Hi !!

(!) ARMA - equation $\neq$ ARMA-model

1. ARMA-equation
2. $P(\ell)$ and $Q(\ell)$ have no common roots
3. $(u_t) \sim$ white noise
$E(u_t)=0 \quad Var(u_t)=\sigma^2$
$Cov(u_t, u_s)=0$ for $t \neq s$ $(\forall t, \forall s)$

3+ (common)
$(u_t) \sim N(0;\sigma^2)$, indep.

4. y_t can be represented as
$y_t = \mu + u_t + c_1 \cdot u_{t-1} + c_2 u_{t-2} + \dots$ $(\forall t)$

5. All roots of $Q(\ell)$ are $|\ell|>1$ with.

$(\forall t)$
$$y_t = c + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \dots + \alpha_p y_{t-p} + u_t + \beta_1 u_{t-1} + \dots + \beta_q u_{t-q}$$

1.

1.

$\Updownarrow$

$P(L) y_t = c + Q(L) \cdot u_t$
where $P(L)$ is a polynom of degree p,
$Q(L)$ —//— degree q.
$P(0)=1$, $Q(0)=1$

e.g. Aad van der Vaart „Time series"

Problem 1

Can $(x_t) \sim$ ARMA-model have $ACF_1 = \frac{1}{2}$, $ACF_2 = \frac{1}{3}$, $ACF_3 = ACF_4 = \dots = 0$?

ACF = autocorrelation function

$Cov(u_1+u_2, u_1-u_2)=0$
$Cov(u_1+u_2, 0u_1+0u_2)=0$

$$ACF_1 = Corr(y_t, y_{t-1}) = \frac{1}{2}$$
$$ACF_2 = Corr(y_t, y_{t-2}) = 1/3$$
$$ACF_3 = Corr(y_t, y_{t-3}) = 0$$

$u_{t-4} \quad u_{t-3} \quad u_{t-2} \quad u_{t-1} \quad u_t$

$y_{t-1} \quad y_t$

$$y_t = \mu + u_t + c_1 \cdot u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + \dots$$
$$y_{t-1} = \mu + u_{t-1} + c_1 u_{t-2} + c_2 u_{t-3} + \dots$$
$$y_{t-3} = \mu + u_{t-3} + c_1 u_{t-4} + c_2 u_{t-5} + \dots$$

guess! $c_3=0 \quad c_4=0 \quad c_5=0 \ldots$

$u_t \sim$ white noise $E(u_t)=0$

$Var(u_t)=\sigma^2$

$Cov(u_t, u_s)=0$ for $t \neq s$

$$\boxed{y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2}}$$

$$ACF_3 = Corr(y_t, y_{t-3}) =$$

$$= Corr(\mu + u_t + c_1 u_{t-1} + c_2 u_{t-2},\ \mu + u_{t-3} + c_1 u_{t-4} + c_2 u_{t-5}) = 0$$

$$\begin{cases} ACF_1 = \frac{1}{2} \\ ACF_2 = \frac{1}{3} \end{cases}$$

$$Var(y_t) \overset{?}{=} Var(u_t + c_1 u_{t-1} + c_2 u_{t-2}) = \sigma^2 + c_1^2\sigma^2 + c_2^2\sigma^2$$

$$Cov(y_t, y_{t-1}) = Cov(u_t + c_1 u_{t-1} + c_2 u_{t-2},\ u_{t-1} + c_1 u_{t-2} + c_2 u_{t-3})$$

$$= c_1 \cdot \sigma^2 + c_1 \cdot c_2 \cdot \sigma^2$$

$$Cov(y_t, y_{t-2}) = Cov(u_t + c_1 u_{t-1} + c_2 u_{t-2},\ u_{t-2} + c_1 u_{t-3} + c_2 u_{t-4}) = c_2 \cdot \sigma^2$$

① $$ACF_1 = \frac{Cov(y_t, y_{t-1})}{\sqrt{\underbrace{Var(y_t)\,Var(y_{t-1})}_{=}}} = \frac{Cov(y_t, y_{t-1})}{Var(y_t)} = \boxed{\frac{c_1 + c_1 c_2}{1 + c_1^2 + c_2^2} = \frac{1}{2}}$$

