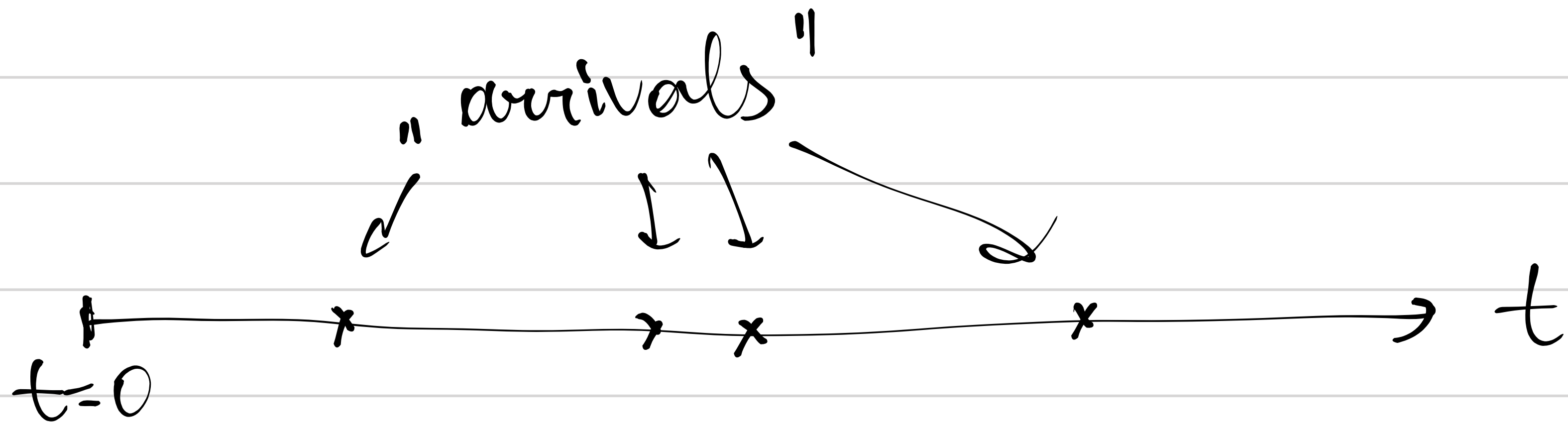
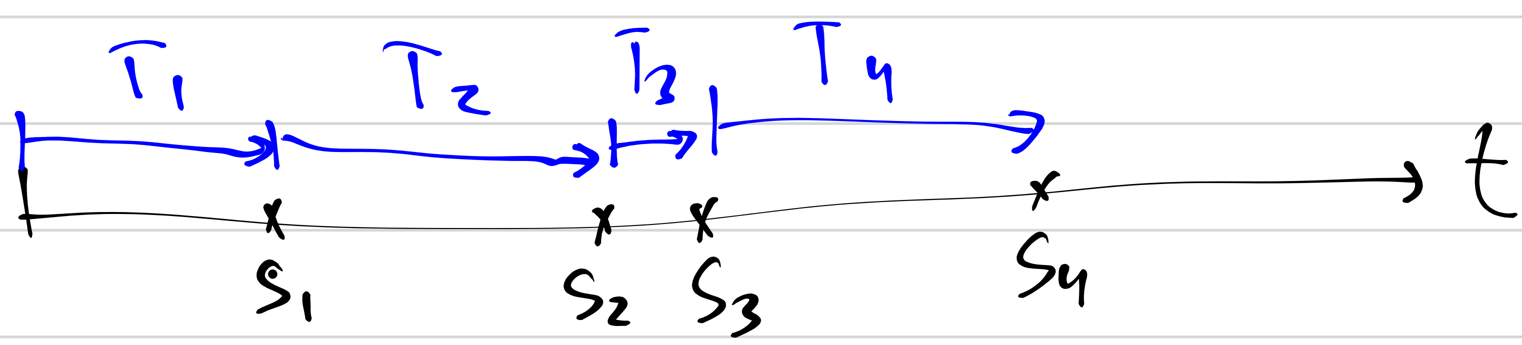
Hi

→ Poisson Arrival Process

→ Problem solving.

