Hi :)   2021-12-17

→ Ito's integration
→ BS model

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3. (10 points) The process Y_t is defined by

$$dY_t = W_t^2 dt + W_t dW_t,\ Y_0 = 0.$$

(a) (6 points) Find $\mathbb{E}(Y_t)$, $\mathbb{E}(Y_t W_t)$, $\mathbb{E}(Y_t W_t^2)$.

(b) (4 points) Find $\mathrm{Var}(Y_t)$.

$$Y_t = \underset{Y_0}{0} + \int_0^t W_u^2\, du + \int_0^t W_u\, dW_u$$

$$E(Y_t) = E\int_0^t W_u^2 du + E\int_0^t W_u\, dW_u = \int_0^t E(W_u^2)\, du = \int_0^t u\, du = t^2/2$$

$E\int_0^t A_u\, dW_u = 0$

$W_u \sim N(0,u)$

technique

$E(\text{tricky cont. process})$

1) Use Ito's lemma to write it as a sum $\int_0^t \ldots du + \int_0^t \ldots dW_u$

2) calculate $E(\cdot)$

$E(Y_t W_t)$?

$$Z_t = Y_t W_t = h(Y_t, W_t)$$

$h''_{YW} = 1$

$h''_{YY} = 0$

$h''_{WW} = 0$

$$dZ_t = dY_t \cdot W_t + Y_t\, dW_t + \frac{1}{2}\cdot 2\, dY_t \cdot dW_t =$$

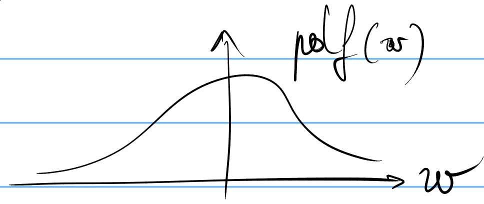
$$= (W_t^2 dt + W_t\, dW_t)\cdot W_t + Y_t \cdot dW_t + (W_t^2 dt + W_t dW_t)\cdot dW_t = W_t^3 dt + W_t^2 dW_t + Y_t dW_t + W_t \cdot dt;$$

$$Z_t = Z_0 + \int_0^t W_u^3 + W_u\, du + \int_0^t (W_u^2 + Y_u)\, dW_u$$

$Z_0 = Y_0 \cdot W_0 = 0$

$$E(Z_t) = 0 + \int_0^t \underline{E(W_u^3 + W_u)}\, du + 0 = 0$$

$E(W_t) = 0$ $\qquad W_t \sim N(0; t)$



$E(W_t^2) = t$

$E(W_t^3) = 0$

$E(W_t^4) =$

$$\boxed{MGF_{W_t}(u) = \exp\left(\frac{u^2 t}{2}\right)} =$$

$\exp(\beta) = 1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \dots$

$$= 1 + \frac{u^2 t}{2} + \frac{u^4 t^2/4}{2!} + \frac{u^6 t^3/8}{3!} + \frac{u^8 t^4/16}{4!} + \dots$$

$+0u+ \qquad +0u^3+ \qquad +0u^5+$

$$= 1 + E(W_t)\cdot u + \frac{E(W_t^2)}{2!}\cdot u^2 + \frac{E(W_t^3)}{3!} u^3 + \frac{E(W_t^4)}{4!} u^4 + \dots$$

$$E(W_t^4) = 4! \cdot \frac{t^2/4}{2!} = 3t^2$$

$$E(W_t^6) = 6! \, t^3/8/3! = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1\cdot t^3}{8\cdot3\cdot2\cdot1} = 15t^3$$

$E(Y_t \cdot W_t^2)$? $\qquad$ $Q_t = Y_t \cdot W_t^2$ $\qquad$ $\underline{E(Q_t)\,?}$

$$dQ_t = ?\,dt + ?\,dW_t$$

$$Q_t = Q_0 + \int_0^t ?\,du + \int_0^t ?\,dW_u$$

$Q_0 \to 0$

$$E(Q_t) = 0 + \int_0^t E(?)\,du + 0$$

2. (10 points) Simplify as much as possible the integral

$$\int_0^t \boxed{\exp(-W_u - u/2)}\, dW_u.$$

$$X_t = \exp\left(-W_t - \frac{t}{2}\right) = h(W_t, t)$$

$X_0 = 1$

Are you in cont time?

y → Apply Itô's lemma → ...

n → Find a martingale → ...

