Class 3.1.6 Hi !! TSSP 2022-01-25

(Weak) Stationarity.

struct → $\boxed{\text{ETS (A A A) model}}$

non struct. → solutions of ARMA eq-n.

Forecast-ng Prin-s and Practice

ETS(A A A)

E T S — seasonality (-//-)

trend (-//-)

error (A - additive)

y_t (observed) TS

$\beta = 0 \iff$ stable slope of trend

unicorns

u_t - source of randomness

l_t ← trend - level (smooth. y_t) (deseasoned y_t)

b_t - trend | growth rate | slope of trend

S_t - seasonal component

ETS(AAA)

$$\begin{cases} u_t \sim N(0; \sigma^2) \text{ independent.} \\ S_t = S_{t-12} + \gamma \cdot u_t \\ b_t = b_{t-1} + \beta \cdot u_t \\ l_t = l_{t-1} + b_{t-1} + \alpha \cdot u_t \\ y_t = l_{t-1} + b_{t-1} + S_{t-12} + u_t \end{cases}$$

S_{t-12}: same month of past year

ETS(AAA)

$[S_0, S_{-1}, S_{-2}, \ldots S_{-11}]$

$S_0 + S_{-1} + S_{-2} + \ldots + S_{-11} = 0$

← b_0

← l_0

init. cond

$$\begin{cases} u_t \sim N(0;\sigma^2) \text{ independent.} \\ S_t = S_{t-12} + \gamma \cdot u_t \\ \quad \text{same month of past year} \\ b_t = b_{t-1} + \beta \cdot u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha \cdot u_t \\ y_t = \ell_{t-1} + b_{t-1} + S_{t-12} + u_t \end{cases}$$

ETS(AAA)

$S_0, S_{-1}, S_{-2}, \dots S_{-11}$

$S_0 + S_{-1} + S_{-2} + \dots + S_{-11} = 0$

$\leftarrow b_0$

$\leftarrow \ell_0$

ETS(AAN)

init. cond

18 par + 1 constraint on them

17 free param

Ex.

a) How many param..s are there? $\sigma^2, S_0 \dots S_{-11}, \ell_0, b_0, \alpha, \beta, \gamma.$

b) $E(y_1), E(y_2), E(y_t)$ c) $Var(y_1), Var(y_2)$

d) Is (y_t) stationary?

const: $\ell_0, b_0, S_0, S_{-1} \dots S_{-11}$

$b_1 = b_0 + \beta \cdot u_1 \leftarrow$ random var-le

$\underbrace{\ell_1}_{\text{random}} = \underbrace{\ell_0 + b_0}_{\text{const}} + \underbrace{\alpha \cdot u_1}_{\text{random}}$

$E(y_1) = E(\ell_0 + b_0 + S_{-11} + u_1) = \ell_0 + b_0 + S_{-11} + 0$

$Var(y_1) = Var(\ell_0 + b_0 + S_{-11} + u_1) = \sigma^2$

$E(y_2) \overset{?}{=} E(\ell_1 + b_1 + S_{-10} + u_2) =$

$= E((\ell_0 + b_0 + \alpha \cdot u_1) + (b_0 + \beta \cdot u_1) + S_{-10} + u_2) =$

$Var(y_2) = Var((\alpha+\beta) \cdot u_1 + u_2) = (\alpha+\beta)^2 \sigma^2 + \sigma^2$

$= \ell_0 + b_0 + 0 + b_0 + S_{-10} + 0 =$

$= \ell_0 + 2b_0 + S_{-10}$

exp trend exp seas

$$\boxed{E(y_{100}) \overset{?}{=} \ell_0 + 100 b_0 + S_{-8}}$$

S_{99} - r.v.

$S_0, S_{-1} \dots S_{-11}$ const

$y_{100} \to S_{88} \to S_{76} \to \dots \to S_{-8}$

def (y_t) is stationary

$$\boxed{\begin{gathered} E(y_t) = \mu \\ Var(y_t) = \gamma_0 \\ Cov(y_t, y_{t-1}) = \gamma_1 \\ Cov(y_t, y_{t-2}) = \gamma_2 \\ Cov(y_t, y_{t-k}) = \gamma_k \end{gathered}}$$

d) $y_t \sim ETS(AAA)$

$E(y_t) \neq const$

$Var(y_t) \neq const$

not stationary

e) $\ell_{100} = 10 \quad b_{100} = 3 \quad S_{89} = -2 \quad S_{90} = 4 \quad \sigma^2 = 9$

$\alpha = \beta = \gamma = \frac{1}{2}$

$\mathcal{F}_{100} = \sigma(y_{100}, y_{99}, \dots)$

95% PI for y_{101} given $\mathcal{F}_{100}$

95% PI for y_{102} given $\mathcal{F}_{100}$

indep

$$E(y_{101} | \mathcal{F}_{100}) = E(\ell_{100} + b_{100} + S_{89} + u_{101} | \mathcal{F}_{100}) =$$

$$= \underbrace{\ell_{100}}_{rv} + \underbrace{b_{100}}_{rv} + \underbrace{S_{89}}_{rv} + \underbrace{E(u_{101})}_{0} =$$

$$= 10 + 3 - 2 = 11$$

we know only params of the model

difference

$E(y_{101})$?

