Stochastic Processes problems

https://github.com/bdemeshev/stochastic_pro

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Contents

1 First step analysis

1.1 Biden and Trump alternately throw a fair dice infinite number of times. Biden throws first. The person who obtains the first 6 wins the game.

a) What is the probability that Biden will win?

b) What is expected number of turns?

c) What is variance of the number of turns?

d) What is expected number of turns given that Biden won?

e) Find the transition matrix of this four state Markov chain.

1.2 Elon throws an unfair coin until "head" appears. The probability of "head" is $p \in (0; 1)$. Let N be the total number of throws.

a) Find $\mathbb{E}(N)$, $\mathrm{Var}(N)$, $\mathbb{E}(N^3)$, $\mathbb{E}(\exp(tN))$.

b) What is the probability than N will be even?

1.3 Alice and Bob throw a fair coin until the sequence HTT or THT appears. Alice wins if HTT appears first, Bob wins if THT appears first.

a) Find the probability that Alice wins.

b) Find the expected value and variance of the total number of throws.

c) Using any open source software find the probability that Alice wins for all possible combinations of three coins sequences for Alice and Bob.

d) Now Alice and Bob play the following game. Alice chooses her three coins winning sequence first. Next Bob, knowing the choice of Alice, chooses his three coins winning sequence. Than they throw a fair coin until either of their sequences appears. What is the best strategy for Alice? For Bob? What is the probability that Alice wins this game?

1.4 You throw a dice unbounded number of times. If it shows 1, 2 or 3 then the corresponding amount of dollars is added in the pot. It it shows 4 or 5 the game stops and you get the pot with money. If it shows 6 the game ends and you get nothing. Initially the pot is empty.

a) What is probability that the game will end by 6?

b) What is expected duration of the game?

c) What is your expected payoff?

d) What is your payoff variance?

e) Consider variation-A of the game. Rules are the same, but initially the pot contains 100 dollars. How will the answers to questions (a)-(d) change?

f) Consider variation-B of the game. Initially the pot is empty. One rule is changed. If the dice shows 5 the content of the pot is burned and the game continues. How will the answers to questions (a)-(d) change?

1.5 Boris Johnson throws a fair coin until 1 appears or until he says "quit". His payoff is the value of the last throw. Boris optimizes his expected payoff. If many strategies gives the same expected payoff he chooses the strategy that minimizes the expected duration of the game.

a) What is the optimal strategy and the corresponding expected payoff?

b) What is the expected duration?

c) How the answers to points (a) and (b) will change if Boris should pay 0.3 dollars for each throw?

1.6 Winnie-the-Pooh starts wandering from the point $x = 1$. Every minute he moves one unit left or one right with equal probabilities.

Let T be the random moment of time when he reaches $x = 0$.

a) Find the generating function $g(u) = \mathbb{E}(u^T)$.

b) Extract all probabilities $\mathbb{P}(T = k)$ from the function $g(u)$.

1.7 Gleb Zheglov catches one criminal every day. With probability 0.2 the catched criminal is replaced by w new criminals. Initially there are n criminals in the town.

What is the expected time to the ultimate crime eradication in the town?

a) (4 points) Solve the problem for $w = 1$ and $n = 1$.

b) (6 points) Solve the problem for arbitrary w and n.

1.8 Consider infinite ladder with steps numbered from 0 to infinity. I start at step 0. Every day with probability u I go one step up. With probability d I go one step down. With probability $1 - u - d$ I stay on the same step.

If I am at step 0 then I stay there with probability $1 - u$ because it's impossible to go down.

Consider the case $d > u$.

What is the probability that I will be at step 0 after 10^{1000} days?

1.9 I throw a fair die until the sequence 6-2-6 appears. Let N be the number of throws.

a) What is the expected value $\mathbb{E}(N)$?

b) Write down the system of linear equations for the moment generating function of N. You don't need to solve it!

1.10 Consider a symmetric random walk $S_t = S_0 + X_1 + X_2 + \cdots + X_t$ where (X_t) are independent identically distributed and take values $+1$ or -1 with equal probabilities.

Random time T is the first moment when we reach the value -3 or 7. Let's denote by p_k the conditional probability to reach 7 before 3 given that we start at $S_0 = k$.

a) Find p_{-3} and p_7.

b) Find equation on p_k, p_{k-1} and p_{k+1} using first step analysis.

c) Guess the unique solution of this recurrence equation and hence find p_0.

Now let's denote by e_k the conditional expected value of T given that we start at $S_0 = k$.

d) Find e_{-3} and e_7.

e) Find equation on e_k, e_{k-1} and e_{k+1} using first step analysis.

f) Guess the unique solution of this recurrence equation and hence find e_0.

g) Generalize the formulas for p_0 and e_0 by replacing -3 and 7 by arbitrary $-a < 0$ and $b > 0$.

1.11 Consider a non-symmetric random walk $S_t = S_0 + X_1 + X_2 + \cdots + X_t$ where (X_t) are independent identically distributed and take values $+1$ or -1 with $\mathbb{P}(X_t = +1) = r$

Random time T is the first moment when we reach the value -3 or 7. Let's denote by p_k the conditional probability to reach 7 before 3 given that we start at $S_0 = k$.

a) Find p_{-3} and p_7.

b) Find equation on p_k, p_{k-1} and p_{k+1} using first step analysis.

c) Guess the unique solution of this recurrence equation and hence find p_0.

Now let's denote by e_k the conditional expected value of T given that we start at $S_0 = k$.

d) Find e_{-3} and e_7.

e) Find equation on e_k, e_{k-1} and e_{k+1} using first step analysis.

f) Guess the unique solution of this recurrence equation and hence find e_0.

g) Generalize the formulas for p_0 and e_0 by replacing -3 and 7 by arbitrary $-a < 0$ and $b > 0$.

1.12 Consider a symmetric random walk $S_t = X_1 + X_2 + \cdots + X_t$ where (X_t) are independent identically distributed and take values $+1$ or -1 with equal probabilities. We start at zero, $S_0 = 0$.

Let N_k the number of visits of point k before the first return of a random walk to 0.

a) What is the probability that a random walk will eventually return to 0?

b) What is the probability $\mathbb{P}(N_k = 0)$?

c) What is the probability $\mathbb{P}(N_k = n)$ for arbitrary natural number n?

d) Compare $\mathbb{E}(N_2)$ and $\mathbb{E}(N_{\text{undecillion}})$.

1.13 You roll a pair of fair dice infinitely many times.

a) What is the probability that you will obtain the sum equal to 10 before the sum equal to 8?

b) What is the probability that you will obtain the sum equal to 10 before the sum equal to 8 but after the sum equal to 11?

1.14 How many coin tosses on average you need to obtain n heads?

1.15 Consider a symmetric random walk S_n with $S_0 = 0$. One may write $S_n = X_1 + X_2 + \cdots + X_n$ where the steps X_n are independent and take values $+1$ or -1 with equal probabilities.

Consider the first moment of time τ when S_n will hit or intersect the line $n - 100$.

a) Find $\mathbb{E}(\tau)$ and $\operatorname{Var}(\tau)$.

b) Find the probability $\mathbb{P}(\tau = k)$.

c) Find the moment generating function $\operatorname{mgf}(u)$ of τ.

d) Solve the point (a), (b) and (c) for the line $2n - 100$.

e) Solve the point (a) for the line $1/2n - 100$.

1.16

1.17

First step into fractal

1.18 The Cantor set fractal C is constructed in the following way. We start with unit segment $I_0 = [0;1]$. At the first stage we remove the middle third from the segment. Hence after the first stage we obtain $I_1 = [0;1/3] \cup [2/3;1]$. At the second stage we remove the middle third from every segment. Hence after the second stage we obtain $I_2 = [0;1/9] \cup [2/9;3/9] \cup [6/9;7/9] \cup [8/9;9/9]$. We continue this process infinitely many times and in the limit we obtain the Cantor set C, see https://en.wikipedia.org/wiki/Cantor_set.

a) Find the length of the Cantor set C.

Now let's choose a point X in the Cantor set randomly uniformly.

b) Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

c) How one may randomly choose a point in the Cantor set using a fair coin?

d) Find at least one point in the Cantor set that does is not the endpoint of any segment.

e) Let F the cumulative distribution function of X. Find the values $F(1/2)$, $F(1/3)$, $F(1/4)$ and $F(1/5)$.

1.19 The Sierpiński triangle S is constructed in the following way. We start with an equilateral triangle T_0 with unit sides. At the first stage we subdivide the triangle into four smaller congruent equilateral triangles and remove the central triangle. Hence after the first stage the set T_1 consists of three triangles. At the second stage we we subdivide every triangle into four smaller congruent equilateral triangles and remove the central triangle. Hence after the second stage we obtain the set T_2 that consists of nine triangles. We continue this process infinitely many times and in the limit we obtain the Sierpiński triangle S, see https://en.wikipedia.org/wiki/Sierpiński_triangle.

a) Find the are of the Sierpiński triangle S.

Now let's choose a point (X, Y) in the Sierpiński triangle with vertices $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$ randomly uniformly.

b) Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

c) Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.

d) Find $\text{Cov}(X, Y)$.

e) How one may randomly choose a point in the Sierpiński triangle using a fair coin?

1.20

1.21

2 Markov chains

2.1 HSE student lives in two states: "sleep" and "study" and tries to change the state every 1 hour. After the sleep state the student continues sleeping with probability equal to 0.25, otherwise a student starts studying. If the student is studying, the probabilities to continue studying and to start sleeping are equal.

a) Write down the transition matrix of this Markov chain.

b) Draw the graph representation.

c) What is the probability that a Sleeping Student will be a Studying Student after 1 hour? After 2 hours?

d) We know that initially student is sleeping with probability $p = \frac{2}{3}$. Find the probabilities of sleep and study states after 1 and 2 hours.

e) Find the probabilities of sleep and study states after 20 and 100 hours (do it with **matrix** operations and any soft). Is there any difference and why?

2.2 HSE student has three states: pre-coffee, with-coffee and over-coffee. He goes to Jeffrey's each break seeking for a cup of coffee. The line is usually too long, so probability to stay pre-coffee is equal to 60% and to be over-coffee — is zero. Caffeinated students can stay in lines longer, so for with-coffee student the probability to become over-coffee is 20% and to become pre-coffee — 30%. Over-coffee student runs to coffeeshop very fast and able to stay over-coffeed with $p = 0.70$ and can suddenly become pre-coffee with $p = 0.10$.

a) Draw the graph representation of this Markov chain.

b) Write the transition matrix of this Markov chain.

c) What is the probability that morning pre-coffee student will be with-coffee after 1 break? After 3 breaks?

d) What is the probability that morning pre-coffee student will be with-coffee after 1 break? After 3 breaks? After 200 breaks?

2.3 Unteachable students in NOTHSE University try to pass the exams. Students cheat successfully and pass the exams with probability 10%. In the case of a failure students are allowed to infinite number of retakes. All students are unteachable so the amount of knowledge is always the same and doesn't depend of the number of retakes.

a) Draw the graph representation of this Markov chain.

b) What is the probability to graduate using no more than 5 retakes?

c) What is the probability to graduate eventually?

d) Use **first step analysis** to find the average number of retakes per student in this University.

2.4 Every month the real estate Galina agent has two options: to increase her commission and to ask an owner to increase the rent. If the agent has increased the commission, on the next step she increases the commission again with probability $5/8$. If she has asked the owner, she decides to increase the commission with probability equal to $3/4$ on the next step.

a) Write the transition matrix of this Markov chain.

b) Draw the graph representation.

c) Use **first step analysis** to find how many steps the agent does between asking the owner to increase the price.

2.5 Alice and Bob toss a coin, writing down the results. If the last 3 tosses are Head, Head and Tail, Alice wins. If the last 3 tosses are Tail, Head and Head, Bob wins.

a) Is it easy to work with matrix representation in this case?

b) Draw the graph representation. Who is more likely to win the game?

c) Use **first step analysis** to find the probability of Alice's win.

d) Find the probability that the game ends in exactly 4 tosses.

e) Find the expected value and variance of the total number of coin throws in the game.

2.6 HSE student has an unusually caring granny who cooks one pie with probability 0.7 every weekend. Granny's pies are so tasty that HSE student can't resist and he gains 1 kilo for each pie eaten. Without pies the student with more than 70 kilos weight looses 1 kilo per week, yeah, he has a lot of studies! At the beginning of the study year student's weight is $W_0 = 70$ kilos.

Let W_t be the weight of the student t weeks later.

a) Find the probability $\mathbb{P}(W_3 \geqslant 71)$ and expected value $\mathbb{E}(W_3)$.

b) Find the limit weight after infinitely many study weeks $\lim_{t\to\infty} W_t$.

c) Explain whether the chain (W_t) has a stationary distribution.

2.7 The fair price of Sborbank in discrete stock market is somewhere between 100 and 101 rubles. If the price is equal to 100, then the price grows up by 1 ruble with probability $\frac{9}{10}$, otherwise it goes down by 1 ruble. If the price is greater then 100, it grows by 1 ruble with probability $\frac{1}{3}$ or declines by 1 ruble. If the price is lower than 100, it grows by 1 ruble with probability $\frac{2}{3}$ or declines by 1 ruble.

a) Draw the graph representation of the corresponding Markov chain.

b) Do you think this chain has some stationary distribution?

c) Find the average time for the stock price to fall from 102 rubles to 98 rubles.

Hint: you may to decompose the long path into smaller ones and to use the first step analysis.

2.8 The hedgehog Melissa starts at the vertex A of a triangle ΔABC. Each minute she randomly moves to an adjacent vertex with probabilities $\mathbb{P}(A \to B) = 0.7$, $\mathbb{P}(A \to C) = 0.3$, $\mathbb{P}(B \to C) = \mathbb{P}(B \to A) = 0.5$, $\mathbb{P}(C \to B) = \mathbb{P}(C \to A) = 0.5$.

a) What is the probability that she will be in vertex B after 3 steps?

b) Write down the transition matrix of this Markov chain.

c) What proportion of time Melissa will spend in each state in the long run?

2.9 A Hedgehog starts at the point $x = 2$ on the real line. Every minute he moves one step left with probability 0.3 or one step right with probability 0.7. There are two exceptions from this rule: the absorbing point $x = 0$ and the reflecting barrier at $x = 3$.

If the Hedgehog reaches the absorbing point $x = 0$ then he stops moving and stays there. If the Hedgehog reaches the reflecting barrier $x = 3$ then his next move will be one step left with probability 1.

a) [2] Write the transition matrix of this Markov chain.

b) [3] What is the probability that Hedgehog will be at $x = 1$ after exactly 3 steps?

c) [5] What is the expected time to reach the absorbing state?

2.10 Vampire Petr drinks blood of a new victim every day. Unfortunately 20% of the population are vaccinated against vampires. If more than one victim of the last three victims are vaccinated then Petr will be instantaneously cured and will return to the normal life.

For simplicity let's assume that the last three victims were not vaccinated.

a) What is the probability that vampire Petr will be cured in the next three days?

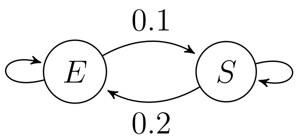
b) How many victims will be bitten by vampire Petr on average?

2.11 A hedgehog moves at random on the vertices A, B, C and D of a regular tetrahedron (тетраэдр). She start at the vertex A and every minute changes her position to one of the adjacent vertices with probability $1/3$ independently of past moves.

a) Write down the transition matrix of this Markov chain.

b) What is the expected time of the first return to the starting vertex A?

2.12 The Cat can be only in two states: Sleeping (S) and Eating (E). Cat's mood depends only on the previous state. The transition probabilities are given below:



a) Compute the missing probabilities on the graph.

b) Write down the transition matrix.

c) Compute $\mathbb{P}(X_3 = \text{Eating} \mid X_0 = \text{Eating})$.

2.13 Cowboy Joe enters the Epsilon Bar and orders one pint of beer. He drinks it and orders one pint more. And so on and so on and so on... The problem is that the barmaid waters down each pint with probability 0.2 independently of other pints. Joe does not like watered down beer. He will blow the Epsilon Bar to hell if two or more out of the last three pints are watered down.

We point out that Joe never drinks less than 3 pints in a bar.

a) What is the expected number of pints of beer Joe will drink?

Let Y_t be the indicator that the pint number t was watered down. Consider the Markov chain $S_t = (y_{t-2}, y_{t-1}, y_t)$. For example, $S_t = (100)$ means that the pint number $t-2$ was watered down while pints number $t-1$ and t are good.

b) What are the possible values of S_3 and their probabilities?

c) Write down the transition matrix of this Markov chain.

2.14 Pavel Durov starts at the point $X_0 = 3$ on the real line. Each minute he moves left with probability 0.4 or right with probability 0.6 independently of past moves. The points 0 and 5 are absorbing. If Pavel reaches 0 or 5 he stays there forever. Let X_t be the coordinate of Pavel after t minutes.

a) Write down the transition matrix of this Markov chain.

b) Calculate the distribution of X_7 [list all values of the random variable X_7 and estimate the probabilities].

Hint: you are free to use python for this problem :)

2.15 Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCD$. Each minute each hedehog simulteneously and independently chooses to go clockwise to the adjacent point, to go counter-clockwise to the adjacent point or to stay at his location.

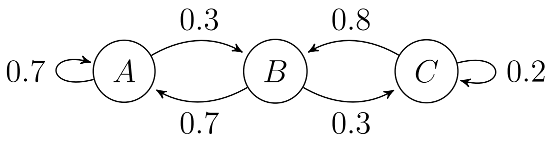
Thus the brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.

a) Draw the graph for the brotherhood Markov chain and calculate all transition probabilities.

b) Write down the transition matrix of the brotherhood Markov chain.

c) What is the probability that they will be in one vertex after 3 steps?

2.16 Consider the following Markov chain:



a) Find the stationary distribution of this Markov chain.

The Markov chains starts at the vertex A. Let N be the first moment when the state C will be reached.

b) Find the expected value $\mathbb{E}(N)$.

c) Find the variance $\text{Var}(N)$.

2.17 Design a Markov chain with 3 states and unique stationary distribution $\pi = (0.1, 0.2, 0.7)$.

2.18 Consider three games:

Game A: You toss a biased coin with probability 0.48 of H. You get $+1$ dollar for H and -1 dollar for T.

Game B: If your wellfare is divisible by three you toss a coin that lands on H with probability 0.09. If your wellfare is not divisible by three you toss a coin that lands on H with probability 0.74. You get $+1$ dollar for H and -1 dollar for T.

Game C: You toss an unbiased coin. If it lands on H you play Game A. If it lands on T you play Game B.

Your initial capital is 10000 dollars.

a) Generate and plot two random trajectories of your wellfare if you play Game A 10^6 times.

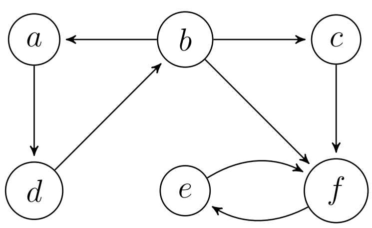
b) Generate and plot two random trajectories of your wellfare if you play Game B 10^6 times.

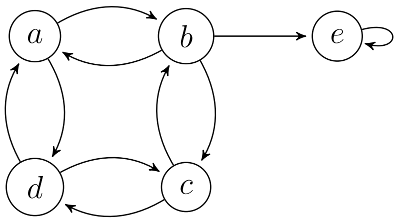
c) Generate and plot two random trajectories of your wellfare if you play Game C 10^6 times.

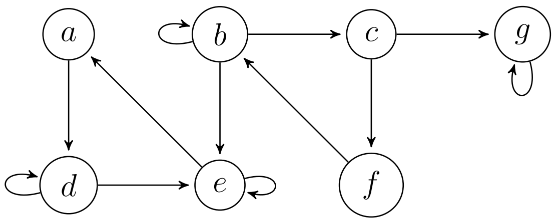
2.19

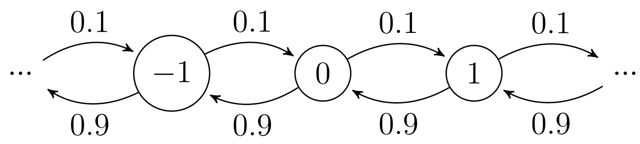
3 Classification of states

3.1 We randomly wander on the graph choosing at each moment of time one of the possible directions. If probabilities are not given we choose equiprobably.









a) Split each Markov chain into communicating classes.
b) Find the period of every state.
c) Classify each state as transient, null-recurrent and positive recurrent.
d) For positive recurrent states find the expected return time.
e) Find all stationary distributions.

3.2 A Knight randomly wanders on the chessboard. At each step he randomly chooses one of the possible Knight-moves with equal probabilities.

a) Find the stationary distribution.
b) Find the expected return time for every square.
c) Find the period of every square.

3.3 Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

a) (3 points) Split the chain in classes and classify them into closed or not closed.
b) (2 points) Classify the states into recurrent or transient.
c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.
What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

3.4 Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) (3 points) Split the chain in classes and classify them into closed or not closed.

b) (2 points) Classify the states into recurrent or transient.

c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.

What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

3.5

3.6

4 Generating functions

4.1 The MGF (moment generating function) of the random variable X is give by $M(t) = 0.3\exp(2t) + 0.2\exp(3t) + 0.5\exp(7t)$.

Recover the distribution of the random variable X.

4.2 The random variable Y takes values 1, 2 or 3 with equal probabilities.

Find the MGF of the random variable Y.

4.3 The MGF of the random variable W has a Taylor expansion that starts with $M(t) = 1 + 2t + 7t^2 + 20t^3 + \ldots$.

a) Find $\mathbb{E}(W)$, $\text{Var}(W)$, $\mathbb{E}(W^3)$.

b) Find the starting terms of the Taylor expansion of the moment generating function for the sum $S = W + W^*$, where W and W^* are independent and identically distributed.

4.4 The random variable X takes non-negative integer values. The generating function $g(u) = \mathbb{E}(u^X)$ has a Taylor expansion that starts with $g(u) = 0.1 + 0.2u + 0.15u^2 + \ldots$.

Find probabilities $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.

4.5 Random variables X_i are mutually independent and X_i has Gamma distribution $\text{Gamma}(\alpha_i, \beta_i)$.

I sum up the random number N of terms,

$$S = X_1 + X_2 + \ldots + X_N.$$

The number N has Poisson distribution $\text{Pois}(\lambda)$ and is independent of the sequence (X_i).

a) Find the MGF of S. You may the MGF formula for Gamma distribution as known.

b) Find $\mathbb{E}(S)$ and $\text{Var}(S)$.

4.6 The random variable X take non-negative integer values. Its moment generating function is equal to $M(t) = (2 - \exp(t))^{-7}$.

a) Find the probability generating function $g(u) = \mathbb{E}(u^X)$.

b) Find $\mathbb{E}(X)$, $\text{Var}(X)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.

c) Find $\mathbb{P}(X = k)$.

4.7 The number of players N who will win the lottery is a random variable with probability mass function $\mathbb{P}(N = k) = 7 \cdot 0.3^k/3$ for $k \geqslant 1$. Each player will get a random prize $X_i \sim U[0;1]$. All random variables are independent. Let S be the sum of all the prizes.

a) Find $\mathbb{E}(S \mid N)$ and conditional moment generating function $M_{S|N}(u)$.
b) Find the unconditional moment generating function $M_S(u)$.
c) What is the probabilistic meaning of $M''_S(0) - (M'_S(0))^2$?

4.8 Prince Myshkin throws a fair coin until two consecutive heads appear. Let N be the number of throws.

Find the moment generating function of N.

4.9 The moment generating function of a random variable X is $1/(1-2t)$.

a) Find the moment generating function of $2X$.
b) Find the moment generating function of $X+Y$ where X and Y are independent and identically distributed.
c) Do you remember the sum of geometric progression? Find $\mathbb{E}(X^{2021})$.

4.10 The ultimate goal of this exercise is to prove the good upper bound for tail probability of a normal distribution: if $X \sim \mathcal{N}(0;\sigma^2)$ then $\mathbb{P}(X > c) \leqslant \exp(-c^2/2\sigma^2)$.

Here are the guiding hints (you free to use not use them):

a) State the MGF of X. You may derive it or simply write it if you remember.
b) Consider $Y = \exp(uX)$. Using Markov inequality provide the upper bound for $\mathbb{P}(Y > \exp(uc))$.
c) Prove that $\mathbb{P}(X > c) \leqslant MGF_X(u)\exp(-uc)$ for any u.
d) Find the value of u that makes the upper bound as tight as possible.

4.11 I have an unfair coin with probability of heads equal to $h \in (0;1)$.

a) Let N be the number of tails before the first head. Find the MGF of N.
b) Let S be the number of tails before k heads (not necessary consecutive). Find the MGF of S.
c) What is the limit of $MGF_S(t)$ when $k \to \infty$ and $k \times h \to 0.5$?
d) What is the limit of $MGF_S(t)$ when $k \to \infty$ and $k \times (1-h) \to 0.5$?

4.12 I have three problems in the home assignment. Time spent on each problem is modelled by independend exponentially distributed random variables with rate λ: X_1, X_2, X_3.

a) Find the moment generating function of X_i and hence the moment generating function of $S = X_1 + X_2 + X_3$.
b) Find $\mathbb{E}(S^3)$.
c) Find the joint density of $R = X_1/(X_1 + X_2 + X_3)$ and S.

4.13 Recognise the distribution family and its parameters by looking at the moment-generating function:

a) $0.7 + 0.3\exp(t)$;
b) $\exp(2024\exp(t))/\exp(2024)$;

c) $\frac{\exp(3t)-1}{3t\exp(-2t)}$;

d) $\exp(6t + 2024t^2)$;

e) $1/(5t-1)^{2024}$.

You may use the table from the article

https://en.wikipedia.org/wiki/Moment-generating_function.

4.14 Consider the moment-generating function of a random variable X:

$$g(t) = \frac{\exp(3t)-1}{3t\exp(-2t)}.$$

a) Expand the function $g(t)$ as Taylor series up to t^4 included.

b) Find $\mathbb{E}(X)$, $\mathbb{E}(X^2)$, $\mathbb{E}(X^3)$, $\mathbb{E}(X^4)$.

4.15 The moment-generating function of the pair of random variables (X, Y) is given by $\exp(6t_1 + 5t_2 + t_1^2 + 20t_2^2 - 2t_1t_2)$.

Find $\mathbb{E}(X)$, $\operatorname{Var}(Y)$, $\mathbb{E}(XY)$.

