

# Time Series

Peter Lukianchenko

17 January 2022

# ARMA models

# Autoregressive Moving-Average Models

$\{\varepsilon_t\}$  is a white noise

- ARMA(1, 1)

$$Y_t = \theta Y_{t-1} + \varepsilon_t + \varphi \varepsilon_{t-1}$$

- ARMA(**p**, **q**)

$$Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots \theta_p Y_{t-p} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \cdots + \varphi_q \varepsilon_{t-q}$$

- More general ARMA(p, q)

$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots \theta_p Y_{t-p} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \cdots + \varphi_q \varepsilon_{t-q}$$

# Properties of ARMA(p,q) models

# White Noise

- **Stationarity**

- ›  $E[\varepsilon_t] = 0$  for all  $t$
- ›  $\text{Var}(\varepsilon_t) = \sigma^2$  for all  $t$
- ›  $\text{Cov}(\varepsilon_t, \varepsilon_{t+j}) = 0$  for all  $t$  and  $j \neq 0$

- **Autocovariances**

- ›  $\gamma(0) = \sigma^2$
- ✓ ›  $\gamma(k) = 0$  for all  $k \neq 0$

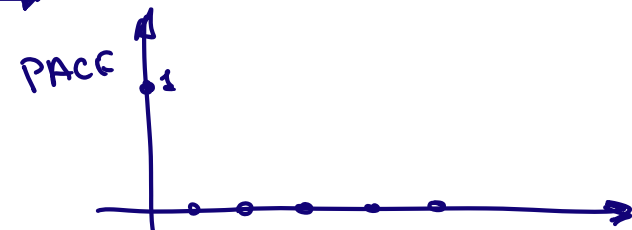
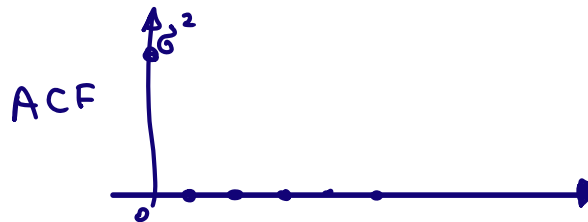
- **Autocorrelation**

- ›  $\rho(0) = 1$
- ›  $\rho(k) = 0$  for all  $k \neq 0$

- **PACF**

- ›  $\alpha(0) = 1$
- ›  $\alpha(k) = 0$  for all  $k > 0$

$$\gamma(k) = \text{Cov}(y_t, y_{t-k}) : \text{cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$$



$$\text{MA}(1): Y_t = \varepsilon_t + \varphi \varepsilon_{t-1}$$

$$\varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$$

### Stationarity

$$\triangleright E[Y_t] = E[\varepsilon_t] + \varphi E[\varepsilon_{t-1}] \stackrel{=0}{=} 0 \text{ for all } t$$

$$\triangleright \text{Var}(Y_t) = \text{Var}(\varepsilon_t) + \varphi^2 \text{Var}(\varepsilon_{t-1}) + 2\varphi \text{Cov}(\varepsilon_t, \varepsilon_{t-1}) = \underline{(1 + \varphi^2)\sigma^2} \text{ for all } t$$

$$\triangleright \text{Cov}(Y_t, Y_{t+1}) = \underline{\varphi\sigma^2} \text{ for all } t. \text{Cov}(Y_t, Y_{t+k}) = 0, \text{ for all } |k| > 1$$

$$\begin{aligned} \text{Cov}(Y_t, Y_{t+1}) &= \text{Cov}(\varepsilon_t + \varphi \varepsilon_{t-1}, \varepsilon_{t+1} + \varphi \varepsilon_t) \\ &= \text{Cov}(\varepsilon_t, \varepsilon_{t+1}) + \varphi \text{Cov}(\varepsilon_{t-1}, \varepsilon_{t+1}) + \text{Cov}(\varepsilon_t, \varepsilon_t) \cdot \varphi + \varphi^2 \text{Cov}(\varepsilon_{t-1}, \varepsilon_t) \\ &\stackrel{=0}{=} 0 + 0 + \text{Var}(\varepsilon_t) \cdot \varphi + 0 = \varphi \sigma^2 \end{aligned}$$

### Autocovariances

$$\triangleright \gamma(0) = \text{Var}(Y_t) = \sigma^2(1 + \varphi^2), \gamma(1) = \text{Cov}(Y_t, Y_{t+1}) = \varphi\sigma^2$$

$$\triangleright \gamma(k) = 0 \text{ for all } |k| > 1$$

$$\gamma(2) = \text{Cov}(Y_t, Y_{t+2})$$

### ACF

$$\triangleright \rho(0) = 1, \rho(1) = \frac{\varphi}{1 + \varphi^2}$$

$$\triangleright \rho(k) = 0 \text{ for all } |k| > 1$$

### PACF

$$\triangleright \text{complicated, but does not become 0 at some lag}$$

$$\text{MA}(q): Y_t = \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \dots + \varphi_q \varepsilon_{t-q}$$

- **Stationarity**

- › automatically follows from stationarity of  $\{\varepsilon_t\}$

- **Autocovariances**

- ›  $\gamma(0) = \text{Var}(Y_t) = \sigma^2(1 + \varphi_1^2 + \dots + \varphi_q^2),$
- ›  $\gamma(k) = \sigma^2(\varphi_k + \varphi_{k+1}\varphi_1 + \varphi_{k+2}\varphi_2 + \dots + \varphi_q\varphi_{q-k})$  for  $k = 1, \dots, q$
- ›  $\gamma(k) = 0$  for  $|k| > q$

# AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

- Plug in the expression for  $Y_{t-1}$ ,  $Y_{t-2}$ , and so on:

$$Y_t = \theta Y_{t-1} + \varepsilon_t$$

$$Y_{t-1} = \theta Y_{t-2} + \varepsilon_{t-1} \longrightarrow Y_{t-2} = \theta Y_{t-3} + \varepsilon_{t-2}$$

$$Y_t = \theta(\theta Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \theta^2 Y_{t-2} + \theta \varepsilon_{t-1} + \varepsilon_t$$

$$Y_t = \theta^n Y_{t-n} + \sum_{j=0}^{n-1} \theta^j \varepsilon_{t-j}$$

if  $n=t$

$$Y_t = \theta^t Y_0 + \sum_{j=0}^{t-1} \theta^j \varepsilon_{t-j}$$

- If  $|\theta| \geq 1$ , as  $n \rightarrow \infty$ ,  $\theta^n \rightarrow \infty$ , and  $Y_t$  explodes.

- So we need  **$|\theta| < 1$  for stationarity.**

$$\hookrightarrow \theta^t \xrightarrow{t \rightarrow \infty} 0$$

$$E[Y_t] = E[\theta^t Y_0] + \sum_{j=0}^{t-1} \theta^j E[\varepsilon_{t-j}]$$

$$\begin{aligned} \text{Var } Y_t &= \text{Var}(\theta^t Y_0) + \sum_{j=0}^{t-1} \theta^{2j} \text{Var}(\varepsilon_{t-j}) + \sum_{j=0}^{t-1} \sum_{i:j \neq i} \theta^j \theta^i \text{Cov}(\varepsilon_{t-j}, \varepsilon_{t-i}) \\ &= \sum_{j=0}^{t-1} \theta^{2j} \sigma^2 \end{aligned}$$



# AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

- **Stationarity:** stationary if  $|\theta| < 1$ . Then

- ›  $E[Y_t] = 0$  for all  $t$

- ›  $\text{Var}(Y_t) = \theta^2 \text{Var}(Y_{t-1}) + \text{Var}(\varepsilon_t) = \frac{\sigma^2}{1-\theta^2}$  for all  $t$