$$dX_t \overset{?}{=} h'_w \cdot dW_t + h'_t \cdot dt + \frac{1}{2} \cdot h''_{ww} \cdot dt =$$

$$= (-1) \cdot \exp\left(-W_t - \frac{t}{2}\right) \cdot dW_t - \frac{1}{2} \cdot \exp\left(-W_t - \frac{t}{2}\right) + \frac{1}{2}(-1)\cdot(-1)\cdot \exp\left(-W_t - \frac{t}{2}\right) \cdot dt = -\exp\left(-W_t - \frac{t}{2}\right) dW_t$$

$$X_t = X_0 + \int_0^t -\exp\left(-W_u - \frac{u}{2}\right) dW_u$$

$X_0 = 1$

$$1 - X_t = \int_0^t \exp\left(-W_u - \frac{u}{2}\right) dW_u = 1 - \exp\left(-W_t - \frac{t}{2}\right)$$

$$\int_0^t c \cdot dW_u = c W_t$$

$$\int_0^t W_u \, dW_u = \frac{W_t^2 - t}{2}$$

$$\int_0^t \exp\left(-W_u - \frac{u}{2}\right) dW_u = 1 - \exp\left(-W_t - \frac{t}{2}\right)$$

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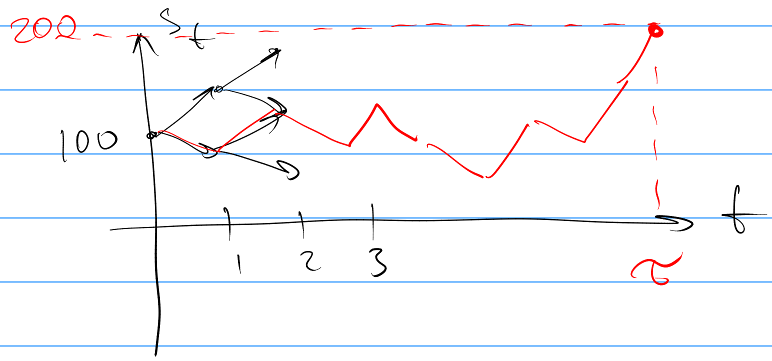
3. Today the price of a share is $S_0 = 100$ roubles. Each day the price S_t goes up by one rouble with probability $p \in (0; 1)$ or by two roubles with probability $1 - p$.

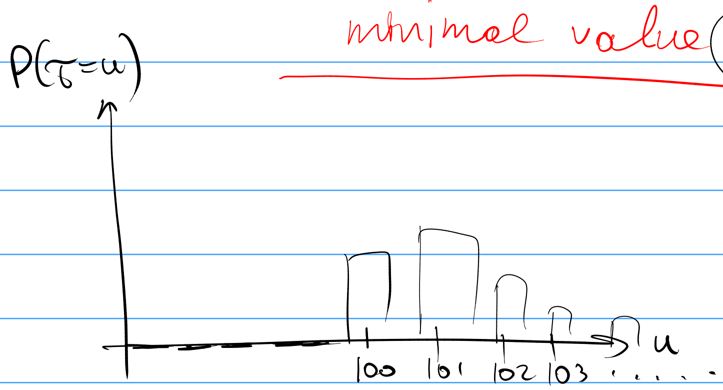
   (a) Find a number a such that $M_t = a^{S_t}$ is a martingale.

   (b) Let τ be the first moment of time when the price will be greater or equal to 200 roubles. Find $\mathbb{E}(\tau)$.



$\tau = \min_t \{ t \mid S_t = 200 \}$ $\qquad E(\tau)$? $\qquad \geqslant 100$

minimal value τ ? $\qquad$ 100



a) $\quad a$? $\qquad M_t = a^{S_t}$ — mart-gale

$$S_t = 100 + Z_1 + Z_2 + \ldots + Z_t$$

$Z_t \sim$ are indep.

