

## Home Assignment 1

1. Let  $\Omega = \mathbb{R}$ . Explicitly find the sigma-algebras  $\mathcal{F}_1 = \sigma(A)$ ,  $\mathcal{F}_2 = \sigma(B)$ ,  $\mathcal{F}_3 = \sigma(A, B)$  where  $A = [-10; 5]$  and  $B = (0; 10)$ .
2. I throw a die once. Let  $X$  be the result of the toss. Count the number of events in sigma-algebras  $\mathcal{F}_1 = \sigma(X)$ ,  $\mathcal{F}_2 = \sigma(\{X > 3\})$ ,  $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$ .
3. Let  $\Omega = \mathbb{R}$ . The sigma-algebra  $\mathcal{F}$  is generated by all the sets of the form  $(-\infty, t]$ ,

$$\mathcal{F} = \sigma(\{(-\infty; t] \mid t \in \mathbb{R}\})$$

Check whether  $A_1 = (0; 10) \in \mathcal{F}$ ,  $A_2 = \{5\} \in \mathcal{F}$ ,  $A_3 = \mathbb{N} \in \mathcal{F}$ .

4. Prove the following statements or provide a counter-example:
  - (a) If  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are sigma-algebras then  $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$  is sigma-algebra.
  - (b) If  $X$  and  $Y$  are independent random variables then  $\text{card } \sigma(X, Y) = \text{card } \sigma(X) + \text{card } \sigma(Y)$ .

For finite sets  $\text{card}$  denotes just the number of elements.

5. I throw a die infinite number of times. Let the random variable  $X_n$  be equal to 1 if the  $n$ -th toss is head and 0 otherwise. Consider a pack of sigma-algebras:  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$  and  $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \dots)$ .

Where possible provide an example of a non-trivial event (neither  $\Omega$  nor  $\emptyset$ ) such that

- (a)  $A_1 \in \mathcal{F}_{2020}$ ;
- (b)  $A_2 \in \mathcal{H}_{2020}$ ;
- (c)  $A_3 \in \mathcal{F}_{2020}$  and  $A_3 \in \mathcal{H}_{2020}$ ;
- (d)  $A_4 \in \mathcal{F}_n$  for all  $n$ ;
- (e)  $A_5 \in \mathcal{H}_n$  for all  $n$ .

Deadline: 25 September 2020, 21:00 MSK.

## Home Assignment 2

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0 & 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)
  - (b) Classify the states of the Markov chain.
2. The Lonely Knight is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.
- (a) How many moves on average will it take to go back to A1?
  - (b) What proportion of her eternal life will she spend on every field?
3. Donald Trump throws a die until one appears or until he says «Stop». The payoff is equal to the last thrown number. Donald maximizes the expected payoff.

- (a) What is the best strategy and the corresponding expected payoff?

How do the answers change in the following modifications of the original game?

- (b) Donald is also required to stop at 3 and to continue on 4.
  - (c) Donald should pay 0.3 for every throw.
4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed Zmei Gorynich. Yes, there are infinitely many Zmei Gorynich.
- What is the probability that there is an Eternal Peaceful Journey?
5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTT or TTH appears. Ilya wins if HTT appears and Zmei wins if TTH appears.
- (a) What is the probability that Ilya wins?
  - (b) What is the expected number of throws?
  - (c) What is the expected number of throws given that Ilya won?

Deadline: 9 October 2020, 21:00 MSK.

## Home Assignment 3

1. Consider the Vasicek interest rate model,

$$dR_t = a(b - R_t) dt + \sigma dW_t.$$

Here  $R_t$  is the interest rate and  $a$ ,  $b$  and  $\sigma$  are positive constants.

- (a) Using the substitution  $Y_t = e^{at} R_t$  find the solution of the stochastic differential equation;
  - (b) Find  $\mathbb{E}(R_t)$  and  $\mathbb{V}\text{ar}(R_t)$ .
  - (c) Which value in this model would you call long-term equilibrium rate and why?
2. Let  $X_t$  be the exchange rate measured in roubles per dollar. We suppose that  $dX_t = \mu X_t dt + \sigma X_t dW_t$ . Consider the inverse exchange rate  $Y_t = 1/X_t$  measured in dollars per rouble. Write the stochastic differential equation for  $dY_t$ . The equation may contain  $Y_t$  and constants, but not  $X_t$ .
3. Let  $W_t^a$  and  $W_t^b$  be two independent Wiener processes. Consider the process  $Q_t = \alpha W_t^a + \beta W_t^b$ , where  $\alpha^2 + \beta^2 = 1$ .
- (a) Is  $Q_t$  a Wiener process? Carefully check all the assumptions.
  - (b) Find the limit in  $L^2$  for  $n \rightarrow \infty$  of

$$A_n = \sum_{i=1}^n (W^a(it/n) - W^a((i-1)t/n)) (W^b(it/n) - W^b((i-1)t/n))$$

- (c) Find the limit in  $L^2$  for  $n \rightarrow \infty$  of

$$B_n = \sum_{i=1}^n (Q(it/n) - Q((i-1)t/n)) (W^b(it/n) - W^b((i-1)t/n))$$

- (d) Find  $\mathbb{C}\text{orr}(Q_t, W_t^b)$ .
  - (e) Without formal proof guess the value of  $dQ_t dW_t^b$  in the Ito's lemma for correlated Wiener processes.
4. Consider the Cox-Ingersoll-Ross interest rate model

$$dR_t = a(b - R_t) dt + \sigma \sqrt{R_t} dW_t.$$

Here  $R_t$  is the interest rate and  $a$ ,  $b$  and  $\sigma$  are positive constants.

