Home Assignment 1

1. Consider the Markov chain with the transition matrix:

$$\begin{pmatrix} 0.2 & 0.1 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   (a) Draw the most beautiful graph for this chain. A fox or a cat is ok :)

   (b) Split the chain into classes and classify them as closed and not closed.

   (c) Classify the states as recurrent and transient.

2. The Lonely Queen is standing on the A1 field of the chessboard. She starts moving randomly according to chess rules.

   (a) How many moves on average will it take to go back to A1?

   (b) What proportion of her eternal life will the Queen spend on every field?

3. Joe Biden throws a die until six appears or until he says «Stop». The payoff is equal to the previous thrown number before the last throw. If six appears on the first throw Joe receives nothing. Joe maximizes the expected payoff.

   What is the best strategy and the corresponding expected payoff?

4. Ilya Muromets stands before the first stone. There are three roads behind the stone. And every road ends with a new stone. And there are three new roads behind every new stone. And so on. Every road is guarded with one-third probability by a three headed dragon Zmei Gorynich. Yes, there are infinitely many Zmeis Gorynichs.

   (a) What is the probability that Ilya will never meet Zmei Gorynich if Ilya chooses a road at random?

   (b) What is the probability that **there is** at least one Eternal Peaceful Journey without Zmei Gorynich?

5. Ilya and Zmei finally met and play with a coin. They throw a coin until the sequence HTH or TTH apears. Ilya wins if HTH appears and Zmei wins if TTH appears.

   (a) What is the probability that Ilya wins?

   (b) What is the expected number of throws?

   (c) What is the expected number of throws given that Ilya won?

Deadline: **2021-10-06, 21:00**.

Home Assignment 2

1. I walk in the street during the first snow. Snowflakes falling into my palm is a Poisson process with rate $\lambda = 10$ snowflakes per minute.

   (a) What is the probability that there will be exactly 4 snowflakes in 30 seconds?

   (b) What is the expected value and variance of snowflakes in 2 minutes?

2. Grasshoppers are scattered accross a field according to a Poisson process with rate one grasshopper per two square meters.

   Which area should I search to find at least one grasshopper with probability 0.9?

3. Ilon Mask has two mobile phones. The calls to the first phone are a Poisson process with rate λ_1, the calls to the second one — a Poisson process with rate λ_2. Rate is measured in calls per hour. These processes are independent.

   Ilon turns on the phones simulteneously.

   (a) What is the probability that he receives exactly 2 calls on the first phone and exactly 3 calls on the second in one hour? Ilon Mask is like Bruce Willis and can answer unlimited number of calls simulteneously.

   (b) What is covariance between the total number of calls in the first hour and the total number of calls in the first two hours?

   (c) (harder) What is the probability that the first phone will ring first?

   Hint: there are at least two ways to solve the hard point. You can calculate a double integral for exponentially distributed waiting times. You can use the assumptions of Poisson process and first step approach.

4. I wait on the bus stop. The buses arrive according a Poisson process with rate 2 per hour. The taxis arrive according to a Poisson process with rate 5 per hour.

   (a) What is the probability that at least two taxis will arrive before a bus?

   (b) What is the probability that exactly two taxis will arrive before a bus?

   Hint: in this problem you may use the following fact without a proof. For two independent exponentially distributed variables with rates λ_1 and λ_2: $\mathbb{P}(Y_1 < Y_2) = \lambda_1/(\lambda_1 + \lambda_2)$.

5. (harder) Students arrive to the Grusha caffé according to a Poisson process with rate λ. The service time are independent and exponentially distributed with rate $\mu > \lambda$.

   Let's denote by S_t the number of students in the queue at time t (counting the student who is serviced). Imagine that Grusha is open 24/24 and the arrivals and service go on and go on. The distribution of S_t will stabilize, you don't need to prove it.

   Find the probability $\mathbb{P}(S_t = k)$ for big value of t.

Deadline: **2021-10-29, 21:00**.

Home Assignment 3

1. Let $\Omega = \mathbb{R}$. Explicitely find the sigma-algebras $\mathcal{F}_1 = \sigma(A)$, $\mathcal{F}_2 = \sigma(B)$, $\mathcal{F}_3 = \sigma(A, B)$ where $A = [-10; 5]$ and $B = (0; 10)$.

