Lecture 6
Sigma Algebra

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$\mathcal{B}$ Borel σ-algebra

$\mathcal{H} = \{\emptyset, \Omega, \{5\}, \{3\}$, $\{23456\}, \{1,2,4,5,6\}, \{13\}, \{2456\}\}$

Filtration

- The concept of a σ-field can be used to describe the amount of information available at a given moment.
- Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all natural numbers.

Definition (σ-Field)

A collection $\mathcal{F}$ of subsets of Ω is called a σ-**field** (or a σ-**algebra**) whenever:

1. $\Omega \in \mathcal{F}$,
2. if $A \in \mathcal{F}$ then $A^c := \Omega \setminus A \in \mathcal{F}$,
3. if $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Filtration

- The set of information has to contain all possible states, so that we postulate that Ω belongs to any σ-field.
- Any set $A \in \mathcal{F}$ is interpreted as an observed **event**.
- If an event $A \in \mathcal{F}$ is given, that is, some collection of states is given, then the remaining states can also be identified and thus the complement A^c is also an event.
- The idea of a σ-field is to model a certain level of information.
- In particular, as the σ-field becomes larger, more and more events can be identified.
- We will later introduce a concept of an increasing flow of information, formally represented by an ordered (increasing) family of σ-fields.

Filtration

Definition (Probability Measure)

A map $\mathbb{P} : \mathcal{F} \to [0,1]$ is called a **probability measure** if:

1. $\mathbb{P}(\Omega) = 1$,
2. for any sequence $A_i, i \in \mathbb{N}$ of pairwise disjoint events we have

$$\mathbb{P}\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**.

- By convention, the probability of all possibilities is 1 (see 1).
- Probability should satisfy σ-additivity (see 2)
- Note that $\mathbb{P}(\emptyset) = 0$ and for an arbitrary event $A \in \mathcal{F}$ we have $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

Filtration

We take $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and we define the σ-fields:

$$\begin{aligned}
\mathcal{F}_1 &= \{\emptyset, \Omega\} \\
\mathcal{F}_2 &= \Big\{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_4\}\Big\} \\
\mathcal{F}_3 &= 2^{\Omega} \quad \text{(the class of all subsets of } \Omega\text{).}
\end{aligned}$$

Note that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$, that is, the information increases:

- $\mathcal{F}_1$: no information, except for the set Ω.
- $\mathcal{F}_2$: partial information, since we cannot distinguish between the occurrence of either ω_1 or ω_2.
- $\mathcal{F}_3$: full information, since $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}$ and $\{\omega_4\}$ can be observed.

Filtration

- We define the probability measure $\mathbb{P}$ on the σ-field $\mathcal{F}_2$

$$\mathbb{P}\left(\{\omega_1, \omega_2\}\right) = \frac{2}{3}, \quad \mathbb{P}\left(\{\omega_3\}\right) = \frac{1}{6}, \quad \mathbb{P}\left(\{\omega_4\}\right) = \frac{1}{6}.$$

- The σ-additivity of $\mathbb{P}$ leads to

$$\mathbb{P}\left(\{\omega_1, \omega_2\} \cup \{\omega_3\} \cup \{\omega_4\}\right) = 1 = \mathbb{P}\left(\Omega\right).$$

- Note that $\mathbb{P}$ is not yet defined on the σ-field $\mathcal{F}_3 = 2^{\Omega}$ and in fact the extension of $\mathbb{P}$ from $\mathcal{F}_2$ to $\mathcal{F}_3$ is not unique.
- For any $\alpha \in [0, 2/3]$ we may set

$$\mathbb{P}_\alpha\left(\{\omega_1\}\right) = \alpha = \frac{2}{3} - \mathbb{P}_\alpha\left(\{\omega_2\}\right).$$

Filtration

Definition

Let I be some index set. Assume that we are given a collection $(B_i)_{i\in I}$ of subsets of Ω. Then the smallest σ-field containing this collection is denoted by $\sigma\left((B_i)_{i\in I}\right)$ and is called the σ-field **generated** by $(B_i)_{i\in I}$.

Definition (Partition)

By a **partition** of Ω, we mean any collection $\mathcal{P} = (A_i)_{i\in I}$ of non-empty subsets of Ω such that the sets A_i are pairwise disjoint, that is, $A_i \cap A_j = \emptyset$ whenever $i \neq j$ and $\bigcup_{i\in I} A_i = \Omega$.

