Time Series

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Pseudo-out-of-sample fit

- Can check **pseudo-out-of-sample fit**:
- Take the sample of size T and split into two parts: periods $1, ..., T_1$ (for some T_1) and $T_1 + 1, ..., T$.
- Estimate candidate models:
  - Fixed scheme: on $1, ..., T_1$
  - Rolling scheme: on $i, ..., T_1 + i - 1$
  - Recursive scheme: on $1, ..., T_1 + i - 1$
- Use the estimated model to forecast for period $T_1 + i$, for $i = 1, ..., T - T_1$.
- Compute MSPE and compare models.

Diebold-Mariano test

- Compares two sequences of forecasts: $\{\hat{Y}_{1t}\}$ and $\{\hat{Y}_{2t}\}$
- Forecasts are the primitives, not models
- Look at the loss differential:

  $d_{12t} = L(e_{1t}) - L(e_{2t}) = (Y_t - \hat{Y}_{1t})^2 - (Y_t - \hat{Y}_{2t})^2$
- Assumption DM: $\{d_{12t}\}$ is covariance-stationary
- Two forecasts are equally good if $\mathrm{E}[d_{12t}] = 0$. That's H_0.
- Form the test statistic:

$$t = \frac{\frac{1}{T}\sum_{t=1}^{T} d_{12t}}{\sqrt{\hat{\sigma}_d / T}},$$

  where $\sigma_d = \sum_{j=-\infty}^{+\infty} \gamma_d(j)$
- $t \to^d \mathcal{N}(0,1)$
- If $t < -z_\alpha$, $\{\hat{Y}_{1t}\}$ is preferable; if $t > z_\alpha$, $\{\hat{Y}_{2t}\}$ is preferable.

West and Clark+McCracken

- Use DM test to investigate pseudo-out-of-sample fit for one-step ahead forecasts
- Estimate the model, using one of the schemes
- Be smart about estimating the variance of d_{12t}
- Be careful about whether the compared models are nested
- Be careful about the relative size of in-sample part and pseudo-out-of-sample parts

AR(p)

- $Y_t = \theta Y_{t-1} + \varepsilon_t$
- OLS:

$$\hat{\theta} = \frac{\sum_{t=2}^{T} Y_t Y_{t-1}}{\sum_{t=2}^{T} Y_{t-1}^2} = \theta + \frac{\sum_{t=2}^{T} \varepsilon_t Y_{t-1}}{\sum_{t=2}^{T} Y_{t-1}^2}$$

- $\sqrt{T}(\hat{\theta} - \theta) \to^d \mathcal{N}(0, V)$, where $V = \frac{\mathrm{Var}(Y_{t-1}\varepsilon_t)}{(\mathrm{E}[Y_t^2])^2}$ under no autocorrelation in $Y_{t-1}\varepsilon_t$

Serial correlation

- Sometimes, there is autocorrelation
- Then $V \neq \frac{\mathrm{Var}(Y_{t-1}\varepsilon_t)}{(\mathrm{E}[Y_t^2])^2}$
- Instead, it is equal to

$$V = \left(\mathrm{E}[Y_t^2]\right)^{-2} \lim_{T\to\infty} \frac{1}{T-1} \mathrm{Var}\left(\sum_{t=2}^{T} Y_{t-1}\varepsilon_t\right)$$

- We need HAC variance estimator (was on the board)
- $\hat{V}^{HAC} = \hat{V}\hat{f}$, where

