Привет !!

1. цепи маркова

$E(\ )$, $P(\ )$, $Var(\ )$

самый модный из вер-тей сюжет на соб-ях

2. → технич. про лемму Ито $\int_0^t A_u\, dW_u$

→ посчитать цену евро актива $E^*(e^{-rT}\cdot X_T \mid \mathcal{F}_0)$

0 — T

3. ур-ия модели данн. CI 95% для y_{t+2}

4. MGF !!

6. Vampire Petr and Markov Chains.

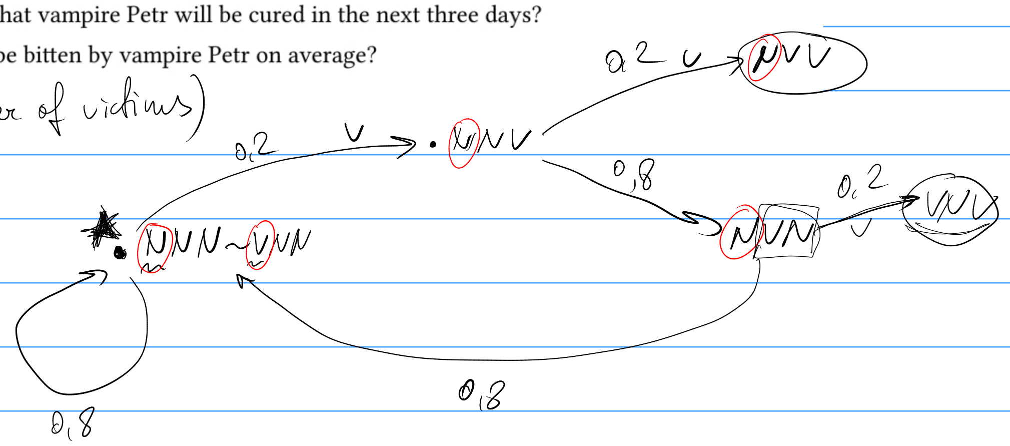
Vampire Petr drinks blood of a new victim every day. Unfortunately 20% of the population are vaccinated against vampires. If more than one victim of the last three victims are vaccinated then Petr will be instantaneously cured and will return to the normal life.

For simplicity let's assume that the last three victims were not vaccinated.

(a) What is the probability that vampire Petr will be cured in the next three days?

(b) How many victims will be bitten by vampire Petr on average?

(c) Var(number of victims)



Траектории
за 2 хода: VV
за 3 хода: NVV
VNV

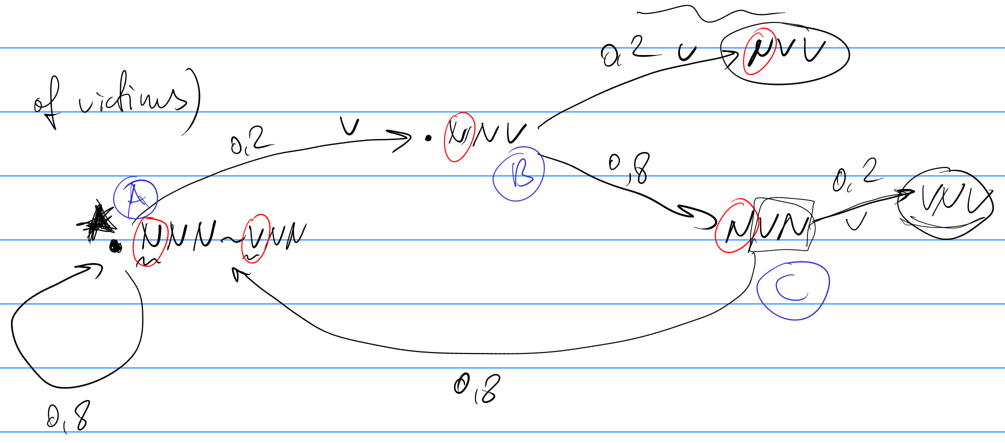
$\boxed{NNN}$ → NNN 0,2
→ NNV 0,8
VNN → NNN 0,2
→ NNV 0,8