u	+1	−2
$P(Z_t = u)$	p	1−p

$$E(M_{t+1} \mid \mathcal{F}_t) \overset{?}{=} M_t \qquad S_{t+1} = S_t + Z_{t+1}$$

$$E(a^{S_{t+1}} \mid \mathcal{F}_t) = a^{S_t}$$

$$E\left(a^{S_t+Z_{t+1}} \mid \mathcal{F}_t\right) = a^{S_t} \qquad E\left(a^{S_t}\cdot a^{Z_{t+1}} \mid \mathcal{F}_t\right) = a^{S_t}$$

a^{S_t} is known at time t

$$\cancel{a^{S_t}} \cdot E\left(a^{Z_{t+1}} \mid \mathcal{F}_t\right) = \cancel{a^{S_t}}$$

$$E\left(a^{Z_{t+1}} \mid \mathcal{F}_t\right) = 1$$

indep t

$$E\left(a^{Z_{t+1}}\right) = 1$$

u	$+1$	-2
$P(Z_{t+1}=u)$	p	$1-p$

$$p\cdot a^{1} + (1-p)a^{-2} = 1 \qquad a?$$

$$pa^3 - a^2 + (1-p) = 0 \qquad a?$$

guess one root! $a=1$ $\qquad p-1+1-p=0$ ok!!

divide by $(a-1)$

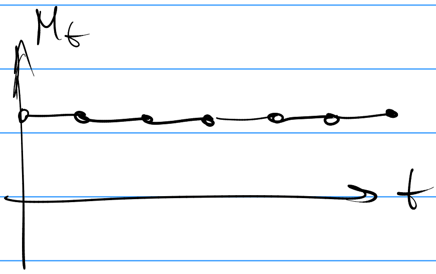
$$p(a^3-1) - a^2 + 1 = 0$$

$$p(a-1)\cdot(a^2+a+1) + (1-a)\cdot(1+a) = 0$$

$$(a-1)\left[\, p(a^2+a+1) - (1+a) \right] = 0$$

$a=1$

$M_t = 1^{S_t} = 1$



$1-\frac{1}{p}<0$

$\frac{1}{p}-1>0$

$$a^2+a+1-\frac{1}{p}-\frac{1}{p}a=0$$

$$a^2+\left(1-\frac{1}{p}\right)a+\left(1-\frac{1}{p}\right)=0$$

$a_1<0 \qquad a_2>0$

a_2

$$a_{1,2} = \frac{\left(\frac{1}{p}-1\right) \pm \sqrt{\left(1-\frac{1}{p}\right)^2 - 4\left(1-\frac{1}{p}\right)}}{2}$$

a) $M_t = a^{S_t}$ — martingale !

$$a = \frac{\left(\frac{1}{p}-1\right) + \sqrt{\left(\frac{1}{p}-1\right)^2 + 4\left(\frac{1}{p}-1\right)}}{2}$$

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b) $E(\tau)$?

$S_t = 100 + Z_1 + Z_2 + \dots + Z_t$

S_t is it a martingale ?

$M_t = f(S_t, t)$

$M_\tau = f(S_\tau, \tau)$

$E(S_{t+1} \mid \mathcal{F}_t) = E(S_t + Z_{t+1} \mid \mathcal{F}_t) =$

$= S_t + E(Z_{t+1} \mid \mathcal{F}_t) = S_t + E(Z_{t+1}) =$

indep

$= S_t + p \cdot 1 + (1-p)\cdot(-2) = S_t - 2 + 3p$

new martingale

$X_t = S_t - t\cdot(3p-2)$

$E(S_{t+1} \mid \mathcal{F}_t) = S_t + (3p-2)$

$E(M_{t+1} - M_t \mid \mathcal{F}_t) = 0$ ⟸ def of mart

can be useful to extract info on τ

$E(X_{t+1} - X_t \mid \mathcal{F}_t) =$

$= E(S_{t+1} - (t+1)(3p-2) - S_t + t\cdot(3p-2) \mid \mathcal{F}_t) =$

$X_\tau = S_\tau - \tau(3p-2)$

$= E(Z_{t+1} - (3p-2) \mid \mathcal{F}_t) = E(Z_{t+1} - (3p-2)) = 0$

Opt. Stop Time theorem

$E(X_\tau) = E(X_0)$

$X_0 = S_0 - 0\cdot(3p-2) = 100$

$E(X_0) = 100$

$X_\tau = S_\tau - \tau\cdot(3p-2)$

$S_\tau = 200$

$E(200 - \tau(3p-2)) = 100$

$$E(\tau) = \begin{cases} 100/(3p-2) & \text{if } 3p-2>0 \\ +\infty & \text{if } 3p-2\le 0 \end{cases} \Longleftarrow E(\tau) = \frac{100}{3p-2}$$

!! ☺

? $p = 0{,}99$ ok !!

$p = 1/2$ $E(\tau) = \frac{100}{-0{,}5} = -200$