Express Y in terms of X_1, X_2 and I.
b) Find $\mathbb{E}(Y \mid I)$, $\text{Var}(Y \mid I)$.
c) Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.

6.7 A Hedgehog in the fog starts in $(0, 0)$ at $t = 0$ and moves randomly with equal probabilities in four directions (north, south, east, west) by one unit every minute.

Let X_t and Y_t be his coordinates after t minutes and $S_t = X_t + Y_t$.

a) Find $\mathbb{E}(X_2 \mid S_2)$;
b) Find $\text{Var}(X_2 \mid S_2)$.

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7 Sigma-algebras and measurability

Sigma-algebra generated by discrete random variable X, $\sigma(X)$ — the list of all events that can be stated using X.

Sigma-algebra generated by arbitrary random variable X, $\sigma(X)$ — the smallest list of events that satisfies two properties:

- The list contains all events of the form $\{X \leqslant t\}$, that means one can compare X with any number;
- If one takes countably many events from this list and does logical operations (union, complement, intersection) then one will obtain an event from the list.

7.1 The random variable X takes values 1, 2 and -2 with equal probabilities.

a) Find the sigma-algebra $\sigma(X)$.

b) How the answer will change if one modifies probability distribution of X?

c) Find the sigma-algebra $\sigma(|X|)$.

d) Foma knows $|X|$ and Yeryoma knows X^2. What can one say about sigma-algebras that model their knowledge?

7.2 Experiment may end by one of the six outcomes:

	$X=-1$	$X=0$	$X=1$
$Y=0$	0.1	0.2	0.3
$Y=1$	0.2	0.1	0.1

a) Find expicitely the sigma-algebras $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$, $\sigma(X^2)$, $\sigma(2X+3)$.

b) How many elements are there in $\sigma(X, Y)$, $\sigma(X+Y)$, $\sigma(X, Y, X+Y)$?

7.3 Let's look at the number of possible elements in a sigma-algebra.

a) The random variable X has five possible values. How many events are there in $\sigma(X)$?

b) Can a sigma-algebra contain exactly 1000 events? Exactly 1024 events?

Maria throws a coin 100 times and remembers very well all the tosses.

c) How many elementary outcomes are there in the probability space Ω?

d) How many events are there in a sigma-algebra that models Maria's knowledge?

7.4 How sigma-algebras $\sigma(X)$ and $\sigma(f(X))$ are related? When they are equal?

7.5 How many different σ-algebras can be created using the set of outcomes Ω has three elements? And if Ω has four elements?

7.6 Provide an example of algebra that is not a σ-algebra.

7.7 Prove a statement or provide a counter-example:

a) The intersection of two sigma-algebras is a sigma-algebra.

b) If the intersection of two sigma-algebras is a sigma-algebra then one of them is contained in the other one.

c) The union of two sigma-algebras is a sigma-algebra.

d) If the union of two sigma-algebras is a sigma-algebra then one of them is contained in the other one.

7.8 Let $\mathcal{F}$ be some σ-algebra of subsets of Ω and $B \subseteq \Omega$. Consider the collection of sets $\mathcal{H} = \{A : A \subseteq B \text{ or } B^c \subseteq A\}$.

Is $\mathcal{H}$ a σ-algebra?

7.9 Будем обозначать количество элементов множества с помощью $\operatorname{card} A$. Рассмотрим подмножества натуральных чисел, $A \subseteq \mathbb{N}$. Определим для подмножества плотность Чезаро (Cesaro density),

$$\gamma(A) = \lim_{n\to\infty} \frac{\operatorname{card}(A \cap \{1, 2, 3, \ldots, n\})}{n}$$

в тех случаях, когда этот предел существует.

Плотность Чезаро показывает, какую «долю» от всех натуральных чисел составляет указанное подмножество. Обозначим с помощью $\mathcal{H}$ все подмножества, имеющие плотность Чезаро.

a) Чему равна плотность Чезаро у нечётных чисел?

b) Приведите пример множества, у которого не определена доля Чезаро.

c) Верно ли, что у натуральных чисел, в записи которых присутствует хотя бы одна единица, есть доля? Если да, то чему она равна?

d) Верно ли, что у натуральных чисел, в записи которых присутствует ровно одна единица, есть доля? Если да, то чему она равна?

e) Верно ли, что $\mathcal{H}$ — алгебра? Сигма-алгебра?

7.10 We throw a fair dice. Let Y be the indicator of a even score and Z be the indicator of score bigger than 2.

a) Find the sigma-algebra $\sigma(Z)$.

b) Find the sigma algebra $\sigma(Y \cdot Z)$.

c) How many elements are there in $\sigma(Y, Z)$?

d) How are related the σ-algebras $\sigma(Y \cdot Z)$ and $\sigma(Y, Z)$?

7.11 We throw a coin infinitely many times. Let X_n be the indicator that the coin landed on Head at toss number n. Consider a pack of σ-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, \ldots, X_n)$, $\mathcal{H}_n := \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$.

a) For each case provide two examples of σ-algebras that contain the corresponding event

  (a) $\{X_{37} > 0\}$;

  (b) $\{X_{37} > X_{2024}\}$;

  (c) $\{X_{37} > X_{2024} > X_{12}\}$;

b) Simplify experessions: $\mathcal{F}_{11} \cap \mathcal{F}_{25}$, $\mathcal{F}_{11} \cup \mathcal{F}_{25}$, $\mathcal{H}_{11} \cap \mathcal{H}_{25}$, $\mathcal{H}_{11} \cup \mathcal{H}_{25}$.

c) For each case provide two non-trivial examples (different from Ω and $\emptyset$) of an event A such that

  (a) $A \in \mathcal{F}_{2024}$;

  (b) $A \notin \mathcal{F}_{2025}$;

  (c) $A \in \mathcal{H}_n$ for all possible n;

7.12 Правда ли равносильны три набора требований к списку множеств $\mathcal{F}$?

Тариф «Классический»:

a) $\Omega \in \mathcal{F}$;

b) Если $A \in \mathcal{F}$, то $A^c \in \mathcal{F}$;

c) Если $A_1, A_2, A_3, \ldots, \in \mathcal{F}$, то $\cup A_i \in \mathcal{F}$.

Тариф «Перевёрнутый»:

a) $\emptyset \in \mathcal{F}$;

b) Если $A \in \mathcal{F}$, то $A^c \in \mathcal{F}$;

c) Если $A_1, A_2, A_3, \ldots, \in \mathcal{F}$, то $\cap A_i \in \mathcal{F}$.

Тариф «Хочу всё»:

a) $\Omega \in \mathcal{F}$, $\emptyset \in \mathcal{F}$;

b) Если $A \in \mathcal{F}$ и $B \in \mathcal{F}$, то $A \backslash B \in \mathcal{F}$;

c) Если $A_1, A_2, A_3, \ldots, \in \mathcal{F}$, то $\cup A_i \in \mathcal{F}$ и $\cap A_i \in \mathcal{F}$.

7.13 Рассмотрим $\Omega = [0; 1]$ и набор множества $\mathcal{F}$ таких, что либо каждое множество не более, чем счётно, либо дополнение к нему не более, чем счётно.

a) Верно ли, что $\mathcal{F}$ — алгебра? σ-алгебра?

b) Придумайте $B \subset \Omega$, такое что $B \notin \mathcal{F}$.

7.14 В лесу есть три вида грибов: рыжики, лисички и мухоморы. Попадаются они равновероятно и независимо друг от друга. Маша нашла 100 грибов. Пусть R — количество рыжиков, L — количество лисичек, а M — количество мухоморов среди найденных грибов.

a) Сколько элементов $\sigma(R)$?

b) Сколько элементов $\sigma(R, M)$?

c) Измерима ли L относительно $\sigma(R)$?

d) Измерима ли L относительно $\sigma(R, M)$?

e) Измерима ли L относительно $\sigma(R + M)$?

f) Измерима ли L относительно $\sigma(R - M)$?

