Class 2.1.6.

$\boxed{\text{RWalk.}}$ → how to forecast given a RW model?

✓ → how to forecast given a more complex model? ....

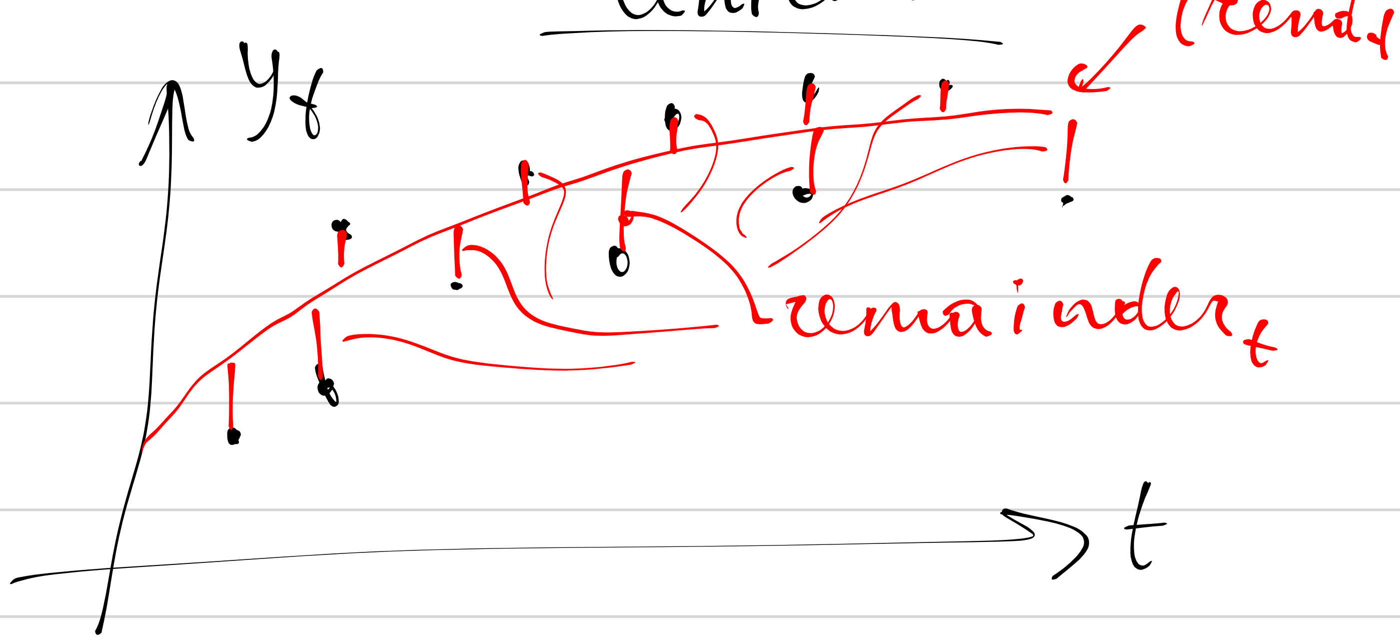
✓ → how to estimate parameters of a model? ....

→ how to select a model?

How to see unicorns?

$y_1, y_2, y_3, \dots y_T$ – observed TS.

$$y_t = \underbrace{Trend_t + Season_t + Remaider_t}_{\text{unicorns.}}$$



ETS - model (class of ≈ 25 models)

error trend seasonality - model

A - additive
M - multiplicative
N - no component

ETS(AAA)
Error, trend, seasonality

ETS(ANM)
E T S

ETS (AAN)

