Q. A.

инверсия

$$W_t \to X_t = \begin{cases} 0 & t = 0 \\ t\,W(1/t) & \text{при } t > 0 \end{cases}$$

Q. $E(W_3 \mid W_9)$?

A. используя инверсию

$$X_t = t \cdot W_{1/t}$$

$$\frac{1}{t} \cdot X_t = W_{1/t}$$

$$1/t = a$$

$$W_a = a \cdot X_{1/a}$$

$\boxed{P^T \cdot \varphi = \varphi}$ φ – с. вектор м-цы P^T $\varphi = \begin{pmatrix} 0 \\ 0 \\ \alpha \\ 1-\alpha \end{pmatrix}$

3. Consider the Hedgehog problem from the exam. The Hedgehog starts at the state one and moves randomly between states with transition matrix

$\boxed{\varphi^T \cdot P = \varphi^T}$

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

$$\lim_{t\to\infty} p(t) = \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}$$

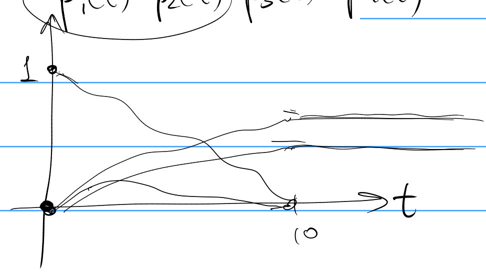
Let $p_1(t)$, $p_2(t)$, $p_3(t)$ and $p_4(t)$ be the probabilities of observing the Hedgehog in each of the four states after exactly t moves.

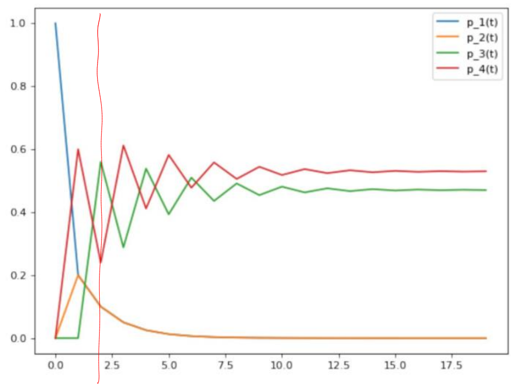
[9/10] (a) Draw these probabilities as the functions of t using any open source software (Python, R, Julia, ...). Provide your code.

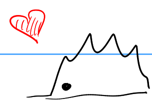
[1/10] (b) Is the number of steps equal to 10^{2021} sufficient for convergence?

Сущ кр

$p_1(t)$ $p_2(t)$ $p_3(t)$ $p_4(t)$







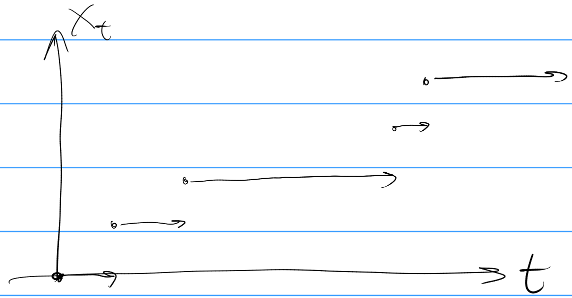
4. Let X_t be the Poisson process with rate $\lambda = 42$.

(a) Find a constant α such that $A_t = X_t - \alpha t$ is a martingale.

(b) Find a constant β such that $B_t = \exp(X_t - \beta t)$ is a martingale.

(X_t) – пуассон процесс

X_t



X_t – сколько „происшествий" было за период от 0 до t

$$E(X_{t+\Delta} \mid \mathcal{F}_t) = E\Big(\underbrace{X_t}_{\text{изв-но в момент } t} + \underbrace{(X_{t+\Delta} - X_t)}_{\text{не зависит от имеющейся инфо}} \Big| \mathcal{F}_t\Big) =$$

$$= X_t + E(X_{t+\Delta} - X_t) = X_t + \lambda\Delta \neq X_t$$

$X_{t+\Delta} - X_t \sim \mathrm{Poiss}(\lambda \cdot \Delta)$

$E(X_{t+\Delta} - X_t) = \lambda \cdot \Delta$

$\mathrm{Var}(X_{t+\Delta} - X_t) = \lambda \cdot \Delta$

у Винер. процесс

$W_t - W_s \sim N(0; t-s)$

$E(W_t - W_s) = 0$

$\mathrm{Var}(W_t - W_s) = t - s$

X_t – не мартг.

$\lambda = 42$

а) $A_t = X_t - \alpha t$

$$E(A_{t+\Delta} \mid \mathcal{F}_t) = A_t$$

$$E(X_{t+\Delta} - \alpha(t+\Delta) \mid \mathcal{F}_t) = X_t - \alpha t$$

$$X_t + \lambda \cdot \Delta - \alpha(t+\Delta) = X_t - \alpha t$$

$$42\Delta - \alpha\Delta = 0 \quad \forall \Delta$$

$$\alpha \overset{?}{=} 42$$

$$A_t = X_t - 42t$$ – мартингал!

