# Time Series

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**This paper is not to be removed from the Examination Hall**

**UNIVERSITY OF LONDON** **ST3134 ZA**

**BSc degrees and Diplomas for Graduates in Economics, Management, Finance and the Social Sciences, the Diplomas in Economics and Social Sciences**

**Advanced Statistics: Statistical Inference**

Tuesday, 8 May 2018: 14:30 to 16:30

Candidates should answer all **FOUR** questions: **QUESTION 1** of Section A (40 marks) and all **THREE** questions from Section B (60 marks in total). **Candidates are strongly advised to divide their time accordingly.**

A calculator may be used when answering questions on this paper and it must comply in all respects with the specification given with your Admission Notice. The make and type of machine must be clearly stated on the front cover of the answer book.

1. (a) Let $X = \{X_1, X_2, \ldots, X_n\}$ be a random sample from a smooth probability density function $f(x \mid \theta)$, where $\theta$ is an unknown parameter.

$\theta$

$X \sim f_X(x;\theta)$

i. Let $s(\theta \mid X)$ denote the score function. Show that $E[s(\theta \mid X)] = 0$. (You may assume sufficient regularity conditions to allow differentiation under integral signs.)

**(5 marks)**

ii. Let $I(\theta)$ denote the Fisher information. Show that:

$$s(\theta|X) = \boxed{\frac{\partial f_X}{\partial \theta}} / f_X$$

$$I(\theta) = -E\left[\frac{\partial}{\partial\theta} s(\theta \mid X)\right].$$

**(5 marks)**

$$E_X[s(\theta|X)] = \int_{-\infty}^{+\infty} s(\theta|X) \cdot f_X \, dx = \int_{-\infty}^{+\infty} \frac{\partial f_X}{\cancel{\partial\theta\, f_X}} \cdot \cancel{f_X}\, dx \overset{!}{=} \int_{-\infty}^{+\infty} \frac{\partial}{\partial\theta} \cdot f_X \, dx = \frac{\partial}{\partial\theta}\left(\int_{-\infty}^{+\infty} f_X \, dx\right) = 0 \quad !!!$$

$$\frac{\partial E[s(\theta|x)]}{\partial\theta} = \frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty} s(\theta|x)\cdot f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{\partial s}{\partial\theta}\cdot f_X(x) + s\,\frac{\partial f}{\partial\theta}\,dx =$$

$= s(\theta|x)\cdot f_X$

$$= \int_{-\infty}^{+\infty}\left(\frac{\partial s}{\partial\theta} + s^2\right) f_X\,dx = E\left[\frac{\partial s}{\partial\theta}\right] + E[s^2] = 0 \Rightarrow E[s^2(\theta|X)] = -E\left[\frac{\partial s}{\partial\theta}\right]$$

(b) Let $\{X_1, \ldots, X_n\}$ be a random sample from a Geometric($\theta$) distribution. Find Fisher's information for $\theta$.

Hint: You may use the fact that if $X \sim \text{Geometric}(\theta)$, then $\mathrm{E}(X) = 1/\theta$.

**(10 marks)**

(c) i. Provide the definition of a sufficient statistic. **(5 marks)**

ii. State the factorisation theorem. **(5 marks)**

$$f_X = \boxed{\frac{\partial f_x}{\partial \theta}} \cdot \boxed{\frac{1}{s(\theta|X)}}$$

X, θ

kernel

$$f_X(x|\theta) = \prod_{i=1}^{n} (1-\theta)^{x_i - 1} \cdot \theta$$

$$\ell = \ln f = \sum_{i=1}^{n} \left[ (x_i - 1) \ln(1-\theta) + \ln\theta \right] = \ln(1-\theta) \cdot \sum_{i=1}^{n} (x_i - 1) + n \cdot \ln\theta$$

$$\frac{\partial \ell}{\partial \theta} = \frac{1}{\theta - 1} \cdot \sum (x_i - 1) + \frac{n}{\theta} \qquad \frac{\partial^2 \ell}{\partial \theta^2} = \frac{-1}{(\theta-1)^2} \cdot \sum_{i=1}^{n} (x_i - 1) - \frac{n}{\theta^2}$$

