# Welcome to the Advanced Statistics!

Peter Lukianchenko

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# Course structure

## Course Plan

- Stochastic Processes
- Time Series
- Advanced Statistics UoL

UNIVERSITY OF LONDON

**Advanced statistics: statistical inference**

J. Penzer

ST2134

**2018**

Undergraduate study in Economics, Management, Finance and the Social Sciences

This subject guide is for a 200 course offered as part of the University of London undergraduate study in Economics, Management, Finance and the Social Sciences. This is equivalent to Level 5 within the Framework for Higher Education Qualifications in England, Wales and Northern Ireland (FHEQ).

For more information about the University of London, see: london.ac.uk

THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

UNIVERSITY OF LONDON

**Advanced statistics: distribution theory**

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# Course structure

| Module | Time period | Control | Weight |
|---|---|---|---|
| **I** | | Fall Mock | 45% |
| **II** | | Winter Exam | 45% |
| **III** | | Spring Mock | 10% |
| **IV** | | UoL Exam | 65% |
| | | Final Exam | 10% |

## Formula for final grade

$$\text{Fall} = \mathbf{0.45} * \text{FallMock} + \mathbf{0.45} * \text{WinterExam} + \mathbf{0.1} * \text{FallHomework}$$

$$\text{Spring} = \mathbf{0.1} * \text{SprMock} + \mathbf{0.1} * \text{FinalExam} + \mathbf{0.65} * \text{UoLExam} + \mathbf{0.1} * \text{SprHomework} + \mathbf{0.05} * \text{Quizzes}$$

$$\textbf{Total} = \mathbf{0.25} * \text{Fall} + \mathbf{0.75} * \text{Spring}$$

if ex. online → { 1. written ⇒ ; 2. oral ⇒ { 1. 2. ⋮ 5. }  "-1" (out of 10)

# Course structure

## Lecturer

Petr Lukianchenko

e-mail:
plukyanchenko@hse.ru or
lukianchenko.pierre@gmail.com

## Class teacher

Boris Demeshev, Office S517

e-mail:
bdemeshev@hse.ru or boris.demeshev@gmail.com



TG

Smart Lms

## Stochastic Processes: Basic Definitions

| | |
|---|---|
| Stochastic process | The value of a variable changes in an uncertain way |
| Discrete vs. continuous time | When can a variable change? Shrève<br>What values can a variable take? |
| Markov property | Only the current value of a variable is relevant for future predictions<br>No information from past prices or path |

"Blacu Swcn"

# Stochastic Processes: Basic Definitions

## Definition

*Stochastic process* $X = \{X(t), t \in \boldsymbol{T}\}$ is a collection of random variables (rvs); one rv for each $X(t)$ for each $t \in \boldsymbol{T}$.

Index set $\boldsymbol{T}$ – set of possible values of $t$; $t$ only means time

$\boldsymbol{T}$: countable – discrete-time process

$\boldsymbol{T}$: real number – continuous-time process

*State space* – set of possible values of $X(t)$

# Stochastic Processes: Basic Definitions (examples)

Consider a teletrafic (or any) system. It typically **evolves** in time **randomly**

- *Example 1*: the number of occupied channels in a telephone link at time $t$ or at the arrival time of the $n^{th}$ customer;
- *Example 2*: the number of packets in the buffer of a statistical multiplexer at time $t$ or at the arrival time of the $n^{th}$ customer;

- This kind of evolution is described by a stochastic processes;
- At any individual time $t$ (or $n$) the system can be described by a random variable;
- Thus, the stochastic processes is a collection of random variables.

# Stochastic Processes: Basic Definitions

## Definition

A (real-valued) *stochastic process* $X = (X_t \mid t \in \boldsymbol{I})$ is a collection of random variables $X_t$

- taking values in some (real-valued) set $\boldsymbol{S}$, $X_t(w) \in \boldsymbol{S}$, and
- indexed by a real-valued (time) parameter $t \in \boldsymbol{I}$.

Stochastic processes are also called *random processes* (or just *processes*).

- The index set $\boldsymbol{I} \subset \Re$ is called the *parameter space* of the process
- The value set $\boldsymbol{S} \subset \Re$ is called the *state space* of the process

Note:

sometimes notation $X_t$ is used to refer to the whole stochastic process
(instead of a single random variable)

# Categories of stochastic processes

Reminder:

- Parameter space: set $\boldsymbol{I}$ of indices $t \in \boldsymbol{I}$
- State space: set $\boldsymbol{S}$ of values $X_t(\omega) \in \boldsymbol{S}$

Categories:

- Based on the parameter space:
  - **Discrete-time processes:** parameter space discrete
  - **Continuous-time processes:** parameter space continuous
- Based on the state space:
  - **Discrete-state processes:** state space discrete
  - **Continuous-state processes:** state space continuous