Lemma

A partition $\mathcal{P} = (A_i)_{i\in I}$ generates a σ-field $\mathcal{F}$ if every set $A \in \mathcal{F}$ can be represented as follows: $A = \bigcup_{j\in J} A_j$ for some subset $J \subset I$.

Filtration

Definition (Partition Associated with $\mathcal{F}$)

A **partition of Ω associated with** a σ-field $\mathcal{F}$ is a collection of non-empty sets $A_i \in \mathcal{F}$ for some $i \in I$ such that

1. $\Omega = \bigcup_{i \in I} A_i$.
2. The sets A_i are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$.
3. For each $A \in \mathcal{F}$ there exists $J \subseteq I$ such that $A = \bigcup_{i \in J} A_i$.

Lemma

For any σ-field $\mathcal{F}$ of subsets of a finite state space Ω, a partition associated with this σ-field always exists and is unique.

Filtration

- Consider the σ-field $\mathcal{F}_2$ introduced in Example 5.1.
- The unique partition associated with $\mathcal{F}_2$ is given by

$$\mathcal{P}_2 = \Big\{ \{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\} \Big\}.$$

- Define the probabilities

$$\mathbb{P}\left(\{\omega_1, \omega_2\}\right) = \frac{2}{3}, \quad \mathbb{P}\left(\{\omega_3\}\right) = \frac{1}{6}, \quad \mathbb{P}\left(\{\omega_4\}\right) = \frac{1}{6}.$$

- Then for each event $A \in \mathcal{F}_2$ the probability of A can be easily evaluated, for instance

$$\mathbb{P}\left(\{\omega_1, \omega_2, \omega_4\}\right) = \mathbb{P}\left(\{\omega_1, \omega_2\}\right) + \mathbb{P}\left(\{\omega_4\}\right) = \frac{5}{6}.$$

Filtration

- Let $\mathcal{F}$ be an arbitrary σ-field of subsets of Ω.
- In the next definition, we do not assume that the sample space is discrete.

Definition ($\mathcal{F}$-Measurability)

A map $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-**measurable** if for every closed interval $[a, b] \subset \mathbb{R}$ the preimage (i.e. the inverse image) under X belongs to $\mathcal{F}$, that is,

$$X^{-1}([a, b]) := \{\omega \in \Omega \mid X(\omega) \in [a, b]\} \in \mathcal{F}.$$

Equivalently, for any real number x

$$X^{-1}((-\infty, x]) := \{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F}.$$

If X is $\mathcal{F}$-measurable then X is called a **random variable** on $(\Omega, \mathcal{F})$.

Filtration

Definition (Filtration)

A family $(\mathcal{F}_t)_{0\leq t\leq T}$ of σ-fields on Ω is called a **filtration** if $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$. For brevity, we denote $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$.

- We interpret the σ-field $\mathcal{F}_t$ as the information available to an agent at time t. In particular, $\mathcal{F}_0$ represents the information available at time 0, that is, the initial information.
- We assume that the information accumulated over time can only grow, so that we never forget anything!

Filtration

Definition (Stochastic Process)

A family $X = (X_t)_{0 \leq t \leq T}$ of random variables is called a **stochastic process**. A stochastic process X is said to be $\mathbb{F}$-**adapted** if for every $t = 0, 1, \ldots, T$ the random variable X_t is $\mathcal{F}_t$-measurable.

