2021-12-17 Hi !! Thumbes !!

HAS n7 (Bonus)

7. (bonus) Find the following limits in L^2.

(a)

$$S_n = \sum_{i=1}^{n} (t/n)\,(W(it/n) - W((i-1)t/n)) . \xrightarrow{L^2} 0$$

mnemonic rule

$dt \cdot dW_t = 0$

$0 \quad \frac{t}{n} \quad \frac{2t}{n} \quad t$

(b)

$$T_n = \sum_{i=1}^{n} (W(it/n) - W((i-1)t/n))^3 . \xrightarrow{L^2} 0$$

$(dW_t)^3 = 0$

$dW \cdot dW \cdot dW = 0$

$E(T_n) = 0$

$Var(T_n) \to (\dots) \to 0$

$T_n = L_1^3 + L_2^3 + \dots L_n^3$

$L_i \sim N(0; \frac{t}{n})$

$E(L_i) = 0 \quad E(L_i^3) = 0$

$S_1, S_2, S_3, \dots$

$S_1 = \frac{t}{1} \cdot (W(t) - W(0))$

$S_2 = \frac{t}{2} \cdot (W(\frac{t}{2}) - W(0)) + \frac{t}{2}(W(t) - W(\frac{t}{2}))$

$S_3 = \frac{t}{3}(W(\frac{t}{3}) - W(0)) + \frac{t}{3}(W(\frac{2t}{3}) - W(\frac{t}{3})) + \frac{t}{3}(W(t) - W(\frac{2t}{3}))$

$\frac{L_i - 0}{\sqrt{\frac{t}{n}}} = Z_i \quad Z_i \sim N(0;1)$

$T_n = (Z_1 \sqrt{\frac{t}{n}})^3 + (Z_2 \sqrt{\frac{t}{n}})^3 + \dots$

$Var(T_n) = n \cdot Var(Z_1^3 \cdot (\sqrt{\frac{t}{n}})^3) =$

$= n \cdot \frac{t^3}{n^3} \cdot \underbrace{Var(Z_1^3)}_{=const} = \frac{t^3 \cdot const}{n^2} \xrightarrow{n\to\infty} 0$

$S_n = \boxed{\frac{t}{n}} \underbrace{(W(\frac{t}{n}) - W(0))}_{N(0; \frac{t}{n})} + \dots$

$\to 0 \qquad \to 0$

$Var(\bar{X}) = \frac{Var(X_1)}{n} = \frac{n Var(X_1)}{n^2}$

limit of S_n in L^2 $\quad S$

$$E((S_n - S)^2) \xrightarrow{n\to\infty} 0$$

lemma If $E(S_n) \to \mu$ and $Var(S_n) \to 0$ then

$$S_n \xrightarrow{L^2} \mu$$

$$E(S_n) = \sum_{i=1}^{n} \frac{t}{n} \cdot E\Big(W(\tfrac{it}{n}) - W(\tfrac{(i-1)t}{n})\Big) = 0.$$

$$Var(S_n) = \frac{Var(t \cdot (W(\frac{it}{n}) - W(\frac{(i-1)t}{n})))}{n} = \frac{const}{n} \xrightarrow{n\to\infty} 0$$

conclusion

$S_n \xrightarrow{L^2} 0$

Q

дано: $b_1 > 0$ $q>1$: какие м. быть q?

$b_n > b_1 + b_2 + \ldots + b_{n-1}$

$b_{n+1} > \sum_{i=1}^{n} b_i = \frac{b_1 - b_{n+1}}{1-q}$ ⟵ $\forall n \geq 1$

$b_1 \cdot q^n > \frac{b_1(1-q^n)}{1-q}$

$q^n > \frac{1-q^n}{1-q}$

$q^n - q^{n+1} < 1 - q^n$

$2q^n - q^{n+1} < 1$

$2 - q < \frac{1}{q^n}$ $\forall n \geq 1$

одновр. $\begin{cases} 2-q < \frac{1}{q} \\ 2-q < \frac{1}{q^2} \\ 2-q < \frac{1}{q^3} \\ \vdots \end{cases}$

$2 - q \leq 0.$

HA6 Q3 B)

$dS_t = \mu S_t dt + \sigma S_t dW_t$

time $= 0$ ⟶ time $= T$

$A \xrightarrow{X_0} B$ $B \xrightarrow{S_T^2} A$

$X_T = S_T^2$

$X_0^{disc} = E^*(X_T^{disc} \mid \mathcal{F}_0)$

in cont. time

$$X_t^{disc} = \frac{X_t}{\exp(rt)} = \exp(-rt) X_t$$

in discr. time

$$X_t^{disc} = \frac{X_t}{(1+r)^t}$$

$$X_0^{disc} = E^*\left(X_T^{disc} \mid \mathcal{F}_0\right)$$

$\tilde{E}$

$$\frac{X_0}{\exp(r\cdot 0)} = E^*\left(\frac{X_T}{\exp(rT)} \,\Big|\, \mathcal{F}_0\right) \qquad X_T = S_T^2$$