$$\sum\left(\frac{1}{\theta} - 1\right) = \frac{n}{\theta} - n = \frac{n - n\theta}{\theta}$$

$$I(\theta) = -\mathrm{E}_X\left[\frac{\partial}{\partial \theta^2} \ell\right] = \mathrm{E}_X\left[\frac{1}{(\theta-1)^2} \cdot \sum (x_i - 1) + \frac{n}{\theta^2}\right] = \frac{1}{(\theta-1)^2} \cdot \sum_{i=1}^{n} \mathrm{E}_X[X_i - 1] + \frac{n}{\theta^2} =$$

$$= \frac{n - n\theta}{\theta(\theta-1)^2} + \frac{n}{\theta^2} = \frac{-n}{\theta(\theta-1)} + \frac{n}{\theta^2}.$$



(d) Let $\{X_1, \ldots, X_n\}$ be a random sample from the Pareto distribution with probability density function:

$$f(x; x_0, \alpha) = \begin{cases} \alpha x_0^\alpha x^{-\alpha-1} & \text{for } x \geq x_0 \\ 0 & \text{otherwise.} \end{cases}$$

i. If $x_0$ is known and $\alpha > 0$ is unknown, find a sufficient statistic for $\alpha$.

**(5 marks)**

ii. If $\alpha$ is known and $x_0$ is unknown, find a sufficient statistic for $x_0$.

**(5 marks)**

2. Let $\{X_1, X_2, \ldots, X_n\}$ be a random sample drawn from a distribution with probability density function for each $X_i$, for $i = 1, \ldots, n$, given by:

indep

$$f(x \mid \lambda) = \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right), \quad \text{for } x > 0,\ \lambda > 0$$

and 0 otherwise.

It is known that $E(X_i) = \lambda$, $Var(X_i) = \lambda^2$ and $F(x \mid \lambda) = 1 - \exp(-x/\lambda)$.

(a) Find the distribution of the random variable $T = \min(X_1, \ldots, X_n)$.

**(5 marks)**

(b) Use the distribution of $T$ to derive an unbiased estimator of $\lambda$. Calculate the variance of the estimator.

**(5 marks)**

(c) Find the method of moments estimator of $\lambda$ and check whether it is an unbiased estimator of $\lambda$.

**(5 marks)**

(d) Consider the unbiased estimator in (b) and the method of moments estimator in (c) and compare the variances of the two estimators. Which one is better? Justify your answer.

**(5 marks)**

"$F_T(A)$"

$F_T = P(T < A) = 1 - P(T \geq A) = 1 - P(\text{all } X_i \geq A) = 1 - \prod_{i=1}^{n} P(X_i \geq A) = 1 - \prod_{i=1}^{n}(1 - P(X_i < A)) = 1 - (1 - F_X(A))^n = 1 - (1 - (1 - e^{-\frac{A}{\lambda}}))^n = 1 - e^{-\frac{A \cdot n}{\lambda}}$

$F_T(\tau) = 1 - e^{-\frac{\tau \cdot n}{\lambda}}$ $\quad T \sim E(\lambda/n)$

$E(T) = \frac{\lambda}{n} \Rightarrow \lambda = n \cdot E(T)$

$\lambda = E[nT]$ $\quad \hat{\lambda} = nT$ - unbiased

$Var(\hat{\lambda}) = Var(nT) = n^2 Var(T) = n^2 \cdot \frac{\lambda^2}{n^2} = \lambda^2$

$E[X] = \bar{X}$

$\hat{\lambda}_{MM} = \bar{X}$

$Var\,\hat{\lambda}_{MM} = Var(\bar{X}) = \frac{Var\,X}{n} = \frac{\lambda^2}{n}$

$\bar{X} = \frac{\sum X_i}{n}$

$\frac{Var \sum X_i}{n^2}$

3. Let $\{X_1, \ldots, X_n\}$ be a random sample drawn from a distribution with probability density function:

$$f(x;\,\theta) = \begin{cases} 2x\theta e^{-\theta x^2} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta > 0$ is an unknown parameter.

