Sigma Algebra

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Sigma Algebra

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
- X, Z two random variables
- elementary conditional probability :

$$\mathbb{P}[X = x \mid Z = z] = \mathbb{P}[X = x, Z = z] / \mathbb{P}[Z = z]$$

- elementary conditional expectation :

$$\mathbb{E}[X \mid Z = z] = \sum_x x \mathbb{P}[X = x \mid Z = z]$$

- $Y = \mathbb{E}[X \mid \sigma(Z)]$?
  - Y is measurable with respect to $\sigma(Z)$
  - $\mathbb{E}[Y 1_{Z=z}] = \mathbb{E}[X 1_{Z=z}]$

Sigma Algebra

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space
- X is a random variable on the probability space with $\mathbb{E}[|X|] < \infty$
- $\mathcal{A} \subset \mathcal{F}$ is a sub σ-algebra

Then there exists a random variable Y such that

- Y is $\mathcal{A}$-measurable with $\mathbb{E}[|Y|] < \infty$
- for any $A \in \mathcal{A}$, we have $\mathbb{E}[Y 1_A] = \mathbb{E}[X 1_A]$.

Moreover, if $\tilde{Y}$ also satisfies the above two properties, then $\tilde{Y} = Y$ a.s.
A random variable Y with the above two properties is called the **conditional expectation** of X given $\mathcal{A}$, and we denote it by $\mathbb{E}[X \mid \mathcal{A}]$.

Remark :

- If $\mathcal{A} = \{\emptyset, \Omega\}$, then $\mathbb{E}[X \mid \mathcal{A}] = \mathbb{E}[X]$.
- ! If X is $\mathcal{A}$-measurable, then $\mathbb{E}[X \mid \mathcal{A}] = X$.
- If $Y = \mathbb{E}[X \mid \mathcal{A}]$, then $\mathbb{E}[Y] = \mathbb{E}[X]$

$A = \{ \{07\text{-}10\text{-}2021\}, \{08\text{-}10\text{-}2021\}, \{05\text{-}10\text{-}2021\} \}$

X_8 - st. price @ 8^{th} Oct $E[X_8]$

if $X_8 \sim A$-measurable $E[X_8 | A] = X_8$

$E[X_{11} | A] = E[X_{11}]$

Sigma Algebra

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where Ω is sample space, $\mathcal{F}$ is set of events (the σ-algebra) and $P : \mathcal{F} \to [0, 1]$ is the probability function.
- σ-**algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.
- **Measure** is function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i.
- Measure μ is **probability measure** if $\mu(\Omega) = 1$.

$\mathcal{G} = \{\{1\}, \{2\}, \ldots \{6\}, \{2,3,4,5,6\}, \{1,3,4,5,6\}, \ldots \{1,2,3,4,5\}\}$

$\mathcal{F} = \{\{1\}, \{2,3,4,5,6\}\} \subseteq \mathcal{G}$

Sigma Algebra

- **monotonicity**: $A \subset B$ implies $\mu(A) \leq \mu(B)$
- **subadditivity**: $A \subset \cup_{m=1}^{\infty} A_m$ implies $\mu(A) \leq \sum_{m=1}^{\infty} \mu(A_m)$.
- **continuity from below:** measures of sets A_i in increasing sequence converge to measure of limit $\cup_i A_i$
- **continuity from above:** measures of sets A_i in decreasing sequence converge to measure of intersection $\cap_i A_i$

Why not all Subsets are Sigma-Algebra?

- Uniform probability measure on $[0,1)$ should satisfy **translation invariance:** If B and a horizontal translation of B are both subsets $[0,1)$, their probabilities should be equal.
- Consider **wrap-around translations** $\tau_r(x) = (x+r) \mod 1$.
- By translation invariance, $\tau_r(B)$ has same probability as B.
- Call x, y "equivalent modulo rationals" if $x - y$ is rational (e.g., $x = \pi - 3$ and $y = \pi - 9/4$). An **equivalence class** is the set of points in $[0,1)$ equivalent to some given point.
- There are uncountably many of these classes.
- Let $A \subset [0,1)$ contain **one** point from each class. For each $x \in [0,1)$, there is **one** $a \in A$ such that $r = x - a$ is rational.
- Then each x in $[0,1)$ lies in $\tau_r(A)$ for **one** rational $r \in [0,1)$.
- Thus $[0,1) = \cup \tau_r(A)$ as r ranges over rationals in $[0,1)$.
- If $P(A) = 0$, then $P(S) = \sum_r P(\tau_r(A)) = 0$. If $P(A) > 0$ then $P(S) = \sum_r P(\tau_r(A)) = \infty$. Contradicts $P(S) = 1$ axiom.

Sigma Algebra

- The **Borel σ-algebra** $\mathcal{B}$ is the smallest σ-algebra containing all open intervals.
- Say that $\mathcal{B}$ is "generated" by the collection of open intervals.
- Why does this notion make sense? If $\mathcal{F}_i$ are σ-fields (for i in possibly uncountable index set I) does this imply that $\cap_{i \in I} \mathcal{F}_i$ is a σ-field?

