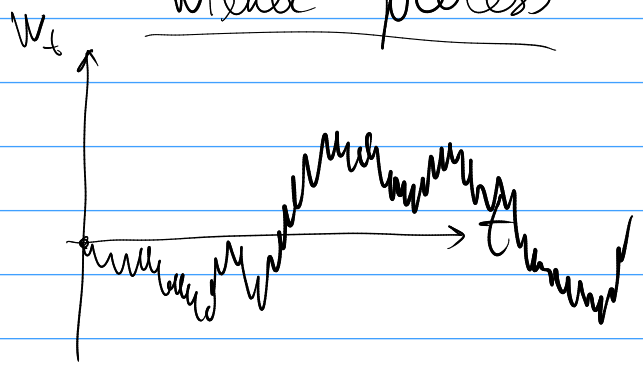
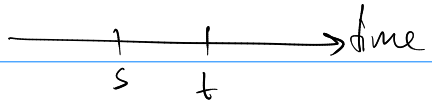
Привет !!

(W_t) – Винеровский процесс !!
Броуновское движение
Wiener process



1. $W_0 = 0$
2. если $t > s$

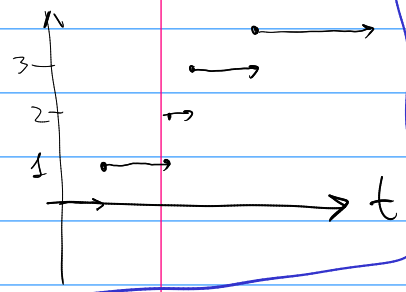


$\Delta = W_t - W_s \sim N(0; t-s)$

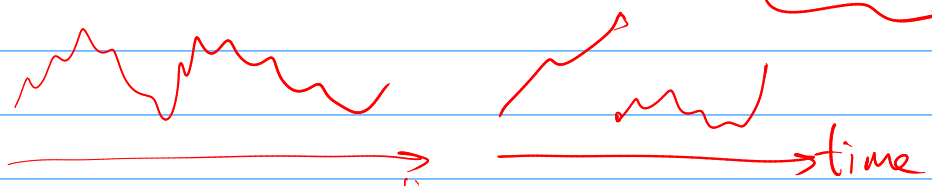
3. $P(\text{траектория } (W_t) \text{ непр}) = 1$

$P(N_t \text{ траект. разрывна}) = 1$

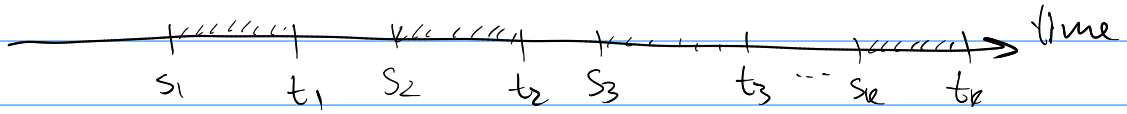
N_t (пуасс. процесс)



$\xi \sim U[0;3]$
$P(\xi = \sqrt{3}) = 0$
$P(\xi \neq \sqrt{3}) = 1$



4. "прошлое не несёт инфо о будущих приращениях"



$\Delta_1 = W(t_1) - W(s_1)$
$\Delta_2 = W(t_2) - W(s_2)$
$\vdots$
$\Delta_k = \dots$

если: $s_1 \le t_1 \le s_2 \le t_2 \le s_3 \le t_3 \dots$

то: $\Delta_1, \Delta_2, \Delta_3 \dots \Delta_k$ независимы

Упр.

(W_t) – Wiener process

a) $E(W_t)$, $Var(W_t)$ ☺

б) $Cov(W_t, W_s)$, $E(W_t \cdot W_s)$

c) $E(W_{t+\Delta} \mid W_t)$, $Var(W_{t+\Delta} \mid W_t)$ если $\Delta \geq 0$

$$W_t = W_t - W_0 \sim N(0; t-0) \qquad E(W_t)=0, \quad Var(W_t) = t$$

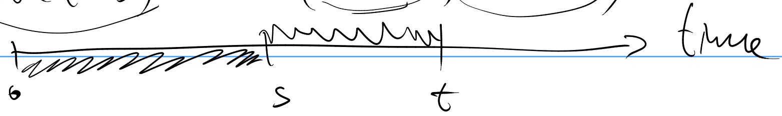
Трюк

с.в. = предск. часть + непр. часть

$s<t$

$$Cov(W_t, W_s) = Cov(W_s + (W_t - W_s), W_s) =$$

$$= Var(W_s) + Cov(W_t - W_s, W_s - W_0) = s + 0$$



$$\underset{\text{новое}}{W_t} = \underset{\text{старое}}{W_s} + \underset{\text{приращение}}{(W_t - W_s)}$$

