Hi !! :)

ARMA-equation

№1

$y_t = 0.4y_{t-1} - 0.03y_{t-2} + 10 + u_t + u_{t-1}$, $t \in \mathbb{Z}$, $u_t \sim$ white noise

∞ non st. solutions

a) how many non-stationary solutions are there? provide initial condions for one of them.

b) — // — stationary solutions are there? Find all of them.

$y_0 = 4 + u_0$
$y_1 = 0$

initial cond-ns

$\to y_2, y_3, y_4, y_5 \ldots$
$\to y_{-1}, y_{-2}, y_{-3}, \ldots$

$E(y_0) = 4$
$E(y_1) = 0$
$E(y_2) = -0.12 + 10$

$y_2 \overset{?}{=} y_2 = 0.4 \cdot 0 - 0.03 \cdot (4 + u_0) + 10 + u_2 + u_1$

$E(u_t) = 0$

(y_t) is not stationary

$y_3 = 0.4y_2 - 0.03 \cdot 0 + 10 + u_3 + u_2$

$Var(y_0) = Var(4 + u_0) = \sigma_u^2$

$0 = 0.4(4 + u_0) - 0.03y_{-1} + u_1 + u_0$

y_1 y_0 $\to y_{-1}$

$Var(y_1) = 0$

$\boxed{y_t \sim \text{stationary}: \quad E(y_t) = \mu, \quad Var(y_t) = \sigma_y^2, \quad Cov(y_t, y_{t-k}) = \gamma_k}$

If the ARMA-equation $P(L) \cdot y_t = Q(L) \cdot u_t + c$ is not reducible {$P(\ell)$ and $Q(\ell)$ have no common roots} then there three possible cases:

$P(\ell)$ has at least one root $|\ell| = 1$
No stationary solutions

all roots of $P(\ell)$ are $|\ell| \neq 1$
Exactly one stationary solution

All roots of $P(\ell)$ are $|\ell| > 1$
The st. solution has the form
$y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + \ldots$

All roots with $|\ell| \neq 1$, and at least one root with $|\ell| < 1$
The stat. sol-n has the form
$y_t = \mu + \nu_t + c_1 \nu_{t-1} + c_2 \nu_{t-2} + \ldots$ where ν_t is a white noise

If the ARMA-equation $P(L)\cdot y_t = Q(L)\cdot u_t + c$ is not reducible { $P(\ell)$ and $Q(\ell)$ have no common roots } then there three possible cases:

① $P(\ell)$ has at least one root $|\ell|=1$

No stationary solutions

all roots of $P(\ell)$ are $|\ell|\neq 1$

Exactly one stationary solution

② All roots of $P(\ell)$ are $|\ell|>1$

The st. solution has the form

$$\boxed{y_t=\mu+u_t+c_1 u_{t-1}+c_2 u_{t-2}+\dots}$$

③ All roots with $|\ell|\neq 1$, and at least one root with $|\ell|<1$

The stat. sol-n has the form $y_t=\mu+v_t+c_1 v_{t+1}+c_2 v_{t+2}+\dots$ where v_t is a white noi

$$y_t = 0.4y_{t-1} - 0.03y_{t-2} + 10 + u_t + u_{t-1}$$

$$(1 - 0.4L + 0.03L^2)y_t = 10 + (1 + L)u_t$$

factorize!

$$1-(0{,}3+0{,}1)\cdot L+0{,}3\cdot 0{,}1L^2=(1-0{,}3L)\cdot(1+0{,}1L)$$

$$\underbrace{(1 - 0.3L)(1 - 0.1L)}_{P(L)}y_t = 10 + \underbrace{(1 + L)}_{Q(L)}u_t$$

$ax^2+bx+c = a(x-x_1)\cdot(x-x_2)$

AR-roots: $P(\ell)=0$

$\ell_1=\frac{1}{0{,}3}\qquad \ell_2=\frac{1}{0{,}1}$

MA-roots: $Q(\ell)=0$

$\ell_1=-1$

no common roots in P and Q

Th.) stationary sol-n is unique and has the form

$$y_t=\mu+u_t+c_1\cdot u_{t-1}+c_2\cdot u_{t-2}+c_3\cdot u_{t-3}+\dots$$

How to find this solution?

Strat 1: plug in (y_t) in original equation and solve for $\mu, c_1, c_2, c_3, \dots$

v Strat 2: use lag polynomials.

Theorem: If we divide/multiply ARMA-equation by $P(L)$ or $P(F)$ with all roots $|\ell|\neq 1$ then the set of stationary solutions does not change.