T – время излечения $P(T\le 3) = P(T=2)+P(T=3) =$

$= P(VV) + P(NVV) + P(VNV) =$

$= 0{,}2^2 + (0{,}2^2\cdot 0{,}8)\times 2$

N — число жертв. $E(N)$



$\mu_A = E(N|A)$ — среднее ост-ся число жертв, если стартуем из A.

$$\begin{cases} \mu_A = 0{,}8\cdot(1+\mu_A) + 0{,}2(1+\mu_B) \\ \mu_B = 0{,}2\cdot 1 + 0{,}8\cdot(1+\mu_C) \\ \mu_C = 0{,}2\cdot 1 + 0{,}8\cdot(1+\mu_A) \end{cases} \Longrightarrow$$

здесь закон п. вер-стей

$$\begin{cases} \mu_A = 1 + 0{,}8\mu_A + 0{,}2\mu_B \\ \mu_B = 1 + 0{,}8\cdot\mu_C \\ \mu_C = 1 + 0{,}8\mu_A \end{cases}$$

$0{,}2\mu_A = 1 + 0{,}2\mu_B$ $\qquad \mu_A = 5 + \mu_B$

$$\begin{cases} \mu_B = 1 + 0{,}8\mu_C \\ \mu_C = 1 + 0{,}8(5+\mu_B) \end{cases}$$

$$\begin{cases} \mu_B = 1 + 0{,}8\mu_C \\ \mu_C = 5 + 0{,}8\mu_B \end{cases}$$

$\mu_C = 5 + 0{,}8(1 + 0{,}8\mu_C)$

$5{,}8 + 0{,}64\mu_C = \mu_C$

$\mu_C = \frac{5{,}8}{0{,}36} = 16.(1)$

$\mu_B \approx 18.85$ $\qquad \boxed{\mu_A \approx 23.85}$

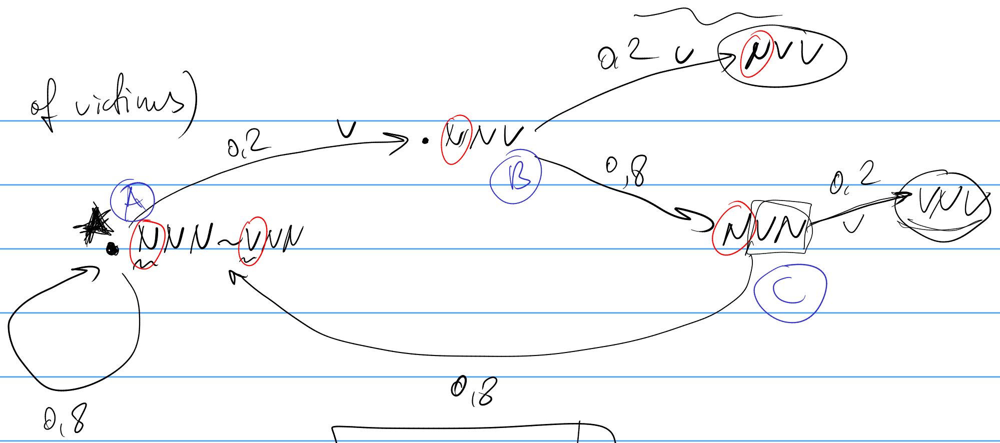
$$\boxed{Var(M) = E(M^2) - (E(M))^2}$$

N — ост-ся число жертв.

$E(N|A)$, $E(N|B)$, $E(N|C)$ — с.в.