4.16 A monkey is randomly typing letters. She types a with probability 0.01, b — with probability 0.02 and c — with probability 0.03. Other probabilities are unknown and not relevant. She stops if starting exactly from the beginning she types one of the words: abc, $bacab$ or $abba$. Let A be the event that she stops.

Consider the generating function $f(a, b, c) = abc + bacab + abba$. What is the combinatorial or probabilistic meaning of the following quantities:

a) $f(1, 1, 1)$;

b) $f(0.01, 0.02, 0.03)$;

c) $g'(1)$ if $g(t) = f(0.01t, 0.02t, 0.03t)$;

d) $g'(1)$ if $g(t) = f(0.01, 0.03, 0.03t)$;

e) $g'(1)/g(1)$ if $g(t) = f(0.01t, 0.02t, 0.03t)$;

f) $g'(1)/g(1)$ if $g(t) = f(0.01, 0.03, 0.03t)$;

g) $g'(0)$ if $g(t) = f(0.01\exp(t), 0.02\exp(t), 0.03\exp(t))$.

h) $g'(0)/g(0)$ if $g(t) = f(0.01, 0.02, 0.03\exp(t))$.

4.17 A monkey is randomly typing letters. She types a with probability 0.01, b — with probability 0.02 and c — with probability 0.03. Other probabilities are unknown and not relevant. She stops if starting exactly from the beginning she types one of the words: abc, $bacab$, $abba$, ... Let A be the event that she stops.

You don't know the exact list of words but you can do any calculations with the generating function function $f(a, b, c) = abc + bacab + abba + \ldots$.

How can you extract the following information:

a) the number of words;

b) the probability that monkey stops;

c) the expected payoff when she gets one dollar for every letter typed in the case of finite game;

d) the expected payoff when she gets one dollar for every letter c typed in the case of finite game;

e) the conditional expected payoff when she gets one dollar for every letter typed if it is known that the game ended in a finite number of steps;

f) the conditional expected payoff when she gets one dollar for every letter c typed if it is known that the game ended in a finite number of steps;

g) $\mathbb{E}(N_a N_b I(A))$ where N_a is the number of a letters, N_b — the number of b letters and $I(A)$ — the indicator of a finite game.

h) $\mathbb{E}(N_a(N_a-1)N_b(N_b-1)(N_b-2)I(A))$ where N_a is the number of a letters, N_b — the number of b letters and $I(A)$ — the indicator of a finite game.

i) $\mathbb{E}(N_a^2 N_b^3 I(A))$ where N_a is the number of a letters, N_b — the number of b letters and $I(A)$ — the indicator of a finite game.

Hint: in points (c) and (d) try to write the expression with and without exponent.

4.18

4.19

5 Inequalities

5.1 I have 100 numbers written on small sheets of paper: x_1, x_2, ..., x_{100}. The sum of these numbers is 1.

Find the possible values of the sum

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \ldots + \frac{x_{100}}{\sqrt{1-x_{100}}}.$$

Hint: consider a randomly selected number X and apply the Jensen's inequality.

5.2 Ordinary non-random sequences also have alternative limit consept! The real number c is a Cesaro limit of a sequence (a_n) if this number is a limit of a sequence of cumulative averages, $c = \lim \bar{a}_n$.

a) Consider a sequence $(a_n) = (1, 0, 1, 0, 1, 0, \ldots)$. Does it have ordinary limit? Does it have Cesaro limit?

b) Construct an aperiodic sequence that does not have ordinary limit but has Cesaro limit.

c) Is it true that any ordinary convergent sequence in also Cesaro-converges?

You now that (a_n) Cesaro-converges to a and (b_n) Cesaro-converges to b.

d) Is it true that $(a_n + b_n)$ is Cesaro-convergent?

e) Is it true that $(a_n \cdot b_n)$ is Cesaro-convergent?

5.3

5.4

6 Limits, convergence

6.1 Polina loves sweet chestnuts. She has infinite sequence of baskets before her. In the basket number n there are n chestnuts in total. Unfortunately only one chestnut in every basket is a sweet one.

She picks chestnuts one by one at random from all the buskets sequentially. First she picks the unique chestnut from the basket number one, than she picks in a random order two chestnuts from the basket number two and so on.

The random variable S_t indicates whether the chestnut number t was a sweet one.

a) Find $\lim S_t$ or prove that the limit does not exist.

b) Find $\operatorname{plim} S_t$ or prove that the limit does not exist.

6.2 Let (X_n) be independent, each variable X_n has exponential distribution with rate $\lambda_n = n$.

a) Find the probability limit $\operatorname{plim} X_n$ or prove that it does not exist.

Let (Y_n) be independent, each variable Y_n has exponential distribution with rate $\lambda_n = n/(n+1)$.

b) Find the probability limit $\operatorname{plim} Y_n$ or prove that it does not exist.

6.3 Let (X_n) be independent normally distributed $\mathcal{N}(5; 10)$.

a) Find the probability limit

$$\operatorname{plim} \frac{X_1 + X_2 + \cdots + X_n}{7n}.$$

b) Find the probability limit

$$\operatorname{plim} \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{7n}.$$

c) Find the probability limit

$$\operatorname{plim} \ln(X_1^2 + X_2^2 + \cdots + X_n^2) - \ln n.$$

6.4 Let (X_n) be independent uniform on $[0; 1]$. Let $Y_n = X_n^2 + X_n^3$.

a) Find the probability limit $\operatorname{plim} V_n$ for

$$V_n = \max\{Y_1, Y_2, \ldots, Y_n\}.$$

b) Find the probability limit $\operatorname{plim} W_n$ for

$$W_n = \max\{X_1 + Y_1, X_1 + Y_2, \ldots, X_1 + Y_n\}.$$

6.5 Consider the random variable X and the sequence of random variables Y_n with $\mathbb{E}(Y_n) = \frac{1}{n}$ and $\operatorname{Var}(Y_n) = \frac{\sigma^2}{n}$. Let $W_n = X + Y_n$.

a) Find the probability limit $\operatorname{plim} Y_n$;

b) Find the probability limit $\operatorname{plim} W_n$.

6.6 The random variables X_i are independent and uniformly distributed on $[0; 1]$. Let $Y_n = \min X_1, \ldots X_n$.

a) Find the almost sure limit of Y_n;

b) Find the probability limit of Y_n;

c) Find the limiting distribution of Y_n.

6.7 Let X and Y be independent and uniformly distributed on $[0;1]$. Let $V_n = n^2 Y \cdot I(X \leqslant 1/n)$ and $W_n = Y \cdot I(X > 1/n)$.

a) Find $\operatorname{plim} V_n$ and $\operatorname{plim} W_n$.

b) Does (V_n) converge in mean squared?

c) Does (W_n) converge in mean squared?

6.8 a) As a warm-up find the limit

$$\lim_{n\to\infty} \frac{2n^2+3n+6}{5n^2+2n+9}.$$

Now consider the sequence with parameters:

$$X_n = \frac{Ln^2+3n+6}{Rn^2+2n+9}$$

b) For each value of parameters L and R find the limit $\lim X_n$.

c) Find the almost surely limit of X_n if L and R are independent and Unif$[0;1]$. Does the pointwise limit exist?

d) Random variables L and Q be independent and take values 0 or 1 with equal probabilities. Let $R = L + Q$. Find the almost surely limit of X_n in terms of L and R. Does the pointwise limit exist?

6.9 Consider the sequence $Y_n = U^n$ with parameter U.

a) Find the ordinary limit of Y_n for all values of U for which the sequence converges.

b) Find the almost surely limit of Y_n if $U \sim$ Unif$[0;1]$.

c) What is the probability that Y_n converges if $U \sim$ Unif$[0;2]$?

d) What is the probability that Y_n converges if U takes values $+1$ or -1 with equal probabilities?

e) Does Y_n converges in distribution if U takes values $+1$ or -1 with equal probabilities?

6.10 The random variables X_i are independend and uniformly distributed on $[0;2]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^{10} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^3}{n+1}\right\}.$$

6.11 The random variables X_i are independend and uniformly distributed on $[0;1]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^{n} X_i}{n}, \frac{2\sum_{i=1}^{n} X_i^2}{n}\right\}.$$

6.12 The random variables X_i are independend and uniformly distributed on $[0;2]$. Find

$$\operatorname*{plim}_{n\to\infty} \frac{(X_1-\bar{X})^3+(X_2-\bar{X})^3+\ldots+(X_n-\bar{X})^3}{n+2022}.$$

6.13 Consider the stochastic process (X_n), where X_0 is uniform on $[0;2]$ and $X_n = (1+X_{n-1})/2$.

a) Find $\mathbb{E}(X_n)$ and $\operatorname{Var}(X_n)$.

b) Find the probability limit $\operatorname{plim} X_n$.

6.14 The random variables X_i are independent and exponentially distributed with rate $\lambda = 1$.

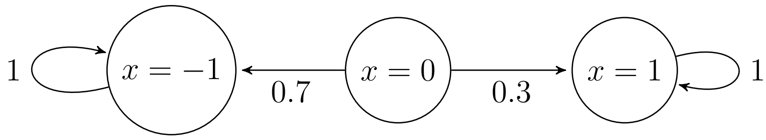
a) Find the probability limit
$$\operatorname{plim} \frac{X_1 + X_2 + X_3 + \cdots + X_n}{2n+7}.$$

b) Find the probability limit
$$\operatorname{plim} \frac{X_1^2 + X_2^2 + X_3^2 + \cdots + X_n^2}{2n+7}.$$

c) Find the probability limit
$$\operatorname{plim} \min\{X_1, X_2, X_3, \ldots, X_n\}.$$

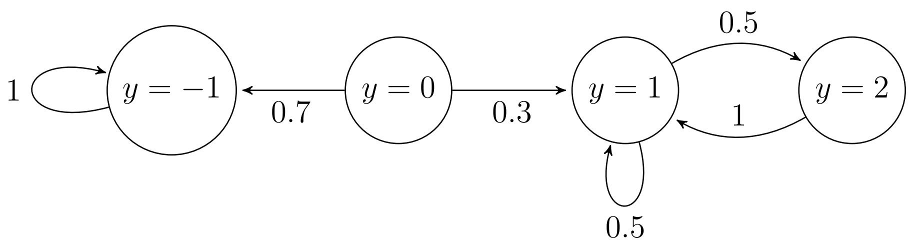
d) Find the probability limit
$$\operatorname{plim} \sqrt[n]{\exp(2X_1 + 2X_2 + \cdots + 2X_n)}.$$

6.15 Consider two Markov chains, (X_t) and (Y_t):



with $X_0 = 0$;

and



with $Y_0 = 0$.

a) Find $\mathbb{P}(\lim X_n \text{ exists})$ and $\mathbb{P}(\lim Y_n \text{ exists})$.

b) Find the limiting distribution of (X_n) and the limiting distribution of (Y_n).
Hint: here you need to calculate all limits $\lim \mathbb{P}(X_n = k)$, $\lim \mathbb{P}(Y_n = k)$.

c) Does (X_n) converges almost surely? In distribution? In probability?

d) Does (Y_n) converges almost surely? In distribution? In probability?

6.16 Let X_n be a discrete time stochastic process that converges in probability to a random number X as $n \to \infty$.

a) Does this condition imply that X_n converges to X in mean? Almost surely? In distribution? Support your answers with a proof or counterexample.

b) Give a definition for each type of convergence.

6.17

6.18

6.19

7 Sigma-algebras and measurability

Sigma-algebra generated by discrete random variable X, $\sigma(X)$ — the list of all events that can be stated using X.

Sigma-algebra generated by arbitrary random variable X, $\sigma(X)$ — the smallest list of events that satisfies two properties:

- The list contains all events of the form $\{X \leqslant t\}$, that means one can compare X with any number;
- If one takes countably many events from this list and does logical operations (union, complement, intersection) then one will obtain an event from the list.

7.1 The random variable X takes values 1, 2 and -2 with equal probabilities.

a) Find the sigma-algebra $\sigma(X)$.

b) How the answer will change if one modifies probability distribution of X?

c) Find the sigma-algebra $\sigma(|X|)$.

d) Foma knows $|X|$ and Yeryoma knows X^2. What can one say about sigma-algebras that model their knowledge?

7.2 Experiment may end by one of the six outcomes:

	$X=-1$	$X=0$	$X=1$
$Y=0$	0.1	0.2	0.3
$Y=1$	0.2	0.1	0.1

a) Find expicitely the sigma-algebras $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$, $\sigma(X^2)$, $\sigma(2X+3)$.

b) How many elements are there in $\sigma(X,Y)$, $\sigma(X+Y)$, $\sigma(X,Y,X+Y)$?

7.3 Let's look at the number of possible elements in a sigma-algebra.

a) The random variable X has five possible values. How many events are there in $\sigma(X)$?

b) Can a sigma-algebra contain exactly 1000 events? Exactly 1024 events?

Maria throws a coin 100 times and remembers very well all the tosses.

c) How many elementary outcomes are there in the probability space Ω?

d) How many events are there in a sigma-algebra that models Maria's knowledge?

7.4 How sigma-algebras $\sigma(X)$ and $\sigma(f(X))$ are related? When they are equal?

7.5 How many different σ-algebras can be created using the set of outcomes Ω has three elements? And if Ω has four elements?

7.6 Provide an example of algebra that is not a σ-algebra.

7.7 Prove a statement or provide a counter-example:

a) The intersection of two sigma-algebras is a sigma-algebra.

b) If the intersection of two sigma-algebras is a sigma-algebra then one of them is contained in the other one.

c) The union of two sigma-algebras is a sigma-algebra.

d) If the union of two sigma-algebras is a sigma-algebra then one of them is contained in the other one.

7.8 Let $\mathcal{F}$ be some σ-algebra of subsets of Ω and $B \subseteq \Omega$. Consider the collection of sets $\mathcal{H} = \{A : A \subseteq B \text{ or } B^c \subseteq A\}$.

Is $\mathcal{H}$ a σ-algebra?

7.9 Будем обозначать количество элементов множества с помощью $\operatorname{card} A$. Рассмотрим подмножества натуральных чисел, $A \subseteq \mathbb{N}$. Определим для подмножества плотность Чезаро (Cesaro density),

$$\gamma(A) = \lim_{n\to\infty} \frac{\operatorname{card}(A \cap \{1, 2, 3, \ldots, n\})}{n}$$

в тех случаях, когда этот предел существует.

Плотность Чезаро показывает, какую «долю» от всех натуральных чисел составляет указанное подмножество. Обозначим с помощью $\mathcal{H}$ все подмножества, имеющие плотность Чезаро.

a) Чему равна плотность Чезаро у нечётных чисел?

b) Приведите пример множества, у которого не определена доля Чезаро.

c) Верно ли, что у натуральных чисел, в записи которых присутствует хотя бы одна единица, есть доля? Если да, то чему она равна?

d) Верно ли, что у натуральных чисел, в записи которых присутствует ровно одна единица, есть доля? Если да, то чему она равна?

e) Верно ли, что $\mathcal{H}$ — алгебра? Сигма-алгебра?

7.10 We throw a fair dice. Let Y be the indicator of a even score and Z be the indicator of score bigger than 2.

a) Find the sigma-algebra $\sigma(Z)$.

b) Find the sigma algebra $\sigma(Y \cdot Z)$.

c) How many elements are there in $\sigma(Y, Z)$?

d) How are related the σ-algebras $\sigma(Y \cdot Z)$ and $\sigma(Y, Z)$?

7.11 We throw a coin infinitely many times. Let X_n be the indicator that the coin landed on Head at toss number n. Consider a pack of σ-algebras: $\mathcal{F}_n := \sigma(X_1, X_2, \ldots, X_n)$, $\mathcal{H}_n := \sigma(X_n, X_{n+1}, X_{n+2}, \ldots)$.

a) For each case provide two examples of σ-algebras that contain the corresponding event

   (a) $\{X_{37} > 0\}$;

   (b) $\{X_{37} > X_{2024}\}$;

   (c) $\{X_{37} > X_{2024} > X_{12}\}$;

b) Simplify experessions: $\mathcal{F}_{11} \cap \mathcal{F}_{25}$, $\mathcal{F}_{11} \cup \mathcal{F}_{25}$, $\mathcal{H}_{11} \cap \mathcal{H}_{25}$, $\mathcal{H}_{11} \cup \mathcal{H}_{25}$.

c) For each case provide two non-trivial examples (different from Ω and $\emptyset$) of an event A such that

   (a) $A \in \mathcal{F}_{2024}$;

(b) $A \notin \mathcal{F}_{2025}$;

(c) $A \in \mathcal{H}_n$ for all possible n;

7.12 Правда ли равносильны три набора требований к списку множеств $\mathcal{F}$?

Тариф «Классический»:

a) $\Omega \in \mathcal{F}$;

b) Если $A \in \mathcal{F}$, то $A^c \in \mathcal{F}$;

c) Если $A_1, A_2, A_3, \ldots, \in \mathcal{F}$, то $\cup A_i \in \mathcal{F}$.

Тариф «Перевёрнутый»:

a) $\emptyset \in \mathcal{F}$;

b) Если $A \in \mathcal{F}$, то $A^c \in \mathcal{F}$;

c) Если $A_1, A_2, A_3, \ldots, \in \mathcal{F}$, то $\cap A_i \in \mathcal{F}$.

Тариф «Хочу всё»:

a) $\Omega \in \mathcal{F}$, $\emptyset \in \mathcal{F}$;

b) Если $A \in \mathcal{F}$ и $B \in \mathcal{F}$, то $A \backslash B \in \mathcal{F}$;

c) Если $A_1, A_2, A_3, \ldots, \in \mathcal{F}$, то $\cup A_i \in \mathcal{F}$ и $\cap A_i \in \mathcal{F}$.

7.13 Рассмотрим $\Omega = [0; 1]$ и набор множества $\mathcal{F}$ таких, что либо каждое множество не более, чем счётно, либо дополнение к нему не более, чем счётно.

a) Верно ли, что $\mathcal{F}$ — алгебра? σ-алгебра?

b) Придумайте $B \subset \Omega$, такое что $B \notin \mathcal{F}$.

7.14 В лесу есть три вида грибов: рыжики, лисички и мухоморы. Попадаются они равновероятно и независимо друг от друга. Маша нашла 100 грибов. Пусть R — количество рыжиков, L — количество лисичек, а M — количество мухоморов среди найденных грибов.

a) Сколько элементов $\sigma(R)$?

b) Сколько элементов $\sigma(R, M)$?

c) Измерима ли L относительно $\sigma(R)$?

d) Измерима ли L относительно $\sigma(R, M)$?

e) Измерима ли L относительно $\sigma(R + M)$?

f) Измерима ли L относительно $\sigma(R - M)$?

7.15 Сейчас либо солнечно, либо дождь, либо пасмурно без дождя. Соответственно, множество Ω состоит из трёх исходов, $\Omega = \{\text{солнечно}, \text{дождь}, \text{пасмурно}\}$. Джеймс Бонд пойман и привязан к стулу с завязанными глазами, но он может на слух отличать, идёт ли дождь.

a) Как выглядит σ-алгебра событий, которые различает агент 007?

b) Как выглядит минимальная σ-алгебра, содержащая событие $A = \{\text{не видно солнце}\}$?

c) Сколько различных σ-алгебр можно придумать для данного Ω?

7.16 The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

a) [3] Explicitely list all the events in sigma-algebra $\sigma(X_1 \cdot X_2)$.

b) [3] Pavel says that he knows only whether X_1 and X_3 are equal. How will you describe his knowledge with sigma-algebra?

c) [4] How many events are in the sigma-algebra $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3)$?

7.17 Vincenzo Peruggia makes attempts to steal the Mona Lisa painting until the first success. Each attempt is successful with probability 0.1.

Let X be the number of attempts and $Z = \min\{X, 5\}$.

a) (5 points) How many events are in sigma-algebras $\sigma(Z)$ and $\sigma(X)$?

b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(Z)$ but $A \notin \sigma(X)$; $B \in \sigma(X)$ but $B \notin \sigma(Z)$.

c) (10 points) Find $\mathbb{E}(Z \mid X)$ and $\mathbb{E}(X \mid Z)$.

7.18 Variables X_1, X_2, ...X_{100} are independent and identically distributed with mean 1 and variance 2. Each X_i has only three possible values: 0, 1, and 2.

a) (5 points) How many events are in sigma-algebras $\sigma(X_1, X_2)$ and $\sigma(X_1 - X_2)$?

b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(X_1, X_2)$ but $A \notin \sigma(X_1 - X_2)$; $B \in \sigma(X_1 - X_2)$ but $B \notin \sigma(X_1, X_2)$.

c) (10 points) Find $\mathbb{E}(X_1 + \ldots + X_{100} \mid X_1 + \ldots + X_{50})$ and $\mathbb{E}(X_1 + \ldots + X_{50} \mid X_1 + \ldots + X_{100})$.

7.19 HSE student rolls the dice once. Find the σ-algebras that model the following situations:

a) she only knows that the dice was rolled once;

b) she knows the result of the roll;

c) she observes the result of the roll but she is able to count only up to two.

7.20

7.21

7.22

7.23

8 Conditional expected value

8.1 We randomly uniformly select a point inside triangle $A = (6, 0)$, $B = (0, 2)$ and $O = (0, 0)$. Let (X, Y) be coordinates of this random point.

a) Find conditional expected values $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

b) Find conditional variances $\operatorname{Var}(Y \mid X)$ and $\operatorname{Var}(X \mid Y)$.

8.2 The pair of random variables X and Y has joint probability density

$$f(x, y) = \begin{cases} x + y, & \text{if } x \in [0; 1], y \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

a) Find the marginal densities $f_X(x)$ and $f_Y(y)$.
b) Find the conditional densities $f(x \mid y)$ and $f(y \mid x)$.
c) Find the conditional expected values $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.
d) Find the conditional variances $\mathrm{Var}(Y \mid X)$ and $\mathrm{Var}(X \mid Y)$.

8.3 The random variables X and Y are independend with Poisson distribution with rate $\lambda = 1$. Let $S = X + Y$.

a) Find conditional probabilities $\mathbb{P}(X = x \mid S = s)$ and $\mathbb{P}(Y = y \mid S = s)$.
b) Find conditional expected values $\mathbb{E}(X \mid S)$ and $\mathbb{E}(Y \mid S)$.
c) Find conditional variances $\mathrm{Var}(X \mid S)$ and $\mathrm{Var}(Y \mid S)$.
d) How the answers will change if $X \sim \mathrm{Pois}(\lambda_x)$ and $Y \sim \mathrm{Pois}(\lambda_y)$?

8.4 Let X and Y be independent and exponentially distributed with rate $\lambda = 1$ and $S = X + Y$.

a) Find conditional densities $f(x \mid s)$ and $f(y \mid s)$.
b) Find conditional expected values $\mathbb{E}(X \mid S)$ and $\mathbb{E}(Y \mid S)$.
c) Find conditional variances $\mathrm{Var}(X \mid S)$ and $\mathrm{Var}(Y \mid S)$.
d) Find $\mathrm{Cov}(X, Y \mid S)$ and $\mathrm{Corr}(X, Y \mid S)$.
e) How the answers will change if $X \sim \mathrm{Expo}(\lambda_x)$ and $Y \sim \mathrm{Expo}(\lambda_y)$?

8.5 The random variable X has Poisson distribution with rate $\lambda = 1$. The random variable Y has uniform distribution on $[1; 2]$. Random variables X and Y are independent.

Find $\mathbb{E}(XY \mid X)$, $\mathrm{Var}(XY + X^3 \mid X)$, $\mathrm{Cov}(X, Y \mid X)$, $\mathrm{Cov}(XY, X^2Y \mid X)$.

8.6 The random variables X_1 and X_2 are independent and normally distributed, $X_1 \sim \mathcal{N}(1; 1)$, $X_2 \sim \mathcal{N}(2; 2)$. I choose X_1 with probability 0.3 and X_2 with probability 0.7 without knowing their values.

Casino pays me the value Y that is equal to the chosen random variable.

Let the indicator I be equal to 1 if I choose X_1 and 0 otherwise.

a) Express Y in terms of X_1, X_2 and I.
b) Find $\mathbb{E}(Y \mid I)$, $\mathrm{Var}(Y \mid I)$.
c) Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

8.7 A Hedgehog in the fog starts in $(0, 0)$ at $t = 0$ and moves randomly with equal probabilities in four directions (north, south, east, west) by one unit every minute.

Let X_t and Y_t be his coordinates after t minutes and $S_t = X_t + Y_t$.

a) Find $\mathbb{E}(X_2 \mid S_2)$;
b) Find $\mathrm{Var}(X_2 \mid S_2)$.

8.8 Bonnie and Clyde start at the points $(5, 0)$ and $(-5, 0)$ of the plane. Each minute each of them simulteneously and independently makes one step in one of the four possible directions (south, north, east, west).

Each of them does n steps. Let X be the number of times they will be at the same point.

a) Estimate the probability $\mathbb{P}(X \geqslant 1)$ for $n = 50$ using $B = 10000$ simulations.

b) Estimate $\mathbb{E}(X)$ and $\text{Var}(X)$ for $n = 50$ using $B = 10000$ simulations.

c) Plot the estimated value of $\mathbb{E}(X)$ as a function of n for n from 1 to 200 using $B = 10000$ simulations.

8.9 Albert Nikolayevich Shiryaev randomly selects a natural number N from 1 to 7. Let Y be the remainder after division of N by 2 and X be the remainder after division of N by 3.

a) Write the joint probability table for (X, Y).

b) Find $\mathbb{E}(Y \mid X)$. Is it linear in X?

c) Find $\mathbb{E}(X \mid Y)$. Is it linear in Y?

d) Find $\mathbb{E}(\mathbb{E}(Y \mid X))$ and $\text{Var}(\mathbb{E}(Y \mid X))$.

e) Find $\text{Var}(Y \mid X)$.

f) Find $\mathbb{E}(\text{Var}(Y \mid X))$.

g) Find $\mathbb{E}(\text{Var}(Y \mid X)) + \text{Var}(\mathbb{E}(Y \mid X))$.

8.10 Albert Nikolayevich selects a random point uniformly inside a quadrilateral $ABCD$ where $A = (0, 0)$, $B = (0, 2)$, $C = (4, 4)$, $D = (4, 0)$.

a) Find $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

b) Find $\text{Var}(Y \mid X)$ and $\text{Var}(X \mid Y)$.