In this course we will concentrate on the discrete-state processes (with either a discrete or a continuous parameter space). Typical processes describe the number of customers in a queueing system (the state space being thus $\boldsymbol{S} = \{0,1,2,\ldots\}$)

## Examples

- **Discrete-time, discrete-state processes**

*Example 1:* the number of occupied channels in a telephone link at the arrival time of the $n^{th}$ customer, $n = 1, 2 \dots$

*Example 2:* the number of packets in the buffer of a statistical multiplexer at the arrival time of the $n^{th}$ customer, $n = 1, 2 \dots$

- **Continuous-time, discrete-state processes**

*Example 3:* the number of occupied channels in a telephone link at time $t > 0$

*Example 4:* the number of packets in the buffer of a statistical multiplexer at time $t > 0$

# Markov Chain

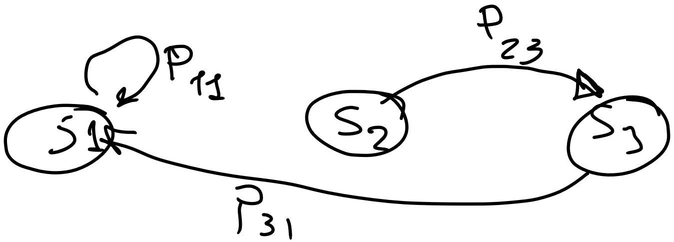
- A stochastic process $\{X_t\}$ is a Markov chain if it has Markovian property.

- Markovian property:
  - $P\{ X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i \}$
    $= P\{ X_{t+1} = j \mid X_t = i \}$

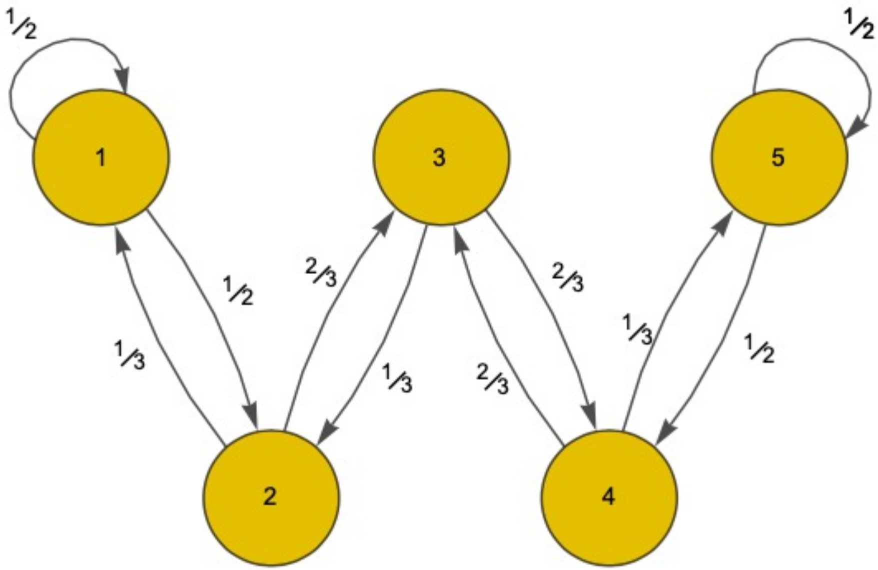
$= P_{ij}$

- $P\{ X_{t+1} = j \mid \underbrace{X_t = i}_{\text{current state is } i} \}$ is called the transition probability. $i \rightarrow j$

next state $j$



# Markov Chain



- Stationary transition probability:
  - If ,for each i and j, $P\{ X_{t+1} = j \mid X_t = i \} = P\{ X_1 = j \mid X_0 = i \}$, for all t, then the transition probability are said to be stationary.

# Markov Chain

Transition matrix:

j target

$$
\mathbf{P} = \begin{array}{ccccc}
\text{state} & 0 & 1 & 2 & 3 \\
0 & p_{00} & p_{01} & p_{02} & p_{03} \\
1 & p_{10} & p_{11} & p_{12} & p_{13} \\
2 & p_{20} & p_{21} & p_{22} & p_{23} \\
3 & p_{30} & p_{31} & p_{32} & p_{33}
\end{array}
$$

i init.