$E(y_{101} | \mathcal{F}_{100})$

we know $y_1 \dots y_{100}$ + param of the model

$$Var(y_{101} | \mathcal{F}_{100}) = \qquad \ell_0 + 10 b_0 + S_{-7}$$

$$= Var(\ell_{100} + b_{100} + S_{89} + u_{101} | \mathcal{F}_{100}) = \qquad \ell_{100} + b_{100} + S_{89}$$

$$= Var(u_{101} | \mathcal{F}_{100}) = Var(u_{101}) = 9$$

PI $\quad [11 - 1.96\sqrt{9};\ 11 + 1.96\sqrt{9}]$

! Warning !

! Achtung !

Equations can have many solutions

Ex.

$u_t \sim N(0;9)$ indep.

$y_t = 0.5 y_{t-1} + u_t + u_{t-1}$ ← ARMA equation

[!] equation can have ∞ solutions [!]

a) $y_0 = 10$ $E(y_1), E(y_2), E(y_t)$? $Var(y_1), Var(y_2)$? is (y_t) stat-ry?

b) $y_0 = u_0$ $E(y_t)$? $Var(y_1)$? $Var(y_2)$ is (y_t) stat-ry?

c*) find y_0 such that (y_t) is stat-nary.

$y_0 = 10$

$y_1 = 5 + u_1 + u_0$ $E(y_1) = 5$

$y_2 = 0.5(5 + u_1 + u_0) + u_2 + u_1$ $E(y_2) = 2.5$

$y_3 = 0.5(=\boxed{y_2}=) + u_3 + u_2$

$E(y_t) = 0.5E(y_{t-1}) + 0 + 0$ $E(y_t) = \frac{10}{2^t}$

$\boxed{(y_t) \text{ is not stat-ry}}$

$Var(y_1) = Var(5 + u_1 + u_0) = 9 + 9 = 18$

$Var(y_2) = Var(0.5u_1 + 0.5u_0 + u_2 + u_1) = Var(\frac{1}{2}u_0 + 1.5u_1 + u_2)$

$= (\frac{1}{2})^2 \cdot 9 + (1.5)^2 \cdot 9 + 1^2 \cdot 9$

b) $E(y_0) = E(u_0) = 0$ => $E(y_1) = 0$ => $E(y_2) = 0$ ....

$$\mathrm{Var}(y_1) = \mathrm{Var}(0.5 \cdot u_0 + u_1 + u_0) =$$

$$= \mathrm{Var}(u_1 + 1.5 u_0) = 9 + 1.5^2 \cdot 9$$

$$\mathrm{Var}(y_2) = \mathrm{Var}(0.5\, y_1 + u_2 + u_1) = \ldots \neq \mathrm{Var}(y_1)$$

(y_t) is not stationary.

c) $y_0 = ?$ to make (y_t) stationary?

Strategy I. sol

$$y_t = u_t + \alpha_1 \cdot u_{t-1} + \alpha_2 \cdot u_{t-2} + \ldots \quad \leftarrow MA(\infty)$$

$$y_1 = u_1 + \alpha_1 \cdot u_0 + \alpha_2 \cdot u_{-1} + \ldots$$

$$y_2 = u_2 + \alpha_1 \cdot u_1 + \alpha_2 \cdot u_0 + \ldots$$

...

$$\mathrm{Var}(y_1) = 1^2 \cdot 9 + \alpha_1^2 \cdot 9 + \alpha_2^2 \cdot 9 + \ldots$$

$$\mathrm{Var}(y_2) = 1^2 \cdot 9 + \alpha_1^2 \cdot 9 + \ldots$$

$$y_t = 0.5\, y_{t-1} + u_t + u_{t-1} \quad \text{orig equat.}$$

$$u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \alpha_3 u_{t-3} = 0.5 \left(u_{t-1} + \alpha_1 \cdot u_{t-2} + \alpha_2 \cdot u_{t-3} + \ldots \right) + u_t + u_{t-1}$$

coef	LHS	RHS
u_t	1	1
u_{t-1}	α_1	$0.5 + 1$
u_{t-2}	α_2	$0.5 \cdot \alpha_1$
u_{t-3}	α_3	$0.5 \cdot \alpha_2$
$\vdots$	$\vdots$	$\vdots$

