Home Assignment 1

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0.2 & 0.1 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)

   (b) Split the chain into classes and classify them as closed and not closed.

   (c) Classify the states as recurrent and transient.

2. The Lonely Queen is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.

   (a) How many moves on average will it take to go back to A1?

   (b) What proportion of her eternal life will the Queen spend on every field?

3. Joe Biden throws a die until six appears or until he says «Stop». The payoff is equal to the previous thrown number before the last throw. If six appears on the first throw Joe receives nothing. Joe maximizes the expected payoff.

   What is the best strategy and the corresponding expected payoff?

4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed dragon Zmei Gorynich. Yes, there are infinitely many Zmeis Gorynichs.

   (a) What is the probability that Ilya will never meet Zmei Gorynich if Ilya chooses a road at random?

   (b) What is the probability that **there is** at least one Eternal Peaceful Journey without Zmei Gorynich?

5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTH or TTH apears. Ilya wins if HTH appears and Zmei wins if TTH appears.

   (a) What is the probability that Ilya wins?

   (b) What is the expected number of throws?

   (c) What is the expected number of throws given that Ilya won?

Deadline: **2021-10-06, 21:00**.

Home Assignment 2

1. I walk in the street during the first snow. Snowflakes falling into my palm is a Poisson process with rate $\lambda = 10$ snowflakes per minute.

   (a) What is the probability that there will be exactly 4 snowflakes in 30 seconds?

   (b) What is the expected value and variance of snowflakes in 2 minutes?

2. Grasshoppers are scattered accross a field according to a Poisson process with rate one grasshopper per two square meters.

   Which area should I search to find at least one grasshopper with probability 0.9?

3. Ilon Mask has two mobile phones. The calls to the first phone are a Poisson process with rate λ_1, the calls to the second one — a Poisson process with rate λ_2. Rate is measured in calls per hour. These processes are independent.

   Ilon turns on the phones simulteneously.

   (a) What is the probability that he receives exactly 2 calls on the first phone and exactly 3 calls on the second in one hour? Ilon Mask is like Bruce Willis and can answer unlimited number of calls simulteneously.

   (b) What is covariance between the total number of calls in the first hour and the total number of calls in the first two hours?

   (c) (harder) What is the probability that the first phone will ring first?

   Hint: there are at least two ways to solve the hard point. You can calculate a double integral for exponentially distributed waiting times. You can use the assumptions of Poisson process and first step approach.

4. I wait on the bus stop. The buses arrive according a Poisson process with rate 2 per hour. The taxis arrive according to a Poisson process with rate 5 per hour.

   (a) What is the probability that at least two taxis will arrive before a bus?

   (b) What is the probability that exactly two taxis will arrive before a bus?

   Hint: in this problem you may use the following fact without a proof. For two independent exponentially distributed variables with rates λ_1 and λ_2: $\mathbb{P}(Y_1 < Y_2) = \lambda_1/(\lambda_1 + \lambda_2)$.

5. (harder) Students arrive to the Grusha caffé according to a Poisson process with rate λ. The service time are independent and exponentially distributed with rate $\mu > \lambda$.

   Let's denote by S_t the number of students in the queue at time t (counting the student who is serviced). Imagine that Grusha is open 24/24 and the arrivals and service go on and go on. The distribution of S_t will stabilize, you don't need to prove it.

   Find the probability $\mathbb{P}(S_t = k)$ for big value of t.

Deadline: **2021-10-29, 21:00**.

Home Assignment 3

1. Let $\Omega = \mathbb{R}$. Explicitely find the sigma-algebras $\mathcal{F}_1 = \sigma(A)$, $\mathcal{F}_2 = \sigma(B)$, $\mathcal{F}_3 = \sigma(A, B)$ where $A = [-10; 5]$ and $B = (0; 10)$.

2. I throw a die once. Let X be the result of the toss. Count the number of events in sigma-algebras $\mathcal{F}_1 = \sigma(X)$, $\mathcal{F}_2 = \sigma(\{X > 3\})$, $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$.

3. Let $\Omega = \mathbb{R}$. The sigma-algebra $\mathcal{F}$ is generated by all the sets of the form $(-\infty, t]$,
$$\mathcal{F} = \sigma\left(\{(-\infty; t] \mid t \in \mathbb{R}\}\right)$$
Check whether $A_1 = (0; 10) \in \mathcal{F}$, $A_2 = \{5\} \in \mathcal{F}$, $A_3 = \mathbb{N} \in \mathcal{F}$.

