Brownian Motion

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Ito's Lemma

3.1. **First variation.** Take $(X_t)_{t \in \mathbb{R}_+}$ to be any stochastic process. (This, of course, includes the case of deterministic functions of time.) Imagine that X_t denotes the position at time $t \in \mathbb{R}_+$ of a particle in motion. We wish to define here a quantification of the distance travelled by the particle on the interval $[0, t]$. If we split the interval $[0, t]$ into n equally-spaced subintervals, an approximation for the distance travelled would be

$$\sum_{i=1}^{n} \left| X_{ti/n} - X_{t(i-1)/n} \right| .$$

One then gets the total distance travelled by taking the limiting case where the number of subintervals converges to infinity. We define

$$\mathsf{Var}(X)_t := \lim_{n \to \infty} \sum_{i=1}^{n} \left| X_{ti/n} - X_{t(i-1)/n} \right| .$$

It is easy to see, using the definition of the Riemann integral, that *if X is continuously differentiable with derivative X', then*

$$\mathsf{Var}(X)_t = \int_0^t |X'_u| \mathrm{d}u.$$

Ito's Lemma

3.2. **Quadratic variation.** Even through Brownian motion has infinite first variation, we shall see now that its second-order variation is finite. It is exactly this fact that allows for a nice development of stochastic calculus. Take $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (X_t)_{t\in\mathbb{R}_+}$ to be any stochastic processes. (As before, this includes the case of deterministic functions of time.) Define the path-wise *covariation of X and Y over* $[0, t]$ to be

$$[X, Y]_t := \lim_{n\to\infty} \sum_{i=1}^{n} \left(X_{ti/n} - X_{t(i-1)/n}\right)\left(Y_{ti/n} - Y_{t(i-1)/n}\right), \quad t \in \mathbb{R}_+,$$

provided that the previous limit exists. In particular, when $X = Y$ we have

$$[X, X]_t := \lim_{n\to\infty} \sum_{i=1}^{n} \left(X_{ti/n} - X_{t(i-1)/n}\right)^2, \quad t \in \mathbb{R}_+,$$

and $[X, X]_t$ is called the *quadratic variation of X over* $[0, t]$. In a very informal way we can write $[X, Y]_t = \int_0^t (\mathrm{d}X_u)(\mathrm{d}Y_u)$ and $[X, X]_t = \int_0^t (\mathrm{d}X_u)^2$; equivalently (and equally informally), $\mathrm{d}[X, Y]_t = (\mathrm{d}X_t)(\mathrm{d}Y_t)$ and $\mathrm{d}[X, X]_t = (\mathrm{d}X_t)^2$ holds for all $t \in \mathbb{R}_+$.

Proposition 3.2. *Suppose that $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$ be two stochastic processes with continuous paths, one of which has finite first variation over $[0, t]$ for some $t \in \mathbb{R}_+$. (In particular, one of them can be continuously differentiable in $[0, t]$.) Then, $[X, Y]_t = 0$ holds.*

Proof. Without loss of generality, suppose that $\mathsf{Var}(Y)_t < \infty$. For each $n \in \mathbb{N}$, define the random variable $\chi_n := \sup_{i=1,\dots,n} \left|X_{ti/n} - X_{t(i-1)/n}\right|$. Since X has continuous paths, and since a continuous function defined on a compact interval is uniformly continuous, we obtain that $\lim_{n\to\infty} \chi_n = 0$. Continuing, observe that

$$\begin{aligned} \left|\sum_{i=1}^{n} \left(X_{ti/n} - X_{t(i-1)/n}\right)\left(Y_{ti/n} - Y_{t(i-1)/n}\right)\right| &\leq \sum_{i=1}^{n} \left|\left(X_{ti/n} - X_{t(i-1)/n}\right)\left(Y_{ti/n} - Y_{t(i-1)/n}\right)\right| \\ &\leq \chi_n \sum_{i=1}^{n} \left|Y_{ti/n} - Y_{t(i-1)/n}\right| = \chi_n \mathsf{Var}(Y)_t. \end{aligned}$$

Since $\mathsf{Var}(Y)_t < \infty$ and $\lim_{n\to\infty} \chi_n = 0$ hold, we obtain that $[X, Y]_t = 0$. □

In particular, if $X_t = Y_t = t$ for all $t \in \mathbb{R}_+$, Proposition 3.2 implies that $(\mathrm{d}t)^2 = 0$, in our informal notation. Furthermore, taking $X_t = t$ for all $t \in \mathbb{R}_+$ and $Y = W$ to be a standard Brownian motion, we obtain, in an informal notation again, $(\mathrm{d}t)(\mathrm{d}W_t) = 0$.