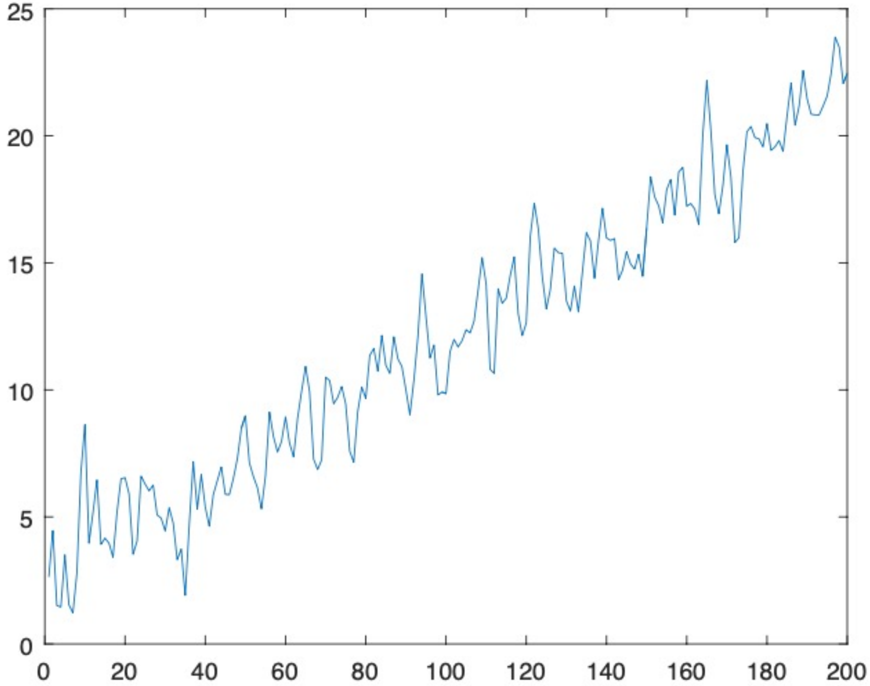
$$\hat{f} = 1 + 2\sum_{j=1}^{m} \frac{m-j}{m}\hat{\rho}(j),$$

and

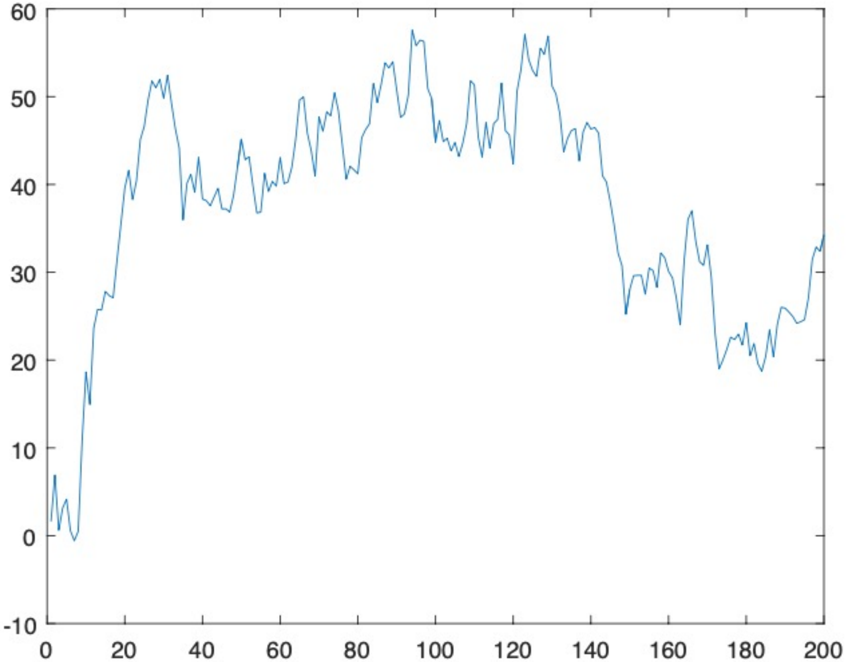
$$m = CT^{1/3}, \text{ where } C = \left(\frac{6\rho^2}{(1-\rho^2)^2}\right)^{1/3},$$

for the case when $\varepsilon_t Y_{t-1}$ is AR(1) with parameter ρ.