- ›  $\text{Cov}(Y_t, Y_{t-k}) = \theta^k \frac{\sigma^2}{1-\theta^2}$  for all  $t$ , for all  $k$

$$V(y) = \theta^2 V(y) + \sigma^2$$

$$V(y) \cdot (1 - \theta^2) = \sigma^2$$

$$V(y) = \frac{\sigma^2}{1 - \theta^2}$$

- **Autocovariances**

- ›  $\gamma(k) = \theta^k \frac{\sigma^2}{1-\theta^2}$  for all  $k$



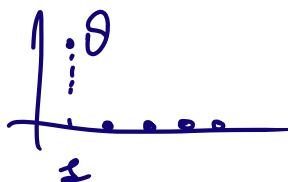
- **ACF**

- ›  $\rho(k) = \theta^k$  for all  $k$

- **PACF**

- ›  $\alpha(1) = \theta$

- ›  $\alpha(k) = 0$  for all  $|k| > 1$



$$\text{AR}(1): Y_t = \theta Y_{t-1} + \varepsilon_t$$

$$Y_{t-1} = L \cdot Y_t \Rightarrow Y_t = \theta \cdot L \cdot Y_t + \varepsilon_t$$

$$Y_t \cdot (1 - \theta \cdot L) = \varepsilon_t$$

- Can be derived in a different way:  $(1 - \theta L)Y_t = \varepsilon_t$ , so if  $(1 - \theta L)$  has an inverse,  $Y_t$  can be written as

$$Y_t = (1 - \theta L)^{-1} \varepsilon_t = \sum_{j=0}^{\infty} \theta^j L^j \varepsilon_t \quad \frac{1}{1 - \theta L} = (1 - \theta L)^{-1} = 1 + \theta L + \frac{(\theta L)^2}{2!} + \frac{(\theta L)^3}{3!} + \dots$$

$\sum \theta^j \cdot \varepsilon_t$

- So it is covariance-stationary, if  $\sum_{j=0}^{\infty} |\theta^j| < \infty$ , i.e., whenever  $|\theta| < 1$ .
- Now,  $\text{Cov}(\varepsilon_t, Y_{t-1}) = \sum_{j=0}^{\infty} \theta^j \text{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ , if  $\text{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$  for all  $j > 0$ . So, if  $\{\varepsilon_t\}$  is a white noise, it holds.
- Also,  $E[\varepsilon_t | Y_{t-1}] = E[\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \dots]$ , so if  $\{\varepsilon_t\}$  is an MDS, the regression assumption is satisfied.

Taylor's Series  $(1+x)^n \sim 1 + n \cdot x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$

# AR(1)

- Can be derived in a different way:  $(1 - \theta L)Y_t = \varepsilon_t$ , so if  $(1 - \theta L)$  has an inverse,  $Y_t$  can be written as

$$Y_t = (1 - \theta L)^{-1} \varepsilon_t = \sum_{j=0}^{\infty} \theta^j L^j \varepsilon_t$$

- So it is covariance-stationary and ergodic, if  $\sum_{j=0}^{\infty} |\theta^j| < \infty$ , i.e., whenever  $|\theta| < 1$ .
- Now,  $\text{Cov}(\varepsilon_t, Y_{t-1}) = \sum_{j=1}^{\infty} \theta^j \text{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ , if  $\text{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$  for all  $j > 0$ . So, if  $\{\varepsilon_t\}$  is a white noise, it holds.
- Also,  $E[\varepsilon_t | Y_{t-1}] = E[\varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2}, \dots]$ , so if  $\{\varepsilon_t\}$  is an MDS, the regression assumption is satisfied.

# AR(p)

$$y_t = \theta_1 y_{t-1} + \theta_2 y_{t-2} + \dots + \theta_p y_{t-p} + \varepsilon_t$$

**Stationarity:**  $y_t \cdot [1 - \theta_1 L - \theta_2 L^2 - \dots - \theta_p L^p] = \varepsilon_t \Rightarrow y_t = \frac{\varepsilon_t}{[ \dots ]}$

- AR(p) process is stationary, if  $\Theta(L) = 1 - \theta_1 L - \dots - \theta_p L^p$  can be inverted.
- Holds, if the roots of the (characteristic) polynomial  $1 - \theta_1 x - \theta_2 x^2 \dots - \theta_p x^p$  lie *outside* the unit circle.
- AR(1):  $1 - \theta x = 0 \Rightarrow |x| = 1/|\theta| > 1$ , if  $|\theta| < 1$ .
- Equivalent formulation: the process is stationary if the roots of the **inverse characteristic** polynomial  $\lambda^p - \theta_1 \lambda^{p-1} - \dots - \theta_{p-1} \lambda - \theta_p$  lie **inside** the unit circle
- AR(1):  $\lambda - \theta = 0 \Rightarrow |\lambda| = |\theta| < 1$ .
- **Necessary condition:** the coefficients of  $\Theta(L)$  should add up to less than 1, i.e.  $\sum_{j=1}^p \theta_j < 1$ . *AR(2):  $y_t = 1.2 \cdot y_{t-1} - 0.4 \cdot y_{t-2} + \varepsilon_t$*
- **Sufficient condition:** the absolute values of coefficients of  $\Theta(L)$  should add up to less than 1, i.e.  $\sum_{j=1}^p |\theta_j| < 1$ .

# AR(p)

- **Stationarity:**

- AR(p) process is stationary, if the roots of the (characteristic) polynomial  $1 - \theta_1 x - \theta_2 x^2 \dots - \theta_p x^p$  lie *outside* the unit circle.

- **ACF:** can be computed recursively (***Yule-Walker equations***): for  $k = 1, 2, \dots$

$$\rho(k) = \theta_1 \rho(k-1) + \dots + \theta_p \rho(k-p)$$

- **PACF:** First  $p$   $\alpha(k)$  are (in general) nonzero, and  $\alpha(k) = 0$ , for  $|k| > p$ .

# ARMA(p,q)

- Can be written as

$$Y_t = \Psi(L) \varepsilon_t,$$

if  $\Theta(L)$  is invertible (i.e., has inverse), where  $\Psi(L) = \Theta(L)^{-1}\Phi(L)$ ,

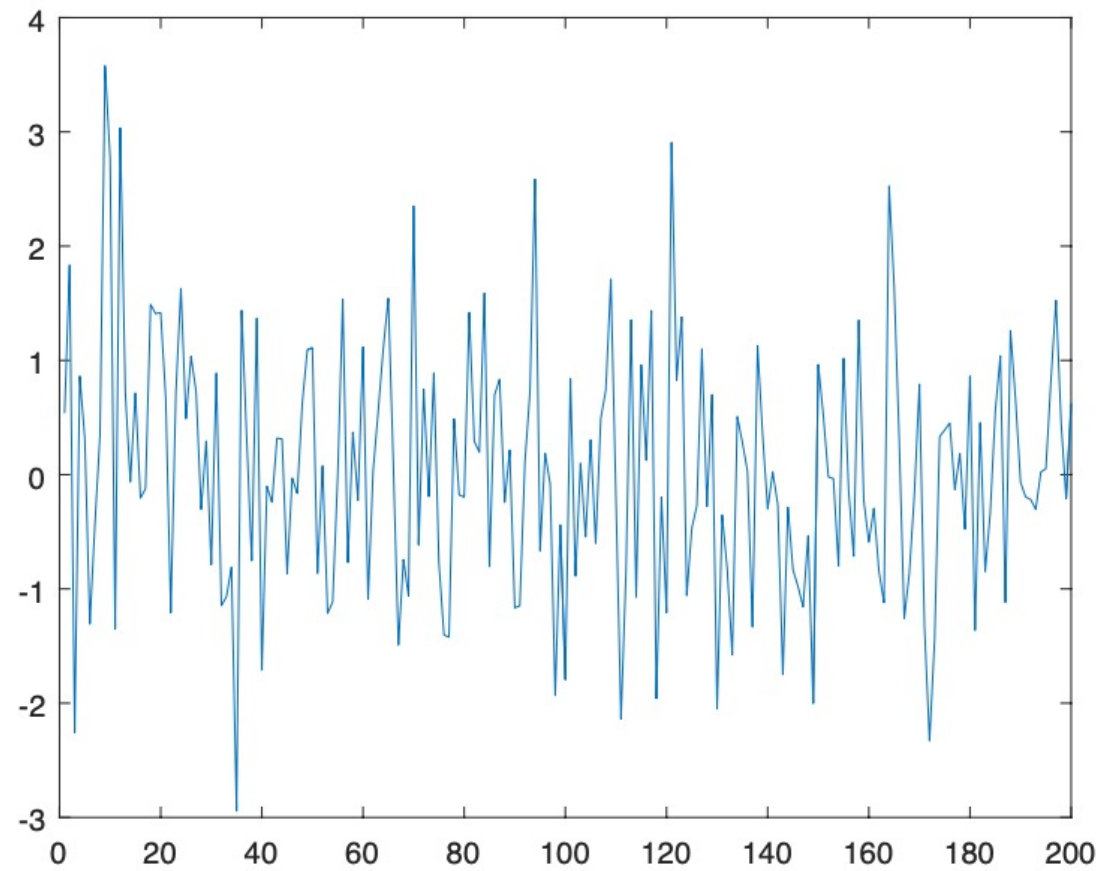
$\Theta(L) = 1 - \theta_1 L - \dots - \theta_p L^p$  and  $\Phi(L) = 1 + \varphi_1 L - \dots + \varphi_q L^q$ .