The informal rules we have established above can be used to compute quadratic variations of more complicated processes; more importantly, the results we obtain using these informal rules can be shown to be formally correct as well, using the definition of quadratic (co)variation. For example, let X be a $\mathsf{BM}(x, \mu, \sigma^2)$. Then, we have $X_t = x + \mu t + \sigma W_t$ for all $t \in \mathbb{R}_+$, which implies that $\mathrm{d}X_t = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$. Then,

$$(\mathrm{d}X_t)^2 = (\mu\mathrm{d}t + \sigma\mathrm{d}W_t)^2 = \mu^2(\mathrm{d}t)^2 + 2\mu\sigma(\mathrm{d}t)(\mathrm{d}W_t) + \sigma^2(\mathrm{d}W_t)^2 = \sigma^2\mathrm{d}t,$$

following since $(\mathrm{d}t)^2 = 0 = (\mathrm{d}t)(\mathrm{d}W_t)$ and $(\mathrm{d}W_t)^2 = \mathrm{d}t$. The relationship $(\mathrm{d}X_t)^2 = \sigma^2\mathrm{d}t$ really means that $[X, X]_t = \sigma^2 t$ holds for all $t \in \mathbb{R}_+$, and this is how it should be read.

The following exercise will be of use when we discuss the Itô-Doeblin formula.

Ito’s Lemma

We finally conclude that

$$X_tY_t = X_0Y_0 + \int_0^t X_u \mathrm{d}Y_u + \int_0^t Y_u \mathrm{d}X_u + [X,Y]_t \tag{2.1}$$

Equation (2.1) is call the *integration-by-parts formula*, and it is usually given in differential form:

$$\mathrm{d}\,(X_tY_t) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t + \mathrm{d}X_t\mathrm{d}Y_t, \quad t \in \mathbb{R}_+. \tag{2.2}$$

BS Model

- Consider a stock with spot price S
- Assume in a short period of time of length Δt, the return on the stock is normally distributed:

$$\frac{\Delta S}{S} \approx \varphi(\mu \Delta t, \sigma^2 \Delta t)$$

where μ is expected return and σ is volatility.

Below example for Apple stocks (AAPLUS)

* - *yahoo.finance.com*

BS Model

- It follows from assumption of normality of returns that stock prices (S_T) are lognormally distributed

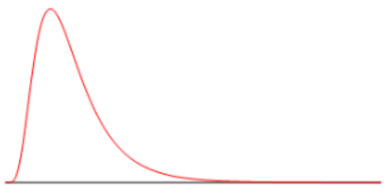
$$\ln S_T \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

where S_T is stock price at time T, S_0 is stock price at time 0, μ is expected return on stock per year, σ is volatility of the stock price per year; $N(\mu, \sigma)$ is normal distribution with mean μ and standard deviation σ

- Continuously compounded annual return of a stock price (x) is normally distributed with mean $\mu - \frac{\sigma^2}{2}$ and a standard deviation $\frac{\sigma}{\sqrt{T}}$

- $x \sim \varphi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$

- Expected value of S_T is $ES_T = S_0 e^{\mu T}$



$E(S_T) = S_0\, e^{\mu T}$

$\mathrm{var}(S_T) = S_0^{\,2} e^{2\mu T}(e^{\sigma^2 T} - 1)$

- Expected Return in Δt is μ whereas in T is $\mu - \frac{\sigma^2}{2}$

- Note that mean return will always be slightly less than the expected return μ

BS Model

- What is volatility? The volatility is a standard deviation of continuously compounded rate of return in 1 year (recall $\frac{\sigma}{\sqrt{T}}$ from previous slide)

- The volatility for short periods of time can be scaled to longer periods in time by multiplying it by the square root of the number of periods ($\sigma \Rightarrow \sigma\sqrt{N}$)

- The volatility estimation process is a matter of collecting daily price data (S_i) and then computing the standard deviation of the series of corresponding continuously compounded returns $ln(\frac{S_i}{S_{i-1}})$

- The annualized volatility is simply the estimated volatility multiplied by the square root of the number of trading days in a year

