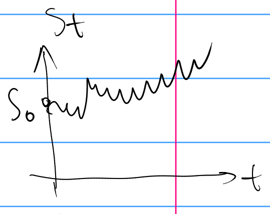
Привет!! 2021-12-07

BS.

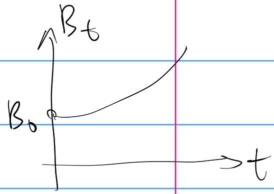


цена акции S_t

$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \cdot t + \sigma \cdot W_t\right)$$

через $\tilde{W}_t$

$$\boxed{dS_t = \mu \cdot S_t \, dt + \sigma \cdot S_t \, dW_t}$$



цена облигации B_t

$$B_t = B_0 \cdot \exp(rt)$$

$$\boxed{dB_t = r \cdot B_t \, dt}$$

$$\boxed{g'(t) = 7 \cdot g(t)} \Longleftrightarrow g(t) = c \cdot \exp(7t)$$

Задача

X_0? T X_T

пример X_0?

$$X_T = \begin{cases} S_T, & \text{если } S_T \leq 100 \\ 0, & \text{если } S_T > 100 \end{cases}$$

В модели Б.Ш. определим

$$\tilde{W}_t = W_t + \frac{\mu - r}{\sigma} \cdot t$$

и $(\tilde{W}_t)$ – это винер. процесс отн-но вер-ти $\tilde{P}$.

Дисконтир-ая цена любого актива – мартингал отн-но $\tilde{P}$.

$$S_t^{disc} = \frac{S_t}{e^{rt}} \qquad X_t^{disc} = \frac{X_t}{e^{rt}}$$

$$\boxed{X_t^{disc} = \tilde{E}\left(X_{t+s}^{disc} \mid \mathcal{F}_0\right)}$$

$t = 0$ $s = T$

$$X_0^{disc} = \widetilde{E}\left(X_T^{disc} \middle| \mathcal{F}_0\right)$$

$P, \widetilde{P}$

Brown. W_t Brown. $\widetilde{W}_t$

$$S_t = S_0 \cdot \exp(\mu t) \cdot \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right) \qquad \frac{X_0}{e^{r \cdot 0}} = \widetilde{E}\left(\frac{X_T}{e^{rT}} \middle| \mathcal{F}_0\right)$$

$$\boxed{S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$$

$$W_t = \widetilde{W}_t + \frac{r-\mu}{\sigma} t$$

$$\boxed{\sigma W_t = \sigma \widetilde{W}_t + (r-\mu)t}$$

$$\boxed{S_t = S_0 \cdot \exp(rt) \cdot \exp\left(\sigma \widetilde{W}_t - \frac{\sigma^2}{2} t\right)} \qquad \boxed{S_t = S_0 \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \widetilde{W}_t\right)}$$

$$B_t = B_0 \cdot \exp(rt)$$

$\widetilde{P}$ – risk neutral probability measure

$$\boxed{X_T = S_T \cdot I(S_T \le 100)} \qquad I(S_T \le 100) \to 1, 0$$

$$X_0 = \widetilde{E}\left(\frac{X_T}{\exp(rT)} \middle| \mathcal{F}_0\right) = \exp(-rT) \cdot \widetilde{E}\left[S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \widetilde{W}_T\right) \cdot I \middle| \mathcal{F}_0\right]$$

$$= \exp(-rT) \cdot S_0 \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)T\right) \cdot \widetilde{E}\left(\exp(\sigma \widetilde{W}_T) \cdot I(S_T \le 100) \middle| \mathcal{F}_0\right)$$

$$X_0 = S_0 \cdot \exp\left(-\frac{\sigma^2}{2} T\right) \cdot \widetilde{E}\left(\dots \middle| \mathcal{F}_0\right)$$

$$\widetilde{W}_T \overset{?}{\underset{\widetilde{P}}{\sim}} N(0; T) \qquad \widetilde{W}_T = \sqrt{T} \cdot Z$$

$$\frac{\widetilde{W}_T - 0}{\sqrt{T}} = Z \sim N(0; 1)$$

$$P(Z \le t) = \int_{-\infty}^{t} f(u)\,du = \boxed{F(t)}$$

$$\{S_T \le 100\} = \left\{S_0 \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \cdot \widetilde{W}_T\right) \le 100\right\} =$$

