Time Series

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Markov process

Definition

$\{x(t,\omega)\}_{t\in J}$ is called a **Markov process (MP)**, if the following **Markov property** holds: for any $t_0 \leq \tau \leq t \leq T$ an all $A \in \mathcal{B}^n$

$$\boxed{\mathrm{P}\left\{x(t,\omega) \in A \mid \mathcal{F}_{[t_0,\tau]}\right\} \overset{a.s.}{=} \mathrm{P}\left\{x(t,\omega) \in A \mid x(\tau,\omega)\right\}}$$

Markov process

Let the *phase space* of a Markov process $\{x(t,\omega)\}_{t\in\mathcal{T}}$ be *discrete*, that is,

$$\boxed{\begin{array}{c} x(t,\omega) \in \mathcal{X} := \{(1,2,\dots,N) \text{ or } \mathbb{N} \cup \{0\}\} \\ \mathbb{N} = 1,2,\dots \text{ is a countable set, or finite} \end{array}}$$

Definition

A Markov process $\{x(t,\omega)\}_{t\in\mathcal{T}}$ with a discrete phase space X is said to be a **Markov chain (or Finite Markov Chain** if $\mathbb{N}$ is finite)

a) in continuous time if

$[\ )[\ )[\ )\dots$

$t_0 \quad t_1 \quad t_2 \quad \dots$

$$\mathcal{T} := [t_0, T)\,,\quad T \text{ is admitted to be } \infty$$

b) in discrete time if

$$\mathcal{T} := \{t_0, t_1, \dots, t_T\}\,,\quad T \text{ is admitted to be } \infty$$

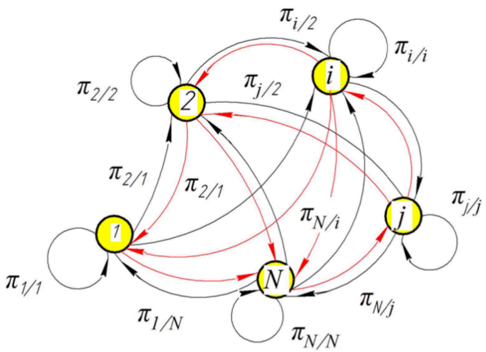
Markov process

Homogeneous

Definition

A Markov Chain is said to be Homogeneous **(Stationary)** if the transition probabilities are constant, that is,

$$\boxed{\pi_{j|i}\left(n\right) = \pi_{j|i} = \text{const for all } n = 0, 1, 2, \ldots}$$

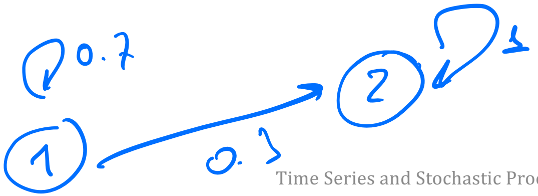


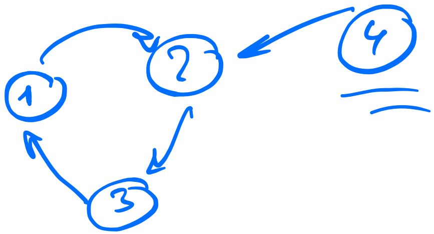
Markov process

Definition

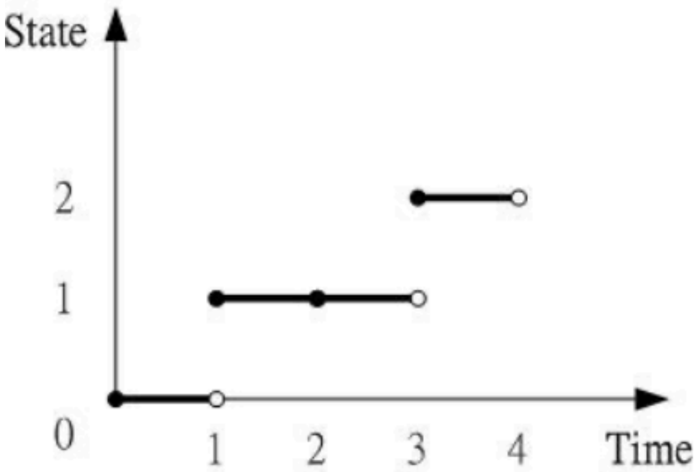
A Markov Chain is called **ergodic** if all its states are returnable.