Hint: you may use the formula for the variance of uniform distribution :)

8.11 Albert Nikolayevich selects a random point (X, Y) with joint probability density

$$f(x, y) = \begin{cases} (3x^2 + 4y^3)/2, & \text{if } x \in [0; 1], y \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

a) For the random variable x find the marginal probability density function $f(x)$.

b) Find the conditional density $f(y \mid x)$.

c) Find the conditional expected value $\mathbb{E}(Y \mid X)$. Is it linear in X?

d) Find $\text{Var}(Y \mid X)$. Is it constant?

e) Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\text{Cov}(X, Y)$ and $\text{Var}(X)$.

f) Find the best linear approximation $Y^* = \beta_0 + \beta_1 X$ of the random variable Y.
Hint: Here you should minimize $\mathbb{E}((Y - Y^*)^2)$ with respect to the true constants β_0 and β_1.

8.12 Consider a fair dice. In the experiment we throw the dice until the first six appears.

a) Simulate $B = 100000$ experiments. For every experiment number i record the total number of throws, y_i, and the number of even faces appeared, x_i.

b) For all values of x where you have more than 100 records estimate $\hat{\mu}(x) = \hat{\mathbb{E}}(y_i \mid x_i = x)$ and $\hat{v}(x) = \widehat{\text{Var}}(y_i \mid x_i = x)$.

c) Explain intuitively why $\hat{\mu}(0)$ is less than 3.

d) Randomly select 100 experiments out of all B experiments. Draw the scatter plot (x_i, y_i) for randomly selected experiments. Add the line $\hat{\mu}(x)$ with bands $\hat{\mu}(x) \pm 2\sqrt{\hat{v}(x)}$ to the scatter plot.

e) Is it reasonable to assume that $\hat{\mu}(x)$ is linear?

f) Is it reasonable to assume that $\hat{v}(x)$ is constant?

No formal tests are required for the last two questions, graphical analysis is sufficient.

8.13 The joint distribution of vector (X, Y) is given by $\mathbb{P}(X = i, Y = j) = 0.1$ for $1 \leqslant i \leqslant j \leqslant 4$.

Find $\mathbb{E}(Y \mid X)$.

8.14 The random variable X is exponentially distributed with parameter λ. The random variable Y is exponentially distributed with parameter $1/X$.

Find $\mathbb{E}(Y \mid X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

8.15 In the first bag there balls numbered from 0 to 9, in the second bag there are balls numbered from 1 to 10. One ball was selected from the first bag and one ball from the second one. You will select at random one ball from these two and you will know only its number. Let's denote its number by X and the number of the other of the two balls by Y.

Find $\mathbb{E}(Y \mid X)$.

8.16 The random variable X is uniformly distributed on $[0; a]$. Conditionally of X the random variable Y is uniformly distributed on $[0; X]$.

Find $\mathbb{E}(Y \mid X)$, $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

8.17 Random variables X and Y are independent with finite expected values and finite variances.

Elon Musk knows that $\mathbb{E}(X \mid X + Y) = X + Y$.

a) Find $\mathbb{E}(Y \mid X + Y)$ and $\mathbb{E}(Y)$.

b) Find $\mathrm{Var}(Y)$ and $\mathrm{Var}(X \mid X + Y)$.

c) What is the distribution of the random variable Y?

8.18 Let's sum up independent identically distributed random variables $X_t \sim \mathrm{Unif}[0; 1]$. Denote the cumulative sum by $S_t = X_1 + X_2 + \cdots + X_t$.

a) Find $\mathbb{E}(S_9 \mid S_8)$, $\mathbb{E}(S_9 \mid S_{10})$, $\mathbb{E}(S_9 \mid S_8, S_{10})$, $\mathbb{E}(S_9 \mid S_5, S_{10}, S_{20})$.

b) Find $\mathrm{Var}(S_9 \mid S_8)$.

8.19 At time moment $t = 0$ in the casino there are countably many players with perfect memory. Let's number them as Miss First, Mister Second, etc.

Time is discrete. Random variables X_t are independent and take values $+1$ or -1 with equal probabilities. At each moment of time $t > 0$ everybode gets X_t roubles and than the player number t leaves the casiono.

The cumulative sum $S_t = X_1 + \cdots + X_t$ reaches its first local maximum at the random time T. At time $T + 1$ the dealer calls his friend Black Jack and says «It's time!» They have agreed beforehand on the call time.

Black Jack chases the player number T and steals all his information before the police can intervent. Let's describe the information of Black Jack by sigma-algebra $\mathcal{F}_J$ and the information of every player t at the last moment in casino by $\mathcal{F}_t$.

a) Which sigma-algebras contain the event $\{T = 10\}$?

b) Provide an example of two events from $\mathcal{F}_J$ that do not enter in neither $\mathcal{F}_t$.

c) Find conditional expected values $\mathbb{E}(T \mid \mathcal{F}_J)$, $\mathbb{E}(X_T \mid \mathcal{F}_J)$, $\mathbb{E}(X_{T+1} \mid \mathcal{F}_J)$.

d) Find conditional expected values $\mathbb{E}(S_{T-1} \mid \mathcal{F}_J)$, $\mathbb{E}(S_T \mid \mathcal{F}_J)$, $\mathbb{E}(S_{T+1} \mid \mathcal{F}_J)$, $\mathbb{E}(S_{T+2} \mid \mathcal{F}_J)$.

Let's define Y_{T-k} as

$$Y_{T-k} = \begin{cases} X_{T-k}, \text{ if } T-k > 0; \\ 0, \text{ otherwise.} \end{cases}$$

e) Find $\mathbb{E}(Y_{T-10} \mid \mathcal{F}_J)$.

f) Find conditional expected values $\mathbb{E}(X_T \mid \mathcal{F}_{10})$, $\mathbb{E}(X_{T+1} \mid \mathcal{F}_{10})$.

g) Find $\mathbb{E}(T \mid \mathcal{F}_{10})$ and $\mathbb{E}(S_T \mid \mathcal{F}_{10})$.

8.20 Bad police officers operate in groups of 1, 2 or 3 people with probabilities 0.5, 0.2 and 0.3. If you cross the road in the wrong place, they will catch you and demand a bribe X of 1, 5 or 10 thousand rubles respectively.

For each of the following cases write down the σ-algebra $\mathcal{F}$ that models your information and calculate $\mathbb{E}(X \mid \mathcal{F})$.

a) you can see how many officers are going to stop you;

b) they are sitting in the car and you don't know their number;

c) it is dark and you can only say if it is one policeman or more than one.

8.21 The joint distribution of the random vector (X, Y) is given by its probability density function

$$f(x,y) = \begin{cases} ce^{x-y}, \text{ for } 0 \leqslant x, y \leqslant 1 \\ 0, \text{ otherwise} \end{cases}$$

where c is a normalization constant.

Find $\mathbb{E}(X \mid Y)$, $\operatorname{Var}(X \mid Y)$.

8.22 Donald Trump throws a fair coin infinitely many times. Let X be the number of a toss when the first Tail appears. Let Y be the result of the second toss, 1 for Tail, 0 for Head.

a) Find $\mathbb{E}(X \mid Y)$, $\mathbb{E}(Y \mid X)$.

b) Find $\operatorname{Var}(X \mid Y)$, $\operatorname{Var}(Y \mid X)$.

8.23 Let $\Omega = \{a, b, c\}$ and the distribution of Y is given by the table

w	a	b	c
$Y(w)$	5	4	3
$\mathbb{P}(\{w\})$	0.2	0.3	0.5

The sigma-algebra $\mathcal{F}$ is defined by $\mathcal{F} = \sigma(A)$ where $A = \{Y \neq 4\}$, and $\mathcal{H} = \sigma(Y)$.

a) Find $\mathbb{E}(Y \mid \mathcal{F})$, $\operatorname{Var}(Y \mid \mathcal{F})$, $\mathbb{P}(Y = 3 \mid \mathcal{F})$.

b) Find $\mathbb{E}(Y \mid \mathcal{H})$, $\operatorname{Var}(Y \mid \mathcal{H})$, $\mathbb{P}(Y = 3 \mid \mathcal{H})$.

8.24 Masha picked up 100 mushrooms. There are three types of mushrooms, l-mushrooms, m-mushrooms and r-mushrooms. The types are independent and may by encountered with probabilities 0.2, 0.3 and 0.5 correspondingly. Let's denote the total number of picked up mushrooms of each type by L, M and R.

a) Find $\mathbb{E}(R+L \mid M)$, $\mathbb{E}(M \mid R+L)$, $\mathrm{Var}(R+L \mid M)$.

b) Find $\mathbb{E}(R \mid L)$ and $\mathrm{Var}(R \mid L)$.

c) Find $\mathbb{E}(R+L \mid L+M)$.

d) Find $\mathbb{P}(\mathbb{E}(R \mid L)=0)$ and $\mathbb{P}(R=0 \mid L)$.

e) Find $\mathbb{E}\left(\left(\frac{0.3}{0.8}\right)^{100-L}\right)$.

8.25 At stage one I choose the random value X uniformly on $[1;2]$. At stage two I choose the value of Y randomly exponentially with rate X conditionally on X.

a) Find $\mathbb{E}(Y \mid X)$, $\mathrm{Var}(Y \mid X)$.

b) Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

c) Find $\mathbb{E}(X \mid Y)$ and $\mathrm{Var}(X \mid Y)$.

8.26 We select a point (X,Y) randomly uniformly inside the square with vertices $(0,0)$, $(4,0)$, $(4,4)$, and $(0,4)$.

a) [2 + 3] Find $\mathbb{E}(Y \mid Y-X)$ and $\mathrm{Var}(Y \mid Y-X)$.

b) [2] Consider the σ-algebra $\mathcal{F}=\sigma(Y-X)$. Provide an example of two non-trivial events from $\mathcal{F}$ and two events not from $\mathcal{F}$.

c) [3] Find $\mathrm{Cov}(X,Y \mid Y-X)$.

Hint: by definition $\mathrm{Cov}(X,Y \mid \mathcal{F})=\mathbb{E}(XY \mid \mathcal{F})-\mathbb{E}(X \mid \mathcal{F})\,\mathbb{E}(Y \mid \mathcal{F})$.

Source: Stochastic Processes December 2025 exam.

8.27 The joint distribution of X and Y is given by the table:

	$Y=-1$	$Y=0$	$Y=1$
$X=-1$	$1/8$	$1/16$	$1/8$
$X=0$	$1/16$	$1/4$	$1/16$
$X=1$	$1/8$	$1/16$	$1/8$

Three students are building statistical models with different information sets:

- Roman Bokhyan: $\mathcal{F}_1=\sigma(X,Y)$, the σ-algebra generated by X and Y.
- Mikhail N: $\mathcal{F}_2=\sigma(X+Y, X-Y)$.
- Andrey L: $\mathcal{F}_3=\sigma(X^2+Y^2)$.

a) [3] For each pair of σ-algebras above determine whether one is contained in the other, they are equal, or neither. Justify your claims.

b) [2] Andrey and Mikhail decide to share their information. Compute the number of elements in the smallest σ-algebra containing both $\mathcal{F}_2$ and $\mathcal{F}_3$.

c) [5] Help Andrey and compute $\mathbb{E}(X \mid \mathcal{F}_3)$, $\mathrm{Var}(X \mid \mathcal{F}_3)$.

Source: Stochastic Processes December 2025 exam.

8.28

8.29

8.30

9 Martingales

- *Natural filtration* of a process (X_n) is given by $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.
- A process (X_n) is a *martingale* if $\mathbb{E}(X_{n+k} \mid X_n, X_{n-1}, \ldots, X_1) = X_n$ for all $k \geqslant 1$.
- A process (X_n) is *adapted* to filtration $(\mathcal{F}_n)$ if every random variable X_n is measurable wrt to sigma-algebra $\mathcal{F}_n$.
- A process (X_n) is a *martingale wrt to filtration* $(\mathcal{F}_n)$ if $\mathbb{E}(X_{n+k} \mid \mathcal{F}_n) = X_n$.

9.1 Consider the sequence (X_t) of independent identically distributed random variables with mean $\mathbb{E}(X_t) = 2$ and variance $\mathrm{Var}(X_t) = 3$. Let's work with natural filtration $\mathcal{F}_t = \sigma(X_1, X_2, \ldots, X_t)$.

On the base of (X_t) let's create more sequences: $S_t = X_1 + X_2 + \cdots + X_t$, $W_t = S_t - 2t$ and $Y_t = W_t^2 - 3t$.

a) Is (X_t) a martingale with respect to $(\mathcal{F}_t)$?

b) Is (S_t) a martingale with respect to $(\mathcal{F}_t)$?

c) Is (W_t) a martingale with respect to $(\mathcal{F}_t)$?

d) Is (Y_t) a martingale with respect to $(\mathcal{F}_t)$?

e) Is (W_t) a martingale with respect to $(\mathcal{F}_{t-1})$?

f) Is (W_t) a martingale with respect to $(\mathcal{F}_{t+1})$?

9.2 Vasiliy has found three non-random infinite sequences in his garage: $a_n = n$, $b_n = -n$ and $c_n = 0$. He randomly selects one of these sequences with equal probabilities and hence obtain a sequence of random variables (X_n).

a) What is the distribution of X_7?

b) Find $\mathbb{E}(X_n)$ and $\mathrm{Var}(X_n)$.

c) Is (X_n) a Markov chain?

d) Is (X_n) a martingale?

e) Explicitely find the σ-algebra $\sigma(X_1, X_2, X_3, \ldots, X_{1000})$.

f) Find the probability that limit of (X_n) exists.

g) Does $\operatorname{plim} X_n$ exist?

9.3 Consider the sequence (X_t) of independent identically distributed random variables that take values 0 or 1 with equal probabilities. Let's work with natural filtration $\mathcal{F}_t = \sigma(X_1, X_2, \ldots, X_t)$.

On the base of (X_t) let's create more sequences: $S_t = X_1 + X_2 + \cdots + X_t$, $W_t = S_t - at$, $M_t = \exp(bS_t)$.

a) Is (X_t) a martingale?

b) Is (S_t) a martingale?

c) For which values of a the process (W_t) is a martingale?

d) For which values of b the process (M_t) is a martingale?

9.4 Consider a well-mixed standard deck of 52 cards. James Bond in an elegant outfit[1] opens cards one by one. Let the sigma-algebra $(\mathcal{F}_n)$ model his information and (X_n) be the proportion of Queens in the closed part of the deck after opening n cards.

a) Find the marginal distribution of X_0, X_1 and X_{51}.

b) Find the joint distribution of X_{50} and X_{51}.

c) Is (X_n) a martingale with respect to $(\mathcal{F}_n)$?

9.5 If possible create a martingale (X_n) such that simulteneously $\mathbb{P}(X_n = 0 \text{ infinitely often}) = 1$ and $\mathbb{P}(X_n = 1 \text{ infinitely often}) = 1$.

9.6 At time $t = 0$ there is one black and one white ball in the vase. At each moment of time we take out randomly one ball from the vase and put back two balls of the same color. Let (W_t) be the proportion of white balls in the vase after t extractions and (Q_t) be the number of times when white ball was extracted.

a) What is the distribution of W_1? Of W_2?

b) Is (W_t) a martingale?

c) Consider a fixed parameter $p \in (0;1)$ and the process $M_t = (t+1)C_t^{Q_t} p^{Q_t}(1-p)^{t-Q_t}$. Is (M_t) a martingale?

d) What is the limiting distribution of (W_t)?

9.7 Consider non-random sequence of numbers (a_n). How can this sequence be a martingale?

9.8 Let (M_n) be a martingale and $a < b < c < d$.

a) Find covariance $\operatorname{Cov}(M_d - M_c, M_b - M_a)$.

b) Are $(M_d - M_c)$ and $(M_b - M_a)$ independent?

9.9 Let (M_t) be a process adapted to filtration $(\mathcal{F}_t)$.

Is it true that in discrete time conditions

$$\mathbb{E}(M_{t+1} \mid \mathcal{F}_t) = M_t$$

and

$$\mathbb{E}(M_{t+k} \mid \mathcal{F}_t) = M_t \text{ for all } k \geqslant 1$$

are equivalent?

9.10 Initial wealth of a player is equal to $W_0 = 1$. At each moment of time she can bet any proportion of her wellfare on the toss of a coin. If she guesses wrong she loses her bet. If she guesses right she gets profit equal to hear bet. The coin is not fair lands on head with probability 0.8.

a) Find the bet that maximises one period log interest rate $\mathbb{E}(\ln(W_{t+1}/W_t) \mid \mathcal{F}_t)$.

b) Assume that the player maximises one period log interest rate every time. Find a constant a such that $\ln W_n - an$ is a martingale.

9.11 For each case provide an example of a process.

a) (X_n) is a Markov chain and a martingale.

[1] Sponsors are wellcome to contact us for product placement!

b) (X_n) is a Markov chain but not a martingale.

c) (X_n) is a martingale but not a Markov chain.

d) (X_n) is neither a Markov chain nor a martingale.

9.12 Let (X_n) be a simple symmetric random walk and $(\mathcal{F}_n)$ its natural filtration.

Find a deterministic (non-random) sequence a_n such that $M_n = X_n^3 + a_n X_n$ is a martingale with respect to $(\mathcal{F}_n)$.

9.13 The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

a) [3] Find $\mathbb{E}(X_3 \mid X_1, X_2)$, $\mathbb{E}(X_3 \mid X_1 + X_3)$.

b) [3] Find $\text{Var}(X_3 \mid X_1, X_2, X_3)$, $\text{Var}(X_3 \mid X_1 + X_3)$.

c) [4] Let Y_n be equal to $\mathbb{E}(X_1 + \ldots + X_{2022} \mid X_1, X_2, \ldots, X_n)$.
Is the process Y_1, Y_2, ..., Y_{2022} a martingale?

9.14 Let $S_0 = 0$, $S_t = X_1 + X_2 + \ldots + X_t$. The increments X_t are independent and identically distributed:

x	-1	0	1
$\mathbb{P}(X_t = x)$	0.2	0.2	0.6

a) If possible find all constants a such that $M_t = S_t + at$ is a martingale.

b) If possible find all constants b such that $R_t = b^{S_t}$ is a martingale.

9.15 Let X_i be independent identically distributed with $\mathbb{P}(X_i = 1) = 0.9$, $\mathbb{P}(X_i = -1) = 0.1$.

Find all constants a and b such that $Y_t = a \exp\left(b \sum_{i=1}^{t} X_i\right)$ is a martingale.

9.16 The random variables (Z_t) are independent identically distributed with moment generating function given by $M_Z(u) = 1/(1-5u)^3$.

We define X_t as $X_t = \exp(Z_1 + 2Z_2 + 3Z_3 + \ldots + tZ_t)$ with $X_0 = 0$.

If possible find a martingale of the form $Y_t = h(t)X_t$ where $h()$ is a non-random function.

9.17 The process (Z_t) in discrete time is called *stationary* if it has constant expected value and constant covariances γ_k that do not depend on t.

$$\begin{cases} \mathbb{E}(Z_t) = \mu; \\ \text{Cov}(Z_t, Z_t) = \gamma_0; \\ \text{Cov}(Z_t, Z_{t+1}) = \gamma_1; \\ \text{Cov}(Z_t, Z_{t+2}) = \gamma_2; \\ \ldots \end{cases}$$

a) If possible provide an example of a martingale that is not stationary.

b) If possible provide an example of a stationary process that is not a martingale.

9.18 The population starts with one microbe Eve. So the size of the initial generations is $G_0 = 1$. After one minute every microbe either dies with probability 0.2, remains alive with probability 0.5 or splits in two copies with probability 0.3. Let G_n be the size of microbe population after n minutes.

a) Draw a pretty picture of Eve :)

b) Find the distribution of G_2.

c) Find a constant a such that $M_n = G_n/a^n$ is a martingale.

d) Let D be the event of eventual death of the microbe civilization. Check whether the process $K_n = \mathbb{E}(I_D \mid G_n, G_{n-1}, \ldots, G_0)$ is martingale. Here I_D is the indicator of the event D.

e) Using first step analysis find $\mathbb{P}(D)$.
Hint: you may obtain a quadratic equation for $\mathbb{P}(D)$, the smallest root is your friend :)

9.19 The random variables X_n are independent and take values $+1$ with probability 0.7 or 2 with probability 0.3. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the cumulative sum.

a) Find the constant a such that $M_n = S_n - an$ is a martingale.

b) Find all constants b such that $K_n = \exp(bS_n)$ is a martingale.

9.20

9.1 Stopping time

Doob's optional stopping time theorem. If (M_t) is a martingale and τ is a stopping time then $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ provided at least one of the following conditions hold:

- $\mathbb{P}(\tau < \infty) = 1$, the stopped process $X_t = M_{t\wedge\tau}$ is bounded by some constant.
- $\mathbb{E}(\tau) < \infty$, the process $D_t = \mathbb{E}(M_{(t+1)\wedge\tau} - M_{t\wedge\tau} \mid \mathcal{F}_t)$ is bounded by some constant.
- $\mathbb{P}(\tau < \infty) = 1$, the process M_t is uniformly integrable,

$$\lim_{a\to\infty} \sup_t \mathbb{E}(M_t \cdot I(M_t > a)) = 0.$$

9.21 A gambler wins or looses one rouble in each round in the casino with equal probabilities and independently. Let's denote the result of the n-th round by X_n.

The gambler starts with initial fortune $S_0 = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the wealth at time n. She can have negative balance up to $-a$ roubles.

She quits the casino when she either reaches the target of $+b$ roubles or the credit limit of $-a$ roubles.

a) Is (S_n) a martingale?

b) Use optional-stopping theorem to find probabilities of reaching $+b$ or $-a$.

c) Is $M_n = S_n^2 - n$ a martingale?

d) Find the expected number of rounds before she will stop gambling.

9.22 A gambler wins or looses one rouble in each round in the casino with unequal probabilities and independently. Let's denote the result of the n-th round by X_n, $\mathbb{P}(X_n = 1) = p$, $\mathbb{P}(X_n = -1) = q = 1 - p$.

The gambler starts with initial fortune $S_0 = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ be the wealth at time n. She can have negative balance up to $-a$ roubles.

She quits the casino when she either reaches the target of $+b$ roubles or the credit limit of $-a$ roubles.

a) Is (S_n) a martingale?

b) Is $K_n = (q/p)^{S_t}$ a martingale?

c) Use optional-stopping theorem to find probabilities of reaching $+b$ or $-a$.

d) Is $M_n = S_n - (p-q)n$ a martingale?

e) Find the expected number of rounds before she will stop gambling.

9.23 Famous «ABRACADABRA» problem.

A monkey types randomly letters on a typewriter choosing each time one of the 26 letters with equal probabilities. Let T be the number of keypresses required to write the word «ABRACADABRA» for the first time.

a) Organise a casino to calculate $\mathbb{E}(T)$.

b) Organise a casino to calculate $\mathbb{E}(T^2)$ and hence $\mathrm{Var}(T)$.

9.24 To survive vampire Boris needs to bite 70 talented students.

These 70 talented students have formed a secret group. They have written their emails on small pieces of paper and have randomly distributed these pieces among them. Each student has exactly one piece of paper with an email[2].

Initially vampire Boris knows contacts of just two persons from the group. Today he will contact them, drink their blood and get the emails they have. Then vampire Boris will contact new victims and so on.

a) For $t \geqslant 1$ consider the process M_t, the proportion of non bitten students after the day t. Is this process a martingale?

b) Using martingale stopping theorem or otherwise find the probability that vampire Boris will bite all 70 students.

9.25 Let $S_0 = 0$. Each hour independently of previous history the process S_n increases by 1 with probability 0.25, decrease by 1 with probability 0.25 or does not change. Consider the first moment of time T when $|S_T| = 4$.

a) [3] If possible find α such that $Q_t = S_t^2 + \alpha t$ is a martingale.

b) [3] If possible find β such that $Q_t = \exp(S_t + \beta t)$ is a martingale.

c) [4] Find $\mathbb{E}(T)$.

Source: Stochastic Processes December 2025 exam.

9.26

9.26 The process S_n models the «accumulated wrath of the heavens». Let $S_0 = 0$. Each hour independently of previous history S_n increases by 1 with probability $1/3$, decrease by 1 with probability $1/6$ or does not change.

The witches fall from the sky when S_n first reaches the level of 4. Let $T = \min\{n : S_n = 4\}$.

a) [4] For which $\alpha \neq 1$ is the process $M_n = \alpha^{S_n}$ a martingale?

b) [2] Check whether $Q_n = S_n - n/6$ is a martingale.

[2] The group is so secret that it is possible that a student has his own email on his piece of paper

c) [4] Find $\mathbb{E}(T)$.

Source: Stochastic Processes December 2025 exam.

9.27

9.28

10 Poisson process

The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent: If $t_1 < t_2 < t_3 < \cdots < t_k$ then random increments $N(t_2) - N(t_1)$, $N(t_3) - N(t-2)$, ... are independent.
- Increments have Poisson distribution:

$$N_b - N_a \sim \text{Pois}(\lambda(b-a));$$

Alternative definition. The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent;
- Increments are stationary:
  The distribution of $N_b - N_a$ depends only on $(b-a)$.
- Probability of observing two or more points in a small interval is negligible:

$$\mathbb{P}(N_{t+\Delta} - N_t > 1) = o(\Delta).$$

- Probability of observing one point is approximately proportional to the length of time interval:

$$\mathbb{P}(N_{t+\Delta} - N_t = 1) = \lambda + \Delta o(\Delta).$$

10.1 Two cashiers Alice and Bob simulteneously started to service their clients. The service times X_a and X_b are independent and exponentially distributed with rates $\lambda_a = 1$ and $\lambda_b = 2$.

a) Find the probability $\mathbb{P}(X_a < X_b)$.

b) Find the density of $S = X_a + X_b$.

c) Find the density of $L = \min\{X_a, X_b\}$.

d) Find the density of $R = \max\{X_a, X_b\}$.

e) Solve all the previous points for general rates λ_a and λ_b.

10.2 Let X_t and Y_t be two independent Poisson processes. Is it true that $S_t = X_t + Y_t$ is also a Poisson process?

10.3 Hedgehogs are scattered in a big forest according Poisson process with rate $\lambda = 1$ per 100 squared meters.

What should be the edge of a square such that the probability of finding a hedgehog there is 0.7?

10.4 Let N_t be a Poisson process with rate λ.

a) Is the process $A_t = N_t - \lambda t$ a martingale?

b) Is the process $B_t = A_t^2 - \lambda t$ a martingale?

10.5 Students arrive in the Grusha café according to the Poisson arrival process (X_t) with constant rate λ. The probability of no visitors during 5 minutes is 0.05.

a) Find the value of λ.

b) Find the variance and expected number of arrivals between 5 pm and 8 pm.

c) What is the probability of exactly 5 arrivals between 5 pm and 8 pm?