7.15 Сейчас либо солнечно, либо дождь, либо пасмурно без дождя. Соответственно, множество Ω состоит из трёх исходов, $\Omega = \{\text{солнечно}, \text{дождь}, \text{пасмурно}\}$. Джеймс Бонд пойман и привязан к стулу с завязанными глазами, но он может на слух отличать, идёт ли дождь.

a) Как выглядит σ-алгебра событий, которые различает агент 007?

b) Как выглядит минимальная σ-алгебра, содержащая событие $A = \{\text{не видно солнце}\}$?

c) Сколько различных σ-алгебр можно придумать для данного Ω?

7.16 The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

a) [3] Explicitely list all the events in sigma-algebra $\sigma(X_1 \cdot X_2)$.

b) [3] Pavel says that he knows only whether X_1 and X_3 are equal. How will you describe his knowledge with sigma-algebra?

c) [4] How many events are in the sigma-algebra $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3)$?

7.17 Vincenzo Peruggia makes attempts to steal the Mona Lisa painting until the first success. Each attempt is successful with probability 0.1.

Let X be the number of attempts and $Z = \min\{X, 5\}$.

a) (5 points) How many events are in sigma-algebras $\sigma(Z)$ and $\sigma(X)$?

b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(Z)$ but $A \notin \sigma(X)$; $B \in \sigma(X)$ but $B \notin \sigma(Z)$.

c) (10 points) Find $\mathbb{E}(Z \mid X)$ and $\mathbb{E}(X \mid Z)$.

7.18 Variables X_1, X_2, ...X_{100} are independent and identically distributed with mean 1 and variance 2. Each X_i has only three possible values: 0, 1, and 2.

a) (5 points) How many events are in sigma-algebras $\sigma(X_1, X_2)$ and $\sigma(X_1 - X_2)$?

b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(X_1, X_2)$ but $A \notin \sigma(X_1 - X_2)$; $B \in \sigma(X_1 - X_2)$ but $B \notin \sigma(X_1, X_2)$.

c) (10 points) Find $\mathbb{E}(X_1 + \ldots + X_{100} \mid X_1 + \ldots + X_{50})$ and $\mathbb{E}(X_1 + \ldots + X_{50} \mid X_1 + \ldots + X_{100})$.

7.19

7.20

7.21

7.22

8 Sigma-algebras and conditional expected value

8.1 At time moment $t = 0$ in the casiono there are countably many players with perfect memory. Let's number them as Miss First, Mister Second, etc.

Time is discrete. Random variables X_t are independent and take values $+1$ or -1 with equal probabilities. At each moment of time $t > 0$ everybode gets X_t roubles and than the player number t leaves the casiono.

The cumulative sum $S_t = X_1 + \cdots + X_t$ reaches its first local maximum at the random time T. At time $T + 1$ the dealer calls his friend Black Jack and says «It's time!» They have agreed beforehand on the call time.

Black Jack chases the player number T and steals all his information before the police can intervent. Let's describe the information of Black Jack by sigma-algebra $\mathcal{F}_J$ and the information of every player t at the last moment in casino by $\mathcal{F}_t$.

a) Which sigma-algebras contain the event $\{T = 10\}$?

b) Provide an example of two events from $\mathcal{F}_J$ that do not enter in neither $\mathcal{F}_t$.

c) Find conditional expected values $\mathbb{E}(T \mid \mathcal{F}_J)$, $\mathbb{E}(X_T \mid \mathcal{F}_J)$, $\mathbb{E}(X_{T+1} \mid \mathcal{F}_J)$.

d) Find conditional expected values $\mathbb{E}(S_{T-1} \mid \mathcal{F}_J)$, $\mathbb{E}(S_T \mid \mathcal{F}_J)$, $\mathbb{E}(S_{T+1} \mid \mathcal{F}_J)$, $\mathbb{E}(S_{T+2} \mid \mathcal{F}_J)$.

Let's define Y_{T-k} as

$$Y_{T-k} = \begin{cases} X_{T-k}, & \text{if } T - k > 0; \\ 0, & \text{otherwise.} \end{cases}$$

e) Find $\mathbb{E}(Y_{T-10} \mid \mathcal{F}_J)$.

f) Find conditional expected values $\mathbb{E}(X_T \mid \mathcal{F}_{10})$, $\mathbb{E}(X_{T+1} \mid \mathcal{F}_{10})$.

g) Find $\mathbb{E}(T \mid \mathcal{F}_{10})$ and $\mathbb{E}(S_T \mid \mathcal{F}_{10})$.

8.2 Bad police officers operate in groups of 1, 2 or 3 people with probabilities 0.5, 0.2 and 0.3. If you cross the road in the wrong place, they will catch you and demand a bribe X of 1, 5 or 10 thousand rubles respectively.

For each of the following cases write down the σ-algebra $\mathcal{F}$ that models your information and calculate $\mathbb{E}(X \mid \mathcal{F})$.

a) you can see how many officers are going to stop you;

b) they are sitting in the car and you don't know their number;

c) it is dark and you can only say if it is one policeman or more than one.

8.3 HSE student rolled the dice once. Find the σ-algebras that model the following situations:

a) she only knows that the dice was rolled once;

b) she knows the result of the roll;

c) she observes the result of the roll but she is able to count only up to two.

9 Martinales

- *Natural filtration* of a process (X_n) is given by $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.
- A process (X_n) is a *martingale* if $\mathbb{E}(X_{n+k} \mid X_n, X_{n-1}, \ldots, X_1) = X_n$ for all $k \geqslant 1$.
- A process (X_n) is *adapted* to filtration $(\mathcal{F}_n)$ if every random variable X_n is measurable wrt to sigma-algebra $\mathcal{F}_n$.
- A process (X_n) is a *martingale wrt to filtration* $(\mathcal{F}_n)$ if $\mathbb{E}(X_{n+k} \mid \mathcal{F}_n) = X_n$.

9.1 Consider the sequence (X_t) of independent identically distributed random variables with mean $\mathbb{E}(X_t) = 2$ and variance $\operatorname{Var}(X_t) = 3$. Let's work with natural filtration $\mathcal{F}_t = \sigma(X_1, X_2, \ldots, X_t)$.

On the base of (X_t) let's create more sequences: $S_t = X_1 + X_2 + \cdots + X_t$, $W_t = S_t - 2t$ and $Y_t = W_t^2 - 3t$.

a) Is (X_t) a martingale with respect to $(\mathcal{F}_t)$?

b) Is (S_t) a martingale with respect to $(\mathcal{F}_t)$?

c) Is (W_t) a martingale with respect to $(\mathcal{F}_t)$?

d) Is (Y_t) a martingale with respect to $(\mathcal{F}_t)$?

e) Is (W_t) a martingale with respect to $(\mathcal{F}_{t-1})$?

f) Is (W_t) a martingale with respect to $(\mathcal{F}_{t+1})$?

9.2 Vasiliy has found three non-random infinite sequences in his garage: $a_n = n$, $b_n = -n$ and $c_n = 0$. He randomly selects one of these sequences with equal probabilities and hence obtain a sequence of random variables (X_n).

a) What is the distribution of X_7?

b) Find $\mathbb{E}(X_n)$ and $\operatorname{Var}(X_n)$.

c) Is (X_n) a Markov chain?

d) Is (X_n) a martingale?

e) Explicitely find the σ-algebra $\sigma(X_1, X_2, X_3, \ldots, X_{1000})$.

f) Find the probability that limit of (X_n) exists.

g) Does plim X_n exist?