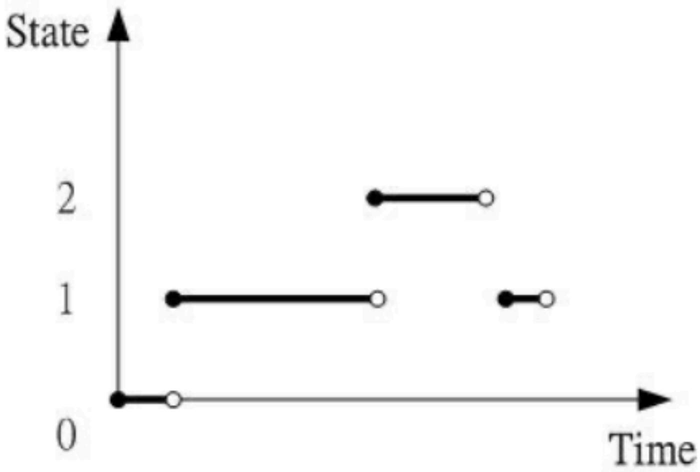
The result below shows that homogeneos ergodic Markov chains posses some additional property:
after a long time such chains "forget" the initial states from which they have started.



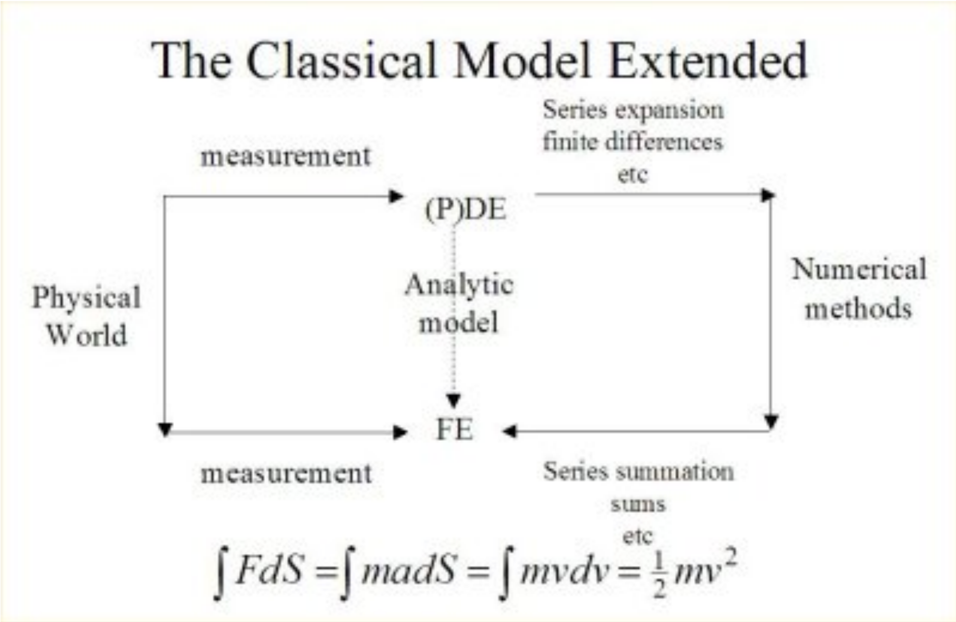


Markov process

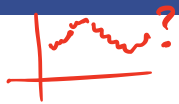


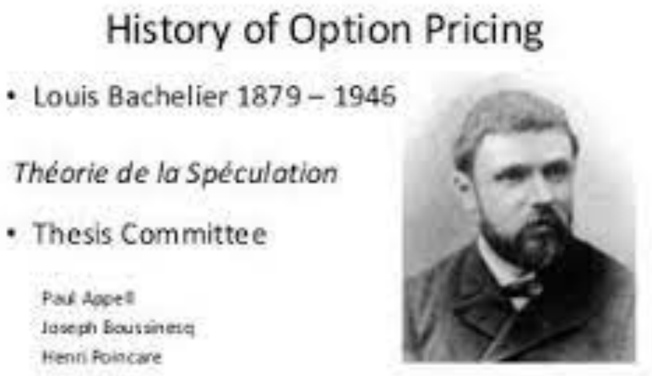


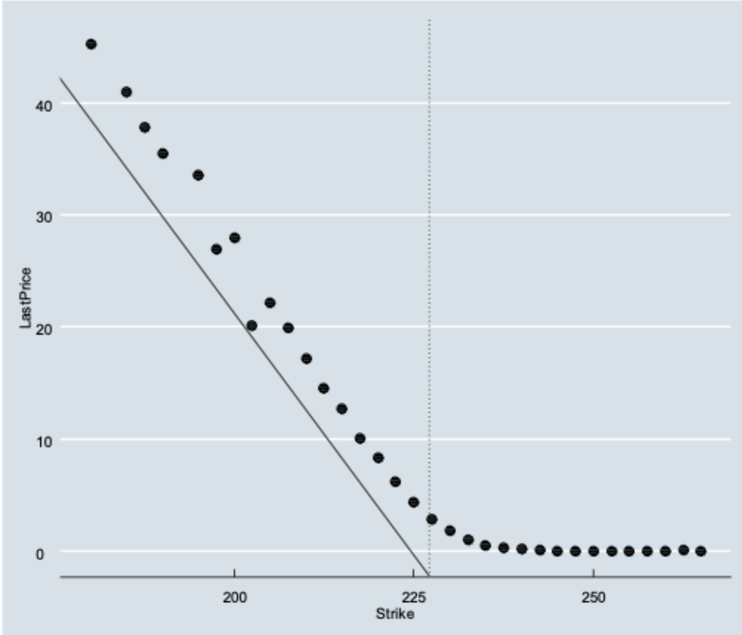
Math Modelling