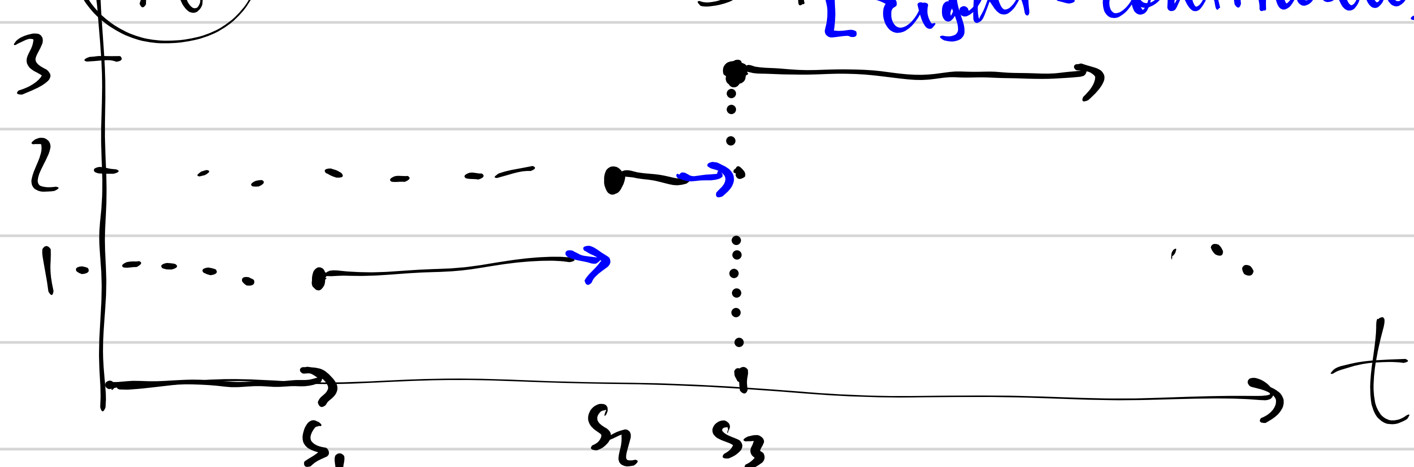



$S_n = T_1 + T_2 + T_3 + \dots + T_n$

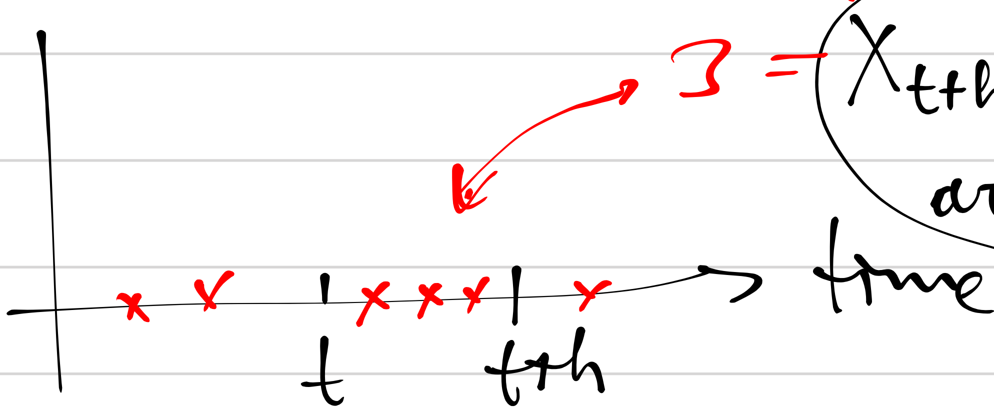
$X_{S_3} \overset{?}{=} 3$

$X_t$ – counting process [right-continuous]




$(X_t)$ $t \in [0 ; \infty)$

$X_t \in \{0, 1, 2, 3, \dots\}$




$3 = X_{t+h} - X_t$ – the number of arrivals $(t ; t+h]$

$X_{t+h} \leftarrow$ the number of arrivals in $[0 ; t+h]$

$X_t \leftarrow$ — // — in $[0 : t]$

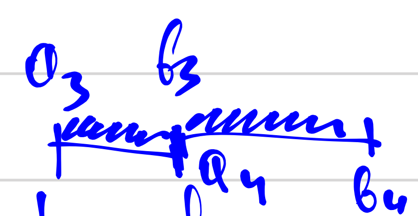
# Assumptions.

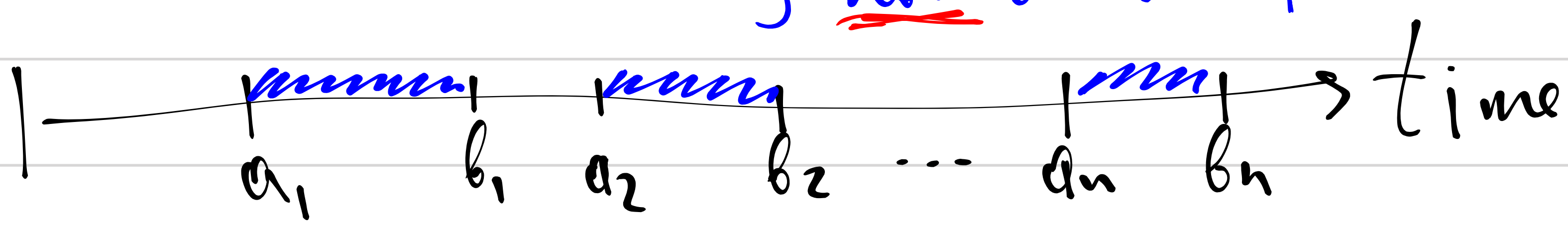
A1. $\boxed{X_0 = 0}$

A2: Independent increments.

may ~~not~~ overlap

may touch




$N_1 = X_{b_1} - X_{a_1}$

$N_2 = X_{b_2} - X_{a_2}$

$\vdots$

$N_k$ - the number of arrivals in interval $k$.

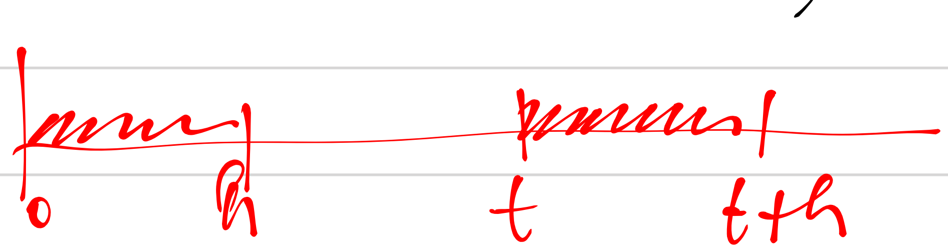
RVs $N_1, N_2 \ldots\ldots N_k$ are independ.

A3. Stationary increments

The distribution of $N_k$ depends only on the length of the time interval.

$$(X_{t+h} - X_t) \sim (X_h - X_0) \qquad \forall t$$

$$N_k \sim \text{Law}(b_k - a_k)$$



$h$ – the length of time interval

for small $h$:

$\boxed{\text{1 arriv}}$: $P(X_{t+h} - X_t = 1)$ is approx. by proportional to $h$.

$\boxed{\geq 2 \text{ arriv}}$: $P(X_{t+h} - X_t \geq 2)$ is negligible compared to $h$.

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Ex. $P(X_{1.002} - X_1 = 1) = 0.01$ $h = 0.002$

$P(X_{1.006} - X_1 = 1) \overset{?}{\approx} 0.03$ $h = 0.006$

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$\boxed{\text{1 arriv}}$ $P(X_{t+h} - X_t = 1) = \lambda \cdot h + \underbrace{o(h)}$

rate $\lambda = 2/10$

on average 2 arrivals in 10 min.

