

Home Assignment 1

1. Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCDE$. Each minute they simultaneously and independently choose to go clockwise or counter-clockwise in the next vertex. The brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.
 - (a) What is the probability that they will be in one vertex after 3 steps?
 - (b) Write down the transition matrix of the brotherhood Markov chain.
 - (c) What proportion of time the brotherhood will spend in each state in the long run?
 - (d) Find the expected time until the hedgehogs meet in one vertex.

2. Each day the Random Restaurant is independently closed with probability p . If the restaurant is open then the number of clients has Poisson distribution with mean μ .

After N days (working or non-working) the Random Restaurant will permanently close and you are right, N is random and has Poisson distribution with mean n .

- (a) Find the moment generating function of the number of clients during day 1, assuming $N \geq 1$.
 - (b) Find the moment generating function of the total number of clients served in the Random Restaurant.
3. Find the probability limit $\text{plim } X_n$, where

$$X_n = \frac{Y_1 + 2Y_2 + 3Y_3 + \dots + nY_n}{n^2}$$

and Y_1, Y_2, \dots are independent uniform on $[0; 1]$.

Hint: try to calculate $\mathbb{E}(X_n), \text{Var}(X_n)$. You may google the formulas for $1+2+\dots+n$ and $1^2+2^2+\dots+n^2$ or ask ChatGPT.

4. Consider the Poisson arrival process X_t with constant rate λ . Now let's scale the time in a non-linear fashion, $Y_t = X_{t^2}$.
 - (a) Find $\mathbb{E}(Y_t), \text{Var}(Y_t), \mathbb{P}(Y_t = 0)$.
 - (b) Find $\mathbb{E}(Y_{t+5} | Y_t)$ and $\text{Var}(Y_{t+5} | Y_t)$.
5. Let's toss a dice until the first six appears. Let X be the result of the first toss and Y — the total number of tosses.
 - (a) Find $\mathbb{E}(X | Y), \mathbb{E}(Y | X)$.
 - (b) Find $\text{Var}(X | Y), \text{Var}(Y | X)$.

6. The joint distribution of X and Y is given in the table

	$X = -1$	$X = 0$	$X = 1$
$Y = 0$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

- (a) Explicitely find the σ -algebras $\sigma(X), \sigma(Y), \sigma(X \cdot Y)$.
 - (b) How many elements are there in $\sigma(X, Y), \sigma(X + Y), \sigma(X, Y, X + Y)$?

Home Assignment 2

Hereinafter (W_t) is a standard Wiener process.

1. Some questions about Wiener process!

(a) Find $\mathbb{E}(W_7 \mid W_5)$, $\mathbb{V}\text{ar}(W_7 \mid W_5)$, $\mathbb{E}(W_7 W_6 \mid W_5)$.

(b) Find $\mathbb{E}(W_5 \mid W_7)$, $\mathbb{V}\text{ar}(W_5 \mid W_7)$.

2. Let (W_t) be a standard Wiener process and $Y_t = W_t^3 + t^2 W_t^2$.

(a) Find $\mathbb{E}(Y_t)$ and $\mathbb{V}\text{ar}(Y_t)$.

(b) Is Y_t a martingale?

(c) Find $\mathbb{E}(Y_t \mid W_s)$ for $t \geq s$.

3. Consider two independent Wiener processes A_t and B_t . Check whether these processes are Wiener processes:

(a) $X_t = (A_t + B_t)/2$.

(b) $Y_t = (A_t + B_t)/\sqrt{2}$.

4. Using Ito's lemma find dX and the corresponding full form.

(a) $X_t = W_t^6 \cos t$.

(b) $X_t = Y_t^3 + t^2 Y_t$ where $dY_t = W_t^2 dW_t + t W_t dt$.

5. Consider $I_t = \int_0^t W_u^2 u^2 du$. Find $\mathbb{E}(I_t)$, $\mathbb{V}\text{ar}(I_t)$ and $\mathbb{C}\text{ov}(I_t, W_t)$.

6. Let $X_t = 42 + t^2 W_t^3 + t W_t^2 + \int_0^t 3W_u du + \int_0^t W_u^3 dW_u$.

(a) Find dX_t .

(b) Is X_t a martingale? Is $Y_t = X_t - \mathbb{E}(X_t)$ a martingale? Provide a short argument for your answer.