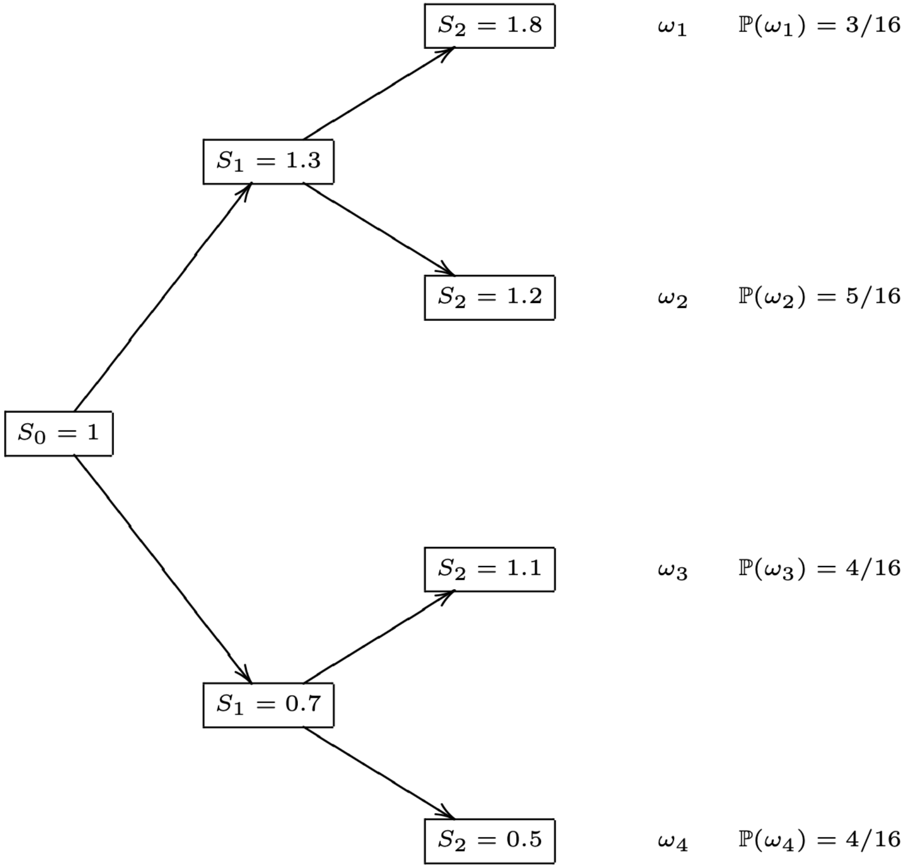
Filtration

- The initial information at time 0 is usually given by the **trivial** σ-field $\mathcal{F}_0 = \{\emptyset, \Omega\}$ since all prices are known at 0, so that there is no uncertainty.
- Let $X = (X_t)_{0 \le t \le T}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- For a fixed t, we define the σ-field $\mathcal{F}_t^X$ by setting

$$\mathcal{F}_t^X = \sigma\big(X_s^{-1}([a,b]) \,\big|\, 0 \le s \le t, a \le b\big).$$

  Then the filtration $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \le t \le T}$ is called the filtration **generated** by the process X.

Filtration



$S_0 = 1$
$S_1 = 1.3$
$S_1 = 0.7$
$S_2 = 1.8$
$S_2 = 1.2$
$S_2 = 1.1$
$S_2 = 0.5$
ω_1 $\mathbb{P}(\omega_1) = 3/16$
ω_2 $\mathbb{P}(\omega_2) = 5/16$
ω_3 $\mathbb{P}(\omega_3) = 4/16$
ω_4 $\mathbb{P}(\omega_4) = 4/16$

Filtration

Example (5.4 Continued)

- $\mathcal{F}_0^S = \{\emptyset, \Omega\}$ since S_0 is deterministic.
- At $t = 1$, we have

$$S_1^{-1}([a,b]) = \begin{cases} \Omega & \text{if } a \leq 0.7 \text{ and } b \geq 1.3 \\ \{\omega_1, \omega_2\} & \text{if } 0.7 < a \leq 1.3 \text{ and } b \geq 1.3 \\ \{\omega_3, \omega_4\} & \text{if } a \leq 0.7 \text{ and } 0.7 \leq b < 1.3 \\ \emptyset & \text{otherwise} \end{cases}$$

  and thus

$$\mathcal{F}_1^S = \Big\{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\Big\}.$$

- $\mathcal{F}_2^S = 2^\Omega$ since the partition generating $\mathcal{F}_2$ is

$$\mathcal{P}_2 = \Big\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\Big\}.$$

Filtration

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite (or countable) probability space.
- Let X be an arbitrary $\mathcal{F}$-measurable random variable.
- Assume that $\mathcal{G}$ is a σ-field which is contained in $\mathcal{F}$.
- Let $(A_i)_{i \in I}$ be the unique partition associated with $\mathcal{G}$.
- Our next goal is to define the **conditional expectation** $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$, that is, the conditional expectation of a random variable X with respect to a σ-field $\mathcal{G}$.
- The expected value $\mathbb{E}_{\mathbb{P}}(X)$ will be obtained from $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ by setting $\mathcal{G} = \mathcal{F}_0$, that is, $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{F}_0)$.

