2.1. QA

Ex.:- **AR(2)** autoregression → forecast → estimation.

$$y_t = 0.6\,y_{t-1} - 0.05\,y_{t-2} + u_t + 1$$

$u_t \sim N(0;16)$, indep of $y_{t-1}, y_{t-2}, y_{t-3} \dots$

$F_t = \sigma(y_t, y_{t-1}, y_{t-2}, \dots)$

$\vdots$

$y_{99} = 5$

$y_{100} = 6$

a) PI for y_{101} (95%)

b) PI for y_{102} (95%)

$$y_{101} = \underbrace{\boxed{0.6\,y_{100} - 0.05\,y_{99}} + \boxed{1}}_{\text{known}} + \underbrace{u_{101}}_{\text{unpredict.}}$$

$$E(y_{101} \mid F_{100}) = 0.6\,y_{100} - 0.05\,y_{99} + 1 = 0.6 \cdot 6 - 0.05 \cdot 5 + 1 = 0.36 - 0.25 + 1 = 1.11$$

$$Var(y_{101} \mid F_{100}) = Var(u_{101} \mid F_{100}) = Var(u_{101}) = 16$$

PI for y_{101} : $[1.11 - 1.96\sqrt{16}\,;\, 1.11 + 1.96\sqrt{16}]$

QS exam

$$\begin{cases} S_t = S_0 \cdot \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right) \\ dS_t = \mu S_t\, dt + \sigma \cdot S_t\, dW_t \end{cases}$$

$$\begin{cases} S_t = S_0 \cdot \exp\left(\sigma W_t^* + \left(r - \frac{\sigma^2}{2}\right) t\right) \\ dS_t = r S_t\, dt + \sigma \cdot S_t \cdot dW_t^* \end{cases}$$

$$W_t^* = W_t + \frac{\mu - r}{\sigma} t$$

W_t – Wiener Pr. wrt $P(\cdot)$

W_t^* – " — $P^*(\cdot)$

under P^* the discounted price of every asset is a martingale

$$\boxed{X_0 = E_*\left(\exp(-rt) \cdot X_t \mid \mathcal{F}_0\right)}$$

$$M_0 = E_*\left(M_T \mid \mathcal{F}_0\right)$$

$M_0 = X_0$ $\quad M_T = \exp(-rT) X_T$

0 ———————— T

$$X_T = \begin{cases} 1/S_T, & S_T < 1 \\ 0, & S_T \geq 1 \end{cases}$$

$$X_0 = E_*\left(\exp(-rT) \cdot X_T \mid \mathcal{F}_0\right)$$

X_t - price $\quad \frac{X_t}{\exp(rT)}$ discounted price

$$\frac{X_t}{(1+r)^t}$$ in discrete time models.

$$\exp(-rt)\cdot X_t = \frac{X_t}{\exp(rt)}$$ in cont. time models

$$r \approx 0 \quad . \quad \exp(r) \approx 1+r$$

$$\exp(rt) \approx (1+r)^t$$

$$X_0 = \exp(-rT)\cdot E_*(X_T \mid \mathcal{F}_0) =$$

$$= \exp(-rT)\cdot E_*\left(\boxed{\frac{1}{S_T}}\cdot \boxed{I(S_T<1)} \,\Big|\, \mathcal{F}_0\right)$$

indicator

$$S_T = S_0\cdot \exp\left(\sigma\cdot W_T^* + \left(r-\frac{\sigma^2}{2}\right)T\right)$$

$$= \exp(-rT)\cdot E_*\left(\frac{1}{S_0}\cdot \exp\left(-\sigma W_T^* - \left(r-\frac{\sigma^2}{2}\right)T\right)\cdot I \,\Big|\, \mathcal{F}_0\right)$$

$$= \exp(-rT)\cdot\frac{1}{S_0}\cdot \exp\left(\left(\frac{\sigma^2}{2}-r\right)T\right)\cdot \underline{E_*\left(\exp(-\sigma W_T^*)\cdot I \mid \mathcal{F}_0\right)}$$

$$I(S_T<1) = I\left(S_0\cdot \exp\left(\sigma W_T^* + \left(r-\frac{\sigma^2}{2}\right)T\right)<1\right) =$$

$$= I\left(\ln S_0 + \sigma W_T^* + \left(r-\frac{\sigma^2}{2}\right)T<0\right) =$$

$$= I\left(W_T^* < \frac{\left(\frac{\sigma^2}{2}-r\right)T - \ln S_0}{\sigma}\right) = I\left(\frac{W_T^*-0}{\sqrt{T}} < \frac{\dots}{\sigma\sqrt{T}}\right)$$

$W_T^* \sim_{P^*} N(0;T)$

$W_T^* = \sqrt{T}\cdot Z \qquad Z\sim N(0;1)$

$$= \exp(-rT)\cdot\frac{1}{S_0}\cdot\exp\left(\left(\frac{\sigma^2}{2}-r\right)T\right)\cdot E_*\left(\exp(-\sigma W_1^*)\cdot I \mid \mathcal{F}_0\right) =$$

$$= \frac{1}{S_0}\cdot\exp\left(-2rT+\frac{\sigma^2}{2}T\right)\cdot$$

$$\cdot E_*\left(\exp\left(\underbrace{-\sigma\sqrt{T}}_{\alpha}\cdot Z\right)\cdot I\left(Z<\underbrace{\frac{-\ln S_0-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\cdot\sqrt{T}}}_{\beta}\right)\middle|\mathcal{F}_0\right)=$$

$$=\frac{1}{S_0}\exp\left(-2rT+\frac{\sigma^2}{2}T\right)\cdot\exp\left(\frac{\sigma^2T}{2}\right)\cdot F\left(\frac{\left(\frac{\sigma^2}{2}-r\right)T-\ln S_0}{\sigma\sqrt{T}}+\sigma\sqrt{T}\right)$$

$$Z \overset{P^*}{\sim} N(0;1)$$

$$E_*\left(\exp(\alpha Z)\cdot I(Z<\beta)\mid\mathcal{F}_0\right)=\exp\left(\frac{\alpha^2}{2}\right)\cdot F(\beta-\alpha)$$

(check!)

$$=\int_{-\infty}^{\beta}\exp(\alpha\cdot z)\cdot\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{z^2}{2}\right)dz$$

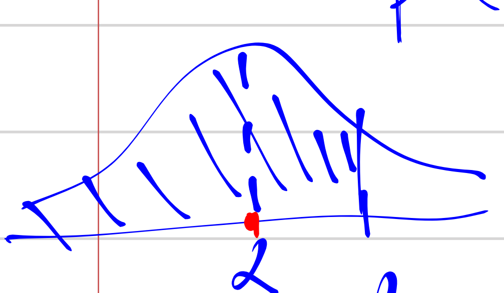
$$\int_{-\infty}^{\beta}\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{z^2-2\alpha z}{2}\right)dz=$$

$$=\int_{-\infty}^{\beta}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2-2\alpha z+\alpha^2-\alpha^2}{2}\right)dz=$$

$$=\int_{-\infty}^{\beta}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z-\alpha)^2}{2}\right)\cdot\exp\left(\frac{\alpha^2}{2}\right)dz= \boxed{\exp\left(\frac{\alpha^2}{2}\right)\cdot F(\beta-\alpha)}$$

$$=\exp\left(\frac{\alpha^2}{2}\right)\cdot\int_{-\infty}^{\beta}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z-\alpha)^2}{2}\right)dz$$

pdf $N(\alpha;1)$



$$P(N(\alpha;1)<\beta)=P(N(0;1)<\beta-\alpha)$$