Math Modelling







Math Modelling

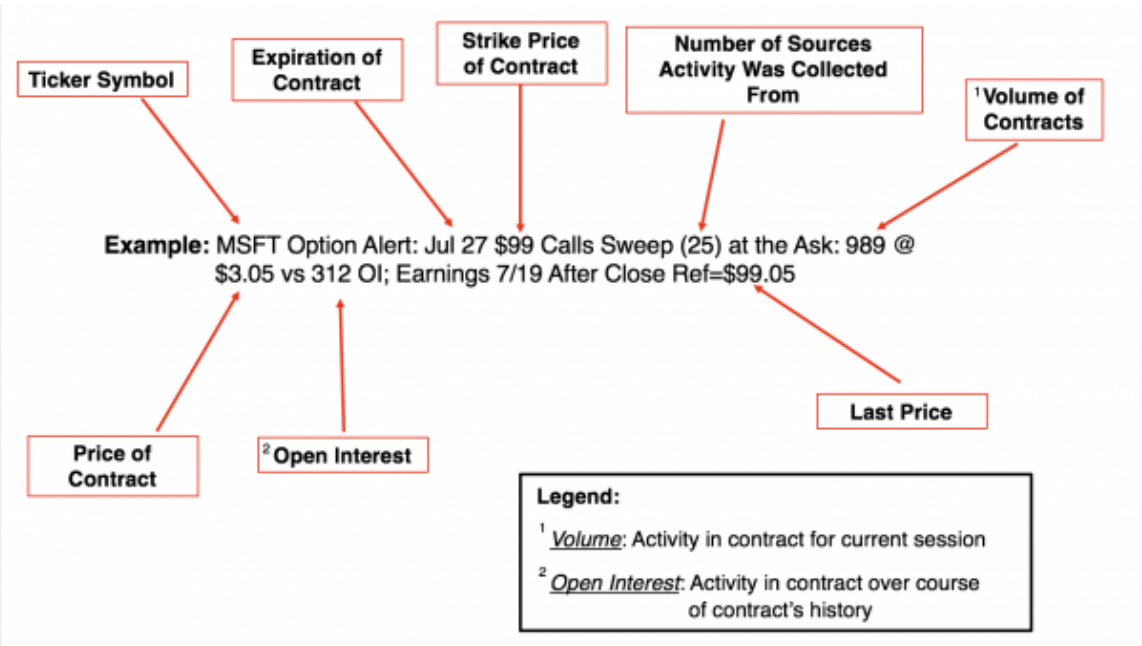
Lévy Process

Paul Levy

Math Modelling

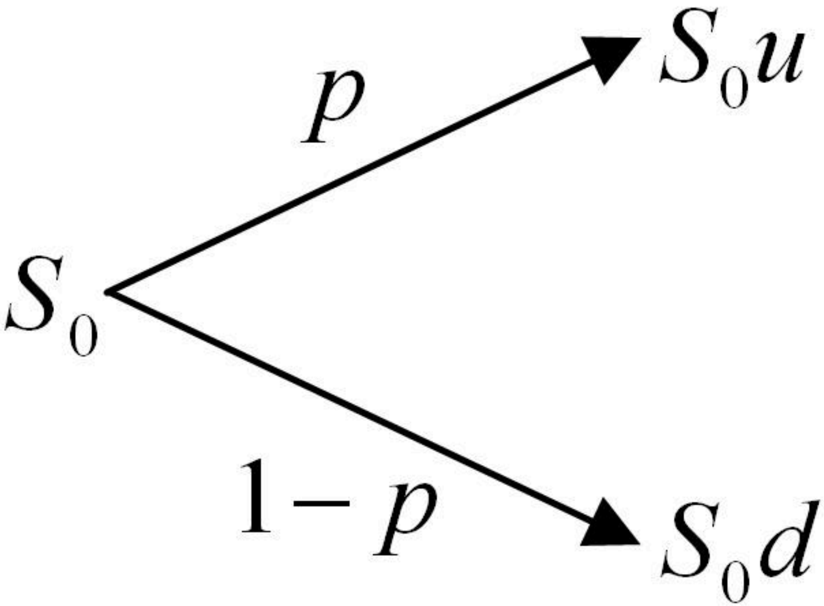
Option — Call C_0 $C_T = \max(S_T - K; 0)$

