

Hi !!

2021-11-26

Ito's integral properties !!

X_t - martingale

martingale

(W_t) - Wiener process

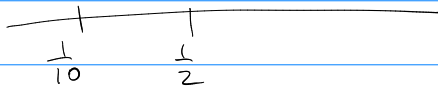
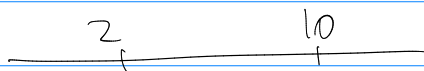
$X_t = t \cdot W(1/t)$ - Wiener process.

$$E(W_2 | W_{10})$$

$$W_{1/t} = \frac{1}{t} \cdot X_t$$

$$W_a = a \cdot X_{1/a}$$

$$W_2 = 2 \cdot X_{1/2}$$



$$\int_0^t 7 \cdot dW_u = -7 \cdot W_0 + 7 \cdot W_t = 7 \cdot W_t$$

$$\int_0^t W_u dW_u = \text{(long limit)} = \frac{W_t^2}{2} - \frac{t}{2}$$

Assumptions.

(A_u) - cont. st. process

$$\int_0^t E(A_u^2) du < \infty$$

① Boring !! Linearity

$$\int_0^t A_u + B_u dW_u = \int_0^t A_u dW_u + \int_0^t B_u dW_u$$

$$\int_0^t \alpha \cdot A_u dW_u = \alpha \int_0^t A_u dW_u$$

$$\int_a^b A_u dW_u + \int_b^c A_u dW_u \stackrel{?}{=} \int_a^c A_u dW_u$$

2. Ito's integral is a martingale

$$I_t = \int_0^t A_u dW_u$$

(W_t) is a Wiener Process
wrt (\mathcal{F}_t)

(I_t) is a martingale wrt (\mathcal{F}_t)

$$E(I_{t+\Delta} | \mathcal{F}_t) = I_t$$

$$E(I_{t+\Delta} | I_t) = I_t$$

$$E(LHS) = E(RHS)$$

$$E(I_{t+\Delta} - I_t | I_t) = 0$$

$$\forall t, \Delta$$

$$E(I_{t+\Delta}) = E(I_t) = \text{const}$$

$$\int_0^0 A_u dW_u = 0$$

$$E(I_t) = 0$$

$$I_0 = 0 \quad E(I_0) = 0$$

(2+) comparison with $\int_0^t B_u du$

$$E\left(\int_0^t B_u du\right) = \int_0^t E(B_u) du$$

$R_t = \int_0^t B_u du$ is never a martingale
(if B_u is non-zero)

3. Var, Covariance

Ito's isometry

$$\text{Var}\left(\int_0^t A_u dW_u\right) = \int_0^t E(A_u^2) du$$

$$\text{Cov}\left(\int_0^t A_u dW_u, \int_0^t B_u dW_u\right) = \int_0^t E(A_u B_u) du$$

Shreve
Skeels

Ex $\text{Cov}\left(\int_0^t W_u^2 dW_u, W_t\right) = ?$

$\text{Cov}\left(\int_0^t W_u^2 dW_u, \int_0^t 1 \cdot dW_u\right) = \int_0^t \mathbb{E}(W_u^2 \cdot 1) du \stackrel{?}{=}$

$\mathbb{E}(W_u^2) = \text{Var}(W_u) = u$

$\Rightarrow \int_0^t u du = \frac{t^2}{2}$

$t > s$
 $W_t - W_s \sim N(0, t-s)$

Ex $\mathbb{E}\left(42 + \underbrace{\int_0^t \cos W_u \cdot dW_u}_0 + \int_0^t u^2 \cdot W_u^2 du\right) = ?$

$= 42 + 0 + \int_0^t \mathbb{E}(u^2 W_u^2) du = 42 + \int_0^t u^3 du = 42 + \frac{t^4}{4}$

$\mathbb{E}(u^2 W_u^2) = u^2 \cdot \mathbb{E}(W_u^2) = u^2 \cdot u = u^3$

Ito's lemma

→ baby-version
 → st. version
 → multiv-fc extension
 short notation

$\mathbb{E}(X_t)$
 $\text{Var}(X_t)$ → number

short notation
 $dx_t = A_t dW_t + B_t dt$

full notation
 $X_t = X_0 + \int_0^t A_u dW_u + \int_0^t B_u du$

R.V. Const R.V. R.V.

Agreement

does not exist
 dW_t is not a R.V.

a way to write less

X_t - Ito's process if it can be written in this form

Ito's lemma.

body-version

If $X_t = h(W_t, t)$

h_{ww}'' , h_t' are continuous

then [short]

$$dX_t = h_w' \cdot dW_t + h_t' \cdot dt + \frac{1}{2} h_{ww}'' \cdot dt$$

[full]

$$X_t = X_0 + \int_0^t h_w' dW_u + \int_0^t h_t' du + \frac{1}{2} \int_0^t h_{ww}'' du$$

Ex

a) $X_t = W_t^2 \cdot t$

dX_t ? full form?

b) $X_t = W_t^4$

dX_t ? full form? $E(X_t)$?

c) $X_t = W_t^2$

dX_t ? full form?