$q_A = E(N^2|A)$, $\underbrace{E(N^2|B)}_{q_B}$, $\underbrace{E(N^2|C)}_{q_C}$

of victims)



осторожно!

$$q_A = E(N^2|A) = 0{,}2 \cdot E((N+1)^2|\ B\ ) + 0{,}8 \cdot E((N+1)^2|\ A\ )$$

$$\cancel{0.2 \cdot E(N^2+1|B) + 0{,}8\, E(N^2+1|\ A)}$$

(1) $$q_A = 0{,}2\left(1 + 2E(N|B) + E(N^2|B)\right) + 0{,}8\left(1 + 2E(N|A) + E(N^2|A)\right)$$

$$q_A = 0{,}2\left(1 + 2\mu_B + q_B\right) + 0{,}8\left(1 + 2\mu_A + q_A\right)$$

знаем

$$q_B = 0{,}2 \cdot 1^2 + 0{,}8 \cdot E\left((N+1)^2|C\right)$$

(2) $$q_B = 0{,}2 + 0{,}8(1 + 2\mu_C + q_C)$$

(3) $$q_C = 0{,}2 \cdot 1^2 + 0{,}8(1 + 2\mu_A + q_A)$$

$$\begin{cases}(1)\\(2)\\(3)\end{cases} \Rightarrow q_a, q_B, q_C$$

(d) Выпишите систему на MGF для N

$$MGF_N(t) = E\left(e^{tN}\right)$$

$$M_A(t) = E(e^{tN}/A)$$
$$M_B(t) = E(e^{tN}|B)$$

$$M_A(t) = E(e^{tN}|A) = 0{,}2 \cdot E\left(e^{t(N+1)}|B\right) + 0{,}8\, E\left(e^{t(N+1)}|A\right)$$

$$M_A(t) = 0{,}2 \cdot E(e^t \cdot e^{tN}|B) + 0{,}8\, E(e^t \cdot e^{tN}/A)$$

$$M_A(t) = e^t \cdot \left[0{,}2\, M_B(t) + 0{,}8\, M_A(t)\right]$$

$$M_B(t) = 0{,}2 \cdot e^t + 0{,}8\, E\left(e^{t(N+1)} \mid C\right)$$

$$M_B(t) = 0{,}2e^t + 0{,}8e^t \cdot M_C(t)$$

Упр. Наблюдаем a_{t}, b_t $\quad c_t = a_t + b_t$

$a_t = 0{,}5a_{t-1} + u_t$ $\quad u_t$ – б. шум. u_t не зав. от прошлых $a_{t-1}, a_{t-2}, \ldots$, $b_{t-1}, b_{t-2}, \ldots$

$b_t = 0{,}8b_{t-1} + w_t$ $\quad w_t$ – б. шум. w_t – не зависит от прошлого $a_{t-1}, a_{t-2}, \ldots$, $b_{t-1}, b_{t-2}, \ldots$

$$Var\left(\begin{pmatrix} u_t \\ w_t \end{pmatrix}\right) = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}$$

$\begin{pmatrix} u_t \\ w_t \end{pmatrix} \sim N$

$a_{100} = 1$

$b_{100} = 2$ $\quad$ 95% PI для c_{102}

идея:

будущее значение = известная часть + часть, не зависящая от известной

отвечает за точечный прогноз

отвечает за ширину предиктивного интервала

$$c_{102} = a_{102} + b_{102} = 0{,}5a_{101} + u_{102} + 0{,}8b_{101} + w_{102} =$$

$$= 0{,}5(0{,}5a_{100} + u_{101}) + u_{102} + 0{,}8(0{,}8b_{100} + w_{101}) + w_{102} =$$

$$= \boxed{0{,}25a_{100} + 0{,}64b_{100}} + (0{,}5u_{101} + 0{,}8w_{101}) + (u_{102} + w_{102})$$

$$E(c_{102} \mid \mathcal{F}_{100}) = 0{,}25a_{100} + 0{,}64b_{100} = 0{,}25 + 1{,}28 = 1{,}53$$

$$Var(c_{102} \mid \mathcal{F}_{100}) = (0{,}5^2 \cdot 4 + 0{,}64 \cdot 9 + 2 \cdot 0{,}5 \cdot 0{,}8 \cdot 1) + (4 + 9 + 2 \cdot 1)$$

