

Let's start with SVD decomposition of  $X = UDV^T$ .

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1k} \\ x_{21} & x_{22} & \dots & x_{2k} \\ x_{31} & x_{32} & \dots & x_{3k} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ u_{31} & u_{32} & \dots & u_{3n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \cdot \begin{pmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & d_{kk} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \cdot \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1k} \\ v_{21} & v_{22} & \dots & v_{2k} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \dots & v_{kk} \end{pmatrix}^T$$

The singular values on the diagonal of  $D$  are positive and are sorted from largest to lowest,  $d_{11} > d_{22} > \dots > d_{kk}$ .

The columns of the matrix  $P = UD = XV$  are called principal components,  $p_j = d_{jj}u_j$ .

Let  $\hat{D}$  be the modified  $D$  matrix, where we keep only the first  $r$  positive elements on the diagonal,  $d_{11}, \dots, d_{kk}$  and replace the other elements by 0.

For  $r = 2$  the approximation of the original  $X$  matrix may be written as  $\hat{X} = U\hat{D}V^T$ ,

$$\begin{pmatrix} \hat{x}_{11} & \hat{x}_{12} & \dots & \hat{x}_{1k} \\ \hat{x}_{21} & \hat{x}_{22} & \dots & \hat{x}_{2k} \\ \hat{x}_{31} & \hat{x}_{32} & \dots & \hat{x}_{3k} \\ \vdots & \vdots & & \vdots \\ \hat{x}_{n1} & \hat{x}_{n2} & \dots & \hat{x}_{nk} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ u_{31} & u_{32} & \dots & u_{3n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \cdot \begin{pmatrix} d_{11} & 0 & 0 & \dots & 0 \\ 0 & d_{22} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \cdot \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1k} \\ v_{21} & v_{22} & \dots & v_{2k} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \dots & v_{kk} \end{pmatrix}^T$$

The approximation  $\hat{X} = U\hat{D}V^T$  may be also written as  $\hat{X} = U_*D_*V_*^T$ ,

$$\begin{pmatrix} \hat{x}_{11} & \hat{x}_{12} & \dots & \hat{x}_{1k} \\ \hat{x}_{21} & \hat{x}_{22} & \dots & \hat{x}_{2k} \\ \hat{x}_{31} & \hat{x}_{32} & \dots & \hat{x}_{3k} \\ \vdots & \vdots & & \vdots \\ \hat{x}_{n1} & \hat{x}_{n2} & \dots & \hat{x}_{nk} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \\ \vdots & \vdots \\ u_{n1} & u_{n2} \end{pmatrix} \cdot \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} \cdot \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ \vdots & \vdots \\ v_{k1} & v_{k2} \end{pmatrix}^T$$

The matrix  $U_*$  consists of the leftmost  $r$  columns of  $U$ . The matrix  $D_*$  consists of the top-left  $r \times r$  corner of  $D$ . The matrix  $V_*^T$  consists of the top  $r$  rows of  $V^T$ .

Errors of approximation are stored in the matrix:

$$X - \hat{X} = UDV^T - U\hat{D}V^T = U(D - \hat{D})V^T$$

Let's calculate the quality of this approximation.

We'll use two facts from linear algebra:  $\text{trace}(LR) = \text{trace}(RL)$  and  $\text{trace}(XX^T) = \text{trace}(X^TX) = \sum x_{ij}^2$ .

$$\sum_{ij} x_{ij}^2 = \text{trace}(X^TX) = \text{trace}(VD^TU^TUDV^T) = \text{trace}(DD^TU^TU) = \text{trace}(D^TD) = \sum_i d_{ii}^2$$

By the same argument

$$\sum_{ij} p_{ij}^2 = \text{trace}(P^TP) = \text{trace}(D^TU^TUD) = \text{trace}(D^TD) = \sum_i d_{ii}^2$$

Recall that all vectors in the  $X$  matrix are standardized. Sample variance of a typical vector  $x_j$  is given by  $\sum_i (x_{ij} - 0)^2 / (n - 1) = 1$ . Hence  $\|x_j\|^2 = \sum_i x_{ij}^2 = (n - 1)$ .

