Short rules: 120 minutes, online without proctoring. You may use any source you want but don't cheat.

Pledge:

Start exam by writing the following honor pledge and signing it.

I pledge on my honor that I will not give nor receive any unauthorized assistance on this exam.

Problems:

1. (10 points) Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) (3 points) Split the chain in classes and classify them into closed or not closed.

   (b) (2 points) Classify the states into recurrent or transient.

   (c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.

   What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

   Note: state number is the row (or column) number.

2. (10 points) Consider infinite ladder with steps numbered from 0 to infinity. I start at step 0. Every day with probability u I go one step up. With probability d I go one step down. With probability $1-u-d$ I stay on the same step.

   If I am at step 0 then I stay there with probability $1-u$ because it's impossible to go down.

   Consider the case $d > u$.

   What is the probability that I will be at step 0 after 10^{1000} days?

3. (10 points) The random variables X_i are independend and uniformly distributed on $[0;2]$. Find the probability limit

$$\operatorname{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^{10} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^3}{n+1}\right\}.$$

4. (10 points) Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes.

   Let Y_t be the number of taxis that will arrive between 0 and t minutes.

   (a) (5 points) Sketch the probability $\mathbb{P}(Y_{t+3} = 1 \mid Y_t = 0)$ as a function of t.

   (b) (5 points) Sketch the covariance $\mathbb{C}\mathrm{ov}(Y_t, Y_{60})$ as a function of t.

   Note: special points like intercepts or extrema should be explicitely marked.

5. (10 points) The moment generating function of a random variable X is $1/(1-2t)$.
   (a) Find the moment generating function of $2X$.
   (b) Find the moment generating function of $X+Y$ where X and Y are independent and identically distributed.
   (c) Do you remember the sum of geometric progression? Find $\mathbb{E}(X^{2021})$.

6. (20 points) Variables X_1, X_2, ...X_{100} are independent and identically distributed with mean 1 and variance 2. Each X_i has only three possible values: 0, 1, and 2.
   (a) (5 points) How many events are in sigma-algebras $\sigma(X_1, X_2)$ and $\sigma(X_1 - X_2)$?
   (b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(X_1, X_2)$ but $A \notin \sigma(X_1 - X_2)$; $B \in \sigma(X_1 - X_2)$ but $B \notin \sigma(X_1, X_2)$.
   (c) (10 points) Find $\mathbb{E}(X_1 + \ldots + X_{100} \mid X_1 + \ldots + X_{50})$ and $\mathbb{E}(X_1 + \ldots + X_{50} \mid X_1 + \ldots + X_{100})$.