$\alpha_1 = 1.5$

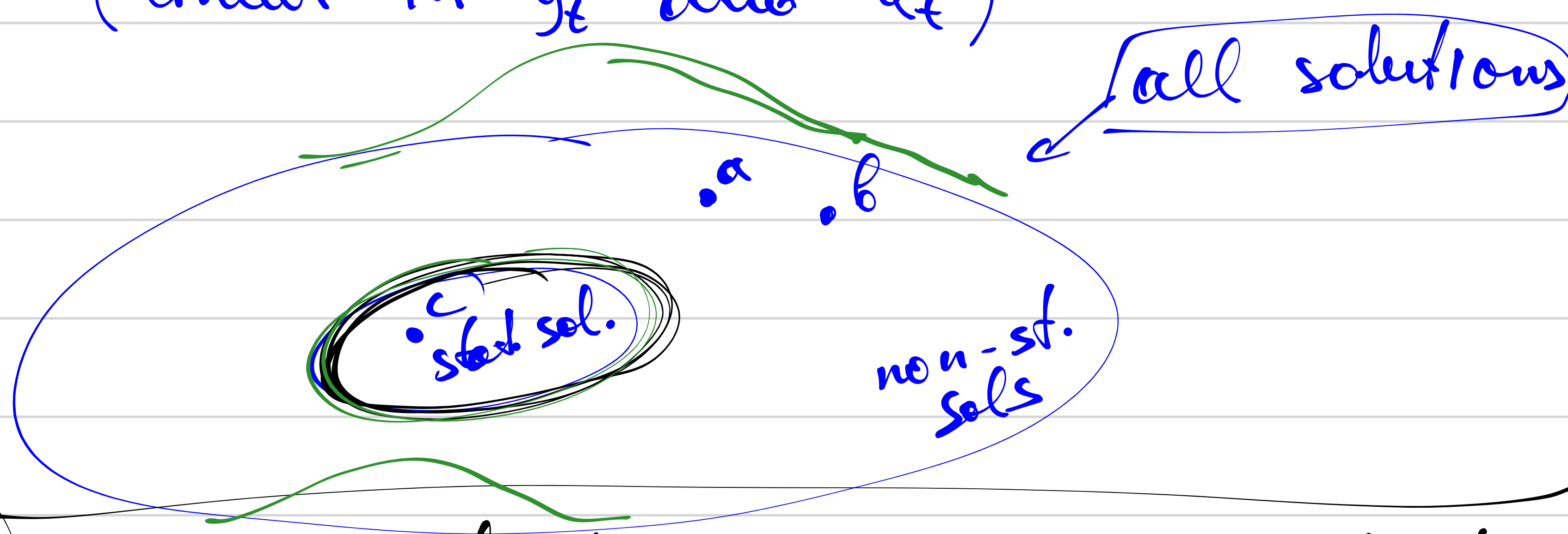
$\alpha_2 = 0.75$

$\vdots$

ARMA-type equation

$$y_t = 0{,}5\,y_{t-1} + u_t + u_{t-1}$$

(linear in y_t and u_t)



Th: If we multiply or divide an ARMA-type eq-n by $P(L)$ with no unit roots then the set of stationary solutions will not change.

$$\boxed{(1-0{,}5L)\cdot y_t = (1+L)\cdot u_t}$$

(*) $$y_t = \frac{1+L}{1-0{,}5L}\,u_t = (1+L)\cdot\frac{1}{1-0{,}5L}\cdot u_t =$$

$$\frac{1}{1-q} = 1+q+q^2+q^3+q^4+\ldots$$

$$= (1+L)\cdot\left(1+0{,}5L+(0{,}5L)^2+(0{,}5L)^3+\ldots\right)\cdot u_t$$

$$= \left(1+L\cdot(1+0{,}5)+L^2\cdot(0{,}5+0{,}5^2)+L^3(0{,}5^2+0{,}5^3)+\ldots\right)u_t$$

$$\boxed{y_t = u_t + 1.5\,u_{t-1} + (0{,}5+0{,}5^2)\,u_{t-2} + (0{,}5^2+0{,}5^3)\,u_{t-3} + \ldots}$$

(*) $$y_0 = u_0 + 1{,}5\,u_{-1} + (0{,}5+0{,}5^2)\cdot u_{-2} + \ldots\ldots$$

init cond

$$\Rightarrow \quad y_1 = u_1 + 1.5\,u_0 + \ldots$$

$$y_2 = u_2 + 1.5\,u_1 + \ldots$$