б) $B_t = \exp(X_t - \beta t)$

мартг.

$$E(\exp(X_{t+\Delta} - \beta(t+\Delta)) \mid \mathcal{F}_t) = \exp(X_t - \beta t)$$

$$E(M_{t+\Delta} \mid \mathcal{F}_t) = M_t$$

$$X_{t+\Delta} = \underbrace{X_t}_{\text{изв-но}} + \underbrace{(X_{t+\Delta} - X_t)}_{\text{не зависит от инфо}}$$

$$E\left[\exp(X_t)\cdot\exp(X_{t+\Delta}-X_t)\cdot\exp(-\beta(t+\Delta)) \,\middle|\, \mathcal{F}_t\right] = \exp(X_t-\beta t)$$

независимы

$$\exp(X_t)\cdot\exp(-\beta t-\beta\Delta)\cdot E\left(\exp(X_{t+\Delta}-X_t) \,\middle|\, \mathcal{F}_t\right) = \exp(X_t)\exp(-\beta t)$$

$R=-1$ $R=+1$

$1/2$ $1/2$

$E(\exp R)$ $\quad \exp(E(R)) = \exp(0) = 1$

$$\frac{1}{2}\cdot\exp(-1)+\frac{1}{2}\cdot\exp(1) = \frac{1}{2}\cdot\frac{1}{e}+\frac{1}{2}\cdot e = \frac{1+e^2}{2e} \approx 1.54$$

$$\exp(-\beta\Delta)\cdot E(\exp(X_{t+\Delta}-X_t)) = 1$$

$\text{Poiss}(\lambda\Delta)$

$\text{Poiss}(42\Delta)$

$S \sim \text{Poiss}(42\Delta)$

цель

$$\boxed{E(\exp(S))} \overset{?}{=}$$

способ 1

$$\exp(0)\cdot P(S=0)+\exp(1)\cdot P(S=1)+\exp(2)\cdot P(S=2)+\ldots$$

$$\sum_{i=0}^{\infty}\exp(i)\cdot P(S=i) = \sum_{i=0}^{\infty}\exp(i)\cdot\exp(-42\Delta)\cdot\frac{(42\Delta)^i}{i!} = (*)$$

$\exp(i) = e^i$

способ 2

$$MGF_S(u) = E(\exp(uS)); \qquad \text{Poiss}(\ldots)$$

Вики

$$E(\exp(S)) = MGF(1) = \exp[42\Delta(e^u-1)]\Big|_{u=1} = \exp(42\Delta(e-1))$$

$$(*) = \sum_{i=0}^{\infty}\exp(-42\Delta)\cdot\frac{(42\Delta e)^i}{i!} =$$

$$= \sum_{i=0}^{\infty}\exp(-42\Delta)\cdot\frac{\exp(-42\Delta e)}{\exp(-42\Delta e)}\cdot\frac{(42\Delta e)^i}{i!} =$$

$$= \exp(42\Delta e-42\Delta)\cdot \boxed{\sum \exp(-42\Delta e)\frac{(42\Delta e)^i}{i!}} \quad \ln$$

$$\sum e^{-\lambda}\cdot\frac{\lambda^i}{i!} = 1$$

$$\exp(-\beta\Delta)\cdot\exp(42\Delta(e-1)) = 1 \qquad \forall\Delta$$

$$-\beta\Delta+42\Delta(e-1) = 0$$

$$\beta = 42\cdot(e-1)$$

☺

Notation	$\mathrm{Pois}(\lambda)$
Parameters	$\lambda \in (0, \infty)$ (rate)
Support	$k \in \mathbb{N}_0$ (Natural numbers starting from 0)
PMF	$\frac{\lambda^k e^{-\lambda}}{k!}$
CDF	$\frac{\Gamma(\lfloor k+1 \rfloor, \lambda)}{\lfloor k \rfloor !}$, or $e^{-\lambda} \sum_{i=0}^{\lfloor k \rfloor} \frac{\lambda^i}{i!}$, or $Q(\lfloor k+1 \rfloor, \lambda)$ (for $k \geq 0$, where $\Gamma(x, y)$ is the upper incomplete gamma function, $\lfloor k \rfloor$ is the floor function, and Q is the regularized gamma function)
Mean	λ
Median	$\approx \lfloor \lambda + 1/3 - 0.02/\lambda \rfloor$
Mode	$\lceil \lambda \rceil - 1, \lfloor \lambda \rfloor$
Variance	λ
Skewness	$\lambda^{-1/2}$
Ex. kurtosis	λ^{-1}
Entropy	$\lambda[1 - \log(\lambda)] + e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k \log(k!)}{k!}$ (for large λ) $\frac{1}{2} \log(2\pi e \lambda) - \frac{1}{12\lambda} - \frac{1}{24\lambda^2} - \frac{19}{360\lambda^3} + O\left(\frac{1}{\lambda^4}\right)$
MGF	$\exp[\lambda(e^t - 1)]$
CF	$\exp[\lambda(e^{it} - 1)]$
PGF	$\exp[\lambda(z - 1)]$
Fisher information	$\frac{1}{\lambda}$

$P(S=k) =$

$S = X_{t+\Delta} - X_t \sim \mathrm{Poiss}(42\Delta)$

$M(t) = E(\exp(tS)) =$