

You have 100 minutes. You can use A4 cheat sheet and calculator. Be brave!

1. I throw a fair die until the sequence 626 appears. Let  $N$  be the number of throws.

(a) What is the expected value  $\mathbb{E}(N)$ ?

(b) Write down the system of linear equations for the moment generating function of  $N$ . You don't need to solve it!

2. Consider the following stationary process

$$y_t = 1 + 0.5y_{t-2} + u_t + u_{t-1},$$

where random variables  $u_t$  are independent  $\mathcal{N}(0; 4)$ .

(a) Find the 95% predictive interval for  $y_{101}$  given that  $y_{100} = 2$ ,  $y_{99} = 3$ ,  $y_{98} = 1$ ,  $u_{99} = -1$ .

(b) Find the point forecast for  $y_{101}$  given that  $y_{100} = 2$ .

3. I have an unfair coin with probability of heads equal to  $h \in (0; 1)$ .

(a) Let  $N$  be the number of tails before the first head. Find the MGF of  $N$ .

(b) Let  $S$  be the number of tails before  $k$  heads (not necessary consecutive). Find the MGF of  $S$ .

(c) What is the limit of  $MGF_S(t)$  when  $k \rightarrow \infty$  and  $k \times h \rightarrow 0.5$ ? What is the name of the corresponding distribution?

4. Consider the stochastic process  $X_t = f(t) \cos(2021W_t)$ .

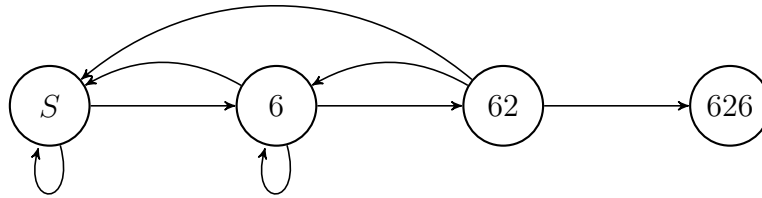
(a) Find  $dX_t$ .

(b) Find any  $f(t) \neq 0$  such that  $X_t$  is a martingale.

(c) Using  $f(t)$  from the previous point find  $\mathbb{E}(\cos(2021W_t))$ .

# 1 Solution

1. Let's draw the chain



The system of equations for expected values:

$$\begin{cases} x_s = 1 + \frac{1}{6}x_6 + \frac{5}{6}x_s \\ x_6 = 1 + \frac{1}{6}x_6 + \frac{1}{6}x_{62} + \frac{4}{6}x_s \\ x_{62} = 1 + \frac{1}{6} \cdot 0 + \frac{1}{6}x_6 + \frac{4}{6}x_s \end{cases}$$

The system of equations for moment generating functions:

$$\begin{cases} m_s(t) = \exp(t) \left( \frac{1}{6}m_6(t) + \frac{5}{6}m_s(t) \right) \\ m_6(t) = \exp(t) \left( \frac{1}{6}m_6(t) + \frac{1}{6}m_{62}(t) + \frac{4}{6}m_s(t) \right) \\ m_{62}(t) = \exp(t) \left( \frac{1}{6} \cdot 1 + \frac{1}{6}m_6(t) + \frac{4}{6}m_s(t) \right) \end{cases}$$

2. (a) Let's denote by  $x$  all available information,

$$x = \begin{pmatrix} y_{100} \\ y_{99} \\ y_{98} \\ u_{99} \end{pmatrix}$$

Let's use  $t = 100$ :

$$y_{100} = 1 + 0.5y_{98} + u_{100} + u_{99}$$

Using all available information we obtain  $u_{100} = 1.5$  and hence

$$y_{101} \mid x \sim \mathcal{N}(1 + 0.5y_{99} + u_{100}; 4)$$

(b) Here we work with true betas:

$$\mathbb{E}(y_{101} \mid y_{100}) = \mu_y + \frac{\text{Cov}(y_{100}, y_{101})}{\text{Var}(y_{100})}(y_{100} - \mu_y)$$

3. (a) Moment generating function

$$m_N(t) = \sum_{j=0}^{\infty} \exp(tj)(1-h)^jh = h \sum_{j=0}^{\infty} (\exp(t)(1-h))^j = \frac{h}{1 - \exp(t)(1-h)}$$

(b) As  $S = N_1 + N_2 + \dots + N_k$ :

$$m_S(t) = \left( \frac{h}{1 - \exp(t)(1 - h)} \right)^k$$

(c) Due to my mistake the limit is easy, 0.

In my dream it was  $k \rightarrow \infty$ ,  $k \cdot (1 - h) \rightarrow 0.5$  and that would be fun!

4. (a) Let's use Ito's lemma

$$dX_t = f'(t) \cos(2021W_t)dt - 2021f(t) \sin(2021W_t)dW_t + \frac{1}{2}2021^2 f(t) \cos(2021W_t)dt$$

(b) To make  $X_t$  a martingale we should kill  $dt$  term.

(c) As  $X_t$  is martingale  $\mathbb{E}(X_t) = \mathbb{E}(X_0) = f(0)$ . So  $\mathbb{E}(\cos(2021W_t)) = f(0)/f(t)$ .