!! October 20 → 22 oct.

σ- algebras

Intuition.

X - Random variable.

$\boxed{\sigma(X) = \text{the list of all the events that can be „stated" (?) in terms of } X.}$

„σ-algebra generated by X"

Ex

	Y = 0	Y = 1
X = -2	0.2	0.1
X = 5	0.1	0.3
X = 2	0.1	0.2

a) σ(Y) ? !!

b) $\sigma(X^2)$?

c) $\sigma(|X|\cdot Y)$?

d) how many events are there in σ(X,Y) ?

σ(Y):

$\{Y=0\}, \{Y=1\}, \{Y>-5\}, \{Y<6\}, \{Y>100\} \dots\dots \{\cos Y < \frac{1}{2}\} \quad \{Y^2+Y^3>15\} \dots$

$$\{Y>-5\} = \Omega = \{Y<6\}$$
$$\{Y>100\} = \phi = \{Y^2+Y^3>15\}$$

$$\boxed{\sigma(Y) = \{\{Y=0\}, \{Y=1\}, \Omega, \phi\}}$$

$\{Y<\frac{1}{2}\}$ ⋮ $\quad \{Y^2=1\}$ ⋮ $\quad \{Y>-7\}$ ⋮ $\quad \{Y^2>10\}$ ⋮

Ω = the trivial event that always happens.

φ = the trivial even that never happens.

Ex

	$Y=0$	$Y=1$
$X=-2$	0.2	0.1
$X=5$	0.1	0.3
$X=2$	0.1	0.2

a) $\sigma(Y)$? !!

b) $\sigma(X^2)$? !!

c) $\sigma(|X|\cdot Y)$?

d) how many events are there in $\sigma(X,Y)$

$\sigma(X^2)$ $\qquad \{X=-2\} \notin \sigma(X^2)$

$$\{X^2=4\} = \{X=2\} \cup \{X=-2\}$$
$$\{X^2=25\} = \{X=5\}$$

$$\{X^2>-5\} = \Omega \in \sigma(X^2)$$
$$\{X^2<-5\} = \emptyset \in \sigma(X^2)$$

$$\sigma(X^2) = \{\{X^2=4\}, \{X^2=25\}, \Omega, \emptyset\}$$

c) $\sigma(|X|\cdot Y)$ $\qquad |X|\cdot Y \to 0, 2, 5$

$$\{|X|\cdot Y>1\} = \{|X|\cdot Y>\tfrac{1}{2}\} = \{|X|\cdot Y=2 \text{ OR } 5\}$$

$$\{|X|\cdot Y=0\} = \{\sin(|X|\cdot Y)\le 0\} = \{\log(|X|\cdot Y+1)=0\}$$

$$\boxed{\sigma(|X|\cdot Y) = \{\Omega, \emptyset, \{|X|\cdot Y=0\}, \{|X|\cdot Y=2\}, \{|X|\cdot Y=5\}, \{|X|\cdot Y\neq 0\}, \{|X|\cdot Y\neq 2\}, \{|X|\cdot Y\neq 5\}\}}$$

$$\{|X|\cdot Y=0\} = \{|X|\cdot Y<1\} = \{|X|\cdot Y<1.1\} = \ldots$$

$$\{|X|\cdot Y\neq 0\} = \{W\in\{2,5\}\}$$

$\Omega = $ | $W=0$ | $W=2$ | $W=5$ |

$W=|X|\cdot Y$ $\quad \Omega$ is a partition of 3 disjoint events.

$$\Omega = \{W=0\} \cup \{W=2\} \cup \{W=5\}$$

$W = |X| \cdot Y$

$L = \{\{W=0\}, \{W=2\}, \{W=5\}\}$

$\sigma(W) = 2^L$

8 events in the σ-algebra $\sigma(W)$

$\pm$	$\pm$	$\pm$	$\to \Omega$, $\to \emptyset$
$+$	$-$	$+$	$\to \{|X| \cdot Y \neq 2\}$
$-$	$+$	$+$	
$+$	$+$	$-$	
$-$	$-$	$+$	$\to \{|X| \cdot Y = 5\}$
$-$	$+$	$-$	
$+$	$-$	$-$	