Additive error
Additive trend
No seasonal component

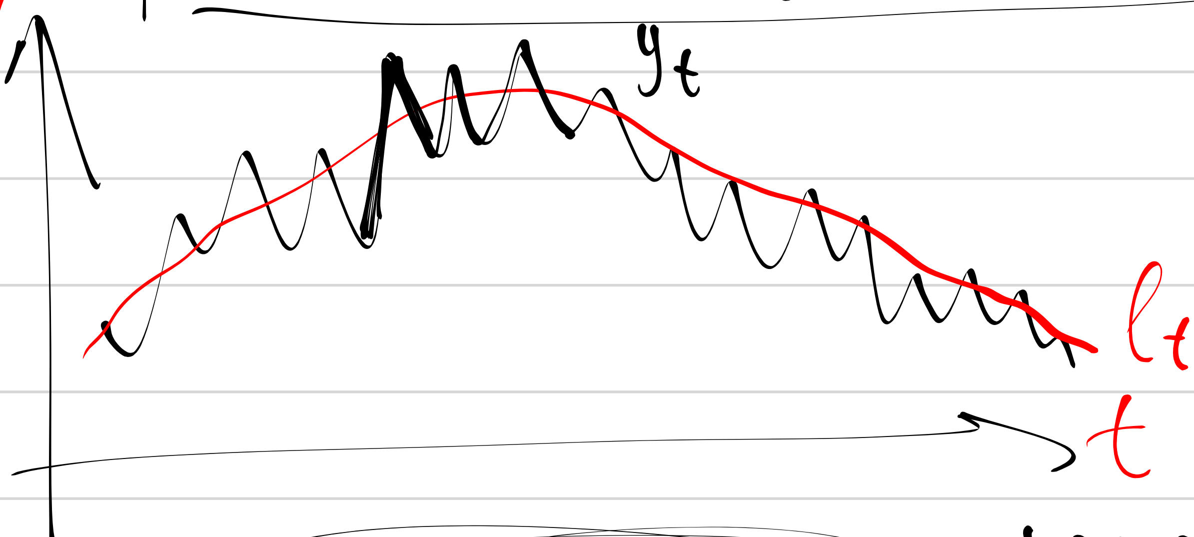
observed TS : y_t

Unicorns:

$u_t \sim N(0; \sigma^2)$, independ.

l_t - level, trend, smoothed y_t

b_t - local growth rate of trend



(1) $b_t = b_{t-1} + \beta \cdot u_t$, source of randomness

small constant

growth rate is approx. constant.

param-s
α, β,
σ^2,
b_0, l_0

(2) $l_t = l_{t-1} + b_{t-1} + \alpha \cdot u_t$

Kalman Filter

$$\begin{pmatrix} l_t \\ b_t \end{pmatrix} = f\begin{pmatrix} l_{t-1} \\ b_{t-1} \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot u_t$$

(3) $u_t \sim N(0; \sigma^2)$ indep

(4) $y_t = l_{t-1} + b_{t-1} + u_t$

→ how to forecast?

$\alpha = \frac{1}{2}$ $\beta = \frac{1}{2}$ $\sigma^2 = 4$ $\ell_0 = 12$ $b_0 = 1$

(how to see unicorns?)

$\ell_1(y_1)$
$\ell_2(y_1, y_2)$
$\ell_3(y_1, y_2, y_3)$

$\ell_0 = 12$ $b_0 = 1$

$y_1 = 14$ $\ell_1 = 13.5$ $b_1 = 1.5$ $u_1 = 1$

$y_2 = 16$ $\ell_2 = 15.5$ $b_2 = 2$ $u_2 = 1$

$y_3 = 17$ $\ell_3 = 17.25$ $b_3 = 1.75$ $u_3 = -0.5$

$+2$, $+1$, $+0.5$, -0.25

a) $\ell_1, \ell_2, \ell_3, b_1, b_2, b_3$

b) PI (95%) for y_4 given $\mathcal{F}_3$

c) PI (95%) for y_5 given $\mathcal{F}_3$

$$\mathcal{F}_t = \sigma(y_t, y_{t-1}, y_{t-2}, \ldots)$$

1. $u_t = y_t - \ell_{t-1} - b_{t-1}$
2. $b_t = b_{t-1} + \beta \cdot u_t$
3. $\ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t$

b) $E(y_4 | \mathcal{F}_3)$, $Var(y_4 | \mathcal{F}_3)$, PI for y_4

$$E(y_4 | \mathcal{F}_3) = E(\ell_3 + b_3 + \overset{\text{indep}}{u_4} | \mathcal{F}_3) =$$

can be calculated with info from $\mathcal{F}_3$

$$= \ell_3 + b_3 + E(u_4 | \mathcal{F}_3) = \ell_3 + b_3 + E(u_4) =$$
$$= \ell_3 + b_3 + 0 = 19$$

$$Var(y_4 | \mathcal{F}_3) = Var(\ell_3 + b_3 + u_4 | \mathcal{F}_3) =$$

indep of $\mathcal{F}_3$

known given $\mathcal{F}_3$

$$= Var(u_4) = 4$$

PI: $[19 - 1.96\sqrt{4}; 19 + 1.96\sqrt{4}]$

c) y_5 ?

