$E(\cdot|\cdot)$ $\quad$ $Var(\cdot|\cdot)$

Two „sources" of randomness

- random choice of a random variable

$$\begin{array}{ccc} X_1 & & X_2 \\ & {\searrow}^{p} \quad {\swarrow}^{1-p} & \\ & X & \end{array}$$

Quest: $E(X)$...

- random number of terms in a sum

$$S = X_1 + X_2 + X_3 + \dots + X_N$$

N - is a random variable

Quest: $E(S)$ ...

$\boxed{P4}$

N - the number of questions.

Q: 0.7 → know the answer; 0.3 → you guess

$\boxed{\text{3 options for question.}}$

a) warm-up $\quad$ $P(\text{you will answer correctly}) =$

$$\overset{?}{=} \underbrace{0.7}_{\text{you know}} + \underbrace{0.3}_{\text{you guess}} \cdot \underbrace{\tfrac{1}{3}}_{\text{you are lucky}}$$

b) $N = 3$ [N is not random yet]

$P(\text{you know just one quest} \mid \text{you have answered 3 questions correctly}) =$

$$= \frac{P(A \cap B)}{P(B)} = \frac{P(\text{I know only one Q} \mid \text{I answer 3 qs correctly})}{P(\text{I answer 3 qs corr})} =$$

$$= \frac{C_3^1 \cdot 0{,}7^1 \cdot \left(0{,}3 \cdot \frac{1}{3}\right)^2}{\left(0{,}7 + 0{,}3 \cdot \frac{1}{3}\right)^3}$$

similar to binomial $C_n^k \cdot p^k (1-p)^{n-k}$

c) $$\boxed{f_T(t) = \begin{cases} \lambda \cdot \exp(-\lambda t) & \text{if you know the answer} \\ 5\lambda \cdot \exp(-5\lambda t) & \text{if you guess.} \end{cases}}$$

$$T \xleftarrow{0{,}7} R_1 \qquad R_1 \sim \operatorname{Exp}(\text{rate} = \lambda) \Rightarrow E(T_1) = \frac{1}{\lambda}$$

$$T \xleftarrow{0{,}3} R_2 \qquad R_2 \sim \operatorname{Exp}(\text{rate} = 5\lambda) \quad E(T_2) = \frac{1}{5\lambda}$$

random choice of a random variable

$$T = I \cdot R_1 + (1 - I) \cdot R_2$$

$A:$	$I = 1$	$I = 0$
$P(A)$	$0{,}7$	$0{,}3$

$\boxed{I, R_1, R_2 \text{ are indep.}}$

$$E(I) = 1 \cdot 0.7 + 0 \cdot 0{,}3 = 0.7$$

$I = 1 \Rightarrow T = R_1$

$I = 0 \Rightarrow T = R_2$

$$E(T)? = E\left(I \cdot R_1 + (1 - I) \cdot R_2\right) =$$

$$= E(I R_1) + E((1 - I) R_2) =$$

$$= E(I) \cdot E(R_1) + E(1 - I) \cdot E(R_2) =$$

$$= 0{,}7 \cdot \frac{1}{\lambda} + 0{,}3 \cdot \frac{1}{5\lambda} = \frac{3.5 + 0{,}3}{5\lambda} = \frac{3.8}{5\lambda}$$

(c) ii $\quad \underline{T_1, T_2, T_3 \dots T_N}$ $\qquad E(N) = 10$

$Var(N) = 10$

$W = T_1 + T_2 + T_3 + \dots + T_N$ ← random number of terms

$T_i = I_i \cdot R_{1i} + (1 - I_i) \cdot R_{2i}$ ← random choice of a random variable

$E(W)$? $\quad Var(W)$?

* $E(W) = E(\boxed{E(W|N)})$ $\quad$ * law of total expected value * Adam's law * Tower property

* $Var(W) = E(\boxed{Var(W|N)}) + Var(E(W|N))$ $\quad$ * Eve's law

$E(W|N)$ ← fix N, think that N is a constant!

$\boxed{E(W|N)} = E(T_1 + T_2 + \dots + T_N | N) =$ ← N is fixed!