4. Prove the following statements or provide a counter-example:
   (a) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are sigma-algebras then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is sigma-algebra.
   (b) If X and Y are independent random variables then $\operatorname{card} \sigma(X, Y) = \operatorname{card} \sigma(X) + \operatorname{card} \sigma(Y)$.

   For finite sets card denotes just the number of elements.

5. I throw a coin infinite number of times. Let the random variable X_n be equal to 1 if the n-th toss is head and 0 otherwise. Consider a pack of sigma-algebras: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \ldots)$.
   Where possible provide and example of a non-trivial event (neither Ω nor $\emptyset$) such that
   (a) $A_1 \in \mathcal{F}_{2020}$;
   (b) $A_2 \in \mathcal{H}_{2020}$;
   (c) $A_3 \in \mathcal{F}_{2020}$ and $A_3 \in \mathcal{H}_{2020}$;
   (d) $A_4 \in \mathcal{F}_n$ for all n;
   (e) $A_5 \in \mathcal{H}_n$ for all n.

Deadline: **2021-11-08, 21:00**.

Home Assignment 4

1. The random variables Z_n are independent and identically distributed with probabilities $\mathbb{P}(Z_n = 1) = 0.2$, $\mathbb{P}(Z_n = -1) = 0.8$.

   (a) Find a constant α such that $A_t = \sum_{n=1}^{t} Z_n - \alpha t$ is a martingale.

   (b) Find all constants β such that $B_t = \beta^{\sum_{n=1}^{t} Z_n}$ is a martingale.

2. Consider two classes of random processes in discrete time:

   - Markov chains, $\mathbb{P}(X_{n+1} = k \mid X_n, X_{n-1}, \ldots, X_1) = \mathbb{P}(X_{n+1} = k \mid X_n)$ for all n.
   - Martingales, $\mathbb{E}(X_{n+1} \mid X_n, X_{n-1}, \ldots, X_1) = X_n$ for all n.

   Provide an example or prove that the case is impossible.

   (a) The process X_t is a Markov chain but not a martingale.

   (b) The process X_t is a martingale but not a Markov chain.

3. Consider the Hedgehog problem from the exam. The Hedgehog starts at the state one and moves randomly between states with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

   Let $p_1(t)$, $p_2(t)$, $p_3(t)$ and $p_4(t)$ be the probabilities of observing the Hedgehog in each of the four states after exactly t moves.

   (a) Draw these probabilities as the functions of t using any open source software (Python, R, Julia, ...). Provide your code.

   (b) Is the number of steps equal to 10^{2021} sufficient for convergence?

4. Let X_t be the Poisson process with rate $\lambda = 42$.

   (a) Find a constant α such that $A_t = X_t - \alpha t$ is a martingale.

   (b) Find a constant β such that $B_t = \exp(X_t - \beta t)$ is a martingale.

5. Ilon Mask draws cards one by one from a well-shuffled deck of 52 cards. Let X_i be the indicator that the i-th card is an Ace.

   He remembers only whether each drawn card was an Ace or not, so his filtration is $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Initial information is trivial, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

   Let Y_n for $n \in \{0, 1, \ldots, 51\}$ be his probability estimate that the last card is an Ace $Y_n = \mathbb{E}(X_{52} \mid \mathcal{F}_n)$.

   (a) Express Y_n in terms of X_1, ..., X_n.

   (b) Is Y_n a martingale?

   (c) Find the joint probabilities for Y_{50}, Y_{51}.

   (d) Using any open source software draw 5 random trajectories of Y_n. Provide your code.

Deadline: **2021-11-16, 21:00**.

Home Assignment 5

Hereinafter (W_t) is a standard Wiener process.

1. Some questions about Wiener process!

   (a) Find $\mathbb{E}(W_7 \mid W_5)$, $\mathbb{V}\text{ar}(W_7 \mid W_5)$, $\mathbb{E}(W_7 W_6 \mid W_5)$.

   (b) Find $\mathbb{E}(W_5 \mid W_7)$, $\mathbb{V}\text{ar}(W_5 \mid W_7)$.

