Here (W_t) denotes the standard Wiener process.

1. For $r < s < t < u$ find the following expected values
   (a) $\mathbb{E}((W_u - W_t)^2(W_s - W_r)^2)$;
   (b) $\mathbb{E}((W_u - W_s)(W_t - W_r))$;
   (c) $\mathbb{E}((W_t - W_r)(W_s - W_r)^2)$;
   (d) $\mathbb{E}(W_r W_s W_t)$;
   (e) $\mathbb{E}(W_r W_s W_t \mid W_s)$;

2. Consider Ito process X_t

$$dX_t = \exp(t) W_t \, dt + \exp(2W_t) \, dW_t, \quad X_0 = 1.$$

   Consider two processes, $A_t = 1 + t^2 + X_t^3$ and $B_t = 1 + t^2 + X_t^3 W_t^4$.
   (a) Find dA_t and dB_t.
   (b) Write the corresponding explicit expressions for A_t and B_t:

$$const + \int_0^t \ldots dW_u + \int_0^t \ldots du$$

   (c) Check whether X_t is a martingale.

3. Let $S_0 = 0$, $S_t = X_1 + X_2 + \ldots + X_t$. The increments X_t are independent and identically distributed:

x	-1	0	1
$\mathbb{P}(X_t = x)$	0.2	0.2	0.6

   (a) If possible find all constants a such that $M_t = S_t + at$ is a martingale.
   (b) If possible find all constants b such that $R_t = b^{S_t}$ is a martingale.

4. Consider the process X_t

$$X_t = tW_t + \int_0^t u W_u^2 \, dW_u.$$

   (a) Find $\mathbb{E}(X_t)$, $\mathbb{V}\mathrm{ar}(X_t)$.
   (b) Find dX_t.
   (c) Check whether X_t is a martingale.

5. A Hedgehog in the fog starts in $(0, 0)$ at $t = 0$ and moves randomly with equal probabilities in four directions (north, south, east, west) by one unit every minute.

   Let X_t and Y_t be his coordinates after t minutes and $S_t = X_t + Y_t$.
   (a) Find $\mathbb{E}(X_2 \mid S_2)$;
   (b) Find $\mathbb{V}\mathrm{ar}(X_2 \mid S_2)$.

   Hint: $\mathbb{V}\mathrm{ar}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - (\mathbb{E}(Y \mid X))^2$.

6. Vampire Petr and Markov Chains.

   Vampire Petr drinks blood of a new victim every day. Unfortunately 20% of the population are vaccinated against vampires. If more than one victim of the last three victims are vaccinated then Petr will be instantaneously cured and will return to the normal life.

   For simplicity let's assume that the last three victims were not vaccinated.

   (a) What is the probability that vampire Petr will be cured in the next three days?

   (b) How many victims will be bitten by vampire Petr on average?

7. Vampire Boris and Martingales.

   To survive vampire Boris needs to bite 70 talented students.

   These 70 talented students have formed a secret group. They have written their emails on small pieces of paper and have randomly distributed these pieces among them. Each student has exactly one piece of paper with an email[1].

   Initially vampire Boris knows contacts of just two persons from the group. Today he will contact them, drink their blood and get the emails they have. Then vampire Boris will contact new victims and so on.

   (a) For $t \geq 1$ consider the process M_t, the proportion of non bitten students after the day t.
   Is this process a martingale?

   (b) Using martingale stopping theorem or otherwise find the probability that vampire Boris will bite all 70 students.

[1]The group is so secret that it is possible that a student has his own email on his piece of paper