

Home Assignment 1

1. Let $\Omega = \mathbb{R}$. Explicitly find the sigma-algebras $\mathcal{F}_1 = \sigma(A)$, $\mathcal{F}_2 = \sigma(B)$, $\mathcal{F}_3 = \sigma(A, B)$ where $A = [-10; 5]$ and $B = (0; 10)$.
2. I throw a die once. Let X be the result of the toss. Count the number of events in sigma-algebras $\mathcal{F}_1 = \sigma(X)$, $\mathcal{F}_2 = \sigma(\{X > 3\})$, $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$.
3. Let $\Omega = \mathbb{R}$. The sigma-algebra \mathcal{F} is generated by all the sets of the form $(-\infty, t]$,

$$\mathcal{F} = \sigma(\{(-\infty; t] \mid t \in \mathbb{R}\})$$

Check whether $A_1 = (0; 10) \in \mathcal{F}$, $A_2 = \{5\} \in \mathcal{F}$, $A_3 = \mathbb{N} \in \mathcal{F}$.

4. Prove the following statements or provide a counter-example:
 - (a) If \mathcal{F}_1 and \mathcal{F}_2 are sigma-algebras then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is sigma-algebra.
 - (b) If X and Y are independent random variables then $\text{card } \sigma(X, Y) = \text{card } \sigma(X) + \text{card } \sigma(Y)$.

For finite sets card denotes just the number of elements.

5. I throw a die infinite number of times. Let the random variable X_n be equal to 1 if the n -th toss is head and 0 otherwise. Consider a pack of sigma-algebras: $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \dots)$.

Where possible provide an example of a non-trivial event (neither Ω nor \emptyset) such that

- (a) $A_1 \in \mathcal{F}_{2020}$;
- (b) $A_2 \in \mathcal{H}_{2020}$;
- (c) $A_3 \in \mathcal{F}_{2020}$ and $A_3 \in \mathcal{H}_{2020}$;
- (d) $A_4 \in \mathcal{F}_n$ for all n ;
- (e) $A_5 \in \mathcal{H}_n$ for all n .

Deadline: 25 September 2020, 21:00 MSK.

Home Assignment 2

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0 & 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)
 - (b) Classify the states of the Markov chain.
2. The Lonely Knight is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.
- (a) How many moves on average will it take to go back to A1?
 - (b) What proportion of her eternal life will she spend on every field?
3. Donald Trump throws a die until one appears or until he says «Stop». The payoff is equal to the last thrown number. Donald maximizes the expected payoff.

- (a) What is the best strategy and the corresponding expected payoff?

How do the answers change in the following modifications of the original game?

- (b) Donald is also required to stop at 3 and to continue on 4.
 - (c) Donald should pay 0.3 for every throw.
4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed Zmei Gorynich. Yes, there are infinitely many Zmei Gorynich.
- What is the probability that there is an Eternal Peaceful Journey?
5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTT or TTH appears. Ilya wins if HTT appears and Zmei wins if TTH appears.
- (a) What is the probability that Ilya wins?
 - (b) What is the expected number of throws?
 - (c) What is the expected number of throws given that Ilya won?

Deadline: 9 October 2020, 21:00 MSK.

Home Assignment 3

1. Consider the Vasicek interest rate model,

$$dR_t = a(b - R_t) dt + \sigma dW_t.$$

Here R_t is the interest rate and a , b and σ are positive constants.

- (a) Using the substitution $Y_t = e^{at} R_t$ find the solution of the stochastic differential equation;
 - (b) Find $\mathbb{E}(R_t)$ and $\mathbb{Var}(R_t)$.
 - (c) Which value in this model would you call long-term equilibrium rate and why?
2. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$. Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble. Write the stochastic differential equation for dY_t . The equation may contain Y_t and constants, but not X_t .
3. Let W_t^a and W_t^b be two independent Wiener processes. Consider the process $Q_t = \alpha W_t^a + \beta W_t^b$, where $\alpha^2 + \beta^2 = 1$.
- (a) Is Q_t a Wiener process? Carefully check all the assumptions.
 - (b) Find the limit in L^2 for $n \rightarrow \infty$ of

$$A_n = \sum_{i=1}^n (W^a(it/n) - W^a((i-1)t/n)) (W^b(it/n) - W^b((i-1)t/n))$$

- (c) Find the limit in L^2 for $n \rightarrow \infty$ of

$$B_n = \sum_{i=1}^n (Q(it/n) - Q((i-1)t/n)) (W^b(it/n) - W^b((i-1)t/n))$$

- (d) Find $\mathbb{Corr}(Q_t, W_t^b)$.
 - (e) Without formal proof guess the value of $dQ_t dW_t^b$ in the Ito's lemma for correlated Wiener processes.
4. Consider the Cox-Ingersoll-Ross interest rate model

$$dR_t = a(b - R_t) dt + \sigma \sqrt{R_t} dW_t.$$

Here R_t is the interest rate and a , b and σ are positive constants.

