Time Series

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Today

- Time series: definition
- Structural/non-structural modeling
- Stationarity, weak and strong
- Autocovariance, autocorrelation, partial autocorrelation
- Lag operator
- ARMA models (beginning)
- Tsay "Analysis of Financial Time Series." (1.2, 2.1-2.6) V

  Hamilton "Time Series Analysis" (2.1, 3.1-3.5)

  Stock and Watson "Introduction to Econometrics" (14.1, 14.2)

  ✓ Diebold "Forecasting" (online version:

  http://www.ssc.upenn.edu/ fdiebold/Teaching221/Forecasting.pdf

  (6.5, 7.1, 7.2)

Time Series

Cross-sectional data:

- The sample is **i.i.d.** (or at least independent)
- Useful for answering questions about **causal effects** of one variable on another

Time Series:

- The sample is **not i.id.**, observe variable(s) over time
- Useful for answering questions about **dynamic causal effects**
- Useful for forecasting **future** values of a variable

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Time Series

Useful for forecasting future values of a variable

We will study quantitative (non-structural) models of time series to use them later for forecasting:

Structural

- Has some **economic theory** behind it
- Parameters **have meaning and causal interpretation**
- Will briefly touch in VAR and ADL topic

Non-structural

- Models based on **fitting data**
- Coefficients **do not have causal interpretation**
- Will be the main topic of the course

Time Series: definition

- **Informaly:** a set of realizations of a random variable ordered according to time
- **Formally:**

Definition 1

Collection of random variables defined on the sample space $\{Y_t,\ t \in \boldsymbol{T}\}$ is called a *stochastic process*

We will consider $\boldsymbol{T} = \{\ldots, -1, 0, 1, 2, \ldots\} = \mathbf{Z}$

Definition 2

A *time series* is a realization of a stochastic process: $\{y_t,\ t \in \boldsymbol{Z}\}$

Definition 3

A *time series sample* is $\{y_t,\ t = 1, \ldots, T\}$ for some $T < \infty$.

But ‘time series’ can be used as a synonym of ‘stochastic process’

Important concepts

- **Goal:** forecast values of a random variable using the time series sample
- So, we need the future to be like the past
- Reflected in the concept of ***stationarity***

Stationarity

Definition 4

A process $\{Y_t,\ t \in Z\}$ is *strictly stationary* if, for any k, s and any $t_1,\ \ldots,\ t_k$, the *distributions* of $(Y_{t_1}, Y_{t_2},\ \ldots, Y_{t_k})$ and $(Y_{t_1+\ s}, Y_{t_2+s},\ \ldots, Y_{t_k+s})$ are *the same* .

In other words, the following distributions are the same:

- of Y_1 and Y_{100}
- of $(Y_1,\ Y_2)$ and$(Y_5,\ Y_6)$
- of $(Y_3,\ Y_{10},\ Y_{22})$ and $(Y_{13},\ Y_{20},\ Y_{32})$
- and so on ...

Stationarity

Strict stationarity is a complicated concept

Very often people consider *weak stationarity*

Definition 5

A process $\{Y_t,\ t \in Z\}$ is *weakly, or covariance-, stationary* if, for any $t_1, t_2,\ s \in \boldsymbol{Z}$

1) $E\left[Y_{t_1}\right] = E\left[Y_{t_2}\right]$,

2) $Cov\left(Y_{t_1},\ Y_{t_1+\ s}\right) = Cov\left(Y_{t_2},\ Y_{t_2+s}\right)$ $\neq f(t_1)\ \forall s$

So, only the following has to be the same:

- mean of all Y_t $E[Y_t]$
- variance of all Y_t $Var(Y_t)$
- covariances between all of the possible pairs of Y_t that are fixed number of periods away from each other $Cov(Y_t, Y_{t+s})\ \forall s$

MUST BE indep. t on t

Stationarity

Question

If $\{Y_t,\ t \in \boldsymbol{Z}\}$ is *weakly stationary*, is it also *strictly stationary*?

Stationarity

Question

If $\{Y_t, \ t \in \mathbb{Z}\}$ is *strictly stationary*, is it also *weakly stationary*?