~~Put~~



Math Modelling

u=1.1

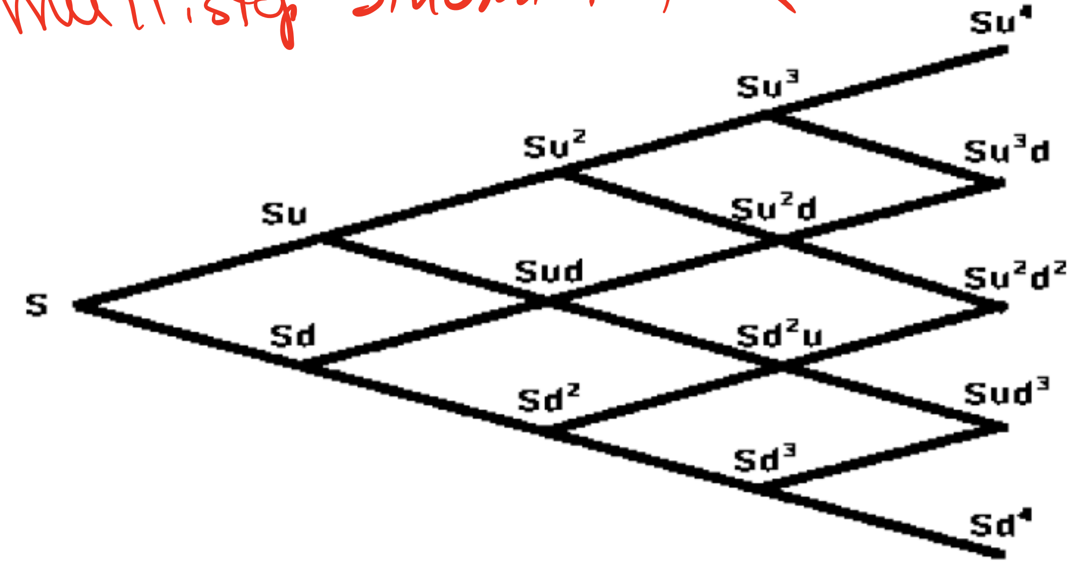


d=0.9

Math Modelling

multistep binomial tree

Payoff
$\max(Su^4 - K; 0)$



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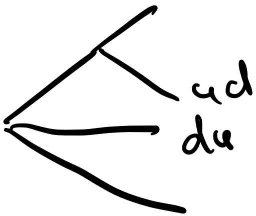
Expiration



Math Modelling

Risk Neutral

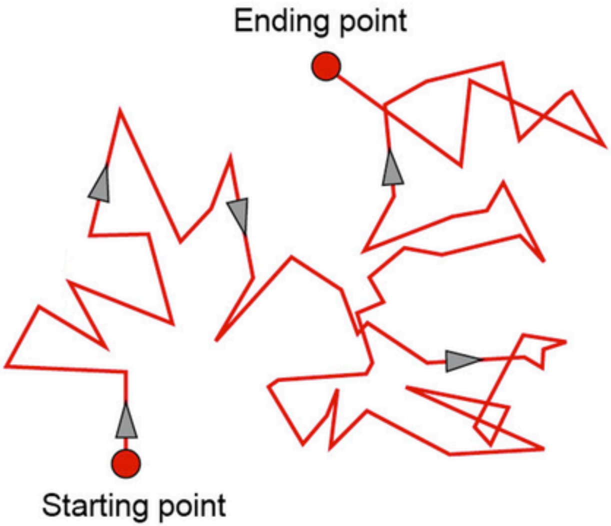
$$
\begin{tikzcd}
 & C^{+} = \max(0, uS - P_x) \\
c = PV(\pi C^{+} + (1-\pi)C^{-}) \arrow[ur, "\pi"] \arrow[dr, "1 - \pi"'] & \\
 & C^{-} = \max(0, dS - P_x)
\end{tikzcd}
$$

Math Modelling

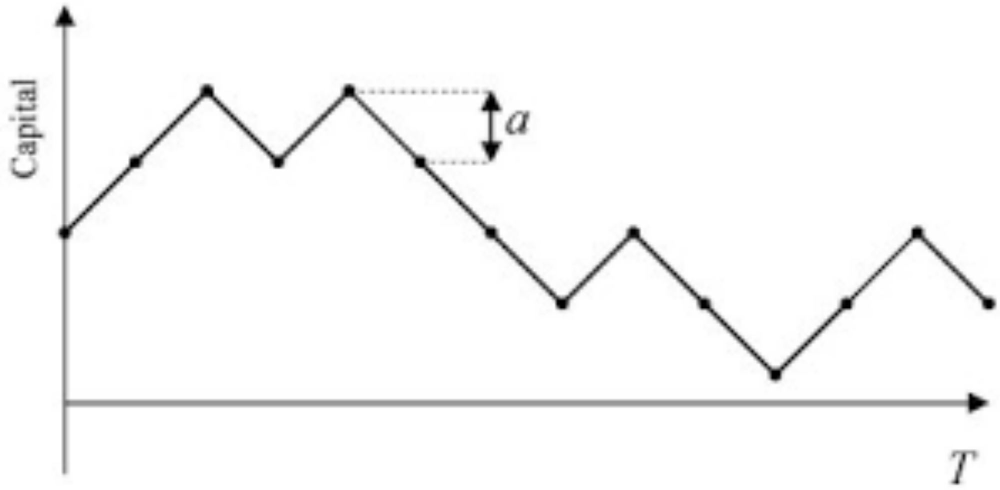
$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$$