Find  $\mathbb{E}(R_t)$  and  $\mathbb{V}\text{ar}(R_t)$ .

5. The share price  $S_t$  satisfies the Black and Scholes model and  $dX_t = t dS_t$ . Find  $\mathbb{E}(X_t)$  and  $\mathbb{V}\text{ar}(X_t)$ .

1. Consider the Black and Scholes model. At time  $T > 1$  the asset pays you

$$X_T = \begin{cases} \ln S_T, & \text{if } S_T > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Today is  $t = 1$ . Find the current price  $X_1$  of this asset.

2. Пусть  $y_t$  — стационарный процесс. Проверьте стационарность процессов:

(a)  $a_t = \Delta^2 y_t$ ;

(b)  $b_t = 2y_t + 3y_{t-1} + 18$ .

3. Правильный кубик подбрасывают три раза, обозначим результаты подбрасываний  $X_1$ ,  $X_2$  и  $X_3$ . Также введём обозначения для сумм  $L = X_1 + X_2$ ,  $R = X_2 + X_3$  и  $S = X_1 + X_2 + X_3$ .

(a) С помощью качественных рассуждений (без вычислений) определите знаки частных корреляций<sup>1</sup>  $\text{pCorr}(L, R; S)$ ,  $\text{pCorr}(L, S; R)$ ,  $\text{pCorr}(X_1, R; S)$ .

(b) Найдите точное значение каждой частной корреляции.

4. У эконометрессы Ефросиньи был стационарный ряд  $(y_t)$ ,  $t \geq 1$  с  $\mathbb{E}(y_t) = 5$ .  $\text{Var}(y_t) = 16$  и  $\text{Cov}(y_t, y_{t-1}) = 4$ .

Ефросинье было скучно и она подбрасывала неправильную монетку, выпадающую орлом с вероятностью 0.7. Если выпадал орёл, она оставляла очередной  $y_t$ , если решка — то зачёркивала.

Обозначим полученную новую последовательность  $(z_t)$ .

(a) Является ли  $(z_t)$  стационарным?

(b) Найдите  $\mathbb{E}(z_t)$ ,  $\text{Var}(z_t)$  и  $\text{Cov}(z_t, z_{t-1})$ .

5. Рассмотрим стационарное решение  $(y_t)$  уравнения  $y_t = 6 + 0.5y_{t-1} + u_t - 0.3u_{t-1}$ , где  $(u_t)$  — белый шум.

(a) Найдите  $\mathbb{E}(y_t)$  и  $\text{Var}(y_t)$ .

(b) Найдите первые три значения автокорреляционной функции.

(c) Найдите первые три значения частной автокорреляционной функции.

Hint: для данного случая есть теорема, которая гарантирует, что у стационарного решения  $\text{Cov}(y_t, u_{t+k}) = 0$  при  $k > 0$ .

<sup>1</sup>Запись  $\text{pCorr}(X, Y; Z)$  означает частную корреляцию между  $X$  и  $Y$ , «очищенных» от эффекта  $Z$ .

1. Consider  $y_t$  described by  $ETS(MNM)$  model. You can find all the equations in <https://otexts.com/fpp3/>. Is it true that  $z_t = \ln y_t$  is exactly described by  $ETS(ANA)$  model? Approximately?
2. Consider  $ETS(AA_dN)$  model with  $\phi = 0.9$ ,  $\alpha = 0.3$ ,  $\beta = 0.1$  and  $\sigma^2 = 16$ . Express 95% predictive intervals for  $y_{t+1}$  and  $y_{t+2}$  in terms of  $\ell_t$ ,  $b_t$ ,  $y_t$  and  $u_t$ .
3. Find  $\mathbb{E}(y_t)$ ,  $\text{Var}(y_t)$ ,  $\text{Cov}(y_t, y_{t+1})$  in the  $ETS(AAN)$  model with given  $\ell_0$ ,  $b_0$ ,  $\alpha$ ,  $\beta$  and  $\sigma^2$ .
4. Consider stationary  $ARMA(1, 1)$  process,  $y_t = 0.7y_{t-1} + u_t + 0.2u_{t-1}$ , where  $\text{Var}(u_t) = 16$ .
  - (a) Find  $\mathbb{E}(y_{t+1} \mid y_t, u_t)$  and  $\text{Var}(y_{t+1} \mid y_t, u_t)$ ;
  - (b) Find  $\mathbb{E}(y_{t+1} \mid y_t)$  and  $\text{Var}(y_{t+1} \mid y_t)$ .
5. Consider the equation  $y_t - 2.5y_{t-1} + y_{t-2} = u_t$ , where  $u_t$  is a white noise.
  - (a) Does it have any stationary solution of the form  $y_t = u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \dots$ ?  
If yes then find  $\alpha_1, \alpha_2, \alpha_3$ .
  - (b) (\*) Does it have any stationary solution of the form  $y_t = \dots + \alpha_{-1} u_{t+1} + \alpha_0 u_t + \alpha_1 u_{t-1} + \dots$ ?  
If yes then find  $\alpha_{-1}, \alpha_0, \alpha_1$ .  
Hint:  $(1 - 2.5L - 0.5L^2) = (1 - 2L)(1 - 0.5L)$ .

You can find more problems in the problem book draft, [https://github.com/bdemeshev/ts\\_pset](https://github.com/bdemeshev/ts_pset).