2. Using Ito's lemma find dX and the corresponding full form.

   (a) $X_t = W_t^6 \cos t$.

   (b) $X_t = Y_t^3 + t^2 Y_t$ where $dY_t = W_t^2 dW_t + tW_t dt$.

3. Consider a standard Wiener process (W_t).

   (a) Find the moment generating function of W_t, $M(u) = \mathbb{E}(\exp(uW_t))$.

   (b) Using the Taylor expansion for $M(u)$ find the $\mathbb{E}(W_t^k)$ for $k \in \mathbb{N}$.

4. Consider two independent Wiener processes A_t and B_t. Check whether these processes are Wiener processes:

   (a) $X_t = (A_t + B_t)/2$.

   (b) $Y_t = (A_t + B_t)/\sqrt{2}$.

5. Consider $I_t = \int_0^t W_u^2 u^2 du$. Find $\mathbb{E}(I_t)$, $\mathbb{V}\text{ar}(I_t)$ and (bonus) $\mathbb{C}\text{ov}(I_t, W_t)$.

6. (bonus) Let's split the time segment $[0; 10]$ into $n = 10^5$ sub-segments of equal lentgh. Let Δ_i be equal to the corresponding increment of Wiener process, $\Delta_i = W(10i/n) - W(10(i-1)/n)$.

   (a) What is the distribution of Δ_i?

   (b) Using any open source software simulate five approximate trajectories of Wiener process and plot them on the same plot.

   (c) Now simulate $n_{sim} = 10^4$ trajectories but do not plot them. Using these trajectories estimate the probability $p = \mathbb{P}(\max_{t\in[0;10]} W_t > 7)$.

   Do not forget to provide your code.

7. (bonus) Find the following limits in L^2.

   (a)
   $$S_n = \sum_{i=1}^{n} (t/n) \left(W(it/n) - W((i-1)t/n) \right).$$

   (b)
   $$T_n = \sum_{i=1}^{n} \left(W(it/n) - W((i-1)t/n) \right)^3.$$

Deadline: **2021-12-04, 21:00**.

Home Assignment 6

1. Let $X_t = 42 + t^2 W_t^3 + tW_t^2 + \int_0^t 3W_u du + \int_0^t W_u^3 dW_u$.

   (a) Find dX_t.

   (b) Is X_t a martingale? Is $Y_t = X_t - \mathbb{E}(X_t)$ a martingale?

   Hint: only the binary answer for (b) is not sufficient but the argument is very-very short if you solve (a).

2. Consider $I_t = \int_0^t W_u^2 u^2 dW_u$.

   (a) Find dI_t, $\mathbb{E}(I_t)$, $\mathbb{V}\text{ar}(I_t)$ and $\mathbb{C}\text{ov}(I_t, W_t)$.

   (b) Find $\mathbb{E}(I_5 \mid I_3)$.

3. In the framework of Black and Scholes model find the price at $t = 0$ of the following two financial assets, $dS_t = \mu S_t dt + \sigma S_t dW_t$ is the share price equation.

   (a) The asset pays you at time T exactly one dollar if $S_T < K$ where K is a constant specified in the contract.

   (b) The asset pays you at time T exactly S_T^2 dollars.

4. Consider the Vasicek interest rate model,

$$dR_t = 5(0.06 - R_t)\, dt + 3\, dW_t, \quad R_0 = 0.07.$$

   Here R_t is the interest rate.

   (a) Using the substitution $Y_t = e^{at} R_t$ find the solution of the stochastic differential equation. Start by finding dY_t.

   (b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\text{ar}(R_t)$.

   (c) Which value in this model would you call long-term equilibrium rate and why?

   Hint: you may have integrals in you expression for R_t, but no R_t.

5. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$.

   Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble.

   Write the stochastic differential equation for dY_t. The equation may contain Y_t and constants, but not X_t.

Deadline: **2021-12-12, 21:00**.

Nine home assignments till the end of the course, three every month. You may receive 10/10 for each home assignment without bonus problems.

And your have three more honey days :) Stay safe, be happy!

