Hi :) 2021-11-26

Ito's integral properties !!

martingale

(W_t) – Wiener process

X_t – martingale

$$X_t = t \cdot W(1/t)$$ – Wiener process.

$$E(W_2 \mid W_{10})$$

$$\Updownarrow$$

$$W_{1/t} = \frac{1}{t} \cdot X_t$$

$$W_a = a X_{1/a}$$

$$W_2 = 2 \cdot X_{1/2}$$

2 10

$\frac{1}{10}$ $\frac{1}{2}$

$$\int_0^t 7 \cdot dW_u = -7 \cdot W_0 + 7 \cdot W_t = 7 \cdot W_t$$

$$\int_0^t W_u \, dW_u = (\text{long limit}) = \frac{W_t^2}{2} - \frac{t}{2}$$

Assumptions.

(A_u) – cont. st. process

$$\int_0^t E(A_u^2) \, du < \infty$$

① Boring !! Linearity

$$\int_0^t A_u + B_u \, dW_u = \int_0^t A_u \, dW_u + \int_0^t B_u \, dW_u$$

$$\int_0^t \alpha \cdot A_u \, dW_u = \alpha \int_0^t A_u \, dW_u$$

$$\int_a^b A_u \, dW_u + \int_b^c A_u \, dW_u \overset{?}{=} \int_a^c A_u \, dW_u$$

2. Ito's integral is a martingale

$$I_t = \int_0^t A_u \, dW_u$$

(W_t) - is a Wiener Process wrt $(\mathcal{F}_t)$

(I_t) is a martingal wrt $(\mathcal{F}_t)$

$$E(I_{t+\Delta} | \mathcal{F}_t) = I_t$$

$$E(I_{t+\Delta} | I_t) = I_t$$

$E(LHS) = E(RHS)$

$$E(I_{t+\Delta} - I_t | I_t) = 0$$

$\forall t, \Delta$

$$E(I_{t+\Delta}) = E(I_t) = \text{const}$$

$$\int_0^0 A_u \, dW_u = 0$$

$$E(I_t) = 0$$

$I_0 = 0 \quad E(I_0) = 0$

(2+) comparison with $\int_0^t B_u \, du$

$$E\left(\int_0^t B_u \, du\right) = \int_0^t E(B_u) \, du$$

$R_t = \int_0^t B_u \, du$ is <u>never</u> a martingale (if B_u is non-zero)

3. Var, Covariance

Ito's isometry.

$$\text{Var}\left(\int_0^t A_u \, dW_u\right) = \int_0^t E(A_u^2) \, du$$

$$\text{Cov}\left(\int_0^t A_u \, dW_u, \int_0^t B_u \, dW_u\right) = \int_0^t E(A_u B_u) \, du$$

Shreve

Steele

$$\int_0^t 1 \cdot dW_u$$

$\boxed{\text{Ex}}$

$$\operatorname{Cov}\left(\int_0^t W_u^2 \, dW_u, W_t\right) \overset{?}{=}$$

$$\operatorname{Cov}\left(\int_0^t W_u^2 \, dW_u, \int_0^t 1 \cdot dW_u\right) = \int_0^t E\left(W_u^2 \cdot 1\right) \underset{\approx}{du} \;(=)$$

$$E\left(W_u^2\right) = \operatorname{Var}\left(W_u\right) = u$$

$$\boxed{\overset{t > s}{W_t - W_s \sim N(0; t-s)}}$$

$$(=) \int_0^t u \, du = \frac{t^2}{2}$$

Ex

$$E\left(42 + \int_0^t \cos W_u \cdot dW_u + \int_0^t u^2 \cdot W_u^2 \, du\right) \overset{?}{=}$$

$$= 42 + 0 + \int_0^t E\left(u^2 W_u^2\right) du = 42 + \int_0^t u^3 \, du = 42 + \frac{t^4}{4}$$

$$E\left(u^2 W_u^2\right) = u^2 \cdot E\left(W_u^2\right) = u^2 \cdot u = u^3$$

Ito's lemma.

