Lecture 3
Convergence

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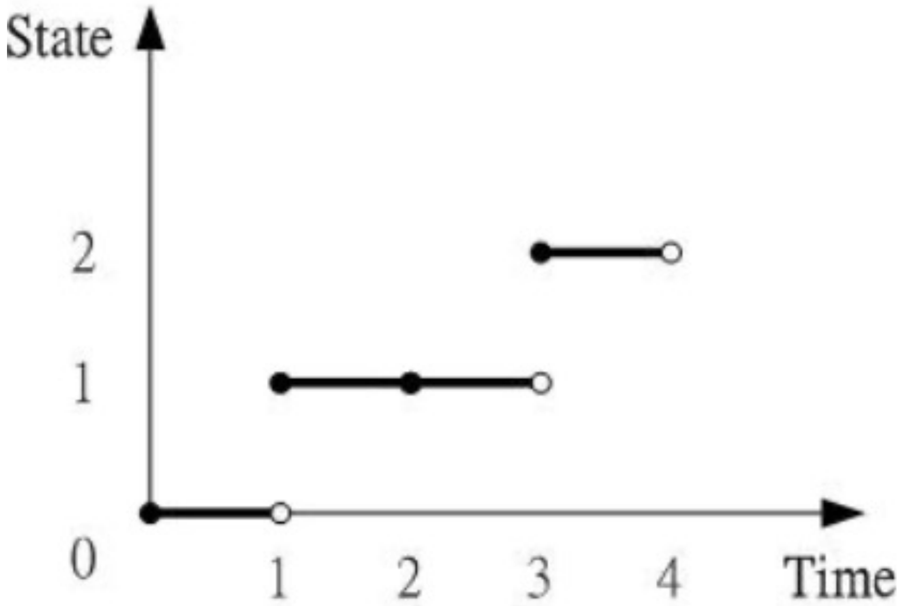
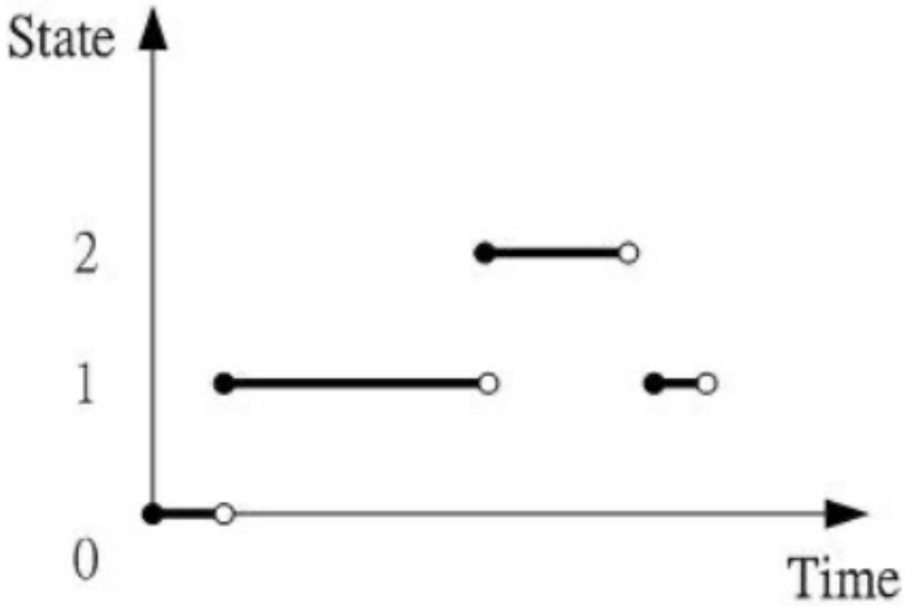
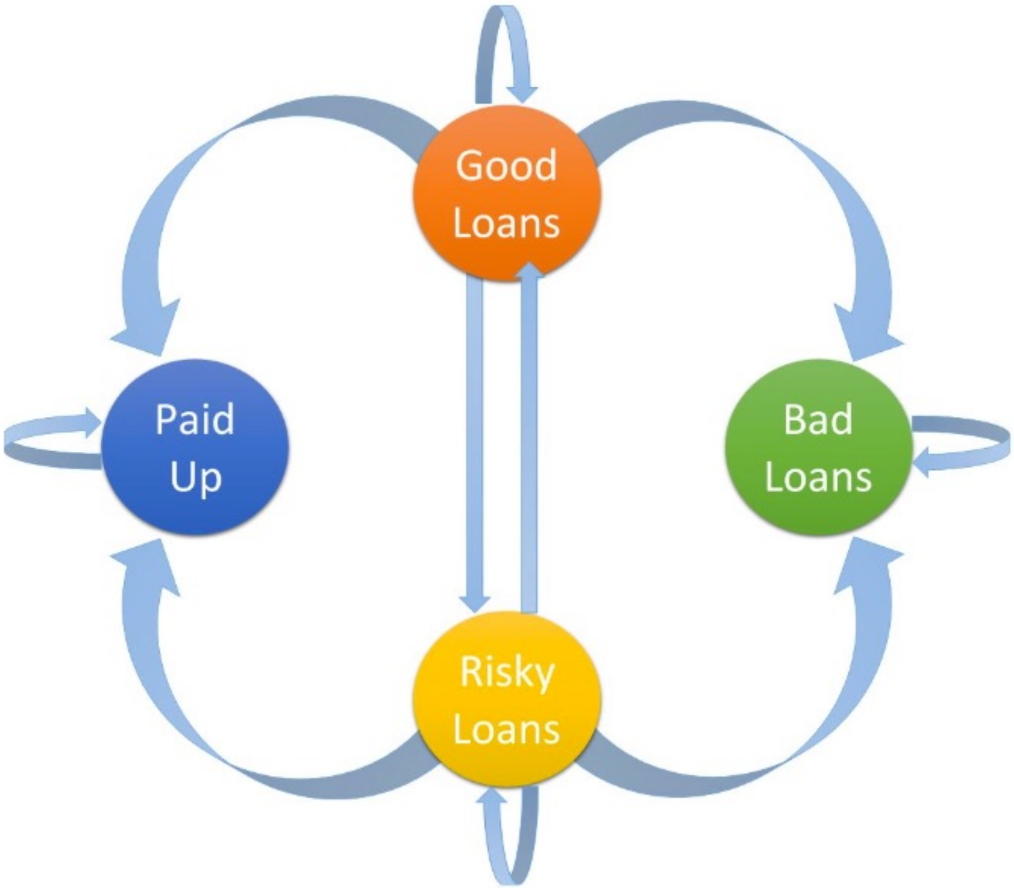
Markov process