Stationarity: extra remarks

Not all *weakly stationary* process *strictly stationary.*

But if $\{Y_t\}$ is gaussian, then it is *weakly stationary*, it is also *strictly stationary*.

Autocovariance and autocorrelation function

- Want to forecast future by exploring the relation between r.v. corresponding to consecutive periods of time
- Autocovariance is a way to quantify this relation

$\gamma(0) = Var(y_t)$
$\gamma(1) = Cov(y_t, y_{t+1})$

Definition 6

- Autocovariance of order k is $\gamma(k) = Cov\left(Y_t,\ Y_{t+k}\right)$
- Autocorrelation of order k is $\rho(k) = corr\ \left(Y_t, Y_{t+k}\right) = \dfrac{\gamma(k)}{Var\left(Y_t\right)}$ $\Rightarrow \rho(k) = \dfrac{\gamma(k)}{\gamma(0)}$.
- $\gamma(\bullet)$ is called *autocovariance function* (ACF)
- $\rho(\bullet)$ is called *autocorrelation function* (also ACF)

Estimated ACF

$$\bar{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t$$

$$\hat{\gamma}(0) = \frac{1}{T}\sum_{t=1}^{T} (Y_t - \bar{Y}_T)^2$$

$$\hat{\gamma}(k) = \frac{1}{T}\sum_{t=k+1}^{T} (Y_t - \bar{Y}_T)(Y_{t-k} - \bar{Y}_T)$$

Sample autocorrelation function:

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)}$$

Correlogram: a graph of sample ACF

$\mathrm{Corr}(y_t, y_{t-1}) >> 1$

y_{t-2} y_{t-1} y_t

$\mathrm{Corr}(y_t, y_{t-2}) >> 1$

Partial Autocorrelation Function (PACF)

- Autocorrelation measures how dependent the data is
- If Y_1 and Y_2 are related, and Y_2 and Y_3 are related, then Y_1 and Y_3 have to be related at least indirectly
- *Partial Autocorrelation Function (PACF)* measures direct relation between

different Y_t.

Definition 6

Partial Autocorrelation Function (PACF) at lag k is

$$\alpha(k) = corr\Big(Y_1 - P(1, Y_2, \ldots, Y_k)Y_1, \quad Y_{k+1} - P(1, Y_2, \ldots, Y_k)Y_{k+1} \Big),$$

where $P(1, Y_2, \ldots, Y_k)Y_j$ is the linear projection of Y_j on a constant, $Y_2, \ldots,$ and Y_k.

Partial Autocorrelation Function (PACF)

Write the linear projection of $Y_{t+k+1} - \mu$ on $Y_{t+k} - \mu, \ \ldots, \ Y_{t+1} - \mu$:

$$\hat{Y}_{t+k+1} - \mu = \alpha(1)\left(Y_{t+k} - \mu\right) + \ \ldots + \alpha(k)Y_{t+1} - \mu)$$

PACF of order 1 to k can be found as

$$\begin{pmatrix} \alpha(1) \\ \alpha(2) \\ \cdots \\ \alpha(k) \end{pmatrix} = \begin{pmatrix} \gamma(0) & \gamma(1) & \ldots & \gamma(k) \\ \gamma(1) & \gamma(0) & \ldots & \gamma(k-2) \\ \cdots & \cdots & \cdots & \cdots \\ \gamma(k-1) & \gamma(k-2) & \ldots & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \cdots \\ \gamma(k) \end{pmatrix}$$

Partial Autocorrelation Function (PACF)

Write the linear projection of $Y_{t+k+1} - \mu$ on $Y_{t+k} - \mu,\ \ldots,\ Y_{t+1} - \mu$:

$$\hat{Y}_{t+k+1} - \mu = \alpha(1)\left(Y_{t+k} - \mu\right) + \ \ldots + \alpha(k)(Y_{t+1} - \mu)$$

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Sample PACF: estimate by OLS

$$Y_{t+k+1} - \mu = \alpha(1)\left(Y_{t+k} - \mu\right) + \ \ldots + \alpha(k)(Y_{t+1} - \mu) + \varepsilon_{t+k+1}$$

Lag operator (a.k.a back-shift operator)

Definition 7

The lag operator L is a linear operator such that for all t

$$LY_t = Y_{t-1}$$

$L(y_t) = y_{t-1}$

Properties:

$LLy_t = y_{t-2}$

- $L^2Y_t = L\left(LY_t\right) = LY_{t-1} = Y_{t-2}$
- $L^jL^iY_t = L^{j+i}Y_t = Y_{t-i-j}$
- $Lc = c$
- $(L^j + L^i)Y_t = Y_{t-i} + Y_{t-j}$
- L to a negative power is a *lead operator*: $L^{-i}Y_t = Y_t$
- For $|a| < 1$,

$y_t - y_{t-1} + y_{t-2} - y_{t-3} + y_{t-4} =$

$= y_t \cdot (1 - L + L^2 - L^3 + L^4)$

? if $1 - aL = 0$

$$y_t \cdot \left[\underbrace{1 + aL + a^2L^2 + \ \ldots + a^kL^k}_{y_t + a y_{t-1} + a^2 y_{t-2} + \ldots + a^k \cdot y_{t-k}} + \ldots\right] = (1 - aL)^{-1}Y_t = \frac{Y_t}{1 - aL}$$

ARMA models

Starting with basics

- $\{\varepsilon_t,\ t \in Z\} : \varepsilon_t$ **are iid**
    - is it stationary?
- MDS (Martingale Difference Sequence): $\{\varepsilon_t,\ t \in Z\}$:

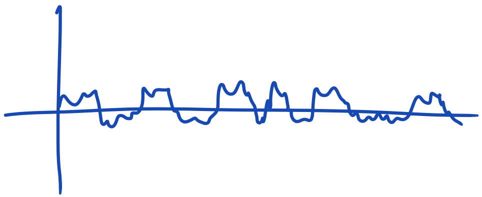
$$E[\varepsilon_t \mid \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots] = 0$$

- A process like that is closer to economics: many dynamic optimization problems result in a condition of this type.
- White noise: (A more statistical description of innovations) $\{\varepsilon_t,\ t \in Z\}$:

s.t.

$Cov(\varepsilon_t, \varepsilon_s) = 0,\ for\ any\ t \neq s$

$E[\varepsilon_t] = 0,$ $\cancel{Var(\varepsilon_t)\sigma^2 < \infty}$ $Var(\varepsilon_t) = \sigma^2 < \infty$



Moving-Average Models

- Start with $\{\varepsilon_t\}$, a white noise.

- **MA(1) – Moving average of order 1**

$$Y_t = \varepsilon_t + \varphi\varepsilon_{t-1}$$

$y_t = \varepsilon_t + \varphi \cdot \varepsilon_{t-1}$

- **MA(q) – Moving average of order q**

$$Y_t = \varepsilon_t + \varphi_1\varepsilon_{t-1} + \varphi_2\varepsilon_{t-2} \ldots + \varphi_q\varepsilon_{t-q}$$

Linear regression: review

$$Y_i = X_i\beta + \varepsilon_i$$

- Is it a model?
- Need assumption: $\boldsymbol{E\left[\varepsilon_i \mid X_i\right] = 0}$
- Equivalent to assumption that $E\left[Y_i \middle| X_i\right] = X_i\beta$
- This is a model for ***conditional mean of Y given X***
- That is what's called a ***linear regression***

Linear regression: review

$Y_i = X_i\beta + \varepsilon_i$

- Need assumption: $E\left[\varepsilon_i \mid X_i\right] = 0$, equivalent to assumption that $E\left[Y_i \middle| X_i\right] = X_i\beta$. Then it's a linear regression.
- Sometimes, people are interested in the linear projection of Y on X only. Then they assume $Cov\left(X_i, \varepsilon_i\right) = 0$.
- It's not a regression. But people call it regression anyway...
- Given a sample $\left\{ \left(X_i,\ Y_i\right) \right\}_{i=1}^{n}$, the relation between X and Y is estimated by the OLS (ordinary least squares): $\hat{\beta}_n = (\boldsymbol{X}'\boldsymbol{X})^{-\mathbf{1}}\boldsymbol{X}'\boldsymbol{Y}$
- For consistency of $\hat{\beta}_n$ ($\hat{\beta}_n \xrightarrow{p} \beta$), $Cov\left(X_i, \varepsilon_i\right) = 0$ is enough.