если $s \geq t$ то: $Cov(W_t, W_s) = t$

$$Cov(W_s, W_t) = \min(s,t)$$

$Cov(W_t, W_s) = E(W_t \cdot W_s) - E(W_t)\cdot E(W_s)$

$$E(W_s \cdot W_t) = \min(s,t)$$

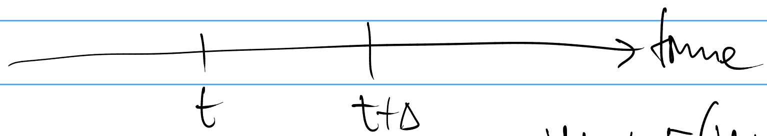
$Cov(W_s - W_0, W_t - W_0)$

$[\Delta \geq 0]$

c) $E(W_{t+\Delta} \mid W_t) =$

$$= E(W_t + (W_{t+\Delta} - W_t) \mid W_t) = E(W_t \mid W_t) + E(W_{t+\Delta} - W_t \mid W_t) =$$

известное; незав



$$= W_t + E(W_{t+\Delta} - W_t) = W_t$$

$E(Z \mid Z) = Z$

$E(Z \mid L) = E(Z)$ незав

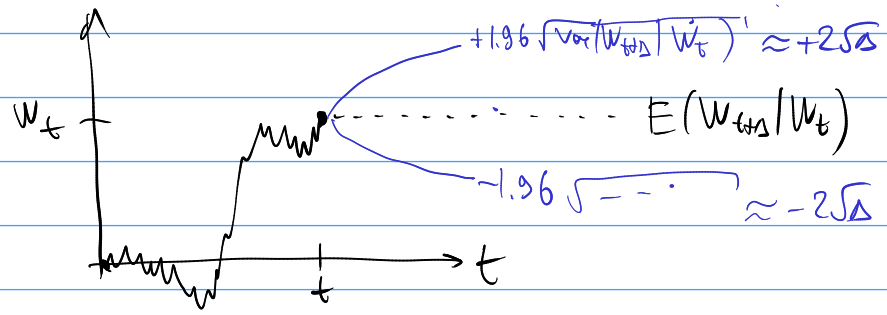
$Var(Z \mid Z) = 0$

$Var(Z \mid L) = Var(Z)$ незав

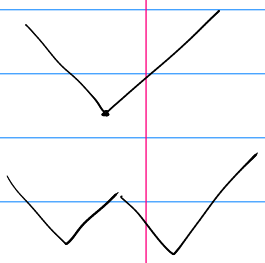
$$\mathrm{Var}(W_{t+\Delta} \mid W_t) = \mathrm{Var}\big(W_t + (W_{t+\Delta} - W_t) \mid W_t\big) = \mathrm{Var}\big(W_{t+\Delta} - W_t \mid W_t\big) =$$

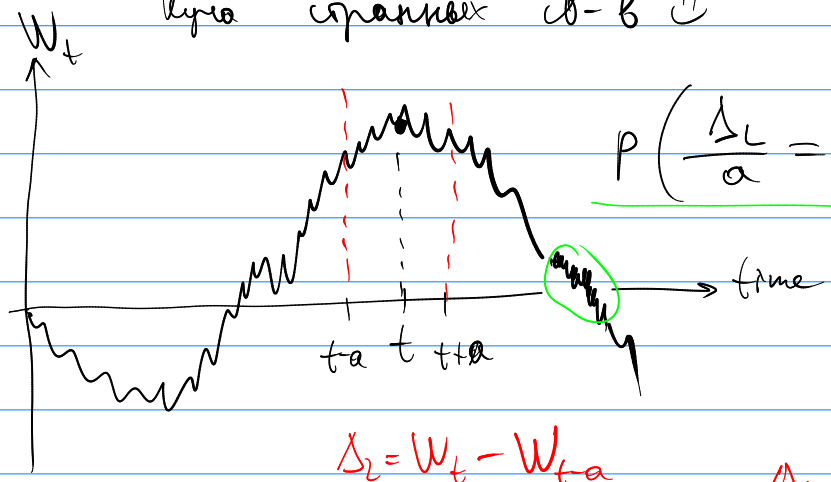
W_t — сл. величина; $W_{t+\Delta} - W_t$ — не зав

$$= \mathrm{Var}(W_{t+\Delta} - W_t) = t+\Delta - t = \Delta$$



Куча страшных св-в !!