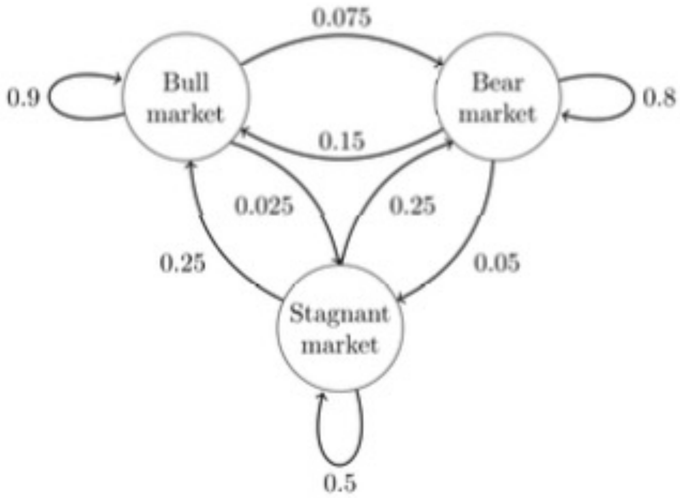




Markov process



Markov process



Convergence

Definition *Let $\{x_n, n \geq 1\}$ be a real-valued sequence, i.e., a map from $\mathbb{N}$ to $\mathbb{R}$. We say that the sequence $\{x_n\}$ converges to some $x \in \mathbb{R}$ if there exists an $n_0 \in \mathbb{N}$ such that for all $\epsilon > 0$,*

$$|x_n - x| < \epsilon, \; \forall \; n \geq n_0.$$

We say that the sequence $\{x_n\}$ converges to $+\infty$ if for any $M > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n > M$.
We say that the sequence $\{x_n\}$ converges to $-\infty$ if for any $M > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $x_n < -M$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of real-valued random variables defined on this probability space.

