1. Consider $ETS(ANN)$ model, $\begin{cases} y_t = \ell_{t-1} + u_t \\ \ell_t = \ell_{t-1} + \alpha u_t \\ u_t \sim \mathcal{N}(0;\sigma^2). \end{cases}$ Let $\ell_{99} = 50$, $\alpha = 1/2$, $\sigma^2 = 16$, $y_{98} = 48$, $y_{99} = 52$, $y_{100} = 55$. Calculate 95% predictive interval for y_{101}.

2. Young investor Winnie-the-Crypto compares two trading strategies: buying bitcoins from good bees and from bad bees. Let d_t be the price difference at day t (bad minus good). Winnie-the-Crypto would like to test H_0: $\mathbb{E}(d_t) = 0$ against H_a: $\mathbb{E}(d_t) \neq 0$ at 5% significance level.

   Winnie assumed that (d_t) can be approximated by a $MA(1)$ process and estimated the parameters using $T = 400$ observations, $\hat{d}_t = 2 + u_t + 0.7u_{t-1}$ with $\hat{\sigma}_u^2 = 4$.

   (a) Estimate $\mathbb{E}(d_t)$, $\mathbb{V}\mathrm{ar}(d_t)$ and $\mathbb{C}\mathrm{ov}(d_t, d_{t-1})$.

   (b) Estimate $\mathbb{E}(\bar{d})$, $\mathbb{V}\mathrm{ar}(\bar{d})$ and help Winnie by considering $Z = \frac{\bar{d}-0}{se(\bar{d})}$.

3. The variables X_1, ..., X_n are independent and uniformly distributed on $[0; 2a]$ for some positive a.

   (a) Find any sufficient statistic for a.

   (b) How the answer will change if $X_i \sim U[-a; 2a]$?

4. Consider an estimator $\hat{a}$ with $\mathbb{E}(\hat{a}) = 0.5a + 3$. For the given sample size the Fisher information is $I_F(a) = 400/a^2$.

   (a) What is the theoretical minimal variance of $\hat{a}$?

   (b) Assume that $\hat{a}$ attains the minimal variance boundary and is asymptotically normal. Given that $\hat{a} = 2022$ provide 95% CI for a.

5. You observe X_1, ..., X_{400} and Y_1, ..., Y_{400}, $\bar{X} = 5$, $\bar{Y} = 6$. All variables are independent.

   Consider the null hypothesis that all random variables are exponentially distributed with common parameter λ against alternative that parameter is λ_X for every X_i and λ_Y for every Y_j.

   (a) Estimate common λ using maximum likelihood for the restricted model.

   (b) Estimate both λ_X and λ_Y using maximum likelihood in the unrestricted model.

   (c) Use LR-test to test the null hyphotesis at 5% significance level.

6. The ultimate goal of this exercise is to prove the good upper bound for tail probability of a normal distribution: if $X \sim \mathcal{N}(0;\sigma^2)$ then $\mathbb{P}(X > c) \leq \exp(-c^2/2\sigma^2)$.

   Here are the guiding hints (you free to use not use them):

   (a) State the MGF of X. You may derive it or simply write it if you remember.

   (b) Consider $Y = \exp(uX)$. Using Markov inequality provide the upper bound for $\mathbb{P}(Y > \exp(uc))$.

   (c) Prove that $\mathbb{P}(X > c) \leq MGF_X(u)\exp(-uc)$ for any u.

   (d) Find the value of u that makes the upper bound as tight as possible.

7. (bonus) Draw good bees and bad bees selling crypto. Any funny statistics/math joke is also ok!