Math Modelling

Brownian Motion

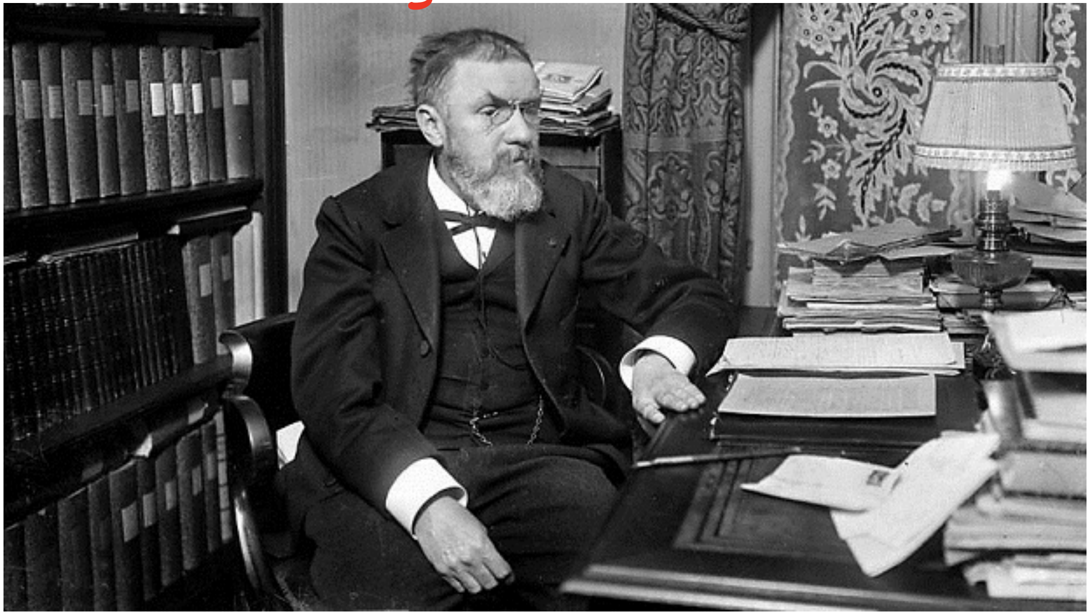


Math Modelling



Math Modelling

Henry Poincaré



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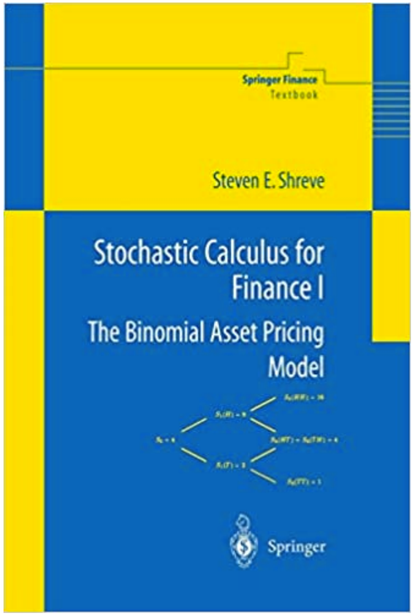
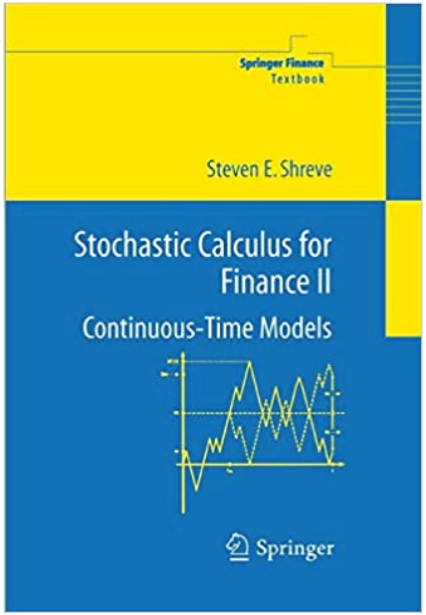
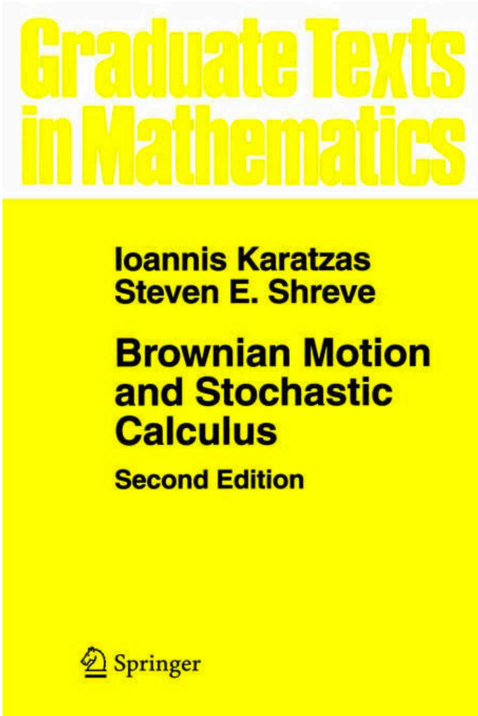