$= E(T_1|N) + \dots + E(T_N|N) =$

$= E(T_1) \cdot N = \boxed{N \cdot \frac{3.8}{5\lambda}}$

$E(W) = E(E(W|N)) = E\left(N \cdot \frac{3.8}{5\lambda}\right) =$

$= \underbrace{E(N)}_{10} \cdot \frac{3.8}{5\lambda} = \frac{38}{5\lambda}$

$$\operatorname{Var}(W) = \underline{E(\operatorname{Var}(W|N))} + \operatorname{Var}(E(W|N))$$

$\operatorname{Var}(W|N) = N \cdot \operatorname{Var}(T_i)$

$N \cdot t$

Eve's law

$t = E(T_i)$

$t^2 \cdot \operatorname{Var}(N)$

Intuition:

case 1: N is random, but $T_i = t$

$$W = \underbrace{t + t + t + \dots + t}_{N \text{ terms}} = N \cdot t$$

$$\operatorname{Var}(W) = t^2 \cdot \operatorname{Var}(N)$$

case 2: n is fixed, but $T_i \sim$ iid

$$\operatorname{Var}(W) = \operatorname{Var}(T_1 + \dots + T_n) = n \cdot \operatorname{Var}(T_i)$$

$$E(W|N) = \frac{3.8}{5\lambda} \cdot N$$

$$\operatorname{Var}(W|N) = N \cdot \operatorname{Var}(T_1) = N \cdot \operatorname{Var}(I_1 \cdot R_{11} + (1 - I_1) \cdot R_{21}) =$$

N is fixed

$$\boxed{= N \cdot \frac{0.9304}{\lambda^2}}$$

$$= N \cdot \left(\operatorname{Var}(I_1 \cdot R_{11}) + \operatorname{Var}((1 - I_1) \cdot R_{21}) \right);$$

$$\operatorname{Var}(I_1 \cdot R_{11}) = E(I_1^2 \cdot R_{11}^2) - (E(I_1 \cdot R_{11}))^2 =$$

$$= E(I_1^2) \cdot E(R_{11}^2) - (E(I_1) \cdot E(R_{11}))^2 =$$

$$= E(I_1) \cdot (\operatorname{Var}(R_{11}) + (E(R_{11}))^2) - (E(I_1) \cdot E(R_{11}))^2 =$$

$R_{11} \sim \operatorname{Exp}(\text{rate} = \lambda)$

$E(R_{11}) = \frac{1}{\lambda}$

$\operatorname{Var}(R_{11}) = \frac{1}{\lambda^2}$

$$= \boxed{0{,}7 \cdot \left(\frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 \right) - \left(0{,}7 \cdot \frac{1}{\lambda}\right)^2} =$$

$$= \frac{1{,}4 - 0{,}49}{\lambda^2} = \frac{0.91}{\lambda^2}$$

By analogy:

$$\operatorname{Var}((1 - I_1) \cdot R_{21}) = 0{,}3 \cdot \left(\frac{1}{(5\lambda)^2} + \frac{1}{(5\lambda)^2} \right) - \left(0{,}3 \cdot \frac{1}{5\lambda}\right)^2 =$$

$$= \frac{0{,}6 - 0{,}09}{25\lambda^2} = \frac{0.51}{25\lambda^2} = \frac{2.04}{100\lambda^2} = \frac{0.0204}{\lambda^2}$$

$$E(W|N) = \frac{3.8}{5\lambda}\cdot N \qquad Var(W|N) = N\cdot\frac{const}{\lambda^2} \leftarrow 0.9304$$

Eve's law:

$$Var(W) = Var\left(\frac{3.8}{5\lambda}N\right) + E\left(N\cdot\frac{const}{\lambda^2}\right) =$$

$$= \left(\frac{3.8}{5\lambda}\right)^2\cdot \underbrace{Var(N)}_{10} + \underbrace{E(N)}_{10}\cdot\frac{const}{\lambda^2}$$

$$= \left(\frac{3.8}{5\lambda}\right)^2\cdot 10 + 10\cdot\frac{const}{\lambda^2}$$

!!

ZA 20-21 p3

N3

$$W = \sum_{i=1}^{N} X_i = X_1 + X_2 + \dots + X_N \qquad N \sim Poiss \quad E(N) = \mu$$

$$X_i \sim exp(rate = \lambda)$$

$$f(x) = \begin{cases} \lambda\cdot exp(-\lambda x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$X_1, X_2, \dots, N$ are indep.

a) $\boxed{\text{MGF for } W \text{ given } N}$

MGF $\quad M_W(u) = E(exp(W\cdot u))$ ← here N has Poisson distr.

final target → conditional MGF $\quad M_{W|N}(u) = E(exp(W\cdot u)/N)$

$\boxed{\text{here } N \text{ is constant}}$

$$M_{W|N}(u) = E(exp(X_1\cdot u + X_2\cdot u + \dots + X_N u)/N) =$$

$$= E(exp(X_1\cdot u)|N)\cdot E(exp(X_2\cdot u)|N)\cdot \dots \cdot E(exp(X_N u)|N)$$

$E(R_1 R_2) = E(R_1)\cdot E(R_2)$ if R_1 and R_2 are uncorrelated

$$M_{W|N}(u) = \left(\underbrace{E(\exp(X_1 \cdot u))}_{\text{moment generating function for } X_1}\right)^N$$

