# Home Assignment 1

1. Consider two identical hedgehogs starting at the vertices $A$ and $B$ of a polygon $ABCDE$. Each minute they simulteneously and independently choose to go clockwise or counter-clockwise in the next vertex.

   The brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.

   (a) What is the probability that they will be in one vertex after 3 steps?
   (b) Write down the transition matrix of the brotherhood Markov chain.
   (c) What proportion of time the brotherhood will spend in each state in the long run?
   (d) Find the expected time until the hedgehogs meet in one vertex.

2. Each day the Random Restaurant is independently closed with probability $p$. If the restaurant is open then the number of clients has Poisson distribution with mean $\mu$.

   After $N$ days (working or non-working) the Random Restaurant will permanently close and you are right, $N$ is random and has Poisson distribution with mean $n$.

   (a) Find the moment generating function of the number of clients during day 1, assuming $Ngeq1$.
   (b) Find the moment generating function of the total number of clients served in the Random Restaurant.

3. Find the probability limit plim $X_n$, where

$$X_n = \frac{Y_1 + 2Y_2 + 3Y_3 + \ldots + nY_n}{n^2}$$

   and $Y_1$, $Y_2$, ... are independent uniform on $[0; 1]$.

   Hint: try to calculate $\mathbb{E}(X_n)$, $\mathbb{V}\mathrm{ar}(X_n)$. You may google the formulas for $1+2+\ldots+n$ and $1^2+2^2+\ldots+n^2$ or ask ChatGPT.

4. Consider the Poisson arrival process $X_t$ with constant rate $\lambda$.

   Now let's scale the time in a non-linear fashion, $Y_t = X_{t^2}$.

   (a) Find $\mathbb{E}(Y_t)$, $\mathbb{V}\mathrm{ar}(Y_t)$, $\mathbb{P}(Y_t = 0)$.
   (b) Find $\mathbb{E}(Y_{t+5} \mid Y_t)$ and $\mathbb{V}\mathrm{ar}(Y_{t+5} \mid Y_t)$.

5. Let's toss a dice until the first six appears. Let $X$ be the result of the first toss and $Y$ — the total number of tosses.

   (a) Find $\mathbb{E}(X \mid Y)$, $\mathbb{E}(Y \mid X)$.
   (b) Find $\mathbb{V}\mathrm{ar}(X \mid Y)$, $\mathbb{V}\mathrm{ar}(Y \mid X)$.

6. The joint distribution of $X$ and $Y$ is given in the table

| | $X = -1$ | $X = 0$ | $X = 1$ |
|---|---|---|---|
| $Y = 0$ | 0.1 | 0.2 | 0.3 |
| $Y = 1$ | 0.2 | 0.1 | 0.1 |

   (a) Explicitely find the $\sigma$-algebras $\sigma(X)$, $\sigma(Y)$, $\sigma(X \cdot Y)$.
   (b) How many elements are there in $\sigma(X, Y)$, $\sigma(X + Y)$, $\sigma(X, Y, X + Y)$?