Convergence

Definition **[Definition 0 (Point-wise convergence or sure convergence)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge point-wise or surely to* X *if*

$$X_n(\omega) \to X(\omega), \quad \forall\, \omega \in \Omega.$$

Convergence

Definition **Definition 1 (Almost sure convergence or convergence with probability 1)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to converge almost surely or with probability 1 (denoted by a.s. or w.p. 1) to X if

$$\mathbb{P}\left(\{\omega | X_n(\omega) \to X(\omega)\}\right) = 1.$$

$\frac{1}{100n}$

X_n : 2 ; 1 $\frac{1}{2} + \frac{1}{10n} - \frac{1}{100n}$; 0 $\frac{1}{2} - \frac{1}{10n}$

X : 1 $1/2$; 0 $1/2$

1 $\frac{1}{2} + \frac{1}{10n} \to \frac{1}{2}$

Convergence

Definition **[Definition 2 (convergence in probability)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge in probability (denoted by i.p.) to* X *if*

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) = 0, \quad \forall\, \epsilon > 0.$$

Convergence

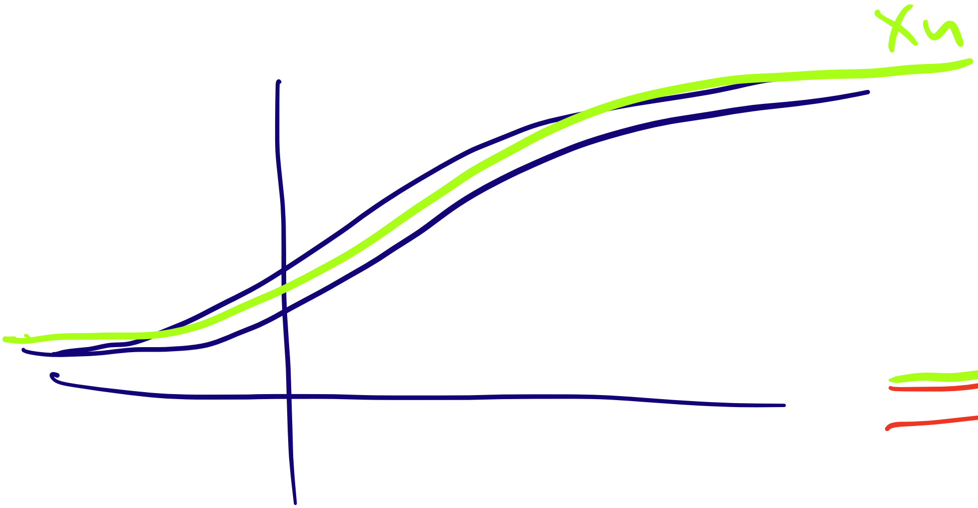
Definition **[Definition 3 (convergence in r^{th} mean)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ *is said to converge in* r^{th} *mean to* X *if*

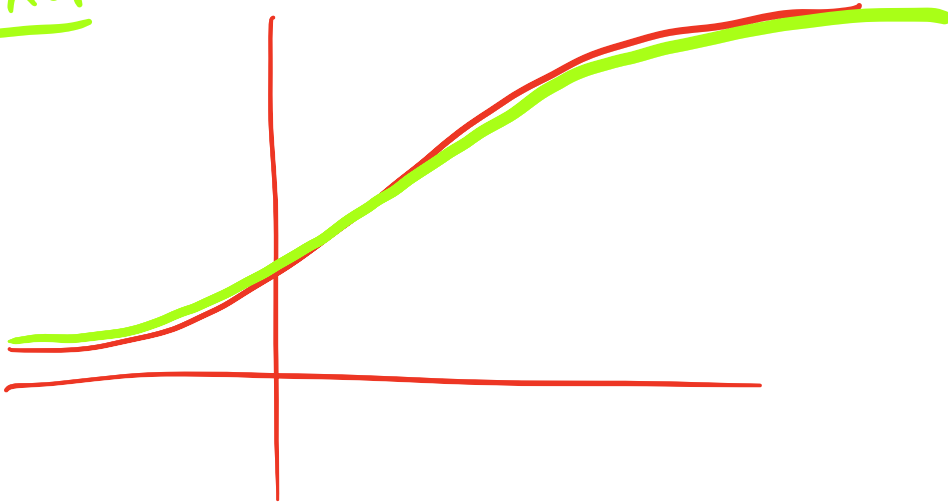
$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X|^r\right] = 0.$$

Convergence

Definition **[Definition 4 (convergence in distribution or weak convergence)]**
A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to converge in distribution to X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \quad \forall\ x \in \mathbb{R} \text{ where } F_X(\cdot) \text{ is continuous.}$$





Convergence

(1) *Point-wise Convergence:* $X_n \xrightarrow{\text{p.w.}} X$.