W given N like P(A|B)
W if N is fixed
W if we drop randomness in N

$$M_{X_1}(u) = E(\exp(X_1 \cdot u)) = \int_0^\infty \exp(xu) \cdot \underbrace{\lambda \cdot \exp(-\lambda x)}_{\text{pdf}} \, dx =$$

$$= \int_0^\infty \lambda \cdot \exp(-x \cdot (\lambda - u)) \, dx = \left[\frac{\lambda \cdot \exp(-x(\lambda - u))}{-(\lambda - u)}\right]_{x=0}^{x=\infty} =$$

$$= 0 - \frac{\lambda}{-(\lambda - u)} = \frac{\lambda}{\lambda - u} \quad !!$$

$$\boxed{M_{W|N}(u) = \left(\frac{\lambda}{\lambda - u}\right)^N}$$ ← conditional MGF

Adam's law
/ Tower property

b) $M_W(u) = E(\exp(W \cdot u)) =$

$$= E\left(E(\exp(Wu) \mid N)\right) = E\left(\left(\frac{\lambda}{\lambda - u}\right)^N\right);$$

$E(N) = \mu$

$N \sim \text{Poiss}(\text{rate} = \mu)$

$$P(N = k) = \exp(-\mu) \cdot \frac{\mu^k}{k!}$$

$$P(N=0) = \exp(-\mu)$$
$$P(N=1) = \exp(-\mu) \cdot \mu$$
$$P(N=2) = \exp(-\mu) \cdot \mu^2/2!$$
$$\ldots$$

b) $M_W(u) = E(\exp(W \cdot u)) =$

$= E\left(\underbrace{E(\exp(Wu) \mid N)}\right) = E\left(\left(\frac{\lambda}{\lambda - u}\right)^N\right);$

$E(N) = \mu$

$$\boxed{\begin{gathered} N \sim \text{Poiss}(\text{rate} = \mu) \\ P(N = k) = \exp(-\mu) \cdot \frac{\mu^k}{k!} \\ P(N=0) = \exp(-\mu) \\ P(N=1) = \exp(-\mu) \cdot \mu \\ P(N=2) = \exp(-\mu) \cdot \mu^2/2! \\ \dots \end{gathered}}$$

$= P(N=0) \cdot \left(\frac{\lambda}{\lambda-u}\right)^0 + P(N=1) \cdot \left(\frac{\lambda}{\lambda-u}\right)^1 + P(N=2)\left(\frac{\lambda}{\lambda-u}\right)^2 + \dots$

$= \exp(-\mu) \cdot \left[1 + \mu \cdot \frac{\lambda}{\lambda-u} + \frac{\mu^2}{2!} \cdot \left(\frac{\lambda}{\lambda-u}\right)^2 + \frac{\mu^3}{3!} \cdot \left(\frac{\lambda}{\lambda-u}\right)^3 + \dots\right] =$


← rabbit

$\exp(t) = \boxed{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots}$

$t = \frac{\mu \cdot \lambda}{\lambda - u}$

$= \exp(-\mu) \cdot \exp\left(\frac{\mu \cdot \lambda}{\lambda - u}\right) = \boxed{\exp\left(\frac{\mu\lambda}{\lambda-u} - 1\right) = M_W(u)}$

c) $$\boxed{\begin{gathered} E(W) = M_W'(0) \\ Var(W) = E(W^2) - (E(W))^2 \\ E(W^2) = M''(0) \end{gathered}}$$

$\boxed{\text{strategy 1}}$

$E(W) = E(\underbrace{E(W|N)}) =$ $\boxed{\text{Strategy 2.}}$

$= E\left(\underbrace{N \cdot \frac{1}{\lambda}}\right) = E(N) \cdot \frac{1}{\lambda} = \mu \cdot \frac{1}{\lambda}$!!

$Var(W) = E(Var(W|N)) + Var(E(W|N)) =$

$= E\left(\underbrace{N} \cdot \frac{1}{\lambda^2}\right) + \underbrace{Var\left(N \cdot \frac{1}{\lambda}\right)} = \mu \cdot \frac{1}{\lambda^2} + \mu \cdot \frac{1}{\lambda^2}$!!

d)

$$Z = N \cdot X_1$$

$$E(Z) = E(N) \cdot E(X_1) = \mu \cdot \frac{1}{\lambda}$$

$$Var(Z) = E(N^2 X_1^2) - (E(N \cdot X_1))^2 =$$

$$= E(N^2) \cdot E(X_1^2) - (E(N) \cdot E(X_1))^2 =$$

$$= (Var N + E(N)^2) \cdot (Var X_1 + (E(X_1))^2) -$$

$$- (E(N) \cdot E(X_1))^2 =$$

$$= (\mu + \mu^2) \cdot \left(\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\right) - \left(\mu \cdot \frac{1}{\lambda}\right)^2 \quad !!$$