Filtration

Definition (Conditional Expectation)

The **conditional expectation** $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ of X with respect to $\mathcal{G}$ is defined as the random variable which satisfies, for every $\omega \in A_i$,

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega) = \frac{1}{\mathbb{P}(A_i)} \sum_{\omega_l \in A_i} X(\omega_l)\mathbb{P}(\omega_l) = \sum_{x_k} x_k\, \mathbb{P}(X = x_k\,|A_i)$$

where the summation is over all possible values of X and

$$\mathbb{P}(X = x_k\,|\,A_i) = (\mathbb{P}(A_i))^{-1}\mathbb{P}(\{X = x_k\} \cap A_i)$$

is the conditional probability of the event $\{\omega \in \Omega\,|\,X(\omega) = x_k\}$ given A_i. Hence

$$\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G}) = \sum_{i \in I} \frac{1}{\mathbb{P}(A_i)}\, \mathbb{E}_{\mathbb{P}}(X\mathbb{1}_{A_i})\,\mathbb{1}_{A_i}.$$

Filtration

- $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is well defined by equation and, by Proposition 5.1, the conditional expectation $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is a $\mathcal{G}$-measurable r.v.
- $\mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})$ is the best estimate of X given the information represented by the σ-field $\mathcal{G}$.
- The following identity uniquely characterises the conditional expectation (in addition to $\mathcal{G}$-measurability):

$$\sum_{\omega \in G} X(\omega)\mathbb{P}(\omega) = \sum_{\omega \in G} \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})(\omega)\mathbb{P}(\omega), \quad \forall\, G \in \mathcal{G}.$$

- One can represent this equality using (discrete) integrals: for every $G \in \mathcal{G}$,

$$\int_G X\,d\mathbb{P} = \int_G \mathbb{E}_{\mathbb{P}}(X|\,\mathcal{G})\,d\mathbb{P}, \quad \forall\, G \in \mathcal{G}.$$

Filtration

Proposition (5.2)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be endowed with sub-σ-fields $\mathcal{G}$ and $\mathcal{G}_1 \subset \mathcal{G}_2$ of $\mathcal{F}$. Then

1. **Tower property:** *If $X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. then*

$$\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}_1) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}_2) \,|\, \mathcal{G}_1\big) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}_1) \,|\, \mathcal{G}_2\big).$$

2. **Taking out what is known:** *If $X : \Omega \to \mathbb{R}$ is a $\mathcal{G}$-measurable r.v. and $Y : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. then*

$$\mathbb{E}_{\mathbb{P}}(XY \,|\, \mathcal{G}) = X \, \mathbb{E}_{\mathbb{P}}(Y \,|\, \mathcal{G}).$$

3. **Trivial conditioning:** *If $X : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable r.v. independent of $\mathcal{G}$ then*

$$\mathbb{E}_{\mathbb{P}}(X \,|\, \mathcal{G}) = \mathbb{E}_{\mathbb{P}}(X).$$

Filtration

The underlying probability space is given by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

- At time $t = 0$, the stock price is known and unique value is $S_0 = 5$. Hence the σ-field $\mathcal{F}_0^S$ is the trivial σ-field.
- At time $t = 1$, the stock can take two possible values and

$$S_1^{-1}([a,b]) = \begin{cases} \Omega & \text{if } a \leq 4 \text{ and } 8 \leq b \\ \{\omega_1, \omega_2\} & \text{if } 4 < a \text{ and } 8 \leq b \\ \{\omega_3, \omega_4\} & \text{if } a \leq 4 \text{ and } b < 8 \\ \emptyset & \text{if } a < 4 \text{ and } b < 4 \end{cases}$$

  so that $\mathcal{F}_1^S = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$.
- At time $t = 2$, we have $\mathcal{F}_1^S = 2^\Omega$.

Filtration

- Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent probability measures on $(\Omega, \mathcal{F})$.
- Let the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ be

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) = L(\omega), \quad \mathbb{P}\text{-a.s.}$$

meaning that L is $\mathcal{F}$-measurable and, for every $A \in \mathcal{F}$,

$$\int_A X \, d\mathbb{Q} = \int_A XL \, d\mathbb{P}.$$

- If Ω is finite then this equality becomes

$$\sum_{\omega \in A} X(\omega)\, \mathbb{Q}(\omega) = \sum_{\omega \in A} X(\omega) L(\omega)\, \mathbb{P}(\omega).$$

- The r.v. L is strictly positive $\mathbb{P}$-a.s. and $\mathbb{E}_{\mathbb{P}}(L) = 1$.
- Equality $\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(XL)$ holds for any $\mathbb{Q}$-integrable random variable X (it suffices to take $A = \Omega$).

Filtration

Lemma (5.1: Bayes Formula)

Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ and let X be a $\mathbb{Q}$-integrable random variable. Then the Bayes formula holds

$$\mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})}.$$

Proof.