10.6 Masha receives on average 10 sms per minute. Sms arrival is well described by the Poisson process.

a) What is the probability that Masha receives exactly 10 sms in the next 40 seconds?

b) Masha just received an sms. What is the probability that she will wait more that 2.5 seconds before the next one?

c) Find the covariance between the number of sms in the first 3 minutes and the number of sms in the first 10 minutes.

10.7 Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes. Let Y_t be the number of taxis that will arrive between 0 and t minutes.

a) Sketch the expected value of Y_t as a function of t.

b) Sketch the probability $\mathbb{P}(Y_t = Y_{60})$ as a function of t.

10.8 A company gets fines for non-removal of quadrobics video content. What is the probability that the total amount will exceed two undecillion roubles in 1000 days for each case?

a) Fines arrive according to Poisson process with rate 1 fine per day and each fine has the size 10^{33} roubles. Fines are summing up without additional penalties.

b) Initial fine is 10^5 roubles but it doubles according to Poisson process with rate 1 doubling per 10 days.

10.9 Let (N_t) be a Poisson process with intensity rate 2. Consider the vector $Y = (N_1, N_2, N_{10})$.

Find $\mathbb{E}(Y)$ and covariance matrix $\text{Var}(Y)$.

10.10 Customers order coffee according to Poisson process with rate 1 cup per minute. The owner will close the shop if no one orders a coffee in 7 minutes.

Let X be the closure time.

Find $\mathbb{E}(X)$ and $\text{Var}(X)$.

10.11 The arrival of buses at a given stop follows Poisson process with rate 3. The arrival of taxis at same stop follows independent Poisson process with rate 5.

a) What is the probability that two or more taxis will arrive before next bus?

b) What is the probability that exactly two taxis will arrive before next bus?

10.12 Customers order coffee according to Poisson process with rate 1 cup per minute. Let N_t be the number of orders up to time t.

Find the probability $\mathbb{P}(N_t \text{ is even})$.

10.13 Arrivals of buses at a stop follow Poisson point process with rate $\lambda = 2$ buses per hour.

a) Find the probability that you will wait for the bus more than 30 minutes.

b) Find the probability to wait for the bus more than 30 minutes, if you have already waited 10 minutes.

10.14 When light bulb burns out it's immediately replaced by a new similar one. The number or burn-outs during any interval of t hours has Poisson distribution with parameter λt.

Find the distribution of the first burn-out time Y_1. Prove your result.

10.15 Prove that two definitions of Poisson process are equivalent.

Definition A. The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent: If $t_1 < t_2 < t_3 < \cdots < t_k$ then random increments $N(t_2) - N(t_1)$, $N(t_3) - N(t-2)$, ... are independent.
- Increments have Poisson distribution:

$$N_b - N_a \sim \text{Pois}(\lambda(b-a));$$

Definition B. The process (N_t) is called *Poisson process* with intensity λ if

- $N_0 = 0$;
- Increments are independent;
- Increments are stationary:
  The distribution of $N_b - N_a$ depends only on $(b-a)$.
- Probability of observing two or more points in a small interval is negligible:

$$\mathbb{P}(N_{t+\Delta} - N_t > 1) = o(\Delta).$$

- Probability of observing one point is approximately proportional to the length of time interval:

$$\mathbb{P}(N_{t+\Delta} - N_t = 1) = \lambda\Delta + o(\Delta).$$

10.16 Class teacher solves three exercises per class on average. Exercise solution times are independent exponentially distributed with the same rate. Class duration is two academic hours.

a) Find the probability that only one exercise will be solved during the first academic hour.

b) Find the probability that one exercise will be completed during the first first academic hour and the second exercise will be completed during the second hour.

10.17 Jeanne Antoinette Poisson receives letters from Diderot and d'Alembert. Each of them independently may be in a good mood or bad mood equiprobably. Every person sends letters to Jeanne Antoinette Poisson according to an independent Poisson process with intensity $\lambda_{\text{good}} = 2$ or $\lambda_{\text{bad}} = 1$ [letters per week] depending on his mood.

Let (M_t) be the number of letters received by Madame de Pompadour in t weeks.

a) [2 + 3] Find $\mathbb{E}(M_t)$ and $\text{Var}(M_t)$.

b) [2] Is (M_t) a Poisson process? If so, find its intensity.

c) [3] Let T_1 be the time of the first letter. Find its cumulative distribution function $F(t)$.

Source: Stochastic Processes December 2025 exam.

10.18 The Legendary November Midterm Grader grades tests according to Poisson point process with variable intensity rate. The first 5 weeks the rate is equal to $\lambda_1 = 1$ test per week. Later the rate increases to $\lambda_2 = 100$ tests per week. Let X_t be the number of tests graded after t weeks.

a) [2] Sketch the curve $v(t) = \text{Var}(X_t)$.

b) [4] Using normal approximation find the probability that LNMG will check not less than 120 works at time $t = 6$.

c) [4] Now $t = 4.5$ weeks have already passed. Find the expected time for the next test checked.

Note: in point (b) you may use the standard normal cdf $F()$ in your answer.

Source: Stochastic Processes December 2025 exam.

10.19

10.20

10.1 Poisson approximation

If $X \sim \text{Bin}(n, p)$, $Y \sim \text{Pois}(\lambda)$ with $\lambda = np$ and $A \subset \mathbb{R}$ then

$$|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \leqslant \min\{p, np^2\}$$

10.21 Random variable X has binomial distribution $\text{Bin}(n, p = \lambda/n)$.

a) Find $\mathbb{E}(X)$.

b) Find $\lim_{n\to\infty} \text{Var}(X)$.

c) Find $\lim_{n\to\infty} \mathbb{P}(X = 0)$, $\lim_{n\to\infty} \mathbb{P}(X = 1)$.

d) Find $\lim_{n\to\infty} \mathbb{P}(X = k)$.

10.22 A manufacturer produces light-bulbs that are packed into boxes of 100. Quality control studies indicate that 0.5% of the light-bulbs produced are defective.

a) Estimate the probability of no defective light-bulbs in a box using Poisson distribution.

b) Estimate the probability of two or more defective light-bulbs in a box using Poisson distribution.

c) Provide bounds for the estimation error in points (a) and (b).

10.23 The home assignment has three exercises. Alice makes no errors in an exercise with probability 0.9 and one arithmetic error with probability 0.1. The errors in different exercises are independent.

a) What is the probability that Alice will make exactly two errors in the home assignment?

b) What is the expected number of errors in Alice home assignment?

The number of errors Bob will make in the home assignment follows Poisson distribution. The expected number of errors per home assignment for Bob and Alice are equal.

c) What is the rate of Bob's Poisson distribution?

d) What is the probability that Bob will make exactly two errors?

e) Solve points (a)-(d) if Alice and Bob write an online test that consists of 40 questions with Alice error probability 0.01 for each question.

10.24 Clients enter the Café according to Poisson process (X_t) with intensity $\lambda = 10$ clients per hour. Let (Y_i) be the inter-arrival times.

a) What is the distribution of Y_1?

b) What is the distribution of Y_1 given that $X_1 = 1$?

c) What is the distribution of Y_1 given that $X_t = 1$?

d) What is the distribution of Y_5 given that $X_1 = 12$?

e) What is the density function of Y_5 given that $X_t = 12$?

10.25

10.26

10.27

11 Wiener Process

11.1 Consider a Wiener process (W_t).

a) Find $\mathbb{E}(W_5)$ and $\operatorname{Var}(W_5)$.

b) Find $\mathbb{E}(W_t)$ and $\operatorname{Var}(W_t)$.

c) Find $\mathbb{E}(W_5 \mid W_3)$ and $\operatorname{Var}(W_5 \mid W_3)$.

d) Find $\mathbb{E}(W_t \mid W_s)$ and $\operatorname{Var}(W_t \mid W_s)$ for $s \leqslant t$.

e) Find $\operatorname{Cov}(W_5, W_3)$ and $\operatorname{Cov}(W_s, W_t)$.

f) Find $\operatorname{Cov}(W_5, W_3 \mid W_2)$ and $\operatorname{Cov}(W_s, W_t \mid W_r)$ for $r \leqslant s \leqslant t$.

g) Find $\mathbb{E}(W_5 - 2W_3)$ and $\operatorname{Var}(W_5 - 2W_3)$.

h) Find $\mathbb{P}(W_5 - 2W_3 > 0)$ and $\mathbb{P}(W_5 - 2W_3 > 1)$ in terms of standard normal cumulative distribution function.

i) Find $\mathbb{E}(W_5 - 2W_3 \mid W_2 = 1)$ and $\operatorname{Var}(W_5 - 2W_3 \mid W_2 = 1)$.

j) Find $\mathbb{P}(W_5 - 2W_3 > 1 \mid W_2 = 1)$ in terms of standard normal cumulative distribution function.

k) What is the distribution of random vector $R = (W_1, W_3, W_5)$?

l) Find the value of a such that $W_3 = aW_5 + U$ where U and W_5 are independent.

m) Find $\mathbb{E}(W_3 \mid W_5)$ and $\text{Var}(W_3 \mid W_5)$.

11.2 Consider a Wiener process (W_t).

a) [4] Let $Y_t = tW_{2t}$. What is the distribution of $Y_t - Y_s$ for $t \geqslant s$? Is Y_t a Wiener process?

b) [6] Find a constant α such that $M_t = W_t^3 + \alpha t W_t$ is a martingale.

11.3 Consider a Wiener process (W_t). For $r < s < t < u$ find the following expected values

a) $\mathbb{E}((W_u - W_t)^2 (W_s - W_r)^2)$;

b) $\mathbb{E}((W_u - W_s)(W_t - W_r))$;

c) $\mathbb{E}((W_t - W_r)(W_s - W_r)^2)$;

d) $\mathbb{E}(W_r W_s W_t)$;

e) $\mathbb{E}(W_r W_s W_t \mid W_s)$;

11.4 Here (W_t) is a Wiener process.

a) Find $\mathbb{E}(W_5 W_4 \mid W_4)$, $\text{Var}(W_5 W_4 \mid W_4)$.

b) Find covariance $\text{Cov}(W_4 W_5, W_5 W_6)$.

11.5 For Wiener process (W_t) find $\mathbb{E}(W_1 W_2 W_3)$ and $\mathbb{E}(W_2 W_3 \mid W_1)$.

11.6 Let (W_t) be the Wiener process.

a) Is (W_t) a martingale?

b) Is $Q_t = W_t^2 - t$ a martingale?

11.7 Let (W_t) be a Wiener process and $Y_t = W_{4t}/2$.

a) Find Y_0.

b) What is the distibution of $Y_t - Y_s$ for $s \leqslant t$?

c) Doest Y_t has independent increments?

d) Is the trajectory of (Y_t) almost surely continous?

e) Is (Y_t) a Wiener process?

f) Complete the statement of the theorem: «If (W_t) is a Wiener process, then $Y_t = W_{a^2 t}/\ldots$ is …».

11.8 Let (W_t) be a Wiener process and

$$Y_t = \begin{cases} 0, & \text{if } t = 0; t \cdot W_{1/t}, & \text{if } t > 0. \end{cases}$$

a) Find Y_0.

b) What is the distibution of $Y_t - Y_s$ for $s \leqslant t$?

c) Doest Y_t has independent increments?

d) Is the trajectory of (Y_t) almost surely continous for $t > 0$?

e) Is the trajectory of (Y_t) almost surely continous at $t = 0$?

f) Is (Y_t) a Wiener process?

g) Complete the statement of the theorem: «If (W_t) is a Wiener process, then $Y_t = \begin{cases}0, & \text{if } t = 0; t \cdot W_{1/t}, & \text{i}\end{cases}$ is ...».

11.9 Let $Y_t = W_t^3 - tW_t^4$.

Find $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.

11.10 Consider the standard Wiener process $(W_t)_{t\geqslant 0}$.

a) [4] Find $\mathrm{Corr}(W_4 - W_2, W_3 - W_1)$.
b) [6] Find $\mathbb{E}(W_3 - W_1 \mid W_4 - W_2)$ and $\mathrm{Var}(W_3 - W_1 \mid W_4 - W_2)$.

Source: Stochastic Processes December 2025 exam.

11.11 Consider the standard Wiener process $(W_t)_{t\geqslant 0}$.

a) [2 + 3] Find $\mathbb{E}(W_3^4)$, $\mathbb{E}(W_3^4 \mid W_2 = 10)$.
b) [5] Find $\mathrm{Cov}(W_1W_2, W_5W_6)$.

Source: Stochastic Processes December 2025 exam.

11.12

11.13

12 Ito's integral

12.1 Let (W_t) be a Wiener process and

$$X_t = \begin{cases}5, & \text{if } t \in [0;6) \\ 7, & \text{if } t \geqslant 6.\end{cases}$$

Let $J = \int_0^9 X_t dW_t$.

a) Express J without integrals.
b) Find $\mathbb{E}(J)$ and $\mathrm{Var}(J)$.

Let $I_t = \int_0^t X_u dW_u$.

c) Express (I_t) without integrals.
d) Find $\mathbb{E}(I_t)$.
e) Find $\mathrm{Var}(I_t)$.
f) Find $\int_0^t \mathbb{E}(X_u^2) du$.
g) Find $\mathrm{Cov}(I_3, I_9)$ and $\mathrm{Cov}(I_s, I_t)$.
h) Is (I_t) a martingale?
i) Find $R_t = \int_0^t X_u du$.
j) Find $\mathbb{E}(R_t)$ and $\mathrm{Var}(R_t)$.

12.2 Let (W_t) be a Wiener process and

$$X_t = \begin{cases} 1, & \text{if } t \in [0;3) \\ 4W_2, & \text{if } t \geqslant 3. \end{cases}$$

Let $J = \int_0^5 X_t dW_t$.

a) Express J without integrals.
b) Find $\mathbb{E}(J)$ and $\text{Var}(J)$.

Let $I_t = \int_0^t X_u dW_u$.

c) Express (I_t) without integrals.
d) Find $\mathbb{E}(I_t)$.
e) Find $\text{Var}(I_t)$.
f) Find $\int_0^t \mathbb{E}(X_u^2) du$.
g) Find $\text{Cov}(I_2, I_8)$ and $\text{Cov}(I_s, I_t)$.
h) Is (I_t) a martingale?
i) Find $R_t = \int_0^t X_u du$.
j) Find $\mathbb{E}(R_t)$ and $\text{Var}(R_t)$.

12.3 Let (W_t) be a Wiener process,

$$X_t = \begin{cases} 1, & \text{if } t \in [0;3) \\ 4W_2, & \text{if } t \geqslant 3. \end{cases}, \quad Y_t = \begin{cases} 5, & \text{if } t \in [0;6) \\ 7, & \text{if } t \geqslant 6. \end{cases}$$

a) Find $\text{Cov}(\int_0^5 X_u dW_u, \int_0^5 Y_u dW_u)$.
b) Find $\text{Cov}(\int_0^5 X_u dW_u, \int_0^5 Y_u du)$.
c) Find $\text{Cov}(\int_0^5 X_u du, \int_0^5 Y_u du)$.

12.4 Consider Ito process X_t

$$dX_t = \exp(t) W_t \, dt + \exp(2W_t) \, dW_t, \quad X_0 = 1.$$

Consider two processes, $A_t = 1 + t^2 + X_t^3$ and $B_t = 1 + t^2 + X_t^3 W_t^4$.

a) Find dA_t and dB_t.
b) Write the corresponding explicit expressions for A_t and B_t:

$$const + \int_0^t \ldots dW_u + \int_0^t \ldots du$$

c) Check whether X_t is a martingale.

12.5 Consider the process X_t

$$X_t = tW_t + \int_0^t u W_u^2 \, dW_u.$$

a) Find $\mathbb{E}(X_t)$, $\text{Var}(X_t)$.

b) Find dX_t.

c) Check whether X_t is a martingale.

12.6 Consider $X_t = \int_0^t W_u^3 dW_u + \int_0^t (W_u^3 + 3W_u u)du - W_t^3 \cdot t$.

a) Find dX_t and the corresponding full form.

b) Is X_t a martingale?

12.7 Consider $X_t = \exp(-2W_t - 2t)$.

a) Find dX_t. Is X_t a martingale?

b) Find $\mathbb{E}(X_t)$ and $\text{Var}(X_t)$.

c) Find $\int_0^t X_u dW_u$.

12.8 Consider an Ito's process $I_t = 2022 + W_t t^2 + \int_0^t W_u^3 dW_u + \int_0^t W_u^2 du$.

a) Find dI_t and check whether I_t is a martingale.

b) Check whether $J_t = I_t - \mathbb{E}(I_t)$ is a martingale.

12.9 Martingales are everywhere :)

Consider the process $Y_t = \exp(-uW_t)$.

a) Find a multiplier $h(u,t)$ such that $M_t = h(u,t) \cdot Y_t$ is a martingale.

b) Find dY_t, $\mathbb{E}(Y_t)$ and $\text{Var}(Y_t)$.

c) Consider M_t that you have found as a function of u. Find the Taylor approximation of the function $M_t(u)$ up to u^4.

d) Consider the coefficient before u^4 in the Taylor expansion of $M_t(u)$. Is it a martingale?

12.10 Consider the process $X_t = \int_0^t W_u^2 dW_u + \int_0^t (W_u^2 + 2W_u u)du - W_t^2 \cdot t$.

a) Find dX_t and the corresponding full form.

b) Is X_t a martingale?

c) Find $\mathbb{E}(X_t)$.

12.11 Consider the stochastic process $X_t = f(t)\cos(2021W_t)$.

a) Find dX_t.

b) Find any $f(t) \neq 0$ such that X_t is a martingale.

c) Using $f(t)$ from the previous point find $\mathbb{E}(\cos(2021W_t))$.

12.12 Let $Y_t = W_t^3 - 3tW_t$.

a) Using Ito's lemma find dY_t.

b) Using your previous result find $\mathbb{E}(Y_t)$ and $\text{Var}(Y_t)$.

12.13 Let $Y_t = \exp(-aW_t - a^2t/2$.

a) Using Ito's lemma find dY_t.

b) Using your previous result find $\mathbb{E}(Y_t)$ and $\text{Var}(Y_t)$.

12.14 Once teachers proposed the following problem on the exam: «Let S_n be a symmetric random walk with $S_0 = 0$. Check whether $S_n^2 - n$ is a martingale».

One student wrongly confused symmetric random walk in discrete time (S_n) with Wiener process in continuous time (W_t). He proposed the wrong solution: «Let's find dS_n by Ito's lemma: $dS_n = 2S_n dS_n - dn + 0.5 \cdot 2(dS_n)^2 = 2S_n dS_n$. The dn coefficient is zero, hence S_n is a martingale».

a) Provide another example of a function $h(x, t)$ such that $h(W_t, t)$ is a martingale in continuous time and $h(S_n, n)$ is a martingale in discrete time.

b) Why this coincidence is possible?

c) Provide an example of a function $h(x, t)$ such that $h(W_t, t)$ is a martingale in continuous time but $h(S_n, n)$ is not a martingale in discrete time.

12.15 Consider a complex-valued stochastic process, $C_t = \exp(iW_t)$.

a) Simulate three trajectories of this process and draw them on a complex plane.

Hint: one may consider a fixed $T = 10$ and split time interval $[0; T]$ into $n = 10000$ small segments. The Wiener process increments on these segments will be independent normal distributed random variables.

b) According to you plot is C_t a martingale or it has a systematic drift?

Consider the process $M_t = h(t) \exp(iW_t)$ where $h(t)$ is a non-random function.

c) Find dM_t using Ito's lemma.

d) Find a function $h(t)$ such that (M_t) is a martingale and $M_0 = 1$.

e) Is Re M_t a martingale?

f) Is Im M_t a martingale?

12.16 Consider the process $Q_t = \cos W_t$.

a) [6] Find dQ_t and represent Q_t as a sum of a constant, an Ito integral and a Riemann integral.

b) [2] Is (Q_t) a martingale?

c) [2] Find an ordinary differential equation for $h(t) = \mathbb{E}(Q_t)$. You don't neet to solve this equation.

Source: designed for Stochastic Processes December 2025 exam, but no used.

12.17

12.17 Consider three processes, $A_t = \int_0^t u\,dW_u$, $\quad B_t = \int_0^t uW_u^4\,dW_u$, $\quad C_t = \int_0^t uW_u^2\,du$.

a) [2 + 2] Find $\mathbb{E}(A_t)$, $\mathbb{E}(C_t)$.

b) [2 + 4] Find $\text{Var}(B_t)$, $\text{Cov}(A_t, C_t)$.

Source: Stochastic Processes December 2025 exam.

12.18 Consider the process $X_t = A_t B_t$ where $A_t = \int_0^t u\,dW_u$ and $B_t = \int_0^t W_u^2\,dW_u$.

a) [6] Find dX_t and hence represent X_t as a sum of a constant, an Ito integreal and a Riemann integral.

b) [2] Is (X_t) a martingale?

c) [2] Find $\mathbb{E}(X_t)$.

Source: designed for Stochastic Processes December 2025 exam, but no used.

12.19 Consider the stochastic process $X_t = g(t)(\exp(2026W_t) + \exp(-2026W_t))$ where $g(t)$ is some non-random function.

a) [6] Find dX_t and hence represent X_t as a sum of a constant, an Ito integreal and a Riemann integral.

b) [4] If possible find any $g(t) \neq 0$ such that X_t is a martingale.

Source: Stochastic Processes December 2025 exam.

12.20 Consider the process

$$R_t = \exp\left(\int_0^t u^3 dW_u + t^2\right).$$

a) [6] Write R_t as a sum of a constant, an Ito's integral and a Riemann integral.

b) [2] Is (R_t) a martingale?

c) [2] Provide an ordinary (non-stochastic) differential equation for $h(t) = \mathbb{E}(R_t)$. You don't need to solve this equation.

Hint: you may write R_t as $R_t = \exp(Q_t)$ and apply Ito's lemma :)

Source: Stochastic Processes December 2025 exam.

12.21 Consider three processes,

$$A_t = \int_0^t W_u^2 \, dW_u, \quad B_t = \int_0^t W_u^4 \, dW_u, \quad C_t = \int_0^t W_u^2 \, du.$$

a) [2 + 2] Find $\mathbb{E}(A_t)$, $\mathbb{E}(C_t)$.

b) [2 + 2 + 2] Find $\mathrm{Var}(A_t)$, $\mathrm{Cov}(A_t, B_t)$, $\mathrm{Var}(C_t)$.

Source: Stochastic Processes December 2025 exam.

12.22

12.23

13 Stochastic Differential Equations

13.1 Let's consider the following system of stochastic differential equations

$$\begin{cases} dX_t = aX_t dt - Y_t dW_t \\ dY_t = aY_t dt + X_t dW_t \end{cases}$$

with initial conditions $X_0 = x_0$ and $Y_0 = 0$.

a) Find the solution of the form $X_t = f(t)\cos W_t$ and $Y_t = g(t)\sin W_t$.

b) Prove that for any solution $D_t = X_t^2 + Y_t^2$ is nonstochastic.

13.2 Let X_t be a stochastic process such that $dX_t = \frac{X_\infty - X_t}{\tau} dt + \sigma dW_t$, where X_∞ and τ are non-random constants, W_t is a Wiener process, and let $Y_t = X_t e^{t/\tau}$.

a) Use Ito Lemma to find both differential and integral expressions for Y_t and use them to express X_t (a.s.) in terms of X_∞, X_0, τ, σ and t. Here X_0 is the value of X_t at time $t = 0$.

b) Find $\mathbb{E}(X_t)$ and $\operatorname{Var}(X_t)$. Sketch the graph of $\mathbb{E}(X_t)$ as a function of t for $X_\infty = 1$, $\tau = 1$, and $X_0 = 0$, 1, and 2. Plot a possible trajectory of X_t in each case. Is there any name or names associated with X_t?

13.3

14 Binomial asset pricing model

14.1 Consider two-period binomial model with initial share price $S_0 = 600$, Up and down multipliers are $u = 1.2$, $d = 0.9$, risk-free interest rate is $r = 0.05$ per period.

Consider an option that pays you $X_2 = 100$ at $T = 2$ if $S_2 > S_1$ and nothing otherwise.

a) Find the risk neutral probabilities.

b) Find the current price X_0 of the asset.

c) How much shares should I have at $t = 1$ in the «up» state of the world to replicate the option?

14.2

14.3

14.4

15 Black and Scholes model

15.1 Consider Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

Find the current price X_0 of an option that pays you $X_2 = S_1^3$ at time $T = 2$.

15.2 Ded Moroz would like to receive $X_T = S_T^{-1}$ at time T if $S_T < 1$ and nothing otherwise.

Assume the framework of Black and Scholes model, S_t is the share price, r is the risk free rate, σ is the volatility.

How much Ded Moroz should pay now at $t = 0$?

15.3 Consider the Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

Find the current price X_0 of an option that pays you one dollar at time $T = 2$ only if $S_2 > \exp(3r) S_0$.

15.4 In the framework of the Black and Scholes model find the price at $t = 0$ of an asset that pays $\min\{M, \ln S_t\}$ at time T. Here S_T denotes the price of one share at time T, M — arbitrary constant, specified at the moment of the issue.

15.5 The price of a share in euros is driven by the equation $dS = \sigma S dW + \alpha S dt$, the dollar/euro exchange rate is driven by the equation $dU = bU dW + cU dt$.

Find the current price in dollars of a European call option with maturity date T, strike price K.

15.6 Consider the Black and Scholes model. The risk-free interest rate is equal to r. The volatility of the share is equal to σ. You have an option that pays you $X_T = \min\{1, \max\{\ln S_T, 0\}\}$ at fixed time moment T.

Find an arbitrage free price X_0 of the option.

Hint: you may use standard normal cdf $F()$ in your answer.

15.7

Lag notation

15.8 Rewrite the equation $y_t = 4 + 0.4y_{t-1} + 0.3u_{t-1} + u_t$ using lag operator.

15.9 Consider the process (x_t) with $t \geqslant 0$ and $y_t = (1+L)^t x_t$. Write x_t in terms of y_t using lag operator L.

15.10 Let (F_n) be the sequence of Fibonacci numbers.

a) Write the recurrence relation of Fibonacci numbers using lag operator.
b) Simplify the expression $L^3(1+L)^3 F_n$.
c) Simplify the expression $L^{13}(1+L)^{10} F_n$.
d) Simplify the fraction

$$\frac{F_{101} + C_5^1 F_{102} + C_5^2 F_{103} + C_5^3 F_{104} + C_5^4 F_{105} + C_5^5 F_{106}}{F_{111}}$$

e) Simplify the value of a.