9.3 Consider the sequence (X_t) of independent identically distributed random variables that take values 0 or 1 with equal probabilities. Let's work with natural filtration $\mathcal{F}_t = \sigma(X_1, X_2, \ldots, X_t)$.

On the base of (X_t) let's create more sequences: $S_t = X_1 + X_2 + \cdots + X_t$, $W_t = S_t - at$, $M_t = \exp(bS_t)$.

a) Is (X_t) a martingale?

b) Is (S_t) a martingale?

c) For which values of a the process (W_t) is a martingale?

d) For which values of b the process (M_t) is a martingale?

9.4 Consider a well-mixed standard deck of 52 cards. James Bond in an elegant outfit[1] opens cards one by one. Let the sigma-algebra $(\mathcal{F}_n)$ model his information and (X_n) be the proportion of Queens in the closed part of the deck after opening n cards.

a) Find the marginal distribution of X_0, X_1 and X_{51}.

b) Find the joint distribution of X_{50} and X_{51}.

c) Is (X_n) a martingale with respect to $(\mathcal{F}_n)$?

9.5 If possible create a martingale (X_n) such that simulteneously $\mathbb{P}(X_n = 0 \text{ infinitely often}) = 1$ and $\mathbb{P}(X_n = 1 \text{ infinitely often}) = 1$.

9.6 At time $t = 0$ there is one black and one white ball in the vase. At each moment of time we take out randomly one ball from the vase and put back two balls of the same color. Let (W_t) be the proportion of white balls in the vase after t extractions and (Q_t) be the number of times when white ball was extracted.

a) What is the distribution of W_1? Of W_2?

b) Is (W_t) a martingale?

c) Consider a fixed parameter $p \in (0; 1)$ and the process $M_t = (t+1)C_t^{Q_t} p^{Q_t}(1-p)^{t-Q_t}$. Is (M_t) a martingale?

d) What is the limiting distribution of (W_t)?

9.7 Consider non-random sequence of numbers (a_n). How can this sequence be a martingale?

9.8 Let (M_n) be a martingale and $a < b < c < d$.

a) Find covariance $\operatorname{Cov}(M_d - M_c, M_b - M_a)$.

b) Are $(M_d - M_c)$ and $(M_b - M_a)$ independent?

[1] Sponsors are wellcome to contact us for product placement!

9.9 Let (M_t) be a process adapted to filtration $(\mathcal{F}_t)$.

Is it true that in discrete time conditions

$$\mathbb{E}(M_{t+1} \mid \mathcal{F}_t) = M_t$$

and

$$\mathbb{E}(M_{t+k} \mid \mathcal{F}_t) = M_t \text{ for all } k \geqslant 1$$

are equivalent?

9.10 Initial wealth of a player is equal to $W_0 = 1$. At each moment of time she can bet any proportion of her wellfare on the toss of a coin. If she guesses wrong she loses her bet. If she guesses right she gets profit equal to hear bet. The coin is not fair lands on head with probability 0.8.

a) Find the bet that maximises one period log interest rate $\mathbb{E}(\ln(W_{t+1}/W_t) \mid \mathcal{F}_t)$.

b) Assume that the player maximises one period log interest rate every time. Find a constant a such that $\ln W_n - an$ is a martingale.

9.11 For each case provide an example of a process.

a) (X_n) is a Markov chain and a martingale.

b) (X_n) is a Markov chain but not a martingale.

c) (X_n) is a martingale but not a Markov chain.

d) (X_n) is neither a Markov chain nor a martingale.

9.12 Let (X_n) be a simple symmetric random walk and $(\mathcal{F}_n)$ its natural filtration.

Find a deterministic (non-random) sequence a_n such that $M_n = X_n^3 + a_n X_n$ is a martingale with respect to $(\mathcal{F}_n)$.

9.13 The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

a) [3] Find $\mathbb{E}(X_3 \mid X_1, X_2)$, $\mathbb{E}(X_3 \mid X_1 + X_3)$.

b) [3] Find $\mathrm{Var}(X_3 \mid X_1, X_2, X_3)$, $\mathrm{Var}(X_3 \mid X_1 + X_3)$.

c) [4] Let Y_n be equal to $\mathbb{E}(X_1 + \ldots + X_{2022} \mid X_1, X_2, \ldots, X_n)$.
Is the process Y_1, Y_2, ..., Y_{2022} a martingale?

9.14 Let $S_0 = 0$, $S_t = X_1 + X_2 + \ldots + X_t$. The increments X_t are independent and identically distributed:

x	-1	0	1
$\mathbb{P}(X_t = x)$	0.2	0.2	0.6

a) If possible find all constants a such that $M_t = S_t + at$ is a martingale.

b) If possible find all constants b such that $R_t = b^{S_t}$ is a martingale.

9.15 Let X_i be independent identically distributed with $\mathbb{P}(X_i = 1) = 0.9$, $\mathbb{P}(X_i = -1) = 0.1$.

Find all constants a and b such that $Y_t = a \exp\left(b \sum_{i=1}^{t} X_i\right)$ is a martingale.

9.16 The random variables (Z_t) are independent identically distributed with moment generating function given by $M_Z(u) = 1/(1-5u)^3$.

We define X_t as $X_t = \exp(Z_1 + 2Z_2 + 3Z_3 + \ldots + tZ_t)$ with $X_0 = 0$.

If possible find a martingale of the form $Y_t = h(t)X_t$ where $h()$ is a non-random function.

9.17 The process (Z_t) in discrete time is called *stationary* if it has constant expected value and constant covariances γ_k that do not depend on t.

$$\begin{cases} \mathbb{E}(Z_t) = \mu; \\ \text{Cov}(Z_t, Z_t) = \gamma_0; \\ \text{Cov}(Z_t, Z_{t+1}) = \gamma_1; \\ \text{Cov}(Z_t, Z_{t+2}) = \gamma_2; \\ \ldots \end{cases}$$

a) If possible provide an example of a martingale that is not stationary.

b) If possible provide an example of a stationary process that is not a martingale.

9.18

9.19

9.20

9.1 Stopping time

Doob's optional stopping time theorem. If (M_t) is a martingale and τ is a stopping time then $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ provided at least one of the following conditions hold:

- $\mathbb{P}(\tau < \infty) = 1$, the stopped process $X_t = M_{t\wedge\tau}$ is bounded by some constant.
- $\mathbb{E}(\tau) < \infty$, the process $D_t = \mathbb{E}(M_{(t+1)\wedge\tau} - M_{t\wedge\tau} \mid \mathcal{F}_t)$ is bounded by some constant.
- $\mathbb{P}(\tau < \infty) = 1$, the process M_t is uniformly integrable, ie

$$\lim_{a\to\infty} \sup_t \mathbb{E}(M_t \cdot I(M_t > a)) = 0.$$

9.21 A gambler wins or looses one rouble in each round in the casino with equal probabilities and independently. Let's denote the result of the n-th round by X_n.

The gambler starts with initial fortune $S_0 = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the wealth at time n. She can have negative balance up to $-a$ roubles.

She quits the casino when she either reaches the target of $+b$ roubles or the credit limit of $-a$ roubles.

a) Is (S_n) a martingale?

b) Use optional-stopping theorem to find probabilities of reaching $+b$ or $-a$.

c) Is $M_n = S_n^2 - n$ a martingale?

d) Find the expected number of rounds before she will stop gambling.

9.22 A gambler wins or looses one rouble in each round in the casino with unequal probabilities and independently. Let's denote the result of the n-th round by X_n, $\mathbb{P}(X_n = 1) = p$, $\mathbb{P}(X_n = -1) = q = 1 - p$.

The gambler starts with initial fortune $S_0 = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the wealth at time n. She can have negative balance up to $-a$ roubles.