$y_5 = \mu + u_5 + c_1 u_4 + c_2 u_3$

$y_6 = \mu + u_6 + c_1 u_5 + c_2 u_4$

② $$ACF_2 = \ldots = \frac{Cov(y_t, y_{t-2})}{Var(y_t)} = \boxed{\frac{c_2}{1 + c_1^2 + c_2^2} = \frac{1}{3}}$$

(σ^2 … σ^2)

→ express $c_2(c_1)$ and plug-in in the other eq-n.

→ guess a solution

$c_1 = 0$? no !!

$c_1 = 1$?

$$\frac{1 + c_2}{1 + 1 + c_2^2} = \frac{1}{2} \qquad \frac{c_2}{1 + 1 + c_2^2} = \frac{1}{3}$$

$c_2 = 2$

$c_1 = -1$?

$c_1 = 2$?

MA(2)

ARMA(0,2)

$$\boxed{y_t = 77 + u_t + u_{t-1} + 2 \cdot u_{t-2}}$$

$u_t \sim$ white noise with $\sigma^2 = 146$

$c_1 = \frac{1}{2} \quad c_2 = \frac{1}{2}$

Problem 2   2019

$$x_t = 1.3\,x_{t-1} - 0.4\,x_{t-2} + u_t - 0.5\,u_{t-1} \quad (*)$$

$u_t \sim$ white noise $E(u_t)=0$

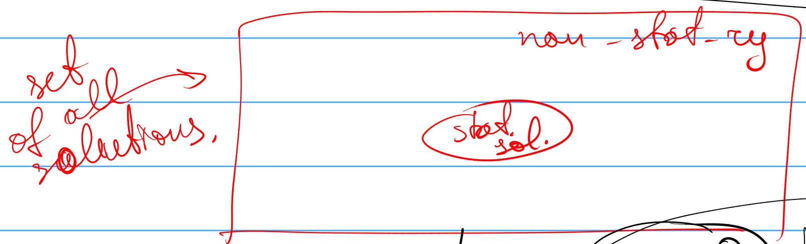
$x^2 = 100$
$x > 1$?

a) Is this (x_t) stationary?

b) if yes, then calculate ACF, ACF$_2$, ACF$_3$, ...

[! non rigorous !]
translate it in 5 minutes

! (*) is an equation ! It has ∞ many solutions.



Just take any x_0, x_1 !

$x_0 = 0$
$x_1 = 1$
init. cond-s

$x_2 = 1.3\cdot 1 - 0.4\cdot 0 + u_2 - 0.5\,u_1$

$x_3 = 1.3\cdot(\ldots x_2) - 0.4\cdot 1 + u_3 - 0.5\,u_2$

one solution

$E(x_0)=0$ $E(x_2)=1.3$
$E(x_1)=1$ ...

non-stat-ry solut-n.

...
x_{-2}
x_{-1}

$x_0 = u_0 + 7$ $Var(x_0) = 8^2$

$x_1 = 8u_0 + 2 + 5u_1$ $Var(x_1) = (64+25)\,8^2$

$x_2 \overset{?}{=} 1.3\,(8u_0 + 2 + 5u_1) - 0.4\,(u_0+7) + u_2 - 0.5\,u_1$

...

another solution.

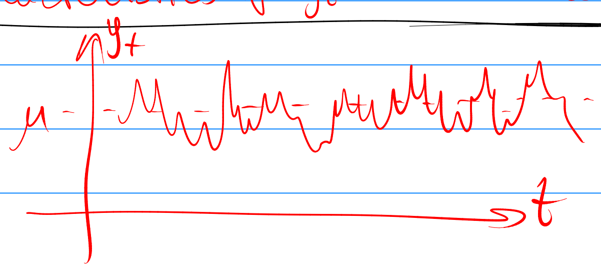
def. stat-ry process.