Autoregressive Models

Now, let $X = Y_{t-1}$. Then $Y_t = \theta Y_{t-1} + \varepsilon_t$

- It's a regression model, if we assume $E\left[\varepsilon_t \,\middle|\, Y_{t-1}\right] = 0$.
- But again, people often assume only that $corr\left(\varepsilon_t,\ Y_{t-1}\right) = 0$.

So, start with $\{\varepsilon_t\}$ being a white noise.

$y_t = \theta \cdot y_{t-1} + \varepsilon_t$

$\ne$ ARMA(1;1)

AR(1) - Autoregressive Model of order 1

$Y_t = \theta Y_{t-1} + \varepsilon_t$

ARMA(1;1) = AR(1) + MA(1)

$y_t = \underbrace{\theta \cdot y_{t-1}}_{AR(1)} + \underbrace{\varepsilon_t + \varphi \cdot \varepsilon_{t-1}}_{MA(1)}$

AR(p) - Autoregressive Model of order p

ARMA(p;q) = AR(p) + MA(q)

$$Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \ \ldots + \theta_p Y_{t-p} + \varepsilon_t$$

Autoregressive Moving-Average Models

$\{\varepsilon_t\}$ is a white noise

- ARMA(1, 1)

$$Y_t = \theta Y_{t-1} + \varepsilon_t + \varphi \varepsilon_{t-1}$$

- **ARMA(p, q)**

$$Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \ldots \theta_p Y_{t-p} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \ldots + \varphi_q \varepsilon_{t-q}$$

- More general ARMA(p, q)

$$Y_t = c + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \ldots \theta_p Y_{t-p} + \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \varphi_2 \varepsilon_{t-2} + \ldots + \varphi_q \varepsilon_{t-q}$$

Properties of ARMA(p,q) models

White Noise

- **Stationarity**
  - $E\left[\varepsilon_t\right] = 0$ for all t
  - $\mathrm{Var}(\varepsilon_t) = \sigma^2$ for all t
  - $Cov(\varepsilon_t, \varepsilon_{t+j}) = 0$ for all t and $j \neq 0$
- **Autocovariances**
  - $\gamma(0) = \sigma^2$
  - $\gamma(k) = 0$ for all $k \neq 0$
- **Autocorrelation**
  - $\rho(0) = 1$
  - $\rho(k) = 0$ for all $k \neq 0$
- **PACF**
  - $\alpha(0) = 1$
  - $\alpha(k) = 0$ for all $k > 0$

MA(1): $Y_t = \varepsilon_t + \varphi\varepsilon_{t-1}$

- **Stationarity**
  - $E[Y_t] = E[\varepsilon_t] + \varphi E[\varepsilon_{t-1}]$ for all t
  - $\mathbf{Var}(\boldsymbol{Y_t}) = Var(\varepsilon_t) + \varphi^2 Var(\varepsilon_{t-1}) + 2\varphi Cov(\varepsilon_t,\ \varepsilon_{t-1}) = (\mathbf{1} + \boldsymbol{\varphi}^2)\boldsymbol{\sigma}^2$ for all t
  - $\boldsymbol{Cov(Y_t, Y_{t+1}) = \varphi\sigma^2}$ for all t. $\boldsymbol{Cov(Y_t, Y_{t+k})} = 0$, for all |k| > 1
- **Autocovariances**
  - $\boldsymbol{\gamma(0)} = Var(Y_t) = \boldsymbol{\sigma^2(1 + \varphi^2)}, \boldsymbol{\gamma(1)} = Cov(Y_t, Y_{t+1}) = \boldsymbol{\varphi\sigma^2}$
  - $\boldsymbol{\gamma(k) = 0}$ **for all** $|k| > 1$
- **ACF**
  - $\rho(0) = 1, \rho(1) = \dfrac{\varphi}{1 + \varphi^2}$
  - $\boldsymbol{\rho(k) = 0}$ **for all** $\left|k\right| > 1$
- **PACF**
  - complicated, but does not become 0 at some lag

MA(q): $Y_t = \varepsilon_t + \varphi_1 \varepsilon_{t-1} + \ .. \ + \varphi_q \varepsilon_{t-q}$