(2) *Almost sure Convergence:* $X_n \xrightarrow{\text{a.s}} X$ or $X_n \xrightarrow{\text{w.p.}\,1} X$.

(3) *Convergence in probability:* $X_n \xrightarrow{\text{i.p.}} X$.

(4) *Convergence in r^{th} mean:* $X_n \xrightarrow{r} X$. When $r = 2$, $X_n \xrightarrow{\text{m.s.}} X$.

(5) *Convergence in Distribution:* $X_n \xrightarrow{\text{D}} X$.

Convergence

Example: Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ and a sequence of random variables $\{X_n, n \geq 1\}$ defined by

$$X_n(\omega) = \begin{cases} n, & \text{if } \omega \in \left[0, \frac{1}{n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Convergence

$$X_n = \begin{cases} n, & \text{with probability } \frac{1}{n}, \\ 0, & \text{with probability } 1 - \frac{1}{n}. \end{cases}$$

$X = 0$

✓ a.s. $\quad n \quad |X_n - X| = n \quad pr = 0$

$0 \quad |X_n - X| = 0 \quad pr = 1$

$P(\omega : |X_n - X| < \varepsilon) = 1$

✓ P $\quad P(|X_n - X| > \varepsilon) = P(X_n(n) - X(n) > \varepsilon) = P(X_n = n) = \frac{1}{n} \to 0$

✓ D : $F_{X_n} = \begin{cases} 0 & X_n < 0 \\ 1 - \frac{1}{n} & X_n < n \\ 1 & X_n < \infty \end{cases} \qquad F_X = \begin{cases} 0 & X < 0 \\ 1 & X < n \\ 1 & X < \infty \end{cases}$

Clearly, when $\omega \neq 0$, $\lim_{n\to\infty} X_n(\omega) = 0$ but it diverges for $\omega = 0$. This suggests that the limiting random variable must be the constant random variable 0. Hence, except at $\omega = 0$, the sequence of random variables converges to the constant random variable 0. Therefore, this sequence does not converge surely, but converges almost surely.

Convergence

For some $\epsilon > 0$, consider

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(|X_n| > \epsilon\right) &= \lim_{n\to\infty} \mathbb{P}\left(X_n = n\right), \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\right), \\
&= 0.
\end{aligned}$$

Hence, the sequence converges in probability.

Convergence

Consider the following two expressions:

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[|X_n|^2\right] &= \lim_{n\to\infty}\left(n^2 \times \frac{1}{n} + 0\right), \\
&= \infty.
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[|X_n|\right] &= \lim_{n\to\infty}\left(n \times \frac{1}{n} + 0\right), \\
&= 1.
\end{aligned}
$$

Convergence

$$
\begin{tikzcd}
\text{p.w.} \arrow[r, Rightarrow] & \text{a.s.} \arrow[dr, Rightarrow] & & \\
& & \text{i.p.} \arrow[r, Rightarrow] & \text{D} \\
r^{th} \text{ mean } (r \geq 1) \arrow[urr, Rightarrow] \arrow[rrr, Rightarrow] & & & s^{th} \text{ mean } (r > s \geq 1)
\end{tikzcd}
$$

$x \geq 0$

Markov's Ir-ty

$$E(X) = \sum p \cdot x = \sum_{x<a} p_i x_i + \sum_{x>a} p_i x_i \geq \sum_{x<a} p_i x_i + a \cdot \sum_{x>a} p_i \geq a \cdot \sum_{x>a} p_i$$

$$E(X) \geq a \sum_{x>a} p_i = P(X > a)$$

$$P(X > a) \leq \frac{EX}{a}$$

a

Convergence

Theorem $X_n \xrightarrow{r} X \implies X_n \xrightarrow{\text{i.p.}} X, \quad \forall\, r \geq 1.$