$$= \left\{\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \sigma \cdot \underbrace{\widetilde{W}_T}_{\sqrt{T} \cdot Z} \le \ln 100\right\} = \left\{\sigma \sqrt{T} \cdot Z \le \ln 100 - \ln S_0 + \left(\frac{\sigma^2}{2} - r\right)T\right\}$$

$$= \left\{Z \le \underbrace{\frac{\ln 100 - \ln S_0 + \left(\frac{\sigma^2}{2} - r\right) \cdot T}{\sigma \sqrt{T}}}_{\beta}\right\} = \{Z \le \beta\}$$

$$\widetilde{P}(S_T \le 100) = F(\beta)$$

$$X_0 = S_0 \cdot \exp\left(-\frac{\sigma^2}{2}T\right)\cdot \tilde{E}\Big(\exp(\sigma\overbrace{\sqrt{T}\cdot z}^{W_T})\cdot \underbrace{I(S_T \le 100)}_{\parallel \atop I(z\le\beta)} \Big| \mathcal{F}_0\Big)$$

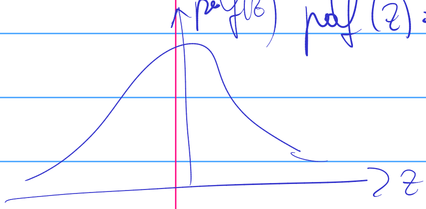
$\beta = \overline{}$ ← конст

$\{S_T \le 100\} = \{z \le \beta\}$ ← event (соб) $\qquad \beta = \dfrac{\ln 100 - \ln S_0 + \left(\frac{\sigma^2}{2} - r\right)\cdot T}{\sigma\sqrt{T}}$

$I(S_T \le 100) = I(z \le \beta)$ ← случ. величин

$$\boxed{\tilde{E}\big(\exp(\alpha\cdot z)\cdot I(z\le\beta)\big)} = \int_{-\infty}^{\beta} \exp(\alpha\cdot z)\cdot pdf(z)\, dz =$$

$z \sim N(0;1)$



$$pdf(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$$

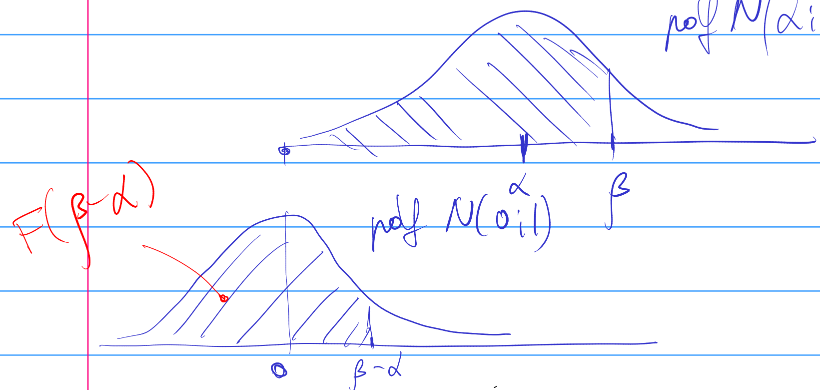
$$= \int_{-\infty}^{\beta} \exp(\alpha z)\cdot \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right) dz =$$

$$= \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2 - 2\alpha z}{2}\right) dz = \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2 - 2\alpha z + \alpha^2 - \alpha^2}{2}\right) dz$$

$$= \exp\left(\frac{\alpha^2}{2}\right)\cdot \int_{-\infty}^{\beta} \boxed{\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{(z-\alpha)^2}{2}\right)} dz = \boxed{\exp\left(\frac{\alpha^2}{2}\right)\cdot F(\beta-\alpha)}$$

pdf $N(\alpha;1)$ ← pdf $N(\alpha;1)$

$$\frac{1}{\sqrt{2\pi\sigma^2}}\cdot\exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right)$$