Hence,

$$\sum_{ij} x_{ij}^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_k\|^2 = k(n - 1) = \sum_{i=1}^k d_{ii}^2;$$

Or,

$$\sum_{ij} p_{ij}^2 = \|p_1\|^2 + \|p_2\|^2 + \cdots + \|p_k\|^2 = k(n - 1) = \sum_{i=1}^k d_{ii}^2;$$

The sum of squares of all approximation errors:

$$Q = \sum_{ij} (x_{ij} - \hat{x}_{ij})^2 = \text{trace}((X - \hat{X})(X - \hat{X})^T)$$

What is inside the trace?

$$(X - \hat{X})(X - \hat{X})^T = V(D - \hat{D})^T U^T U (D - \hat{D}) V^T = V(D - \hat{D})^T (D - \hat{D}) V^T$$

Let's finalize the calculation of total approximation error  $Q$ :

$$\begin{aligned} Q &= \text{trace}((X - \hat{X})(X - \hat{X})^T) = \text{trace}(V(D - \hat{D})^T (D - \hat{D}) V^T) = \text{trace}((D - \hat{D})^T (D - \hat{D}) V^T V) = \\ &= \text{trace}((D - \hat{D})^T (D - \hat{D})) = \sum_{i=r+1}^k d_{ii}^2 = k(n - 1) - \sum_{i=1}^r d_{ii}^2; \end{aligned}$$

Let's regress all the columns of  $X$  onto predictors  $U_*$  and obtain predictions.

We use the standard formula  $\hat{y} = X(X^T X)^{-1} X^T y$  with  $U_*$  instead of  $X$  and  $X$  instead of  $y$ :

$$U_*(U_*^T U_*)^{-1} U_*^T X = U_*(I)^{-1} U_*^T U \hat{D} V^T = U_* D_* V_*^T = \hat{X}$$

So! The  $\hat{X}$  may be viewed as low-rank approximation of  $X$  from SVD or as matrix of forecasts of regressions of every original regressor  $x_j$  on  $r$  first principal components  $p_1, \dots, p_r$ . Hence the sum of all squared approximation errors  $Q$  is the sum of all sum of squared residuals from these regressions

$$Q = \sum_{ij} (x_{ij} - \hat{x}_{ij})^2 = \|x_1 - \hat{x}_1\|^2 + \cdots + \|x_k - \hat{x}_k\|^2 = \text{SS}_1^{\text{res}} + \cdots + \text{SS}_k^{\text{res}}.$$

Let's calculate the  $R^2$  in the first regression:

$$R_1^2 = 1 - \frac{\text{SS}_1^{\text{res}}}{\text{SST}_1} = 1 - \frac{\|x_1 - \hat{x}_1\|^2}{\|x_1\|^2} = 1 - \frac{\|x_1 - \hat{x}_1\|^2}{n - 1}$$

The sum of all  $R_j^2$  hence is

$$\sum R_j^2 = k - \frac{\sum (x_{ij} - \hat{x}_{ij})^2}{n - 1}.$$

The average  $R_j^2$ , also called proportion of variance explained, is

$$\frac{\sum R_j^2}{k} = 1 - \frac{\sum (x_{ij} - \hat{x}_{ij})^2}{k(n - 1)} = 1 - \frac{Q}{k(n - 1)} = \frac{\sum_{i=1}^r d_{ii}^2}{\sum_{i=1}^k d_{ii}^2}.$$

How can we extract all these  $R_j^2$  for different number of principal components? For simplicity let's consider  $x_1$  and the first two principal components. The coefficient  $R_1^2$  comes from regression of  $x_1$  onto first  $r = 2$  principal components  $p_1, p_2$ .

$$\hat{x}_1 = v_{11}p_1 + v_{12}p_2.$$

Artificial regressors  $p_1$  and  $p_2$  are orthogonal, hence  $R_1^2$  may be decomposed into

$$R_1^2 = R_{11}^2 + R_{12}^2,$$

where  $R_{11}^2$  comes from the regression of  $x_1$  onto  $p_1$  and  $R_{12}^2$  – from the regression of  $x_1$  onto  $p_2$ . The orthogonality of regressors also means that coefficient estimate  $v_{11}$  may be obtained from a univariate regression of  $x_1$  onto  $p_1$  alone. And the regression coefficient  $v_{12}$  may be also obtained from a univariate regression of  $x_1$  onto  $p_2$  alone.