95% $\quad c_{102} \in \left[1{,}53 - 1{,}96\sqrt{Var(c_{102} \mid \mathcal{F}_{100})};\ 1{,}53 + 1{,}96\sqrt{Var(c_{102} \mid \mathcal{F}_{100})}\right]$

3. (10 points) The process Y_t is defined by

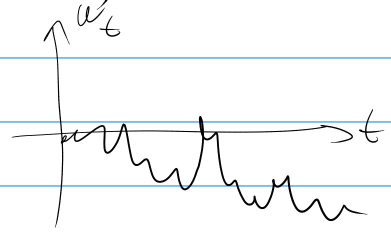
$$dY_t = W_t^2 dt + W_t dW_t, \ Y_0 = 0.$$

(a) (6 points) Find $\mathbb{E}(Y_t)$, $\mathbb{E}(Y_t W_t)$, $\mathbb{E}(Y_t W_t^2)$.

(b) (4 points) Find $\mathrm{Var}(Y_t)$.

Минибег $\int_0^t A_u \, dW_u$ ← прибыль от инвестиций в W_u

(при выполнении мягких предпосылок на A_u)



1. $E\left(\int_0^t A_u \, dW_u\right) = 0$

Isometry 2. $\mathrm{Var}\left(\int_0^t A_u \, dW_u\right) =$

$= \int_0^t E(A_u^2) \, du$

2bis. $\mathrm{Cov}\left(\int_0^t A_u \, dW_u, \int_0^t B_u \, dW_u\right) = \int_0^t E(A_u B_u) \cdot du$

3. $E\left(\int_0^t A_u \, du\right) = \int_0^t E(A_u) \, du$

4. Лемма Ито.

опр X_t - процесс Ито

$X_t = \int_0^t A_u \, dW_u + \int_0^t B_u \, du + X_0$ полная ф. записи

$dX_t = A_t \, dW_t + B_t \, dt$ краткая ф. записи

Лемма Ито для краткой формы записи.

Если $Y_t = f(X_t, W_t, t)$ W_t - вин. процесс, X_t - процесс Ито

(мягкие тех предпосылки), то

dY_t можно найти так!

То: $\begin{cases} dY_t = f'_x \cdot dX_t + f'_w \cdot dW_t + f'_t \cdot dt + \\ \frac{1}{2}\left(f''_{xx} \cdot (dX_t)^2 + f''_{ww} \cdot (dW)^2 + 2 \cdot f''_{xw} \cdot dX_t \cdot dW_t\right) \end{cases}$

$dt \cdot dt = 0$

$dt \cdot dW_t = 0$

причем правая часть упрощается по принципу $(dW_t)^2 = dt$

3. (10 points) The process Y_t is defined by

$$\boxed{dY_t = W_t^2 dt + W_t dW_t,\ Y_0 = 0.}$$

(a) (6 points) Find $\mathbb{E}(Y_t)$, $\mathbb{E}(Y_tW_t)$, $\mathbb{E}(Y_tW_t^2)$.

(b) (4 points) Find $\mathrm{Var}(Y_t)$.

a) $$Y_t = \int_0^t W_u^2\,du + \int_0^t W_u\,dW_u + Y_0$$

$$E(Y_t) = E\left(\int_0^t W_u^2\,du + \int_0^t W_u\,dW_u + 0\right) =$$

$$= \int_0^t E(W_u^2)\,du + 0 + 0 =$$

$$= \int_0^t u^{\cancel{2}}\,du = \boxed{\frac{t^2}{2}}$$

Characterization: W_t

$W_t - W_s \sim N(0; t-s)$

$W_t \sim N(0;t)$

$\boxed{E(W_t^2) = Var(W_t) = t}$

$$E(Y_tW_t) = E\left[W_t\cdot\int_0^t W_u^2\,du + W_t\cdot\int_0^t W_u\,dW_u + 0\right]$$

