

Home Assignment 1

1. Let $\Omega = \mathbb{R}$. Explicitly find the sigma-algebras $\mathcal{F}_1 = \sigma(A)$, $\mathcal{F}_2 = \sigma(B)$, $\mathcal{F}_3 = \sigma(A, B)$ where $A = [-10; 5]$ and $B = (0; 10)$.
2. I throw a die once. Let X be the result of the toss. Count the number of events in sigma-algebras $\mathcal{F}_1 = \sigma(X)$, $\mathcal{F}_2 = \sigma(\{X > 3\})$, $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$.
3. Let $\Omega = \mathbb{R}$. The sigma-algebra \mathcal{F} is generated by all the sets of the form $(-\infty, t]$,

$$\mathcal{F} = \sigma(\{(-\infty; t] \mid t \in \mathbb{R}\})$$

Check whether $A_1 = (0; 10) \in \mathcal{F}$, $A_2 = \{5\} \in \mathcal{F}$, $A_3 = \mathbb{N} \in \mathcal{F}$.

4. Prove the following statements or provide a counter-example:
 - (a) If \mathcal{F}_1 and \mathcal{F}_2 are sigma-algebras then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is sigma-algebra.
 - (b) If X and Y are independent random variables then $\text{card } \sigma(X, Y) = \text{card } \sigma(X) + \text{card } \sigma(Y)$.

For finite sets card denotes just the number of elements.

5. I throw a die infinite number of times. Let the random variable X_n be equal to 1 if the n -th toss is head and 0 otherwise. Consider a pack of sigma-algebras: $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \dots)$.

Where possible provide an example of a non-trivial event (neither Ω nor \emptyset) such that

- (a) $A_1 \in \mathcal{F}_{2020}$;
- (b) $A_2 \in \mathcal{H}_{2020}$;
- (c) $A_3 \in \mathcal{F}_{2020}$ and $A_3 \in \mathcal{H}_{2020}$;
- (d) $A_4 \in \mathcal{F}_n$ for all n ;
- (e) $A_5 \in \mathcal{H}_n$ for all n .

Deadline: 25 September 2020, 21:00 MSK.

Home Assignment 2

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0 & 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)
 - (b) Classify the states of the Markov chain.
2. The Lonely Knight is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.
- (a) How many moves on average will it take to go back to A1?
 - (b) What proportion of her eternal life will she spend on every field?
3. Donald Trump throws a die until one appears or until he says «Stop». The payoff is equal to the last thrown number. Donald maximizes the expected payoff.

- (a) What is the best strategy and the corresponding expected payoff?

How do the answers change in the following modifications of the original game?

- (b) Donald is also required to stop at 3 and to continue on 4.
 - (c) Donald should pay 0.3 for every throw.
4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed Zmei Gorynich. Yes, there are infinitely many Zmei Gorynich.
- What is the probability that there is an Eternal Peaceful Journey?
5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTT or TTH appears. Ilya wins if HTT appears and Zmei wins if TTH appears.
- (a) What is the probability that Ilya wins?
 - (b) What is the expected number of throws?
 - (c) What is the expected number of throws given that Ilya won?

Deadline: 9 October 2020, 21:00 MSK.

Home Assignment 3

1. Consider the Vasicek interest rate model,

$$dR_t = a(b - R_t) dt + \sigma dW_t.$$

Here R_t is the interest rate and a , b and σ are positive constants.

- (a) Using the substitution $Y_t = e^{at} R_t$ find the solution of the stochastic differential equation;
 - (b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\text{ar}(R_t)$.
 - (c) Which value in this model would you call long-term equilibrium rate and why?
2. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$. Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble. Write the stochastic differential equation for dY_t . The equation may contain Y_t and constants, but not X_t .
3. Let W_t^a and W_t^b be two independent Wiener processes. Consider the process $Q_t = \alpha W_t^a + \beta W_t^b$, where $\alpha^2 + \beta^2 = 1$.
- (a) Is Q_t a Wiener process? Carefully check all the assumptions.
 - (b) Find the limit in L^2 for $n \rightarrow \infty$ of

$$A_n = \sum_{i=1}^n (W^a(it/n) - W^a((i-1)t/n)) (W^b(it/n) - W^b((i-1)t/n))$$

- (c) Find the limit in L^2 for $n \rightarrow \infty$ of

$$B_n = \sum_{i=1}^n (Q(it/n) - Q((i-1)t/n)) (W^b(it/n) - W^b((i-1)t/n))$$

- (d) Find $\mathbb{C}\text{orr}(Q_t, W_t^b)$.
 - (e) Without formal proof guess the value of $dQ_t dW_t^b$ in the Ito's lemma for correlated Wiener processes.
4. Consider the Cox-Ingersoll-Ross interest rate model

$$dR_t = a(b - R_t) dt + \sigma \sqrt{R_t} dW_t.$$

Here R_t is the interest rate and a , b and σ are positive constants.

Find $\mathbb{E}(R_t)$ and $\mathbb{V}\text{ar}(R_t)$.

5. The share price S_t satisfies the Black and Scholes model and $dX_t = t dS_t$. Find $\mathbb{E}(X_t)$ and $\mathbb{V}\text{ar}(X_t)$.