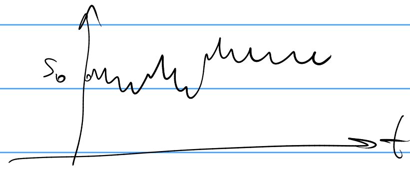
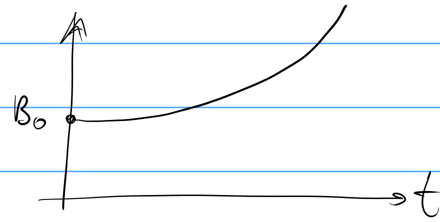
TSSP (Hi) 2021-12-03

BS model !!

S_t – share price (risky)

B_t – bond price





$$dS_t = \mu S_t\,dt + \sigma \cdot S_t \cdot dW_t \rightarrow Y = \ln S_t$$

$$dB_t = r \cdot B_t\,dt$$

$$\frac{dB_t}{dt} = r \cdot B_t \longrightarrow \int \frac{dB_t}{B_t} = \int r\,dt$$

$$\ln B_t = rt + c$$

$$B_t = e^c \cdot e^{rt}$$

$$\boxed{B_t = B_0 \cdot e^{rt}} \qquad \ln B_t = \ln B_0 + rt$$

$$dS_t = \mu \cdot S_t\,dt + \sigma S_t \cdot dW_t$$

$$\boxed{\begin{array}{l} dW_t\,dW_t = dt \\ dt \cdot d\,\text{any} = 0 \end{array}}$$

$Y_t = \ln S_t \qquad Y_t = h(S_t)$

(Ito's lemma)

$$dY_t \overset{?}{=} \underbrace{\frac{1}{S_t}}_{h'_s} dS_t + \underbrace{0}_{h'_t}\,dt +$$

$$+ \frac{1}{2}\left(\underbrace{-\frac{1}{S_t^2}}_{h''_{ss}} \cdot dS_t\,dS_t + 2 \cdot \underbrace{0}_{h''_{ts}} \cdot \underbrace{dt \cdot dS_t}_{=0} + \underbrace{0}_{h''_{tt}} \cdot \underbrace{(dt)^2}_{0} \right) =$$

$$= \frac{1}{S_t}\left(\mu S_t\,dt + \sigma S_t\,dW_t\right) - \frac{1}{2S_t^2} \cdot \left(\mu S_t\,dt + \sigma S_t\,dW_t\right)^2 = \mu\,dt + \sigma\,dW_t - \frac{1}{2S_t^2} \cdot \sigma^2 S_t^2\,dt$$

$$\boxed{dY_t = \left(\mu - \frac{\sigma^2}{2}\right) \cdot dt + \sigma\,dW_t}$$

$(dt)^2 = 0 \qquad dt \cdot dW_t = 0$

$dW_t \cdot dW_t = dt$

$$dY_t = \left(\mu - \frac{\sigma^2}{2}\right) \cdot dt + \sigma dW_t$$ ← short form

dY_t does not exist!

$$\underbrace{Y_t}_{\ln S_t} = \underbrace{Y_0}_{\ln S_0} + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) du + \int_0^t \sigma \cdot dW_u$$

R.V.

δ - delta

σ - sigma

$$\ln S_t = \ln S_0 + \boxed{\int_0^t \mu - \frac{\sigma^2}{2} du} + \boxed{\int_0^t \sigma dW_u}$$

$$\boxed{\int_0^t \mu - \frac{\sigma^2}{2} du} = \left(\mu - \frac{\sigma^2}{2}\right) t \qquad \boxed{\int_0^t \sigma dW_u} = \sigma \cdot W_t$$

$$\ln S_t = \ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma \cdot W_t$$

$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) =$$

W_t - Wiener process.

$$= S_0 \cdot \underbrace{\exp(\mu t)}_{\text{growth rate}} \cdot \underbrace{\exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)}_{\text{martingale}}$$

Hard: Two different probabilities. $P(\text{event})$, $P_*(\text{event})$

Theorem (Cameron - Martin - Girsanov)

If (W_t) is a Wiener process wrt P and $W_t^* = W_t + \alpha \cdot t$ then there is a probability P_* such that (W_t^*) is a Wiener process wrt P_*.