# Markov Chain

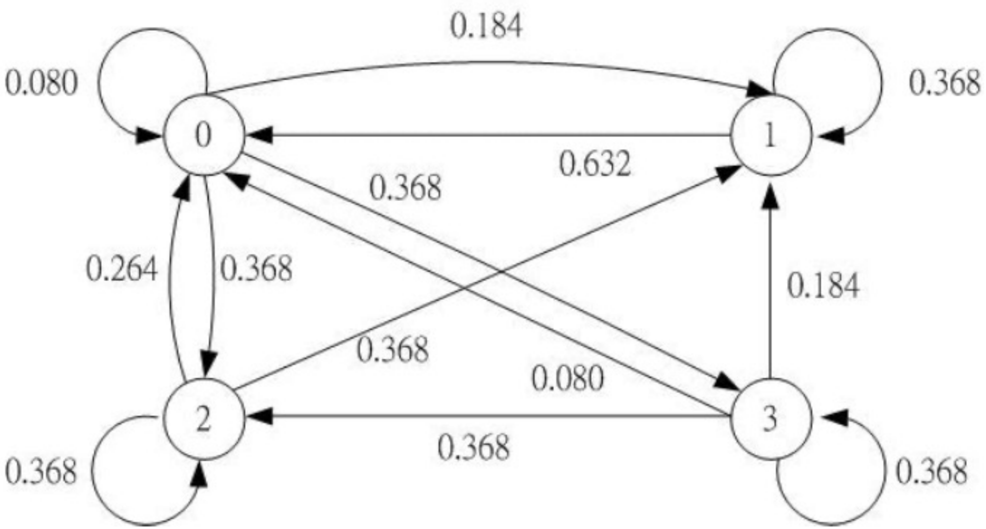
- $X_{t+1} = \max\{ 3 - D_{t+1}, 0 \}$ if $X_t = 0$
  $\max\{ X_t - D_{t+1}, 0 \}$ if $X_t \geq 1$

- $p_{03} = P\{ D_{t+1} = 0 \} = 0.368$
- $p_{02} = P\{ D_{t+1} = 1 \} = 0.368$
- $p_{01} = P\{ D_{t+1} = 2 \} = 0.184$
- $p_{00} = P\{ D_{t+1} \geq 3 \} = 0.080$

$$
\mathbf{P} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & 2 & 3 \\
\hline
0 & 0.080 & 0.184 & 0.368 & 0.368 \\
1 & 0.632 & 0.368 & 0.000 & 0.000 \\
2 & 0.264 & 0.368 & 0.368 & 0.000 \\
3 & 0.080 & 0.184 & 0.368 & 0.368
\end{array}
$$

# Markov Chain

- The state transition diagram:



# Markov Chain

- n-step transition probability :
  - $p_{ij}^{(n)} = P\{ X_{t+n} = j \mid X_t = i \}$

- n-step transition matrix :

$$
\mathbf{P}^{(n)} = \begin{array}{c|cccc}
\text{state} & 0 & 1 & \cdots & M \\
\hline
0 & P_{00}^{(n)} & P_{01}^{(n)} & \cdots & P_{0M}^{(n)} \\
1 & P_{10}^{(n)} & P_{11}^{(n)} & \cdots & P_{1M}^{(n)} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
M & P_{M0}^{(n)} & P_{M1}^{(n)} & \cdots & P_{MM}^{(n)}
\end{array}
$$

# Markov Chain

- Chapman-Kolmogorove Equation :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(m)} p_{kj}^{(n-m)}$$

for all i = 0, 1, …, M,
j = 0, 1, …, M,
and any m = 1, 2, …, n-1,
n = m+1, m+2, …

- The special cases of m = 1 leads to :

$$p_{ij}^{(n)} = \sum_{k=0}^{M} p_{ik}^{(1)} p_{kj}^{(n-1)}$$

for all i and j

- Thus the n-step transition probability can be obtained from one-step transition probability recursively.

# Markov Chain

- Conclusion :
  - $\mathbf{P}^{(n)} = \mathbf{P}\mathbf{P}^{(n-1)} = \mathbf{P}\mathbf{P}\mathbf{P}^{(n-2)} = \ldots = \mathbf{P}^{n}$

- n-step transition matrix for the inventory example :

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}$$

$$\mathbf{P}^{(4)} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.289 & 0.286 & 0.261 & 0.164 \\ 1 & 0.282 & 0.285 & 0.268 & 0.166 \\ 2 & 0.284 & 0.283 & 0.263 & 0.171 \\ 3 & 0.289 & 0.286 & 0.261 & 0.164 \end{array}$$

- Long-Run Properties of Markov Chain
  - Steady-State Probability

$$\mathbf{P} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.080 & 0.184 & 0.368 & 0.368 \\ 1 & 0.632 & 0.368 & 0.000 & 0.000 \\ 2 & 0.264 & 0.368 & 0.368 & 0.000 \\ 3 & 0.080 & 0.184 & 0.368 & 0.368 \end{array}$$

$$\mathbf{P}^{(8)} = \begin{array}{c|cccc} \text{state} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.286 & 0.285 & 0.264 & 0.166 \\ 1 & 0.286 & 0.285 & 0.264 & 0.166 \\ 2 & 0.286 & 0.285 & 0.264 & 0.166 \\ 3 & 0.286 & 0.285 & 0.264 & 0.166 \end{array}$$

## Markov Chain

- The steady-state probability implies that there is a limiting probability that the system will be in each state j after a large number of transitions, and that this probability is independent of the initial state.