- $\mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})$ is strictly positive so that the RHS is well defined.
- By our assumption, the random variable XL is $\mathbb{P}$-integrable.
- Therefore, it suffices to show that the equality

$$\mathbb{E}_{\mathbb{P}}(XL \,|\, \mathcal{G}) = \mathbb{E}_{\mathbb{Q}}(X \,|\, \mathcal{G})\, \mathbb{E}_{\mathbb{P}}(L \,|\, \mathcal{G})$$

holds for every random variable X.

Sigma Algebra

- **Probability space** is triple $(\Omega, \mathcal{F}, P)$ where Ω is sample space, $\mathcal{F}$ is set of events (the σ-algebra) and $P : \mathcal{F} \to [0, 1]$ is the probability function.
- σ-**algebra** is collection of subsets closed under complementation and countable unions. Call $(\Omega, \mathcal{F})$ a measure space.
- **Measure** is function $\mu : \mathcal{F} \to \mathbb{R}$ satisfying $\mu(A) \geq \mu(\emptyset) = 0$ for all $A \in \mathcal{F}$ and countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint A_i.
- Measure μ is **probability measure** if $\mu(\Omega) = 1$.

Sigma Algebra

A **filtration** is a non-decreasing family of sub σ–algebras of $\mathcal{F}$ indexed by time, i.e. a family $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{T}}$ such that

$$\mathcal{F}_s \subseteq \mathcal{F}_t,$$

for $s \leq t$, where $t, s \in \mathbb{T}$.

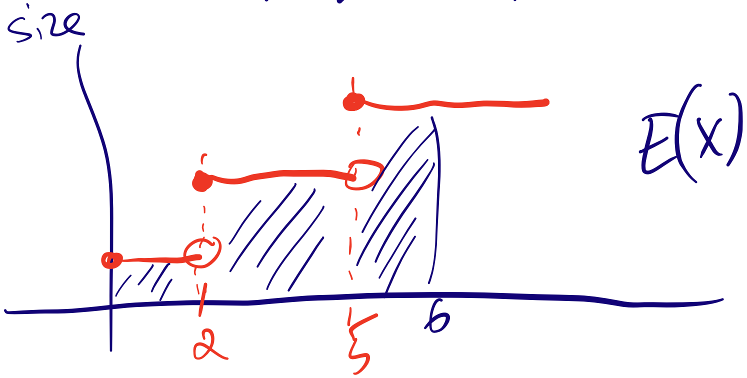
Sigma Algebra

Let $\mathbb{F}$ be a (continuous time) filtration. We say that $\mathbb{F}$ is the **right-continuous filtration** if for any $t \in \mathbb{T}$ we get

$$\mathcal{F}_t = \mathcal{F}_{t_+} \,,$$

$[\ ;\ ), [\ ;), [\ ;)$ Hausdorf

where $\mathcal{F}_{t_+} := \bigcap_{s>t, s\in\mathbb{T}} \mathcal{F}_s$.



t

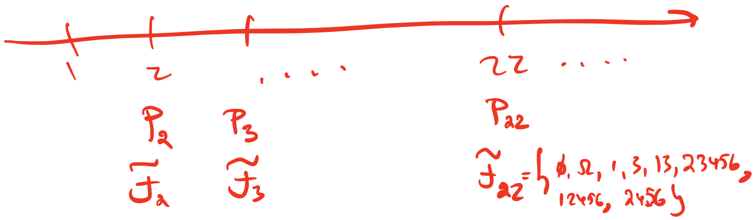
$$E(X) = \int X \, dF_X = \int x f_x \, dx$$

$f' dx = df$

$$\mathcal{F}_t \longrightarrow X_t$$

$$\mathcal{F}_{t-1} \longrightarrow X$$

process X is said to be **adapted** to filtration $\mathbb{F}$ (or $\mathbb{F}$**-adapted**) if X_t is $\mathcal{F}_t$–measurable for any $t \in \mathbb{T}$.



Sigma Algebra

Let X be a stochastic process. We say that $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\in\mathbb{T}}$, where

$$\mathcal{F}_t^X = \sigma(X_s, s \leq t, s \in \mathbb{T})$$

is a filtration **generated** by stochastic process X.

