Start exam by writing the following honor **pledge** and **signing it**:

I pledge on my honor that I will not give nor receive any unauthorized assistance on this exam.

Problems:

1. (10 points) Consider an Ito's process $I_t = 2022 + W_t^3 t^2 + \int_0^t W_u^4 dW_u + \int_0^t W_u^3 du$.
   (a) Find dI_t and check whether I_t is a martingale.
   (b) Check whether $J_t = I_t / \mathbb{E}(I_t)$ is a martingale.

2. (10 points) The random variables (Z_t) are independent identically distributed with moment generating function given by $M_Z(u) = 1/(1-3u)^5$.

   We define X_t as $X_t = \exp(Z_1 + 2Z_2 + 3Z_3 + \ldots + tZ_t)$ with $X_0 = 0$.

   If possible find a martingale of the form $Y_t = h(t)X_t$ where $h()$ is a non-random function.

3. (10 points) The process (Z_t) in discrete time is called *stationary* if it has constant expected value and constant covariances γ_k that do not depend on t.

$$\begin{cases} \mathbb{E}(Z_t) = \mu; \\ \mathbb{C}\mathrm{ov}(Z_t, Z_t) = \gamma_0; \\ \mathbb{C}\mathrm{ov}(Z_t, Z_{t+1}) = \gamma_1; \\ \mathbb{C}\mathrm{ov}(Z_t, Z_{t+2}) = \gamma_2; \\ \ldots \end{cases}$$

   (a) If possible provide an example of a Markov chain that is not stationary.
   (b) If possible provide an example of a stationary process that is not a Markov chain.

4. (10 points) Find $\mathbb{E}(W_2 W_2 W_3)$ and $\mathbb{E}(W_2 W_2 \mid W_1)$.

5. (10 points) Ded Moroz would like to receive $X_T = S_T^{-2}$ at time T if $S_T > 1$ and nothing otherwise.

   Assume the framework of Black and Scholes model, S_t is the share price, r is the risk free rate, σ is the volatility.

   How much Ded Moroz should pay now at $t = 0$?

6. (20 points) Martingales are everywhere :)

   Consider the process $Y_t = \exp(-uW_t)$.
   (a) Find a multiplier $h(u,t)$ such that $M_t = h(u,t) \cdot Y_t$ is a martingale.
   (b) Find dY_t treating u as a parameter, $\mathbb{E}(Y_t)$ and $\mathbb{V}\mathrm{ar}(Y_t)$.
   (c) Consider M_t that you have found as a function of u. Find the Taylor approximation of the function $M_t(u)$ up to u^4.
   (d) Consider the coefficient before u^4 in the Taylor expansion of $M_t(u)$. Is it a martingale?

7. Bonus point. Guess your exam result (out of 70 possible points).