You have 100 minutes. You can use A4 cheat sheet and calculator. Be brave!

1. I throw a fair die until the sequence 626 appears. Let N be the number of throws.
   (a) What is the expected value $\mathbb{E}(N)$?
   (b) Write down the system of linear equations for the moment generating function of N. You don't need to solve it!

2. Consider the following stationary process

$$y_t = 1 + 0.5y_{t-2} + u_t + u_{t-1},$$

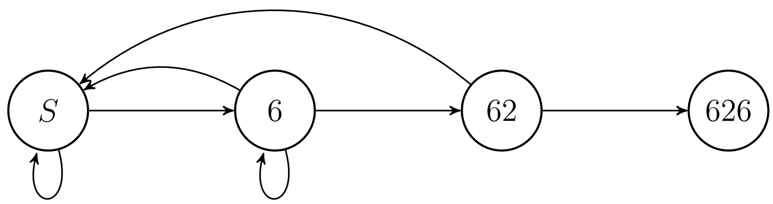
   where random variables u_t are independent $\mathcal{N}(0; 4)$.
   (a) Find the 95% predictive interval for y_{101} given that $y_{100} = 2$, $y_{99} = 3$, $y_{98} = 1$, $u_{99} = -1$.
   (b) Find the point forecast for y_{101} given that $y_{100} = 2$.

3. I have an unfair coin with probability of heads equal to $h \in (0; 1)$.
   (a) Let N be the number of tails before the first head. Find the MGF of N.
   (b) Let S be the number of tails before k heads (not necessary consecutive). Find the MGF of S.
   (c) What is the limit of $MGF_S(t)$ when $k \to \infty$ and $k \times h \to 0.5$? What is the name of the corresponding distribution?

4. Consider the stochastic process $X_t = f(t)\cos(2021W_t)$.
   (a) Find dX_t.
   (b) Find any $f(t) \neq 0$ such that X_t is a martingale.
   (c) Using $f(t)$ from the previous point find $\mathbb{E}(\cos(2021W_t))$.

1 Solution

1. Let's draw the chain



The system of equations for expected values:

$$\begin{cases} x_s = 1 + \frac{1}{6}x_6 + \frac{5}{6}x_s \\ x_6 = 1 + \frac{1}{6}x_6 + \frac{1}{6}x_{62} + \frac{4}{6}x_s \\ x_{62} = 1 + \frac{1}{6} \cdot 0 + \frac{5}{6}x_s \end{cases}$$

The system of equations for moment generating functions:

$$\begin{cases} m_s(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{5}{6}m_s(t)\right) \\ m_6(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{1}{6}m_{62}(t) + \frac{4}{6}m_s(t)\right) \\ m_{62}(t) = \exp(t)\left(\frac{1}{6} \cdot 1 + \frac{5}{6}m_s(t)\right) \end{cases}$$

2. (a) Let's denote by x all available information,

$$x = \begin{pmatrix} y_{100} \\ y_{99} \\ y_{98} \\ u_{99} \end{pmatrix}$$

Let's use $t = 100$:

$$y_{100} = 1 + 0.5y_{98} + u_{100} + u_{99}$$

Using all available information we obtain $u_{100} = 1.5$ and hence

$$y_{101} \mid x \sim \mathcal{N}(1 + 0.5y_{99} + u_{100}; 4)$$

(b) Here we work with true betas:

$$\mathbb{E}(y_{101} \mid y_{100}) = \mu_y + \frac{\mathbb{C}\mathrm{ov}(y_{100}, y_{101})}{\mathbb{V}\mathrm{ar}(y_{100})}(y_{100} - \mu_y)$$

3. (a) Moment generating function

$$m_N(t) = \sum_{j=0} \exp(tj)(1-h)^j h = h \sum_{j=0} (\exp(t)(1-h))^j = \frac{h}{1 - \exp(t)(1-h)}$$

(b) As $S = N_1 + N_2 + \ldots + N_k$:

$$m_S(t) = \left(\frac{h}{1 - \exp(t)(1-h)}\right)^k$$

(c) Due to my mistake the limit is easy, 0.
In my dream it was $k \to \infty$, $k \cdot (1-h) \to 0.5$ and that would be fun!

4. (a) Let's use Ito's lemma

$$dX_t = f'(t)\cos(2021W_t)dt - 2021f(t)\sin(2021W_t)dW_t + \frac{1}{2}2021^2 f(t)\cos(2021W_t)dt$$

(b) To make X_t a martingale we should kill dt term.

(c) As X_t is martingale $\mathbb{E}(X_t) = \mathbb{E}(X_0) = f(0)$. So $\mathbb{E}(\cos(2021W_t)) = f(0)/f(t)$.