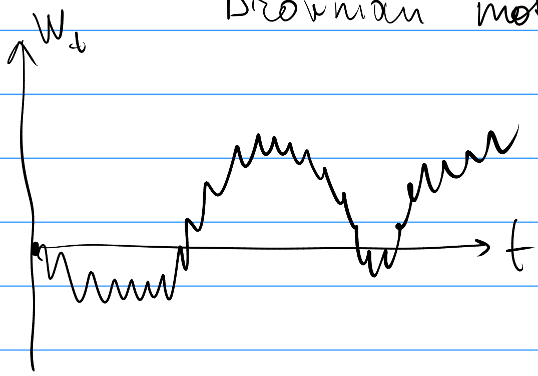
Hi

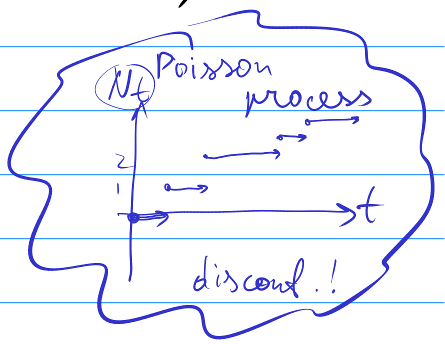
Wiener process.

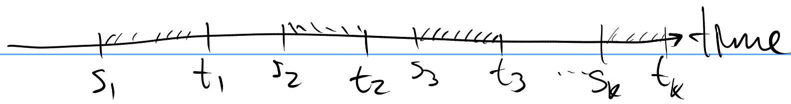
Brownian motion.



Axioms.

1. $W_0 = 0$
2. $W_t - W_s \sim N(0; t-s)$ for $t>s$
3. $\mathbb{P}$(trajectory of (W_t) is continuous) $= 1$
4. future increment is independent of past increments





$\Delta_1 = W(t_1) - W(s_1)$

$\Delta_2 = W(t_2) - W(s_2)$

$\vdots$

Δ_k

If $s_1 \le t_1 \le s_2 \le t_2 \le \ldots$

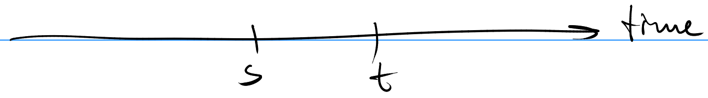
the $\Delta_1, \Delta_2, \Delta_3, \ldots$ are independent RV-s.

Exer.

a) $E(W_t)$, $Var(W_t)$ ☺

b) $Cov(W_t, W_s)$, $E(W_t \cdot W_s)$

c) for $s<t$ $\quad E(W_t \mid W_s) \quad Var(W_t \mid W_s)$



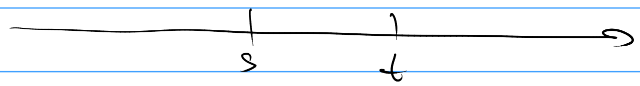
$W_t - W_0 \sim N(0; t-0)$

a) $$E(W_t) = E(\underbrace{W_t - W_0}_{\text{free}}) = 0$$

$$Var(W_t) = Var(W_t - W_0) = t - 0 = t$$

b) $s<t$ $\quad Cov(W_s, W_t) \overset{?}{=} (*)$

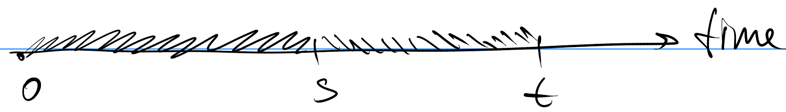
$\boxed{R.V.}$ = predictable part + unpred part



$$\underset{\text{future value}}{W_t} = \underset{\text{past value}}{W_s} + \underset{\text{increment}}{(W_t - W_s)}$$

$$(*) = Cov(W_s, W_s + (W_t - W_s)) = \underline{Var(W_s)} + Cov(W_s, W_t - W_s)$$

$$= s + Cov(\overbrace{W_s - \underset{\overset{\shortparallel}{0}}{W_0}}^{\Delta_L}, \overbrace{W_t - W_s}^{\Delta_R}) = \underline{s} + 0 \quad \left(Cov(W_s, W_t) \text{ if } \underline{s<t} \right)$$