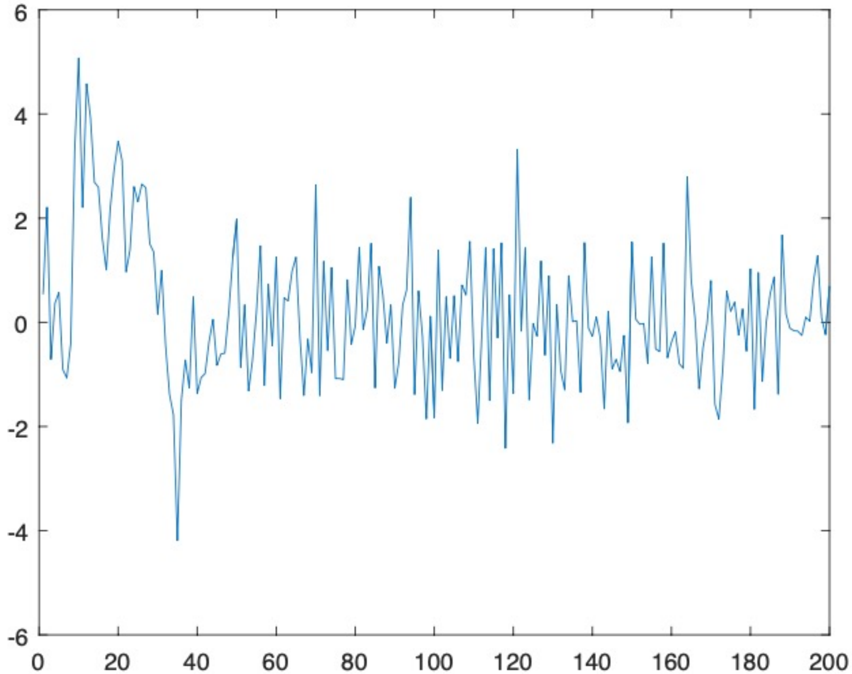
Series 1



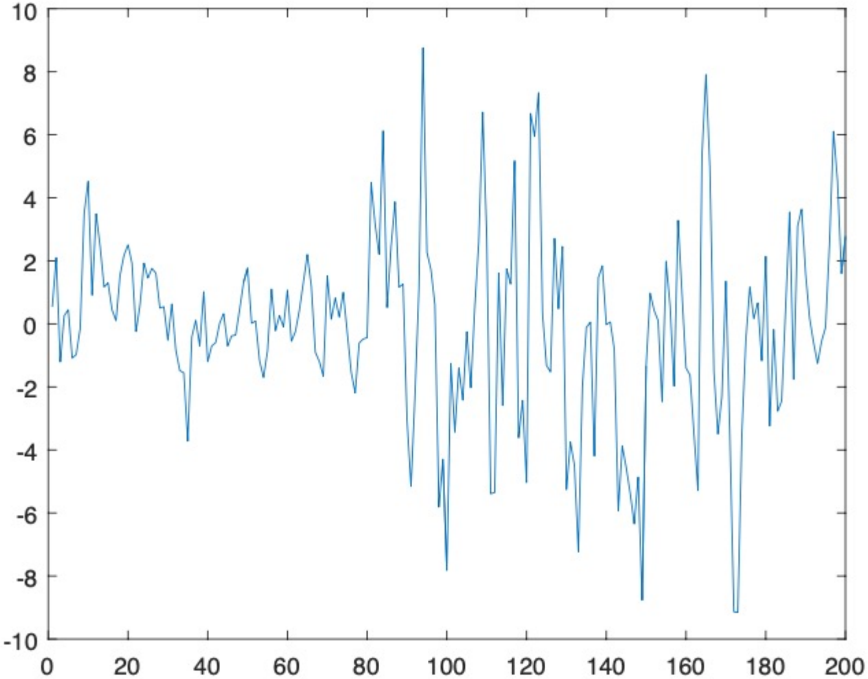
Series 2



Series 3



Series 4



Type of Non-Stationary TimeSeries

- Time trend
- Unit root
- Structural break in levels
- Structural break in variance