She quits the casino when she either reaches the target of $+b$ roubles or the credit limit of $-a$ roubles.

a) Is (S_n) a martingale?

b) Is $K_n = (q/p)^{S_t}$ a martingale?

c) Use optional-stopping theorem to find probabilities of reaching $+b$ or $-a$.

d) Is $M_n = S_n - (p-q)n$ a martingale?

e) Find the expected number of rounds before she will stop gambling.

9.23 Famous «ABRACADABRA» problem.

A monkey types randomly letters on a typewriter choosing each time one of the 26 letters with equal probabilities. Let T be the number of keypresses required to write the word «ABRACADABRA» for the first time.

a) Organise a casino to calculate $\mathbb{E}(T)$.

b) Organise a casino to calculate $\mathbb{E}(T^2)$ and hence $\mathrm{Var}(T)$.

9.24 To survive vampire Boris needs to bite 70 talented students.

These 70 talented students have formed a secret group. They have written their emails on small pieces of paper and have randomly distributed these pieces among them. Each student has exactly one piece of paper with an email[2].

Initially vampire Boris knows contacts of just two persons from the group. Today he will contact them, drink their blood and get the emails they have. Then vampire Boris will contact new victims and so on.

a) For $t \geqslant 1$ consider the process M_t, the proportion of non bitten students after the day t.
Is this process a martingale?

b) Using martingale stopping theorem or otherwise find the probability that vampire Boris will bite all 70 students.

9.25

9.26

9.27

9.28

10 Poisson process

The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent: If $t_1 < t_2 < t_3 < \cdots < t_k$ then random increments $N(t_2) - N(t_1)$, $N(t_3) - N(t-2)$, ... are independent.
- Increments have Poisson distribution:

$$N_b - N_a \sim \mathrm{Pois}(\lambda(b-a));$$

[2]The group is so secret that it is possible that a student has his own email on his piece of paper

The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent;
- Increments are stationary:
  The distribution of $N_b - N_a$ depends only on $(b - a)$.
- Probability of observing two or more points in a small interval is negligible:

$$\mathbb{P}(N_{t+\Delta} - N_t > 1) = o(\Delta).$$

10.1 Two cashiers Alice and Bob simulteneously started to service their clients. The service times X_a and X_b are independent and exponentially distributed with rates $\lambda_a = 1$ and $\lambda_b = 2$.

a) Find the probability $\mathbb{P}(X_a < X_b)$.

b) Find the density of $S = X_a + X_b$.

c) Find the density of $L = \min\{X_a, X_b\}$.

d) Find the density of $R = \max\{X_a, X_b\}$.

e) Solve all the previous points for general rates λ_a and λ_b.

10.2 Let X_t and Y_t be two independent Poisson processes. Is it true that $S_t = X_t + Y_t$ is also a Poisson process?

10.3 Hedgehogs are scattered in a big forest according Poisson process with rate $\lambda = 1$ per 100 squared meters.

What should be the edge of a square such that the probability of finding a hedgehog there is 0.7?

10.4 Let N_t be a Poisson process with rate λ.

a) Is the process $A_t = N_t - \lambda t$ a martingale?

b) Is the process $B_t = A_t^2 - \lambda t$ a martingale?

10.5 Students arrive in the Grusha café according to the Poisson arrival process (X_t) with constant rate λ. The probability of no visitors during 5 minutes is 0.05.

a) Find the value of λ.

b) Find the variance and expected number of arrivals between 5 pm and 8 pm.

c) What is the probability of exactly 5 arrivals between 5 pm and 8 pm?

10.6 Masha receives on average 10 sms per minute. Sms arrival is well described by the Poisson process.

a) What is the probability that Masha receives exactly 10 sms in the next 40 seconds?

b) Masha just received an sms. What is the probability that she will wait more that 2.5 seconds before the next one?

c) Find the covariance between the number of sms in the first 3 minutes and the number of sms in the first 10 minutes.

10.7 Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes. Let Y_t be the number of taxis that will arrive between 0 and t minutes.

a) Sketch the expected value of Y_t as a function of t.
b) Sketch the probability $\mathbb{P}(Y_t = Y_{60})$ as a function of t.

10.8 A company gets fines for non-removal of quadrobics video content. What is the probability that the total amount will exceed two undecillion roubles in 1000 days for each case?

a) Fines arrive according to Poisson process with rate 1 fine per day and each fine has the size 10^{33} roubles. Fines are summing up without additional penalties.
b) Initial fine is 10^5 roubles but it doubles according to Poisson process with rate 1 doubling per 10 days.

10.9

10.10 Customers order coffee according to Poisson process with rate 1 cup per minute. The owner will close the shop if no one orders a coffee in 7 minutes.

Let X be the closure time.

Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

10.11 The arrival of buses at a given stop follows Poisson process with rate 3. The arrival of taxis at same stop follows independent Poisson process with rate 5.

a) What is the probability that two or more taxis will arrive before next bus?
b) What is the probability that exactly two taxis will arrive before next bus?

10.12 Customers order coffee according to Poisson process with rate 1 cup per minute. Let N_t be the number of orders up to time t.

Find the probability $\mathbb{P}(N_t \text{ is even})$.

10.13 Prove that two definitions of Poisson process are equivalent.

Definition A. The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent: If $t_1 < t_2 < t_3 < \cdots < t_k$ then random increments $N(t_2) - N(t_1)$, $N(t_3) - N(t - 2)$, … are independent.
- Increments have Poisson distribution:

$$N_b - N_a \sim \mathrm{Pois}(\lambda(b - a));$$

Definition B. The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent;
- Increments are stationary:
  The distribution of $N_b - N_a$ depends only on $(b - a)$.
- Probability of observing two or more points in a small interval is negligible:

$$\mathbb{P}(N_{t+\Delta} - N_t > 1) = o(\Delta).$$

11 Wiener Process

11.1 Consider a Wiener process (W_t).

a) [4] Let $Y_t = tW_{2t}$. What is the distribution of $Y_t - Y_s$ for $t \geqslant s$? Is Y_t a Wiener process?

b) [6] Find a constant α such that $M_t = W_t^3 + \alpha t W_t$ is a martingale.

11.2 Consider a Wiener process (W_t). For $r < s < t < u$ find the following expected values

a) $\mathbb{E}((W_u - W_t)^2(W_s - W_r)^2)$;

b) $\mathbb{E}((W_u - W_s)(W_t - W_r))$;

c) $\mathbb{E}((W_t - W_r)(W_s - W_r)^2)$;

d) $\mathbb{E}(W_r W_s W_t)$;

e) $\mathbb{E}(W_r W_s W_t \mid W_s)$;

11.3 Here (W_t) is a Wiener process.

a) Find $\mathbb{E}(W_5 W_4 \mid W_4)$, $\operatorname{Var}(W_5 W_4 \mid W_4)$.

b) Find covariance $\operatorname{Cov}(W_4 W_5, W_5 W_6)$.

11.4 For Wiener process (W_t) find $\mathbb{E}(W_1 W_2 W_3)$ and $\mathbb{E}(W_2 W_3 \mid W_1)$.

11.5

11.6

11.7

11.8

12 Ito's integral

12.1 Consider Ito process X_t

$$dX_t = \exp(t) W_t \, dt + \exp(2W_t) \, dW_t, \quad X_0 = 1.$$

Consider two processes, $A_t = 1 + t^2 + X_t^3$ and $B_t = 1 + t^2 + X_t^3 W_t^4$.

a) Find dA_t and dB_t.

b) Write the corresponding explicit expressions for A_t and B_t:

$$const + \int_0^t \ldots dW_u + \int_0^t \ldots du$$

c) Check whether X_t is a martingale.

12.2 Consider the process X_t

$$X_t = tW_t + \int_0^t u W_u^2 \, dW_u.$$

a) Find $\mathbb{E}(X_t)$, $\operatorname{Var}(X_t)$.

b) Find dX_t.

c) Check whether X_t is a martingale.

