Stochastic Processes problems

https://github.com/bdemeshev/stochastic_pro

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Contents

1 First step analysis

1.1 Biden and Trump alternately throw a fair dice infinite number of times. Biden throws first. The person who obtains the first 6 wins the game.

a) What is the probability that Biden will win?
b) What is expected number of turns?
c) What is variance of the number of turns?
d) What is expected number of turns given that Biden won?
e) Find the transition matrix of this four state Markov chain.

1.2 Elon throws an unfair coin until "head" appears. The probability of "head" is $p \in (0; 1)$. Let N be the total number of throws.

a) Find $\mathbb{E}(N)$, $\operatorname{Var}(N)$, $\mathbb{E}(N^3)$, $\mathbb{E}(\exp(tN))$.
b) What is the probability than N will be even?

1.3 Alice and Bob throw a fair coin until the sequence HTT or THT appears. Alice wins if HTT appears first, Bob wins if THT appears first.

a) Find the probability that Alice wins.
b) Find the expected value and variance of the total number of throws.
c) Using any open source software find the probability that Alice wins for all possible combinations of three coins sequences for Alice and Bob.
d) Now Alice and Bob play the following game. Alice chooses her three coins winning sequence first. Next Bob, knowing the choice of Alice, chooses his three coins winning sequence. Than they throw a fair coin until either of their sequences appears. What is the best strategy for Alice? For Bob? What is the probability that Alice wins this game?

1.4 You throw a dice unbounded number of times. If it shows 1, 2 or 3 then the corresponding amount of dollars is added in the pot. It it shows 4 or 5 the game stops and you get the pot with money. If it shows 6 the game ends and you get nothing. Initially the pot is empty.

a) What is probability that the game will end by 6?
b) What is expected duration of the game?
c) What is your expected payoff?
d) What is your payoff variance?
e) Consider variation-A of the game. Rules are the same, but initially the pot contains 100 dollars. How will the answers to questions (a)-(d) change?
f) Consider variation-B of the game. Initially the pot is empty. One rule is changed. If the dice shows 5 the content of the pot is burned and the game continues. How will the answers to questions (a)-(d) change?

1.5 Boris Johnson throws a fair coin until 1 appears or until he says "quit". His payoff is the value of the last throw. Boris optimizes his expected payoff. If many strategies gives the same expected payoff he chooses the strategy that minimizes the expected duration of the game.

a) What is the optimal strategy and the corresponding expected payoff?

b) What is the expected duration?

c) How the answers to points (a) and (b) will change if Boris should pay 0.3 dollars for each throw?

1.6 Winnie-the-Pooh starts wandering from the point $x = 1$. Every minute he moves one unit left or one right with equal probabilities.

Let T be the random moment of time when he reaches $x = 0$.

a) Find the generating function $g(u) = \mathbb{E}(u^T)$.

b) Extract all probabilities $\mathbb{P}(T = k)$ from the function $g(u)$.

1.7

1.8

2 Markov chains

2.1 HSE student lives in two states: "sleep" and "study" and tries to change the state every 1 hour. After the sleep state the student continues sleeping with probability equal to 0.25, otherwise a student starts studying. If the student is studying, the probabilities to continue studying and to start sleeping are equal.

a) Write down the transition matrix of this Markov chain.

b) Draw the graph representation.

c) What is the probability that a Sleeping Student will be a Studying Student after 1 hour? After 2 hours?

d) We know that initially student is sleeping with probability $p = \frac{2}{3}$. Find the probabilities of sleep and study states after 1 and 2 hours.

e) Find the probabilities of sleep and study states after 20 and 100 hours (do it with **matrix** operations and any soft). Is there any difference and why?