$\lim_{h \to 0} \frac{o(h)}{h} = 0$

$h^2 = o(h)$

$h^3 = o(h)$

$h^{17} = o(h)$

$\boxed{\text{2 arrivals}}$ $P(X_{t+h} - X_t \geq 2) = o(h)$

Theorem. $\boxed{\text{If}}$ $\boxed{X_0 = 0}$, $\boxed{\text{Ind Incr}}$, $\boxed{\text{Stat. Incr}}$, $\boxed{\text{1 arreiv}}$, $\boxed{\geq 2 \text{ arr.}}$

then: $\underbrace{T_k}$ ~ $\boxed{\text{Expon}(\text{rate} = \lambda)}$ [proof]

interarrival — indep $\forall t, h$

$\underbrace{X_{t+h} - X_t}$ ~ $\text{Poiss}(\text{rate} = \lambda h)$ [partially]

number of arrivals in $[t : t+h]$

$\boxed{\text{Partial proof}}$

$$X_{t+h} - X_t \sim \text{Poisson}(\text{rate} = h\cdot\lambda)$$

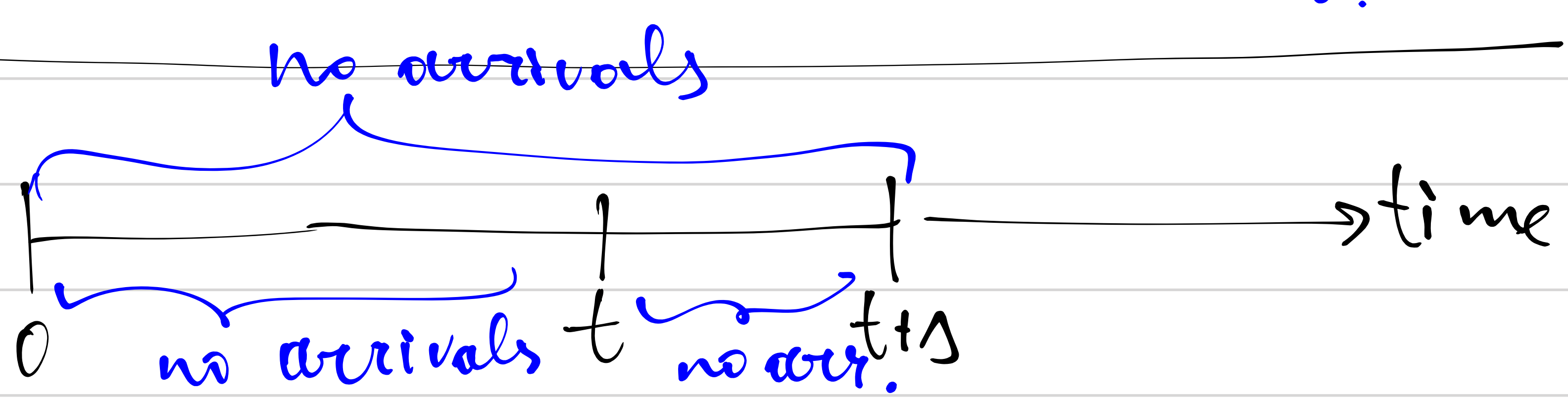
$$\boxed{P(X_{t+h} - X_t = 0) = \exp(-\lambda h)} \leftarrow \text{proof}$$

$$P(X_{t+h} - X_t = 1) = \exp(-\lambda h)\cdot\frac{\lambda h}{1!}$$

$$P(X_{t+h} - X_t = 2) = \exp(-\lambda h)\cdot\frac{(\lambda h)^2}{2!}$$

$$\dots$$

$$P(X_{t+h} - X_t = b) = \exp(-\lambda h)\cdot\frac{(\lambda h)^b}{b!}$$




$$P(X_{t+\Delta} - X_0 = 0) \underset{\text{Ind Incr}}{\overset{?}{=}} P(X_t - X_0 = 0)\cdot$$

$$\cdot\underbrace{P(X_{t+\Delta} - X_t = 0)}$$

idea $\Delta \approx 0$

$$\begin{cases} P(X_{t+\Delta} - X_t = 1) = \lambda\cdot\Delta + o(\Delta) \\ P(X_{t+\Delta} - X_t \geq 2) = o(\Delta) \end{cases}$$

$$\to P(X_{t+\Delta} - X_t = 0) = 1 - \lambda\Delta - 2o(\Delta)$$

$$2o(\Delta) = o(\Delta)$$

$\boxed{X_0 = 0}$

$$\underbrace{P(X_{t+\Delta} = 0)}_{=p_0(t+h)} = \underbrace{P(X_t = 0)}_{=p_0(t)}\cdot(1 - \lambda\Delta - o(\Delta))$$

$$p_0(t+\Delta) = p_0(t) - \lambda\cdot\Delta\cdot p_0(t) - o(\Delta)\cdot p_0(t)$$

$$p_0(t+\Delta) = p_0(t) - \lambda\cdot\Delta\cdot p_0(t) - o(\Delta)\cdot p_0(t)$$

$$p_0(t+\Delta) - p_0(t) = -\lambda\Delta\, p_0(t) - o(\Delta)\cdot p_0(t)$$

$$\frac{p_0(t+\Delta) - p_0(t)}{\Delta} = -\lambda\, p_0(t) - \frac{o(\Delta)}{\Delta}\cdot p_0(t)$$

$\Delta \to 0$ $\qquad\qquad$ $\Delta \to 0$

$\forall t$ $$\boxed{p_0'(t) = -\lambda\, p_0(t) + 0}$$

der. is proportional to the fun.

$$p_0(t) = \text{const}\cdot \exp(-\lambda t)$$

$$p_0(0) = P(X_0 = 0) = 1$$

$$p_0(t) = \exp(-\lambda t)$$

$$\boxed{P(X_t = 0) = \exp(-\lambda t)} \quad \forall t$$

$$P(X_h = 0) = \boxed{\exp(-\lambda h)}$$

Stat. Incr.

$$P(X_{t+h} - X_t = 0)$$

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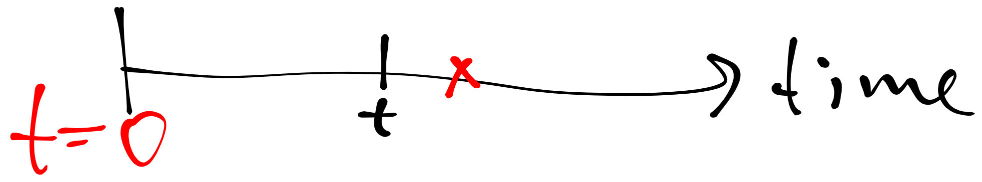
$$P(T_1 \le t) = \text{cdf of } T_1 =$$

$$= 1 - P(T_1 > t) = 1 - P(X_t = 0) = 1 - \exp(-\lambda t)$$

no students at time $t$



$$f_{T_1}(t) = \text{pdf of } T_1 = \left(1-\exp(-\lambda t)\right)' =$$
$$= \lambda \cdot \exp(-\lambda t)$$

$$T_1 \sim Exp(rate = \lambda)$$

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Ex.