12.3 Consider $X_t = \int_0^t W_u^3 dW_u + \int_0^t (W_u^3 + 3W_u u) du - W_t^3 \cdot t$.

a) Find dX_t and the corresponding full form.

b) Is X_t a martingale?

12.4 Consider $X_t = \exp(-2W_t - 2t)$.

a) Find dX_t. Is X_t a martingale?

b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$.

c) Find $\int_0^t X_u dW_u$.

12.5 Consider an Ito's process $I_t = 2022 + W_t t^2 + \int_0^t W_u^3 dW_u + \int_0^t W_u^2 du$.

a) Find dI_t and check whether I_t is a martingale.

b) Check whether $J_t = I_t - \mathbb{E}(I_t)$ is a martingale.

12.6 Martingales are everywhere :)

Consider the process $Y_t = \exp(-uW_t)$.

a) Find a multiplier $h(u,t)$ such that $M_t = h(u,t) \cdot Y_t$ is a martingale.

b) Find dY_t, $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.

c) Consider M_t that you have found as a function of u. Find the Taylor approximation of the function $M_t(u)$ up to u^4.

d) Consider the coefficient before u^4 in the Taylor expansion of $M_t(u)$. Is it a martingale?

12.7 Consider the process $X_t = \int_0^t W_u^2 dW_u + \int_0^t (W_u^2 + 2W_u u) du - W_t^2 \cdot t$.

a) Find dX_t and the corresponding full form.

b) Is X_t a martingale?

c) Find $\mathbb{E}(X_t)$.

12.8 Consider the stochastic process $X_t = f(t) \cos(2021 W_t)$.

a) Find dX_t.

b) Find any $f(t) \neq 0$ such that X_t is a martingale.

c) Using $f(t)$ from the previous point find $\mathbb{E}(\cos(2021 W_t))$.

12.9

12.10

12.11

12.12

13 Binomial asset pricing model

13.1 Consider two-period binomial model with initial share price $S_0 = 600$, Up and down multipliers are $u = 1.2$, $d = 0.9$, risk-free interest rate is $r = 0.05$ per period.

Consider an option that pays you $X_2 = 100$ at $T = 2$ if $S_2 > S_1$ and nothing otherwise.

a) Find the risk neutral probabilities.

b) Find the current price X_0 of the asset.

c) How much shares should I have at $t = 1$ in the «up» state of the world to replicate the option?

13.2

13.3

13.4

14 Black and Scholes model

14.1 Consider Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

Find the current price X_0 of an option that pays you $X_2 = S_1^3$ at time $T = 2$.

14.2 Ded Moroz would like to receive $X_T = S_T^{-1}$ at time T if $S_T < 1$ and nothing otherwise.

Assume the framework of Black and Scholes model, S_t is the share price, r is the risk free rate, σ is the volatility.

How much Ded Moroz should pay now at $t = 0$?

14.3 Consider the Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

Find the current price X_0 of an option that pays you one dollar at time $T = 2$ only if $S_2 > \exp(3r)S_0$.

14.4

15 Stationarity

15.1 The variables x_t take values 0 or 1 with equal probabilities. The variables u_t are normal $\mathcal{N}(0; 1)$. All variables are independent.

Consider the process $z_t = x_t(1 - x_{t-2})u_t$.

a) Find the covariance $\mathrm{Cov}(z_t, z_s)$. Is the process z_t stationary?

b) Given that $z_{100} = 2.3$ find shërtest predictive intervals for z_{101} and z_{102} with probability of coverage at least 95%.

15.2

15.3

15.4

15.5

15.6

16 ARMA

16.1 Consider stationary $AR(2)$ model, $y_t = 2 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$, where (u_t) is a white noise with $\mathrm{Var}(u_t) = 4$.

The last two observations are $y_{100} = 2$, $y_{99} = 1$.

a) Find 95% predictive interval for y_{102}.
b) Find the first two values of the autocorrelation function, ρ_1, ρ_2.
c) Find the first two values of the partial autocorrelation function, ϕ_{11}, ϕ_{22}.

16.2 Snegurochka studies a stochastic analog of the Fibonacci sequence

$$y_t = y_{t-1} + y_{t-2} + u_t,$$

where (u_t) is a white noise process.

a) How many non-stationary solutions are there?
b) What can you say about the number and the structure of the stationary solutions?
c) Can Snëgurochka find two starting constants $y_0 = c_0$ and $y_1 = c_1$ in such a way to make a solution stationary?

16.3 Stochastic process X_t is defined by $X_t = 7 + u_t + 0.3u_{t-1}$, where (u_t) is a white noise with variance σ^2.

a) Is (X_t) stationary?
b) Find the autocorrelation function of (X_t).
c) Find $\mathbb{E}(X_{t+2} \mid X_t, X_{t-1}, \ldots)$.

16.4 Young investor Winnie-the-Crypto compares two trading strategies: buying bitcoins from good bees and from bad bees. Let d_t be the price difference at day t (bad minus good). Winnie-the-Crypto would like to test H_0: $\mathbb{E}(d_t) = 0$ against H_a: $\mathbb{E}(d_t) \neq 0$ at 5% significance level.

Winnie assumed that (d_t) can be approximated by a $MA(1)$ process and estimated the parameters using $T = 400$ observations, $\hat{d}_t = 2 + u_t + 0.7u_{t-1}$ with $\hat{\sigma}^2_u = 4$.

a) Estimate $\mathbb{E}(d_t)$, $\mathrm{Var}(d_t)$ and $\mathrm{Cov}(d_t, d_{t-1})$.
b) Estimate $\mathbb{E}(\bar{d})$, $\mathrm{Var}(\bar{d})$ and help Winnie by considering $Z = \frac{\bar{d}-0}{se(\bar{d})}$.

16.5 Consider the following stationary process

$$y_t = 1 + 0.5y_{t-2} + u_t + u_{t-1},$$

where random variables u_t are independent $\mathcal{N}(0; 4)$.

a) Find the 95% predictive interval for y_{101} given that $y_{100} = 2$, $y_{99} = 3$, $y_{98} = 1$, $u_{99} = -1$.
b) Find the point forecast for y_{101} given that $y_{100} = 2$.

16.6

16.7

16.8

16.9

16.10

17 ETS

17.1 The semi-annual y_t is modelled by $ETS(AAA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0;4) \\ s_t = s_{t-2} + 0.1u_t \\ b_t = b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-2} + u_t \end{cases}$$

a) Given that $s_{100} = 2$, $s_{99} = -1.9$, $b_{100} = 0.5$, $\ell_{100} = 4$ find 95% prëdictive interval for y_{102}.

b) In this problem particular values of parameters are specified. And how many parameters are estimated in semi-annual $ETS(AAA)$ model before real forecasting?

17.2 The $ETS(AAdN)$ model is given by the system

$$\begin{cases} u_t \sim \mathcal{N}(0;20) \\ b_t = 0.9b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + 0.9b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + 0.9b_{t-1} + u_t \end{cases}$$

with $\ell_{100} = 20$ and $b_{100} = 2$.

a) Find conditional probability $\mathbb{P}(y_{102} > 30 \mid \ell_{100}, b_{100})$.

b) Approximately find the best point forecast for y_{10000}.

17.3 Consider $ETS(ANN)$ model,

$$\begin{cases} y_t = \ell_{t-1} + u_t \\ \ell_t = \ell_{t-1} + \alpha u_t \\ u_t \sim \mathcal{N}(0;\sigma^2). \end{cases}$$

Let $\ell_{99} = 50$, $\alpha = 1/2$, $\sigma^2 = 16$, $y_{98} = 48$, $y_{99} = 52$, $y_{100} = 55$.

Calculate 95% predictive interval for y_{101}.