- ARMA(p,q) process is **stationary**, **if and only if** the lag polynomial corresponding to the AR part is invertible.
- Stationary ARMA(p,q) can be written as MA( $\infty$ ).
- ACF and PACF: combination of ACFs and PACFs for AR(p) and MA(q) (none is zero after a certain lag, but decays exponentially fast)

# ARMA(p,q)

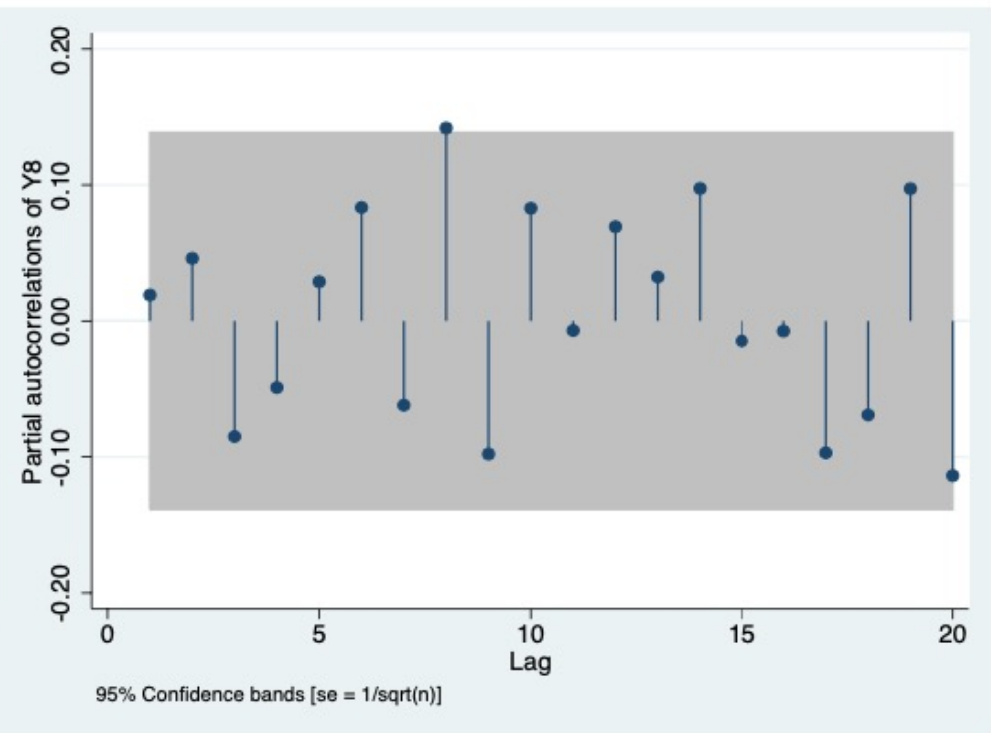
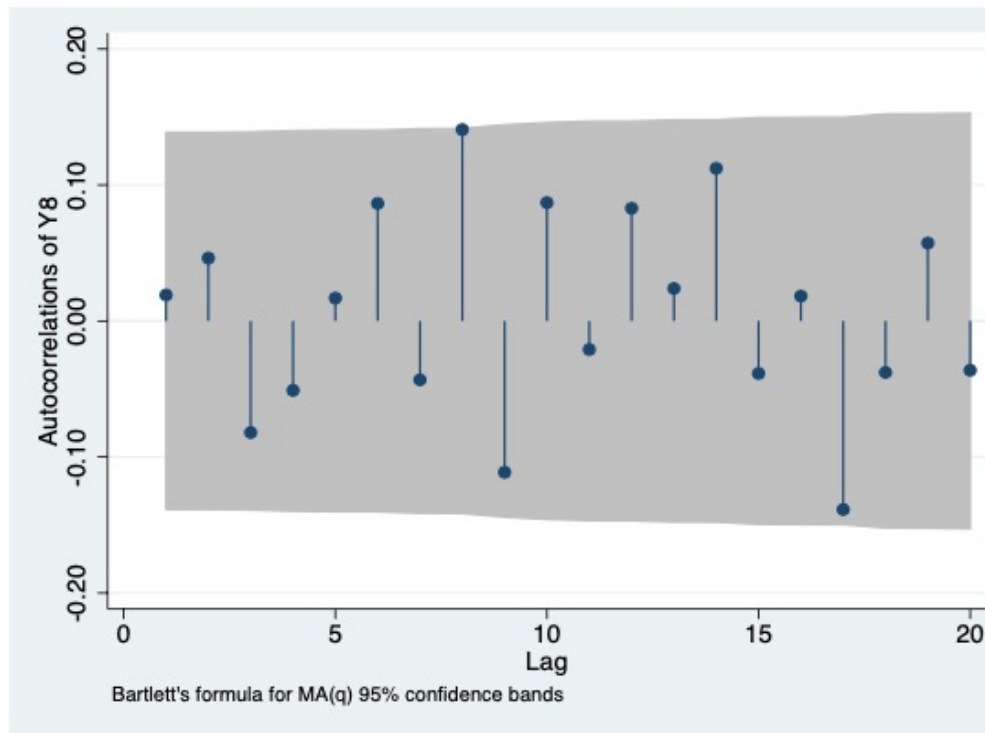
Process	ACF	PACF
WN	$\rho(k) = 0$	$\alpha(k) = 0$
AR(1)	$\rho(k) = \theta^k$	$\alpha(1) = \theta, \alpha(k) = 0$ for $k > 1$
AR(p)	Exponentially decays to 0, may oscillate	First $p$ are non-zero; $\alpha(k) = 0$ , for $k > p$
MA(1)	$\rho(1) = \varphi, \rho(k) = 0$ for $k > 1$	Exp. decays to 0, may oscillate; $\text{sign}(\alpha(1)) = \text{sign}(\varphi)$
MA(q)	First $q$ $\rho(k)$ are non-zero, $\rho(k) = 0$ , for $k > q$	Exp. decays to 0, may oscillate
ARMA(1,1)	$\text{sign}(\rho(1)) = \text{sign}(\theta + \varphi)$ ; exp. decays (oscillating if $\theta < 0$ )	$\alpha(1) = \rho(1)$ ; exp. decays (oscillating if $\theta > 0$ )
ARMA(p,q)	Starts exp. decaying (may oscillate) at lag $q$	Starts exp. decaying (may oscillate) at lag $p$

