Short rules: 120 minutes, online without proctoring. You may use any source you want but don't cheat.

Pledge:

Start exam by writing the following honor pledge and signing it.

I pledge on my honor that I will not give nor receive any unauthorized assitance on this exam.

Problems:

1. (10 points) Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

   (a) (3 points) Split the chain in classes and classify them into closed or not closed.

   (b) (2 points) Classify the states into recurrent or transient.

   (c) (5 points) A hedgehog starts in the state one and moves randomly between states according to the transition matrix.

   What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

2. (10 points) Gleb Zheglov catches one criminal every day. With probability 0.2 the catched criminal is replaced by w new criminals. Initially there are n criminals in the town.

   What is the expected time to the ultimate crime eradication in the town?

   (a) (4 points) Solve the problem for $w = 1$ and $k = 1$.

   (b) (6 points) Solve the problem for arbitrary w and k.

3. (10 points) The random variables X_i are independend and uniformly distributed on $[0; 1]$. Find the probability limit

$$\operatorname{plim}_{n\to\infty} \max\left\{ \frac{\sum_{i=1}^n X_i}{n}, \frac{2\sum_{i=1}^n X_i^2}{n} \right\}.$$

4. (10 points) Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes.

   Let Y_t be the number of taxis that will arrive between 0 and t minutes.

   (a) (2 points) Sketch the expected value of Y_t as a function of t.

   (b) (8 points) Sketch the probability $\mathbb{P}(Y_t = Y_{60})$ as a function of t.

   Note: special points like intercepts or extrema should be explicitely marked.

5. (10 points) Prince Myshkin throws a fair coin until two consecutive heads appear. Let N be the number of throws.

   Find the moment generating function of N.

   Hint: you may use the first step approach.

6. (20 points) Vincenzo Peruggia makes attempts to steal the Mona Lisa painting until the first success. Each attempt is successful with probability 0.1.

   Let X be the number of attempts and $Z = \min\{X, 5\}$.

   (a) (5 points) How many events are in sigma-algebras $\sigma(Z)$ and $\sigma(X)$?

   (b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(Z)$ but $A \notin \sigma(X)$; $B \in \sigma(X)$ but $B \notin \sigma(Z)$.

   (c) (10 points) Find $\mathbb{E}(Z \mid X)$ and $\mathbb{E}(X \mid Z)$.