On average there 2 taxi cabs per 10 minutes.
What is the prob-ty that I will wait more than 7 minutes for a taxi?
[All 5 assumptions are sat.].

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$$E(X_{t+10} - X_t) = 2 \qquad N_k \sim Exp(\lambda)$$

$$X_{t+10} - X_t \sim Poiss(\lambda \cdot 10)$$

$$\boxed{E(X_{t+10} - X_t) = \lambda \cdot 10} = 2$$

$$\lambda = \frac{2}{10} \text{ [rate]}$$

Fact

$$W \sim Poiss(rate = \tau)$$
$$E(W) = \tau$$
$$Var(W) = \tau$$
$$P(W = k) = \exp(-\tau) \cdot \frac{\tau^k}{k!}$$

$$Y \sim Expon(rate = \lambda)$$
$$E(Y) = \frac{1}{\lambda}$$
$$Var(Y) = 1/\lambda^2$$
$$f(t) = \begin{cases} \lambda \cdot \exp(-\lambda t) \\ 0 & t < 0 \end{cases}$$

$$P(T_{n+1} > 7) = \exp(-\lambda \cdot 7) = \exp\left(-\frac{2}{10}\cdot 7\right) = \exp\left(-\frac{7}{5}\right) \approx 0.247$$

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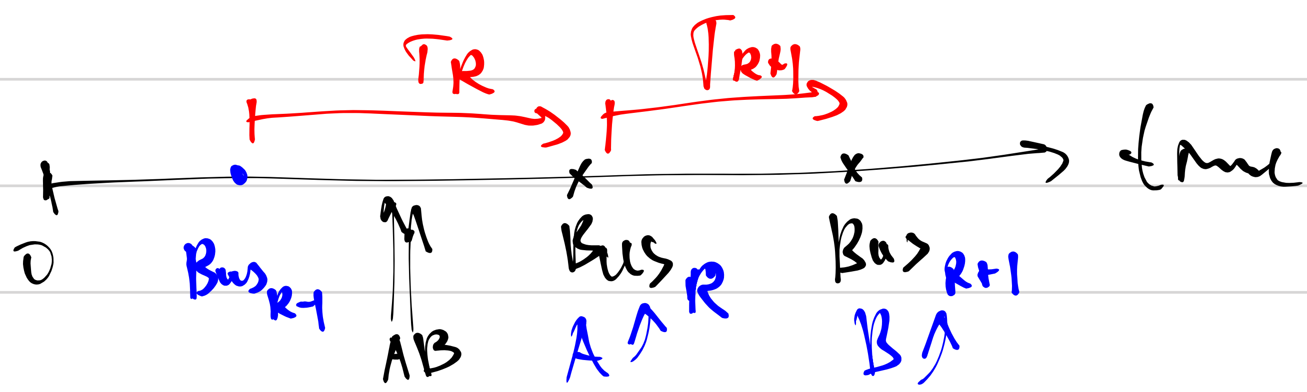
Inspection paradox.

A and B arrive at random mom. of time 5 ass-ns

A : waits for the first bus

B : waits for the second bus



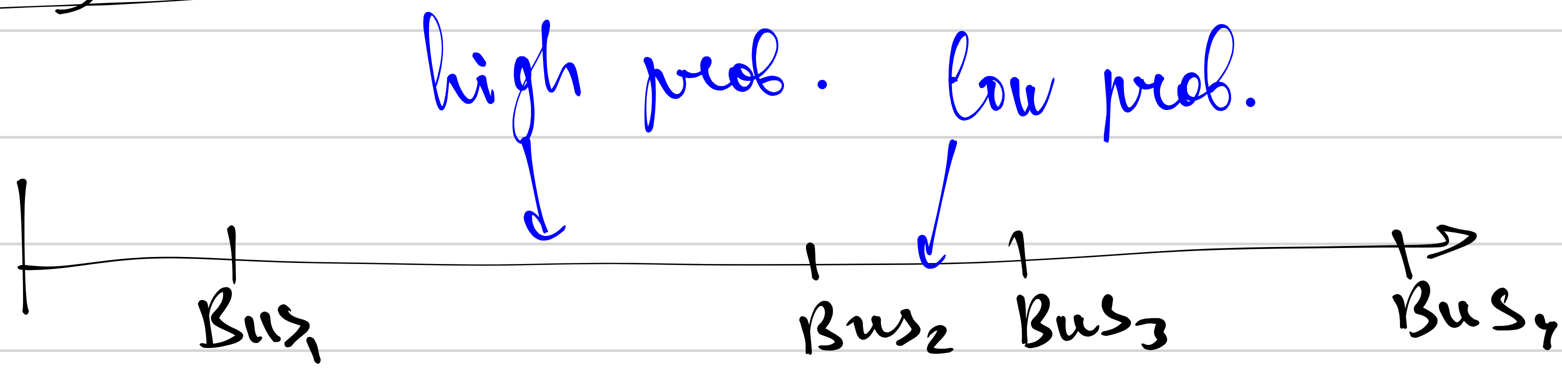

$\boxed{\text{Which bus is more crowded and by how much?}}$

$$E(T_k) \qquad\qquad E(T_{k+1})$$

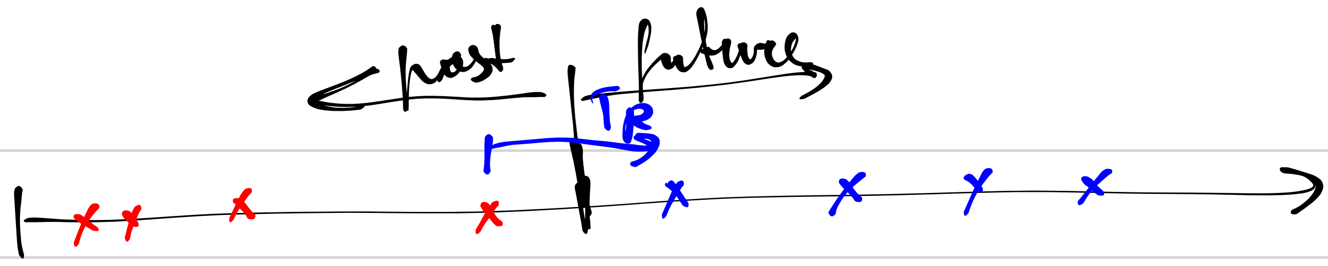
k - random!