Math Modelling



Math Modelling

$$
\begin{aligned}
&\lim_{n\to\infty} \mathrm{P}(|X_n - X| > \varepsilon) \\
&= \lim_{n\to\infty} \mathrm{P}(|X_n - 0| > \varepsilon) \\
&= \lim_{n\to\infty} [1 - \mathrm{P}(-\varepsilon \le X_n \le \varepsilon)] \\
&= 1 - \lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} f_{X_n}(x)\,dx \\
&= 1 - \lim_{n\to\infty} \int_{\max(-\varepsilon,-1/n)}^{\min(\varepsilon,1/n)} \frac{n}{2}\,dx \\
&= 1 - \lim_{n\to\infty} \int_{-1/n}^{1/n} \frac{n}{2}\,dx \qquad \text{(when } n \text{ becomes large } \tfrac{1}{n} < \varepsilon\text{)} \\
&= 1 - \lim_{n\to\infty} 1 \\
&= 0
\end{aligned}
$$

Math Modelling

Example 1: Let the random variable U be uniformly distributed on $[0, 1]$. Consider the sequence defined as:

$$X(n) = \frac{(-1)^n U}{n}.$$

1. *Almost sure convergence:* Suppose

$$U = a.$$

   The sequence becomes

$$\begin{aligned} X_1 &= -a, \\ X_2 &= \frac{a}{2}, \\ X_3 &= -\frac{a}{3}, \\ X_4 &= \frac{a}{4}, \\ &\vdots \end{aligned}$$

   In fact, for any $a \in [0, 1]$

$$\lim_{n \to \infty} X_n = 0,$$

   therefore, $X_n \xrightarrow{a.s.} 0$.

Math Modelling

Example 1: Let the random variable U be uniformly distributed on $[0,1]$. Consider the sequence defined as:

$$X(n) = \frac{(-1)^n U}{n}.$$

In order to answer this question, we need to prove that

$$\lim_{n\to\infty} E\left[|X_n - 0|^2\right] = 0.$$

We know that,

$$\begin{aligned}
\lim_{n\to\infty} E\left[|X_n - 0|^2\right] &= \lim_{n\to\infty} E\left[X_n^2\right], \\
&= \lim_{n\to\infty} E\left[\frac{U^2}{n^2}\right], \\
&= \lim_{n\to\infty} \frac{1}{n^2} E\left[U^2\right], \\
&= \lim_{n\to\infty} \frac{1}{n^2} \int_0^1 u^2 du, \\
&= \lim_{n\to\infty} \frac{1}{n^2} \left.\frac{u^3}{3}\right]_0^1, \\
&= \lim_{n\to\infty} \frac{1}{3n^2}, \\
&= 0.
\end{aligned}$$

Hence, $X_n \xrightarrow{m.s.} 0$.

Math Modelling