Home Assignment 1

1. Let $\Omega = \mathbb{R}$. Explicitely find the sigma-algebras $\mathcal{F}_1 = \sigma(A)$, $\mathcal{F}_2 = \sigma(B)$, $\mathcal{F}_3 = \sigma(A, B)$ where $A = [-10; 5]$ and $B = (0; 10)$.

2. I throw a die once. Let X be the result of the toss. Count the number of events in sigma-algebras $\mathcal{F}_1 = \sigma(X)$, $\mathcal{F}_2 = \sigma(\{X > 3\})$, $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$.

3. Let $\Omega = \mathbb{R}$. The sigma-algebra $\mathcal{F}$ is generated by all the sets of the form $(-\infty, t]$,
$$\mathcal{F} = \sigma\left(\{(-\infty; t] \mid t \in \mathbb{R}\}\right)$$
Check whether $A_1 = (0; 10) \in \mathcal{F}$, $A_2 = \{5\} \in \mathcal{F}$, $A_3 = \mathbb{N} \in \mathcal{F}$.

4. Prove the following statements or provide a counter-example:
   (a) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are sigma-algebras then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is sigma-algebra.
   (b) If X and Y are independent random variables then $\operatorname{card} \sigma(X, Y) = \operatorname{card} \sigma(X) + \operatorname{card} \sigma(Y)$.

   For finite sets card denotes just the number of elements.

5. I throw a die infinite number of times. Let the random variable X_n be equal to 1 if the n-th toss is head and 0 otherwise. Consider a pack of sigma-algebras: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \ldots)$.

   Where possible provide and example of a non-trivial event (neither Ω nor $\emptyset$) such that
   (a) $A_1 \in \mathcal{F}_{2020}$;
   (b) $A_2 \in \mathcal{H}_{2020}$;
   (c) $A_3 \in \mathcal{F}_{2020}$ and $A_3 \in \mathcal{H}_{2020}$;
   (d) $A_4 \in \mathcal{F}_n$ for all n;
   (e) $A_5 \in \mathcal{H}_n$ for all n.

Deadline: 25 September 2020, 21:00 MSK.

Home Assignment 2

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0 & 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)

   (b) Classify the states of the Markov chain.

2. The Lonely Knight is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.

   (a) How many moves on average will it take to go back to A1?

   (b) What proportion of her eternal life will she spend on every field?

3. Donald Trump throws a die until one appears or until he says «Stop». The payoff is equal to the last thrown number. Donald maximizes the expected payoff.

   (a) What is the best strategy and the corresponding expected payoff?

   How do the answers change in the following modifications of the orignal game?

   (b) Donald is also required to stop at 3 and to continue on 4.

   (c) Donald should pay 0.3 for every throw.

4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed Zmei Gorynich. Yes, there are infinitely many Zmei Gorynich.

   What is the probability that there is an Eternal Peaceful Journey?

5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTT or TTH apears. Ilya wins if HTT appears and Zmei wins if TTH appears.

   (a) What is the probability that Ilya wins?

   (b) What is the expected number of throws?

   (c) What is the expected number of throws given that Ilya won?

Deadline: 9 October 2020, 21:00 MSK.

Home Assignment 3

1. Consider the Vasicek interest rate model,

$$dR_t = a(b - R_t)\,dt + \sigma\,dW_t.$$

Here R_t is the interest rate and a, b and σ are positive constants.

   (a) Using the substitution $Y_t = e^{at}R_t$ find the solution of the stochastic differential equation;
   (b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\mathrm{ar}(R_t)$.
   (c) Which value in this model would you call long-term equilibrium rate and why?

2. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$.
Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble.
Write the stochastic differential equation for dY_t. The equation may contain Y_t and constants, but not X_t.

3. Let W_t^a and W_t^b be two independent Wiener processes. Consider the process $Q_t = \alpha W_t^a + \beta W_t^b$, where $\alpha^2 + \beta^2 = 1$.