Home Assignment 7: stationarity, white noise, simple models

1. The process (u_t) is a white noise. Consider four processes: $a_t = (1+L)^3 u_t$, $b_t = \frac{1}{1-0.3L} u_t$, $c_t = \frac{1-0.3F}{1-0.3L} u_t$, $d_t = \frac{1-0.4F}{1-0.3L} u_t$.
   (a) Write explicit expression of these processes without lag L nor forward F operators.
   (b) Check whether these processes are stationary.
   (c) Check whether these processes are white noises.
2. The process (y_t) is stationary. Check whether these processes are stationary:
   (a) $a_t = \Delta^2 y_t$;
   (b) $b_t = 4y_t - 3y_{t-1} + 18$.
   (c) $c_t = 3 + t y_t$.
3. Sasha and Masha have five observations: $y_1 = 5$, $y_2 = 4$, $y_3 = 7$, $y_4 = 9$, $y_5 = 8$.
   Masha assumes the model $y_t = \mu + u_t$, $u_t \sim \mathcal{N}(0; \sigma^2)$ and independent.
   Sasha assumes the model $y_t = \mu + y_{t-1} + u_t$, $u_t \sim \mathcal{N}(0; \sigma^2)$ and independent.
   (a) Estimate the parameters of Masha's model using unconditional maximum likelihood,
   $$\ln f(y_1, \ldots, y_5) \to \max .$$
   (b) Estimate the parameters of Sasha's model using conditional maximum likelihood,
   $$\ln f(y_2, \ldots, y_5 \mid y_1) \to \max .$$
4. Let (y_t) be a stationary process with autocovariance function $\gamma_k = 20 \cdot 0.5^k$ and expected value 100. You remove all odd observations and hence your have a new process $z_t = y_{2t}$.
   (a) Is the new process (z_t) stationary?
   (b) Draw the autocorrelation function of (z_t) if it is stationary.
5. Provide an example of two dependent processes (a_t) and (b_t) such that each of them is stationary, but their sum is not stationary.
6. (bonus) Variables u_1 and u_2 are independent $\mathcal{N}(0; 1)$. Consider the process $y_t = u_1 \cos(\pi t/2) + u_2 \sin(\pi t/2)$.
   (a) Is (y_t) stationary? Is it a white noise process?
   (b) Your know that $y_{100} = 0.2022$. What is your prediction for y_{104}? What about predictive interval?

Deadline: **2022-02-12, 21:00**.

Home Assignment 8: ACF and PACF and forecasting

1. A dice is thrown three times, let's denote the results by X_1, X_2 and X_3. Consider three sums $L = X_1 + X_2$, $R = X_2 + X_3$ and $S = X_1 + X_2 + X_3$.
   (a) Using common sense find the sign of these partial correlations[1] $\mathrm{p}\mathbb{C}\mathrm{orr}(L, R; S)$, $\mathrm{p}\mathbb{C}\mathrm{orr}(L, S; R)$, $\mathrm{p}\mathbb{C}\mathrm{orr}(X_1, R; S)$.
   (b) Calculate these partial correlations.
2. For $MA(2)$ process $y_t = 5 + u_t + 3u_{t-2}$ find all values of the autocorrelation function ρ_k and first two values of the partial autocorrelation function ϕ_{kk}.
3. For stationary $AR(1)$ process with equation $y_t = 5 + 0.3y_{t-1} + u_t$ find all values of the autocorrelation function ρ_k and all values of the partial autocorrelation function ϕ_{kk}.
4. For stationary $AR(2)$ process with equation $y_t = 5 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$ find first two values of the autocorrelation function ρ_k and all values of the partial autocorrelation function ϕ_{kk}.
5. Consider stationary $AR(2)$ model, $y_t = 2 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$, where (u_t) is a white noise with normal distribution and $\mathbb{V}\mathrm{ar}(u_t) = 4$.

   The last two observations are $y_{100} = 2$, $y_{99} = 1$.

   Find 95% predictive interval for y_{101} and y_{102}.
6. (bonus) Consider $MA(1)$ process $y_t = u_t + 3u_{t-1} + 7$, where $u_t \sim \mathcal{N}(0; 16)$ and independent. You know that $y_{100} = 6$.

   Find 95% predictive interval for y_{101} and y_{102}.

Deadline: **2022-02-19, 21:00**.

[1] $\mathrm{p}\mathbb{C}\mathrm{orr}(X, Y; Z)$ denotes partial correlation between X and Y with «fixed» Z.