$$P\left(\frac{\Delta_L}{a} = \frac{\Delta_R}{a}\right) = 0$$

$\Delta_L = W_t - W_{t-a}$

$\Delta_R = W_{t+a} - W_t$

Δ_L и Δ_R неуав

Ищем в последней (!) точке.

Упр.

(W_t) — винер процесс

$Y_t = \alpha \cdot W_{t/4}$ (раст по гор в 4 раза)

α? так, чтобы (Y_t) тоже был броуновским движением?

1. $Y_0 = 0$ ок
2. $Y_t - Y_s = \alpha(W_{t/4} - W_{s/4})$ $\leftarrow N(0; \frac{t}{4} - \frac{s}{4})$

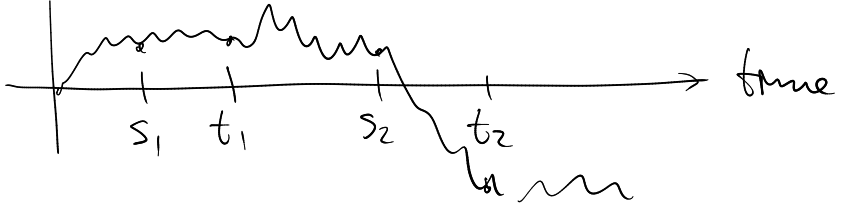
$\alpha = \pm 2$

$\alpha = 2$

$\mathrm{Var}(2(W_{t/4} - W_{s/4})) = 4 \cdot \mathrm{Var}(W_{t/4} - W_{s/4}) = t - s$

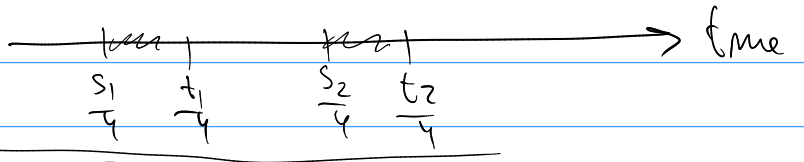
3. ☺

4.



$$\Delta_1^Y = Y(t_1) - Y(s_1) = 4 \cdot (W(t_1/4) - W(s_1/4))$$
$$\Delta_2^Y = Y(t_2) - Y(s_2) = 4 (W(t_2/4) - W(s_2/4))$$

Δ независ



$Y_t = 2 W_{t/4}$ — винеров процесс

$$P(W_6 > 3) = P(Y_6 > 3)$$

обращать время вспять!

инверсия

d) $E(W_s | W_t)$ $Var(W_s | W_t)$ $s<t$ (!)

Упр W_t — винеров процесс

$$Y_t = \begin{cases} 0, & \text{если } t=0 \\ t \cdot W_{1/t}, & \text{если } t>0 \end{cases}$$

① ?
② ?
③ ④] поверим

$\overset{?}{Y_0 = 0}$ $t>s$

$Y_t - Y_s = t W_{1/t} - s W_{1/s}$

$E(Y_t - Y_s) = t \cdot E(\dots) - sE(\dots) = t \cdot 0 - s \cdot 0 = 0$

$Var(Y_t - Y_s) = t^2 \cdot Var(W_{1/t}) + s^2 Var(W_{1/s}) - 2st \cdot Cov(W_{1/t}, W_{1/s}) =$

$= \frac{t^2}{t} + \frac{s^2}{s} - 2 \cdot s t \cdot \frac{1}{t} = t + s - 2s = t - s$