Trend-Stationary TimeSeries

$$Y_t = \mu + \delta t + \Psi(L)\,\varepsilon_t = \mu + \delta t + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$

where $\sum_{j=1}^{\infty} |\psi_j| < \infty$

- $Y_t - \delta t$ is stationary
- Forecasts:
  - $\hat{Y}_{t+h|t} = \mu + \delta(t+h) + \psi_h \varepsilon_t + \psi_{h+1}\,\varepsilon_{t-1} + \ldots$
  - Forecast error: $e_{t+h|t} = \varepsilon_{t+h} + \psi_1\,\varepsilon_{t+h-1} + \ldots + \psi_{h-1}\,\varepsilon_{t+1}$
  - Variance of the forecast error: $\mathrm{Var}(e_{t+h|t}) = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2 < \infty$
- Impulse response to a shock: $\frac{\partial Y_{t+h}}{\partial \varepsilon_t} = \psi_h \to 0$, as $h \to \infty$

Trend-Stationary TimeSeries

- Estimate the trend + arma
- If there is no trend, $\hat{\delta} \rightarrow^{p} 0$
- If there is a trend, but just estimate arma, you'll get something close to a unit root (model is misspecified)
- Trends might be logarithmic or quadratic

Difference stationary TS

$$Y_t = \mu + Y_{t-1} + \Psi(L)\,\varepsilon_t = \mu + Y_{t-1} + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j},$$

where $\sum_{j=1}^{\infty} |\psi_j| < \infty$

- $(1-L)\,Y_t = Y_t - Y_{t-1}$ is stationary
- Forecasts (for simplicity, let $\Psi(L) = I$):
  - $\hat{Y}_{t+h|t} = \mu h + Y_t$
  - Forecast error: $e_{t+h|t} = \sum_{j=1}^{h} \varepsilon_{t+j}$
  - Variance of the forecast error: $\mathrm{Var}(e_{t+h|t}) = \sigma^2 h \to \infty$, as $h \to \infty$
- Impulse response to a shock: $\frac{\partial Y_{t+h}}{\partial \varepsilon_t} = 1$

Difference Stationary TS

- Work with $Z_t = (1 - L)Y_t = Y_t - Y_{t-1}$, which is stationary
- Need to determine if there is a unit root
- Look at ACF (but might confuse with just large $\theta < 1$)
- Do statistical testing

Dickey Fuller Test

- Model:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

- True process: $Y_t = Y_{t-1} + \varepsilon_t$
- The null: $H_0 : \theta = 1$ vs $H_1 : |\theta| < 1$
- Estimate by OLS, form the test statistic:

$$t_n = \frac{\hat{\theta} - 1}{s.e.(\hat{\theta})}$$

- What's the distribution?
- Test with significance level α: Reject H_0 if $t_n < DF_n^\alpha$

Augmented Dickey Fuller Test

- Model:

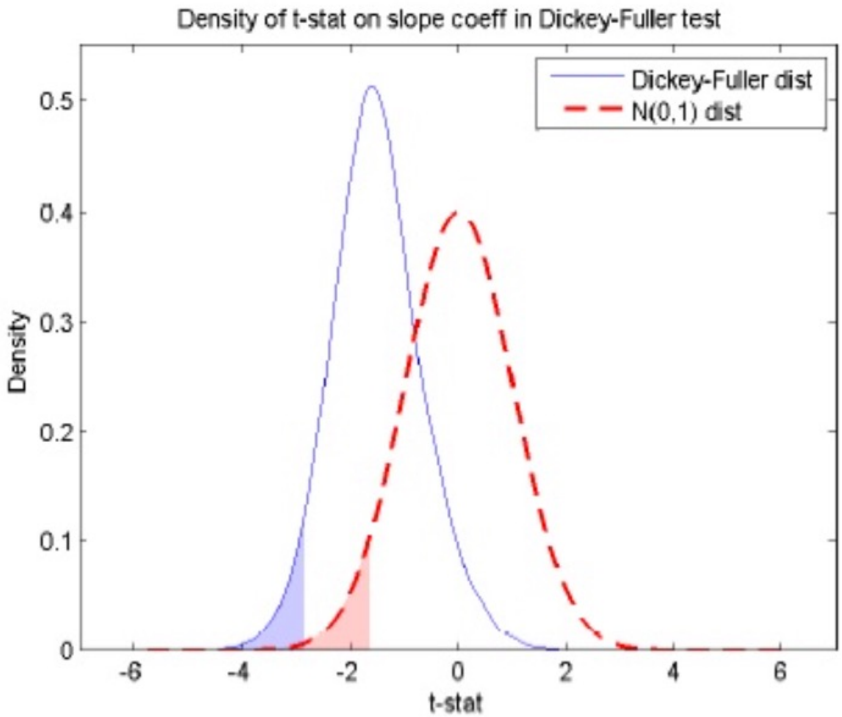
$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \varepsilon_t$$