$$E(y_5 | \mathcal{F}_3) = E(\ell_4 + b_4 + u_5 | \mathcal{F}_3) =$$

$$= E\left(\left(\ell_3 + b_3 + \tfrac{1}{2} \cdot u_4\right) + \left(b_3 + \tfrac{1}{2} \cdot u_4\right) + u_5 \middle| \mathcal{F}_3 \right)$$

$$= \boxed{\ell_3 + 2b_3}$$

$$Var(y_5 | \mathcal{F}_3) = Var(u_4 + u_5 | \mathcal{F}_3) = 2\sigma^2$$

....

How param-s are estimated?

Write likelihood function (condit-l on first values of y_t) and maximize it.

RW
$$\begin{cases} y_t = y_{t-1} + \alpha + u_t \\ u_t \sim N(0; \sigma^2), \text{ normally dist-d and indep-'t of } y_{t-1}, y_{t-2}, y_{t-3} \ldots \end{cases}$$

$y_1 = 8$
$y_2 = 10$
$y_3 = 16$

$\hat{\alpha}, \hat{\sigma}^2$ using cond-l max likelihood?

$$\begin{bmatrix} P(A \cap B) = P(A|B) \cdot P(B) \\ P(A \cap B | C) = P(A | B, C) \cdot P(B|C) \end{bmatrix}$$

$$L(y_3, y_2 | y_1) = f(y_3, y_2 | y_1) =$$

$$= f(y_3 | y_2, y_1) \cdot f(y_2 | y_1)$$

$$\boxed{\ln L(y_3, y_2 | y_1) = \ln f(y_3 | y_2, y_1) + \ln f(y_2 | y_1)} \to \max_{\alpha, \sigma^2}$$

$$(y_2 | y_1) \sim N(y_1 + \alpha ; \sigma^2)$$

$$y_t = \boxed{y_{t-1}} + \boxed{\alpha} + \boxed{u_t}$$

$$u_t \sim N(0 ; \sigma^2)$$

$$(y_3 | y_2, y_1) \sim N(y_2 + \alpha ; \sigma^2)$$

$$\boxed{(y_3 | y_1) \sim N(y_1 + 2\alpha ; \ldots)}$$

$$Var(y_3 | y_2, y_1) =$$
$$= Var(y_2 + \alpha + u_3 | y_2, y_1) =$$
$$= Var(u_3) = \sigma^2$$

pdf $N(\mu : \sigma^2)$:
(wiki)
$$\frac{1}{\sqrt{2\pi \sigma^2}} \cdot \exp\left(-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}\right)$$

$$\boxed{\ln pdf = -\frac{1}{2} \ln(2\pi \sigma^2) - \frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}}$$

$$\ln L(y_3, y_2 | y_1) = \ln f(y_3 | y_2, y_1) + \ln f(y_2 | y_1) =$$

$$= \left(-\frac{1}{2} \ln(2\pi\sigma^2) - \frac{1}{2} \frac{(y_3 - (y_2+\alpha))^2}{\sigma^2}\right) + \left(-\frac{1}{2} \ln(2\pi\sigma^2) - \frac{1}{2} \frac{(y_2 - (y_1+\alpha))^2}{\sigma^2}\right)$$

$$\to \max_{\alpha, \sigma^2}$$

$$\max_{\alpha} \left[-\frac{1}{2\sigma^2}\left((y_3 - y_2 - \alpha)^2 + (y_2 - y_1 - \alpha)^2\right)\right]$$

! Hint!

$$\alpha^* = \frac{y_3 - y_2 + y_2 - y_1}{2} =$$

$$(\sigma^2)^* = \frac{\sum_{t=2}^{3} (y_t - y_{t-1} - \alpha^*)^2}{2}$$

$$\alpha^* = \frac{y_3 - y_1}{2}$$

$$\frac{\partial \ln z^2}{\partial(\underbrace{z^2})} = \frac{1}{z^2}$$

(trick)

$z^2 = v$

Likelik $= P(\ldots)$
$= f(\ldots)$