- Not all Markov chains have this property.

| state | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ |
| 1 | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ |
| 2 | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ |
| 3 | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ |

## Markov Chain

- Steady-State Equations :

$A \cdot A = A$

$$\boxed{\pi_j = \sum_{i=0}^{M} \pi_i p_{ij}} \quad \text{for i = 0, 1, …, M}$$

$\vec{\Pi} = \vec{\Pi} \cdot P \Rightarrow \vec{\Pi} P = \vec{\Pi} \Rightarrow P^T \cdot \vec{\Pi}^T = \vec{\Pi}^T$

$$\sum_{j=0}^{M} \pi_j = 1$$

- , which consists of M+2 equations in M+1 unknowns.

$A \cdot \vec{V} = \vec{V}$ eigenvectors
eigenvalue = 1

# Review

## Bayes theorem

Suppose $B_1, \dots B_n$ be collectively exhaustive events such that $\mathrm{P}(\mathrm{B_i}) \neq 0$ for any $i$, then for any event A such that $\mathrm{P(A)} \neq 0$, the following holds

$$\mathrm{P(B_k|A)} = \frac{\mathrm{P(A|B_k) * P(B_k)}}{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{P(A|B_i) * P(B_i)}}$$

for any possible $k$

# Review

## Definition

Events $A_1, \dots A_n$ are **mutual independent** if for any collection $A_{k_1}, A_{k_2} \dots A_{k_m}$ it holds that

$$P(A_{k_1} \cap \dots \cap A_{k_m}) = P(A_{k_1}) * \dots * P(A_{k_m})$$

where $k_1, \dots, k_m$ are distinct indices.

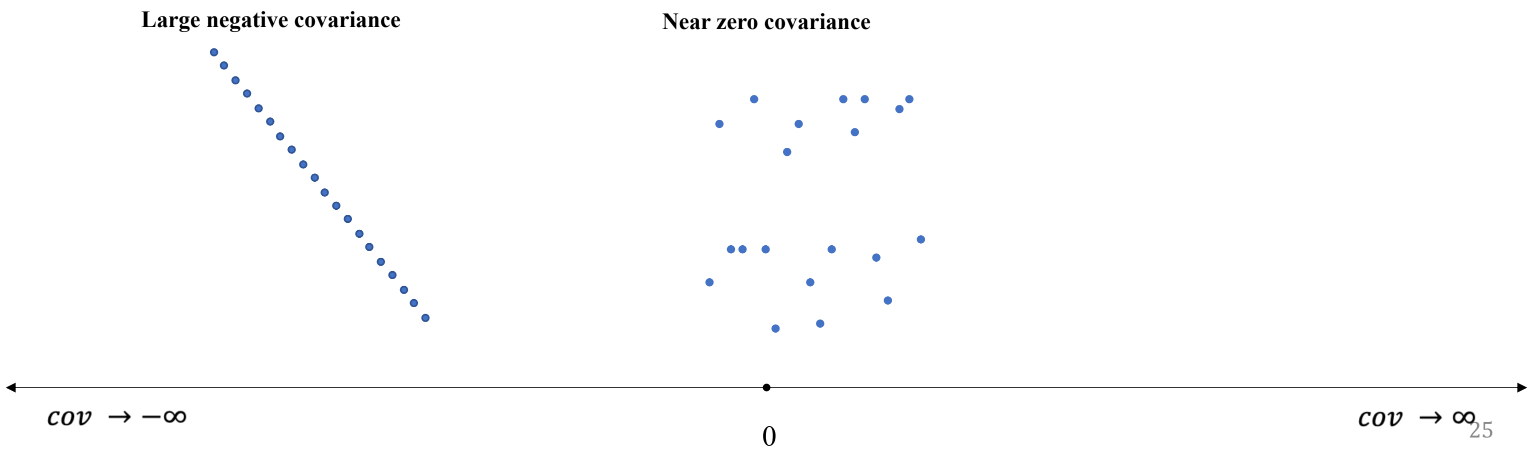
## Example:

A multiple-choice test has 10 questions, each with 4 answers where only one is correct. Suppose a student guesses the answers, what is the probability to answer all questions correctly? What is the probability to answer at least one question correctly?

# Review

## Dimension

The **covariance** of two random variables X and Y can take positive, negative values, or values close to zero.



Thank you for your attention!

See next week!