# What is this process?



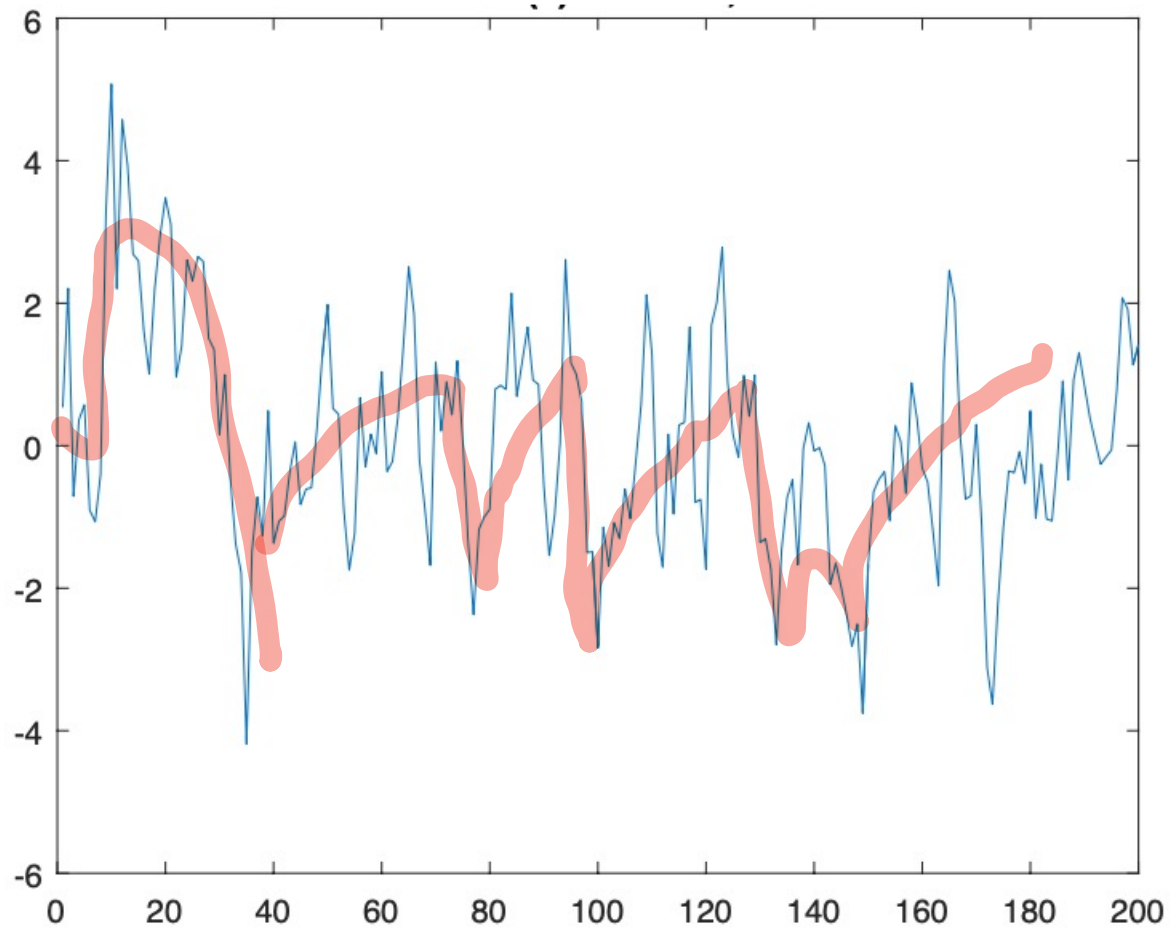


# What is this process?



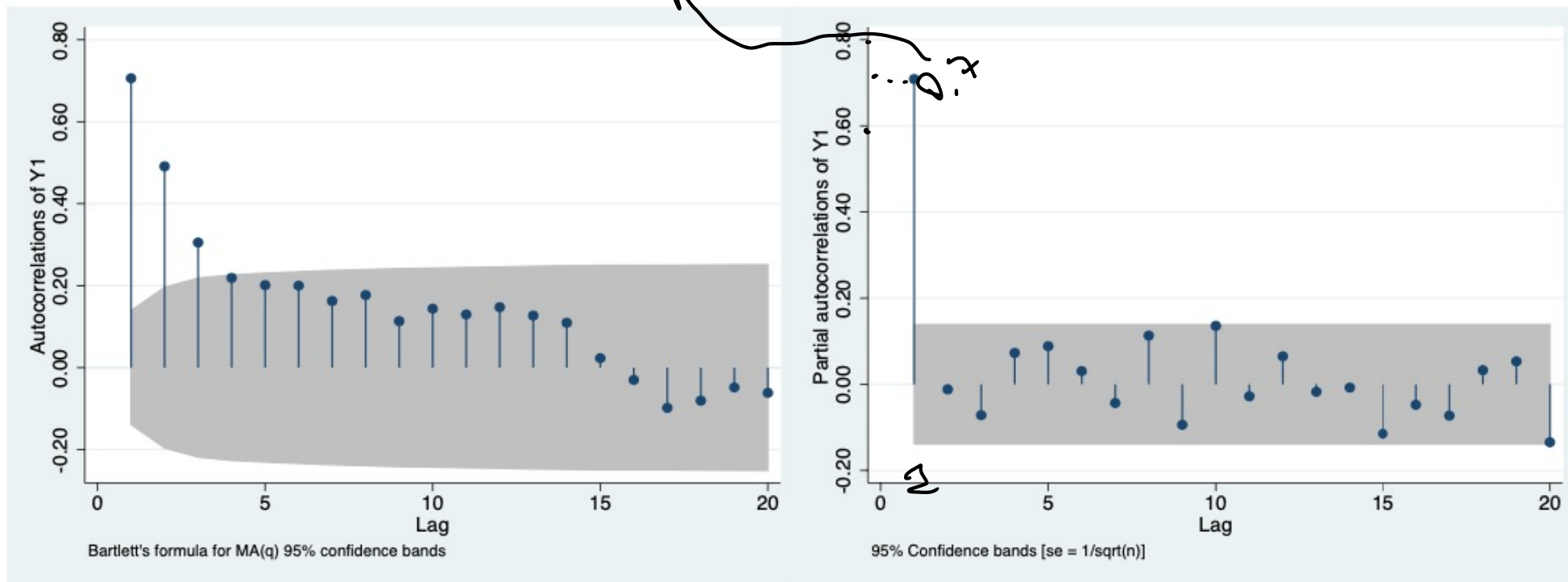
White noise

# What is this process?



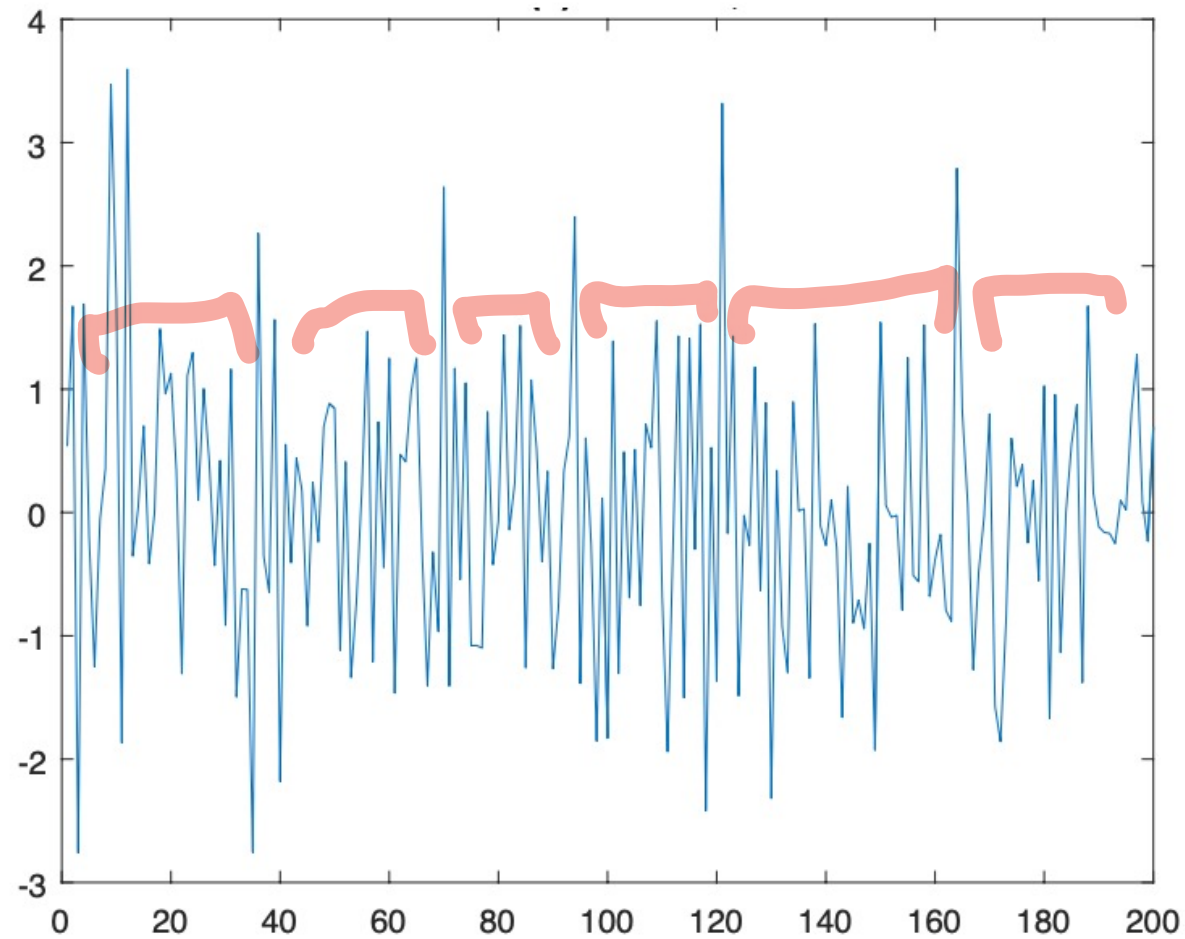
# What is this process?