2.2 HSE student has three states: pre-coffee, with-coffee and over-coffee. He goes to Jeffrey's each break seeking for a cup of coffee. The line is usually too long, so probability to stay pre-coffee is equal to 60% and to be over-coffee — is zero. Caffeinated students can stay in lines longer, so for with-coffee student the probability to become over-coffee is 20% and to become pre-coffee — 30%. Over-coffee student runs to coffeeshop very fast and able to stay over-coffeed with $p = 0.70$ and can suddenly become pre-coffee with $p = 0.10$.

a) Draw the graph representation of this Markov chain.

b) Write the transition matrix of this Markov chain.

c) What is the probability that morning pre-coffee student will be with-coffee after 1 break? After 3 breaks?

d) What is the probability that morning pre-coffee student will be with-coffee after 1 break? After 3 breaks? After 200 breaks?

2.3 Unteachable students in NOTHSE University try to pass the exams. Students cheat successfully and pass the exams with probability 10%. In the case of a failure students are allowed to infinite number of retakes. All students are unteachable so the amount of knowledge is always the same and doesn't depend of the number of retakes.

a) Draw the graph representation of this Markov chain.

b) What is the probability to graduate using no more than 5 retakes?

c) What is the probability to graduate eventually?

d) Use **first step analysis** to find the average number of retakes per student in this University.

2.4 Every month the real estate Galina agent has two options: to increase her commission and to ask an owner to increase the rent. If the agent has increased the commission, on the next step she increases the commission again with probability $5/8$. If she has asked the owner, she decides to increase the commission with probability equal to $3/4$ on the next step.

a) Write the transition matrix of this Markov chain.

b) Draw the graph representation.

c) Use **first step analysis** to find how many steps the agent does between asking the owner to increase the price.

2.5 Alice and Bob toss a coin, writing down the results. If the last 3 tosses are Head, Head and Tail, Alice wins. If the last 3 tosses are Tail, Head and Head, Bob wins.

a) Is it easy to work with matrix representation in this case?

b) Draw the graph representation. Who is more likely to win the game?

c) Use **first step analysis** to find the probability of Alice's win.

d) Find the probability that the game ends in exactly 4 tosses.

e) Find the expected value and variance of the total number of coin throws in the game.

2.6 HSE student has an unusually caring granny who cooks one pie with probability 0.7 every weekend. Granny's pies are so tasty that HSE student can't resist and he gains 1 kilo for each pie eaten. Without pies the student with more than 70 kilos weight looses 1 kilo per week, yeah, he has a lot of studies! At the beginning of the study year student's weight is $W_0 = 70$ kilos.

Let W_t be the weight of the student t weeks later.

a) Find the probability $\mathbb{P}(W_3 \geqslant 71)$ and expected value $\mathbb{E}(W_3)$.

b) Find the limit weight after infinitely many study weeks $\lim_{t\to\infty} W_t$.

c) Explain whether the chain (W_t) has a stationary distribution.

2.7 The fair price of Sborbank in discrete stock market is somewhere between 100 and 101 rubles. If the price is equal to 100, then the price grows up by 1 ruble with probability $\frac{9}{10}$, otherwise it goes down by 1 ruble. If the price is greater then 100, it grows by 1 ruble with probability $\frac{1}{3}$ or declines by 1 ruble. If the price is lower than 100, it grows by 1 ruble with probability $\frac{2}{3}$ or declines by 1 ruble.

a) Draw the graph representation of the corresponding Markov chain.