Sigma Algebra

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space with $\Omega = [0,1]$.[9] Let

$$\mathcal{A} := \sigma(N \subset [0,1] : \#N < \infty)$$

denote the σ–algebra of countable sets (and their complements). For time horizon $\mathbb{T} = [0, +\infty)$ we define filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ by setting

$$\mathcal{F}_t := \begin{cases} \mathcal{A} & \text{for } t \in [0,1); \\ \mathcal{F} & \text{for } t \in [1,\infty). \end{cases}$$

Next, we define a stochastic process $X = (X_t)_{t\in\mathbb{T}}$ by setting

$$X_t(\omega) := \mathbb{1}_\Delta(t,\omega) = \begin{cases} 1 & \text{if } t = \omega \text{ and } t \leq 1/2 \\ 0 & \text{otherwise} \end{cases}, \qquad t \in \mathbb{T}, \omega \in \Omega.$$

where $\Delta := \{(t,t) : t \in [0, \frac{1}{2}]\}$ is a subset of $\mathbb{T} \times \Omega$.

Sigma Algebra

Let $\Omega = \{1, 2, 3, 4\}$ and

$$\mathcal{F} := \{\emptyset, \{1\}, \{3\}, \{1,3\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$
$$\mathcal{G} := \{\emptyset, \{1\}, \{2\}, \{1,3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$
$$\mathcal{H} := \{\emptyset, \{1\}, \{4\}, \{1,4\}, \{2,3\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}$$

(a) Decide, which of the collections $\mathcal{F}$, $\mathcal{G}$ and/or $\mathcal{H}$ are σ-algebras and which are not.
(b) Let $f : \Omega \to \mathbb{R}$ be defined as $f(n) := (-1)^n$. Decide whether f is measurable or not with respect to the σ-algebras identified in question (a).

SOLUTION:
(a) $\mathcal{F}$ and $\mathcal{H}$ are σ-algebras. $\mathcal{G}$ is not a σ-algebra.

(b) f is $\mathcal{F}$-measurable but not $\mathcal{H}$-measurable. $\square$

Sigma Algebra

A stochastic process $(X_t)_{t\geq 0}$ may also be seen as a random system evolving in time. This system carries some information. More precisely, if one observes the paths of a stochastic process up to a time $t \geq 0$, one is able to decide if an event

$$A \in \sigma(X_s, s \leq t)$$

has occured (here and in the sequel $\sigma(X_s, s \leq t)$ denotes the smallest σ-field that makes all the random variables $\{(X_{t_1}, \cdots, X_{t_n}), 0 \leq t_1 \leq \cdots \leq t_n \leq t\}$ measurable). This notion of information carried by a stochastic process is modeled by filtrations.

Definition. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration $(\mathcal{F}_t)_{t\geq 0}$ is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$.*

As a basic example, if $(X_t)_{t\geq 0}$ is a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$,then

$$\mathcal{F}_t = \sigma(X_s, s \leq t)$$

is a filtration. This filtration is called the natural filtration of the process X and will often be denoted by $(\mathcal{F}_t^X)_{t\geq 0}$.

Sigma Algebra

Definition. *A stochastic process* $(X_t)_{t\geq 0}$ *is said to be adapted to a filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if for every* $t \geq 0$*, the random variable* X_t *is measurable with respect to* $\mathcal{F}_t$*.*

Of course, a stochastic process is always adapted with respect to its natural filtration. We may observe that if a stochastic process $(X_t)_{t\geq 0}$ is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ and that if $\mathcal{F}_0$ contains all the subsets of $\mathcal{F}$ that have a zero probability, then every process $(\tilde{X}_t)_{t\geq 0}$ that satisfies

$$\mathbb{P}(\tilde{X}_t = X_t) = 1, \quad t \geq 0,$$

is still adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

We previously defined the notion of measurability for a stochastic process. In order to take into account the dynamic aspect associated to a filtration, the notion of progressive measurability is needed.

Sigma Algebra

Definition. *A stochastic process* $(X_t)_{t\geq 0}$ *that is adapted to a filtration* $(\mathcal{F}_t)_{t\geq 0}$*, is said to be progressively measurable with respect to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ *if for every* $t \geq 0$*,*

$$\forall A \in \mathcal{B}(\mathbb{R}), \{(s, \omega) \in [0, t] \times \Omega, X_s(\omega) \in A\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t.$$

By using the diagonal method, it is possible to construct adapted but not progressively measurable processes. However, the next proposition whose proof is let as an exercise to the reader shows that an adapted and continuous stochastic process is atomically progressively measurable.

Proposition. *A continuous stochastic process* $(X_t)_{t\geq 0}$*, that is adapted with respect to a filtration* $(\mathcal{F}_t)_{t\geq 0}$*, is also progressively measurable with respect to it.*

Thank you for your attention!

See next week!