$\boxed{\text{how many events are there in } \sigma(X,Y) \text{ ?}}$

$\{X < Y\}$ $\quad \{X^2 + Y^2 = 5\}$ $\quad \{\frac{X}{Y+5} = 7\}$

$\{X \geq Y\}$ . . . . . $\{\cos(X) + \cos(Y) > 0\}$

$\{Y=0\} \to \{Y=0, X=-2\}$ $\{Y=0, X=5\}$, $\{Y=0, X=2\}$

Ex

	$Y=0$	$Y=1$
$X=-2$	0.2	0.1
$X=5$	0.1	0.3
$X=2$	0.1	0.2

$\boxed{\{\frac{X}{Y+5} = 7\}} = \boxed{\emptyset}$

$\boxed{\{X^2+Y^2=5\}} = \{|X|=2, Y=1\} =$

$= \boxed{\{X<5, Y>0\}}$

$\sigma(X,Y) = \{\{X=-2, Y=1\}, \ldots\ldots \{X<5, Y>0\}, \ldots\}$

$\{X=-2, Y=0\}$ $\qquad \{X=5, Y=1\}$

$\{Y=0\}$

$\{Y=0, X \neq 2\}$

$\boxed{2^6}$ cells in the table

$\oplus / \ominus$

+	−
−	−
−	−

$\uparrow$

$\{X=-2, Y=0\}$

6 outcomes: $\{X=-2, Y=0\}$ . . . . . . $\{X=5, Y=1\}$

$\emptyset \to$ − − − − − −

$\Omega \to$ + + + + + +

$\{X=-2\} =$ + − − +

⊕	⊕
⊖	⊖
⊖	⊖

$\{X=-2\}$

−	+
−	+
−	+

$\{Y=1\}$

+	−
−	+
+	−

{the parity of X and Y is the same}

in total $\mathrm{card}(\sigma(X,Y)) = 2^6 = 64$ events

$\{X=2\} \to \{X=2, Y=0\}$, $\to \{X=2, Y=1\}$

$E(Y \mid \sigma(X))$

$E(Y \mid \sigma(X, W))$

X, Y are discr.

<u>X and Y are indep ?</u>

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

X and Y <u>are indep</u>
joint density

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

<u>X and Y are Indep:</u>

X and Y are independent if every event A from $\sigma(X)$ and every event $B \in \sigma(Y)$ are independent.

def — $\sigma(X)$ -: σ-algebra generated by RV X.

the smallest ~~lost~~ that satisfies.

idea 1: we can compare X with any real number.

$\{X \le 5\} \in \sigma(X)$, $\{X \le 6.2\} \in \sigma(X)$ ...

$\forall t \in \mathbb{R}$ $\boxed{\{X \le t\} \in \sigma(X)}$

idea 2 if $A \in \sigma(X)$ then $A^c \in \sigma(X)$ $\boxed{\bar{A} = A^c}$

idea 3 if $A_1 \in \sigma(X)$, $A_2 \in \sigma(X)$, ... then $\bigcup_{i=1}^{\infty} A_i \in \sigma(X)$

Ex. $X \sim Bin(n=10, p=0.3)$ $X = 0, 1, 2, \dots 10$

How many events are in $\sigma(x)$?

$\sigma(X) = \{ \{X=5\}, \{X>6\}, \{X>2\}, \{X \ne 4\} \dots \dots \}$

$\text{card } \sigma(X) = 2^{11} = 2048$

card (smth) = the number of objects in smth

def $\mathcal{F}$ is "a σ-algebra" [generated by some RVs]

- idea 1. $\phi \in \mathcal{F}$
- idea 2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
- idea 3 if $A_1, A_2, A_3 \dots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$X=0$	$X=1$	$X=2$	$X=3$
0.2	0.1	0.1	0.6

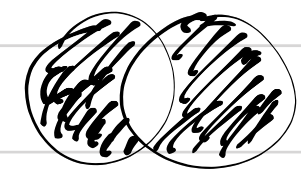
$\mathcal{F} = \{ \{X < 1.5\}, \{X \ge 1.5\}, \phi, \Omega \} \ne \sigma(x)$

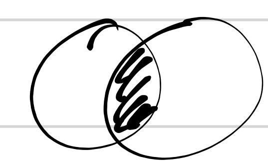
$W = \begin{cases} 1, & X \ge 1.5 \\ 0, & X < 1.5 \end{cases}$ $\mathcal{F} = \sigma(W)$

"list of events"

"Why such a strange name "σ-algebra"?

algebra:

$5+6=11$ | $A \triangle B$ =  $= (A\setminus B)\cup(B\setminus A)$

$5\cdot 7=35$ | $A\cap B$ = 

$5\cdot(3+7)=5\cdot 3+5\cdot 7$ | $A\cap(B\triangle C)=(A\cap B)\triangle(A\cap C)$

	$Y=0$	$Y=1$
$X=-2$	+	+
$X=5$	+	+
$X=2$	+	+

$=\Omega$

	−	+
	−	+
	−	+

$=\{Y=1\}$

	+	+
	−	−
	−	−

$=\{X=-2\}$