Ex W_t - Wiener process wrt $P(\cdot)$

W_t^* - Wiener process wrt $P_*(\cdot)$

$W_t^* = W_t + \frac{1}{2} t \implies W_t = W_t^* - \frac{1}{2} t$

a) $E(W_t^*) =$ $\qquad E_*(W_t) =$ $\qquad E_*(W_t^*) =$

b) $Var(W_t^*) =$ $\qquad Var_*(W_t) =$

c) $E_*(W_t^2 W_t^*) =$

d) $P_*(W_2 > 0)$?

a) $E(W_t^*) = E(W_t + \frac{1}{2}t) =$

$= 0 + \frac{1}{2}t$

$E_*(W_t) = E_*(\underbrace{W_t^* - \frac{1}{2}t}) =$

$= 0 - \frac{1}{2}t$

$E_*(W_t^*) = 0$

	a	b
X	0	1
P	$\frac{1}{2}$	$\frac{1}{2}$
P^*	0.1	0.9

$E(X) = \frac{1}{2}$

$E_*(X) = 0.9$

b) $Var(W_t^*) = Var(W_t + \underbrace{\frac{1}{2}t}_{\text{non-random}}) = t + 0$

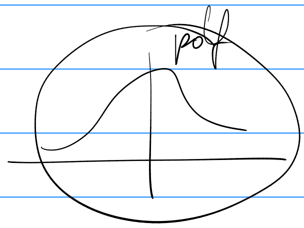
$Var_*(W_t) = Var_*(W_t^* - \frac{1}{2}t) = t + 0$

c) $E_*(W_t^2 W_t^*) = E_*((W_t^* - \frac{1}{2}t)^2 W_t^*) =$

$= E_*((W_t^*)^3 - 2\cdot\frac{1}{2}t\cdot(W_t^*)^2 + \frac{t^2}{4}\cdot W_t^*) =$

$= 0 - t\cdot E_*((W_t^*)^2) + \frac{t^2}{4}\cdot 0 = -t\cdot t = -t^2$

$W_t^* \overset{P^*}{\sim} N(0; t)$



$E_*((W_t^*)^3) =$

$= \int_{-\infty}^{\infty} \underbrace{x^3 \cdot f(x)}_{\text{odd function}} dx = 0$

d) $P_*(W_2 > 0) = P_*(W_2^* - \frac{1}{2}\cdot 2 > 0) =$

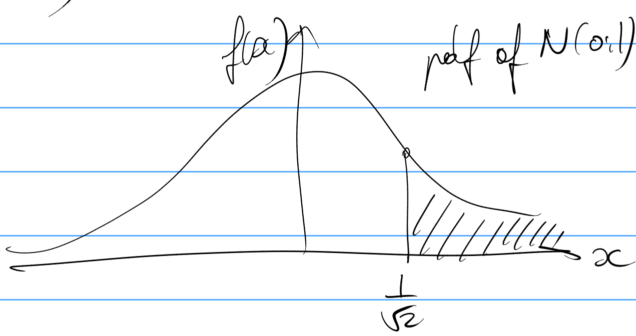
$= P_*(W_2^* > 1) = P_*(\underbrace{\frac{W_2^* - 0}{\sqrt{2}}}_{Z} > \frac{1-0}{\sqrt{2}}) = P_*(Z > \frac{1}{\sqrt{2}}) =$

$W_2^* \overset{P^*}{\sim} N(0; 2)$ $\quad Z \sim N(0;1)$

$= 1 - F(\frac{1}{\sqrt{2}}) \approx 0.24$

cdf

cum distr fun



Rem: Y_t is a martingale if and only if [....]