2. I throw a die once. Let X be the result of the toss. Count the number of events in sigma-algebras $\mathcal{F}_1 = \sigma(X)$, $\mathcal{F}_2 = \sigma(\{X > 3\})$, $\mathcal{F}_3 = \sigma(\{X > 3\}, \{X < 5\})$.

3. Let $\Omega = \mathbb{R}$. The sigma-algebra $\mathcal{F}$ is generated by all the sets of the form $(-\infty, t]$,

$$\mathcal{F} = \sigma\left(\{(-\infty; t] \mid t \in \mathbb{R}\}\right)$$

   Check whether $A_1 = (0; 10) \in \mathcal{F}$, $A_2 = \{5\} \in \mathcal{F}$, $A_3 = \mathbb{N} \in \mathcal{F}$.

4. Prove the following statements or provide a counter-example:

   (a) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are sigma-algebras then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is sigma-algebra.

   (b) If X and Y are independent random variables then $\operatorname{card} \sigma(X, Y) = \operatorname{card} \sigma(X) + \operatorname{card} \sigma(Y)$.

   For finite sets card denotes just the number of elements.

5. I throw a coin infinite number of times. Let the random variable X_n be equal to 1 if the n-th toss is head and 0 otherwise. Consider a pack of sigma-algebras: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{H}_n = \sigma(X_n, X_{n+1}, X_{n+2} \ldots)$.

   Where possible provide and example of a non-trivial event (neither Ω nor $\emptyset$) such that

   (a) $A_1 \in \mathcal{F}_{2020}$;

   (b) $A_2 \in \mathcal{H}_{2020}$;

   (c) $A_3 \in \mathcal{F}_{2020}$ and $A_3 \in \mathcal{H}_{2020}$;

   (d) $A_4 \in \mathcal{F}_n$ for all n;

   (e) $A_5 \in \mathcal{H}_n$ for all n.

Deadline: **2021-11-08, 21:00**.

Home Assignment 4

1. The random variables Z_n are independent and identically distributed with probabilities $\mathbb{P}(Z_n = 1) = 0.2$, $\mathbb{P}(Z_n = -1) = 0.8$.
   (a) Find a constant α such that $A_t = \sum_{n=1}^t Z_n - \alpha t$ is a martingale.
   (b) Find all constants β such that $B_t = \beta^{\sum_{n=1}^t Z_n}$ is a martingale.

2. Consider two classes of random processes in discrete time:
   - Markov chains, $\mathbb{P}(X_{n+1} = k \mid X_n, X_{n-1}, \ldots, X_1) = \mathbb{P}(X_{n+1} = k \mid X_n)$ for all n.
   - Martingales, $\mathbb{E}(X_{n+1} \mid X_n, X_{n-1}, \ldots, X_1) = X_n$ for all n.

   Provide an example or prove that the case is impossible.
   (a) The process X_t is a Markov chain but not a martingale.
   (b) The process X_t is a martingale but not a Markov chain.

3. Consider the Hedgehog problem from the exam. The Hedgehog starts at the state one and moves randomly between states with transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

   Let $p_1(t)$, $p_2(t)$, $p_3(t)$ and $p_4(t)$ be the probabilities of observing the Hedgehog in each of the four states after exactly t moves.
   (a) Draw these probabilities as the functions of t using any open source software (Python, R, Julia, ...). Provide your code.
   (b) Is the number of steps equal to 10^{2021} sufficient for convergence?

4. Let X_t be the Poisson process with rate $\lambda = 42$.
   (a) Find a constant α such that $A_t = X_t - \alpha t$ is a martingale.
   (b) Find a constant β such that $B_t = \exp(X_t - \beta t)$ is a martingale.

5. Ilon Mask draws cards one by one from a well-shuffled deck of 52 cards. Let X_i be the indicator that the i-th card is an Ace.

   He remembers only whether each drawn card was an Ace or not, so his filtration is $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$. Initial information is trivial, $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

   Let Y_n for $n \in \{0, 1, \ldots, 51\}$ be his probability estimate that the last card is an Ace $Y_n = \mathbb{E}(X_{52} \mid \mathcal{F}_n)$.
   (a) Express Y_n in terms of X_1, ..., X_n.
   (b) Is Y_n a martingale?
   (c) Find the joint probabilities for Y_{50}, Y_{51}.
   (d) Using any open source software draw 5 random trajectories of Y_n. Provide your code.

Deadline: **2021-11-16, 21:00**.