Short rules: 120 minutes, one A4 cheat sheet allowed.

1. Consider $ETS(AAN)$ model, $\begin{cases} y_t = \ell_{t-1} + b_{t-1} + u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ b_t = b_{t-1} + \beta u_t \\ u_t \sim \mathcal{N}(0; \sigma^2). \end{cases}$

   Let $\ell_{100} = 50$, $b_{100} = 2$, $\alpha = 0.4$, $\beta = 0.5$, $\sigma^2 = 16$.

   Calculate one step and two steps ahead 95% predictive intervals.

2. Consider the process $y_t = 4 + u_t + u_{t-1} + 2u_{t-2}$, where (u_t) is a white noise with variance 16.

   (a) Is this process stationary? Explain.

   (b) Find the autocorrelation function of this process. Explain the meaning of ρ_2.

   (c) Consider the process $d_t = \Delta y_t$. Is it $ARIMA(p, d, q)$? If yes, then find p, d and q.

3. Consider the stationary $AR(2)$ process $y_t = 5 - 0.9y_{t-1} - 0.2y_{t-2} + u_t$, where (u_t) is a white noise.

   (a) Find the first value of autocorrelation function ρ_1.

   (b) Find the partial autocorrelation function of this process. Explain the meaning of ϕ_{22}.

   (c) What is the relationship between values of autocorrelation function ρ_{100}, ρ_{99} and ρ_{98}.

   Hint: values ϕ_{22}, ϕ_{33} etc may be calculated almost effortlessly :)

4. Consider iid sample from bivariate normal distribution, $\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \theta \\ 2\theta \end{pmatrix}; \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}\right)$.

   Calculate Fischer information for the following cases:

   (a) You observe X_1 only.

   (b) You observe X_1, ..., X_n.

   (c) You observe X_1, ..., X_n, Y_1, ..., Y_n.

   Hint: the multivariate normal density is $f(u) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(u-\mu)^T\Sigma^{-1}(u-\mu)\right)$.

5. Random variables X_1, ..., X_n are independent with density $f(x) = \begin{cases} -\ln(a) \cdot a^x, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

   (a) Estimate a using maximum likelihood.

   (b) Check whether the estimator is unbiased and consistent.

   (c) Check whether the corresponding Cramer-Rao lower bound is attained.

6. Consider the $ARCH(1)$ model, $u_t = \sigma_t \nu_t$, where ν_t are iid $\mathcal{N}(0; 1)$ and $\sigma_t^2 = 1 + 0.3u_{t-1}^2$.

   (a) Find 95% predictive interval for u_{101} if $u_{100} = -2$.

   (b) Find the autocorrelation function of $r_t = u_t^2$.