Home Assignment 1

1. Let $\Omega = \mathbb{R}$. Explicitely find the sigma-algebras $\mathcal{F}_1 = \sigma(A)$, $\mathcal{F}_2 = \sigma(B)$, $\mathcal{F}_3 = \sigma(A, B)$ where $A = [-10; 5]$ and $B = (0; 10)$.

2. I throw a die once. Let X be the result of the toss. Count the number of events in sigma-algebras $\mathcal{F}_1 = \sigma(X)$, $\mathcal{F}_2 = \sigma(\{X > 3\})$, $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$.

3. Let $\Omega = \mathbb{R}$. The sigma-algebra $\mathcal{F}$ is generated by all the sets of the form $(-\infty, t]$,

$$\mathcal{F} = \sigma\left(\{(-\infty; t] \mid t \in \mathbb{R}\}\right)$$

   Check whether $A_1 = (0; 10) \in \mathcal{F}$, $A_2 = \{5\} \in \mathcal{F}$, $A_3 = \mathbb{N} \in \mathcal{F}$.

4. Prove the following statements or provide a counter-example:

   (a) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are sigma-algebras then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is sigma-algebra.

   (b) If X and Y are independent random variables then $\operatorname{card} \sigma(X, Y) = \operatorname{card} \sigma(X) + \operatorname{card} \sigma(Y)$.

   For finite sets card denotes just the number of elements.

5. I throw a die infinite number of times. Let the random variable X_n be equal to 1 if the n-th toss is head and 0 otherwise. Consider a pack of sigma-algebras: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \ldots)$.

   Where possible provide and example of a non-trivial event (neither Ω nor $\emptyset$) such that

   (a) $A_1 \in \mathcal{F}_{2020}$;

   (b) $A_2 \in \mathcal{H}_{2020}$;

   (c) $A_3 \in \mathcal{F}_{2020}$ and $A_3 \in \mathcal{H}_{2020}$;

   (d) $A_4 \in \mathcal{F}_n$ for all n;

   (e) $A_5 \in \mathcal{H}_n$ for all n.

Deadline: 25 September 2020, 21:00 MSK.

Home Assignment 2

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0 & 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)

   (b) Classify the states of the Markov chain.

2. The Lonely Knight is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.

   (a) How many moves on average will it take to go back to A1?

   (b) What proportion of her eternal life will she spend on every field?

3. Donald Trump throws a die until one appears or until he says «Stop». The payoff is equal to the last thrown number. Donald maximizes the expected payoff.

   (a) What is the best strategy and the corresponding expected payoff?

   How do the answers change in the following modifications of the orignal game?

   (b) Donald is also required to stop at 3 and to continue on 4.

   (c) Donald should pay 0.3 for every throw.

4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed Zmei Gorynich. Yes, there are infinitely many Zmei Gorynich.

   What is the probability that there is an Eternal Peaceful Journey?

5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTT or TTH apears. Ilya wins if HTT appears and Zmei wins if TTH appears.

   (a) What is the probability that Ilya wins?

   (b) What is the expected number of throws?

   (c) What is the expected number of throws given that Ilya won?

Deadline: 9 October 2020, 21:00 MSK.

Home Assignment 2

1.

Deadline: 25 September 2020, 21:00 MSK.