$$E(T_1) = E(T_2) = E(T_3) = \dots = E(T_{100}) = \dots$$

(by the theorem)




$$E(T_k) > E(T_{k+1})$$




A, B

$\ldots\ldots\ T_3^- \quad T_2^- \quad T_1^- \quad T_1 \quad T_2 \quad T_3 \quad T_4 \ \ldots\ldots$

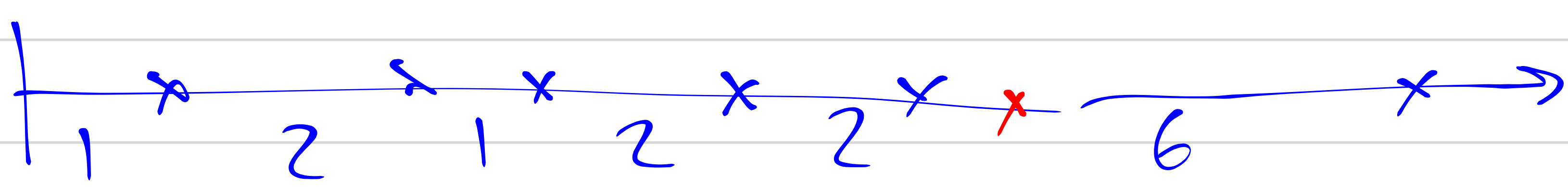
$$T_2^- \sim T_1^- \sim T_1 \sim T_2 \sim T_3 \sim T_4 \sim \mathrm{Exp}(\lambda)$$

$$E(T_1) = \frac{1}{\lambda}$$

$$E(T_1^-) = \frac{1}{\lambda}$$

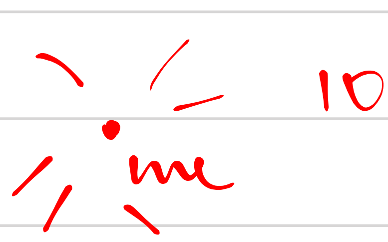
$$E(T_R) = \frac{1}{\lambda} + \frac{1}{\lambda} = \frac{2}{\lambda} \qquad E(T_{R+1}) = \frac{1}{\lambda}$$

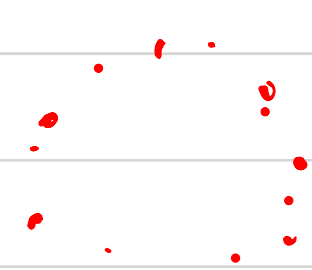
$$\boxed{E(T_R) = 2 \cdot E(T_{R+1})}$$




$$\text{aver} = \frac{1+2+1+2+2+6}{6} = \frac{14}{6}$$

$$\text{aver (from inspector viewpoint)} = \frac{6}{14}\cdot 6 + \frac{6}{14}\cdot 2 + \frac{2}{14}\cdot 1 > \frac{14}{6}$$