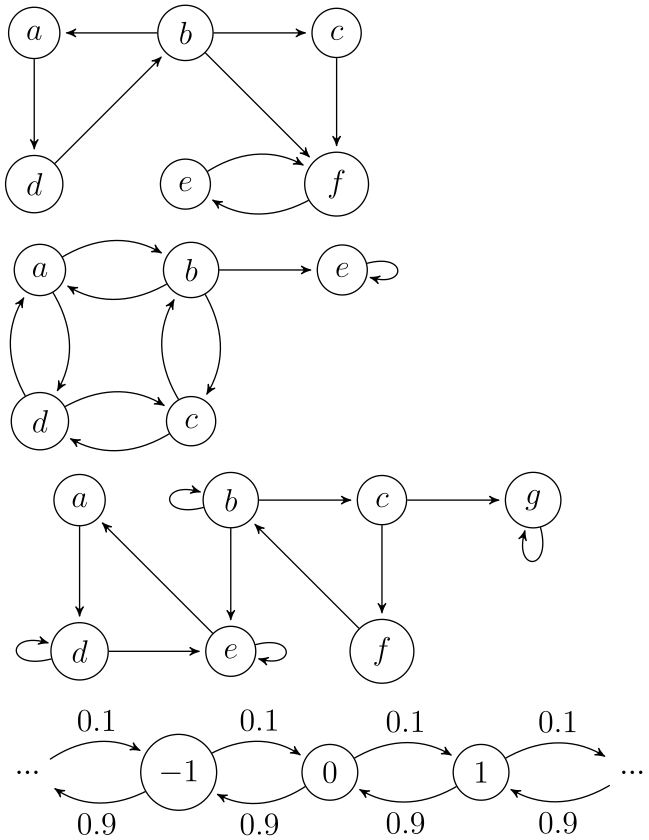
b) Do you think this chain has some stationary distribution?

c) Find the average time for the stock price to fall from 102 rubles to 98 rubles.

Hint: you may to decompose the long path into smaller ones and to use the first step analysis.

3 Classification of states

3.1 We randomly wander on the graph choosing at each moment of time one of the possible directions. If probabilities are not given we choose equiprobably.



a) Split each Markov chain into communicating classes.

b) Find the period of every state.

c) Classify each state as transient, null-recurrent and positive recurrent.

d) For positive recurrent states find the expected return time.

e) Find all stationary distributions.

3.2 A Knight randomly wanders on the chessboard. At each step he randomly chooses one of the possible Knight-moves with equal probabilities.

a) Find the stationary distribution.

b) Find the expected return time for every square.

c) Find the period of every square.

4 Generating functions

4.1 The MGF (moment generating function) of the random variable X is give by $M(t) = 0.3\exp(2t) + 0.2\exp(3t) + 0.5\exp(7t)$.

Recover the distribution of the random variable X.

4.2 The random variable Y takes values 1, 2 or 3 with equal probabilities.

Find the MGF of the random variable Y.

4.3 The MGF of the random variable W has a Taylor expansion that starts with $M(t) = 1 + 2t + 7t^2 + 20t^3 + \ldots$.

Find $\mathbb{E}(W)$, $\text{Var}(W)$, $\mathbb{E}(W^3)$.

4.4 The random variable X takes non-negative integer values. The generating function $g(u) = \mathbb{E}(u^X)$ has a Taylor expansion that starts with $g(u) = 0.1 + 0.2u + 0.15u^2 + \ldots$.

Find probabilities $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.

4.5 Random variables X_i are mutually independent and X_i has Gamma distribution $\text{Gamma}(\alpha_i, \beta_i)$.

I sum up the random number N of terms,

$$S = X_1 + X_2 + \ldots + X_N.$$

The number N has Poisson distribution $\text{Pois}(\lambda)$ and is independent of the sequence (X_i).

a) Find the MGF of S. You may the MGF formula for Gamma distribution as known.
b) Find $\mathbb{E}(S)$ and $\text{Var}(S)$.

4.6 The random variable X take non-negative integer values. Its moment generating function is equal to $M(t) = (2 - \exp(t))^{-7}$.

a) Find the probability generating function $g(u) = \mathbb{E}(u^X)$.
b) Find $\mathbb{E}(X)$, $\text{Var}(X)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$, $\mathbb{P}(X = 2)$.
c) Find $\mathbb{P}(X = k)$.

5 Limits

5.1 Polina loves sweet chestnuts. She has infinite sequence of baskets before her. In the basket number n there are n chestnuts in total. Unfortunately only one chestnut in every basket is a sweet one.

She picks chestnuts one by one at random from all the buskets sequentially. First she picks the unique chestnut from the basket number one, than she picks in a random order two chestnuts from the basket number two and so on.

The random variable S_t indicates whether the chestnut number t was a sweet one.

a) Find $\lim S_t$ or prove that the limit does not exist.
b) Find $\text{plim}\, S_t$ or prove that the limit does not exist.