1

$$X_0 = \exp(-rT)\cdot E^*\left(S_T^2 \mid \mathcal{F}_0\right)$$

express S_T using W_T^*

$$S_T \to S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\cdot T + \sigma W_T\right)$$

$$S_T \to S_0 \cdot \exp\left(\left(r - \frac{\sigma^2}{2}\right)\cdot T + \sigma W_T^*\right) \qquad E^*(W_T^*) \overset{?}{=} 0$$

$$W_T^* = \frac{\mu - r}{\sigma}T + W_T$$

(W_T) is a Wiener Process under $P(.)$

(W_T^*) — " — under $P^*(.)$

real

artificial

S_T^2

$$X_0 = \exp(-rT)\cdot E^*\left(S_0^2 \exp\left((2r - \sigma^2)\cdot T + 2\sigma W_T^*\right) \Big| \mathcal{F}_0\right) =$$

$$= \exp(-rT)\cdot S_0^2 \cdot \exp\left((2r - \sigma^2)T\right)\cdot E^*\left(\exp(2\sigma W_T^*)\right) =$$

$$X_0 = S_0^2 \cdot \exp\left((r - \sigma^2)T\right)\cdot E^*\left(\exp(\underbrace{\boxed{2\sigma\sqrt{T}}}_{\alpha} \cdot Z)\right)$$

$$X_0 = S_0^2 \cdot \exp\left((r - \sigma^2)T\right)\cdot \exp\left(4\sigma^2 T/2\right) \ominus$$

$$\boxed{MGF(u) = E(\exp(uZ))} \quad u = 2\sigma\sqrt{T}$$

→ integral

→ old result

$$\boxed{E\left(\exp(\alpha Z)\cdot I(Z \le \beta)\right) = \exp\left(\frac{\alpha^2}{2}\right)\cdot F(\beta - \alpha)}$$

pdf

cdf

$$E\left(\exp(\alpha Z)\cdot \underbrace{I(Z \le +\infty)}_{1}\right) = \exp\left(\frac{\alpha^2}{2}\right)\cdot \underbrace{F(\infty)}_{1} = \exp\left(\frac{\alpha^2}{2}\right)$$

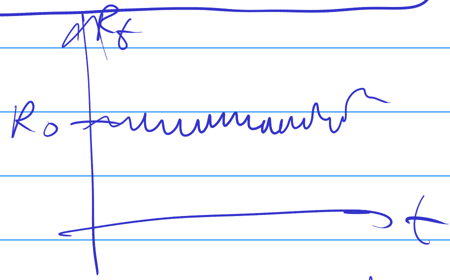
MGF for $N(0;1)$

$$X_0 \ominus S_0^2 \cdot \exp(rT + \sigma^2 T)$$ ▨

HA 6 N 4

$dR_t = 5(0.06 - R_t)\,dt + 3\,dW_t \qquad R_0 = 0.07$

(c)



4. Consider the Vasicek interest rate model,

$$dR_t = 5(0.06 - R_t)\,dt + 3\,dW_t, \qquad R_0 = 0.07.$$

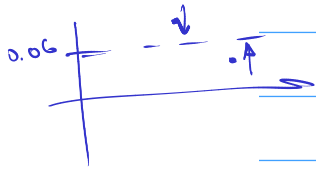
Here R_t is the interest rate.

(a) Using the substitution $Y_t = e^{at}R_t$ find the solution of the stochastic differential equation. Start by finding dY_t.

(b) Find $\mathbb{E}(R_t)$ and $\mathbb{V}\mathrm{ar}(R_t)$.

(c) Which value in this model would you call long-term equilibrium rate and why?

Hint: you may have integrals in you expression for R_t, but no R_t.



guess R_t has e^{-5t} inside

$dR_t = 0.3\,dt - 5R_t\,dt + [\ldots]$

$Y_t = \exp(5t)\cdot R_t$

plug in

$dY_t = \exp(5t)\cdot dR_t + 5\exp(5t)R_t\,dt +$
$+ \frac{1}{2}\cdot 0\cdot (dR_t)^2$

$\frac{dy_t}{dt} = -5y_t$

$y_t = \exp(-5t)c$

$dY_t = \exp(5t)\,(5(0.06 - R_t)\,dt + 3\,dW_t) + 5\exp(5t)R_t\,dt =$

$dY_t = 5\cdot 0.06\cdot \exp(5t)\,dt + 3\exp(5t)\,dW_t$

$Y_t = Y_0 + 5\int_0^t 0.06\cdot \exp(5u)\,du + \int_0^t 3\exp(5u)\,dW_u$

$\frac{0.06(\exp(5t) - 1)}{\cancel{5}}$

$Y_t = \exp(5t)\cdot R_t$

$R_0 = 0.07$

$\exp(5t)\cdot R_t = \exp(5\cdot 0)\cdot 0.07 + \frac{0.06(\exp(5t) - 1)}{\cancel{5}} + 3\int_0^t \exp(5u)\,dW_u$

$$\exp(5t)\cdot R_t = \exp(5\cdot 0)\cdot 0.07 + \frac{0.06\,(\exp(5t)-1)}{\cancel{5}} + 3\int_0^t \exp(5u)\,dW_u$$

$$R_t = \exp(-5t)\cdot 0.07 + \frac{0.06\,(1-\exp(-5t))}{\cancel{5}} + 3\exp(-5t)\cdot\int_0^t \exp(5u)\,dW_u$$

B) $$E(R_t) = \exp(-5t)\cdot 0.07 + 0.06(1-\exp(-5t)) + 0 =$$

$$= 0.06 + 0.01\cdot\exp(-5t)$$