a) $dX_t = 2W_t \cdot t \cdot dW_t + W_t^2 dt + \frac{1}{2} \cdot 2 \cdot t \cdot dt$

$$\boxed{W_t^2 \cdot t = \underbrace{W_0^2 \cdot 0}_0 + \int_0^t 2W_u \cdot u \cdot dW_u + \int_0^t W_u^2 \cdot du + \int_0^t u \cdot du}$$

b) $dX_t = 4W_t^3 \cdot dW_t + 0 \cdot dt + \frac{1}{2} \cdot 4 \cdot 3 \cdot W_t^2 \cdot dt$

$$W_t^4 = W_0^4 + \int_0^t 4W_u^3 dW_u + 6 \cdot \int_0^t W_u^2 du$$

$$E(W_t^4) = 0 + 0 + 6 \int_0^t \underbrace{E(W_u^2)}_u du =$$

$$= 6 \cdot \int_0^t u du = 6 \cdot \frac{t^2}{2} = 3t^2$$

0/1

c) $X_t = W_t^2$

$$W_t^2 = W_0^2 + \int_0^t 2W_u dW_u + \int_0^t 1 \cdot du$$

$$\int_0^t W_u dW_u = \frac{W_t^2 - t}{2}$$



$$dX_t = \underbrace{2 \cdot W_t}_{h_w'} \cdot dW_t + \underbrace{0}_{h_t'} \cdot dt + \frac{1}{2} \cdot \underbrace{2}_{h_{ww}''} \cdot dt$$

Ex Find α such that $M_t = \exp(W_t + \alpha t)$ is a martingale

$$dM_t = \underbrace{\exp(W_t + \alpha t)}_{h'_t} \cdot \underbrace{dW_t}_{h'_t} + \underbrace{\alpha \cdot \exp(W_t + \alpha t)}_{h'_t} \cdot \underbrace{dt}_{h''_t} + \underbrace{\frac{1}{2} \exp(W_t + \alpha t)}_{h''_t} \cdot \underbrace{dt}_{h''_t}$$

$$M_t = \underbrace{M_0}_{\substack{\uparrow \\ 1 = \exp(0 + \alpha \cdot 0)}} + \underbrace{\int_0^t \exp(W_u + \alpha u) dW_u}_{\substack{\uparrow \text{ is a martingale}}} + \underbrace{\int_0^t (\alpha + \frac{1}{2}) \cdot \exp(W_u + \alpha u) \cdot du}_{\substack{\uparrow \text{ is never} \\ \text{a martingale} \\ \text{except one case}}}$$

$$\alpha = -\frac{1}{2}$$

$$M_t = 1 + \int_0^t \exp(W_u - \frac{u}{2}) dW_u + 0$$

M_t is a martingale if $\alpha = -\frac{1}{2}$

$$\begin{aligned} E(M_{t+\Delta} | \mathcal{F}_t) &= E\left(1 + \underbrace{\int_0^{t+\Delta} \dots dW_u}_{\text{martingale}} \middle| \mathcal{F}_t\right) = \\ &= 1 + \int_0^t \dots dW_u = M_t \end{aligned}$$

Ito's lemma

If $X_t = h(Y_t, t)$ where Y_t is an Ito's process.
 $dY_t = A_t dW_t + B_t dt$ h_{YY}, h_t' are cont-ous f-ns

then (short form) $dX_t = h_Y' dY_t + h_t' dt + \frac{1}{2} \cdot h_{YY}'' \cdot A_t^2 \cdot dt$

(full form) $X_t = X_0 + \int_0^t h_Y' dY_u + \int_0^t h_t' du + \frac{1}{2} \int_0^t h_{YY}'' \cdot A_u^2 \cdot du$

(algorithm to remember)

to obtain dX_t :

① write dX_t using second order Taylor expansion

② simplify it using the rules:

$dW_t \cdot dW_t = dt$

$dW_t \cdot dt = 0$

$dt \cdot dt = 0$

$dt \cdot \text{anything} = 0$

Ex $X_t = Y_t^3 + t^2$ where $dY_t = W_t dW_t + W_t^2 dt$

$dX_t = 3Y_t^2 \cdot dY_t + 2t \cdot dt +$

$+ \frac{1}{2} \left[6Y_t \cdot (dY_t)^2 + 2 \cdot 0 \cdot dt \cdot dY_t + 2 \cdot (dt)^2 \right] =$

$(W_t dW_t + W_t^2 dt)^2 = W_t^2 (dW_t)^2 + 2 \cdot W_t \cdot W_t^2 dW_t dt + W_t^4 (dt)^2 = W_t^2 dt$

$dX_t = 3Y_t^2 (W_t dW_t + W_t^2 dt) + 2t dt + 3Y_t \cdot W_t^2 dt$
 $= \text{something} \cdot dW_t + \text{something else} \cdot dt$

(Ex)

$$dY_t = 5W_t^3 dW_t + 6W_t dt.$$

$$X_t = Y_t^5$$

$$dX_t? = 5Y_t^4 \cdot dY_t +$$

$$+ \frac{1}{2} 20 \cdot Y_t^3 \cdot (dY_t)^2 =$$

$$= 5Y_t^4 (5W_t^3 dW_t + 6W_t dt) +$$

$$+ 10Y_t^3 \cdot (5W_t^3 dW_t + 6W_t dt)^2 =$$

$$\begin{aligned} & \boxed{a^2 + 2ab + b^2} \\ & \downarrow \quad \quad \quad \downarrow \\ & 25W_t^6 dt \quad \quad \quad 0 \quad \quad \quad 0 \end{aligned}$$

Step 1. Get 2nd order
Taylor exp-n
for dX_t

Step 2. simplify
 $dt \cdot \text{anything} = 0$
 $dW \cdot dW = dt$

$$\begin{aligned} & 25Y_t^4 \cdot W_t^3 dW_t + \\ & 30Y_t^4 \cdot W_t \cdot dt + \\ & + 250Y_t^3 \cdot W_t^6 dt \end{aligned}$$