1) $E(y_t) = \mu$
2) $Var(y_t) = Cov(y_t, y_t) = \gamma_0$
3) $Cov(y_t, y_{t-1}) = \gamma_1$
4) $Cov(y_t, y_{t-2}) = \gamma_2$

⋮

„Characteristics of y_t are stable"

$Cov(y_5, y_6) =$
$= Cov(y_6, y_7) =$
$= Cov(y_7, y_8) = \ldots$



Is x_t a stationary process?

Are there any stationary solution of the form

$$y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + \dots \quad ?$$

$$x_t = 1{,}3\, x_{t-1} - 0{,}4 \cdot x_{t-2} + u_t - 0{,}5\, u_{t-1}$$

Theorem:

If we divide or multiply an ARMA-equation by a polynomial $P(L)$, then the set of non-stationary solutions changes but the set of stationary solutions (with roots $|\ell| \neq 1$) remains the same.

Eq 1 $\quad y_t = u_t - u_{t-1} \qquad \cdot \underbrace{(1 - 0{,}2L)}_{P(L)}$

Eq 2 $\quad (1-0{,}2L)\cdot y_t = (1-0{,}2L)\, u_t - (1-0{,}2L)\, u_{t-1}$

$$y_t - 0{,}2\, y_{t-1} = u_t - 0{,}2\, u_{t-1} - u_{t-1} + 0{,}2\, u_{t-2}$$

def of division

$$\frac{1}{1-\alpha L} \underset{|\alpha|<1}{\overset{def}{=}} 1 + \alpha L + (\alpha L)^2 + (\alpha L)^3 + \dots$$

$$\frac{1}{1-\alpha L} \underset{|\alpha|>1}{\overset{def}{=}} \frac{L^{-1}}{L^{-1}-\alpha} = \frac{-L^{-1}\cdot\alpha^{-1}}{1-\alpha^{-1}L^{-1}} =$$

$$= \frac{-F/\alpha}{1-\frac{1}{\alpha}F} = -\frac{F}{\alpha}\cdot\left(1 + \frac{1}{\alpha}F + \left(\frac{1}{\alpha}F\right)^2 + \left(\frac{1}{\alpha}F\right)^3 + \dots\right).$$

$$(1 - 1{,}3L + 0{,}4L^2)\cdot x_t = (1-0{,}5L)\cdot u_t$$

$1 - 1{,}3\ell + 0{,}4\ell^2 = 0$

$\hookrightarrow \ell_1 = 2 \quad \ell_2 = 1.25$

$|\ell| \neq 1$

$a_0 (\ell-\ell_1)\cdot(\ell-\ell_2)$

$$0{,}4\cancel{(L-2)}\cdot(L-1{,}25)\cdot x_t = (1-0{,}5L)\cdot u_t$$

$$= 0{,}5\cancel{(2-L)}\cdot u_t$$

$$-0.4\,(L-1{,}25)\, x_t = 0{,}5\, u_t$$

$$-0{,}8\,(L-1{,}25)\, x_t = u_t$$

$$(1-0{,}8L)\, x_t = u_t$$

$$x_t = \frac{1}{1-0{,}8L}\cdot u_t$$

$x_t - 0{,}8\, x_{t-1} = u_t$

AR(1)

$$x_t = (1 + 0{,}8L + (0{,}8L)^2 + (0{,}8L)^3 + \dots)\cdot u_t$$

$$x_t = u_t + 0{,}8\, u_{t-1} + 0{,}8^2 u_{t-2} + 0{,}8^3 u_{t-3} + \dots$$

init cond

$x_0 = u_0 + 0{,}8 \cdot u_{-1} + 0{,}8^2 u_{-2} + 0{,}8^3 u_{-3} + \dots$

$x_1 = u_1 + 0{,}8\, u_0 + 0{,}8^2 u_{-1} + 0{,}8^3 u_{-2} + \dots$

$\Downarrow$

$x_1 =$ ______

$x_2 =$ ______

$x_3 = u_3 + 0{,}8\, u_2 + 0{,}8^2 u_1 + \dots$

Yes there is a stationary solution of the form

$x_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + \dots$