BS Model

Historical volatility calculation involves

- Converting a time series of N+1 prices S_i to N returns: $R_i = \frac{S_i - S_{i-1}}{S_{i-1}}$
- Calculate continuously compounded returns: $R_i^c = \ln(1 + R_i) = \ln(S_i / S_{i-1})$
- Calculate the variance $\sigma^2 = \frac{\sum\left(R_i^c - \overline{R_i^c}\right)^2}{N-1}$
- Take a square root of the variance

$$s = \sqrt{\frac{1}{N-1} \sum (R_i^c - \overline{R_i^c})^2}$$

- It follows that σ can be estimated as $\hat{\sigma}$

$$\hat{\sigma} = s / \sqrt{\tau}$$

Where τ is an interval in years, for monthly data τ =1/12, weekly data τ = 1/52

BS Model

1. Underlying asset follows a **lognormal distribution**

2. **No arbitrage**

3. The volatility of the underlying asset is **constant and known**.

4. **No transaction costs or taxes**, all securities are divisible

5. **No dividends** during life of options

6. Security trading is **continuous**

7. Borrow and lend at **constant risk-free** rate

8. **No early exercise**

NB: some of assumptions above may be relaxed (e.g. r and σ can variable but time dependent)

BS Model

- Option price and stock price depend on the same source of uncertainty (for Δf we may use Ito's lemma)

$$\Delta S = \mu S \, \Delta t + \sigma S \, \Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \, \Delta z$$

- Create a **riskless** portfolio out of stock and option (to eliminate uncertainty)

- Question how many shares per 1 derivative? Answer: $\frac{\partial f}{\partial S}$ shares

  -1 derivative + $\frac{\partial f}{\partial S}$ shares

- The portfolio **instantaneous** return should be risk-free rate

- This leads to BSM **differential** equation – see derivation slides

BS Model

- Value of the portfolio Π

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

- Change in value of the portfolio Π in time Δt

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

- Instantaneous return of this portfolio should be risk free

$$\Delta\Pi = r\,\Pi\Delta t$$

$$-\Delta f + \frac{\partial f}{\partial S}\Delta S = r\left(-f + \frac{\partial f}{\partial S} S\right)\Delta t$$

- Let's substitute Δf and ΔS from previous slide to get **BSM differential equation**

BS Model

- BSM differential equation can be solved for call and put boundary conditions

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

- The formulas for the BSM model are:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT}\left(1 - N(d_2)\right) - S_0\left(1 - N(d_1)\right)$$

Where $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + (0.5\sigma^2)\right)T}{\sigma\sqrt{T}}, d_2 = d_1 - (\sigma\sqrt{T}),$

T is the time to maturity (in % of a 365-day year)

S_0 is the asset price

K is the exercise price

r - continuously compounded risk-free rate

σ – volatility of continuously compounded returns on the stock

N() is the cumulative normal probability

Heat Equation

BS Model

IMPLIED VOLATILITY ESTIMATION

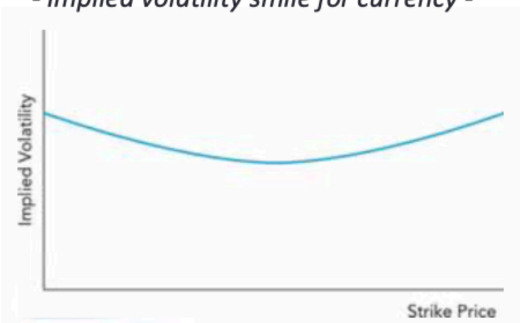
- Implied volatility is the value for standard deviation of continuously compounded rates of return that is “implied” by the market price of the option

- Of the five inputs into the BSM model, four are observable: stock price S_0, exercise price K, risk-free rate r, and time to maturity T. If we use these four inputs in the formula and set the BSM formula equal to market price, we can solve for the volatility σ that satisfies the equality

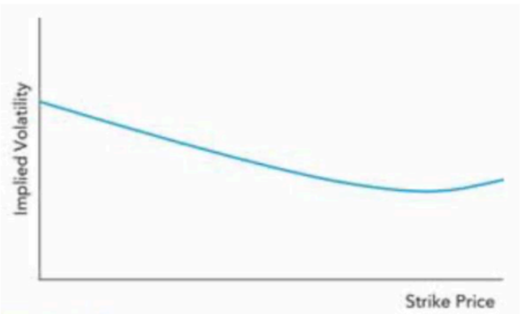
- There is no closed-form solution for the volatility that will satisfy the equation