$$dY_t = A_t \cdot dW_t + 0 \cdot dt$$

BS

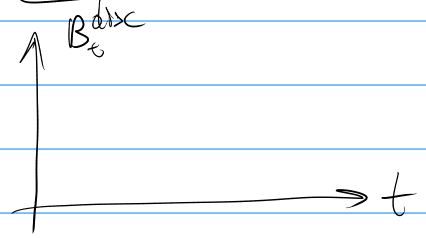
$$B_t = B_0 \cdot \exp(rt)$$

$$S_t = S_0 \cdot \exp(\mu t) \cdot \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

X_t

in continuous	in discrete
$X_t^{disc} = \frac{X_t}{\exp(rt)}$	$X_t^{disc} = \frac{X_t}{(1+r)^t}$

$$B_t^{disc} = \frac{B_0 \cdot \exp(rt)}{\exp(rt)} = B_0 \quad (!)$$

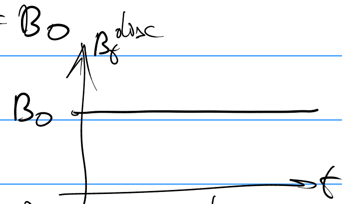


$$S_t^{disc} = S_0 \cdot \frac{\exp(\mu t)}{\exp(rt)} \cdot \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right) =$$

$$S_t^{disc} = S_0 \exp\left(\left(\mu - r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

Ex: Is S_t^{disc} a martingale? | Is B_t^{disc} a martingale?

$B_t^{disc} = B_0$



$$E\left(B_{t+\Delta}^{disc} \mid \mathcal{F}_t\right) = B_0 = B_t^{disc}$$

mart by def

$dt \cdot d\text{any} = 0$

$$dS_t^{disc} = d\Big(\underbrace{\exp(-rt) \cdot S_t}_{h(S_t,\, t)}\Big) =$$

(1)
(2)
(3)
(4)
(5)
(6)
(7)

$$= h'_s \cdot dS_t + h'_t\, dt +$$

$$+ \frac{1}{2}\left(h''_{ss} (dS)^2 + 2 h''_{st}\, dS \cdot dt + h''_{tt} (dt)^2\right) =$$

$$= \exp(-rt) \cdot dS_t + (-r) \cdot \exp(-rt) S_t\, dt + \frac{1}{2}\left(0 (dS)^2 + 2 \cdot \ldots \cdot 0 + \ldots \cdot 0\right)$$

$$dS_t^{disc} = \exp(-rt)\left[\mu S_t\, dt + \sigma S_t \cdot dW_t - r S_t\, dt\right] = \boxed{\exp(-rt) \cdot S_t \cdot \left((\mu - r)dt + \sigma\, dW_t\right)}$$

$$dS_t^{disc} = \exp(-rt)\cdot S_t \cdot \big((\mu - r)dt + \sigma\, dW_t\big)$$

S_t^{disc} is not a martingale wrt P

(1) $\sigma W_t^* = \sigma W_t + (\mu - r)\cdot t$ $\qquad$ $W_t^* = W_t + \frac{(\mu-r)}{\sigma}\cdot t$

(2)

(3) there is a P_* such that W_t^* is a Wien. process.

(4)

(5) $d(\sigma W_t^*) = \sigma\cdot dW_t + (\mu - r)\cdot dt = \sigma\cdot dW_t^*$

(6)

(7) $dS_t^{disc} = \exp(-rt)\cdot S_t \cdot \sigma \cdot dW_t^*$ $\qquad$ no dt term

S_t^{disc} is a mart. wrt $P_*(\ )$

S_t^{disc} is not a mart wrt $P(\ )$

1 bond $\qquad$ share

"shbond" = 1 share + 1 bond

$X_t = S_t + B_t$

$$X_t^{disc} = \underset{\text{.....}}{S_t^{disc}} + \underset{0}{B_t^{disc}}$$

Under P_* the discounted price of every asset (in BS model) is a martingale

under P^* the W_t^* is a Wiener process

$$\sigma W_t^* = \sigma W_t + (\mu - r)t$$
$$\sigma W_t = \sigma W_t^* - (\mu - r)t$$