5.2 Let (X_n) be independent, each variable X_n has exponential distribution with rate $\lambda_n = n$.

a) Find the probability limit $\text{plim}\, X_n$ or prove that it does not exist.

Let (Y_n) be independent, each variable Y_n has exponential distribution with rate $\lambda_n = n/(n+1)$.

b) Find the probability limit $\text{plim}\, Y_n$ or prove that it does not exist.

5.3 Let (X_n) be independent normally distributed $\mathcal{N}(5; 10)$.

a) Find the probability limit

$$\text{plim}\, \frac{X_1 + X_2 + \cdots + X_n}{7n}.$$

b) Find the probability limit

$$\operatorname{plim} \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{7n}.$$

c) Find the probability limit

$$\operatorname{plim} \ln\left(X_1^2 + X_2^2 + \cdots + X_n^2\right) - \ln n.$$

5.4 Let (X_n) be independent uniform on $[0;1]$. Let $Y_n = X_n^2 + X_n^3$.

a) Find the probability limit $\operatorname{plim} V_n$ for

$$V_n = \max\{Y_1, Y_2, \ldots, Y_n\}.$$

b) Find the probability limit $\operatorname{plim} W_n$ for

$$W_n = \max\{X_1 + Y_1, X_1 + Y_2, \ldots, X_1 + Y_n\}.$$

5.5 Consider the random variable X and the sequence of random variables Y_n with $\mathbb{E}(Y_n) = \frac{1}{n}$ and $\operatorname{Var}(Y_n) = \frac{\sigma^2}{n}$. Let $W_n = X + Y_n$.

a) Find the probability limit $\operatorname{plim} Y_n$;

b) Find the probability limit $\operatorname{plim} W_n$.

5.6 The random variables X_i are independent and uniformly distributed on $[0;1]$. Let $Y_n = \min X_1, \ldots X_n$.

a) Find the almost sure limit of Y_n;

b) Find the probability limit of Y_n;

c) Find the limiting distribution of Y_n.

5.7 Let X and Y be independent and uniformly distributed on $[0;1]$. Let $V_n = n^2 Y \cdot I(X \leqslant 1/n)$ and $W_n = Y \cdot I(X > 1/n)$.

a) Find $\operatorname{plim} V_n$ and $\operatorname{plim} W_n$.

b) Does (V_n) converge in mean squared?

c) Does (W_n) converge in mean squared?

6 Conditional expected value without sigma-algebras

6.1 We randomly uniformly select a point inside triangle $A = (6, 0)$, $B = (0, 2)$ and $O = (0, 0)$. Let (X, Y) be coordinates of this random point.

a) Find conditional expected values $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

b) Find conditional variances $\operatorname{Var}(Y \mid X)$ and $\operatorname{Var}(X \mid Y)$.

6.2 The pair of random variables X and Y has joint probability density

$$f(x, y) = \begin{cases} x + y, & \text{if } x \in [0;1], y \in [0;1]; \\ 0, & \text{otherwise.} \end{cases}$$

a) Find the marginal densities $f_X(x)$ and $f_Y(y)$.

b) Find the conditional densities $f(x \mid y)$ and $f(y \mid x)$.

c) Find the conditional expected values $\mathbb{E}(Y \mid X)$ and $\mathbb{E}(X \mid Y)$.

d) Find the conditional variances $\operatorname{Var}(Y \mid X)$ and $\operatorname{Var}(X \mid Y)$.

6.3 The random variables X and Y are independend with Poisson distribution with rate $\lambda = 1$. Let $S = X + Y$.

a) Find conditional probabilities $\mathbb{P}(X = x \mid S = s)$ and $\mathbb{P}(Y = y \mid S = s)$.

b) Find conditional expected values $\mathbb{E}(X \mid S)$ and $\mathbb{E}(Y \mid S)$.

c) Find conditional variances $\operatorname{Var}(X \mid S)$ and $\operatorname{Var}(Y \mid S)$.

d) How the answers will change if $X \sim \operatorname{Pois}(\lambda_x)$ and $Y \sim \operatorname{Pois}(\lambda_y)$?