Find $\mathbb{E}(R_t)$ and $\mathbb{Var}(R_t)$.

5. The share price S_t satisfies the Black and Scholes model and $dX_t = t dS_t$. Find $\mathbb{E}(X_t)$ and $\mathbb{Var}(X_t)$.

1. Consider the Black and Scholes model. At time $T > 1$ the asset pays you

$$X_T = \begin{cases} \ln S_T, & \text{if } S_T > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Today is $t = 1$. Find the current price X_1 of this asset.

2. Пусть y_t — стационарный процесс. Проверьте стационарность процессов:

(a) $a_t = \Delta^2 y_t$;

(b) $b_t = 2y_t + 3y_{t-1} + 18$.

3. Правильный кубик подбрасывают три раза, обозначим результаты подбрасываний X_1 , X_2 и X_3 . Также введём обозначения для сумм $L = X_1 + X_2$, $R = X_2 + X_3$ и $S = X_1 + X_2 + X_3$.

(a) С помощью качественных рассуждений (без вычислений) определите знаки частных корреляций¹ $\text{pCorr}(L, R; S)$, $\text{pCorr}(L, S; R)$, $\text{pCorr}(X_1, R; S)$.

(b) Найдите точное значение каждой частной корреляции.

4. У эконометрессы Ефросиньи был стационарный ряд (y_t) , $t \geq 1$ с $\mathbb{E}(y_t) = 5$. $\text{Var}(y_t) = 16$ и $\text{Cov}(y_t, y_{t-1}) = 4$.

Ефросинье было скучно и она подбрасывала неправильную монетку, выпадающую орлом с вероятностью 0.7. Если выпадал орёл, она оставляла очередной y_t , если решка — то зачёркивала.

Обозначим полученную новую последовательность (z_t) .

(a) Является ли (z_t) стационарным?

(b) Найдите $\mathbb{E}(z_t)$, $\text{Var}(z_t)$ и $\text{Cov}(z_t, z_{t-1})$.

5. Рассмотрим стационарное решение (y_t) уравнения $y_t = 6 + 0.5y_{t-1} + u_t - 0.3u_{t-1}$, где (u_t) — белый шум.

(a) Найдите $\mathbb{E}(y_t)$ и $\text{Var}(y_t)$.

(b) Найдите первые три значения автокорреляционной функции.

(c) Найдите первые три значения частной автокорреляционной функции.

Hint: для данного случая есть теорема, которая гарантирует, что у стационарного решения $\text{Cov}(y_t, u_{t+k}) = 0$ при $k > 0$.

¹Запись $\text{pCorr}(X, Y; Z)$ означает частную корреляцию между X и Y , «очищенных» от эффекта Z .

1. Consider y_t described by $ETS(MMM)$ model. You can find all the equations in <https://otexts.com/fpp3/>. Is it true that $z_t = \ln y_t$ is exactly described by $ETS(AAA)$ model? Approximately?
2. Consider $ETS(AA_dN)$ model with $\phi = 0.9$, $\alpha = 0.3$, $\beta = 0.1$ and $\sigma^2 = 16$. Express 95% predictive intervals for y_{t+1} and y_{t+2} in terms of ℓ_t , b_t , y_t and u_t .
3. Find $\mathbb{E}(y_t)$, $\text{Var}(y_t)$, $\text{Cov}(y_t, y_{t+1})$ in the $ETS(AAN)$ model with given ℓ_0 , b_0 , α , β and σ^2 .
4. Consider stationary $ARMA(1, 1)$ process, $y_t = 0.7y_{t-1} + u_t + 0.2u_{t-1}$, where $\text{Var}(u_t) = 16$.
 - (a) Find $\mathbb{E}(y_{t+1} \mid y_t, u_t)$ and $\text{Var}(y_{t+1} \mid y_t, u_t)$;
 - (b) Find $\mathbb{E}(y_{t+1} \mid y_t)$ and $\text{Var}(y_{t+1} \mid y_t)$.
5. Consider the equation $y_t - 2.5y_{t-1} + y_{t-2} = u_t$, where u_t is a white noise.
 - (a) Does it have any stationary solution of the form $y_t = u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \dots$?
If yes then find $\alpha_1, \alpha_2, \alpha_3$.
 - (b) (*) Does it have any stationary solution of the form $y_t = \dots + \alpha_{-1} u_{t+1} + \alpha_0 u_t + \alpha_1 u_{t-1} + \dots$?
If yes then find $\alpha_{-1}, \alpha_0, \alpha_1$.
Hint: $(1 - 2.5L - 0.5L^2) = (1 - 2L)(1 - 0.5L)$.

You can find more problems in the problem book draft, https://github.com/bdemeshev/ts_pset.