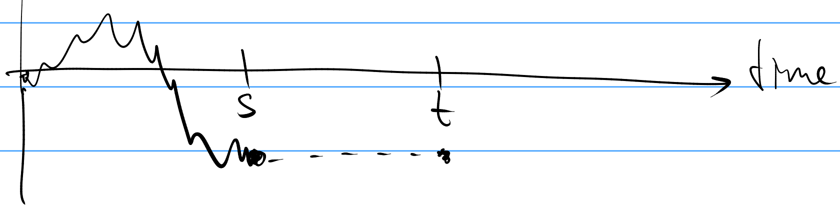


If $\underline{s \geq t}$ then $Cov(W_s, W_t) = \underline{t}$

$$\underline{Cov(W_s, W_t)} \overset{?}{=} \underline{\min(t, s)} = E(W_t \cdot W_s)$$

$$\llcorner E(W_s \cdot W_t) - \underbrace{E(W_s)}_{0} \cdot \underbrace{E(W_t)}_{0} = E(W_t \cdot W_s)$$

c) $E(W_t | W_s)$ if $t \geq s$



$$E(\underbrace{W_t}_{\text{future}} | W_s) = E(\underbrace{W_s + (W_t - W_s)}_{\text{past + increment}} | W_s) =$$

$$= E(W_s | W_s) + E(W_t - W_s | W_s) =$$

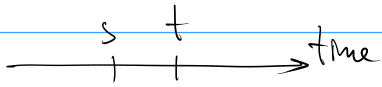
indep

$E(Z|Z) = Z$

$E(Z|I) = E(Z)$

indep

$$= W_s + \underbrace{E(W_t - W_s)}_{0 \text{ (prop 2)}} = W_s + 0 = W_s$$



$$Var(W_t | W_s) = Var(\underbrace{W_s + (W_t - W_s)}_{\text{past + increment}} | W_s) =$$

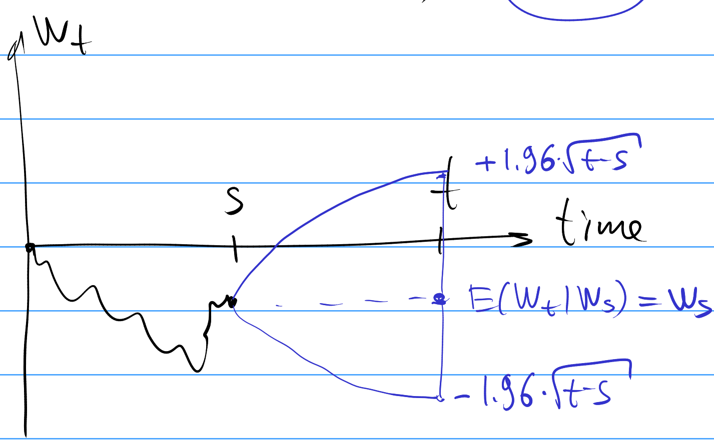
$$= Var(W_s | W_s) + Var(W_t - W_s | W_s) =$$

$$= 0 + Var(W_t - W_s) = t - s \quad !!$$

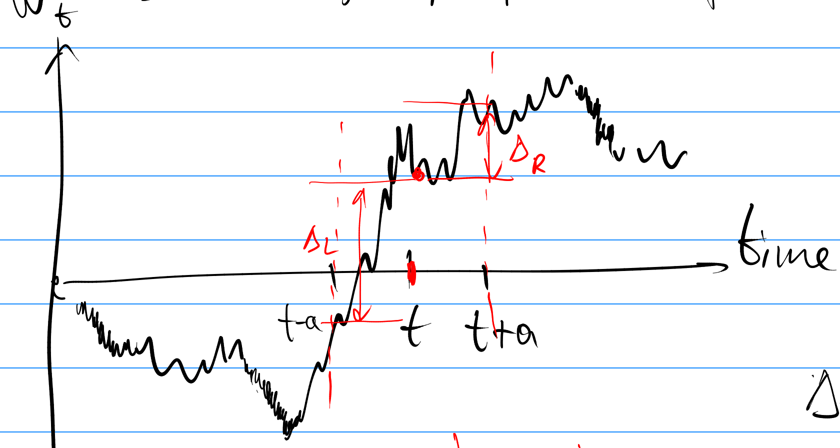
$Var(Z|Z) = 0$

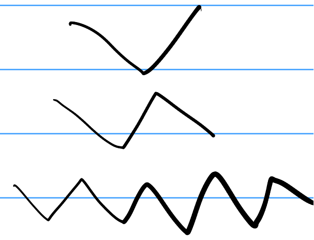
$Var(Z|I) = Var(Z)$

indep

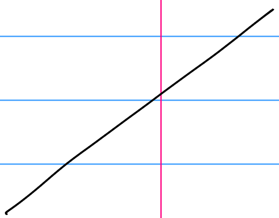


Strange properties of W.P.