- Various numerical techniques may be used to “imply” volatility, examples of implied volatility graphs below. To the left on currency implied vol, right - stock

- Implied volatility smile for currency -



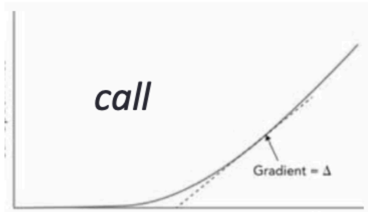
- Implied volatility smile for equity -

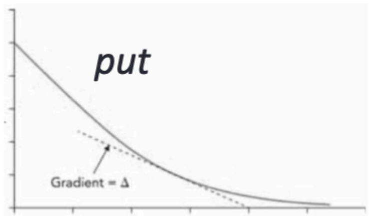


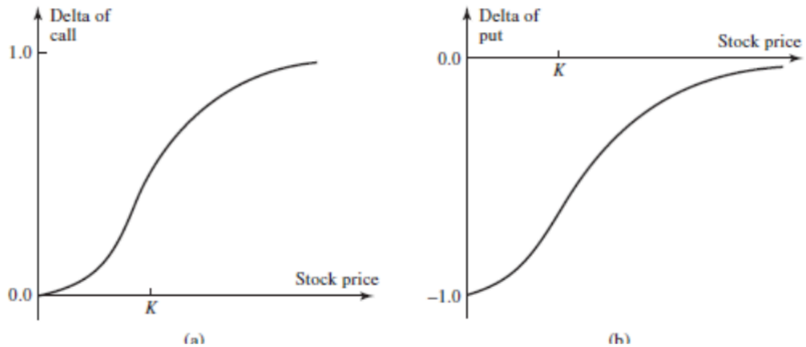
BS Model

- General formula for all types of derivatives $\Delta = \frac{\partial f}{\partial S}$

Derivative	Delta	Properties
Forward	1	The same Δ as for underlying
Future	e^{rT} on a stock/ index w/o dividends $e^{(r-q)T}$ on a stock/index with dividends	Daily settlement
Call option	$N(d_1)$	Positive S ↑ → c ↑
Put option	$N(d_1) - 1$	Negative S ↑ → p ↓





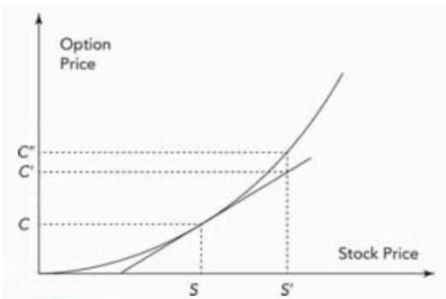


BS Model

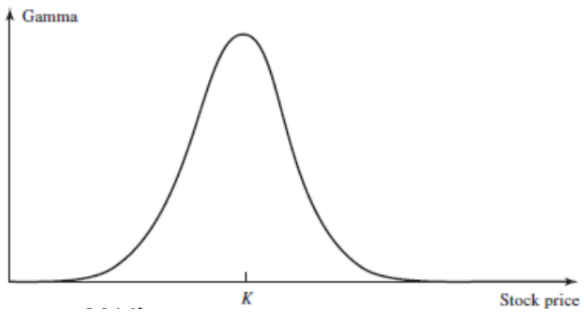
GAMMA

- Gamma measures sensitivity of delta to the stock price $\Gamma = \frac{\partial \Delta}{\partial s}$
- Alternatively $\Gamma = \frac{\partial^2 f}{\partial s^2}$ where f is the derivative price, and s is the price of underlying
- For European call/put options on non-dividend stocks is following: $\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$
- **Why needed?** Delta-neutral positions can hedge the portfolio against small changes in stock price, while gamma can help hedge against relatively large changes in stock price
- Underlying assets/forward instruments have linear payoffs → they have **zero gamma**
- Number of options that must be added to an existing portfolio to generate a gamma-neutral position is $-\left(\frac{\Gamma_p}{\Gamma_T}\right)$, where Γ_p is the gamma of the existing portfolio position, and Γ_T is the gamma of a traded option that can be added