15.11 Consider the process (x_t) with integer time t and $y_t = x_{-t}$.

Which reasoning is good and why?

a) $Ly_t = Lx_{-t} = x_{-t-1}$;
b) $Ly_t = y_{t-1} = x_{-t+1}$;
c) $x_t L y_t = x_t y_{t-1}$;
d) $x_t L y_t = x_{t-1} y_t$.

16 Stationarity

The process (u_t) is called a white noise if $\mathbb{E}(u_t) = 0$, $\mathrm{Var}(u_t) = \sigma^2$ and $\mathrm{Cov}(u_t, u_s) = 0$ for $t \neq s$.

The process (y_t) is called weakly stationary if $\mathbb{E}(y_t) = \mu$ and $\mathrm{Cov}(u_t, u_s) = \gamma_{t-s}$.

The process (y_t) is called strictly stationary if the distribution law of $(y_t, y_{t+1}, \ldots, y_{t+k})$ does not depend on t for all k.

16.1 The variables x_t take values 0 or 1 with equal probabilities. The variables u_t are normal $\mathcal{N}(0; 1)$. All variables are independent.

Consider the process $z_t = x_t(1 - x_{t-2})u_t$.

a) Find the covariance $\text{Cov}(z_t, z_s)$. Is the process z_t stationary?

b) Given that $z_{100} = 2.3$ find shërtest predictive intervals for z_{101} and z_{102} with probability of coverage at least 95%.

16.2 Recall the definition of a martingale, Markov chain and stationary process.

Provide an explicit example for all 8 possible combination of these properties.

16.3 Each process can be weakly stationary or not, strictly stationary or not.

Provide an explicit example for all four possible combination of these properties.

16.4 Let y_t be a stationary process. Check whether the following processes are stationary. For stationary (z_t) express its autocovariance function in terms of an autocovariance function of (y_t).

a) $z_t = 2y_t$;

b) $z_t = y_t + 1$;

c) $z_t = \Delta y_t$;

d) $z_t = 2y_t + 3y_{t-1}$.

16.5 Let u_t be a white noise. Consider the equation $y_t = 2 + 0.5y_{t-1} + u_t$ where $t \geqslant 0$ and each y_t may depend only on u_t, u_{t-1}, ..., u_0.

a) Find $\mathbb{E}(y_t)$ and $\text{Var}(y_t)$ and check the stationarity of the process in the following cases:

   (a) $y_0 = 0$

   (b) $y_0 = 4$

   (c) $y_0 = 4 + u_0$

b) Find initial condition y_0 such that the resulting process is stationary.

c) In some books and internet sources it is written something like «the process $y_t = 2+0.5y_{t-1}+u_t$ is stationary as $|0.5| < 1$». What is the problem with this statement?

16.6 I remove odd numbered observations from a stationary process (x_t) and obtain a new process (y_t).

a) Express the process (y_t) in terms of the process (x_t).

b) Is the resulting process (y_t) stationary?

c) Find the autocovariance function of (y_t) in terms of the autocovariance function of (x_t).

16.7 Elon Musk observes a stationary process (x_t) and is very bored. He throws an fair coin every time he observes a next value. If the coin lands on head then he keeps the observed value as the next value of (y_s), otherwise he discards the value and does not record it.

a) Is the resulting process (y_t) stationary?

b) Find the autocovariance function of (y_t) in terms of the autocovariance function of (x_t).

16.8 Consider the process (u_t) for $t \geqslant 0$ where all u_t are independent and uniformly distributed on $[-1; 1]$.

a) Is the process (u_t) a white noise?

Let's draw the process (u_t) and consider a new process x_t for $t \geqslant 1$. Define $x_t = 1$ if the segment joining u_t and u_{t-1} intersects the abscissa and $x_t = 0$ otherwise.

a) Find $\mathbb{P}(x_t = 1)$, $\mathbb{E}(x_t)$.

b) Find $\operatorname{Cov}(x_t, x_{t+1})$ and $\operatorname{Cov}(x_t, x_{t+2})$.

c) Is (x_t) stationary? Is it a white noise?

Now let's count the cumulative number of abscissa intersetions, $s_t = x_1 + x_2 + \dots + x_t$.

d) Find $\mathbb{E}(s_t)$.

e) What is the distribution of s_t?

f) Find approximately $\mathbb{P}(s_{100} > 55)$.

16.9 The variables x_t are independent and take values 0 or 1 with equal probabilities. The variables y_t are independent and normal $\mathcal{N}(0; 24)$. The processes (x_t) and (y_t) are independent.

Consider two processes:

$$a_t = x_t(1 - x_{t-1})y_t \quad \text{and} \quad b_t = y_{t-1}y_t.$$

a) Are (a_t) identically distributed? And (b_t)?

b) Are (a_t) independent? And (b_t)?

c) Is (a_t) a white noise? And (b_t)?

16.10 The variables (u_t) are independent and identically continuously distributed.

Let's call the time moment t a turning point («поворотная точка») if it is the local extremum, i.e. it is higher than both of his neighbors or is lower than both of his neighbors.

Consider the process (z_t) where $z_t = 1$ if t is a turning point and $z_t = 0$ if not.

a) Find $\mathbb{P}(z_t = 1)$ and $\mathbb{E}(z_t)$.

b) Find $\operatorname{Cov}(z_1, z_2)$, $\operatorname{Cov}(z_1, z_3)$, $\operatorname{Cov}(z_1, z_4)$.

c) Is (z_t) a stationary process?

d) Represent z_t in a form $z_t = \mu + v_t + \alpha_1 v_{t-1} + \alpha_2 v_{t-2}$, where v_t is a white noise.

e) Are (v_t) in point (d) independent?

Now let's also count the number of observed turning points by $s_t = z_2 + \ldots + z_{t-1}$. The values z_1 and z_t do not enter the sum, as we do not count boundary points as turning points.

a) Find $\mathbb{E}(s_t)$ and $\operatorname{Var}(s_t)$.

b) Is (s_t) stationary?

16.11 The processes (a_t) and (b_t) are stationary. The correlation $\operatorname{Corr}(a_t, b_t) = 0$ for all t.

Consider the product $y_t = a_t b_t$ and the sum $x_t = a_t + b_t$.

a) Is the process (x_t) stationary? Provide a proof or a counter-example.

b) Is the process (y_t) stationary? Provide a proof or a counter-example.

c) How the answers to (a) and (b) will change if $\operatorname{Corr}(a_t, b_s) = 0$ arbitrary t and s.

16.12

16.13

Partial correlation

16.14 We throw a fair dice three time, let's denote the results by X_1, X_2 and X_3. Consider the sums $L = X_1 + X_2$, $R = X_2 + X_3$ and $S = X_1 + X_2 + X_3$.

a) Intuitively without calculations find the sign of ordinary and partial correlations $\mathrm{Corr}(L, R)$, $\mathrm{Corr}(L, S)$, $\mathrm{pCorr}(L, R; S)$, $\mathrm{pCorr}(L, S; R)$, $\mathrm{Corr}(X_1, R)$, $\mathrm{pCorr}(X_1, R; S)$, $\mathrm{pCorr}(X_1, R; L)$, $\mathrm{pCorr}(L, R; X_2)$, $\mathrm{pCorr}(L, R; X_1)$;

b) Try to find the exact value of all ordinary and partial calculations intuitively without calculations.

c) Calculate all the values honestly.

16.15 The random variable Z takes values 0 and 1 with equal probabilities. We know the conditional distribution of the vector $X = (X_1, X_2)$ given Z:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mid Z = 0 \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mid Z = 1 \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}\right)$$

a) Find the partial correlation $\mathrm{pCorr}(X_1, X_2; Z)$.

b) Find the conditional correlation $\mathrm{Corr}(X_1, X_2 \mid Z)$.

16.16 The random vector $X = (X_1, X_2, X_3)$ has multivariate normal distribution:

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 10 & 1 & -1 \\ 1 & 10 & 2 \\ -1 & 2 & 10 \end{pmatrix}\right).$$

a) Find the partial correlation $\mathrm{pCorr}(X_1, X_2; X_3)$.

b) Find the conditional correlation $\mathrm{Corr}(X_1, X_2 \mid X_3)$.

17 ARMA

17.1 Consider stationary $AR(2)$ model, $y_t = 2 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$, where (u_t) is a white noise with $\mathrm{Var}(u_t) = 4$.

The last two observations are $y_{100} = 2$, $y_{99} = 1$.

a) Find 95% predictive interval for y_{102}.

b) Find the first two values of the autocorrelation function, ρ_1, ρ_2.

c) Find the first two values of the partial autocorrelation function, ϕ_{11}, ϕ_{22}.

17.2 Snegurochka studies a stochastic analog of the Fibonacci sequence

$$y_t = y_{t-1} + y_{t-2} + u_t,$$

where (u_t) is a white noise process.

a) How many non-stationary solutions are there?

b) What can you say about the number and the structure of the stationary solutions?

c) Can Snëgurochka find two starting constants $y_0 = c_0$ and $y_1 = c_1$ in such a way to make a solution stationary?

17.3 Stochastic process X_t is defined by $X_t = 7 + u_t + 0.3u_{t-1}$, where (u_t) is a white noise with variance σ^2.

a) Is (X_t) stationary?

b) Find the autocorrelation function of (X_t).

c) Find $\mathbb{E}(X_{t+2} \mid X_t, X_{t-1}, \ldots)$.

17.4 Young investor Winnie-the-Crypto compares two trading strategies: buying bitcoins from good bees and from bad bees. Let d_t be the price difference at day t (bad minus good). Winnie-the-Crypto would like to test H_0: $\mathbb{E}(d_t) = 0$ against H_a: $\mathbb{E}(d_t) \neq 0$ at 5% significance level.

Winnie assumed that (d_t) can be approximated by a $MA(1)$ process and estimated the parameters using $T = 400$ observations, $\hat{d}_t = 2 + u_t + 0.7u_{t-1}$ with $\hat{\sigma}_u^2 = 4$.

a) Estimate $\mathbb{E}(d_t)$, $\text{Var}(d_t)$ and $\text{Cov}(d_t, d_{t-1})$.

b) Estimate $\mathbb{E}(\bar{d})$, $\text{Var}(\bar{d})$ and help Winnie by considering $Z = \frac{\bar{d}-0}{se(\bar{d})}$.

17.5 Consider the following stationary process

$$y_t = 1 + 0.5y_{t-2} + u_t + u_{t-1},$$

where random variables u_t are independent $\mathcal{N}(0; 4)$.

a) Find the 95% predictive interval for y_{101} given that $y_{100} = 2$, $y_{99} = 3$, $y_{98} = 1$, $u_{99} = -1$.

b) Find the point forecast for y_{101} given that $y_{100} = 2$.

17.6 Consider $MA(2)$ process given by

$$y_t = 5 + u_t + 2u_{t-1} + 4u_{t-2},$$

where (u_t) is a white noise with $\text{Var}(u_t) = \sigma^2$.

a) [1] Find the expected value $\mathbb{E}(y_t)$.

b) [7] Find the autocorrelation function $\rho_k = \text{Corr}(y_t, y_{t-k})$.

c) [2] Is the process (y_t) stationary?

17.7 Consider $MA(2)$ process given by

$$y_t = 5 + u_t + 2u_{t-1} + 4u_{t-2},$$

where u_t are normal independent random variables with $\text{Var}(u_t) = 4$.

You know that $u_{100} = 2$ and $u_{99} = -1$.

a) [5] Find the 95% predictive interval for y_{101}.

b) [5] Find the 95% predictive interval for $y_{1000001}$.

17.8 The stationary process (y_t) has autocorrelation function $\rho_k = 0.2^k$ and expected value 100.

a) [7] Find the first two values of the partial autocorrelation function, ϕ_{11} and ϕ_{22}.

b) [3] Provide a possible linear recurrence equation for this process. Your equation may include y_t, its lags and a white noise process (u_t).

17.9 Consider the equation $y_t = 5 + 2.5y_{t-1} - y_{t-2} + u_t$, where (u_t) is a white noise process.

a) [3] Find the roots of the corresponding characteristic equation.

b) [4] Rewrite the process as $A(L)(y_t - \mu) = u_t$. You should explicitely write the lag polynomial $A(L)$ and the value of μ.

c) [1] How many non-stationary solutions does the equation have?

d) [1] How many stationary solutions does the equation have?

e) [1] How many stationary solutions of the $MA(\infty)$ form with respect to (u_t) does the equation have?

17.10 Consider the difference equation:

$$y_t = 0.7y_{t-1} - 0.12y_{t-2} + u_t,$$

where (u_t) is a white noise.

a) How many stationary and non-stationary solutions does the difference equation have?

Consider stationary $AR(2)$ process that satisfies the difference equation.

b) Find first two values of autocorrelation function.

c) Find α_1 and α_2 in $MA(\infty)$ representation

$$y_t = u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \alpha_3 u_{t-3} + \dots$$

17.11 Consider the process $y_t = 4 + u_t + u_{t-1} + 2u_{t-2}$, where (u_t) is a white noise with variance 16.

a) Is this process stationary? Explain.

b) Find the autocorrelation function of this process. Explain the meaning of ρ_2.

c) Consider the process $d_t = \Delta y_t$. Is it $ARIMA(p, d, q)$? If yes, then find p, d and q.

17.12 Consider the stationary $AR(2)$ process $y_t = 5 - 0.9y_{t-1} - 0.2y_{t-2} + u_t$, where (u_t) is a white noise.

a) Find the first value of autocorrelation function ρ_1.

b) Find the partial autocorrelation function of this process. Explain the meaning of ϕ_{22}.

c) What is the relationship between values of autocorrelation function ρ_{100}, ρ_{99} and ρ_{98}.

Hint: values ϕ_{22}, ϕ_{33} etc may be calculated almost effortlessly :)

17.13 Random variables x_t are iid with $\mathbb{P}(x_t = 0) = \mathbb{P}(x_t = 1) = 0.5$. Consider the process $r_t = x_t \cdot x_{t-1} - 0.25$.

a) Is (r_t) stationary?

b) Elon Musk states that this is $MA(1)$ that can be rewritten as $r_t = u_t + \alpha u_{t-1}$.
Is Elon Musk right? If yes, then express u_t using x_t and its lagged values.

17.14 The process (u_t) is a white noise, $t \in \mathbb{Z}$. Which of the following equations admit at least one stationary solution?

a) $y_t = 1 + u_t + 0.5u_{t-1} + 0.25u_{t-2}$;

b) $y_t = -2y_{t-1} - 3y_{t-2} + u_t + u_{t-1}$;

c) $y_t = -0.5y_{t-1} + u_t$;

d) $y_t = 1 - 1.5y_{t-1} - 0.5y_{t-2} + u_t - 1.5u_{t-1} - 0.5u_{t-2}$;

e) $y_t = 1 + 0.64y_{t-2} + u_t + 0.64u_{t-1}$;

f) $y_t = 1 + t + u_t$;

g) $y_t = 1 + y_{t-1} + u_t$.

17.15 The process y_t is stationary and is generated by equation $y_t = 1 + 0.5y_{t-1} + u_t$ where (u_t) is a white noise.

We have 10000 observations.

a) What will be the approximate OLS estimates in the regression $\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 y_{t-1}$?

b) What will be the approximate OLS estimates in the regression $\hat{y}_t = \hat{\gamma}_1 + \hat{\gamma}_2 y_{t+1}$?

17.16 Consider the recurrence equation $y_t = 2 + 0.5y_{t-1} + u_t$ for $t \in \{0, 1, 2, \ldots\}$ and independent $u_t \sim \mathcal{N}(0; \sigma^2)$.

a) Find three different stationary processes $(y_t)_{t=0}^{\infty}$ that satisfy this equation.

b) Will the forecasts for these three solutions from (a) be different?

c) How the answer to (a) will change if $t \in \mathbb{Z}$ and we need to find $(y_t)_{t=-\infty}^{\infty}$?

17.17 This exercise provides an elementary proof that a unit root corresponds to the absence of stationary solutions.

Let (y_t) be a stationary process with autocovariance function γ_k and (u_t) — a white noise with variance σ^2.

a) Using trivial bounds for correlation deduce the possible range for $\mathrm{Cov}(y_0, u_1 + u_2 + \cdots + u_k)$.

b) Prove that (y_t) can't satisfy an equation $y_t = y_{t-1} + u_t$.

c) Prove that (y_t) can't satisfy an equation $(1 - L)(1 - 0.5L)y_t = u_t$.

Let (m_t) be an $MA(2)$ process with the autocovariance function c_k.

d) Find $\lim_{k\to\infty} \mathrm{Var}(m_1 + m_2 + \cdots + m_k)/k$.

e) Prove that (y_t) can't satisfy an equation $(1 - L)(1 - 0.5L)y_t = m_t$ if $c_0 + c_1 + c_2 \neq 0$.

f) Provide an example of $MA(2)$ process with autocovariance function c_k that satisfies the equation $c_0 + c_1 + c_2 = 0$.

g) Prove the general theorem: the recurrence equation $a(L)y_t = b(L)u_t$ where (u_t) is a white noise, $a(1) = 0$ and $b(1) \neq 0$ can't have a stationary solution (y_t).

17.18 Is it possible that a stationary (y_t) satisfies the equation $y_t = 5 + 2y_{t-1} - y_{t-2} + u_t + u_{t-2}$ and (u_t) is a white noise process? Provide a proof or a counter-example.

17.19 Is it possible that $\mathrm{Cov}(y_t, u_s) = 0$ for all $t < s$, (y_t) satisfies the equation $y_t = 5 + 2y_{t-1} + u_t$ and (u_t) is a white noise process? Provide a proof or a counter-example.

17.20 Solve the puzzle!

$$AR(p) + MA(?) = ARMA(p, p)$$

18 ETS

18.1 The semi-annual y_t is modelled by $ETS(AAA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0;4) \\ s_t = s_{t-2} + 0.1u_t \\ b_t = b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-2} + u_t \end{cases}$$

a) Given that $s_{100} = 2$, $s_{99} = -1.9$, $b_{100} = 0.5$, $\ell_{100} = 4$ find 95% prëdictive interval for y_{102}.

b) In this problem particular values of parameters are specified. And how many parameters are estimated in semi-annual $ETS(AAA)$ model before real forecasting?

18.2 The $ETS(AAdN)$ model is given by the system

$$\begin{cases} u_t \sim \mathcal{N}(0;20) \\ b_t = 0.9b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + 0.9b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + 0.9b_{t-1} + u_t \end{cases}$$

with $\ell_{100} = 20$ and $b_{100} = 2$.

a) Find the 95% predictive interval for y_{101}.

b) Find conditional probability $\mathbb{P}(y_{102} > 30 \mid \ell_{100}, b_{100})$.

c) Approximately find the best point forecast for y_{10000}.

d) Find the 95% predictive interval for b_{10000}.

18.3 Consider $ETS(ANN)$ model,

$$\begin{cases} y_t = \ell_{t-1} + u_t \\ \ell_t = \ell_{t-1} + \alpha u_t \\ u_t \sim \mathcal{N}(0;\sigma^2). \end{cases}$$

Let $\ell_{99} = 50$, $\alpha = 1/2$, $\sigma^2 = 16$, $y_{98} = 48$, $y_{99} = 52$, $y_{100} = 55$.

Calculate 95% predictive interval for y_{101}.

18.4 The semi-annual (y_t) is modelled by $ETS(ANA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0;4) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \end{cases}$$

Let $s_{100} = 3$, $s_{99} = -2$, $\ell_{100} = 100$.

a) Find 95% predictive interval for y_{102}.

b) Find 95% predictive interval for s_{121}.

c) Write this model in the form $A(L)y_t = B(L)u_t$, where $A(L)$ and $B(L)$ are lag polynomials.

d) Classify this model as $SARIMA(p, d, q)(p^s, d^s, q^s)$: explicitely state p, d, q, p^s, d^s, q^s.

18.5 The semi-annual (y_t) is modelled by $ETS(ANA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \\ \ell_0 = 100, s_0 = -3, s_{-1} = 3 \end{cases}$$

Check whether the process (y_t) is stationary.

18.6 Consider $ETS(AAdN)$ model

$$\begin{cases} u_t \sim \mathcal{N}(0; 20) \\ b_t = 0.9b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + 0.9b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + 0.9b_{t-1} + u_t \end{cases}$$

with $\ell_{100} = 20$ and $b_{100} = 1$.

a) Find 95% prediction interval for y_{102}.

b) Approximately find the best point forecast for y_{10000}.

18.7 Consider $ETS(AAN)$ model, $\begin{cases} y_t = \ell_{t-1} + b_{t-1} + u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ b_t = b_{t-1} + \beta u_t \\ u_t \sim \mathcal{N}(0; \sigma^2). \end{cases}$

Let $\ell_{100} = 50$, $b_{100} = 2$, $\alpha = 0.4$, $\beta = 0.5$, $\sigma^2 = 16$.

Calculate one step and two steps ahead 95% predictive intervals.

18.8

18.9

18.10

18.11

19 GARCH

19.1 The process y_t is described by a simple $GARCH(1, 1)$ model:

$$\begin{cases} y_t = \sigma_t \nu_t \\ \sigma_t^2 = 1 + 0.2y_{t-1}^2 + 0.3\sigma_{t-1}^2 \\ \nu_t \sim \mathcal{N}(0; 1) \end{cases}$$

The variables ν_t are independent of past variables y_{t-k}, ν_{t-k}, σ_{t-k} for all $k \geqslant 1$. The prëcesses y_t, σ_t^2 are stationary.

Given $\sigma_{100} = 1$ and $\nu_{100} = 0.5$ find 95% predictive interval for y_{102}.

19.2 The strictly stationary white noise (u_t) follows $ARCH(1)$ model $\sigma_t^2 = 3 + 0.5u_{t-1}^2$ where $u_t = \sigma_t \nu_t$ and $\nu_t \sim \mathcal{N}(0;1)$.

a) Find 95% prediction interval for u_{101} given that $u_{100} = -1$.

b) Find $\mathbb{E}(u_t)$, $\text{Var}(u_t)$.

c) Find $\text{Corr}(u_t, u_{t-1})$, $\text{Corr}(u_t^2, u_{t-1}^2)$.

19.3 Consider the $ARCH(1)$ model, $u_t = \sigma_t \nu_t$, where ν_t are iid $\mathcal{N}(0;1)$ and $\sigma_t^2 = 1 + 0.3u_{t-1}^2$.

a) Find 95% predictive interval for u_{101} if $u_{100} = -2$.

b) Find the autocorrelation function of $r_t = u_t^2$.

19.4

19.5

19.6

19.7

19.8

20 Method of Moments and maximum likelihood

20.1 The variables X_1, ..., X_n are independent identically distributed with density

$$f(x) = \begin{cases} \lambda \exp(-\lambda(x-\theta)), & \text{if } x \geqslant \theta \\ 0, & \text{otherwise.} \end{cases}$$

a) [5] Find the method of moments estimator of λ for known value $\theta = 1$ using the first moment.

b) [5] Find the method of moments estimator of λ for unknown value θ using the first two moments.

20.2 The variables X_1, ..., X_n are independent and normally distributed $\mathcal{N}(a, 2a)$.

Find the maximum likelihood estimator of a.

Hint: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/2\sigma^2)$.

20.3 The weight of a fish Y_i is a discrete random variables with distribution and observed frequencies given in the table

Weight [kg]	1	2	a
Probability	$0.2 + 0.1a$	$0.3 - 0.1a$	0.5
Observed frequency	N_1	N_2	N_a

Fish weights Y_i are independent, $a > 10$ is unknown.

a) Find the method of moments estimator of the parameter a.

b) Find the maximum likelihood estimator of the parameter a.

20.4 The weight of a fish Y_i is a discrete random variables with distribution and observed frequencies given in the table

Weight [kg]	1	2	4
Probability	$0.2+a$	$0.3-a$	0.5
Observed frequency	N_1	N_2	N_4

Fish weights Y_i are independent.

a) [5] Find the maximum likelihood estimator of the parameter a.

b) [5] Find the method of moments estimator of the parameter a.

20.5 To go to the mountain top I use a gondola lift in the morning. I go back from the top using the same gondola lift in the evening. Cabins are numbered from 1 to a.

I have noticed that the absolute difference of cabin numbers of my two trips was 10.

a) Estimate a using maximum likelihood.

b) Estimate a using method of moments.

20.6 Random variables X_1, X_2, ..., X_n are independent identically distributed with density

$$f(x_i \mid \lambda, a) = \frac{\lambda}{2}\exp(-\lambda|x_i - a|).$$

Observed values for $n = 3$ are -3, 1, 11.

a) Estimate λ using method of moments for fixed $a = 1$.

b) Estimate λ and a using maximum likelihood.

20.7

20.8

20.9

21 LR, LM and Wald tests

21.1 We have two independent random samples $X_1, X_2, \ldots, X_{n_x}$ and $Y_1, Y_2, \ldots, Y_{n_y}$. The random variables X_i follow Poisson distribution with intensity rate λ_x, random variables Y_i follow Poisson distribution with intensity rate λ_y.

We would like to test H_0: $\lambda_x = \lambda_y$ against H_1: $\lambda_x \neq \lambda_y$.

a) [3] Find the maximal value of log-likelihood under H_0.

b) [3] Find the maximal value of log-likelihood under unrestricted model.

c) [2] Construct the likelihood ratio test.

d) [2] Do you reject H_0 if $n_x = 100$, $n_y = 200$, $\sum x_i = 500$, $\sum y_i = 900$ at significance level 5%?

21.2 You observe $X_1, \ldots, X_{400}$ and $Y_1, \ldots, Y_{400}$, $\bar{X} = 5$, $\bar{Y} = 6$. All variables are independent.

Consider the null hypothesis that all random variables are exponentially distributed with common parameter λ against alternative that parameter is λ_X for every X_i and λ_Y for every Y_j.

a) Estimate common λ using maximum likelihood for the restricted model.

b) Estimate both λ_X and λ_Y using maximum likelihood in the unrestricted model.

c) Use LR-test to test the null hyphotesis at 5% significance level.

21.3 Random variables X_1, X_2, ..., X_n are independent identically distributed with density

$$f(x_i \mid \lambda) = \frac{\lambda}{2} \exp(-\lambda|x_i|).$$

For $n = 100$ I have 40 negative values with sum equal to -300 and 60 positive values with sum equal to 500.

a) Test the hypothesis $\lambda = 1$ using LR approach at significance level $\alpha = 0.01$.

b) Test the hypothesis $\lambda = 1$ using LM approach at significance level $\alpha = 0.01$.