Proof: Consider the quantity $\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right)$. Applying Markov's inequaliy, we get

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \epsilon\right) &\leq \lim_{n\to\infty} \frac{\mathbb{E}\left[|X_n - X|^r\right]}{\epsilon^r}, \ \forall \epsilon > 0, \\ &\overset{(a)}{=} 0, \end{aligned}$$

where (a) follows since $X_n \xrightarrow{r} X$. Hence proved.

$$P(T > a) \leq \frac{E(T)}{a}$$

Convergence

Theorem $\quad X_n \xrightarrow{\text{i.p.}} X \implies X_n \xrightarrow{\text{D}} X.$

Proof: Fix an $\epsilon > 0$.

$$\begin{aligned} F_{X_n}(x) &= \mathbb{P}(X_n \leq x), \\ &= \mathbb{P}(X_n \leq x, X \leq x+\epsilon) + \mathbb{P}(X_n \leq x, X > x+\epsilon), \\ &\leq F_X(x+\epsilon) + \mathbb{P}(|X_n - X| > \epsilon). \end{aligned}$$

Similarly,

$$\begin{aligned} F_X(x-\epsilon) &= \mathbb{P}(X \leq x-\epsilon), \\ &= \mathbb{P}(X \leq x-\epsilon, X_n \leq x) + \mathbb{P}(X \leq x-\epsilon, X_n > x), \\ &\leq F_{X_n}(x) + \mathbb{P}(|X_n - X| > \epsilon). \end{aligned}$$

Thus,

$$F_X(x-\epsilon) - \mathbb{P}(|X_n - X| > \epsilon) \leq F_{X_n}(x) \leq F_X(x+\epsilon) + \mathbb{P}(|X_n - X| > \epsilon).$$

As $n \to \infty$, since $X_n \xrightarrow{\text{i.p.}} X$, $\mathbb{P}(|X_n - X| > \epsilon) \to 0$. Therefore,

$$F_X(x-\epsilon) \leq \liminf_{n\to\infty} F_{X_n}(x) \leq \limsup_{n\to\infty} F_{X_n}(x) \leq F_X(x+\epsilon), \ \forall \epsilon > 0.$$

If F is continuous at x, then $F_X(x-\epsilon) \uparrow F_X(x)$ and $F_X(x+\epsilon) \downarrow F_X(x)$ as $\epsilon \downarrow 0$. Hence proved.

Convergence

$\to 0^+$

Theorem $X_n \xrightarrow{r} X \implies X_n \xrightarrow{s} X$, *if* $r > s \geq 1$.

??? $E[(X_n - X)^s] \to 0$

$E[(X_n - X)^r] \to 0$ $\cancel{\le}$ $(\mathbb{E}[|X_n - X|^s])^{1/s} \leq (\mathbb{E}[|X_n - X|^r])^{1/r}$,

Jensen's In-ty

$f(E[x]) \angle E(f(x))$

Convergence

Theorem $X_n \xrightarrow{\text{i.p.}} X \not\Rightarrow X_n \xrightarrow{\text{r}} X$ *in general.*

Proof: Proof by counter-example:
Let X_n be an independent sequence of random variables defined as

$$X_n = \begin{cases} n^3, & \text{w.p. } \frac{1}{n^2}, \\ 0, & \text{w.p. } 1\text{-}\frac{1}{n^2}. \end{cases}$$

$X = \{0, 1\}$

Then, $\mathbb{P}(|X_n| > \epsilon) = \frac{1}{n^2}$ for large enough n, and hence $X_n \xrightarrow{\text{i.p.}} 0$. On the other hand, $\mathbb{E}[|X_n|] = n$, which diverges to infinity as n grows unbounded. ■

$EX_n = n$ $\qquad EX = 0$ $\qquad$ w

$\lim P(|X_n - X| > \varepsilon) = \lim \frac{1}{n^2} = 0 \qquad E[|X_n - X|] = n \to \infty$

Convergence

Theorem $X_n \xrightarrow{\text{D}} X \not\Rightarrow X_n \xrightarrow{\text{i.p.}} X$ *in general.*