Δ_L and Δ_R are indep.

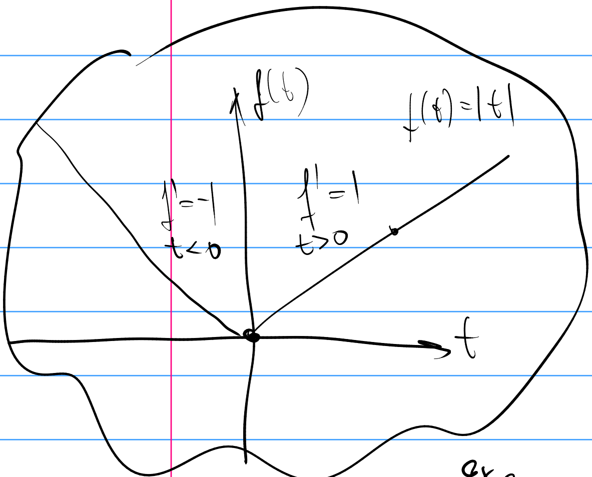


$\Delta_L = W_t - W_{t-a}$

$\Delta_R = W_{t+a} - W_t$

$$P(\Delta_L = \Delta_R) = 0$$

$$\boxed{P\left(\frac{\Delta_L}{a} = \frac{\Delta_R}{a}\right) = 0}$$ strange!



$$\boxed{\frac{dW_t}{dt} \text{ does not exist !!}}$$

exer (W_t) - wiener process

$Y_t = \alpha \cdot W_{t/4}$

α? if (Y_t) is wiener process?

1) $Y_0 = \alpha \cdot W_{0/4} = \alpha \cdot 0 = 0$ ☺

2) $Y_t - Y_s = \alpha \cdot W_{t/4} - \alpha W_{s/4} =$ $\overset{\text{should be}}{\sim} N(0; t-s)$

$= \alpha\underbrace{(W_{t/4} - W_{s/4})}_{\sim N(0;\, \frac{t}{4} - \frac{s}{4})}$

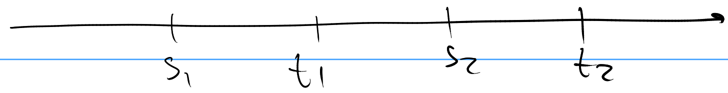
$\alpha = \pm 2$ $\quad Var(\alpha(W_{t/4} - W_{s/4})) = \alpha^2 \cdot \left(\frac{t}{4} - \frac{s}{4}\right) =$

$= 4(t/4 - s/4) = t-s$

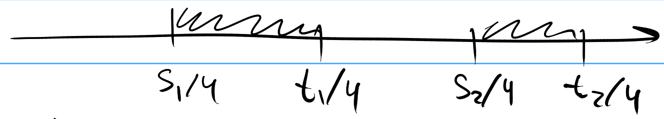
$$Y_t = 2\left(W_{t/4} - W_{s/4}\right) \qquad Y_t = -2\left(W_{t/4} - W_{s/4}\right)$$

3) $P\left((Y_t) \text{ is cont. ous}\right) = 1$ ☺

4)



$$\Delta_1^Y = Y(t_1) - Y(s_1) = 2\cdot\left(\underbrace{W_{t_1/4} - W_{s_1/4}}\right)$$
$$\Delta_2^Y = Y(t_2) - Y(s_2) = 2\cdot\left(\underbrace{W_{t_2/4} - W_{s_2/4}}\right)$$ indep



Δ_1^Y and Δ_2^Y are indep ☺

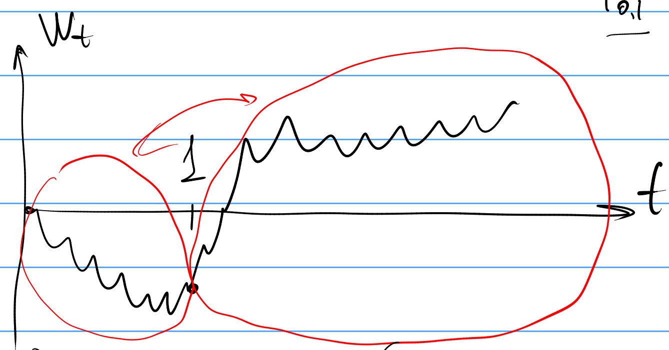
Turn time line backwards!