Home Assignment 3



1. Consider two-period binomial tree model without dividends. Initial stock price is $S_0 = 200$, in each period the stock price is multiplied by $u = 1.15$ or by $d = 0.75$. One period interest rate is $r = 0.05$.
 - (a) Find the risk-neutral probability.
 - (b) Price the following binary option: at time $T = 2$ you get 100\$ if $S_1 > 200$ and nothing otherwise.
 - (c) Price the following chooser option: at $t = 1$ the owner of the option decides whether the option is call or put. The strike price is $K = 200$ and expiry date is $T = 2$.
2. In the framework of Black and Scholes model find the price at $t = 0$ of the following two financial assets, $dS_t = \mu S_t dt + \sigma S_t dW_t$ is the share price equation.
 - (a) The asset pays you at time T exactly one dollar if $S_T < K$ where K is a constant specified in the contract.
 - (b) The asset pays you at time T exactly S_T^2 dollars.
3. Consider the Vasicek interest rate model,

$$dR_t = 5(0.06 - R_t) dt + 3 dW_t, \quad R_0 = 0.07.$$

Here R_t is the interest rate.

- (a) Using the substitution $Y_t = e^{at} R_t$ find the solution of the stochastic differential equation. Start by finding dY_t .
- (b) Find $\mathbb{E}(R_t)$ and $\text{Var}(R_t)$.
- (c) Which value in this model would you call long-term equilibrium rate and why?

Hint: you may have integrals in you expression for R_t , but no R_t .

4. Let X_t be the exchange rate measured in roubles per dollar. We suppose that $dX_t = \mu X_t dt + \sigma X_t dW_t$. Consider the inverse exchange rate $Y_t = 1/X_t$ measured in dollars per rouble. Write the stochastic differential equation for dY_t . The equation may contain Y_t and constants, but not X_t .
5. Solve the stochastic differential equation

$$dY_t = -Y_t dt + dW_t, \quad Y_0 = 1$$

If you are have no clues you may try a substitution $Z_t = f(t)Y_t$. Do not forget that the final answer may contain integrals that can't be calculated explicitly. It's ok.

Home Assignment 4

1. The process (u_t) is a white noise with variance $\text{Var}(u_t) = \sigma^2$. Consider the process $b_t = t^2 + 6t + (1 - 2L)^2 u_t$.
 - (a) Write explicit expression for (b_t) without lag operator L .
 - (b) Find $\mathbb{E}(b_t)$ and $\text{Var}(b_t)$.
 - (c) Find $\text{Cov}(b_t, b_{t-k})$ and $\text{Corr}(b_t, b_{t-k})$.
 - (d) Is the process (b_t) weakly stationary?

2. Let (u_t) be a white noise process with variance $\text{Var}(u_t) = \sigma^2$ and

$$y_t = 1 + u_t + 0.7u_{t-1} + 0.7^2u_{t-2} + 0.7^3u_{t-3} + \dots$$

- (a) Find $\mathbb{E}(y_t)$, $\text{Var}(y_t)$.
 - (b) Find $\text{Cov}(y_t, y_{t-k})$.
 - (c) Is (y_t) weakly stationary?
 - (d) Sketch the autocorrelation function of (y_t) if it is weakly stationary.
3. Provide an example of two dependent processes (a_t) and (b_t) such that each of them is weakly stationary, but their sum is not weakly stationary.
4. Consider three variables (y_1, y_2, y_3) that are jointly normal

$$y \sim \mathcal{N} \left(\begin{pmatrix} 2 \\ 6 \\ 11 \end{pmatrix}; \begin{pmatrix} 16 & 0 & -1 \\ 0 & 4 & 1 \\ -1 & 1 & 4 \end{pmatrix} \right).$$

Find $\text{Corr}(y_1, y_2)$ and $\text{pCorr}(y_1, y_2; y_3)$.

5. Let $y_t = 5 + u_t + u_{t-1} + u_{t-2}$ where (u_t) is a white noise with variance $\text{Var}(u_t) = \sigma^2$.
 - (a) Is the process (y_t) weakly stationary?
 - (b) Find the autocorrelation function ρ_k for this process.
 - (c) Find the first two values of the partial autocorrelation function, ϕ_{11} and ϕ_{22} .
6. [bonus] Variables u_1 and u_2 are independent $\mathcal{N}(0; 1)$. Consider the process $y_t = u_1 \cos(\pi t/2) + u_2 \sin(\pi t/2)$.
 - (a) Find $\mathbb{E}(y_t)$, $\text{Var}(y_t)$, $\gamma_k = \text{Cov}(y_t, y_{t+k})$.
 - (b) Is (y_t) weakly stationary?
 - (c) Can (y_t) be represented as $MA(\infty)$ process with respect to *some* white noise, not necessary (u_t) ?

You know additionally that $y_{100} = 0.2024$.

- (d) What is your best point prediction for y_{104} ?
 - (e) What is the shortest prediction interval that covers y_{104} with at least 95%-probability?