17.4

17.5

18 GARCH

18.1 The process y_t is described by a simple $GARCH(1,1)$ model:

$$\begin{cases} y_t = \sigma_t \nu_t \\ \sigma_t^2 = 1 + 0.2y_{t-1}^2 + 0.3\sigma_{t-1}^2 \\ \nu_t \sim \mathcal{N}(0;1) \end{cases}$$

The variables ν_t are independent of past variables y_{t-k}, ν_{t-k}, σ_{t-k} for all $k \geqslant 1$. The prëcesses y_t, σ_t^2 are stationary.

Given $\sigma_{100} = 1$ and $\nu_{100} = 0.5$ find 95% predictive interval for y_{102}.

18.2

18.3

18.4

19 Method of Moments and maximum likelihood

19.1 The variables X_1, ..., X_n are independent identically distributed with density

$$f(x) = \begin{cases} \lambda \exp(-\lambda(x-\theta)), & \text{if } x \geqslant \theta \\ 0, & \text{otherwise.} \end{cases}$$

a) [5] Find the method of moments estimator of λ for known value $\theta = 1$ using the first moment.

b) [5] Find the method of moments estimator of λ for unknown value θ using the first two moments.

19.2 The variables X_1, ..., X_n are independent and normally distributed $\mathcal{N}(a, 2a)$.

Find the maximum likelihood estimator of a.

Hint: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/2\sigma^2)$.

19.3 The weight of a fish Y_i is a discrete random variables with distribution and observed frequencies given in the table

Weight [kg]	1	2	a
Probability	$0.2 + 0.1a$	$0.3 - 0.1a$	0.5
Observed frequency	N_1	N_2	N_a

Fish weights Y_i are independent, $a > 10$ is unknown.

a) Find the method of moments estimator of the parameter a.

b) Find the maximum likelihood estimator of the parameter a.

19.4

19.5

19.6

19.7

19.8

19.9

20 LR test

20.1 We have two independent random samples $X_1, X_2, ..., X_{n_x}$ and $Y_1, Y_2, ..., Y_{n_y}$. The random variables X_i follow Poisson distribution with intensity rate λ_x, random variables Y_i follow Poisson distribution with intensity rate λ_y.

We would like to test H_0: $\lambda_x = \lambda_y$ against H_1: $\lambda_x \neq \lambda_y$.

a) [3] Find the maximal value of log-likelihood under H_0.

b) [3] Find the maximal value of log-likelihood under unrestricted model.

c) [2] Construct the likelihood ratio test.

d) [2] Do you reject H_0 if $n_x = 100$, $n_y = 200$, $\sum x_i = 500$, $\sum y_i = 900$ at significance level 5%?

20.2 You observe $X_1, ..., X_{400}$ and $Y_1, ..., Y_{400}$, $\bar{X} = 5$, $\bar{Y} = 6$. All variables are independent.

Consider the null hypothesis that all random variables are exponentially distributed with common parameter λ against alternative that parameter is λ_X for every X_i and λ_Y for every Y_j.

a) Estimate common λ using maximum likelihood for the restricted model.

b) Estimate both λ_X and λ_Y using maximum likelihood in the unrestricted model.

c) Use LR-test to test the null hyphotesis at 5% significance level.

20.3

20.4

20.5

20.6

21 Properties of estimators

21.1 The variables $X_1, ..., X_n$ are independent and uniformly distributed Unif$[0; a]$ with $a > 1$. We do not observe X_i directly but we know whether each X_i is larger than 1. Hence we observe the indicators $Y_i = I(X_i > 1)$.

Consider the estimator $\hat{a} = 1/(1 - \bar{Y})$.

a) [5] Is $\hat{a}$ consistent?

b) [5] Is $\hat{a}$ unbiased for $n = 2$?

21.2

21.3

21.4

21.5

22 Fisher information and Cramer — Rao

22.1 The variables X_1, ..., X_n are independent and have Poisson distribution with intensity rate λ. In other words the probability mass function is given by $\mathbb{P}(X_i = k) = \exp(-\lambda)\lambda^k/k!$.

a) [5] Find theoretical Fisher information for λ contained in the sample.

b) [2] Derive the maximum likelihood estimator for λ.

c) [3] Does the maximum likelihood estimator attain the Cramer-Rao lower bound for variance?

22.2 Consider an estimator $\hat{a}$ with $\mathbb{E}(\hat{a}) = 0.5a + 3$. For the given sample size the Fisher information is $I_F(a) = 400/a^2$.

a) What is the theoretical minimal variance of $\hat{a}$?

b) Assume that $\hat{a}$ attains the minimal variance boundary and is asymptotically normal. Given that $\hat{a} = 2022$ provide 95% CI for a.

22.3

22.4

22.5

22.6

23 Sufficiency

23.1 The variables X_1, ..., X_n are independent and gamma distributed with density

$$f(x) = \begin{cases} \lambda^\alpha x^{\alpha-1} \exp(-\lambda x)/\Gamma(\alpha), & \text{if } x \geqslant 0 \\ 0, & \text{otherwise}. \end{cases}$$

a) [5] Find a sufficient statistic for α if we know that $\lambda = 1$.

b) [5] Find a two dimensional sufficient statistic for unknown α and λ.

23.2 The variables X_1, ..., X_n are independent and uniformly distributed on $[0; 2a]$ for some positive a.

a) Find any sufficient statistic for a.

b) How the answer will change if $X_i \sim U[-a; 2a]$?

23.3

23.4

23.5

24 Solutions

1.1.

a) $\mathbb{P}(B) = 6/11$, first step equation for $p = \mathbb{P}(B)$ is $p = 1/6 + (5/6)^2 p$ or $p = 1/6 + 5/6 \cdot (1-p)$.

b) $\mathbb{E}(N) = 6$, first step equation for $m = \mathbb{E}(N)$ is $m = 1/6 + 5/6(m+1)$.

c) $\mathbb{E}(N^2) = 66$, Var$(N) = 30$, first step equation is $\mathbb{E}(N^2) = 1/6 + 5/6\mathbb{E}((N+1)^2)$.

d) $\mathbb{E}(N \mid B) = 61/11$. Start by replacing uncoditional probabilities on the tree by conditional ones. First step equation for $\mu = \mathbb{E}(N \mid B)$ is $\mu = 11/36 + 25/36(\mu + 2)$.

e) $\begin{pmatrix} 0 & 5/6 & 1/6 & 0 \\ 5/6 & 0 & 0 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1.2.

a) $\mathbb{E}(N) = 1/p$, Var$(N) =$, $\mathbb{E}(N^3) =$, $\mathbb{E}(\exp(tN)) =$

b) $a = \mathbb{P}(N \in 2 \cdot \mathbb{N})$, $a = (1-p)(1-a)$, $a = (1-p)/(2-p)$.

1.3.

1.4. Let's denote the throws by (X_t) and the number of throws by T. Thus the last throw is X_T.

a) $\mathbb{P}(X_T = 6) = 1/3$ as we have three possible endings. One may also sum the probability geometric serie or use first step analysis.

b) $\mathbb{E}(T) = 0.5 + 0.5(\mathbb{E}(T) + 1)$;

c) Let $\mu = \mathbb{E}(S)$ and $\gamma = \mathbb{P}(X_T \in \{4, 5\})$.

$$\mu = \frac{3}{6} \cdot 0 + \frac{1}{6}(\mu + 1 \cdot \gamma) + \frac{1}{6}(\mu + 2 \cdot \gamma) + \frac{1}{6}(\mu + 3 \cdot \gamma)$$

d)

e)

f)

$$\mu_B = \frac{2}{6} \cdot 0 + \frac{1}{6}\mu_B + \frac{1}{6}(\mu + 1 \cdot \beta) + \frac{1}{6}(\mu + 2 \cdot \beta) + \frac{1}{6}(\mu + 3 \cdot \beta),$$

with $\beta = 1/3$.