$Y_t - Y_s \sim N(0; t-s)$

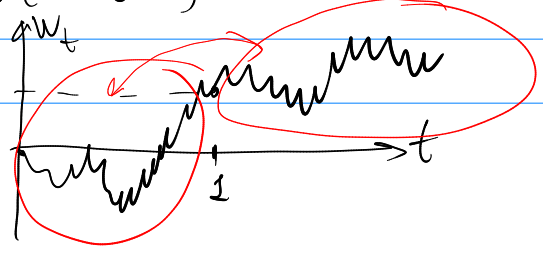
$W_3 = 3 \cdot Y_{1/3}$

$W_{1/2} = \frac{1}{2} Y_2$

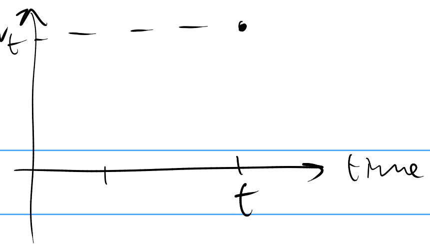
$Y_2 = 2 \cdot W_{1/2}$

$Y_1 = 1 \cdot W_1$

$Y_{0.01} = 0.01 \cdot W_{100}$



$s<t$



$$E(W_s \mid W_t) =$$

$$W_s = s \cdot Y_{1/s}$$
$$W_t = t \cdot Y_{1/t}$$

$$\sigma(Z) = \sigma(2 \cdot Z)$$
$$E(Y \mid Z) = E(Y \mid 2Z)$$

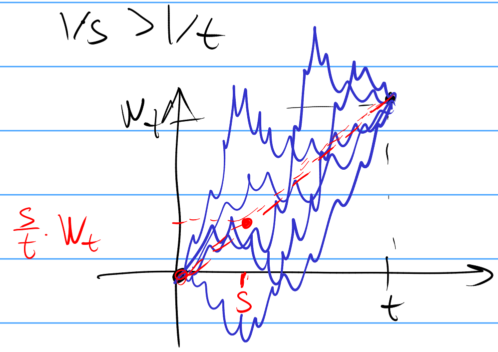
$$= E(s \cdot Y_{1/s} \mid t \cdot Y_{1/t}) =$$

RIP

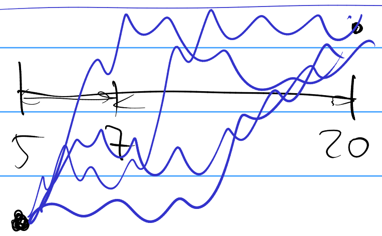
$$= s \cdot E(Y_{1/s} \mid Y_{1/t}) = s \cdot Y_{1/t} =$$

$1/s > 1/t$

$$E(W_s \mid W_t) = s \cdot \frac{1}{t} \cdot W_t = \frac{s}{t} W_t$$



$$E(W_7 \mid W_5, W_{20}) = \frac{13}{15} W_5 + \frac{2}{15} W_{20}$$



$$\mathrm{Var}(W_s \mid W_t) = \mathrm{Var}(s \cdot Y_{1/s} \mid t \cdot Y_{1/t}) = s^2 \cdot \mathrm{Var}(Y_{1/s} \mid Y_{1/t}) =$$

$s<t$ $\qquad$ $1/s > 1/t$

$$= s^2 \cdot \left(\frac{1}{s} - \frac{1}{t}\right) = s^2 \cdot \left(\frac{t-s}{st}\right)$$

def W.P с σ-алгебрами:

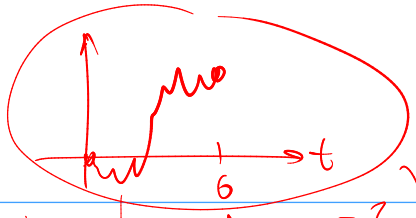
(W_t) – это вин. процесс по отношению к фильтр. $(\mathcal{F}_t)$ если: [1, 2, 3] те же

4olt $(W_t - W_s)$ не зав от $\mathcal{F}_s$ при $t \geq s$

5. W_t изм-ма отн-но $\mathcal{F}_t$ $\forall t$

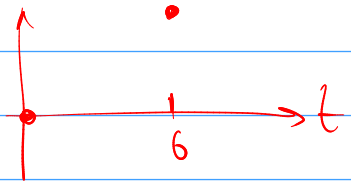
$E(W_t \mid \mathcal{F}_s) = W_s$ мартингал

$W_4 = E(W_4 | \mathcal{F}_6)$



$\sigma(\{W_t \mid t \in [0;6]\})$

$\frac{4}{6} W_6 = E(W_4 | W_6)$



сем бр 1
сем бр 2

?

сем умн
Q+A