Ex **BS-model**

$t=0$ ———————————————— $T=2$

$B \xrightarrow{\$} A$ $\qquad\qquad\qquad\qquad$ $A \xrightarrow{\$} B$

$X_0\,?$

$$X_2 = \begin{cases} 100 & \text{if } S_2 \geq 1000 \\ 0 & \text{if } S_2 < 1000 \end{cases}$$

Th: $\quad X_t^{disc}$ is a martingale under P_*

$$\underbrace{X_0^{disc} = E_*\left(X_2^{disc} \mid \mathcal{F}_0\right)}$$

$$\frac{X_0}{\underbrace{\exp(r\cdot 0)}_{=1}} = E_*\left(\frac{X_2}{\exp(r\cdot 2)} \,\Big|\, \mathcal{F}_0\right)$$

$$X_0 = \exp(-r2)\cdot E_*\left(X_2 \mid \mathcal{F}_0\right)$$

$$?\; X_0 = \exp(-r2)\cdot\left(100\cdot P^*\left(S_2 \geq 1000 \mid \mathcal{F}_0\right) + 0\cdot P^*\left(S_2 < 1000 \mid \mathcal{F}_0\right)\right)$$

$$P^*\left(S_2 \geq 1000 \mid \mathcal{F}_0\right)$$

$$\sigma W_t^* = \sigma W_t + (\mu - r)t$$
$$\sigma W_t = \sigma W_t^* - (\mu - r)t$$

$$A = \{S_2 \geq 1000\} = \left\{S_0\cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\cdot 2 + \sigma\cdot W_2\right) \geq 1000\right\} =$$

$$= \left\{S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\cdot 2 + \sigma W_2^* - (\mu - r)2\right) \geq 1000\right\} =$$

$$= \left\{S_0\cdot \exp\left(2r - \sigma^2 + \sigma\cdot W_2^*\right) \geq 1000\right\} = \left\{\ln S_0 + 2r - \sigma^2 + \sigma W_2^* \geq \ln 1000\right\}$$

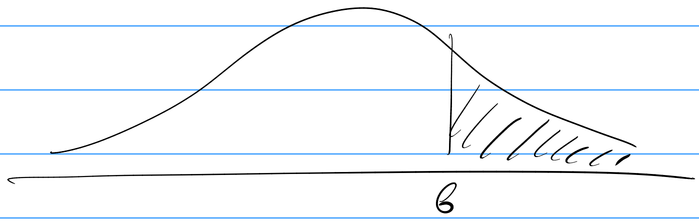
$$W_2^* \overset{P^*}{\sim} N(0\,;\,2)$$

$\boxed{W_2^* - W_0^*}$ is indep of $\mathcal{F}_0$

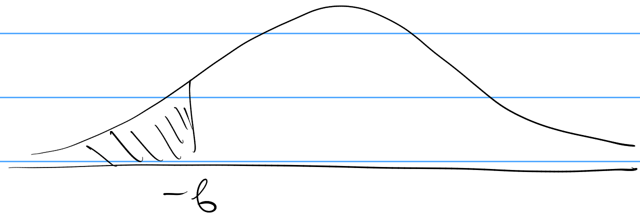
$$P^*(A \mid \mathcal{F}_0) = P^*(A) = P^*\left(\ln S_0 + 2r - \sigma^2 + 2W_2^* \geq \ln 1000\right) =$$

$$= P^*\left(W_2^* \geq \frac{\ln 1000 - \ln S_0 - 2r + \sigma^2}{2}\right) =$$

$$= P^*\left(\frac{W_2^* - 0}{\sqrt{2}} \geq \underbrace{\frac{\text{" " " " " "}}{2\sqrt{2}}}_{=b}\right) =$$



$$= 1 - \underset{\text{cdf}}{F}(b) =$$



$$= F(-b)$$

F - cdf of standard normal

$$\boxed{X_0 = 100 \cdot \exp(-2r) \cdot F\left(\frac{\ln S_0 - \ln 1000 + 2r - \sigma^2}{2\sqrt{2}}\right)}$$