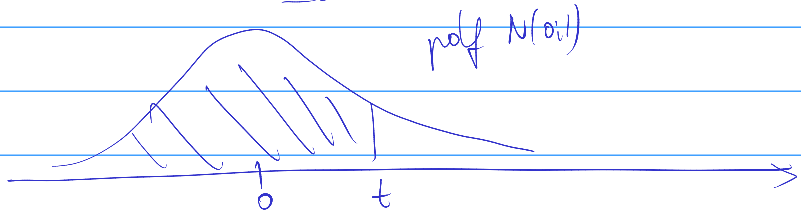
$$X_0 = S_0 \cdot \exp\left(-\frac{\sigma^2}{2}T\right)\cdot \tilde{E}\Big(\exp(\overbrace{\sigma\sqrt{T}}^{\alpha}\cdot z)\cdot I(S_T \le 100) \Big| \mathcal{F}_0\Big)$$

$$X_0 = S_0 \cdot \cancel{\exp\left(-\frac{\sigma^2}{2}T\right)} \cdot \cancel{\exp\left(\frac{\sigma^2\cdot T}{2}\right)} \cdot F\left(\frac{\ln 100 - \ln S_0 + \left(\frac{\sigma^2}{2} - r\right)\cdot T - \sigma^2 T}{\sigma\sqrt{T}}\right)$$

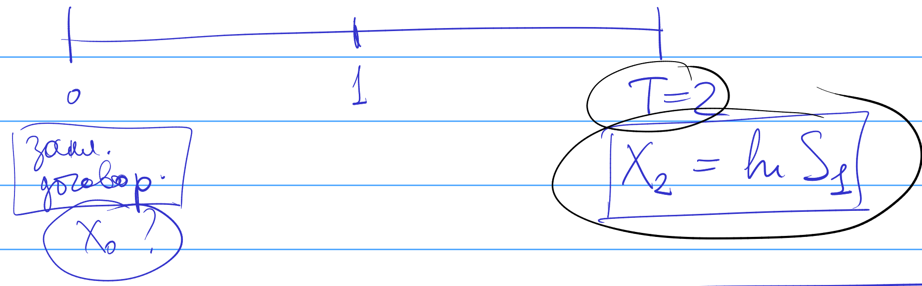
$$X_0 = S_0 \cdot F\left(\frac{\ln 100 - \ln S_0 - \left(\frac{\sigma^2}{2} + r\right)T}{\sigma\sqrt{T}}\right)$$

Где

$$F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2}\, du$$



Ex.



$$X_0 = \widetilde{E}\left(\frac{X_2}{\exp(r2)} \,\Big|\, \mathcal{F}_0\right) =$$

$$= \exp(-r2) \cdot \widetilde{E}\left(\ln S_1 \mid \mathcal{F}_0\right) =$$

$$= \exp(-r2) \cdot \widetilde{E}\left(\ln\left(S_0 \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right) \cdot 1 + \sigma \widetilde{W}_1\right)\right) \,\Big|\, \mathcal{F}_0\right) =$$

?

$$= \exp(-r2) \cdot \widetilde{E}\left(\ln S_0 + r - \frac{\sigma^2}{2} + \sigma \widetilde{W}_1 \,\Big|\, \mathcal{F}_0\right) =$$

$$= \exp(-2r) \cdot \left(\ln S_0 + r - \frac{\sigma^2}{2} + \sigma \underbrace{\widetilde{E}\left(\widetilde{W}_1 \mid \mathcal{F}_0\right)}_{?}\right) = (*)$$

$$\widetilde{W}_0 = 0 \qquad \widetilde{W}_1 - 0 = \underbrace{\widetilde{W}_1 - \widetilde{W}_0}\ \text{нез}\ \mathcal{F}_0$$

$$\widetilde{E}\left(\widetilde{W}_1 - \widetilde{W}_0 \mid \mathcal{F}_0\right) = \widetilde{E}\left(\widetilde{W}_1 - \widetilde{W}_0\right) = 0$$

$$\widetilde{W}_t - \widetilde{W}_s \sim N(0; t-s)$$

$$X_0 \overset{(*)}{=} \exp(-2r) \cdot \left(\ln S_0 + r - \frac{\sigma^2}{2}\right) \qquad 0 < S_0 < \exp\left(\frac{\sigma^2}{2} - r\right)$$

$$\ln S_0 < \frac{\sigma^2}{2} - r$$

$$\ln S_0 + r - \frac{\sigma^2}{2} < 0$$

w	a	b	c
P	0,1	0,3	0,6
$\tilde{P}$	0,5	0,4	0,1
X	2	-3	1