Proof: Proof by counter-example:
Let X be a Bernoulli random variable with parameter 0.5, and define a sequence such that $X_i = X\ \forall\ i$. Let $Y = 1 - X$. Clearly, $X_i \xrightarrow{\text{D}} Y$. But, $|X_i - Y| = 1$, $\forall\ i$. Hence, X_i does not converge to Y in probability. ■

Convergence

Theorem $X_n \xrightarrow{\text{i.p.}} X \not\Rightarrow X_n \xrightarrow{\text{a.s.}} X$ *in general.*

$\boxed{*}$

$P(|X_n - X| > \varepsilon) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

ip

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent random variables defined as

$$X_n = \begin{cases} 1, & \text{w.p. } \frac{1}{n}, \\ 0, & \text{w.p. } 1\text{-}\frac{1}{n}. \end{cases}$$

$X = \{0, 1$

$\lim_{n\to\infty} \mathbb{P}(|X_n| > \epsilon) = \lim_{n\to\infty} \mathbb{P}(X_n = 1) = \lim_{n\to\infty} \frac{1}{n} = 0$. So, $X_n \xrightarrow{\text{i.p.}} 0$.

Let A_n be the event that $\{X_n = 1\}$. Then, A_n's are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. By Borel-Cantelli Lemma 2, w.p. 1 infinitely many A_n's will occur, *i.e.*, $\{X_n = 1\}$ i.o.. So, X_n does not converge to 0 almost surely. ∎

$P(\omega : |X_n - X| < \varepsilon) :$ 0 $\quad 1 - \frac{1}{n} \to 1$

1 $\quad \underline{\underline{|1-0| < \varepsilon}}$ $\quad \frac{1}{n} \to 0$

Convergence

Theorem $X_n \xrightarrow{s} X \not\Rightarrow X_n \xrightarrow{r} X$ *if* $r > s \geq 1$ *in general.*

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent random variables defined as

$$X_n = \begin{cases} n, & \text{w.p. } \frac{1}{n^{\frac{r+s}{2}}}, \\ 0, & \text{w.p. } 1\text{-} \frac{1}{n^{\frac{r+s}{2}}}. \end{cases}$$

Hence, $\mathbb{E}[|X_n^s|] = n^{\frac{s-r}{2}} \to 0$. But, $\mathbb{E}[|X_n^r|] = n^{\frac{r-s}{2}} \to \infty$.

Convergence

Theorem $X_n \xrightarrow{\text{m.s.}} X \;\not\Rightarrow\; X_n \xrightarrow{\text{a.s.}} X$ *in general.*

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent random variables defined as

$$X_n = \begin{cases} 1, & \text{w.p. } \frac{1}{n}, \\ 0, & \text{w.p. } 1\text{-}\frac{1}{n}. \end{cases}$$

$\mathbb{E}[X_n^2]=\frac{1}{n}$. So, $X_n \xrightarrow{\text{m.s.}} 0$. X_n does not converge to 0 almost surely.

Convergence

Theorem $X_n \xrightarrow{\text{a.s.}} X \;\not\Rightarrow\; X_n \xrightarrow{\text{m.s.}} X$ *in general.*

Proof: Proof by counter-example:
Let $\{X_n\}$ be a sequence of independent of random variables defined as

$$X_n(\omega) = \begin{cases} n, & \omega \in (0, \frac{1}{n}), \\ 0, & \text{otherwise.} \end{cases}$$

We know that X_n converges to 0 almost surely. $\mathbb{E}[X_n^2]{=}n \longrightarrow \infty$. So, X_n does not converge to 0 in the mean-squared sense. ■

Before proving the implication $X_n \xrightarrow{\text{a.s.}} X \implies X_n \xrightarrow{\text{i.p.}} X$, we derive a sufficient condition followed by a necessary and sufficient condition for almost sure convergence.

Convergence

Theorem 28.20 [Skorokhod's Representation Theorem]
Let $\{X_n, n \geq 1\}$ and X be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that X_n converges to X in distribution. Then, there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and random variables $\{Y_n, n \geq 1\}$ and Y on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that,

a) $\{Y_n, n \geq 1\}$ *and* Y *have the same distributions as* $\{X_n, n \geq 1\}$ *and* X *respectively.*

b) $Y_n \overset{a.s.}{\to} Y$ *as* $n \to \infty$.