$$y_t = 0.7 \cdot y_{t-1} + \varepsilon_t$$

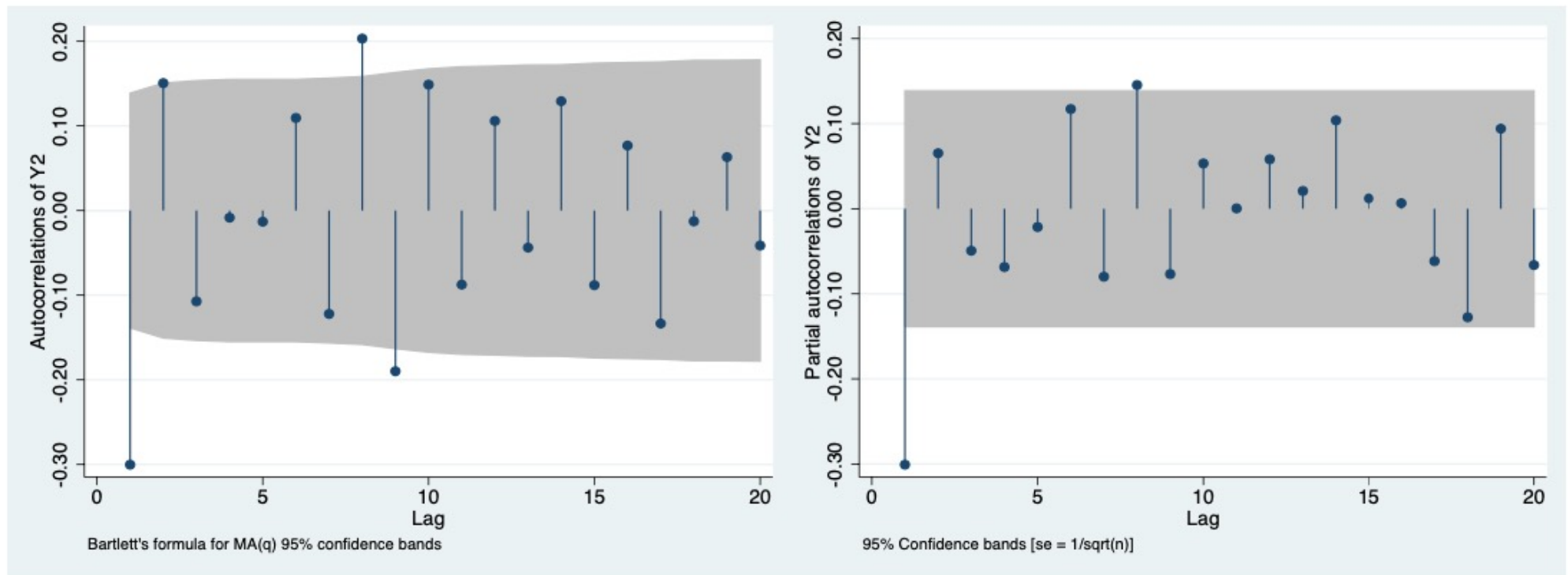


AR(1) with  $\theta_1 = 0.7$

# What is this process?

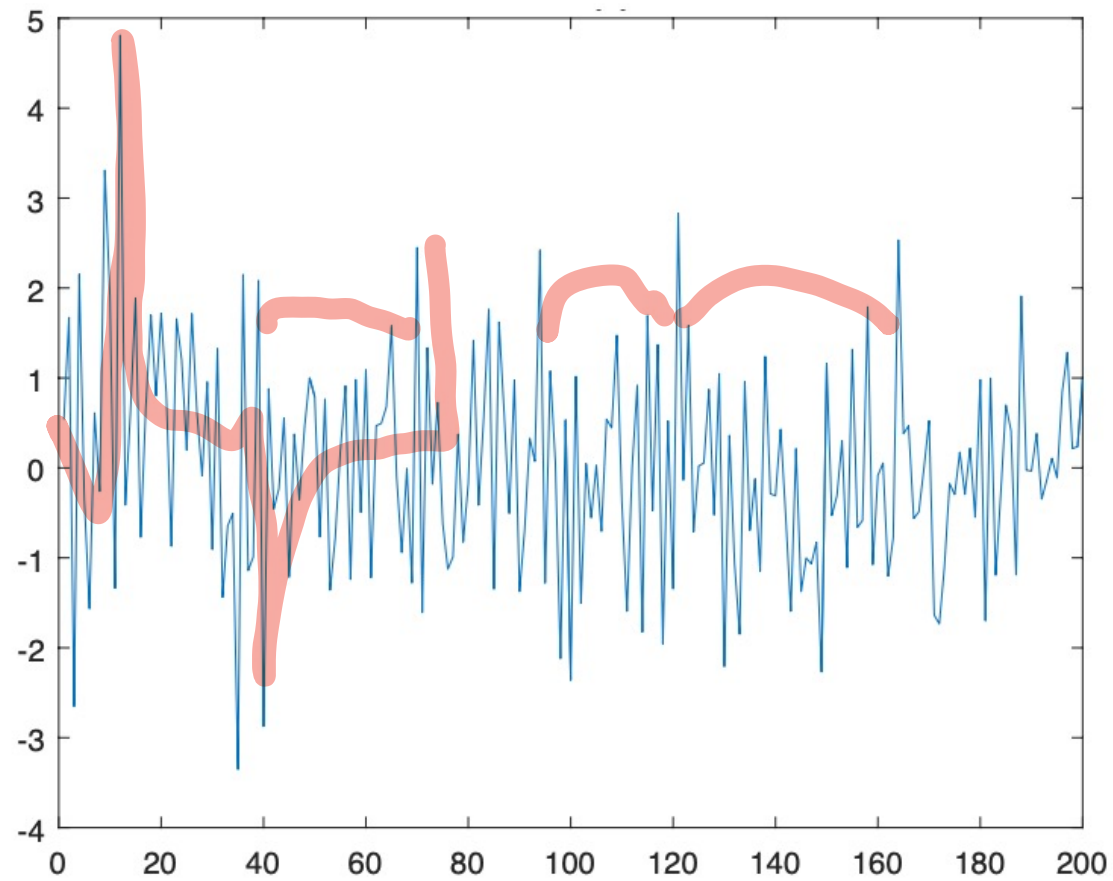


# What is this process?

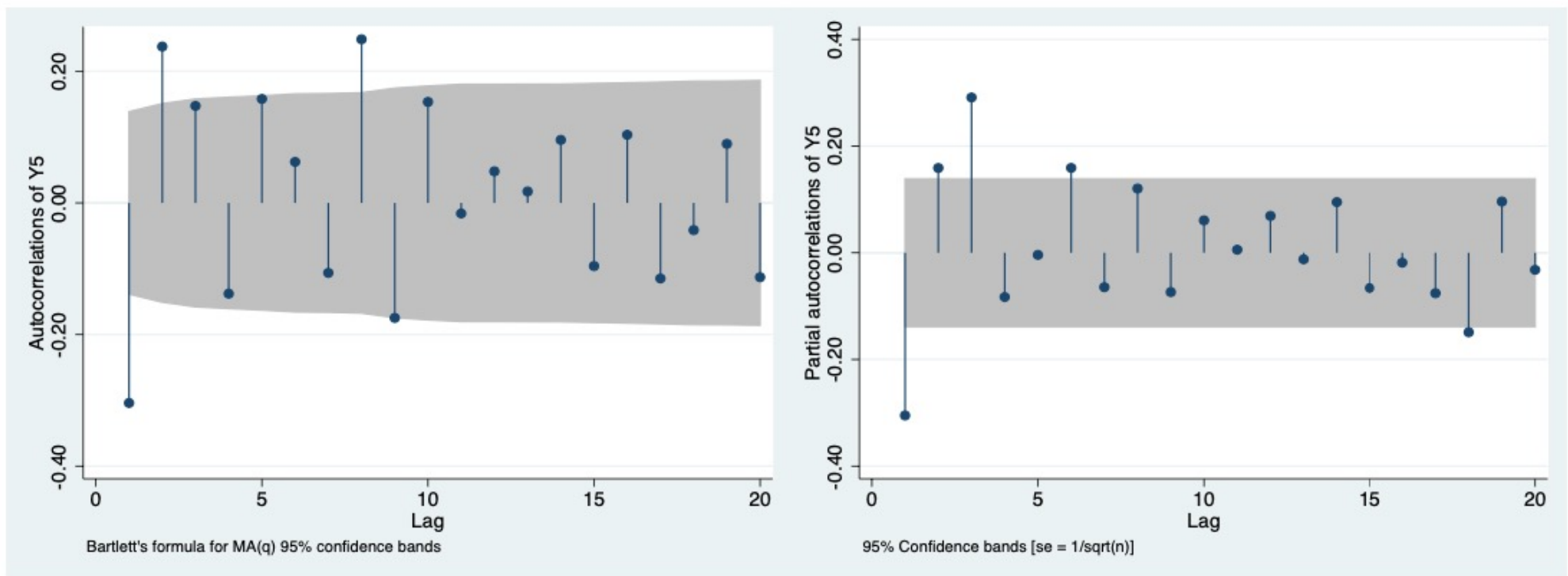