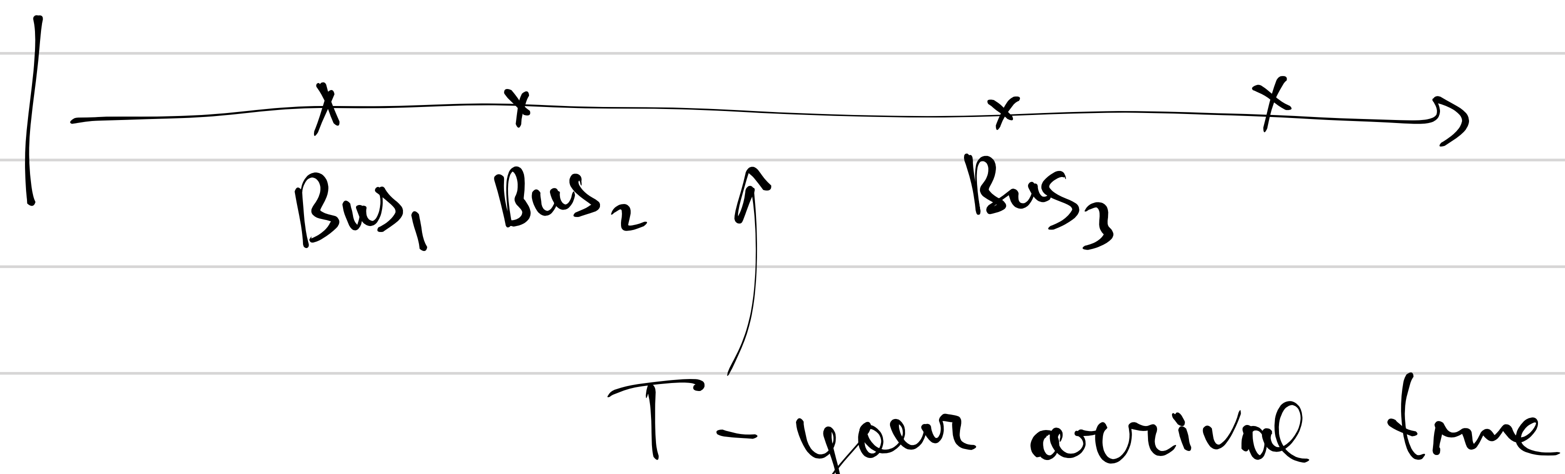


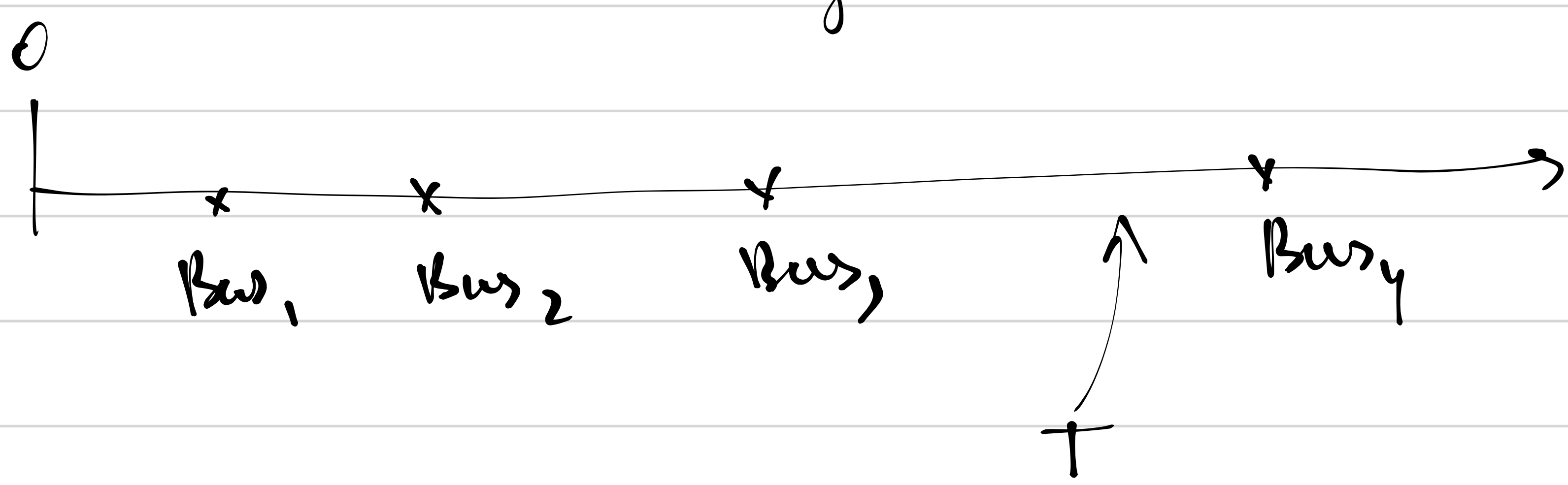



$$\frac{1}{2} < P\left(10 < \frac{X_1 + X_2 + \ldots + X_{10}}{10}\right)$$

1 class + 2 lectures

new practice



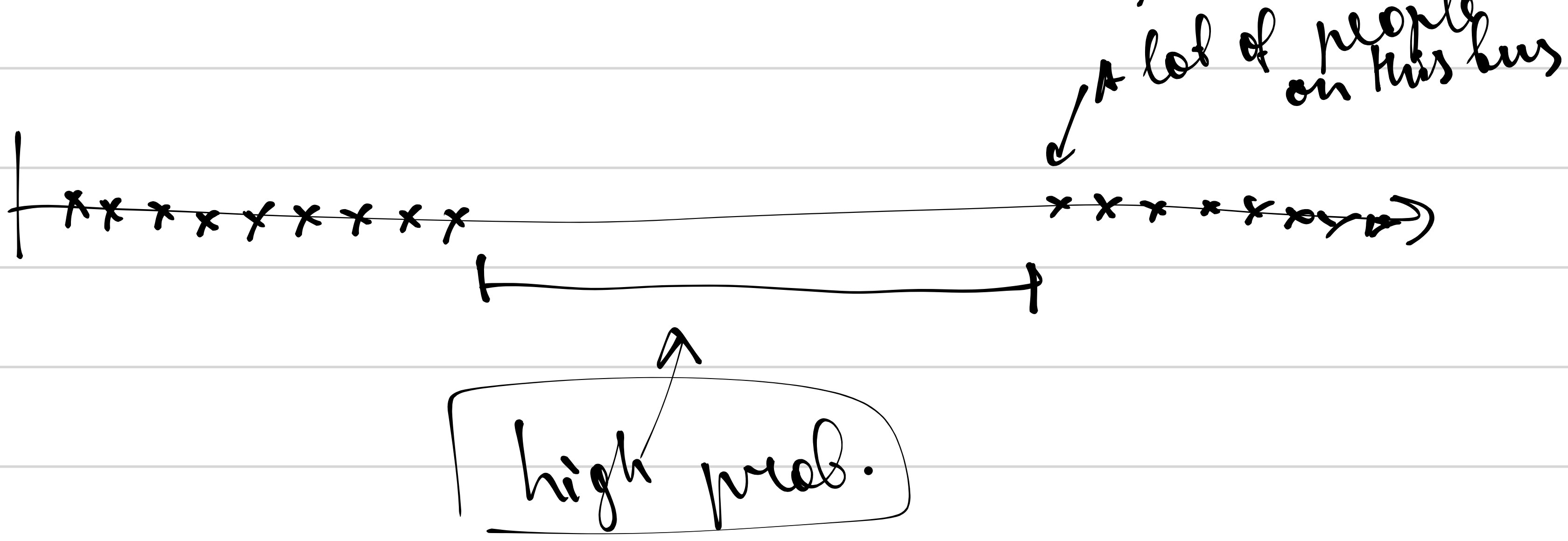

Bus$_R$ R - random number of the first bus you will see.

the first bus you see

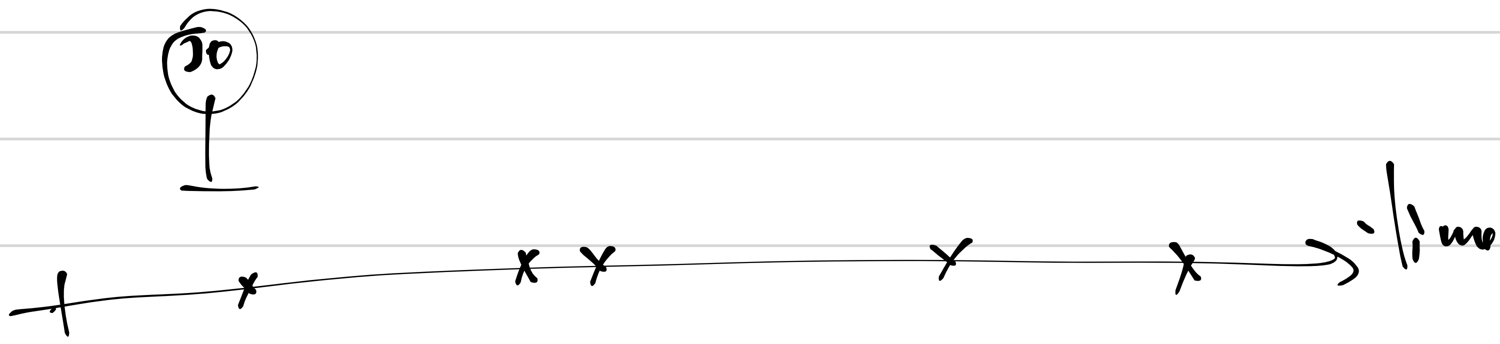
Bus$_{R+1}$ - the second bus you see

$$E(T_R) = 2 \cdot E(T_{R+1})$$

$$E(T_{R+1}) = E(T_{R+2})$$



Problem 1




→ receives on average 2 bribes per hour

→ bribes follow P. Process.