   (a) Is Q_t a Wiener process? Carefully check all the assumptions.
   (b) Find the limit in L^2 for $n \to \infty$ of

$$A_n = \sum_{i=1}^{n} \left(W^a(it/n) - W^a((i-1)t/n)\right)\left(W^b(it/n) - W^b((i-1)t/n)\right)$$

   (c) Find the limit in L^2 for $n \to \infty$ of

$$B_n = \sum_{i=1}^{n} \left(Q(it/n) - Q((i-1)t/n)\right)\left(W^b(it/n) - W^b((i-1)t/n)\right)$$

   (d) Find $\mathbb{C}\mathrm{orr}(Q_t, W_t^b)$.
   (e) Without formal proof guess the value of $dQ_t dW_b^t$ in the Ito's lemma for correlated Wiener processes.

4. Consider the Cox-Ingersoll-Ross interest rate model

$$dR_t = a(b - R_t)\,dt + \sigma\sqrt{R_t}\,dW_t.$$

Here R_t is the interest rate and a, b and σ are positive constants.
Find $\mathbb{E}(R_t)$ and $\mathbb{V}\mathrm{ar}(R_t)$.

5. The share price S_t satisfies the Black and Scholes model and $dX_t = t dS_t$.
Find $\mathbb{E}(X_t)$ and $\mathbb{V}\mathrm{ar}(X_t)$.

1. Consider the Black and Scholes model. At time $T > 1$ the asset pays you

$$X_T = \begin{cases} \ln S_T, \text{ if } S_T > 1; \\ 0, \text{ otherwise.} \end{cases}$$

Today is $t = 1$. Find the current price X_1 of this asset.

2. Пусть y_t — стационарный процесс. Проверьте стационарность процессов:
   (a) $a_t = \Delta^2 y_t$;
   (b) $b_t = 2y_t + 3y_{t-1} + 18$.

3. Правильный кубик подбрасывают три раза, обозначим результаты подбрасываний X_1, X_2 и X_3. Также введём обозначения для сумм $L = X_1 + X_2$, $R = X_2 + X_3$ и $S = X_1 + X_2 + X_3$.
   (a) С помощью качественных рассуждений (без вычислений) определите знаки частных корреляций[1] $\mathrm{p}\mathbb{C}\mathrm{orr}(L, R; S)$, $\mathrm{p}\mathbb{C}\mathrm{orr}(L, S; R)$, $\mathrm{p}\mathbb{C}\mathrm{orr}(X_1, R; S)$.
   (b) Найдите точное значение каждой частной корреляции.

4. У эконометрессы Ефросиньи был стационарный ряд (y_t), $t \geq 1$ с $\mathbb{E}(y_t) = 5$. $\mathbb{V}\mathrm{ar}(y_t) = 16$ и $\mathbb{C}\mathrm{ov}(y_t, y_{t-1}) = 4$.

   Ефросинье было скучно и она подбрасывала неправильную монетку, выпадающую орлом с вероятностью 0.7. Если выпадал орёл, она оставляла очередной y_t, если решка — то зачёркивала.

   Обозначим полученную новую последовательность (z_t).
   (a) Является ли (z_t) стационарным?
   (b) Найдите $\mathbb{E}(z_t)$, $\mathbb{V}\mathrm{ar}(z_t)$ и $\mathbb{C}\mathrm{ov}(z_t, z_{t-1})$.

5. Рассмотрим стационарное решение (y_t) уравнения $y_t = 6 + 0.5y_{t-1} + u_t - 0.3u_{t-1}$, где (u_t) — белый шум.
   (a) Найдите $\mathbb{E}(y_t)$ и $\mathbb{V}\mathrm{ar}(y_t)$.
   (b) Найдите первые три значения автокорреляционной функции.
   (c) Найдите первые три значения частной автокорреляционной функции.