AR(1) with  $\theta_1 = -0.3$

# What is this process?

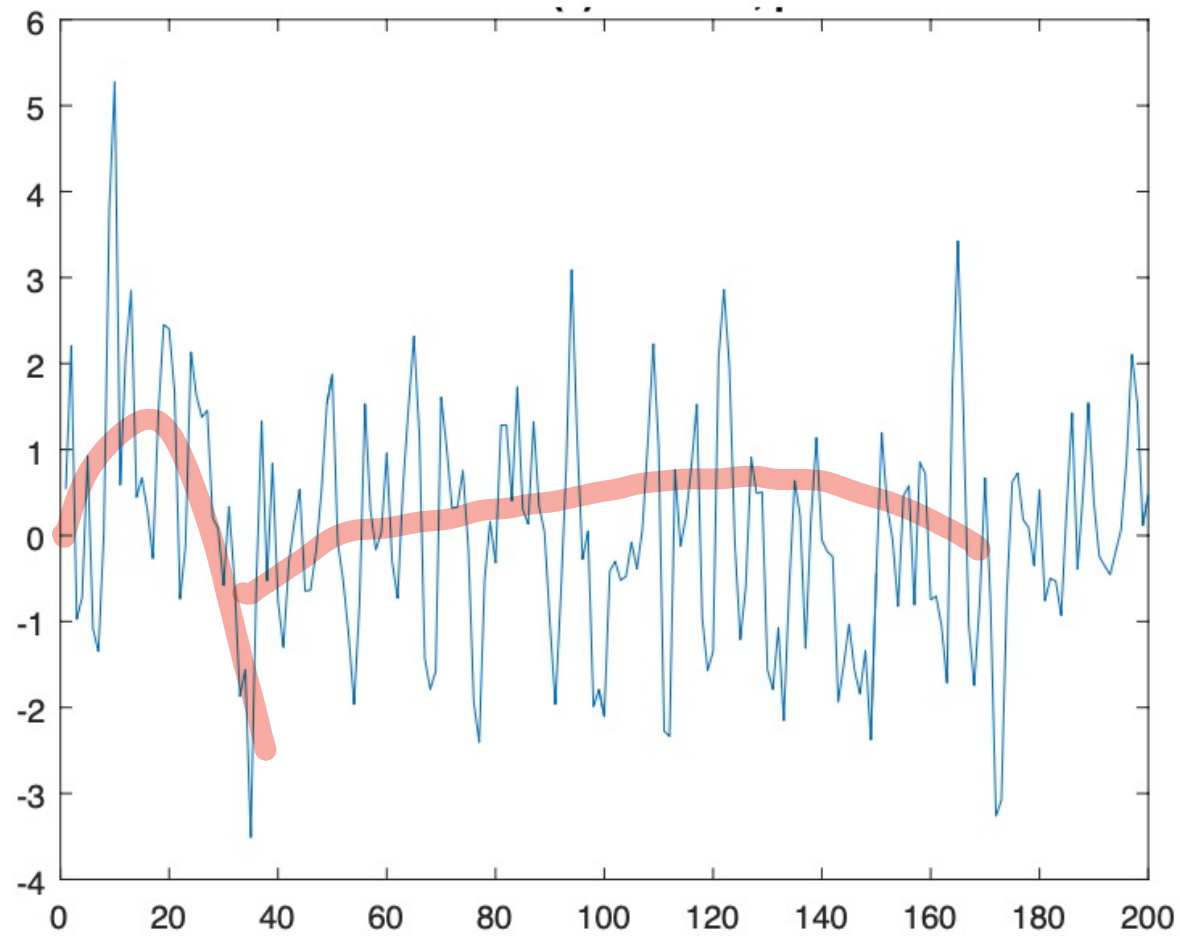


# What is this process?



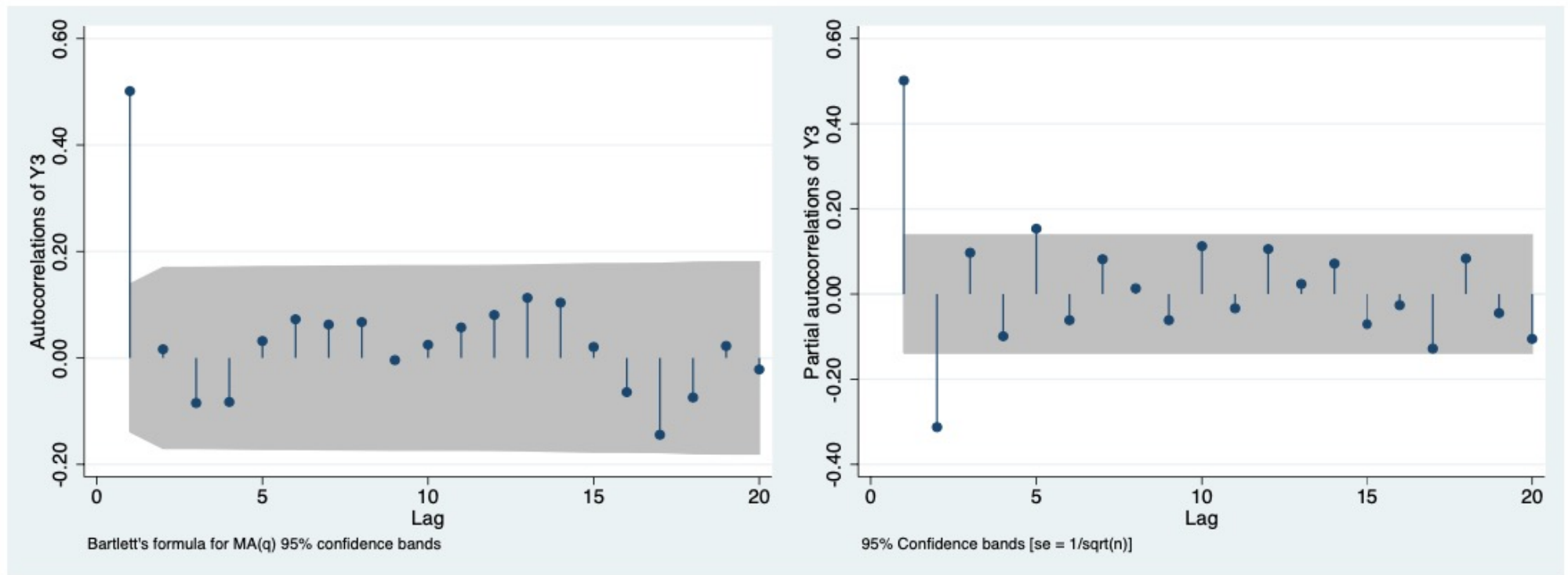
$$\text{AR}(3): Y_t = -0.3Y_{t-1} + 0.2Y_{t-2} + 0.3Y_{t-3} + \varepsilon_t$$