$$L(\theta) = \prod_i f(x_i;\theta)$$

$$\ell(\theta) = \sum_{i=1}^{n} \left[\ln 2 + \ln x_i + \ln\theta - \theta x_i^2\right] =$$

$$= n\ln 2 + n\ln\theta + \sum_{i=1}^{n} \ln x_i - \theta \sum_{i=1}^{n} x_i^2$$

(a) Find the maximum likelihood estimator of $\theta$. You should check that the second derivative of the log-likelihood is negative.

$$\frac{\partial \ell}{\partial\theta} = \frac{n}{\theta} - \sum x_i^2 = 0$$

$$\frac{\partial^2 \ell}{\partial\theta^2} = \frac{-n}{\theta^2} < 0 \qquad \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i^2}$$

$$I(\theta) = -E\left[\frac{\partial^2 \ell}{\partial\theta^2}\right] = E\left[\frac{n}{\theta^2}\right] = \frac{n}{\theta^2}$$

**(6 marks)**

(b) Use derivatives of the log-likelihood function to find the Fisher information $I(\theta)$.

**(4 marks)**

(c) Write down the asymptotic distribution of $\hat{\theta}_{ML}$ and show that the limiting distribution of:

$$\sqrt{n}\left(\frac{\hat{\theta}}{\theta} - 1\right)$$

is $N(0,\,1)$.

$$\hat{\theta}_{ML} \overset{a}{\sim} N\left(\theta; \frac{1}{I(\theta)}\right) \quad n\to\infty \quad !$$

$$\hat{\theta}_{ML} \overset{a}{\sim} N\left(\theta; \frac{\theta^2}{n}\right)$$

$$\frac{\hat{\theta}_{ML}}{\theta} \overset{a}{\sim} N\left(1; \frac{1}{n}\right)$$

**(5 marks)**

$$\sqrt{n}\left(\frac{\hat{\theta}_{ML}}{\theta} - 1\right) \overset{a}{\sim} N(0; 1)$$

4. (a) State the Neyman–Pearson lemma.

**(7 marks)**

(b) Let $Y$ be a random variable with a Binomial(12, $\pi$) distribution. In other words:

$$P(Y = y \,|\, \pi) = \binom{12}{y} \pi^y (1-\pi)^{12-y} \quad \text{for } y = 0, 1, \ldots, 12$$

and 0 otherwise.

$\frac{12!}{9!\,3!} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2}$ $\quad \frac{12!}{10!\,2} = \frac{12 \cdot 11}{2} = 66$

p-value $= P(y \geq 9 \,|\, H_0 \text{ is true}) = P(Y=9) + P(Y=10) + P(11) + P(12)$

$= (C^9_{12} + C^{10}_{12} + C^{11}_{12} + 1) \cdot 0.5^{12} = \frac{739}{2^{12}} = \frac{739}{1024 \cdot 4} = \frac{739}{4096} = 0.18$

660 $\quad$ 66

i. Consider the test of $H_0 : \pi = 0.5$ vs. $H_1 : \pi > 0.5$. What values of $Y$ provide evidence against $H_0$? Find the $p$-value if $y = 9$. Would you reject $H_0$ at the 5% significance level?

DNR

**(5 marks)**

$\ln(LR^2) \sim \chi^2$

$LR = \frac{L(\hat{\pi})}{L(\pi_0)}$

$H_0: \pi = \pi_0$

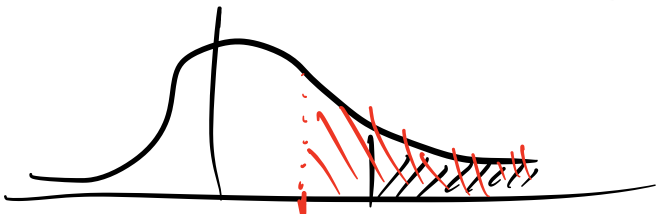
ii. Now consider the test of $H_0 : \pi = 0.5$ vs. $H_1 : \pi \neq 0.5$. Derive the likelihood ratio test statistic for this hypothesis.

**(5 marks)**

iii. Describe how you would construct an asymptotic test based on the likelihood ratio test statistic. Comment on the suitability of this test.

**(3 marks)**





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