6.4 Let X and Y be independent and exponentially distributed with rate $\lambda = 1$ and $S = X + Y$.

a) Find conditional densities $f(x \mid s)$ and $f(y \mid s)$.

b) Find conditional expected values $\mathbb{E}(X \mid S)$ and $\mathbb{E}(Y \mid S)$.

c) Find conditional variances $\operatorname{Var}(X \mid S)$ and $\operatorname{Var}(Y \mid S)$.

d) Find $\operatorname{Cov}(X, Y \mid S)$ and $\operatorname{Corr}(X, Y \mid S)$.

e) How the answers will change if $X \sim \operatorname{Expo}(\lambda_x)$ and $Y \sim \operatorname{Expo}(\lambda_y)$?

6.5 The random variable X has Poisson distribution with rate $\lambda = 1$. The random variable Y has uniform distribution on $[1; 2]$. Random variables X and Y are independent.

Find $\mathbb{E}(XY \mid X)$, $\operatorname{Var}(XY + X^3 \mid X)$, $\operatorname{Cov}(X, Y \mid X)$, $\operatorname{Cov}(XY, X^2 Y \mid X)$.

7 ....

8 Solutions

1.1.

a) $\mathbb{P}(B) = 6/11$, first step equation for $p = \mathbb{P}(B)$ is $p = 1/6 + (5/6)^2 p$ or $p = 1/6 + 5/6 \cdot (1 - p)$.

b) $\mathbb{E}(N) = 6$, first step equation for $m = \mathbb{E}(N)$ is $m = 1/6 + 5/6(m + 1)$.

c) $\mathbb{E}(N^2) = 66$, $\operatorname{Var}(N) = 30$, first step equation is $\mathbb{E}(N^2) = 1/6 + 5/6\mathbb{E}((N + 1)^2)$.

d) $\mathbb{E}(N \mid B) = 61/11$. Start by replacing uncoditional probabilities on the tree by conditional ones. First step equation for $\mu = \mathbb{E}(N \mid B)$ is $\mu = 11/36 + 25/36(\mu + 2)$.

e) $\begin{pmatrix} 0 & 5/6 & 1/6 & 0 \\ 5/6 & 0 & 0 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1.2.

a) $\mathbb{E}(N) = 1/p$, $\text{Var}(N) =$, $\mathbb{E}(N^3) =$, $\mathbb{E}(\exp(tN)) =$

b) $a = \mathbb{P}(N \in 2 \cdot \mathbb{N})$, $a = (1-p)(1-a)$, $a = (1-p)/(2-p)$.

1.3.

1.4. Let's denote the throws by (X_t) and the number of throws by T. Thus the last throw is X_T.

a) $\mathbb{P}(X_T = 6) = 1/3$ as we have three possible endings. One may also sum the probability geometric serie or use first step analysis.

b) $\mathbb{E}(T) = 0.5 + 0.5(\mathbb{E}(T) + 1)$;

c) Let $\mu = \mathbb{E}(S)$ and $\gamma = \mathbb{P}(X_T \in \{4, 5\})$.

$$\mu = \frac{3}{6} \cdot 0 + \frac{1}{6}(\mu + 1 \cdot \gamma) + \frac{1}{6}(\mu + 2 \cdot \gamma) + \frac{1}{6}(\mu + 3 \cdot \gamma)$$

d)

e)

f)

$$\mu_B = \frac{2}{6} \cdot 0 + \frac{1}{6}\mu_B + \frac{1}{6}(\mu + 1 \cdot \beta) + \frac{1}{6}(\mu + 2 \cdot \beta) + \frac{1}{6}(\mu + 3 \cdot \beta),$$

with $\beta = 1/3$.

1.5.

1.6.

1.7.

1.8.

2.1.

2.2.

2.3.

2.4.

2.5.