# What is this process?



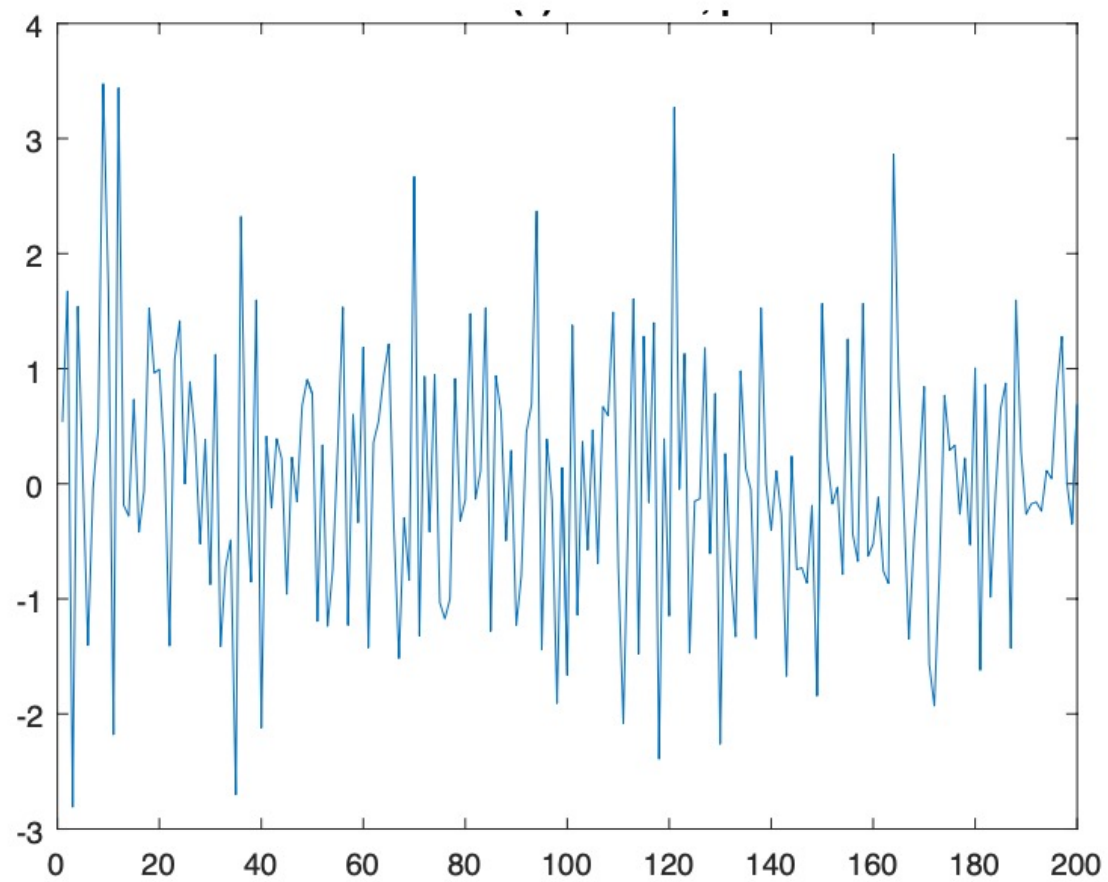


# What is this process?

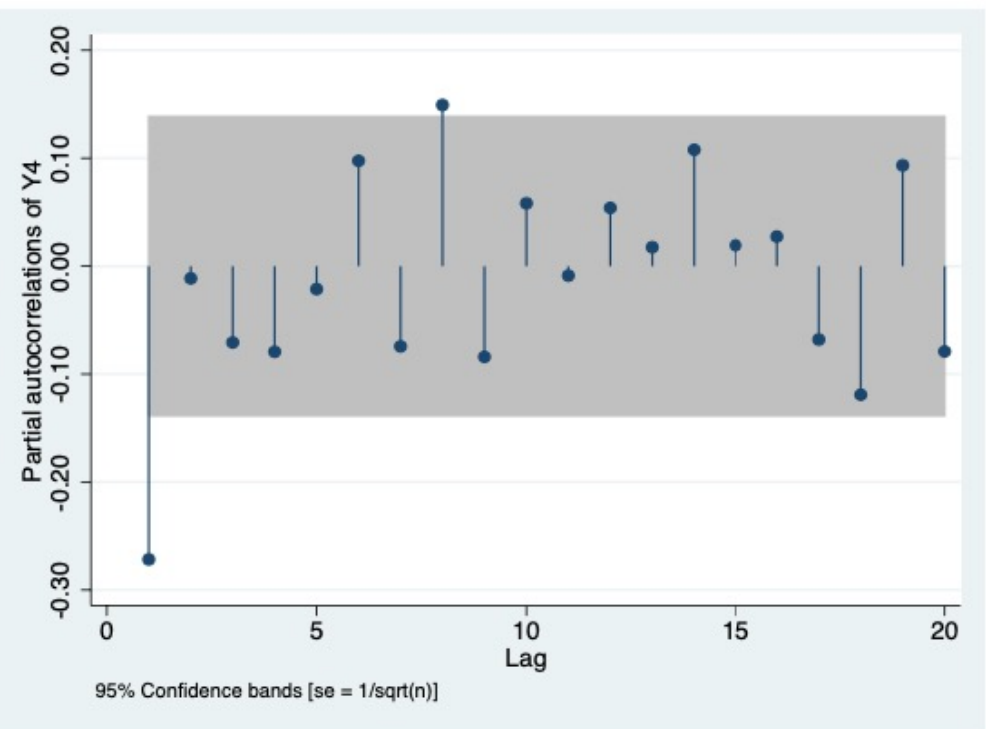
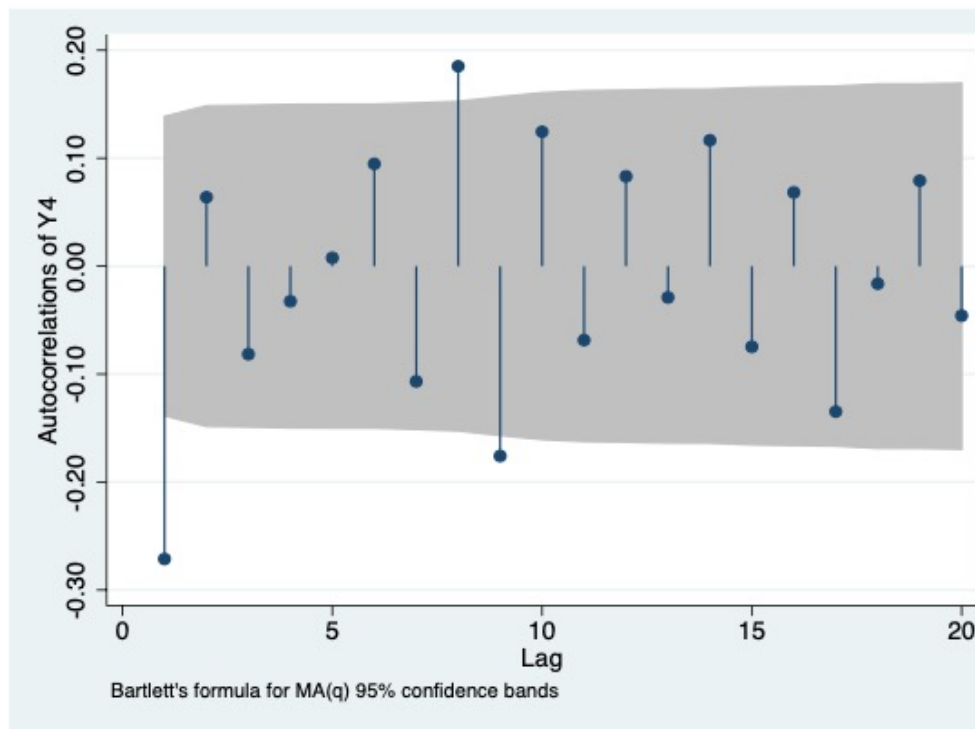