- **Stationarity**
  - automatically follows from stationarity of $\{\varepsilon_t\}$
- **Autocovariances**
  - $\gamma(0) = Var(Y_t) = \sigma^2\left(1 + \varphi_1^2 + \ \ldots + \varphi_q^2\right),$
  - $\gamma(k) = \sigma^2(\varphi_k + \varphi_{k+1}\varphi_1 + \varphi_{k+2}\varphi_2 + \ .. \ + \varphi_q\varphi_{q-k})$ **for** $k = 1, \ldots, q$
  - $\gamma(k) = 0$ **for** $|k| > q$

MA(∞)

$$Y_t = \mu + \sum_{j=0}^{+\infty} \varphi_j \varepsilon_{t-j}$$

- **Well-defined and covariance-stationary**, if sequence $\left\{ \varphi_j \right\}_{j=0}^{\infty}$ is absolutely summable, i.e.

$$\sum_{j=0}^{\infty} \left| \varphi_j \right| < \infty$$

AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

- Plug in the expression for $Y_{t-1},\ Y_{t-2,}$ and so on:
  - $Y_t = \theta Y_{t-1} + \varepsilon_t$
  - $Y_{t-1} = \theta Y_{t-2} + \varepsilon_{t-1}$
  - $Y_t = \theta\left(\theta Y_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t = \theta^2 Y_{t-2} + \theta\varepsilon_{t-1} + \varepsilon_t$

$$\boldsymbol{Y_t = \theta^n Y_{t-n} + \sum_{j=0}^{n-1} \theta^j \varepsilon_{t-j}}$$

- If $\left|\theta\right| \geq 1$,as $n \to \infty,\ \theta^n \to \infty,$ and Y_t explodes.
- So we need $\left|\theta\right| < 1$ **forstationarity**.

AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

- **Stationarity:** stationary if $\left|\theta\right| < 1$. Then
  - $E\left[Y_t\right] = 0$ for all t
  - $\mathbf{Var}(\boldsymbol{Y_t}) = \theta^2 Var(Y_{t-1}) + Var\left(\varepsilon_t\right) = \dfrac{\boldsymbol{\sigma^2}}{\mathbf{1} - \boldsymbol{\theta^2}}$ for all t
  - $\boldsymbol{Cov(Y_t, Y_{t-k}) = \theta^k \dfrac{\sigma^2}{1 - \theta^2}}$ for all t, for all k
- **Autocovariances**
  - $\boldsymbol{\gamma\left(k\right) = \theta^k \dfrac{\sigma^2}{1 - \theta^2}}$ for all k
- **ACF**
  - $\boldsymbol{\rho\left(k\right) = \theta^k}$ for all k
- **PACF**
  - $\alpha(1) = \theta$
  - $\alpha(k) = 0$ for all $|k| > 1$

AR(1): $Y_t = \theta Y_{t-1} + \varepsilon_t$

- Can be derived in a different way: $(1 - \theta L)Y_t = \varepsilon_t$, so if $(1 - \theta L)$ has an inverse, Y_t can be written as

$$Y_t = (1 - \theta L)^{-1}\varepsilon_t = \sum_{j=0}^{\infty} \theta^j L^j \varepsilon_t$$

- So it is covariance-stationary, if $\sum_{j=0}^{\infty} |\theta^j| < \infty$, i.e., whenever $|\theta| < 1$.
- Now, $Cov\left(\varepsilon_t,\ Y_{t-1}\right) = \sum_{j=0}^{\infty} \theta^j Cov\left(\varepsilon_t, \varepsilon_{t-j}\right) = 0$, if $Cov\left(\varepsilon_t, \varepsilon_{t-j}\right) = 0$ for all $j > 0$. So, if $\{\varepsilon_t\}$ is a white noise, it holds.
- Also, $E\left[\varepsilon_t \middle| Y_{t-1}\right] = E[\ \varepsilon_t | \varepsilon_{t-1}, \varepsilon_{t-2},\ \ldots]$, so if $\{\varepsilon_t\}$ is an MDS, the regression assumption is satisfied.

What you should know after today:

- Concepts of weak and strong stationarity
- What are ACF and PACF
- What are white noise, AR, MA, and ARMA processes
- What are their characteristics (ACF, PACF)
- How to write AR(1) model using the lag operator and how to derive MA(∞)