21.4

21.5

21.6

22 Properties of estimators

22.1 The variables X_1, ..., X_n are independent and uniformly distributed Unif$[0; a]$ with $a > 1$. We do not observe X_i directly but we know whether each X_i is larger than 1. Hence we observe the indicators $Y_i = I(X_i > 1)$.

Consider the estimator $\hat{a} = 1/(1 - \bar{Y})$.

a) [5] Is $\hat{a}$ consistent?

b) [5] Is $\hat{a}$ unbiased for $n = 2$?

22.2 You observe time between taxi arrivals on a stop, Y_1, Y_2, ..., Y_n. Assume that Y_i are independent and exponentially distributed with $\mathbb{E}(Y_i) = \theta$, that means the density of each Y_i is $f(y) = \exp(-y/\theta)/\theta$ for $y \geqslant 0$. Consider the following estimator of expected value

$$\hat{\theta} = n \cdot \min\{Y_1, Y_2, \ldots, Y_n\}$$

a) [6] Find the probability density function of $\hat{\theta}$.

b) [2] Is $\hat{\theta}$ unbiased?

c) [2] Is $\hat{\theta}$ consistent?

22.3

22.4

22.5

23 Fisher information and Cramer — Rao

23.1 The variables X_1, ..., X_n are independent and have Poisson distribution with intensity rate λ. In other words the probability mass function is given by $\mathbb{P}(X_i = k) = \exp(-\lambda)\lambda^k/k!$.

a) [5] Find theoretical Fisher information for λ contained in the sample.

b) [2] Derive the maximum likelihood estimator for λ.

c) [3] Does the maximum likelihood estimator attain the Cramer-Rao lower bound for variance?

23.2 Consider an estimator $\hat{a}$ with $\mathbb{E}(\hat{a}) = 0.5a + 3$. For the given sample size the Fisher information is $I_F(a) = 400/a^2$.

a) What is the theoretical minimal variance of $\hat{a}$?

b) Assume that $\hat{a}$ attains the minimal variance boundary and is asymptotically normal. Given that $\hat{a} = 2022$ provide 95% CI for a.

23.3 Consider iid sample from bivariate normal distribution, $\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \theta \\ 2\theta \end{pmatrix}; \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}\right)$.

Calculate Fischer information for the following cases:

a) You observe X_1 only.

b) You observe X_1, ..., X_n.

c) You observe X_1, ..., X_n, Y_1, ..., Y_n.

Hint: the multivariate normal density is $f(u) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp(-\frac{1}{2}(u-\mu)^T\Sigma^{-1}(u-\mu))$.

23.4 Random variables X_1, ..., X_n are independent with density $f(x) = \begin{cases} -\ln(a) \cdot a^x, & \text{if } x \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$

a) Estimate a using maximum likelihood.

b) Check whether the estimator is unbiased and consistent.

c) Check whether the corresponding Cramer-Rao lower bound is attained.

23.5 Random variables X_1, ..., X_n are independent and normally distributed $\mathcal{N}(1, 1/b)$.

a) Estimate b using maximum likelihood.

b) Does the estimator achive the Cramer-Rao lower bound?

c) Is the estimator consistent?

d) Is the estimator unbiased?

23.6

23.7

23.8

24 Sufficiency

24.1 The variables X_1, ..., X_n are independent and gamma distributed with density

$$f(x) = \begin{cases} \lambda^\alpha x^{\alpha-1} \exp(-\lambda x)/\Gamma(\alpha), & \text{if } x \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

a) [5] Find a sufficient statistic for α if we know that $\lambda = 1$.

b) [5] Find a two dimensional sufficient statistic for unknown α and λ.

24.2 The variables X_1, ..., X_n are independent and uniformly distributed on $[0; 2a]$ for some positive a.

a) Find any sufficient statistic for a.

b) How the answer will change if $X_i \sim U[-a; 2a]$?

24.3

24.4

24.5

25 Solutions

1.1.

a) $\mathbb{P}(B) = 6/11$, first step equation for $p = \mathbb{P}(B)$ is $p = 1/6 + (5/6)^2 p$ or $p = 1/6 + 5/6 \cdot (1-p)$.

b) $\mathbb{E}(N) = 6$, first step equation for $m = \mathbb{E}(N)$ is $m = 1/6 + 5/6(m+1)$.

c) $\mathbb{E}(N^2) = 66$, $\mathrm{Var}(N) = 30$, first step equation is $\mathbb{E}(N^2) = 1/6 + 5/6\,\mathbb{E}((N+1)^2)$.

d) $\mathbb{E}(N \mid B) = 61/11$. Start by replacing uncoditional probabilities on the tree by conditional ones. First step equation for $\mu = \mathbb{E}(N \mid B)$ is $\mu = 11/36 + 25/36(\mu + 2)$.

e) $\begin{pmatrix} 0 & 5/6 & 1/6 & 0 \\ 5/6 & 0 & 0 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1.2.

a) $\mathbb{E}(N) = 1/p$, $\mathrm{Var}(N) =$, $\mathbb{E}(N^3) =$, $\mathbb{E}(\exp(tN)) =$

b) $a = \mathbb{P}(N \in 2 \cdot \mathbb{N})$, $a = (1-p)(1-a)$, $a = (1-p)/(2-p)$.

1.3.

1.4. Let's denote the throws by (X_t) and the number of throws by T. Thus the last throw is X_T.

a) $\mathbb{P}(X_T = 6) = 1/3$ as we have three possible endings. One may also sum the probability geometric serie or use first step analysis.

b) $\mathbb{E}(T) = 0.5 + 0.5(\mathbb{E}(T) + 1)$;

c) Let $\mu = \mathbb{E}(S)$ and $\gamma = \mathbb{P}(X_T \in \{4, 5\})$.

$$\mu = \frac{3}{6} \cdot 0 + \frac{1}{6}(\mu + 1 \cdot \gamma) + \frac{1}{6}(\mu + 2 \cdot \gamma) + \frac{1}{6}(\mu + 3 \cdot \gamma)$$

d)

e)

f)

$$\mu_B = \frac{2}{6} \cdot 0 + \frac{1}{6}\mu_B + \frac{1}{6}(\mu + 1 \cdot \beta) + \frac{1}{6}(\mu + 2 \cdot \beta) + \frac{1}{6}(\mu + 3 \cdot \beta),$$

with $\beta = 1/3$.

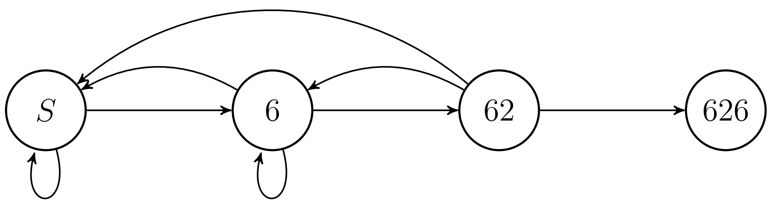
1.5.

1.6.

1.7.

1.8.

1.9. Let's draw the chain



The system of equations for expected values:

$$\begin{cases} x_s = 1 + \frac{1}{6}x_6 + \frac{5}{6}x_s \\ x_6 = 1 + \frac{1}{6}x_6 + \frac{1}{6}x_{62} + \frac{4}{6}x_s \\ x_{62} = 1 + \frac{1}{6} \cdot 0 + \frac{1}{6}x_6 + \frac{4}{6}x_s \end{cases}$$

The system of equations for moment generating functions:

$$\begin{cases} m_s(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{5}{6}m_s(t)\right) \\ m_6(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{1}{6}m_{62}(t) + \frac{4}{6}m_s(t)\right) \\ m_{62}(t) = \exp(t)\left(\frac{1}{6} \cdot 1 + \frac{1}{6}m_6(t) + \frac{4}{6}m_s(t)\right) \end{cases}$$

1.10. Recurrence equation for p_k:

$$p_k = 0.5p_{k+1} + 0.5p_{k-1}, \text{ with } p_{-a} = 0, p_b = 1;$$

Recurrence equation for e_k:

$$e_k = 0.5e_{k+1} + 0.5e_{k-1} + 1, \text{ with } e_{-a} = 0, e_b = 0;$$

In general: $p_0 = a/(a+b)$, $e_0 = ab$.

1.11. Recurrence equation for p_k:

$$p_k = rp_{k+1} + (1-r)p_{k-1}, \text{ with } p_{-a} = 0, p_b = 1;$$

Recurrence equation for e_k:

$$e_k = re_{k+1} + (1-r)e_{k-1} + 1, \text{ with } e_{-a} = 0, e_b = 0;$$

In general:

1.12.

1.13.

a) One may use first step analysis or probability scaling. There 3 options for the sum of 10 and 5 options for the sum of 8. Hence, we scale the answer, $3/(3+5)$.

b) First, 11 should be before 10 or 8, $2/(2+3+8)$. Second, 10 should be before 8, $3/(3+5)$.

The answer, $2/(2+3+8)\cdot 3/(3+5)$.

1.14. Let's denote this number by e_n.

One may consider all the variants from starting point that lead to starting the game anew: T, HT, HHT, ... The final term describes the lucky situation of a win without tails.

$$e_n = 1/2(1+e_n) + 1/4(2+e_n) + \cdots + 1/2^n(n+e_n) + 1/2^n \cdot n.$$

Or one may say, that first one should obtain $(n-1)$ heads and then either just one head or restart anew,

$$e_n = e_{n-1} + 0.5\cdot 1 + 0.5\cdot(1+e_n).$$

1.15.

a) Consider the distance between the point and the line, d_t, and observe that Δd_t is equal to 0 or 1 with equal probabilities.

1.16.

1.17.

1.18.

a) $1 - 1/3 - 2/9 - 4/27 - \cdots = 1 - \frac{1/3}{1-2/3} = 0;$

b) $\mathbb{E}(X) = 1/2$ by symmetry; For the $\mathbb{E}(X^2)$ we may build an equation using the first step approach

$$\mathbb{E}(X^2) = 0.5\,\mathbb{E}((X/3)^2) + 0.5\,\mathbb{E}((2/3 + X/3)^2).$$

From this equation one obtains $\mathbb{E}(X^2) = 3/8$ and hence $\text{Var}(X) = 1/8$.

Or one may also use the formula for the variance

$$\text{Var}(X) = \mathbb{E}(\text{Var}(X \mid Y)) + \text{Var}(\mathbb{E}(X \mid Y)).$$

Here $Y = I(X > 1/2)$ indicates whether X belongs to the left or right side of the fractal.

$$\text{Var}(X \mid Y) = \text{Var}(X)/9$$

Observe that $\mathbb{E}(X \mid Y)$ takes only two values, $1/6$ or $5/6$. And hence $\text{Var}(\mathbb{E}(X \mid Y)) = 1/9$.

$$\text{Var}(X) = \text{Var}(X)/9 + 1/9$$

And, finally $\text{Var}(X) = 1/8$.

c) Let's toss the coin infinitely many times. If we observe H then we choose the left part and if we observe T then we choose the right part during the stage.

d) Let's go left-right-left-right-...

e) $F(1/2) = 1/2$, $F(1/3) = 1/2$, $F(1/4) =$, $F(1/5) =$

1.19.

a) $1 - 1/4 - 3/4^2 - 3^2/4^3 - \cdots = 1 - \frac{1/4}{1-3/4} = 0$;

b) $\mathbb{E}(X) = 1/2$ by symmetry; For the $\mathbb{E}(X^2)$ we may build an equation using the first step approach

$$\mathbb{E}(X^2) = 1/3\,\mathbb{E}((X/2)^2) + 1/3\,\mathbb{E}((1/2 + X/2)^2) + 1/3\,\mathbb{E}((1/4 + X/2)^2).$$

c)

d) $\text{Cov}(X, Y) = 0$.

e) Toss a fair coin infinitely many times. Divide the tosses into pairs. If we observe the pair HH we choose the left smaller triangle, TH — the right one, HT — the upper one, TT — discard the pair of tosses.

1.20.

1.21.

2.1.

2.2.

2.3.

2.4.

2.5.

2.6.

a)

b) $\mathbb{E}(W_t) \to +\infty$;

c) No stationary distribution. For stationary distribution $\mathbb{E}(W_t)$ can't tend to infinity.

2.7.

2.8.

a) $\mathbb{P}(S_3 = B) = \mathbb{P}(A \to B \to A \to B) + \mathbb{P}(A \to C \to A \to B) + \mathbb{P}(A \to B \to C \to B)$

b)

$$\begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

c)

$$\begin{cases} a = 0.5b + 0.5c \\ b = 0.7a + 0.5c \\ c = 0.3a + 0.5b \\ a + b + c = 1 \end{cases}$$

The solution is $a = 15/45$, $b = 17/45$, $c = 13/45$.

2.9. $\mathbb{P}(X_3 = 1) = 0.3 \cdot 0.7 \cdot 0.3 + 0.7 \cdot 1 \cdot 0.3 = 0.21 \cdot 1.3$ Let's denote $\tau_j = \min\{t \mid X_t = 0, X_0 = j\}$, $\mu_j = \mathbb{E}(\tau_j)$.

$$\begin{cases} \mu_0 = 0 \\ \mu_1 = 1 + 0.7\mu_2 \\ \mu_2 = 1 + 0.3\mu_1 + 0.7\mu_3 \\ \mu_3 = \mu_2 + 1 \end{cases}$$

We get $\mu_2 = 200/9$.

2.10.

2.11.

2.12.

2.13.

2.14.

2.15.

2.16.

2.17.

2.18.

2.19.

3.1.

3.2.

3.3.

3.4.

3.5.

3.6.

4.1.

4.2. $M(t) = (\exp(t) + \exp(2t) + \exp(3t))/3$.

4.3. $\mathbb{E}(W) = 2$, $\mathrm{Var}(W) = 7 \cdot 2 - 2^2$, $\mathbb{E}(W^3) = 20 \cdot 3!$.

4.4. $\mathbb{P}(X = 0) = 0.1$, $\mathbb{P}(X = 1) = 0.2$, $\mathbb{P}(X = 2) = 0.15$.

4.5.

4.6. $g(u) = g(\exp(t)) = \mathbb{E}(\exp(tX)) = M(t)$

4.7.

a) Consider N as known fixed value, $\mathbb{E}(S \mid N) = N\,\mathbb{E}(X_1) = N \cdot 0.5$. First, let's find moment generating funciton for X_i:

$$M_X(u) = \int_0^1 \exp(xu) \cdot 1\,dx = \frac{\exp(u) - 1}{u};$$

Hence $M_{S|N}(u) = (M_X(u))^N$ as S is the sum of N independent variables.

b) Random variable N is discrete, $M_S(u) = \mathbb{P}(N = 1)(M_X(u))^1 + \mathbb{P}(N = 2)(M_X(u))^2 + \ldots = \frac{0.7 M_X(u)}{1 - 0.3 M_X(u)}$.

c) Moment generating function is used to calculate moments, $M_S''(0) - (M_S'(0))^2 = \mathrm{Var}(S)$.

4.8.

4.9.

4.10.

4.11.

a) Moment generating function

$$m_N(t) = \sum_{j=0} \exp(tj)(1-h)^j h = h \sum_{j=0} (\exp(t)(1-h))^j = \frac{h}{1-\exp(t)(1-h)}$$

b) As $S = N_1 + N_2 + \ldots + N_k$:

$$m_S(t) = \left(\frac{h}{1-\exp(t)(1-h)}\right)^k$$

c) 0;

d)

4.12.

4.13.

4.14.

4.15.

4.16.

4.17.

4.18.

4.19.

5.1.

5.2.

a) Cesaro-limit is $1/2$. It's like an average long-term salary.

b) Split the sequence into groups of two numbers. In each group put one 0 and one 1 in an aperiodic way.

c) Yes, ordinary convergences implies Cesaro-convergence.

d) Yes, Cesaro-limit of a sum is a sum of Cesaro-limits.

e) No. One sequence has 1 at even places, the other sequence has 1 at odd places.

5.3.

5.4.

6.1.

6.2.

a) $\operatorname{plim} X_n = 0$;

b) $\operatorname{plim} Y_n$ does not exist.

6.3.

a) $\operatorname{plim} \frac{X_1+X_2+\cdots+X_n}{7n} = 5/7$;

b) $\operatorname{plim} \frac{X_1^2+X_2^2+\cdots+X_n^2}{7n} = 5$;

c) $\operatorname{plim} \ln(X_1^2 + X_2^2 + \cdots + X_n^2) - \ln n = \ln 35$.

6.4.

a) $\operatorname{plim} \max\{Y_1, Y_2, \ldots, Y_n\} = 2$;

b) $\operatorname{plim} \max\{X_1 + Y_1, X_1 + Y_2, \ldots, X_1 + Y_n\} = X_1 + 2$.

6.5.

a) $\operatorname{plim} Y_n = 0$;

b) $\operatorname{plim} W_n = X$.

6.6.

a) $\mathbb{P}(\lim Y_n = 0) = 1$;

b) $\operatorname{plim} Y_n = 0$;

c) Limiting distribution is a constant 0.

6.7.

a) $\operatorname{plim} V_n = 0$, $\operatorname{plim} W_n = Y$;

b) The sequence V_n does not converge in mean squared;

c) W_n converges to Y in mean squared.

6.8.

a) $2/5$;

b) L/R or $+\infty$ or $-\infty$ or $3/2$.

c) Almost surely limit is L/R, pointwise limit does not exist.

d) Pointwise and almost surely limits are $\frac{2}{5}I(R > 0) + \frac{3}{2}I(R = 0)$.

6.9.

a) 0, 1, $+\infty$ or does not exist.

b) Almost surely limit is 0;

c) $1/2$;

d) $\mathbb{P}(Y_n \text{ converges }) = 0$;

e) Yes, as every Y_n has the same distribution.

6.10.

6.11.

6.12.

$$\operatorname{plim} \frac{\sum (X_i - \bar{X})^3}{n + 2022} = \operatorname{plim} \frac{\sum X_i^3 - 3\bar{X} \sum X_i^2 + 3\bar{X}^2 \sum X_i - \sum \bar{X}^3}{n + 2022} =$$
$$= \mathbb{E}(X_1^3) - 3\,\mathbb{E}(X_1^2) + 3\,\mathbb{E}(X_1) - 1 = 0;$$

Note that $\mathbb{E}(X_1^2) = 4/3$, $\mathbb{E}(X_1^3) = 2$.

6.13. Start with X_0: $\mathbb{E}(X_0) = 1$, $\operatorname{Var}(X_0) = 4/12 = 1/3$.

a) Expected value is constant, $\mathbb{E}(X_n) = 0.5 + 0.5\,\mathbb{E}(X_{n-1})$, hence $\mathbb{E}(X_n) = 1$. Variance goes to zero, $\operatorname{Var}(X_n) = 0.25 \operatorname{Var}(X_{n-1})$.

b) $\operatorname{plim} X_n = 1$

6.14.

6.15.

6.16.

6.17.

6.18.

6.19.

7.1.

a)

b) Sigma-algebras do not depend on probabilities.

c) $\sigma(|X|) = \{\emptyset, \Omega, \{|X| = 2\}, \{X = 1\}\}$.

d) $\sigma(|X|) = \sigma(X^2)$;

7.2.

a) $\sigma(X^2) = \{\{X^2 = 1\}, \{X = 0\}, \emptyset, \Omega\}$, $\sigma(2X + 3) = \sigma(X)$

b) $\operatorname{card} \sigma(X, Y) = 2^6$, $\operatorname{card} \sigma(X + Y) = 2^4$, $\operatorname{card} \sigma(X, Y, X + Y) = 2^6$.

7.3.

a) 2^5;

b) Only 2^k or infinity;

c) 2^{100};

d) $2^{2^{100}}$.

7.4. In general $\sigma(f(X)) \subseteq \sigma(X)$; If f is a bijection then $\sigma(f(X)) = \sigma(X)$.

7.5. In the finite case sigma-algebra corresponds to partitions. We get five sigma-algebras on a set of three elements and 15 sigma-algebras on a set of four elements. These numbers are known as Bell numbers.

7.6. Let $\Omega = \mathbb{N}$, $\mathcal{A}$ contains all finite sets and sets with finite complement.

7.7.

a) The intersection of two sigma-algebras is always a sigma-algebra.

b) The intersection of two sigma-algebras is always a sigma-algebra.

c) The union of two sigma-algebras is not always a sigma-algebra.

d)

7.8. Yes. This is convinient do draw Ω as a segment. With «пескари» $A \subseteq B$ and «sharks» $A \supseteq B^c$.

7.9. Разобьем натуральный ряд на пары соседних чисел. Можно так подобрать множества A и B, что в каждом из них из каждой пары взято только одно число. Поэтому $\gamma(A) = \gamma(B) = \frac{1}{2}$. Подобрав совпадение-несовпадение в паре, можно сделать так, что $\gamma(A \cap B)$ не существует.

7.10.

a) $\sigma(Z) = \{\{Z = 1\}, \{Z = 0\}, \Omega, \emptyset\}$.

b) $\sigma(YZ) = \{\{YZ = 1\}, \{YZ = 0\}, \Omega, \emptyset\}$.

c) 2^4;

d) $\sigma(Y \cdot Z) \subseteq \sigma(Y, Z)$.

7.11.

7.12. Yes!

7.13. Например, B — Канторово множество, или, гораздо проще, $B = [0; 0,5]$. Оно само более чем счетно и дополнение к нему более чем счетно.

Набор $\mathcal{F}$ действительно σ-алгебра. $\emptyset$ лежит в $\mathcal{F}$, так как имеет ноль элементов.

Если A не более чем счетно, то A^c лежит в $\mathcal{F}$, так как дополнение к A^c содержит не более чем счетное число элементов.

Если дополнение к A не более чем счетно, то A^c лежит в $\mathcal{F}$, так как содержит не более чем счетное число элементов.

Проверяем счетное объединение $\bigcup_i A_i$. Если среди A_i встречаются только не более чем счетные, то и их объединение — не более чем счетно. Если среди A_i встретилось хотя бы одно множество с не более чем счетным дополнением, то $\bigcup_i A_i$ тоже будет обладать не более чем счетным дополнением, так как объединение не может быть меньше ни одного из объединяемых множеств.

7.14. 2^{101}, $2^{101\cdot 51}$,

7.15. $\mathcal{F} = \{\emptyset, \Omega, \{\text{дождь}\}, \{\text{солнечно}, \text{пасмурно}\}$. Всего есть $1 + 1 + 3 = 5$ σ-алгебр.

7.16.

a) $\sigma(X_1 \cdot X_2) = \{\emptyset, \Omega, \{X_1 X_2 = 1\}, \{X_1 X_2 = -1\}\}$;

b) $\sigma(X_1 = X_3)$;

c) Note that $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3) = \sigma(X_1, X_2, X_3)$, the number of events in sigma-algebra is card $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3) = 2^8 = 256$.

7.17.

7.18.

7.19. Here $\Omega = \{1, 2, 3, 4, 5, 6\}$.

a) $\mathcal{F} = \{\emptyset, \Omega\}$;

b) $\mathcal{F} = 2^{\Omega}$, this notation means «all subsets of Ω».

c) $\mathcal{F} = \sigma(\{1\}, \{2\})$, eight events in total;

7.20.

7.21.

7.22.

7.23.

8.1.

a) $\mathbb{E}(Y \mid X) = 1 - X/6$ and $\mathbb{E}(X \mid Y) = 3 - 1.5X$.

b) $\text{Var}(Y \mid X) =$, $\text{Var}(X \mid Y) =$.

8.2.

a)
$$f(x) = \begin{cases} x + 0.5, & \text{if } x \in [0;1]; \\ 0, & \text{otherwise.} \end{cases}$$

b)
$$f(x,y) = \begin{cases} (x+y)/(x+0.5), & \text{if } x \in [0;1], y \in [0;1]; \\ 0, & \text{otherwise.} \end{cases}$$

c)
$$\mathbb{E}(Y \mid X) = \frac{0.5X + 1/3}{X + 0.5}.$$

d)

8.3.

a)

b) $\mathbb{E}(X \mid S) = \mathbb{E}(Y \mid S) = S/2$;

c)

d)

8.4.

a) $X \mid S \sim \text{Unif}[0;S]$, $Y \mid S \sim \text{Unif}[0;S]$.

b) $\mathbb{E}(X \mid S) = \mathbb{E}(Y \mid S) = S/2$;

c) $\text{Var}(X \mid S) = \text{Var}(Y \mid S) = S^2/12$;

d)

8.5. $\mathbb{E}(XY \mid X) = X\,\mathbb{E}(Y) = X/2$, $\text{Var}(XY + X^3 \mid X) = X^2\,\text{Var}(Y) = X^2/12$, $\text{Cov}(X, Y \mid X) = 0$, $\text{Cov}(XY, X^2Y \mid X) = X^3\,\text{Var}(Y) = X^3/12$

8.6.

a) $Y = IX_1 + (1-I)X_2$

b) Consider I as known or fixed variable, $\mathbb{E}(Y \mid I) = I\,\mathbb{E}(X_1) + (1-I)\,\mathbb{E}(X_2)$. Note that $I^2 = I$ and $(1-I)^2 = 1 - I$, hence $\text{Var}(Y \mid I) = I\,\text{Var}(X_1) + (1-I)\,\text{Var}(X_2)$.

c) $\mathbb{E}(Y) = p\mu_1 + (1-p)\mu_2$ and $\text{Var}(Y) = p(1-p)(\mu_1 - \mu_2)^2 + p\sigma_1^2 + (1-p)\sigma_2^2$, where $p = 0.3$, $\mu_1 = \sigma_1^2 = 1$, $\mu_2 = \sigma_2^2 = 2$.

8.7.

8.8.

8.9.

8.10.

8.11.

8.12.

8.13. Joint distribution table of X and Y is:

X/Y	1	2	3	4
1	0.1	0.1	0.1	0.1
2	0	0.1	0.1	0.1
3	0	0	0.1	0.1
4	0	0	0	0.1

Fixing a certain value of X we find the conditional expectation of Y:

$$\mathbb{E}(Y \mid X) = \begin{cases} \frac{1+2+3+4}{4} = 2.5, \text{ if } x = 1 \\ \frac{2+3+4}{3} = 3, \text{ if } x = 2 \\ \frac{3+4}{2} = 3.5, \text{ if } x = 3 \\ \frac{4}{1} = 4, \text{ if } x = 4 \end{cases}$$

Or, simply, $\mathbb{E}(Y \mid X) = 2 + X/2$.