Convergence

Theorem 28.21 [Continuous Mapping Theorem]
If $X_n \xrightarrow{D} X$, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then $g(X_n) \xrightarrow{D} g(X)$.

Proof: By Skorokhod's Representation Theorem, there exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, and $\{Y_n, n \geq 1\}$, Y on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that, $Y_n \xrightarrow{a.s.} Y$. Further, from continuity of g,
$\{\omega \in \Omega' \mid g(Y_n(\omega)) \to g(Y(\omega))\} \supseteq \{\omega \in \Omega' \mid Y_n(\omega) \to Y(\omega)\}$,
$\Rightarrow \mathbb{P}(\{\omega \in \Omega' \mid g(Y_n(\omega)) \to g(Y(\omega))\}) \geq \mathbb{P}(\{\omega \in \Omega' \mid Y_n(\omega) \to Y(\omega)\})$,
$\Rightarrow \mathbb{P}(\{\omega \in \Omega' \mid g(Y_n(\omega)) \to g(Y(\omega))\}) \geq 1$,
$\Rightarrow g(Y_n) \xrightarrow{a.s.} g(Y)$,
$\Rightarrow g(Y_n) \xrightarrow{D} g(Y)$.
This completes the proof since, $g(Y_n)$ has the same distribution as $g(X_n)$, and $g(Y)$ has the same distribution as $g(X)$. ■

Convergence

Theorem 28.23 *If* $X_n \xrightarrow{D} X$, *then* $C_{X_n}(t) \longrightarrow C_X(t)$, $\forall t$.

Proof: If $X_n \xrightarrow{D} X$, from Skorokhod's Representation Theorem, there exist random variables $\{Y_n\}$ and Y such that $Y_n \xrightarrow{a.s.} Y$.
So,

$$\cos(Y_n t) \longrightarrow \cos(Yt),\ \cos(X_n t) \longrightarrow \cos(Xt),\ \ \forall t.$$

As $\cos(\cdot)$ and $\sin(\cdot)$ are bounded functions,

$$\mathbb{E}[\cos(Y_n t)] + i\mathbb{E}[\sin(Y_n t)] \longrightarrow \mathbb{E}[\cos(Yt)] + i\mathbb{E}[\sin(Yt)],\ \ \forall t.$$

$$\Rightarrow C_{Y_n}(t) \longrightarrow C_Y(t),\ \ \forall t.$$

We get,

$$C_{X_n}(t) \longrightarrow C_X(t),\ \ \forall t,$$

since distributions of $\{X_n\}$ and X are same as those of $\{Y_n\}$ and Y respectively, from Skorokhod's Representation Theorem. ■

Convergence

Example 1: Let the random variable U be uniformly distributed on $[0, 1]$. Consider the sequence defined as:

$$X(n) = \frac{(-1)^n U}{n}.$$

1. *Almost sure convergence:* Suppose

$$U = a.$$

The sequence becomes

$$\begin{aligned} X_1 &= -a, \\ X_2 &= \frac{a}{2}, \\ X_3 &= -\frac{a}{3}, \\ X_4 &= \frac{a}{4}, \\ &\vdots \end{aligned}$$

In fact, for any $a \in [0, 1]$

$$\lim_{n\to\infty} X_n = 0,$$

therefore, $X_n \xrightarrow{a.s.} 0$.

Convergence

Convergence in mean square sense:

In order to answer this question, we need to prove that

$$\lim_{n\to\infty} E\left[|X_n - 0|^2\right] = 0.$$

We know that,

$$\begin{aligned}
\lim_{n\to\infty} E\left[|X_n - 0|^2\right] &= \lim_{n\to\infty} E\left[X_n^2\right], \\
&= \lim_{n\to\infty} E\left[\frac{U^2}{n^2}\right], \\
&= \lim_{n\to\infty} \frac{1}{n^2} E\left[U^2\right], \\
&= \lim_{n\to\infty} \frac{1}{n^2} \int_0^1 u^2 du, \\
&= \lim_{n\to\infty} \frac{1}{n^2} \left.\frac{u^3}{3}\right]_0^1, \\
&= \lim_{n\to\infty} \frac{1}{3n^2}, \\
&= 0.
\end{aligned}$$

Hence, $X_n \xrightarrow{m.s.} 0$.

Thank you for your attention!

See next week!