$E(X_t)$, $\operatorname{Var}(X_t)$ — number

- → baby-version
- → sf. version
- → multivariate extension

short notation | full notation

$$dX_t = A_t \, dW_t + B_t \, dt \qquad X_t = X_0 + \int_0^t A_u \, dW_u + \int_0^t B_u \, du$$

X_t: R.V. $\quad X_0$: Const $\quad \int_0^t A_u \, dW_u$: RV $\quad \int_0^t B_u \, du$: R.V

$\boxed{\text{Agreement}}$

a way to write less

dW_t does not exist

dW_t is not a R.V.

def (X_t) – Ito's process if it can be written in this form.

Ito's lemma.

Baby-version

If $X_t = h(W_t, t)$ $\quad h''_{ww}, h'_t$ are continuous

then [short] $\underline{dX_t = h'_w \cdot dW_t + h'_t \cdot dt + \frac{1}{2} h''_{ww} \cdot dt}$

[full] $X_t = X_0 + \int_0^t h'_w \, dW_u + \int_0^t h'_t \, du + \frac{1}{2} \int_0^t h''_{ww} \, du$

Ex

a) $X_t = W_t^2 \cdot t$ $\quad dX_t?$ full form?

b) $X_t = W_t^4$ $\quad dX_t?$ full form? $\underline{E(X_t)}$?

c) $X_t = W_t^2$ $\quad dX_t?$ full form?

a) $dX_t = 2W_t \cdot t \cdot dW_t + W_t^2 \, dt + \frac{1}{2} \cdot 2 \cdot t \cdot dt$

$$\boxed{W_t^2 \cdot t = \underbrace{W_0^2 \cdot 0}_{0} + \int_0^t 2 W_u \cdot u \cdot dW_u + \int_0^t W_u^2 \, du + \int_0^t u \, du}$$

b) $dX_t = 4W_t^3 \cdot dW_t + 0 \cdot dt + \frac{1}{2} 4 \cdot 3 \cdot W_t^2 \cdot dt$

$$W_t^4 = W_0^4 + \int_0^t 4 W_u^3 \, dW_u + 6 \cdot \int_0^t W_u^2 \, du$$

$$E(W_t^4) = 0 + 0 + 6 \int_0^t \underbrace{E(W_u^2)}_{u} du =$$

$$= 6 \cdot \int_0^t u \, du = 6 \cdot \frac{t^2}{2} = 3t^2$$

c) $X_t = W_t^2$ $\quad dX_t = \underbrace{2 \cdot W_t}_{h'_w} \cdot dW_t + \underbrace{0}_{h'_t} \cdot dt + \frac{1}{2} \underbrace{2}_{h''_{ww}} \cdot dt$

$$W_t^2 = \underbrace{W_0^2}_{0} + \int_0^t 2 W_u \, dW_u + \underbrace{\int_0^t 1 \cdot du}_{t}$$

$$\int_0^t W_u \, dW_u = \frac{W_t^2 - t}{2}$$

!!

Ex. Find α such that $M_t = \underbrace{\exp(W_t + \alpha t)}$ is a martingale

$$dM_t = \underset{h'_w}{\exp(W_t+\alpha t)} \cdot dW_t + \underset{h'_t}{\alpha \cdot \exp(W_t+\alpha t)} \cdot dt + \underset{h''_{ww}}{\tfrac{1}{2}\exp(W_t+\alpha t)} \cdot dt$$

$$M_t = M_0 + \int_0^t \exp(W_u + \alpha u)\, dW_u + \int_0^t \underline{\left(\alpha+\tfrac{1}{2}\right)\cdot \exp(W_u+\alpha u)} \cdot du$$

$$M_0 = 1 = \exp(0 + \alpha \cdot 0)$$

is a martingale

is never a martingale except one case

$\alpha = -\frac{1}{2}$!!