- True process has a unit root: $\theta_1 + \theta_2 = 1$
- Write the equation:

$$\begin{aligned}
Y_t - Y_{t-1} &= c + \theta_1 Y_{t-1} - Y_{t-1} + \theta_2 Y_{t-1} + \theta_2 Y_{t-2} - \theta_2 Y_{t-1} + \varepsilon_t \\
\Delta Y_t &= c + (\theta_1 + \theta_2 - 1) Y_{t-1} - \theta_2 \Delta Y_{t-1} + \varepsilon_t \\
\Delta Y_t &= c + \theta^* Y_{t-1} + \theta_2^* \Delta Y_{t-1} + \varepsilon_t
\end{aligned}$$

- Estimate by OLS, form the test statistic $t_n = \frac{\hat{\theta}^*}{s.e.(\hat{\theta}^*)}$
- The same distribution as before
- Test with significance level α: Reject H_0 if $t_n < DF_n^\alpha$

DF Distribution



Augmented DF Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \ldots + \theta_p Y_{t-p} + \varepsilon_t$$

- The process has a unit root: $\theta_1 + \ldots + \theta_p = 1$
-

$$\Delta Y_t = c + \theta^* Y_{t-1} + \theta_2^* \Delta Y_{t-1} + \ldots + \theta_p^* \Delta Y_{t-p+1} + \varepsilon_t$$

- Estimate by OLS, form the test statistic:

$$t_n = \frac{\hat{\theta}^*}{s.e.(\hat{\theta}^*)}$$

- The same distribution as before
- Test with significance level α: Reject H_0 if $t_n < DF_n^\alpha$

Augmented DF Test

- Distribution is different in 4 different cases:

Case number	True Model	Estimated Model
1	$Y_t = Y_{t-1} + \varepsilon_t$	$Y_t = \theta Y_{t-1} + \varepsilon_t$
2	$Y_t = Y_{t-1} + \varepsilon_t$	$Y_t = c + \theta Y_{t-1} + \varepsilon_t$
3	$Y_t = c + Y_{t-1} + \varepsilon_t$	$Y_t = c + \theta Y_{t-1} + \varepsilon_t$
4	$Y_t = c + Y_{t-1} + \varepsilon_t$	$Y_t = c + \theta Y_{t-1} + \delta t + \varepsilon_t$

Philips Perron Test

- Model:

$$Y_t = c + \theta_1 Y_{t-1} + \varepsilon_t$$

- True process has a unit root: $Y_t = Y_{t-1} + \varepsilon_t$
- But allow $\{\varepsilon_t\}$ to be serially correlated
- Two alternative test statistics (ρ and τ)
- Same distribution as DF
- Applicable to cases 1,2,4

Kwiatowski, Philips, Schmidt, Shin Test

- Model:

$$Y_t = trend + \mu_t + \varepsilon_t, \text{ where } \mu_t = \mu_{t-1} + u_t,$$

  ε_t is $I(0)$, possibly heteroskedastic

- $H_0 : \sigma_u^2 = 0$ (i.e. $\mu_t = const$)
- The test is against one-sided alternative
- The distribution depends on which trend is assumed and is non-standard
- Reject the null at 5% level if *KPSS* is larger than 95% quantile of its distribution

Determining d in ARIMA (p,d,q)

- Test whether there is a unit root in $\{Y_t\}$
- If reject, set $d = 0$
- If fail to reject, consider $Z_t = \Delta Y_t$ and test whether there is a unit root in $\{Z_t\}$
- If reject, set $d = 1$
- If fail to reject, consider $W_t = \Delta Z_t = \Delta^2 Z_t$ and test whether there is a unit root in $\{W_t\}$
- ...