Sigma Algebra

A **filtration** is a non-decreasing family of sub σ–algebras of $\mathcal{F}$ indexed by time, i.e. a family $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{T}}$ such that

$$\mathcal{F}_s \subseteq \mathcal{F}_t,$$

for $s \leq t$, where $t, s \in \mathbb{T}$.

Sigma Algebra

Let $\mathbb{F}$ be a (continuous time) filtration. We say that $\mathbb{F}$ is the **right-continuous filtration** if for any $t \in \mathbb{T}$ we get

$$\mathcal{F}_t = \mathcal{F}_{t_+} ,$$

where $\mathcal{F}_{t_+} := \bigcap_{s>t, s\in\mathbb{T}} \mathcal{F}_s$.

Sigma Algebra

$\mathcal{F}_1 \quad \mathcal{F}_2 \quad \mathcal{F}_3 \quad \ldots \quad \mathcal{F}_t$

$X_1 \quad X_2 \quad X_3 \quad \ldots\ldots \quad X_t \quad \ldots\ldots \quad X_{t+\Delta t} \quad \ldots\ldots$

process X is said to be **adapted** to filtration $\mathbb{F}$ (or $\mathbb{F}$**-adapted**) if X_t is $\mathcal{F}_t$–measurable for any $t \in \mathbb{T}$.

Sigma Algebra

Let X be a stochastic process. We say that $\mathbb{F}^X := (\mathcal{F}_t^X)_{t \in \mathbb{T}}$, where

$$\mathcal{F}_t^X = \sigma(X_s, s \leq t, s \in \mathbb{T})$$

is a filtration **generated** by stochastic process X.

Sigma Algebra

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space with $\Omega = [0,1]$.[9] Let

$$\mathcal{A} := \sigma(N \subset [0,1] : \#N < \infty)$$

denote the σ–algebra of countable sets (and their complements). For time horizon $\mathbb{T} = [0, +\infty)$ we define filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ by setting

$$\mathcal{F}_t := \begin{cases} \mathcal{A} & \text{for } t \in [0,1); \\ \mathcal{F} & \text{for } t \in [1,\infty). \end{cases}$$

Next, we define a stochastic process $X = (X_t)_{t \in \mathbb{T}}$ by setting

$$X_t(\omega) := \mathbb{1}_\Delta(t,\omega) = \begin{cases} 1 & \text{if } t = \omega \text{ and } t \leq 1/2 \\ 0 & \text{otherwise} \end{cases}, \qquad t \in \mathbb{T}, \omega \in \Omega.$$

where $\Delta := \{(t,t) : t \in [0, \frac{1}{2}]\}$ is a subset of $\mathbb{T} \times \Omega$.

Sigma Algebra

Definition (Moments)

Let X be a discrete random variable, and let $n \geqslant 1$ be an integer. The number

$$E(X^n) = \sum_x x^n p_X(x)$$

is called the n-th moment of X. Notice that the first moment is the mean.

Sigma Algebra

Definition

Let X be a discrete random variable. The function

$$M_X(t) = E(e^{tX})$$

is called the moment generating function (MGF) of X.

Sigma Algebra

Problem (Geometric)

Let X be geometric with parameter p. Show that

$$M_X(t) = \frac{pe^t}{1 - qe^t}.$$

$\boxed{q = 1 - p}$

Problem (Binomial)

Let X be binomial with parameters n and p. Show that

$$M_X(t) = (pe^t + q)^n.$$

Problem (Poisson)

Let X be Poisson with parameter λ. Show that

$$M_X(t) = e^{\lambda(e^t - 1)}.$$

$$e^y \approx 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

Taylor's series

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{+\infty} e^{tx} \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{+\infty} \frac{(e^t \cdot \lambda)^x}{x!}$$

Sigma Algebra

Theorem

Let X be a discrete random variable. Then

$$M'_X(0) = E(X).$$

In general, for each $n \geqslant 1$,

$$M_X^{(n)}(0) = E(X^n).$$

$$\mathrm{Var}(X) = M''_X(0) - \left(M'_X(0)\right)^2$$

Sigma Algebra

Theorem

1. *Let X be geometric with parameter p. Then*

$$E(X) = 1/p \quad \textit{and} \quad \mathrm{var}(X) = q/p^2.$$

2. *Let X be binomial with parameters n and p. Then*

$$E(X) = np \quad \textit{and} \quad \mathrm{var}(X) = npq.$$

Theorem (Change of Scale Theorem)

Let $Y = aX + b$, where a and b are real numbers and X is a random variable. Then $M_Y(t) = e^{tb} M_X(at)$.

26th Oct

16:20 → $\boxed{\text{2hrs} \quad \text{120 min}}$ IFR 4-5 probl. ! postpapers going to be shared

Sigma Algebra

Theorem (Uniqueness Theorem)

Let X and Y be random variables. If $M_X(t) = M_Y(t)$ for all $t \in [-a, a]$ for some positive real number a, then X and Y have the same distribution, that is, $F_X = F_Y$.

Sigma Algebra