$$d(\underbrace{Y_tW_t}_{f(Y,W)}) = W_t\cdot dY_t + Y_t\cdot dW_t + \frac{1}{2}\left(2\cdot 1\cdot dW_t\cdot dY_t\right) =$$

f''_{YW}

$$= W_t\cdot\left(W_t^2\,dt + W_t\,dW_t\right) + Y_t\cdot dW_t + \underbrace{dW_t}_{dW}\underbrace{(\dots W_t\,dW_t)}_{dY}$$

$$d(Y_tW_t) = W_t^3\,dt + W_t^2\,dW_t + Y_t\,dW_t + W_t\,dt$$

$$Y_tW_t = \int_0^t W_u^3 + W_u\,du + \int_0^t (W_u^2 + Y_u)\,dW_u + Y_0\cdot W_0$$

$$E(Y_tW_t) = \int_0^t E(W_u^3 + W_u)\cdot du + 0 + 0 =$$

$W_u \sim N(0;u)$

$= 0$

(*) $$\boxed{dY_t\,dW_t = (W_t^2dt + W_tdW_t)\cdot dW_t = 0 + W_t\cdot dt}$$

$$d(Y_tW_t^2) = f'_Y\cdot dY + f'_W\cdot dW + \frac{1}{2}\left(2\cdot f''_{YW}\,\boxed{dY\,dW}\right) =$$

$$d(Y_tW_t^2) = W_t^2\cdot dY + 2W_t\cdot Y_t\cdot dW_t + 2W_t\cdot\boxed{W_t\,dt}$$

краткая

$$d(Y_t W_t^2) = W_t^2 \cdot (W_t^2\,dt + W_t\,dW_t) + 2W_t Y_t\,dW_t + 2W_t^2\,dt$$

длинная

$$Y_t W_t^2 = Y_0 W_0^2 + \int_0^t \ldots\ dW_u + \int_0^t (W_u^4 + 2W_u^2)\,du$$

$$E(Y_t W_t^2) = \int_0^t E(W_u^4) + 2E(W_u^2)\,du = \int_0^t 3u^2 + 2u\,du =$$

$$= t^3 + t^2$$

$$W_u \sim N(0; u)$$

$$E(W_u^7) = 0 \qquad E(W_u^8) = u^4 \cdot E(N^8(0;1)) = u^4 \cdot 7 \cdot 5 \cdot 3 \cdot 1$$

$$E(W_u^{10}) = u^5 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$$

$$E(W_u^2) = u^1 \cdot 1$$

б) $Var(Y_t) = E(Y_t^2) - (E(Y_t))^2 = E(Y_t^2) - \left(\cancel{\frac{t^3}{3}}\right)^2 \leftarrow \left(\frac{t^2}{2}\right)^2$

$dW_t \cdot dW_t = dt$

$$d(Y_t^2) = 2Y_t \cdot dY_t + \frac{1}{2}(2 \cdot (dY_t)^2)$$

$$d(Y_t^2) = 2Y_t(W_t^2\,dt + W_t\,dW_t) + (W_t^2\,dt + W_t\,dW_t)^2 =$$

$$d(Y_t^2) = 2Y_t W_t^2\,dt + 2Y_t W_t\,dW_t + (W_t^2\,dt)$$

полная

$$Y_t^2 = Y_0^2 + \int_0^t 2Y_u W_u^2 + W_u^2\,du + \int_0^t 2Y_u \cdot W_u\,dW_u$$

$Y_0^2 = 0$

$$E(Y_t^2) = \int_0^t \left(2E(Y_u W_u^2) + u\right)du + 0 =$$

$$= \int_0^t 2(u^3 + u^2) + u\,du \ \ldots$$