8.14. Variable X has exponential distribution, so

$$f_X(x) = \lambda e^{-\lambda x}$$

And we know conditional distribution of Y:

$$f_{Y|X}(x, y) = \frac{1}{x} e^{-\frac{1}{x} y}$$

So,

$$\mathbb{E}(Y \mid X) = \int_{-\infty}^{+\infty} y f_{Y|X}(X, y) dy = \int_0^{+\infty} y \frac{1}{X} e^{-\frac{1}{X} y} dy = (1/X)^{-1} = X.$$

Expected value:

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y \mid X)) = \mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_0^{+\infty} x \lambda e^{-\lambda x}\, dx = \lambda^{-1}$$

Variance:

$$\mathrm{Var}(Y) = \mathrm{Var}(\mathbb{E}(Y \mid X)) + \mathbb{E}(\mathrm{Var}(Y \mid X)) = \mathrm{Var}(X) + \mathbb{E}((1/X)^{-2}) = \lambda^{-2} + \mathbb{E}(X^2)$$

We know that $\mathbb{E}(X^2) = \mathrm{Var}(X) + (\mathbb{E}(X))^2$, so:

$$\mathrm{Var}(Y) = \lambda^{-2} + \lambda^{-2} + \lambda^{-2} = 3\lambda^{-2}$$

8.15. If $X = 0$ we know that that the chosen ball was in the first basket, consequently Y was in the second one.

$$\mathbb{E}(Y \mid X = 0) = \mathbb{E}(Y \mid Y \text{ from second basket }) = \frac{1 + 2 + \cdots + 10}{10} = 5.5$$

Similarly

$$\mathbb{E}(Y \mid X = 10) = \mathbb{E}(Y \mid Y \text{ from first basket }) = \frac{0 + 1 + \cdots + 9}{10} = 4.5$$

Finaly,

$$\mathbb{E}(Y \mid 0 < X < 10) = \frac{1}{2}\mathbb{E}(Y \mid Y \text{ from first basket }) + \frac{1}{2}\mathbb{E}(Y \mid Y \text{ from second basket }) = 0.5(5.5{+}4.5) = 5$$

Answer:

$$\mathbb{E}(Y \mid X) = \begin{cases} 5.5, \text{ if } X = 0 \\ 5, \text{ if } 0 < X < 10 \\ 4.5, \text{ if } X = 10 \end{cases}$$

8.16. $\mathbb{E}(Y \mid X) = X/2, \mathbb{E}(Y) = \mathbb{E}(X/2) = a/4,$

$$\mathrm{Var}(Y) = \mathbb{E}(\mathrm{Var}(Y \mid X)) + \mathrm{Var}(\mathbb{E}(Y \mid X)) = \mathbb{E}(X^2/12) + \mathrm{Var}(X/2) = \frac{7a^2}{12^2}$$

8.17.

a) $\mathbb{E}(Y \mid X + Y) = 0$ and $\mathbb{E}(Y) = 0$.

b) Using the formula $\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}(X \mid X + Y)) + \mathbb{E}(\mathrm{Var}(X \mid X + Y))$, we obtain that $\mathrm{Var}(Y) + \mathbb{E}(\mathrm{Var}(X \mid X + Y)) = 0$.

Hence, $\mathrm{Var}(Y) = 0$ and $\mathrm{Var}(X \mid X + Y) = 0$.

c) The random variable Y is almost surely zero.

8.18.

8.19.

a) $\mathcal{F}_J, \mathcal{F}_{11}, \mathcal{F}_{12}, \ldots$

b) $\{T \text{ is divisible by } 2\}, \{T \geqslant 3, X_{T-2} = 1\}$.

c) $\mathbb{E}(T \mid \mathcal{F}_J) = T, \mathbb{E}(X_T \mid \mathcal{F}_J) = 1, \mathbb{E}(X_{T+1} \mid \mathcal{F}_J) = -1$.

d) $\mathbb{E}(S_T \mid \mathcal{F}_J) = S_T, \mathbb{E}(S_{T+1} \mid \mathcal{F}_J) = S_T - 1, \mathbb{E}(S_{T+2} \mid \mathcal{F}_J) = S_T - 1$.

e) $\mathbb{E}(Y_{T-k} \mid \mathcal{F}_J) = Y_{T-k}$.

f) $\mathbb{E}(X_T \mid \mathcal{F}_{10}) = 1, \mathbb{E}(X_{T+1} \mid \mathcal{F}_{10}) = -1$.

g) $\mathbb{E}(T \mid \mathcal{F}_{10}) = \ldots, \mathbb{E}(S_T \mid \mathcal{F}_{10}) = \ldots.$

8.20.

8.21. From $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 1$ we infer the value of c, $c = e/(e-1)^2$.
Observe that $f(x,y)$ may be decomposed into a product, hence X and Y are independent, hence $\mathbb{E}(X \mid Y) = \mathbb{E}(X) = 1/(e-1)$ and $\mathrm{Var}(X \mid Y) = \mathrm{Var}(X)$.

8.22.

8.23.

a) Find $\mathbb{E}(Y \mid \mathcal{F})$, $\mathrm{Var}(Y \mid \mathcal{F})$, $\mathbb{P}(Y = 3 \mid \mathcal{F})$.

b) $\mathbb{E}(Y \mid \mathcal{H}) = Y$, $\mathrm{Var}(Y \mid \mathcal{H}) = 0$, $\mathbb{P}(Y = 3 \mid \mathcal{H}) = I_{Y=3}$.

8.24. $\mathbb{E}(R+L \mid M) = n - M$, $\mathbb{E}(M \mid R+L) = n - R - L$, $\mathrm{Var}(R+L \mid M) = 0$, $\mathbb{E}(R \mid L) = (n-L)\frac{r}{1-l}$, $\mathrm{Var}(R \mid L) = (n-L)\frac{r}{1-l}\frac{1-r-l}{1-l}$, $\mathbb{P}(\mathbb{E}(R \mid L) = 0) = l^n$, $\mathbb{P}(R = 0 \mid L) = \left(\frac{r}{r+m}\right)^{n-L}$

8.

8.25.

a) $\mathbb{E}(Y \mid X) = 1/X$, $\mathrm{Var}(Y \mid X) = 1/X^2$.

b) $\mathbb{E}(Y) = \ln 2$, $\mathrm{Var}(Y) = 0.5 - (\ln 2)^2$.

c)

8.26.

a) $\mathbb{E}(Y \mid Y - X) = (4 + Y - X)/2$ and $\mathrm{Var}(Y \mid Y - X) = (4 - |Y - X|)^2/12$.

b) From $\mathcal{F}$: $A_1 = \{X > Y\}$, $A_2 = \{X > Y + 0.5\}$. Not from $\mathcal{F}$: $B_1 = \{X > 2\}$, $B_2 = \{Y < 3\}$.

c) Let's just add a known value $Y - X$ to the first term, $\mathrm{Cov}(X, Y \mid Y - X) = \mathrm{Cov}(X + (Y - X), Y \mid Y - X) = \mathrm{Var}(Y \mid Y - X)$.

8.27.

8.28.

8.29.

8.30.

9.1.

a) (X_t) is not a martingale with respect to $(\mathcal{F}_t)$;

b) (S_t) is not a martingale with respect to $(\mathcal{F}_t)$;

c) (W_t) is a martingale with respect to $(\mathcal{F}_t)$;

d) (Y_t) is a martingale with respect to $(\mathcal{F}_t)$;

e) (W_t) is not a martingale with respect to $(\mathcal{F}_{t-1})$;

f) (W_t) is not a martingale with respect to $(\mathcal{F}_{t+1})$;

9.2.

a) What is the distribution of X_7?

b) $\mathbb{E}(X_n) = 0$ and $\text{Var}(X_n) =$.

c) The process (X_n) is a Markov chain!

d) The process (X_n) is not a martingale.

e) $\sigma(X_1, X_2, X_3, \ldots, X_{1000}) = \sigma(X_1)$

f) $\mathbb{P}(\lim X_n \text{ exists}) = 1/3$.

g) $\text{plim}\, X_n$ does not exist.

9.3.

a) (X_t) is not a martingale;

b) (S_t) is not a martingale;

c) $a = 0.5$;

d) $b = 0$ and $b = \ldots$

9.4.

a) $X_0 = 4/52$; X_1 is $4/51$ with probability $48/52$ or $3/51$ with probability $4/52$; X_{51} is 1 with probability $4/52$ and 0 with probability $48/52$.

b)

c) With probability X_n James Bond will pick up a Queen and the current number of closed Queens $X_n(52-n)$ will decrease by 1. With probability $(1-X_n)$ James Bond will pick up a card different from Queen and the number of closed Queens $X_n(52-n)$ will stay the same.

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n \left(\frac{X_n(52-n)-1}{52-n-1}\right) + (1-X_n)\left(\frac{X_n(52-n)}{52-n-1}\right) = \cdots = X_n.$$

Hence the process (X_n) is a martingale with respect to $(\mathcal{F}_n)$.

9.5. It is possible.

9.6.

a) W_1 is equal to $1/3$ or $2/3$ with equal probabilities; W_2 is equal to $1/4$, $2/4$, $3/4$

b) (W_t) is a martingale;

c) (M_t) is a martingale;

d) The limiting distribution of (W_t) is uniform on $[0; 1]$;

9.7. Only constant non-random sequences are martingales.

9.8. $\operatorname{Cov}(M_d - M_c, M_b - M_a) = 0$;

9.9. Yes.

9.10.

9.11.

9.12. For example, $a_n = -3n$, but one may add any constant.

9.13. $\mathbb{E}(X_3 \mid X_1, X_2) = \mathbb{E}(X_3) = 0$, $\mathbb{E}(X_3 \mid X_1 + X_3) = (X_1 + X_3)/2$, $\operatorname{Var}(X_3 \mid X_1, X_3) = 0$, $\operatorname{Var}(X_3 \mid X_1 + X_3) = 1 - (X_1 + X_3)^2/4$.

Посчитаем ожидание и получим $Y_n = X_1 + X_2 + \ldots + X_n$, the process (Y_n) is a martingale.

9.14.

9.15.

$$\mathbb{E}(Y_{t+1} \mid Y_t) = Y_t \, \mathbb{E}(e^{bX_{t+1}})$$

$$\mathbb{E}(e^{bX_{t+1}}) = 1 \to b = 0 or b = \ln(1/9)$$

Trivial solution: $a = 0$ and any b.

9.16.

9.17.

9.18.

9.19.

9.20.

9.21.

a) (S_n) is a martingale;

b) $\mathbb{P}(S_\tau = b) = a/(a + b)$;

c) $M_n = S_n^2 - n$ is a martingale;

d) $\mathbb{E}(\tau) = ab$;

9.22.

a) (S_n) is not a martingale;

b) $K_n = (q/p)^{S_t}$ is a martingale;

c) $\mathbb{P}(S_\tau = b) = \ldots$

d) $M_n = S_n - (p-q)n$ is a martingale;

e) $\mathbb{E}(\tau) = \ldots$

9.23.

a) $\mathbb{E}(T) = 26^{11} + 26^4 + 26.$

b)

9.24.

9.25.

a) $\alpha = -0.5;$

b) $\beta = \ln 4 - \ln(e + e^{-1} + 2).$

c) $\mathbb{E}(T) = 2 \cdot 4^2 = 32.$

9.26.

9.27.

9.28.

10.1.

a) $\mathbb{P}(X_a < X_b) = \lambda_a/(\lambda_a + \lambda_b)$. There are two possible solutions: double integral and first step analysis.

b)

c)

d)

10.2. Yes.

10.3.

10.4.

a) $A_t = N_t - \lambda t$ is a martingale;

b) $B_t = A_t^2 - \lambda t$ is a martingale;

10.5. Let's measure time in minutes.

a) $\mathbb{P}(X_5 = 0) = \exp(-5\lambda) = 0.05$, so $\lambda = \ln(0.05)/-5 = \ln(20)/5.$

b) $\mathbb{E}(X_{180}) = 180\lambda$, $\mathrm{Var}(X_{180}) = 180\lambda$

c) $\mathbb{P}(X_{180} = 5) = \exp(-180\lambda)(180\lambda)^5/5!$

10.6. $\mathrm{Cov}(N_3, N_{10}) = \mathrm{Cov}(N_3, N_3 + (N_{10} - N_3)) = \mathrm{Var}(N_3) = 3\lambda$.

10.7.

10.8. Here we may approximate Poisson distribution by normal distribution, $\mathcal{N}(\lambda t, \lambda t)$.

10.9.

$$\mathbb{E}(Y) = (2, 4, 20), \quad \mathrm{Var}(Y) = \begin{pmatrix} 2 & ? & ? \\ ? & 4 & ? \\ ? & ? & 20 \end{pmatrix}$$

10.10.

10.11.

10.12. Let's denote $a(t) = \mathbb{P}(N_t \text{ is even})$.

$$a(t + \Delta) = a(t)(1 - \Delta) + (1 - a(t))\Delta + o(\Delta)$$

Hence we get a differential equation $a'(t) = 1 - 2a(t)$ with $a(0) = 1$. The solution is $a(t) = (1 + \exp(-2t))/2$.

10.13.

10.14. $Y_1 \sim \mathrm{Expo}(\lambda)$

10.15. https://math.stackexchange.com/questions/3919517/inter-arrival-time

10.16. The rate is $\lambda = 3/2$ exercises per academic hour.

a) $\mathbb{P}(X_1 = 1) = \exp(-3/2)(3/2)^1/1!$;

b) $\mathbb{P}(X_1 = 1,\ X_2 = 2) = \mathbb{P}(X_1 = 1) \cdot \mathbb{P}(X_2 - X_1 = 1 \mid X_1 = 1) = \exp(-3)(9/4)$;

10.17.

10.18.

10.19.

10.20.

10.21.

a) $\mathbb{E}(X) = \lambda$;

b) $\lim_{n\to\infty} \operatorname{Var}(X) = \lambda$;

c) $\lim_{n\to\infty} \mathbb{P}(X = 0) = \exp(-\lambda)$, $\lim_{n\to\infty} \mathbb{P}(X = 1) = \lambda \exp(-\lambda)$;

d) $\lim_{n\to\infty} \mathbb{P}(X = k) = \exp(-\lambda)\lambda^k/k!$.

10.22.

10.23.

10.24.

a) $Y_1 \sim \text{Expo}(\lambda)$;

b) $(Y_1 \mid X_1 = 1) \sim \text{Unif}[0; 1]$;

c) $(Y_1 \mid X_t = 1) \sim \text{Unif}[0; t]$;

d) $(Y_5 \mid X_1 = 12) \sim \text{Beta}(5; 8)$;

e) $(Y_5/t \mid X_t = 12) \sim \text{Beta}(5; 8)$.

10.25.

10.26.

10.27.

11.1.

a) $\mathbb{E}(W_5) = 0$ and $\operatorname{Var}(W_5) = 5$.

b) $\mathbb{E}(W_t) = 0$ and $\operatorname{Var}(W_t) = t$.

c) $\mathbb{E}(W_5 \mid W_3) = W_3$ and $\operatorname{Var}(W_5 \mid W_3) = 2$.

d) $\mathbb{E}(W_t \mid W_s) = W_s$ and $\operatorname{Var}(W_t \mid W_s) = t - s$.

e) $\operatorname{Cov}(W_5, W_3) = 3$ and $\operatorname{Cov}(W_s, W_t) = \min\{s, t\}$.

f) $\operatorname{Cov}(W_5, W_3 \mid W_2) = 1$ and $\operatorname{Cov}(W_s, W_t \mid W_r) = \min\{s, t\} - r$.

g) $\mathbb{E}(W_5 - 2W_3) = 0$ and $\operatorname{Var}(W_5 - 2W_3)$.

h) $\mathbb{P}(W_5 - 2W_3 > 0) = 1/2$ and $\mathbb{P}(W_5 - 2W_3 > 1)$.

i) $\mathbb{E}(W_5 - 2W_3 \mid W_2 = 1)$ and $\operatorname{Var}(W_5 - 2W_3 \mid W_2 = 1)$.

j) $\mathbb{P}(W_5 - 2W_3 > 1 \mid W_2 = 1)$.

k)

$$R \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{pmatrix}\right)$$

l) $W_3 = 3/5 \cdot W_5 + U$

m) $\mathbb{E}(W_3 \mid W_5) = 3/5 \cdot W_5$ and $\text{Var}(W_3 \mid W_5) = \text{Var}(U) = \text{Var}(W_3 - 3/5W_5)$.

11.2. $\text{Var}(Y_t - Y_s) = \text{Var}(tW_{2t} - sW_{2s}) = 2t^3 + 2s^3 - 4ts^2$. We get $\mathbb{E}(M_{t+u} \mid \mathcal{F}_t) = W_t^3 + 3W_t u + \alpha(t+u)W_t$. From $\mathbb{E}(M_{t+u} \mid \mathcal{F}_t) = W_t^3 + \alpha t W_t$ it follows that $\alpha = -3$.

11.3.

11.4.

a) $\mathbb{E}(W_5 W_4 \mid W_4) = W_4^2$, $\text{Var}(W_5 W_4 \mid W_4) = W_4^2$;

b) $\text{Cov}(W_5 W_4, W_5 W_6) = 40$;

11.5.

11.6.

a) (W_t) is a martingale;

b) $Q_t = W_t^2 - t$ is a martingale.

11.7.

a) $Y_0 = 0$;

b) $Y_t - Y_s \sim \mathcal{N}(0; t - s)$;

c) yes;

d) yes;

e) yes;

f) If (W_t) is a Wiener process, then $Y_t = W_{a^2 t}/a$ is a Wiener process.

11.8.

a) $Y_0 = 0$;

b) $Y_t - Y_s \sim \mathcal{N}(0; t - s)$;

c) yes;

d) yes;

e) yes;

f) If (W_t) is a Wiener process, then $Y_t = \begin{cases} 0, & \text{if } t = 0; t \cdot W_{1/t}, & \text{if } t > 0. \end{cases}$ is a Wiener process.

11.9.

$$\mathbb{E}(Y_t) = \mathbb{E}(W_t^3 - tW_t^4) = \mathbb{E}(W_t^3) - \mathbb{E}(tW_t^4) = \mathbb{E}(W_t^3) - t\,\mathbb{E}(W_t^4) = 0 - 3t^3 = -3t^3,$$

To calculate $\text{Var}(Y_t) = \mathbb{E}(Y_t^2) - (\mathbb{E}(Y))^2$ we start with

$$\mathbb{E}(Y_t^2) = \mathbb{E}(W_t^3 - tW_t^4)^2 = \mathbb{E}(W_t^6 - 2tW_t^7 + t^2W_t^8) = \mathbb{E}(W_t^6) - 2t\,\mathbb{E}(W_t^7) + t^2\,\mathbb{E}(W_t^8) = 15t^3 + 105t^6.$$

11.10.

a) $\text{Corr}(W_4 - W_2, W_3 - W_1) = 1/\sqrt{2 \cdot 2} = 0.5$.

b) Let's find best linear prediction of $W_3 - W_1$ given $W_4 - W_2$, $\beta = \text{Cov}(W_4 - W_2, W_3 - W_1)/Var(W_4 - W_2) = 0.5$, hence $W_3 - W_1 = 0.5(W_4 - W_2) + R$.

   It follows that
   $$\mathbb{E}(W_3 - W_1 \mid W_4 - W_2) = 0.5(W_4 - W_2)$$
   and that
   $$\text{Var}(W_3 - W_1 \mid W_4 - W_2) = \text{Var}(R) = 2 - 0.25 \cdot 2 = 1.5.$$

11.11.

a) $\mathbb{E}(W_3^4) = 3^2 \cdot 3 = 27$, $W_3 = W_2 + I$ where $I \sim \mathcal{N}(0,1)$, hence $W_3^4 = (W_2 + I)^4$ and $\mathbb{E}(W_3^4 \mid W_2 = 10) = 10^4 + 6 \cdot 10^2 + 3$.

b) Using Isserlis theorem, $\text{Cov}(W_1W_2, W_5W_6) = \mathbb{E}(W_1W_2W_5W_6) - \mathbb{E}(W_1W_2)\,\mathbb{E}(W_5W_6) = (5 + 2 + 2) - 5 = 4$.

11.12.

11.13.

12.1.

12.2.

12.3.

12.4.

12.5.

12.6.

a) $dX_t = (W_t^3 - 3W_t^2 \cdot t)dW_t$ (4 points),
   $$X_t = X_0 + \int_0^t W_u^3 - 3W_u^2 \cdot u\, dWu$$

b) A process is a martingale as in short form $A_t dt = 0$.

12.7.

a) $dX_t = -2X_t dW_t$, this process is a martingale;

b) $\mathbb{E}(X) = 1$, $\text{Var}(X) = \exp(4t) - 1$.

c) $\int_0^t X_u \, dWu = \frac{1-X_t}{2}$.

12.8.

12.9.

12.10.

12.11.

a) Let's use Ito's lemma

$$dX_t = f'(t)\cos(2021W_t)dt - 2021f(t)\sin(2021W_t)dW_t + \frac{1}{2}2021^2 f(t)\cos(2021W_t)dt$$

b) To make X_t a martingale we should kill dt term.

c) As X_t is martingale $\mathbb{E}(X_t) = \mathbb{E}(X_0) = f(0)$. So $\mathbb{E}(\cos(2021W_t)) = f(0)/f(t)$.

12.12. Using Ito's lemma $dY_t = (3W_t^2 - 3t)\,dW_t + (-3W_t + \frac{1}{2}6W_t)\,dt = (3W_t^2 - 3t)\,dW_t$. Hence, Y_t is a martingale. $Y_0 = W_0^3 - 3\cdot 0W_0 = 0$.

Now $\mathbb{E}(Y_t) = \mathbb{E}(Y_0) = 0$ and using Ito's isometry:

$$\text{Var}(Y_t) = \int_0^t \mathbb{E}((3W_s^2 - 3s)^2)\,ds = \int_0^t 27s^2 + 9s^2 - 18s^2\,ds = \int_0^t 18s^2\,ds = 6t^3$$

Here we have used the facts that $\mathbb{E}(W_s^2) = s$ and $\mathbb{E}(W_s^4) = 3s^2$.

12.13. $dY_t = -aY_t\,dW_t$, so Y_t is a martingale, and $\mathbb{E}(Y_t) = Y_0 = 1$. To find variance one may use Ito's isometry.

12.14.

a) $h(x,t) = x$;

b) The coincidence is possible because the first, second and third moments of W_t and S_t are the same.

c) $h(x,t) = x^4 - 6tx^2$;

12.15.

$$dM_t = h(t)\exp(iW_t)idW_t + 0.5h(t)\exp(iW_t)i^2(dW_t)^2 + h'(t)\exp(iW_t)dt.$$

For M_t to be a martingale we need $h' = 0.5h$, $h(t) = \exp(t/2)$ and $M_t = \exp(iW_t + t/2)$.

Both $\text{Re}\, M_t = \cos(W_t)\exp(t/2)$ and $\text{Im}\, M_t = \sin(W_t)\exp(t/2)$ are martingales.

12.16.

12.17.

a) $\mathbb{E}(A_t) = 0$, $\mathbb{E}(C_t) = t^3/3$.

b) $\text{Var}(B_t) = 15t^7$, $\text{Cov}(A_t, C_t) = \ldots$.

12.18.

12.19.

12.20. Let $Q_t = \int_0^t u^3\, dW_u + \int_0^t 2u\, du$, so $R_t = \exp(Q_t)$. By Ito's lemma:

$$dR_t = \exp(Q_t) dQ_t + 0.5 \exp(Q_t)(dQ_t)^2 = \exp(Q_t)(t^3\, dW_t + 2t\, dt) + 0.5 \exp(Q_t) t^6\, dt.$$

The differtial equation is $h'(t) = h(t)(2t + t^6/2)$.

12.21.

a) $\mathbb{E}(A_t) = 0$, $\mathbb{E}(C_t) = t^2/2$.

b) $\text{Var}(A_t) = t^3$, $\text{Cov}(A_t, B_t) = 15t^4/4$, $\text{Var}(C_t) = 7t^4/12 - (t^2/2)^2$.

$$\mathbb{E}(C_t^2) = \int_0^t \int_0^t \mathbb{E}(W_u^2 W_s^2)\, du\, ds = 2 \int_{s=0}^t \int_{u=0}^s 2u^2 + us\, du\, ds = 2 \cdot 7/24 t^4.$$

12.22.

12.23.