2.6.

a)

b) $\mathbb{E}(W_t) \to +\infty$;

c) No stationary distribution. For stationary distribution $\mathbb{E}(W_t)$ can't tend to infinity.

2.7.

3.1.

3.2.

4.1.

4.2. $M(t) = (\exp(t) + \exp(2t) + \exp(3t))/3$.

4.3. $\mathbb{E}(W) = 2$, $\text{Var}(W) = 7 \cdot 2 - 2^2$, $\mathbb{E}(W^3) = 20 \cdot 3!$.

4.4. $\mathbb{P}(X = 0) = 0.1, \mathbb{P}(X = 1) = 0.2, \mathbb{P}(X = 2) = 0.15.$

4.5.

4.6. $g(u) = g(\exp(t)) = \mathbb{E}(\exp(tX)) = M(t)$

5.1.

5.2.

a) $\operatorname{plim} X_n = 0$;

b) $\operatorname{plim} Y_n$ does not exist.

5.3.

a) $\operatorname{plim} \frac{X_1+X_2+\cdots+X_n}{7n} = 5/7$;

b) $\operatorname{plim} \frac{X_1^2+X_2^2+\cdots+X_n^2}{7n} = 5$;

c) $\operatorname{plim} \ln(X_1^2 + X_2^2 + \cdots + X_n^2) - \ln n = \ln 35$.

5.4.

a) $\operatorname{plim} \max\{Y_1, Y_2, \ldots, Y_n\} = 2$;

b) $\operatorname{plim} \max\{X_1 + Y_1, X_1 + Y_2, \ldots, X_1 + Y_n\} = X_1 + 2$.

5.5.

a) $\operatorname{plim} Y_n = 0$;

b) $\operatorname{plim} W_n = X$.

5.6.

a) $\mathbb{P}(\lim Y_n = 0) = 1$;

b) $\operatorname{plim} Y_n = 0$;

c) Limiting distribution is a constant 0.

5.7.

a) $\operatorname{plim} V_n = 0$, $\operatorname{plim} W_n = Y$;

b) The sequence V_n does not converge in mean squared;

c) W_n converges to Y in mean squared.

6.1.

a) $\mathbb{E}(Y \mid X) = 1 - X/6$ and $\mathbb{E}(X \mid Y) = 3 - 1.5X$.

b) $\operatorname{Var}(Y \mid X) =$, $\operatorname{Var}(X \mid Y) =$.

6.2.

a)

$$f(x) = \begin{cases} x + 0.5, & \text{if } x \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

b)

$$f(x, y) = \begin{cases} (x + y)/(x + 0.5), & \text{if } x \in [0; 1], y \in [0; 1]; \\ 0, & \text{otherwise.} \end{cases}$$

c)

$$\mathbb{E}(Y \mid X) = \frac{0.5X + 1/3}{X + 0.5}.$$

d)

6.3.

a)

b) $\mathbb{E}(X \mid S) = \mathbb{E}(Y \mid S) = S/2$;

c)

d)

6.4.

a) $X \mid S \sim dUnif[0; S]$, $Y \mid S \sim dUnif[0; S]$.

b) $\mathbb{E}(X \mid S) = \mathbb{E}(Y \mid S) = S/2$;

c) $\operatorname{Var}(X \mid S) = \operatorname{Var}(Y \mid S) = S^2/12$;

d)

6.5. $\mathbb{E}(XY \mid X) = X\mathbb{E}(Y) = X/2$, $\operatorname{Var}(XY + X^3 \mid X) = X^2 \operatorname{Var}(Y) = X^2/12$, $\operatorname{Cov}(X, Y \mid X) = 0$, $\operatorname{Cov}(XY, X^2Y \mid X) = X^3 \operatorname{Var}(Y) = X^3/12$

9 Sources of wisdom

[Buz+15] Nazar Buzun et al. "Stochastic Analysis in Problems, part 1 (in Russian)." In: *arXiv preprint arXiv:1508.03461* (2015). URL: https://arxiv.org/abs/1508.03461.