1.5.

1.6.

1.7.

1.8.

1.9.

1.10.

1.11.

1.12.

1.13.

1.14.

2.1.

2.2.

2.3.

2.4.

2.5.

2.6.

a)

b) $\mathbb{E}(W_t) \to +\infty$;

c) No stationary distribution. For stationary distribution $\mathbb{E}(W_t)$ can't tend to infinity.

2.7.

2.8.

2.9.

2.10.

2.11.

2.12.

2.13.

2.14.

3.1.

3.2.

3.3.

3.4.

3.5.

3.6.

4.1.

4.2. $M(t) = (\exp(t) + \exp(2t) + \exp(3t))/3.$

4.3. $\mathbb{E}(W) = 2$, $\mathrm{Var}(W) = 7 \cdot 2 - 2^2$, $\mathbb{E}(W^3) = 20 \cdot 3!$.

4.4. $\mathbb{P}(X = 0) = 0.1$, $\mathbb{P}(X = 1) = 0.2$, $\mathbb{P}(X = 2) = 0.15$.

4.5.

4.6. $g(u) = g(\exp(t)) = \mathbb{E}(\exp(tX)) = M(t)$

4.7.

4.8.

4.9.

4.10.

4.11.

4.12.

4.13.

4.14.

4.15.

5.1.

5.2.

a) $\operatorname{plim} X_n = 0$;

b) $\operatorname{plim} Y_n$ does not exist.

5.3.

a) $\operatorname{plim} \frac{X_1+X_2+\cdots+X_n}{7n} = 5/7$;

b) $\operatorname{plim} \frac{X_1^2+X_2^2+\cdots+X_n^2}{7n} = 5$;

c) $\operatorname{plim} \ln(X_1^2 + X_2^2 + \cdots + X_n^2) - \ln n = \ln 35$.

5.4.

a) $\operatorname{plim} \max\{Y_1, Y_2, \ldots, Y_n\} = 2$;

b) $\operatorname{plim} \max\{X_1 + Y_1, X_1 + Y_2, \ldots, X_1 + Y_n\} = X_1 + 2$.

5.5.

a) $\operatorname{plim} Y_n = 0$;

b) $\operatorname{plim} W_n = X$.

5.6.

a) $\mathbb{P}(\lim Y_n = 0) = 1$;

b) $\operatorname{plim} Y_n = 0$;

c) Limiting distribution is a constant 0.

5.7.

a) $\operatorname{plim} V_n = 0$, $\operatorname{plim} W_n = Y$;

b) The sequence V_n does not converge in mean squared;

c) W_n converges to Y in mean squared.

5.8.

a) $2/5$;

b) L/R or $+\infty$ or $-\infty$ or $3/2$.

c) Almost surely limit is L/R, pointwise limit does not exist.

d) Pointwise and almost surely limits are $\frac{2}{5}I(R > 0) + \frac{3}{2}I(R = 0)$.

5.9.

a) 0, 1, $+\infty$ or does not exist.

b) Almost surely limit is 0;

c) $1/2$;

d) $\mathbb{P}(Y_n \text{ converges }) = 0$;

e) Yes, as every Y_n has the same distribution.

5.10.

5.11.

5.12.

5.13.

5.14.

5.15.

6.1.

a) $\mathbb{E}(Y \mid X) = 1 - X/6$ and $\mathbb{E}(X \mid Y) = 3 - 1.5X$.

b) $\operatorname{Var}(Y \mid X) =$, $\operatorname{Var}(X \mid Y) =$.

6.2.

a)

$$f(x) = \begin{cases} x + 0.5, & \text{if } x \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

b)

$$f(x, y) = \begin{cases} (x + y)/(x + 0.5), & \text{if } x \in [0; 1], y \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

c)

$$\mathbb{E}(Y \mid X) = \frac{0.5X + 1/3}{X + 0.5}.$$

d)

6.3.

a)

b) $\mathbb{E}(X \mid S) = \mathbb{E}(Y \mid S) = S/2$;

c)

d)

6.4.

a) $X \mid S \sim \text{Unif}[0; S]$, $Y \mid S \sim \text{Unif}[0; S]$.

b) $\mathbb{E}(X \mid S) = \mathbb{E}(Y \mid S) = S/2$;

c) $\text{Var}(X \mid S) = \text{Var}(Y \mid S) = S^2/12$;

d)

6.5. $\mathbb{E}(XY \mid X) = X\mathbb{E}(Y) = X/2$, $\text{Var}(XY + X^3 \mid X) = X^2 \text{Var}(Y) = X^2/12$, $\text{Cov}(X, Y \mid X) = 0$, $\text{Cov}(XY, X^2Y \mid X) = X^3 \text{Var}(Y) = X^3/12$

6.6.

6.7.

6.8.

6.9.

6.10.

6.11.

7.1.

a)

b) Sigma-algebras do not depend on probabilities.

c) $\sigma(|X|) = \{\emptyset, \Omega, \{|X| = 2\}, \{X = 1\}\}$.

d) $\sigma(|X|) = \sigma(X^2)$;

7.2.

a)

b) $\text{card}\,\sigma(X, Y) = 2^6$, $\text{card}\,\sigma(X + Y) = 2^4$, $\text{card}\,\sigma(X, Y, X + Y) = 2^6$.

7.3.

a) 2^5;

b) Only 2^k or infinity;

c) 2^{100};

d) $2^{2^{100}}$.

7.4. In general $\sigma(f(X)) \subseteq \sigma(X)$; If f is a bijection then $\sigma(f(X)) = \sigma(X)$.

7.5. In the finite case sigma-algebra corresponds to partitions. We get five sigma-algebras on a set of three elements and 15 sigma-algebras on a set of four elements. These numbers are known as Bell numbers.

7.6. Let $\Omega = \mathbb{N}$, $\mathcal{A}$ contains all finite sets and sets with finite complement.

7.7.

a) The intersection of two sigma-algebras is always a sigma-algebra.

b) The intersection of two sigma-algebras is always a sigma-algebra.

c) The union of two sigma-algebras is not always a sigma-algebra.

d)

7.8. Yes. This is convinient do draw Ω as a segment. With «пескари» $A \subseteq B$ and «sharks» $A \supseteq B^c$.

7.9. Разобьем натуральный ряд на пары соседних чисел. Можно так подобрать множества A и B, что в каждом из них из каждой пары взято только одно число. Поэтому $\gamma(A) = \gamma(B) = \frac{1}{2}$. Подобрав совпадение-несовпадение в паре, можно сделать так, что $\gamma(A \cap B)$ не существует.

7.10.

a) $\sigma(Z) = \{\{Z = 1\}, \{Z = 0\}, \Omega, \emptyset\}$.

b) $\sigma(YZ) = \{\{YZ = 1\}, \{YZ = 0\}, \Omega, \emptyset\}$.

c) 2^4;

d) $\sigma(Y \cdot Z) \subseteq \sigma(Y, Z)$.

7.11.

7.12. Yes!

7.13. Например, B — Канторово множество, или, гораздо проще, $B = [0; 0,5]$. Оно само более чем счетно и дополнение к нему более чем счетно.

Набор $\mathcal{F}$ действительно σ-алгебра. $\emptyset$ лежит в $\mathcal{F}$, так как имеет ноль элементов.

Если A не более чем счетно, то A^c лежит в $\mathcal{F}$, так как дополнение к A^c содержит не более чем счетное число элементов.

Если дополнение к A не более чем счетно, то A^c лежит в $\mathcal{F}$, так как содержит не более чем счетное число элементов.