a) P( during 40 minutes Ivanov will get more than 3 bribes)?

b) P( at least one violation in the next 10 min | no violations during past hour )

Solution: $\lambda \overset{?}{=} \frac{2}{60}$ [b/min] $\frac{2}{1}$ [b/hour]

a) $P(X_{40} - X_0 > 3) = 1 - P(X_{40} = 3) - P(X_{40} = 2) - P(X_{40} = 1) - P(X_{40} = 0) = 1 - \exp(-\frac{4}{3}) \cdot \left( \frac{(4/3)^3}{3!} + \frac{(4/3)^2}{2!} + \frac{(4/3)^1}{1} + 1 \right)$

for 40 minutes : rate = "$\lambda h$" $= \frac{2}{60} \cdot 40 = \frac{8}{6} = \frac{4}{3}$

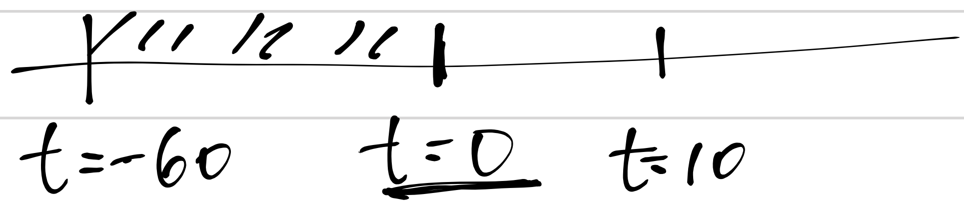
$X_{40} - X_0$ – the number of arrivals

$X_{40} - X_0 \in \{0, 1, 2, 3, \dots\}$

$P(X_{t+h} - X_t = k) = \exp(-\lambda h) \cdot \frac{(\lambda h)^k}{k!}$

probability

no arr.




$$P(X_{10} - X_0 \geq 1 \mid X_0 - X_{-60} = 0) =$$

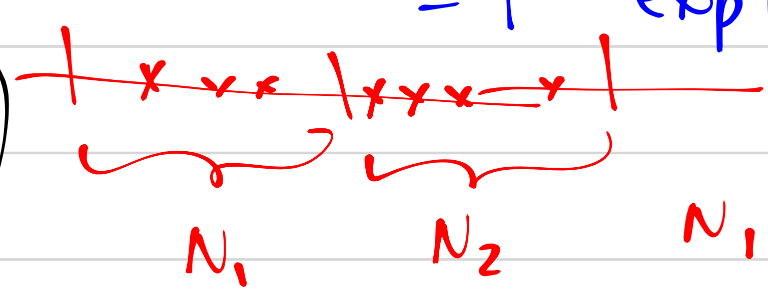
indep ( 5 assumptions of P.P.)

Indep. Increments.

$$= \boxed{P(X_{10} - X_0 \geq 1)} = 1 - P(X_{10} - X_0 = 0) =$$

$$= 1 - \exp\left(-\frac{1}{3}\right)$$

$X_{10} - X_0 \sim \text{Pois}(\text{rate})$




$N_1$ and $N_2$ are indep.

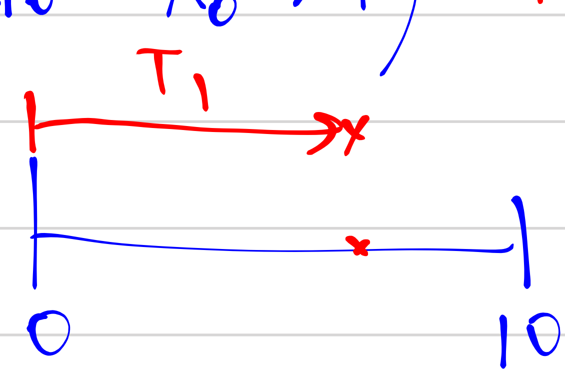
$$\text{rate} = \lambda \cdot 10 = \frac{2}{60} \cdot 10 = \frac{1}{3}$$

$$X_{10} - X_0 \in \{0, 1, 2, 3 \dots\}$$

Solution 2:

$$P(X_{10} - X_0 \geq 1) = P(T_1 < 10) =$$

$T_1 \sim \text{Exp}\left(\frac{2}{60}\right)$




$$= F_{T_1}(10) = 1 - \exp(-\lambda \cdot 10) = 1 - \exp\left(-\frac{1}{3}\right)$$

cdf of exponential

Sol. 3: using pdf

$$= \int_0^{10} \lambda \cdot \exp(-\lambda t)\, dt$$

$\lambda = \frac{1}{3}$

Ex. $(X_t)$ – PP with rate $\lambda_X$ } indep.
$(Y_t)$ – PP with rate $\lambda_Y$

$Z_t = X_t + Y_t$. Is $Z_t$ a P.P.?