13.1. Using Ito's lemma we find dX_t and dY_t

$$dX_t = f_t'(t) \cos W_t dt - f(t) \sin W_t\, dW_t - 0.5 f(t) \cos W_t\, dt = \\ = (f_t'(t) - 0.5 f(t)) \cos W_t dt - f(t) \sin W_t\, dWt$$

$$dY_t = g_t'(t) \sin W_t\, dt + g(t) \cos W_t\, dW_t - 0.5 g(t) \sin W_t\, dt = \\ = (g_t'(t) - 0.5 g(t)) \sin W_t\, dt + g(t) \cos W_t\, dWt$$

Comparing these expressions with the system we receive:

$$\begin{cases} f_t'(t) - 0.5 f(t) = a f(t) \\ f(t) = g(t) \\ g_t'(t) - 0.5 g(t) = a g(t) \\ g(t) = f(t) \end{cases}$$

The general solution has the form

$$f(t) = g(t) = A e^{(a+0.5)t}, \quad A \in \mathbb{R}$$

From initial condition we get $A = x_0$ and

$$f(t) = g(t) = x_0 e^{(a+0.5)t}$$

Answer:

$$X_t = x_0 + x_0 \int_0^t a e^{(a+0.5)u} \cos W_u du - x_0 \int_0^t e^{(a+0.5)t} \sin W_u \, dWu$$
$$Y_t = x_0 \int_0^t a e^{(a+0.5)u} \sin W_u du + x_0 \int_0^t e^{(a+0.5)t} \cos W_u \, dWu$$

Now we use Ito's lemma once again:

$$\begin{aligned} dD_t = 2X_t\, dX_t + 2Y_t\, dY_t = 2X_t((f_t'(t) - 0.5f(t)) \cos W_t\, dt - f(t) \sin W_t\, dWt) + \\ + 2Y_t((g_t'(t) - 0.5g(t)) \sin W_t], dt + g(t) \cos W_t\, dWt) = \\ = 2f(t)(f_t'(t) - 0.5f(t)) \cos^2 W_t\, dt - f^2(t) \cos W_t \sin W_t\, dWt + \\ + 2f(t)(f_t'(t) - 0.5f(t)) \sin^2 W_t\, dt + f^2(t) \cos W_t \sin W_t\, dWt = \\ = 2f(t)(f_t'(t) - 0.5f(t)\, dt = 2af^2(t)\, dt \end{aligned}$$

$$D_t = D_0 + \int_0^t 2ax_0 e^{(2a+1)u} du = x_0^2 + \frac{2ax_0}{2a+1}(e^{(2a+1)t} - 1)$$

13.2.

$$dY_t = \frac{1}{\tau} X_t e^{\frac{t}{\tau}} dt + e^{\frac{t}{\tau}} dX_t = \frac{1}{\tau} X_t e^{\frac{t}{\tau}} dt + \frac{X_\infty - X_t}{\tau} e^{\frac{t}{\tau}} dt + \sigma e^{\frac{t}{\tau}} dW_t = $$
$$= \frac{X_\infty}{\tau} e^{\frac{t}{\tau}} dt + \sigma e^{\frac{t}{\tau}} dW_t$$

$$Y_t = Y_0 + \int_0^t \frac{X_\infty}{\tau} e^{\frac{u}{\tau}} du + \int_0^t \sigma e^{\frac{u}{\tau}} dW_u = X_0 + X_\infty(e^{\frac{t}{\tau}} - 1) + \int_0^t \sigma e^{\frac{u}{\tau}} dW_u$$
$$X_t = e^{-\frac{t}{\tau}} Y_t = e^{-\frac{t}{\tau}}(X_0 + X_\infty(e^{\frac{t}{\tau}} - 1) + \int_0^t \sigma e^{\frac{u}{\tau}} dW_u) = X_0 e^{-\frac{t}{\tau}} + X_\infty(1 - e^{-\frac{t}{\tau}}) + \int_0^t \sigma e^{\frac{u-t}{\tau}} dW_u$$
$$\mathbb{E}(X_t) = X_0 e^{-\frac{t}{\tau}} + X_\infty(1 - e^{-\frac{t}{\tau}})$$
$$\operatorname{Var}(X_t) = \operatorname{Var}(\int_0^t \sigma e^{\frac{u-t}{\tau}} dW_u) = \int_0^t \mathbb{E}(\sigma e^{\frac{u-t}{\tau}})^2 du = \int_0^t \sigma^2 e^{\frac{2u-2t}{\tau}} du = \frac{\tau\sigma^2}{2}(1 - e^{\frac{-2t}{\tau}})$$

13.3.

14.1.

a) $p_u^* = p_d^* = 1/2$;

b) $X_1^u = X_1^d = (0.5 \cdot 100 + 0.5 \cdot 0)/1.05$, hence $X_0 = 50/1.05^2 \approx 45.35$;

c) $\alpha = X_2^{uu} - X_2^{ud}/(S_2^{uu} - S_2^{ud}) = 100/216 \approx 0.46$;

14.2.

14.3.

14.4.

15.1.

$$X_0 = \exp(-2r)\, \mathbb{E}_*(X_2)$$
$$X_2 = S_1^3 = S_0^3 \exp(3r) \exp(3\sigma W_1^* - 9\sigma^2/2)$$

$$X_0 = S_0^3 \exp(r) \exp(3\sigma^2)$$

15.2.

15.3.

15.4.

$$X_0 = \mathbb{E}_{\tilde{p}}(\exp(-rT)X_T \mid \mathcal{F}_0)$$

15.5. We need to find a price of the classic European option in dollars, while stock prices are in euros.

15.6.

15.7.

15.8.

$$(1 - 0.4L)y_t = 4 + (1 + 0.3L)u_t$$

15.9. $x_t = (1 - L)^t y_t$

15.10. Recall that $F_n = L(1 + L)F_n$, hence $F_n = L^k(1 + L)^k F_n$ or $F_{n+k} = (1 + L)^k F_n$ and $a = 1$.

15.11.

a) false;

b) true;

c) true;

d) false.

16.1.

16.2.

16.3.

a) $u_t \sim \mathcal{N}(0; 1)$ and independent;

b) $u_t \sim \mathcal{N}(0; 1)$ for $t > 1$, u_1 takes values -1 or 1 with probability $1/2$. All (u_t) are independent.

c) Variables u_t are independent and identically distributed with infinite $\mathbb{E}(u_t)$.

d) $u_t \sim \mathcal{N}(t; 1)$ and independent.

16.4.

a) stationary, $\gamma_k^z = 4\gamma_k^y$.

b) stationary, $\gamma_k^z = \gamma_k^y$.

c) stationary;

d) stationary;

16.5.

a)

b) Process (y_t) is stationary for $y_0 = 4 + \frac{2}{\sqrt{3}} u_0$.

c) Expression $y_t = 2 + 0.5y_{t-1} + u_t$ is only an equation. There are infinitely many stochastic processes that satisfy this equation. Most of them are non-stationary, but it has a stationary solution.

16.6.

a) $y_t = x_{2t}$;

b) Yes, (y_t) is stationary.

c)

16.7.

a) The process (y_t) is stationary.

b)

16.8.

a) $\mathbb{P}(x_t = 1) = \mathbb{E}(x_t) = 1/2$;

b) $\text{Cov}(x_t, x_s) = 0$ for $t \neq s$;

c) The process (x_t) is stationary but not a white noise as $\mathbb{E}(x_t) \neq 0$.

d) $\mathbb{E}(s_t) = t/2$;

e) $s_t \sim \text{Bin}(t, 1/2)$;

f) Use the central limit theorem.

16.9.

a) (a_t) are identically distributed; (b_t) are identically distributed.

b) (a_t) are dependent, for example if $a_t \neq 0$ then $a_{t-1} = a_{t+1} = 0$; (b_t) are dependent.

c) (a_t) is a white noise; (b_t) is a white noise.

16.10.

a) $\mathbb{P}(z_t = 1) = \mathbb{E}(z_t) = 2/3$;

b) $\text{Cov}(z_1, z_2) =$, $\text{Cov}(z_1, z_3) =$, $\text{Cov}(z_1, z_4) = 0$;

c) (z_t) is stationary;

d)

e)

f) $\mathbb{E}(s_t) = (t-2) \cdot 2/3$, $\text{Var}(s_t) = (16t - 29)/90$;

g) (s_t) is not stationary;

16.11.

a) The process (x_t) may not be stationary;

b) The process (y_t) may not be stationary;

c) The process (x_t) is stationary, but (y_t) may not.

16.12.

16.13.

16.14.

16.15. Projections: $\tilde{X}_1 = X_1 + Z$; $\tilde{X}_2 = X_2 + Z$; $\mathbb{E}(X_i|Z) = 1 - Z$; $\text{Cov}(X_i, Z) = -1/4$;
Variable Z has Bernoulli distribution, hence $\mathbb{E}(Z) = 1/2$ и $\text{Var}(Z) = 1/4$;

$$\text{pCorr}(X_1, X_2; Z) = \frac{-1/2}{12.5} = -\frac{1}{\sqrt{50}}$$

$$\text{Corr}(X_1, X_2|Z) = -Z/6$$

16.16.

17.1.

17.2.

a) Infinitely many non-stationary solutions.

b) $\lambda^2 - \lambda - 1 = 0$, $\lambda = (1 \pm \sqrt{5})/2$; there is exactly one stationary solution, this solution can not be represented as $MA(\infty)$ with respect to (u_t).

c) No;

17.3.

17.4.

17.5.

a) Let's denote by x all available information,

$$x = \begin{pmatrix} y_{100} \\ y_{99} \\ y_{98} \\ u_{99} \end{pmatrix}$$

Let's use $t = 100$:

$$y_{100} = 1 + 0.5y_{98} + u_{100} + u_{99}$$

Using all available information we obtain $u_{100} = 1.5$ and hence

$$y_{101} \mid x \sim \mathcal{N}(1 + 0.5y_{99} + u_{100}; 4)$$

b) Here we work with true betas:

$$\mathbb{E}(y_{101} \mid y_{100}) = \mu_y + \frac{\text{Cov}(y_{100}, y_{101})}{\text{Var}(y_{100})}(y_{100} - \mu_y)$$

17.6.

a) $\mathbb{E}(y_t) = 5$

b) $\rho_3 = \rho_4 = \ldots = 0$

c) The process is stationary.

17.7.

a)

b) Here past information is not useful, $\mathbb{E}(y_t) = 5$, $\text{Var}(y_t) = 4 \cdot (1 + 2^2 + 4^2) = 84$.

$$[5 - 1.96\sqrt{84}; 5 + 1.96\sqrt{84}]$$

17.8.

a) $\phi_{11} = \rho_1 = 0.2$, $\phi_{22} = 0$

b) Possible equation is $y_t = 0.2y_{t-1} + u_t$. Another possibility is $y_t = 5y_{t-1} + u_t$. In the second case the stationary solution will be forward-looking and not $MA(\infty)$ with respect to (u_t).

17.9.

a) $\lambda_1 = 2$, $\lambda_2 = 0.5$, here the roots of the lag polynomial are exactly the same.

b) $(1 - 2L)(1 - 0.5L)(y_t + 10) = u_t$

c) The equation has infinitely many non-stationary solutions.

d) The equation has unique stationary solution.

e) The equation has no stationary solutions that are $MA(\infty)$ with respect to (u_t).

17.10.

a) [2 points] $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, one stationary solution, infinitely many non-stationary solutions.

b) [6 points]: [2 points] for the system + [2 points] for ρ_1 + [2 points] for ρ_2.

$$\begin{cases} \gamma_1 = 0.7\gamma_0 - 0.12\gamma_1 \\ \gamma_2 = 0.7\gamma_1 - 0.12\gamma_0. \end{cases}$$

$$\rho_1 = 70/112 = 0.625, \quad \rho_2 = 49/112 - 0.12 = 0.3175$$

c) [2 points]

$$\alpha_1 = 0.7, \quad \alpha_2 = 0.37$$

17.11.

a) Yes, the process is stationary, that is $MA(2)$ process.

b) $\rho_3 = \rho_4 = \ldots = 0$

c) $d_t = u_t + u_{t-1} + 2u_{t-2} - u_{t-1} - u_{t-2} - 2u_{t-3}$, hence $d_t \sim ARIMA(0, 0, 3)$.

17.12.

a)

b) $\phi_{11} = \rho_1$, $\phi_{22} = -0.2$, $\phi_{33} = \phi_{44} = \ldots = 0$. The partial correlation ϕ_{22} measures how will y_t on average react to the unit change of y_{t-2} given fixed y_{t-1}.

c) $\rho_{100} = -0.9\rho_{99} - 0.2\rho_{98}$

17.13.

17.14.

a) $y_t = 1 + u_t + 0.5u_{t-1} + 0.25u_{t-2}$ admits a stationary solution.

b) $y_t = -2y_{t-1} - 3y_{t-2} + u_t + u_{t-1}$

c) $y_t = -0.5y_{t-1} + u_t$ admits a stationary solution.

d) $y_t = 1 - 1.5y_{t-1} - 0.5y_{t-2} + u_t - 1.5u_{t-1} - 0.5u_{t-2}$

e) $y_t = 1 + 0.64y_{t-2} + u_t + 0.64u_{t-1}$ admits a stationary solution.

f) $y_t = 1 + t + u_t$ does not admit a stationary solution.

g) $y_t = 1 + y_{t-1} + u_t$ does not admit a stationary solution.

17.15. $\hat{\beta}_1 \approx \hat{\gamma}_1 \approx 1$ and $\hat{\beta}_2 \approx \hat{\gamma}_2 \approx 0.5$ as estimates can be recovered from sample autocovariance function and sample mean.

17.16.

a) $y_0 = 4 + 2u_0/\sqrt{3}$, $y_0 = 4 - 2u_0/\sqrt{3}$, ...

b) The forecasts are the same as the distribution of the vector $y = (y_1, y_2, \ldots, y_n)$ is the same.

c) There will be almost surely unique stationary solution.

17.17.

a) $|\mathrm{Cov}(y_0, u_1 + u_2 + \cdots + u_k)| \leqslant \sqrt{k\gamma_0\sigma^2}$.

b) From the recurrence equation it follows that

$$\mathrm{Var}(y_{t+k}) = \mathrm{Var}(u_1 + u_2 + \cdots + u_k + y_0).$$

Hence,

$$0 = k\sigma^2 + 2\,\mathrm{Cov}(y_0, u_1 + u_2 + \cdots + u_k).$$

That is not possible as the covariance is bounded by something proportional to $\sqrt{k}$.

c) Let's denote $(1 - 0.5L)y_t$ by x_t. The process (x_t) is also stationary and can't satisfy equation $x_t = x_{t-1} + u_t$.

d)

$$\mathrm{Var}(m_1 + m_2 + \cdots + m_k) = kc_0 + (k-1)c_1 + (k-2)c_2$$

Hence the limit is equal to the sum of covariances $c_0 + c_1 + c_2$.

e) Now the process $x_t = (1 - 0.5L)y_t$ should satisfy $x_t = x_{t-1} + m_t$. By the same reasoning

$$0 = kc_0 + (k-1)c_1 + (k-2)c_2 + 2\,\mathrm{Cov}(x_0, u_1 + u_2 + \cdots + u_k)$$

If $c_0 + c_1 + c_2 \neq 0$ then covariance can't compensate the linear growth of the first three terms.

f) Any process with unit root in MA-polynomial will do. For example, $m_t = (1-L)(1-0.5L)u_t = u_t - 1.5u_{t-1} + 0.5u_{t-2}$.

g) The same logic applies :)

17.18. If $\mathbb{E}(y_t) = \mu$ then $\mu = 5 + 2\mu - \mu + 0$ and we arrive to a contradiction.

17.19. Observe that $\mathrm{Var}(y_t) = 4\,\mathrm{Var}(y_{t-1}) + \sigma^2$ and we arrive to a contradiction.

17.20. The right answer is $MA(0)$ process or white noise. Explanation: if one adds $AR(p)$ process to an independent white noise then one obtains $ARMA(p,p)$ process. Let's denote the processes by a_t, b_t and $a_t + b_t = c_t$. We know that $A(L)a_t = u_t$, where $A(L)$ is some polynomial of order p. Hence $A(L)a_t + A(L)b_t = A(L)c_t$. All autocorrelations of $A(L)a_t$ are zeros. The process b_t is a white noise, hence all autocorrelations of $A(L)b_t$ are zeros for order $p+1, p+2$, etc. One may deduce that $A(L)c_t$ can be written as $MA(p)$ process and $c_t \sim ARMA(p,p)$.

18.1.

18.2.

18.3.

18.4.

Observe that $(1-L)\ell_t = 0.3u_t$ and $(1-L^2)s_t = 0.1u_t$. Hence it is sufficient to multiply y_t by $(1-L^2)$ to get rid of ℓ_t and s_{t_2} in the right hand side.

18.5. The process is not stationary as $\mathbb{E}(y_1) = 3$ and $\mathbb{E}(y_2) = -3$.

18.6.

a)

$$y_{102} = \ell_{100} + (0.9 + 0.9^2)b_{100} + (0.3 + 0.18)u_{101} + u_{102}$$

$$(y_{102} \mid y_1, \ldots, y_{100}) \sim \mathcal{N}(21.71, 24.608)$$

The interval

$$[21.71 - 1.96 \cdot 4.96; 21.71 + 1.96 \cdot 4.96]$$

b)

$$\lim_{h\to\infty} \mathbb{E}(y_{100+h} \mid y_1, \ldots, y_{100}) = \ell_{100} + (0.9 + 0.9^2 + \ldots)b_{100} = 20 + 9 \cdot 1$$

18.7.

18.8.

18.9.

18.10.

18.11.

19.1.

19.2.

a) [4 points]

$$\sigma^2_{101} = 3 + 0.5(-1)^2 = 3.5$$

$$(u_{101} \mid \sigma_{101}) \sim \mathcal{N}(0; \sigma^2_{101})$$

$$[-1.96\sqrt{3.5}; +1.96\sqrt{3.5}]$$

b) [3 points] [1 point] for $\mathbb{E}(u_t)$ and [2 points] for $\operatorname{Var}(u_t)$ The process (u_t) is a white noise, hence

$$\mathbb{E}(u_t) = 0.$$

$$\sigma^2_u = 3 + 0.5 \cdot \sigma^2_u$$

c) [3 points]: [1 point] for $\text{Corr}(u_t, u_{t-1})$ and [2 points] for $\text{Corr}(u_t^2, u_{t-1}^2)$ The process (u_t) is a white noise, hence

$$\text{Corr}(u_t, u_{t-1}) = 0.$$

$$u_t^2 = 3 + 0.5u_{t-1}^2 + (u_t^2 - \sigma_t^2)$$

We notice that $r_t = u_t^2 - \sigma_t^2$ is a white noise, hence u_t^2 is an $AR(1)$ process. Hence, $\text{Corr}(u_t^2, u_{t-1}^2) = 0.5$.

19.3.

19.4.

19.5.

19.6.

19.7.

19.8.

20.1. Let's observe that we may decompose X_i as a sum $X_i = Y_i + \theta$, where $Y_i \sim \text{Expo}(\lambda)$. Hence, $\mathbb{E}(X_i) = 1/\lambda + \theta$, $\text{Var}(X_i) = \text{Var}(Y_i) = 1/\lambda^2$ and $\mathbb{E}(X_i^2) = 1/\lambda^2 + (1/\lambda + \theta)^2$. There is an alternative solution with direct integration:

$$\mathbb{E}(X_i) = \int_\theta^{+\infty} x f(x)\, dx, \quad \mathbb{E}(X_i^2) = \int_\theta^{+\infty} x^2 f(x)\, dx.$$

a) Solving $1/\hat{\lambda} + 1 = \bar{X}$ we obtain $\hat{\lambda} = 1/(\bar{X} - 1)$.

b) Solving for $\hat{\lambda}$ and $\hat{\theta}$ the system

$$\begin{cases} 1/\hat{\lambda} + \hat{\theta} = \bar{X} \\ 1/\hat{\lambda}^2 + (1/\hat{\lambda} + \hat{\theta})^2 = M_2 \text{ with } M_2 = \sum X_i^2/n \end{cases}$$

we obtain

$$\hat{\lambda} = \frac{1}{\sqrt{M_2 - \bar{X}^2}}, \quad \hat{\theta} = \bar{X} - \sqrt{M_2 - \bar{X}^2}$$

20.2. The log-likelihood function is equal to

$$\ell(a) = \sum_{i=1}^{n} \left((-0.5)\ln(4\pi) - 0.5\ln a - (x_i - a)^2/4a\right).$$

The equation $\ell'(a) = 0$ may be simplified to

$$n\hat{a}^2 + 2n\hat{a} - \sum X_i^2 = 0$$

Hence,

$$\hat{a} = \frac{-2n \pm \sqrt{4n^2 + 4n\sum X_i^2}}{2n}$$

We choose the root $\hat{a} > 0$ as $\operatorname{Var}(X_i) = 2a > 0$.

$$\hat{a} = \sqrt{1 + \sum X_i^2/n} - 1$$

Just for fun. In the case $X_i \sim \mathcal{N}(a, ka)$ the equation would be

$$n\hat{a}^2 + kn\hat{a} - \sum X_i^2 = 0$$

And

$$\hat{a} = \frac{-nk + \sqrt{k^2n^2 + 4n\sum X_i^2}}{2n}.$$

20.3.

20.4.

a)

$$L = const(0.2 + a)^{N_1}(0.3 - a)^{N_2}0.5^{N_3}$$

$$\ell = const + N_1\ln(0.2 + a) + N_2\ln(0.3 - a) + N_3\ln 0.5$$

$$\frac{\partial\ell}{\partial a} = \frac{N_1}{0.2 + a} - \frac{N_2}{0.3 - a}$$

$$\hat{a}_{ML} = \frac{0.3N_1 - 0.2N_2}{N_1 + N_2}$$

We see that $\partial\ell/\partial a$ decreases as a increases, so $\hat{a}_{ML}$ is indeed the point of maximum.

b)

$$\mathbb{E}(Y_i) = (0.2 + a) + 2(0.3 - a) + 4 \cdot 0.5 = 2.8 - a$$

$$\bar{Y} = \frac{N_1 + 2N_2 + 4N_4}{N_1 + N_2 + N_4}$$

$$\hat{a}_{MM} = 2.8 - \frac{N_1 + 2N_2 + 4N_4}{N_1 + N_2 + N_4}$$

20.5.

20.6.

20.7.

20.8.

20.9.

21.1.

a) Under H_0 we have $X_i \sim \text{Pois}(\lambda)$, $Y_i \sim \text{Pois}(\lambda)$.

$$\ell(\lambda) = \sum_{i=1}^{n_x} (-\lambda + X_i \ln \lambda - \ln(X_i!)) + \sum_{i=1}^{n_y} (-\lambda + Y_i \ln \lambda - \ln(Y_i!))$$

The score function is

$$\text{score}(\lambda) = \ell'(\lambda) = \sum_{i=1}^{n_x} (-1 + X_i/\lambda) + \sum_{i=1}^{n_y} (-1 + Y_i/\lambda) \, .$$

The estimator is $\hat{\lambda} = (\sum X_i + \sum Y_i)/(n_x + n_y)$.

$$\max \ell_R = -\hat{\lambda}(n_x + n_y) + \left(\sum X_i + \sum Y_i\right) \ln \hat{\lambda} - \sum \ln X_i! - \sum \ln Y_i!$$

b) In unrestricted model we have two independent estimators,

$$\hat{\lambda}_x = \bar{X}, \quad \hat{\lambda}_y = \bar{Y}$$

$$\max \ell_{UR} = -\hat{\lambda}_x n_x + \sum X_i \ln \hat{\lambda}_x + \sum Y_i \ln \hat{\lambda}_y - \sum \ln X_i! - \sum \ln Y_i!$$

c)

$$LR = 2(\max \ell_{UR} - \max \ell_R) = 2 \sum X_i (\ln \hat{\lambda}_x - \ln \hat{\lambda}) + 2 \sum Y_i (\ln \hat{\lambda}_y - \ln \hat{\lambda})$$

d) Unrestricted model has two parameters, restricted model has one parameter, hence we use chi-squared disribution with $2 - 1 = 1$ degree of freedom, $LR_{\text{crit}} = 3.84$. We calculate estimates, $\hat{\lambda}_x = 5$, $\hat{\lambda}_y = 4.5$, $\hat{\lambda} = 14/3$.

$$LR = 1000(\ln 5 - \ln(14/3)) + 1800(\ln 4.5 - \ln(14/3)) \approx 3.5$$

We do not reject H_0.

21.2.

21.3.

21.4.

21.5.

21.6.

22.1. $\mathbb{E}(Y_i) = \mathbb{P}(X_i > 1) = (a-1)/a = p$.

a) The estimator is consistent as

$$\operatorname{plim} \hat{a} = \frac{1}{1 - \operatorname{plim} \bar{Y}} = \frac{1}{1 - \frac{a-1}{a}} = a$$

b) For $n = 2$ we have the positive probability p^2 that $\bar{Y} = 1$. Hence with positive probability $\hat{a}$ is not defined. The value $\mathbb{E}(\hat{a})$ does not exist for $n = 2$.

22.2.

a)
$$\mathbb{P}(\hat{\theta} > y) = \mathbb{P}(Y_1 > y/n)^n = (\exp(-y/n\theta))^n = \exp(-y/\theta)$$

Hence $\hat{\theta}$ has exponential distribution with rate $1/\theta$ and probability density function

$$f(t) = \begin{cases} \exp(-t/\theta)/\theta, & \text{if } t \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

b) The estimator is unbiased as
$$\mathbb{E}(\hat{\theta}) = 1/(1/\theta) = \theta.$$

c) The estimator is non consistent as its distribution does not depend on n.

22.3.

22.4.

22.5.

23.1.

a) The log-likelihood function is equal to

$$\ell(\lambda) = \sum_{i=1}^{n} (-\lambda + X_i \ln \lambda - \ln(X_i!))$$

The score function is

$$\text{score}(\lambda) = \ell'(\lambda) = \sum_{i=1}^{n} (-1 + X_i/\lambda).$$

And

$$\ell''(\lambda) = \sum_{i=1}^{n} \left(-X_i/\lambda^2\right).$$

Fisher information is

$$I_F = -\mathbb{E}(\ell''(\lambda)) = \sum \mathbb{E}(X_i)/\lambda^2 = n\lambda/\lambda^2 = n/\lambda.$$

b) Solving $\ell' = 0$ we obtain

$$\hat{\lambda} = \bar{X}$$

c) Rewrite $\ell'(\lambda)$ using $\hat{\lambda}$. Be careful! Do not confound λ and $\hat{\lambda}$.

$$\text{score}(\lambda) = \ell'(\lambda) = -n + n\hat{\lambda}/\lambda.$$

Hence the score function is linear function of $\hat{\lambda}$, $\text{Corr}(\text{score}(\lambda), \hat{\lambda}) = 1$ and the Cramer-Rao bound is attained.

One may also find $\mathbb{E}\,\hat{\lambda} = \lambda$, $\text{Var}(\hat{\lambda}) = \lambda/n$ and explicitly check that the general bound

$$\text{Var}(\hat{\lambda}) \geqslant 1/I_F$$

is attained as equality in our case

$$\lambda/n = 1/(n/\lambda).$$

23.2.

23.3.

23.4.

23.5.

23.6.

23.7.

23.8.

24.1. We do not need the formula for $\Gamma(\alpha)$ here.

a) For known $\lambda = 1$ the likelihood is

$$L = \left(\prod X_i\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \cdot \exp(-\sum X_i).$$

If we optimize this function for α the optimal $\hat{\alpha}$ will depend only on $\prod X_i$. Hence $\prod X_i$ is a sufficient statistic for α. There are many other sufficient statistics, $\sum \ln X_i$ is another example.

b) Now the likelihood is

$$L = \left(\prod X_i\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} \exp(-\lambda \sum X_i).$$

If we optimize this function for α and λ the optimal point will depend only on $\prod X_i$ and $\sum X_i$. Hence $\left(\prod X_i \quad \sum X_i\right)$ is a two dimensional sufficient statistic for (α, λ).

24.2.

24.3.

24.4.

24.5.

26 Sources of wisdom

[Buz+15] Nazar Buzun et al. "Stochastic Analysis in Problems, part 1 (in Russian)." In: *arXiv preprint arXiv:1508.03461* (2015). URL: https://arxiv.org/abs/1508.03461.