Проверяем счетное объединение $\bigcup_i A_i$. Если среди A_i встречаются только не более чем счетные, то и их объединение — не более чем счетно. Если среди A_i встретилось хотя бы одно множество с не более чем счетным дополнением, то $\bigcup_i A_i$ тоже будет обладать не более чем счетным дополнением, так как объединение не может быть меньше ни одного из объединяемых множеств.

7.14. 2^{101}, $2^{101\cdot 51}$,

7.15. $\mathcal{F} = \{\emptyset, \Omega, \{\text{дождь}\}, \{\text{солнечно, пасмурно}\}$. Всего есть $1 + 1 + 3 = 5$ σ-алгебр.

7.16.

7.17.

7.18.

7.19.

7.20.

7.21.

7.22.

8.1.

a) $\mathcal{F}_J, \mathcal{F}_{11}, \mathcal{F}_{12}, \ldots$

b) $\{T \text{ is divisible by } 2\}, \{T \geqslant 3, X_{T-2} = 1\}$.

c) $\mathbb{E}(T \mid \mathcal{F}_J) = T, \mathbb{E}(X_T \mid \mathcal{F}_J) = 1, \mathbb{E}(X_{T+1} \mid \mathcal{F}_J) = -1$.

d) $\mathbb{E}(S_T \mid \mathcal{F}_J) = S_T, \mathbb{E}(S_{T+1} \mid \mathcal{F}_J) = S_T - 1, \mathbb{E}(S_{T+2} \mid \mathcal{F}_J) = S_T - 1$.

e) $\mathbb{E}(Y_{T-k} \mid \mathcal{F}_J) = Y_{T-k}$.

f) $\mathbb{E}(X_T \mid \mathcal{F}_{10}) = 1, \mathbb{E}(X_{T+1} \mid \mathcal{F}_{10}) = -1$.

g) $\mathbb{E}(T \mid \mathcal{F}_{10}) = \ldots, \mathbb{E}(S_T \mid \mathcal{F}_{10}) = \ldots$.

8.2.

8.3. Here $\Omega = \{1, 2, 3, 4, 5, 6\}$.

a) $\mathcal{F} = \{\emptyset, \Omega\}$;

b) $\mathcal{F} = 2^{\Omega}$, this notation means «all subsets of Ω».

c) $\mathcal{F} = \sigma(\{1\}, \{2\})$, eight events in total;

9.1.

a) (X_t) is not a martingale with respect to $(\mathcal{F}_t)$;

b) (S_t) is not a martingale with respect to $(\mathcal{F}_t)$;

c) (W_t) is a martingale with respect to $(\mathcal{F}_t)$;

d) (Y_t) is a martingale with respect to $(\mathcal{F}_t)$;

e) (W_t) is not a martingale with respect to $(\mathcal{F}_{t-1})$;

f) (W_t) is not a martingale with respect to $(\mathcal{F}_{t+1})$;

9.2.

a) What is the distribution of X_7?

b) $\mathbb{E}(X_n) = 0$ and $\mathrm{Var}(X_n) =$.

c) The process (X_n) is a Markov chain!

d) The process (X_n) is not a martingale.

e) $\sigma(X_1, X_2, X_3, \ldots, X_{1000}) = \sigma(X_1)$

f) $\mathbb{P}(\lim X_n \text{ exists}) = 1/3$.

g) $\operatorname{plim} X_n$ does not exist.

9.3.

a) (X_t) is not a martingale;

b) (S_t) is not a martingale;

c) $a = 0.5$;

d) $b = 0$ and $b = \ldots$

9.4.

a) $X_0 = 4/52$; X_1 is $4/51$ with probability $48/52$ or $3/51$ with probability $4/52$; X_{51} is 1 with probability $4/52$ and 0 with probability $48/52$.

b)

c) With probability X_n James Bond will pick up a Queen and the current number of closed Queens $X_n(52-n)$ will decrease by 1. With probability $(1-X_n)$ James Bond will pick up a card different from Queen and the number of closed Queens $X_n(52-n)$ will stay the same.

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n \left(\frac{X_n(52-n)-1}{52-n-1}\right) + (1-X_n)\left(\frac{X_n(52-n)}{52-n-1}\right) = \cdots = X_n.$$

Hence the process (X_n) is a martingale with respect to $(\mathcal{F}_n)$.

9.5. It is possible.

9.6.

a) W_1 is equal to $1/3$ or $2/3$ with equal probabilities; W_2 is equal to $1/4$, $2/4$, $3/4$

b) (W_t) is a martingale;

c) (M_t) is a martingale;

d) The limiting distribution of (W_t) is uniform on $[0;1]$;

9.7. Only constant non-random sequences are martingales.

9.8. $\mathrm{Cov}(M_d - M_c, M_b - M_a) = 0$;

9.9. Yes.

9.10.

9.11.

9.12. For example, $a_n = -3n$, but one may add any constant.

9.13.

9.14.

9.15.

9.16.

9.17.

9.18.

9.19.

9.20.

9.21.

a) (S_n) is a martingale;

b) $\mathbb{P}(S_\tau = b) = a/(a+b)$;

c) $M_n = S_n^2 - n$ is a martingale;

d) $\mathbb{E}(\tau) = ab$;

9.22.

a) (S_n) is not a martingale;

b) $K_n = (q/p)^{S_t}$ is a martingale;

c) $\mathbb{P}(S_\tau = b) = \ldots$

d) $M_n = S_n - (p-q)n$ is a martingale;

e) $\mathbb{E}(\tau) = \ldots$

9.23.

a) $\mathbb{E}(T) = 26^{11} + 26^4 + 26$.

b)

9.24.

9.25.

9.26.

9.27.

9.28.

10.1.

a) $\mathbb{P}(X_a < X_b) = \lambda_a/(\lambda_a + \lambda_b)$. There are two possible solutions: double integral and first step analysis.

b)

c)

d)

10.2. Yes.

10.3.

10.4.

a) $A_t = N_t - \lambda t$ is a martingale;

b) $B_t = A_t^2 - \lambda t$ is a martingale;

10.5.

10.6.

10.7.

10.8. Here we may approximate Poisson distribution by normal distribution, $\mathcal{N}(\lambda t, \lambda t)$.

10.9.

10.10.

10.11.

10.12. Let's denote $a(t) = \mathbb{P}(N_t \text{ is even})$.

$$a(t + \Delta) = a(t)(1 - \Delta) + (1 - a(t))\Delta + o(\Delta)$$

Hence we get a differential equation $a'(t) = 1 - 2a(t)$ with $a(0) = 1$. The solution is $a(t) = (1 + \exp(-2t))/2$.

10.13.

11.1.

11.2.

11.3.

11.4.

11.5.

11.6.

11.7.

11.8.

12.1.

12.2.

12.3.

12.4.

12.5.

12.6.

12.7.

12.8.

12.9.

12.10.

12.11.

12.12.

13.1.

13.2.

13.3.

13.4.

14.1.

14.2.

14.3.

14.4.

15.1.

15.2.

15.3.

15.4.

15.5.

15.6.

16.1.

16.2.

16.3.

16.4.

16.5.

16.6.

16.7.

16.8.

16.9.

16.10.

17.1.

17.2.

17.3.

17.4.

17.5.

18.1.

18.2.

18.3.

18.4.

19.1.

19.2.

19.3.

19.4.

19.5.

19.6.

19.7.

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19.9.

20.1.

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20.6.

21.1.

21.2.

21.3.

21.4.

21.5.

22.1.

22.2.

22.3.

22.4.

22.5.

22.6.

23.1.

23.2.

23.3.

23.4.

23.5.

25 Sources of wisdom

[Buz+15] Nazar Buzun et al. "Stochastic Analysis in Problems, part 1 (in Russian)." In: *arXiv preprint arXiv:1508.03461* (2015). URL: https://arxiv.org/abs/1508.03461.