$$M_t = 1 + \int_0^t \exp\left(W_u - \tfrac{u}{2}\right) dW_u + 0$$

M_t is a martingale if $\alpha = -\frac{1}{2}$

$$E\left(M_{t+\Delta} \mid \mathcal{F}_t\right) = E\Big(1 + \underbrace{\int_0^{t+\Delta} \dots dW_u}_{\text{martingale}} \Big| \mathcal{F}_t\Big) =$$

$$= 1 + \int_0^t \dots dW_u = M_t$$

Ito's lemma

If $X_t = h(Y_t, t)$ where Y_t is an Ito's process,
$dY_t = A_t\, dW_t + B_t\, dt$ $\quad h''_{YY}, h'_t$ are cont-ous f-ns

then (short form)
$$dX_t = h'_Y\, dY_t + h'_t \cdot dt + \frac{1}{2} \cdot h''_{YY} \cdot A_t^2 \cdot dt$$

(full form)
$$X_t = X_0 + \int_0^t h'_Y\, dY_u + \int_0^t h'_t \cdot du + \frac{1}{2} \int_0^t h''_{YY} \cdot A_u^2 \cdot du$$

(algorithm to remember)

to obtain dX_t:

1. write dX_t using second order Taylor expansion
2. simplify it using the rules:

$dW_t \cdot dW_t = dt$

$\left.\begin{array}{l} dW_t \cdot dt = 0 \\ dt \cdot dt = 0 \end{array}\right\}$ $dt \cdot \text{danything} = 0$

Ex $\quad X_t = \underbrace{Y_t^3 + t^2}_{h(Y_t,t)}$ where $dY_t = W_t\, dW_t + W_t^2\, dt$

$$dX_t? = \underbrace{3Y_t^2}_{h'_Y} \cdot dY_t + \underbrace{2t}_{h'_t} \cdot dt + \frac{1}{2}\Big[\underbrace{6Y_t}_{h''_{YY}} \cdot (dY_t)^2 + 2 \cdot \underbrace{0}_{h''_{tY}}\, dt \cdot dY_t + \underbrace{2}_{h''_{tt}} (dt)^2\Big] =$$

$dt \cdot dY_t \to 0$, $\quad (dt)^2 \to 0$

$$(W_t\, dW_t + W_t^2\, dt)^2 = W_t^2 (dW_t)^2 + 2 \cdot W_t \cdot W_t^2\, dW_t\, dt + W_t^4 (dt)^2 = W_t^2\, dt$$

$(dW_t)^2 \to dt$, $\quad dW_t\, dt \to 0$, $\quad (dt)^2 \to 0$

$$dX_t = 3Y_t^2 (W_t\, dW_t + W_t^2\, dt) + 2t\, dt + 3Y_t \cdot W_t^2\, dt$$

$= \text{smth} \cdot dW_t + \text{smthelse} \cdot dt$

:)

Ex

$$dY_t = 5W_t^3\, dW_t + 6W_t\, dt.$$

$$X_t = Y_t^5$$

$$dX_t ? = 5Y_t^4 \cdot dY_t +$$

$$+ \frac{1}{2} 20 \cdot Y_t^3 \cdot (dY_t)^2 =$$

Step 1. Get 2nd order Taylor exp-n for dX_t

Step 2. simplify

$dt \cdot d\text{anything} = 0$

$dW \cdot dW = dt$

$$= 5Y_t^4 (5W_t^3\, dW_t + 6W_t\, dt) +$$

$$+ 10Y_t^3 \cdot (5W_t^3\, dW_t + 6W_t\, dt)^2 = \boxed{25Y_t^4 \cdot W_t^3\, dW_t + 30Y_t^4 \cdot W_t \cdot dt + 250Y_t^3 \cdot W_t^6\, dt}$$

$$a^2 + 2ab + b^2$$

$$25W_t^6 \cdot dt + 2 \cdot \underbrace{dW \cdot dt}_{=0} + (dt)^2 \to 0$$