In simple regression of original variable  $x_i$  onto principal component  $p_j$  the  $R_{ij}^2$  may be obtained as a squared value of sample correlation between  $x_i$  and  $p_j$ . The sample correlation matrix of all columns  $X$  with all columns of  $P$  is called loadings matrix,  $L$ . The sample correlation matrix is unaffected by scaling, so  $L$  is also a sample correlation matrix of columns of  $X$  and columns of  $U$ . Sample correlation matrix is just a sample covariance of standardized vectors. The columns in  $X$  are already standardized so we need to standardize only columns of  $P$ , or columns of  $U$ .

$$L = \begin{pmatrix} \text{sCorr}(x_1, p_1) & \text{sCorr}(x_1, p_2) & \dots \\ \text{sCorr}(x_2, p_1) & \text{sCorr}(x_2, p_2) & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} = \frac{X^T P^{st}}{n-1} = \frac{X^T U^{st}}{n-1}$$

As  $U^{st} = \sqrt{n-1}U$  we get

$$L = \frac{X^T U^{st}}{n-1} = \frac{V D^T U^T U \sqrt{n-1}}{n-1} = \frac{1}{\sqrt{n-1}} V D^T.$$

Hence, in the row 1 the sum of the first  $r$  squared values will give the coefficient  $R_1^2$ ,

$$R_1^2 = \ell_{11}^2 + \ell_{12}^2 + \dots + \ell_{1r}^2.$$

The sample correlation matrix of columns of  $X$  is

$$\frac{X^T X}{n-1} = \frac{V D^T U^T U D V^T}{n-1} = \frac{V D^T D V^T}{n-1} = V \Lambda V^T.$$

Here  $\Lambda$  is the diagonalized sample correlation matrix of columns of  $X$ . On its diagonal we have  $\lambda_i = d_{ii}^2/(n-1)$ . The eigenvectors of the sample correlation matrix are just columns of  $V$ .

The loading matrix can be expressed in terms of  $V$  and  $\Lambda$ :

$$L = V \Lambda^{1/2} = V \cdot \begin{pmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\lambda_k} \end{pmatrix}.$$

As  $\lambda_i = d_{ii}^2/(n-1)$  the average  $R_j^2$  is also

$$\frac{\sum R_j^2}{k} = \frac{\sum_{i=1}^r d_{ii}^2}{\sum_{i=1}^k d_{ii}^2} = \frac{\sum_{i=1}^r \lambda_i}{\sum_{i=1}^k \lambda_i}.$$

Two more facts about  $L$  matrix:

$$L^T L = \Lambda^{1/2} V^T V \Lambda^{1/2} = \Lambda^1 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_k \end{pmatrix}$$

Hence  $\ell_i \perp \ell_j$  for  $i \neq j$ ,  $\|\ell_j\|^2 = \lambda_j = d_{jj}/(n-1)$ . The sample correlation matrix of columns of  $X$  may be written as

$$LL^T = \frac{1}{n-1} V D^T D V^T = \frac{X^T X}{n-1}$$

Let's create a new variable  $q$  as a linear combination of columns of  $X$  with weight  $w$ ,  $q = Xw$ .

We would like to maximize the sample variance of  $Xw$  while keeping  $\|w\| = 1$ . As sample variance for centered variable is just  $\|Xw\|^2/(n-1)$  this is equivalent

$$\|Xw\|^2 \rightarrow \max_{\|w\|=1}$$

Solution 1 (no Lagrange, no matrix differential):

We can rewrite  $\|Xw\|^2$  as

$$\|Xw\|^2 = (Xw)^T Xw = w^T X^T Xw = w^T V D^T D V^T w$$

Let's introduce  $a = V^T w$ . Remark that  $\|a\|^2 = a^T a = w^T V V^T w = w^T w = 1$  and

$$\|Xw\|^2 = a^T \cdot \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{kk} \end{pmatrix} \cdot a = d_{11}^2 a_1^2 + \dots + d_{kk}^2 a_k^2$$

For  $k = 3$  this may look like

$$\|Xw\|^2 = 1.3a_1^2 + 0.7a_2^2 + 0.2a_3^2.$$

As  $\|a\|^2 = a_1^2 + \dots + a_k^2 = 1$  we conclude that the optimal point is  $a = (1, 0, \dots, 0)$ . And optimal  $w$  is  $w = Va = v_1$ , the first column of  $V$ .

Solution 2 (with matrix differential):

$$\|Xw\|^2 \rightarrow \max_{\|w\|=1}$$

As  $w^T w = 1$ ,

$$\frac{\|Xw\|^2}{n-1} = \frac{w^T V D^T D V^T w}{n-1} = \frac{w^T V D^T D V^T w}{(n-1)w^T w} = \frac{w^T \Lambda w}{w^T w}$$

Now take differential

$$d \frac{w^T \Lambda w}{w^T w} = \frac{\dots}{(w^T w)^2}$$

Solution 3 (with Lagrange)