Short introduction to ARMA procesess without nonsense. The goal is to state all the theorems rigorously.

Definition 1. The process (u_t) is called white noise if

$$\mathrm{E}(u_t) = 0, \quad \mathrm{Var}(u_t) = \sigma^2, \quad \mathrm{Cov}(u_s, u_t) = 0 \text{ for } s \neq t.$$

This definition does not assume that u_t and u_s are independent. They may be dependent but uncorrelated.

This definition does not assume normality of u_t but normality of white noise is often assumed in maximum likelihood estimation.

Definition 2. Lag operator L transforms a stochastic process (y_t) with $t \in \mathbb{Z}$ into a new stochastic process by shifting the index back in time,

$$Ly_t = y_{t-1}.$$

Definition 3. Forward operator F transforms a stochastic process (y_t) with $t \in \mathbb{Z}$ into a new stochastic process by shifting the index forward in time,

$$Ly_t = y_{t+1}.$$

Simple arithmetic examples are:

$$(1 + 2L + 3L^2)y_t = y_t + 2y_{t-1} + 3y_{t-2},$$

$$(3 + 2F + 5F^2)y_t = 3y_t + 2y_{t+1} + 5y_{t+2},$$

Theorem 4. The operators L and F are linear and $L^{-1} = F$.

Proof. The action LF or FL does nothing with any process (y_t). So operators L and F are mutually inverse. □

Definition 5. The process (y_t) is called stationary in weak sense if

$$\mathrm{E}(y_t) = \mu, \quad \mathrm{Cov}(u_s, u_t) = \gamma(t - s).$$

In particular all variances of stationary process are equal, $\mathrm{Var}(y_t) = \mathrm{Cov}(y_t, y_t) = \gamma_0$.

When infinite sums do exist?

We *define* division by monomials.

Definition 6. For $|\alpha| < 1$ we define

$$\frac{1}{1 - \alpha L} y_t = (1 + \alpha L + \alpha^2 L^2 + \alpha^3 L^3 + \ldots) y_t,$$

and

$$\frac{1}{1 - \alpha F} y_t = (1 + \alpha F + \alpha^2 F^2 + \alpha^3 F^3 + \ldots) y_t.$$

Theorem 7. If (u_t) is a white noise and $|\alpha| < 1$ then $\frac{1-\alpha L}{1-\alpha F} u_t$ and $\frac{1-\alpha F}{1-\alpha L} u_t$ are white noises.

Theorem 8. The equation

$$P(L)y_t = Q(L)u_t + c,$$

where (u_t) is a white noise has infinitely many non-stationary solutions (y_t) if degree of P is higher than one.

Theorem 9. Consider the equation

$$P(L)y_t = Q(L)u_t + c.$$

If polynomials P and Q are coprime then

1. There are no stationary solutions (y_t) at all if P has at least one root ℓ with $|\ell| = 1$.

2. There is exactly one stationary solution (y_t) if all roots ℓ of P have $|\ell| \neq 1$.

There are two subcases when all roots ℓ of P have $|\ell| \neq 1$:

1. All roots ℓ of P have $|\ell| > 1$. In this case the unique stationary solution has the form

$$y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + \ldots,$$

where (u_t) is the white noise from original equation.

2. At least one root of P has $|\ell| < 1$. In this case the unique stationary solution has the form

$$y_t = \mu + \nu_t + c_1 \nu_{t-1} + c_2 \nu_{t-2} + c_3 \nu_{t-3} + \ldots,$$

where (ν_t) is a white noise different from (u_t).

Definition 10. The process (y_t) is called $ARMA(p, q)$ process with equation

$$P(L)y_t = Q(L)u_t + c,$$

if

1. the process (y_t) satisfies this equation;

2. polynomial $P(L)$ has degree p and polynomial $Q(L)$ has degree q;

3. $P(0) = Q(0) = 1$;

4. P and Q are coprime, in other words they have no common roots.

5. the process (y_t) can be represented in $MA(\infty)$ form with respect to (u_t):

$$y_t = \mu + u_t + c_1 u_{t-1} + c_2 u_{t-2} + c_3 u_{t-3} + \ldots$$

From the last requirement in this definition it *follows* that all $ARMA(p, q)$ processes are stationary. By definition. Point.

Not all solutions of equation $P(L)y_t = Q(L)u_t + c$ are called $ARMA$ processes.

Definition 11. The *equation*

$$P(L)y_t = Q(L)u_t + c,$$

is called *invertible* if all roots ℓ of Q have $|\ell| > 1$.

Stationarity is the property of a process, invertibility is the property of an equation. One cannot check whether a given sequence of random variables is invertible. Even more.