MA(1) with  $\varphi_1 = 0.7$

# What is this process?

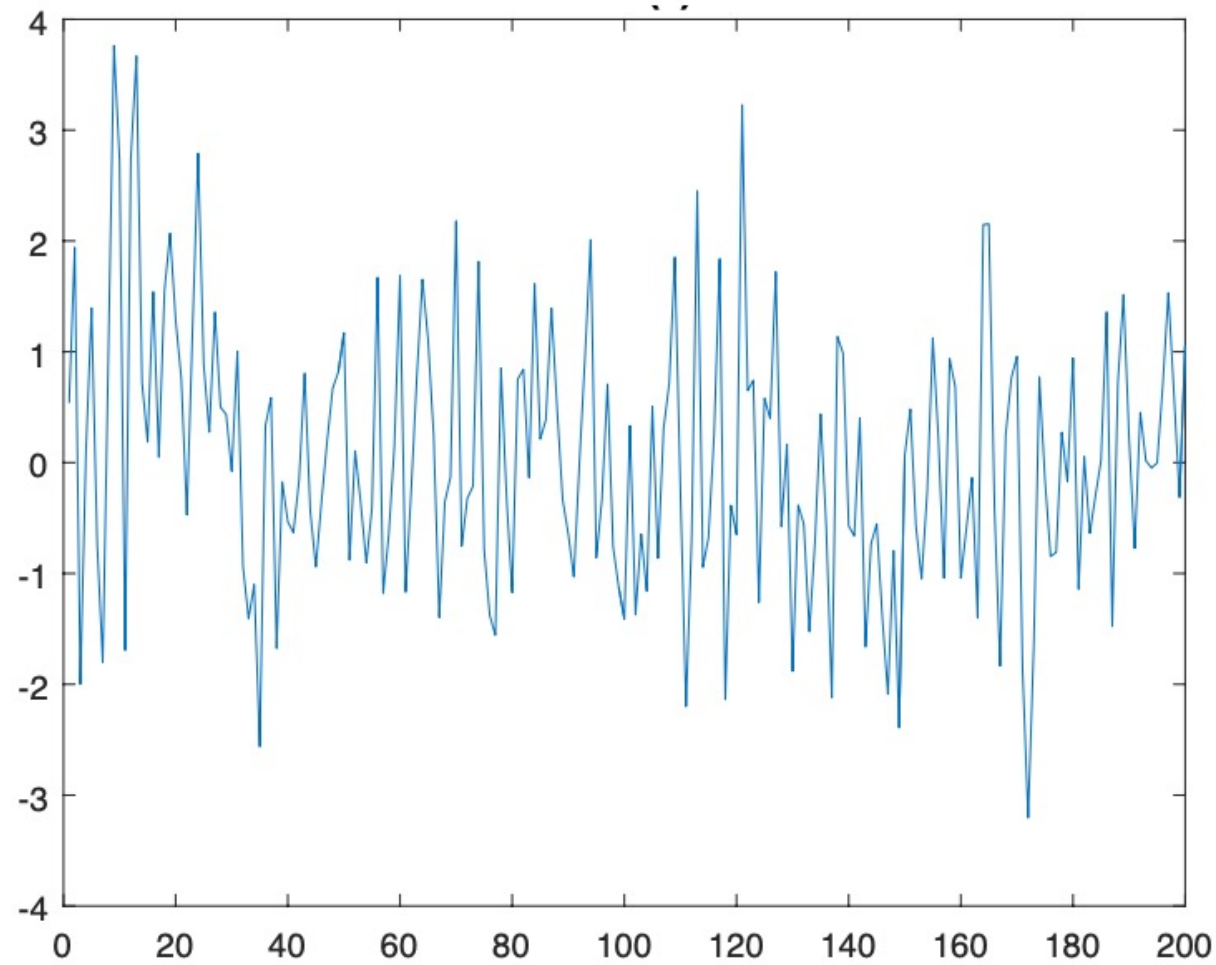


# What is this process?

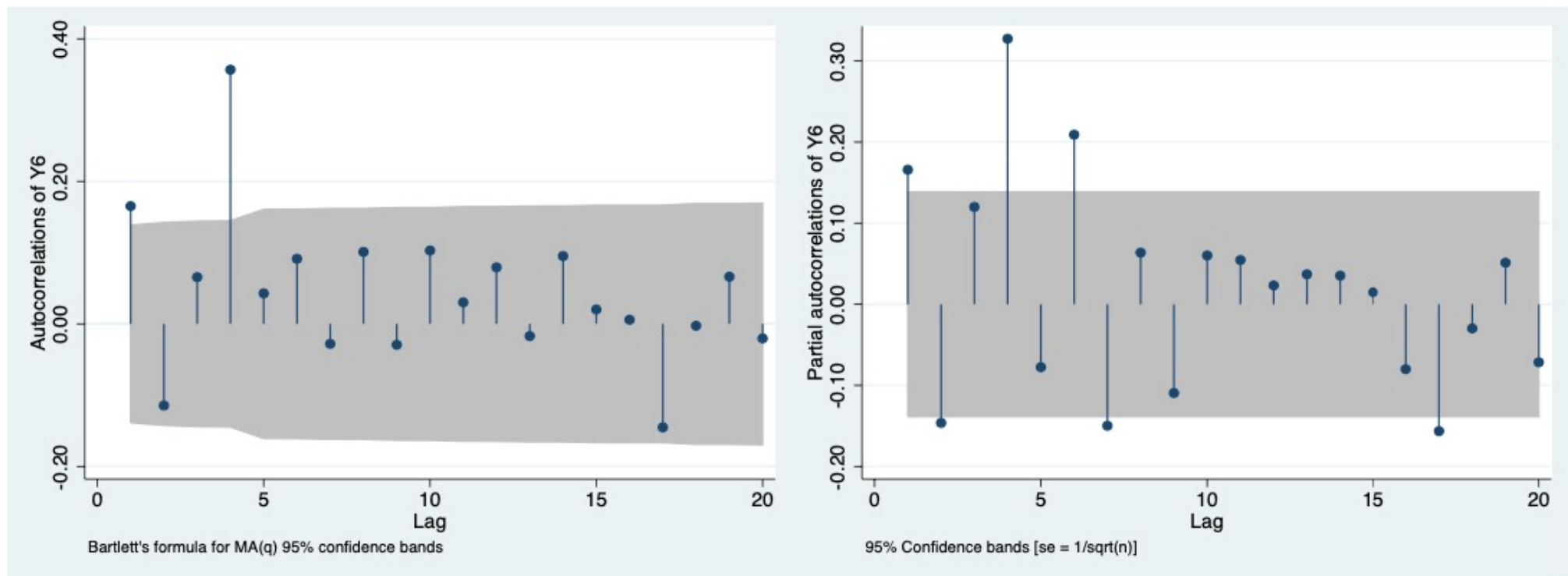


MA(1) with  $\varphi_1 = -0.3$

# What is this process?



# What is this process?



$$\text{MA}(4): Y_t = \varepsilon_t + 0.2\varepsilon_{t-1} - 0.2\varepsilon_{t-2} + 0.1\varepsilon_{t-3} + 0.5\varepsilon_{t-4}$$

# What is this process?