Inversion

(W_t) – wiener process

def $$Y_t = \begin{cases} 0 & \text{if } t = 0 \\ t\cdot W_{1/t} & \text{for } t > 0 \end{cases}$$

$$Y_1 = W_1$$
$$Y_{10} = 10\cdot W_{0.1}$$
$$Y_{0.1} = 0.1\cdot W_{10}$$



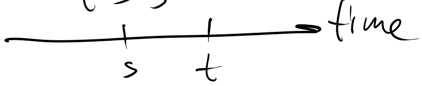
theorem: if (W_t) – is a W.P then (Y_t) is a W.P.

1] check!
2
3] Believe
4

1) $Y_0 = 0$ ☺

2) $Y_t - Y_s = t\cdot W_{1/t} - s\,W_{1/s}$

$t > s$



$$E(Y_t - Y_s) = t\cdot 0 - s\cdot 0 = 0$$

$$\mathrm{Var}(Y_t - Y_s) = \mathrm{Var}(t W_{1/t} - s\cdot W_{1/s}) = t^2 \cdot \boxed{\mathrm{Var}(W_{1/t})} +$$
$$+ s^2 \cdot \mathrm{Var}(W_{1/s}) - 2st\cdot \mathrm{Cov}(W_{1/t}, W_{1/s}) =$$
$$= t^2 \cdot \frac{1}{t} + s^2 \cdot \frac{1}{s} - 2st \cdot \frac{1}{t} = t + s - 2s = t - s$$
$$Y_t - Y_s \sim N(0; t-s)$$

Ex (W_t) - W.P

$$W_3 = 3\cdot Y_{1/3} \qquad W_7 = 7\cdot Y_{1/7}$$

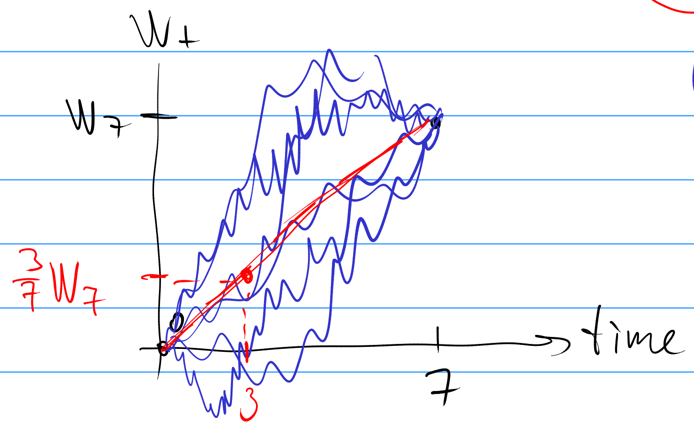
$$E(W_3 \mid W_7) = E(3 Y_{1/3} \mid 7\cdot Y_{1/7}) =$$

will $\quad 1/3 > 1/7$

$s<t$
$E(W_t|W_s) = W_s$

$$= 3\cdot E(Y_{1/3} \mid Y_{1/7}) =$$
$$= 3\cdot Y_{1/7} = 3\cdot \frac{1}{7}\cdot W_7 = \frac{3}{7} W_7$$

$$\sigma(Z) = \sigma(7Z)$$
$$\{Z > 10\} \in \sigma(Z)$$
$$\{7Z > 70\} \in \sigma(7Z)$$
$$E(R|Z) = E(R \mid \sigma(Z))$$



does not influence the info

$$\mathrm{Var}(W_3 \mid W_7) = \mathrm{Var}(3\cdot Y_{1/3} \mid 7 Y_{1/7}) = 3^2 \mathrm{Var}(Y_{1/3} \mid Y_{1/7})$$
$$= 3^2 \cdot \left(\frac{1}{3} - \frac{1}{7}\right) = 3^2 \cdot \frac{7-3}{3\cdot 7} = \frac{4\cdot 3}{7}$$

$t > s$
$\mathrm{Var}(W_t|W_s) = t - s$

def (W_t) is a Wiener process wrt to filtration $(\mathcal{F}_t)$ if
1, 2, 3 (the same)

4 alt for $s<t$ $(W_t - W_s)$ is indep of $\mathcal{F}_s$

s t time

5. At time t you info contains all values up to t.
$$\mathcal{F}_t = \sigma(\{W_u \mid u \in [0;t]\}).$$