Short introduction to ARMA procesess without nonsense. The goal is to state all the theorems rigorously.

Definition 1. The process (u_t) is called white noise if

$$\mathrm{E}(u_t) = 0, \quad \mathrm{Var}(u_t) = \sigma^2, \quad \mathrm{Cov}(u_s, u_t) = 0 \text{ for } s \neq t.$$

This definition does not assume that u_t and u_s are independent. They may be dependent but uncorrelated.

This definition does not assume normality of u_t but normality of white noise is often assumed in maximum likelihood estimation.

Definition 2. Lag operator L transforms a stochastic process (y_t) with $t \in \mathbb{Z}$ into a new stochastic process by shifting the index back in time,

$$Ly_t = y_{t-1}.$$

Definition 3. Forward operator F transforms a stochastic process (y_t) with $t \in \mathbb{Z}$ into a new stochastic process by shifting the index forward in time,

$$Fy_t = y_{t+1}.$$

Simple arithmetic examples are:

$$(1 + 2L + 3L^2)y_t = y_t + 2y_{t-1} + 3y_{t-2},$$

$$(3 + 2F + 5F^2)y_t = 3y_t + 2y_{t+1} + 5y_{t+2},$$

Theorem 4. The operators L and F are linear and $L^{-1} = F$.

Proof. The action LF or FL does nothing with any process (y_t). So operators L and F are mutually inverse. □

Definition 5. The process (y_t) is called stationary in weak sense if

$$\mathrm{E}(y_t) = \mu, \quad \mathrm{Cov}(u_s, u_t) = \gamma(t - s).$$

In particular all variances of stationary process are equal, $\mathrm{Var}(y_t) = \mathrm{Cov}(y_t, y_t) = \gamma_0$.

When infinite sums do exist?

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We *define* division by monomials.

Definition 6. For $|\alpha| < 1$ we define

$$\frac{1}{1 - \alpha L} y_t = (1 + \alpha L + \alpha^2 L^2 + \alpha^3 L^3 + \ldots)y_t,$$

and

$$\frac{1}{1 - \alpha F} y_t = (1 + \alpha F + \alpha^2 F^2 + \alpha^3 F^3 + \ldots)y_t.$$

Theorem 7. If (u_t) is a white noise and $|\alpha| < 1$ then $\frac{1-\alpha L}{1-\alpha F} u_t$ and $\frac{1-\alpha F}{1-\alpha L} u_t$ are white noises.

Theorem 8. The equation for (y_t)

$$P(L)y_t = Q(L)u_t + c,$$

where (u_t) is a white noise has infinitely many non-stationary solutions (y_t) if degree of P is higher than one.

Theorem 9. Consider the equation for (y_t)

$$P(L)y_t = Q(L)u_t + c,$$

where (u_t) is a white noise. If polynomials P and Q are coprime then

1. There are no stationary solutions (y_t) at all if P has at least one root ℓ with $|\ell| = 1$.
2. There is exactly one stationary solution (y_t) if all roots ℓ of P have $|\ell| \neq 1$.

   There are two subcases when all roots ℓ of P have $|\ell| \neq 1$:

1. All roots ℓ of P have $|\ell| > 1$. In this case the unique stationary solution has the form

   $$y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + \ldots,$$

   where (u_t) is the white noise from original equation.
2. At least one root of P has $|\ell| < 1$. In this case the unique stationary solution has the form

   $$y_t = \mu + \nu_t + c_1 \nu_{t-1} + c_2 \nu_{t-2} + c_3 \nu_{t-3} + \ldots,$$

   where (ν_t) is a white noise different from (u_t).

Definition 10. The process (y_t) is called $ARMA(p, q)$ process with equation

$$P(L)y_t = Q(L)u_t + c,$$

if

1. the process (y_t) satisfies this equation;
2. polynomial $P(L)$ has degree p and polynomial $Q(L)$ has degree q;
3. $P(0) = Q(0) = 1$;
4. P and Q are coprime, in other words they have no common roots.
5. the process (y_t) can be represented in $MA(\infty)$ form with respect to (u_t):

   $$y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + \ldots$$

From the last requirement in this definition it *follows* that all $ARMA(p, q)$ processes are stationary. By definition. Point. By the theorem 9 the polynomial P has all roots with $|\ell| > 1$. The equation $P(L)y_t = Q(L)u_t + c$ has infinitely many non stationary solutions. Not all solutions of equation $P(L)y_t = Q(L)u_t + c$ are called $ARMA$ processes. The same process y_t can have *many* $ARMA(p, q)$ representations with the same AR part and different MA parts.

In these examples (u_t) is a white noise.

1. Equation $y_t = 5 + y_{t-1} + u_t$. This equation has no stationary solutions at all. This equation has infinite number of non-stationary solution.

   Polynomials P and Q are coprime, unit root in the polynomial P. We do not use this equation to describe $ARMA$ process.

2. Equation $y_t = y_{t-1} + u_t - u_{t-1}$. This equation has infinite number of stationary solutions. For example $y_t = u_t + 5$. This equation has infinite number of non-stationary solution.

   Polynomials P and Q are not coprime, unit root in both polynomials P and Q. We do not use this equation to describe $ARMA$ process.

3. Equation $y_t = 5 + 2y_{t-1} + u_t - u_{t-1}$. This equation has unique stationary solution. This equation has infinite number of non-stationary solution.

   Polynomials P and Q are coprime, not all roots of P are outside unit circle.

   We do not use this equation to describe $ARMA$ process.

4. Equation $y_t = 7 + 0.5y_{t-1} + u_t - u_{t-1}$. This equation has unique stationary solution. This equation has infinite number of non-stationary solution.

   Polynomials P and Q are coprime, all roots of P are outside unit circle.

   The stationary solution of this equation is called $ARMA(1, 1)$ process with equation $y_t = 0.5y_{t-1} + u_t - u_{t-1}$.

Definition 11. The $ARMA(p, q)$ process with *equation*

$$P(L)y_t = Q(L)u_t + c,$$

is called *invertible* if white noise values can be recovered from past observed (y_t) in a linear form,

$$u_t = b + y_t + d_1 y_{t-1} + d_2 y_{t-2} + d_3 y_{t-3} + \dots$$

Stationarity is the property of the process per se, invertibility is the property of *the process and the equation.* One cannot check whether a given sequence of random variables is invertible.

Example 12. Consider two white noise processes (u_t) and (v_t) linked by equation $(1 - 2L)u_t = (1 - 0.5L)v_t$. This equation may be solved to obtain $u_t = \frac{1-0.5L}{1-0.5F} \frac{F}{-2} v_t$, that shows that the two white noises do exist.

Now consider the process $y_t = (1 - 2L)u_t = (1 - 0.5L)v_t$.

Is (y_t) invertible? This question is meaningless.

The process (y_t) described by equation $y_t = u_t - 2u_{t-1}$ is not invertible.

The same process (y_t) described by equation $y_t = v_t - 0.5v_{t-1}$ is invertible.

Theorem 13. Any $ARMA(p, q)$ process has at most one invertible equation.

Theorem 14. The $ARMA(p, q)$ process with *equation*

$$P(L)y_t = Q(L)u_t + c,$$

is *invertible* if and only if all roots ℓ of P have $|\ell| > 1$.

By our definition of $ARMA$ process, it is stationary, polynomials P and Q are coprime and P has all roots with $|\ell| > 1$.