   Hint: для данного случая есть теорема, которая гарантирует, что у стационарного решения $\mathbb{C}\mathrm{ov}(y_t, u_{t+k}) = 0$ при $k > 0$.

[1] Запись $\mathrm{p}\mathbb{C}\mathrm{orr}(X, Y; Z)$ означает частную корреляцию между X и Y, «очищенных» от эффекта Z.

1. Consider y_t described by $ETS(MNM)$ model. You can find all the equations in `https://otexts.com/fpp3/`. Is it true that $z_t = \ln y_t$ is exactly described by $ETS(ANA)$ model? Approximately?
2. Consider $ETS(AA_dN)$ model with $\phi = 0.9$, $\alpha = 0.3$, $\beta = 0.1$ and $\sigma^2 = 16$. Express 95% predictive intervals for y_{t+1} and y_{t+2} in terms of ℓ_t, b_t, y_t and u_t.
3. Find $\mathbb{E}(y_t)$, $\mathbb{V}\text{ar}(y_t)$, $\mathbb{C}\text{ov}(y_t, y_{t+1})$ in the $ETS(AAN)$ model with given ℓ_0, b_0, α, β and σ^2.
4. Consider stationary $ARMA(1,1)$ process, $y_t = 0.7y_{t-1} + u_t + 0.2u_{t-1}$, where $\mathbb{V}\text{ar}(u_t) = 16$.
   (a) Find $\mathbb{E}(y_{t+1} \mid y_t, u_t)$ and $\mathbb{V}\text{ar}(y_{t+1} \mid y_t, u_t)$;
   (b) Find $\mathbb{E}(y_{t+1} \mid y_t)$ and $\mathbb{V}\text{ar}(y_{t+1} \mid y_t)$.
5. Consider the equation $y_t - 2.5y_{t-1} + y_{t-2} = u_t$, where u_t is a white noise.
   (a) Does it have any stationary solution of the form $y_t = u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \ldots$?
   If yes then find α_1, α_2, α_3.
   (b) (*) Does it have any stationary solution of the form $y_t = \ldots + \alpha_{-1} u_{t+1} + \alpha_0 u_t + \alpha_1 u_{t-1} + \ldots$?
   If yes then find α_{-1}, α_0, α_1.
   Hint: $(1 - 2.5L + L^2) = (1 - 2L)(1 - 0.5L)$.

You can find more problems in the problem book draft, `https://github.com/bdemeshev/ts_pset`.

Beta distribution $Beta(a, b)$ has density function $f(x)$ proportional to $x^{a-1}(1-x)^{b-1}$ on $[0; 1]$. The proportionality constant depends on a and b.

Gamma distribution $Gamma(\lambda, k)$ has density function $f(x)$ proportional to $x^{k-1}\lambda^k \exp(-\lambda x)$ on $[0; +\infty)$. The proportionality constant depends on k.

You can find more information about these distributions on Wikipedia or elsewhere, I believe in you! :)

1. Consider a random sample $Y_1, Y_2, \ldots, Y_n$ from uniform distribution on $[-a; 7a]$.
   (a) Find method of moments estimator for a using $\mathbb{E}(Y_i)$.
   (b) Find method of moments estimator for a using $\mathbb{E}(|Y_i|)$.
   (c) Are these method of moments estimators unbiased?
   (d) Which method of moments estimator has lowest mean squared error?
   (e) Find the maximum likelihood estimator of a.
2. Find the moment generating function for the $Gamma(\lambda, k)$ distribution.
3. Find sufficient statistics for unknown parameters:
   (a) Beta distribution $Beta(a, b)$ with unknown a and b.
   (b) Beta distribution $Beta(a, b)$ with known a and unknown b.
   (c) Gamma distribution $Gamma(\lambda, k)$ with unknown λ and k.
   (d) Gamma distribution $Gamma(\lambda, k)$ with known k and unknown λ.
4. The log-density function has the following form:

$$\ln f(x \mid \theta_1, \theta_2) = a(x) - b(\theta_1, \theta_2) + \theta_1 c_1(x) + \theta_2 c_2(x),$$

   where a, b, c_1 and c_2 are some known functions.
   (a) Find the sufficient statistics for unknown θ_1, θ_2.
   (b) Find the sufficient statistics for unknown θ_1 with known θ_2.
   (c) Express $\mathbb{E}(c_1(X))$ using the function $b(\theta_1, \theta_2)$.
   (d) Express $\mathbb{C}\mathrm{ov}(c_1(X), c_2(X))$ using the function $b(\theta_1, \theta_2)$.

   Hint for the last two points: what are the expected value and the variance of the score-function?
5. The estimator $\hat{\theta}$ is unbiased but not necessary obtained by maximum likelihood.
   Find $\mathbb{C}\mathrm{ov}(\hat{\theta}, \partial\ell/\partial\theta)$ where ℓ is the log-likelihood function.

A little bit of inequalities...

1. You know that $\mathbb{E}((X-20)^6) = 1000$ and $\mathbb{E}(X) = 20$.

   What are the possible values for $\mathbb{P}(|X-20| \geq 20)$?

2. The loose milk price at day t is M_t. The variables (M_t) are independent and identically distributed with $\mathbb{E}(M_t) = 100$ roubles per liter.

   Every day Masha buys one liter of milk. Every day Sasha buys loose milk exactly for 100 roubles.

   After 30 days Masha and Sasha compares their spendings. Let's denote by X^S and X^M the total expenditures by Sasha and Masha. And let's denote by Q^S and Q^M the total volume of milk bought by Sasha and Masha.

   (a) Compare $\mathbb{E}(X^S)$ and $\mathbb{E}(X^M)$.
   (b) Compare $\mathbb{E}(Q^S)$ and $\mathbb{E}(Q^M)$.

A little bit of estimation...

3. Kazimir Malevich draws random black rectangles. One side of a rectangle, A_i, is approximately normally distributed $\mathcal{N}(a; 1)$, the other side, B_i, is approximately $\mathcal{N}(b, 1)$. All variables are independent.

   (a) Calculate $\mathbb{E}(S_i)$ and $\mathbb{E}(S_i^2)$.
   (b) You have recorded data of area for 100 paintings: S_i: $\bar{S} = 36$, $\sum S_i^2 = 162500$. Estimate a and b using method of moments.

4. In one of the offices there are 4 bank teller: Alice, Bob, Carol and Dave. Service times are independent and exponentially distributed with unknown rate λ. Exactly at the opening time exactly 4 clients entered the office.

   Provide maximum likelihood estimate of λ in the following cases:

   (a) Alice serviced her client in 20 minutes. Bob serviced his client in 15 minutes. Carol and Dave forgot to note the service time.
   (b) Alice serviced her client in 15 minutes. Bob serviced his client in 20 minutes. Carol and Dave were servicing their first clients more than 30 minutes.
   (c) The first client was serviced in 15 minutes. The second client was serviced in 20 minutes.

5. The random variables X_1, X_2, ..., X_n are independent and binomially distributed $Bin(100, p)$. Researcher Neznaika tries to estimate the parameter $\theta = \mathbb{P}(X_i = 2)$.

   He has invented an estimator $\hat{\theta} = \begin{cases} 1, & \text{if } X_1 = 2, \\ 0, & \text{otherwise.} \end{cases}$.

   (a) Find the minimal sufficient statistic for p, let's denote it by T.
   (b) Using Rao-Blackwell theorem construct a better estimate $\hat{\theta}'$ using T.
   (c) Check whether the new estimator is unbiased.
   (d) Using delta-method estimate the variance of $\hat{\theta}'$ for $n = 1000$ and $\sum X_i = 42000$ and find 99% confidence interval.