Home Assignment 9: ETS[2].

1. Consider $ETS(AA_dN)$ model with $\phi = 0.9$, $\alpha = 0.3$, $\beta = 0.1$ and $\sigma^2 = 16$. Express 95% predictive intervals for y_{t+1} and y_{t+2} in terms of ℓ_t, b_t, y_t and u_t.

2. Find $\mathbb{E}(y_t)$, $\mathbb{V}\mathrm{ar}(y_t)$, $\mathbb{C}\mathrm{ov}(y_t, y_{t+1})$ in the $ETS(AAN)$ model with given ℓ_0, b_0, α, β and σ^2.

3. Consider $ETS(AAN)$ model with $\ell_0 = 10$, $b_0 = 3$, $\alpha = \beta = 0.1$. Given observed values $y_1 = 15$, $y_2 = 17$, $y_3 = 20$ reconstruct the corresponding ℓ_t, b_t and u_t.

4. The semi-annual y_t is modelled by $ETS(AAA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ b_t = b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-2} + u_t \end{cases}$$

   Given that $s_{100} = 2$, $s_{99} = -1.9$, $b_{100} = 0.5$, $\ell_{100} = 4$ find 95% prëdictive interval for y_{102}.

5. How many free parameters are estimated in $ETS(ANA)$, $ETS(AAdA)$, $ETS(MMM)$ models for monthly time series? Explicitely list all free parameters in each case.

6. (bonus) Consider y_t described by $ETS(MMM)$ model. Is it true that $z_t = \ln y_t$ is exactly described by $ETS(AAA)$ model? Approximately?

Deadline: **2022-03-05, 21:00**.

[2]You can find all the equations in `https://otexts.com/fpp3/ets.html`

Home Assignment 10: recurrence equations and solutions

1. Consider the equation $y_t = 0.8y_{t-1} - 0.12y_{t-2} + u_t + 2u_{t-1}$, where (u_t) is a white noise.
   (a) Write the lag polynomials of AR and MA part. Write the corresponding characterestic polynomials.
   (b) Do lag polynomials of AR and MA part have common roots?

2. Consider the equation $y_t = 0.8y_{t-1} - 0.12y_{t-2} + u_t + 2u_{t-1}$, where (u_t) is a white noise.
   (a) How many non-stationary solutions are there? Provide at least two set of initial values for non-stationary solution.
   (b) How many stationary solutions are there?
   (c) Find constants μ, c_1 and c_2 for the stationary solution of the form

$$y_t = \mu + u_t + c_1u_{t-1} + c_2u_{t-2} + \dots$$

3. Find all stationary solutions of the equation $y_t = -5 + 0.7y_t + u_t - 0.5u_t$, where (u_t) is a white noise. Are there any stationary solutions that are $MA(\infty)$ with respect to u_t?

4. Find the simplest recurrence equation on y_t and u_t with solution

$$y_t = 5 + u_t - 0.1u_{t-1} + 0.1^2u_{t-2} - 0.1^3u_{t-3} + 0.1^4u_{t-4} + \dots$$

5. Consider three equations $a_t = u_t + 2u_{t-2} + 3u_{t-2}$, $b_t = 4b_{t-1} - 4.5b_{t-2} + u_t$, $c_t = c_{t-1} - 0.3c_{t-2} + u_t + 0.7u_{t-1}$, where (u_t) is a white noise.
   (a) Check whether the equation has any stationary solution.
   (b) Check whether the equation has any stationary solution that is $MA(\infty)$ with respect to u_t.

6. (bonus) Consider the equation $y_t = -5 + 2y_{t-1} + u_t$, where (u_t) is a white noise.
   (a) Find all stationary solutions of this equation.
   (b) Provide an example of a process (z_t) that is $MA(\infty)$ with respect to (u_t) and has exactly the same autocorrelation function as the stationary solution (y_t).

Deadline: **2022-03-17, 21:00**.

Home Assignment 11

1.

Deadline: **2022-03-24, 21:00**.

Home Assignment 12

1.

Deadline: **2022-03-31**, **21:00**.

Home Assignment 13

1.

Deadline: **2022-04-14, 21:00**.

Home Assignment 14

1.

Deadline: **2022-04-21**, **21:00**.

Home Assignment 15

1.

Deadline: **2022-04-28, 21:00**.