Curvature of option price profile -



Gamma profile of call/put -



BS Model

THETA

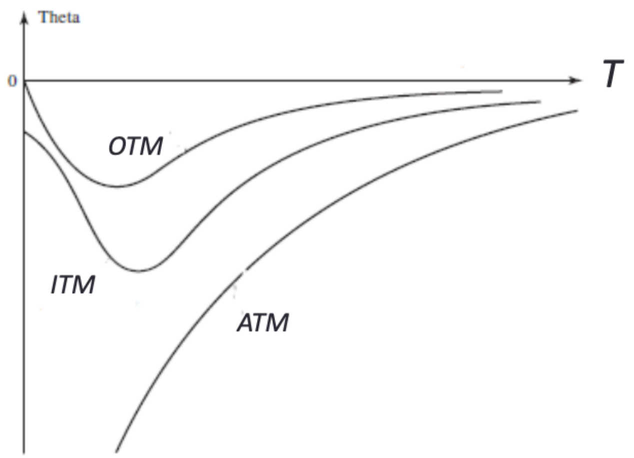
- Theta measures the change in value of derivative (f) over time. Theta is also termed the "time decay" of an option. $\Theta = \frac{\partial f}{\partial t}$
- Note that theta in the below equations is measured in years. It can be converted to a daily basis by dividing by 365
- For European call options on non-dividend-paying stocks, theta can be calculated using the Black-Scholes-Merton formula as follows:

$$\Theta_{call} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

$$\Theta_{put} = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

- Theta profile of call depending on time-

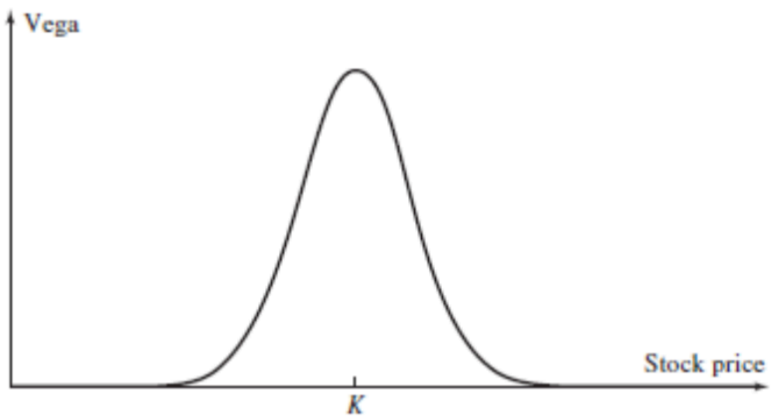


BS Model

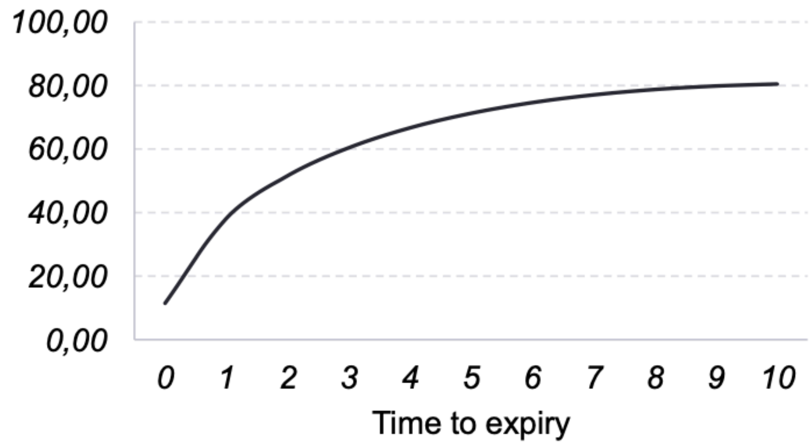
VEGA

- Vega measures the sensitivity of the derivative's price to changes in the volatility of the underlying: $vega = \frac{\partial f}{\partial \sigma}$
- Vega for European calls and puts on non-dividend-paying stocks is calculated as in BSM framework: $vega = S_0 N'(d_1)\sqrt{T}$
- Options are most sensitive to changes in volatility when they are at-the-money. Deep out-of-the-money or deep in-the-money options have little sensitivity to changes in volatility (i.e., vega is close to zero)

- Vega profile of call/put vs Price -



-Vega profile of ATM call vs Time to expiry -



BS Model

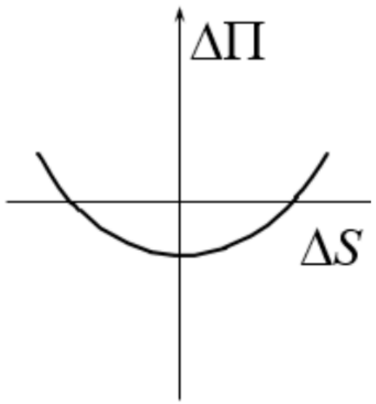
THETA AND GAMMA RELATIONSHIP

- Stock option prices are affected by delta, theta, and gamma as indicated in the following relationship:

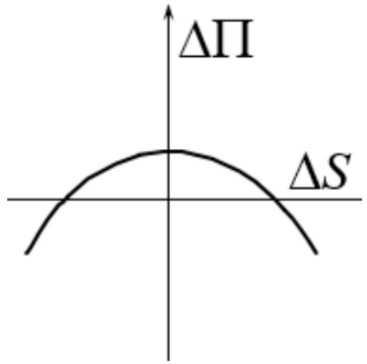
$$r\Pi = \Theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$$

where
r is the risk-neutral risk-free rate of interest
Π = the price of the option
Θ - option theta usually negative for long option position
S = the price of the underlying stock
Δ = the option delta
σ^2 = the variance of the underlying stock
Γ = option gamma, for delta-neutral portfolio negatively related to theta

Γ positive



Γ negative

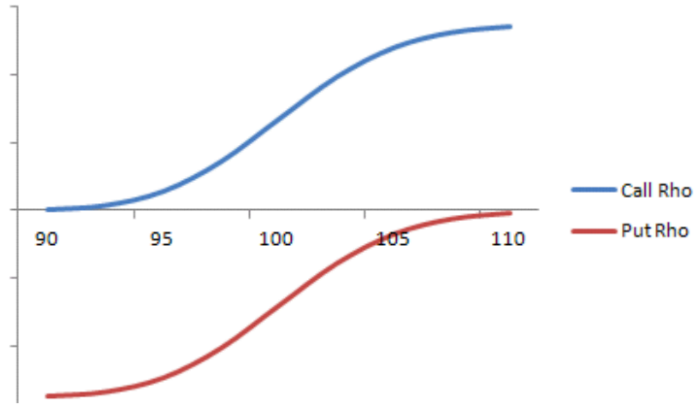


BS Model

RHO

- Rho derivative value f sensitivity to changes in the risk-free rate $r : \rho = \frac{\partial f}{\partial r}$
- For European calls on a non-dividend-paying stock, rho is measured as: $\rho_{call} = KTe^{-rT}N(d_2), \rho_{put} = -KTe^{-rT}N(-d_2)$
- In-the-money calls and puts are more sensitive to changes in rates than out-of-the-money options. Increases in rates cause larger increases for in-the-money call prices (versus out-of-the-money calls) and larger decreases for in-the-money puts (versus out-of-the-money puts)

- Rho profiles of call/put. Rho vs price -



2.2. **Vasicek's interest rate model.** We now present an application of the previous theory to the study of a quite famous model for the stochastic behaviour of the short rate, due to Vasicek. It is assumed that the instantaneous interest rate $(r_t)_{t\in\mathbb{R}_+}$ satisfies the dynamics

$$\mathrm{d}r_t = a(\rho - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{2.3}$$

where W is a standard Brownian motion, and a, ρ and σ are strictly positive real numbers. The above dynamics in (2.3) result in r being a so-called *mean-reverting* process: when far from level ρ, r tends to revert back toward that level with amplitude determined by a.

Ito’s Lemma

3. Itô-Doëblin Formula

3.1. **The formula.** Suppose that $X = (X_t)_{t\in\mathbb{R}_+}$ is an Itô process as in (1.11). If X is actually differentiable (i.e., if $\sigma \equiv 0$), the usual chain rule of calculus implies that $\mathrm{d}f(X_t) = f'(X_t)\mathrm{d}X_t$ holds for all $t \in \mathbb{R}_+$ and continuously differentiable functions f; in other words, $f(X_t) = f(X_0) + \int_0^t f'(X_u)\mathrm{d}X_u$. The Itô-Doeblin formula is a generalisation of this in the case where the "dW" part of an Itô process is not vanishing.

Theorem 3.1. *Suppose that $X = (X_t)_{t\in\mathbb{R}_+}$ is an Itô process, and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function that is twice continuously differentiable. Then, $(f(X_t))_{t\in\mathbb{R}_+}$ is also an Itô process; in fact,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)\mathrm{d}X_u + \frac{1}{2}\int_0^t f''(X_u)\mathrm{d}[X,X]_u \tag{3.1}$$

holds for all $t \in \mathbb{R}_+$.

Thank you for your attention!

See next week!