$L\cdot y_t = y_{t-1}\qquad F\cdot y_t = y_{t+1}$

L = lag

F = forward

def:

$$\frac{1}{1-\alpha L} = 1 + \alpha L + \alpha^2 L^2 + \alpha^3 \cdot L^3 + \dots$$

for $|\alpha| < 1$

$$\frac{1}{1-\alpha F} = 1 + \alpha F + \alpha^2 F^2 + \alpha^3 F^3 + \dots$$

$$(1 - 0.3L)(1 - 0.1L)y_t = 10 + (1 + L)u_t$$

$$y_t = \frac{1}{(1 - 0.3L)(1 - 0.1L)}10 + \frac{1}{(1 - 0.3L)(1 - 0.1L)}(1 + L)u_t$$

$$\frac{1}{1-5L} = \frac{1}{-5L} \cdot \frac{1}{1-\frac{1}{5}F} = \frac{F}{-5} \cdot \frac{1}{1-\frac{1}{5}F}$$

$L \cdot F = 1$

$L \cdot F = F \cdot L = 1$

$L \cdot 10 = 10$ $\quad F \cdot 10 = 10$

$L^2 \, 10 = 10$

$$(L^2 + 7F^3 + 2FL^2) \cdot 10 = (1 + 7 + 2) \cdot 10 = 100$$

$$\frac{1}{(1-0.3L)(1-0.1L)} \cdot 10 = \frac{1}{(1-0.3) \cdot (1-0.1)} \cdot 10 = \frac{10}{0.7 \cdot 0.9} = \frac{1000}{63}$$

$$y_t = \frac{1}{(1 - 0.3L)(1 - 0.1L)}10 + \frac{1}{(1 - 0.3L)(1 - 0.1L)}(1 + L)u_t$$

$$y_t = \frac{1000}{63} + (1 + 0.3L + 0.3^2L^2 + 0.3^3L^3 \dots) \cdot (1 + 0.1L + 0.1^2L^2 + \dots)(1+L) \cdot u_t$$

Here is the unique stationary solution!

$$y_t = \frac{1000}{63} + 1 \cdot u_t + (0.1 + 0.3 + 1) \cdot u_{t-1} +$$

$$+ (0.1^2 + 0.3^2 + 0.3 \cdot 0.1 + 0.3 + 0.1) u_{t-2} +$$

$$+ \dots$$

$y_t - 2y_{t-1} = 10 + u_t + u_{t-1}$, $u_t \sim$ white noise

ⓐ How many non-st-ry sol-s are there?
provide one example of y_0 that gives non-st-ry solution.

∞ many!
$y_0 = 17$ $\quad$ $y_0 = u_0 + 32$

ⓑ — // ——— stationary solut-s are there? Find all of them!

b) $\overbrace{(1-2L)}^{P(L)} y_t = 10 + \overbrace{(1+L)}^{Q(L)} \cdot u_t$

$l = \frac{1}{2}$ $\qquad$ $l = -1$ $\qquad$ no common roots

$\boxed{\text{Unique stationary solution!}}$

$$y_t = \frac{1}{1-2L} \cdot 10 + \frac{1+L}{1-2L} \cdot u_t$$

$$y_t = \frac{1}{1-2} \cdot 10 + (1+L) \cdot \frac{1}{-2L} \cdot \frac{1}{1-\frac{1}{2}F} \cdot u_t$$

$L \cdot F = 1$

$\frac{1}{L} = F$

$$y_t = -10 + \left(-\frac{1}{2}\right) \cdot (1+L) \cdot F \cdot \frac{1}{1-\frac{1}{2}F} \cdot u_t$$

$$y_t = -10 - \frac{1}{2} \cdot (F+1) \cdot \frac{1}{1-\frac{1}{2}F} u_t$$

unique stat-ry solution $\rightarrow$ $y_t = -10 - \frac{1}{2}(1+F)\left(1 + \frac{1}{2}F + \left(\frac{1}{2}\right)^2 F^2 + \left(\frac{1}{2}\right)^3 F^3 + \dots\right) \cdot u_t$

$$y_t = -10 + 1 \cdot u_t + \left(1 + \frac{1}{2}\right) u_{t+1} + \left(\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right) u_{t+2} + \dots + \dots + \left(\left(\frac{1}{2}\right)^{99} + \left(\frac{1}{2}\right)^{100}\right) \cdot u_{t+100} + \dots$$

$$y_t = -10 - \frac{1}{2} \cdot (F+1) \cdot \frac{1}{1 - \frac{1}{2}F} u_t$$

According to the case 3 of the theorem this can be expressed as $y_t = \mu + v_t + c_1 v_{t-1} + c_2 v_{t-2} + \dots$

where v_t is a white noise.

c*) express this (v_t) from (u_t).