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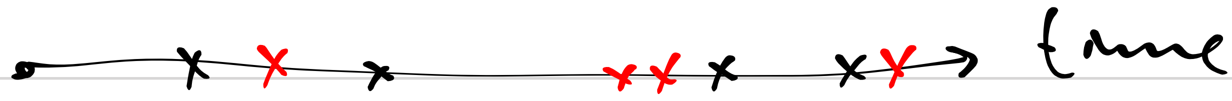
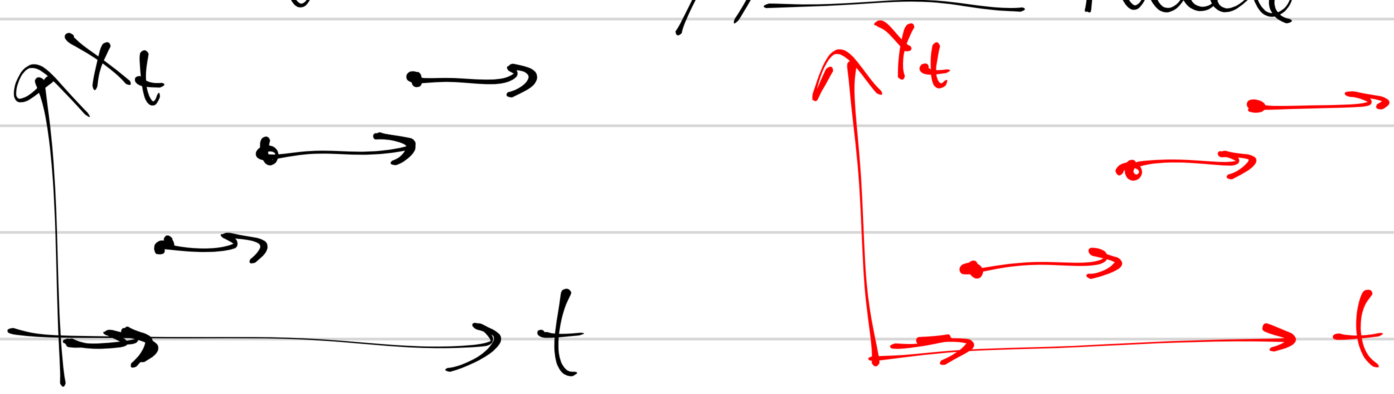
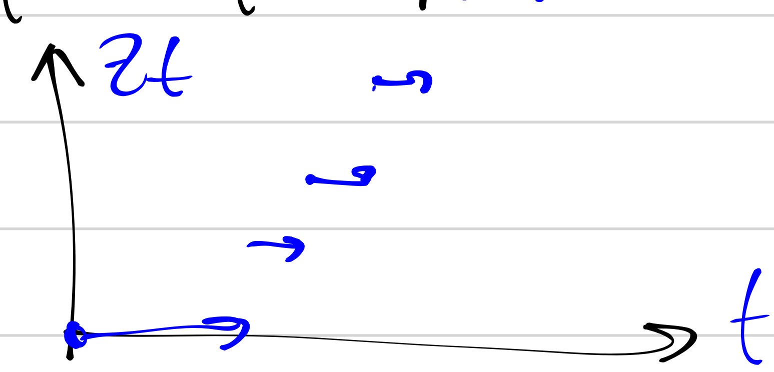



$X_t$ = the number of female students entering Grusha $[0;1]$

$Y_t$ = — // — male students.



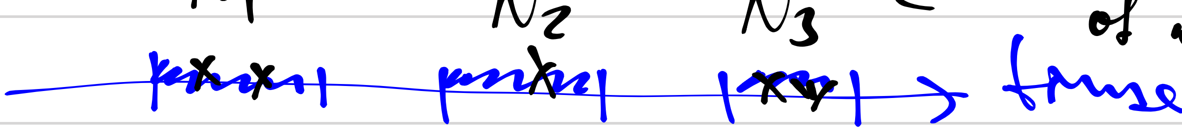

$Z_t = X_t + Y_t$



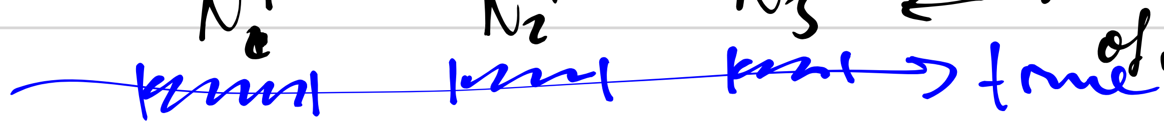

∫ ass-ns for $(X_t)$ ∫ ass-ns for $(Y_t)$

i) $X_0 = 0$ $Y_0 = 0$ $\Rightarrow Z_0 = X_0 + Y_0 = 0$

ii) Ind. Incr. ind: $N_1^X$ $N_2^X$ $N_3^X$ ← number of "X" arrivals

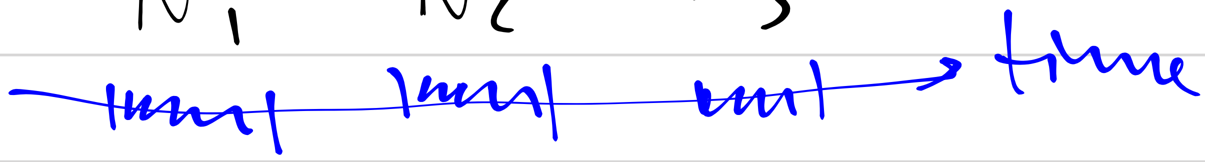



ind: $N_1^Y$ $N_2^Y$ $N_3^Y$ ← number of "Y" arrivals




$\Downarrow$

$N_1^Z$ $N_2^Z$ $N_3^Z$




$\Rightarrow N_1^Z, N_2^Z, N_3^Z$ ind

check: $Z_0=0$, $(Z_t)$ has ind increments.
[3 more ass-ns for $(Z_t)$]

$$\Rightarrow (Z_t) \text{ - P.P.}$$

$$Z_t - Z_0 \sim Poiss(t\cdot\lambda_Z)$$

$$E(Z_t) = t\cdot\lambda_Z = E(X_t)+E(Y_t)$$

$$t\cdot\lambda_Z = t\cdot\lambda_X + t\cdot\lambda_Y$$

$$\lambda_Z = \lambda_X + \lambda_Y$$

Ex.

Lena $\xrightarrow{(X_t)}$ Masha $\xleftarrow{(Y_t)}$ Alex

indep.
Lena sends messages to Masha accord to P.P ($\lambda_L = 10$ mess/hour)
Alex — / ————— // —— ($\lambda_A = 2$ mess/hour)

P( Masha will receive exactly 15 mess in the next 40 minutes)?

$$\lambda_{Total} = 10+2 = 12 \text{ mess/hour}$$

$$\frac{12 \text{ mess}}{60 \text{ minutes}}$$

$$-\frac{12}{1}\cdot\frac{40}{60}$$

$$P(Z_{40} - Z_0 = 15) = P(Z_{40} = 15) =$$

$$= \exp\left(-\frac{12}{60}\cdot 40\right)\cdot\frac{\left(\frac{12}{60}\cdot 40\right)^{15}}{15!}$$

$$P(X_{t+h} - X_t = k) = P(X_h - X_0 = k) = $$
$$= \exp(-\lambda h) \cdot \frac{(\lambda h)^k}{k!}$$

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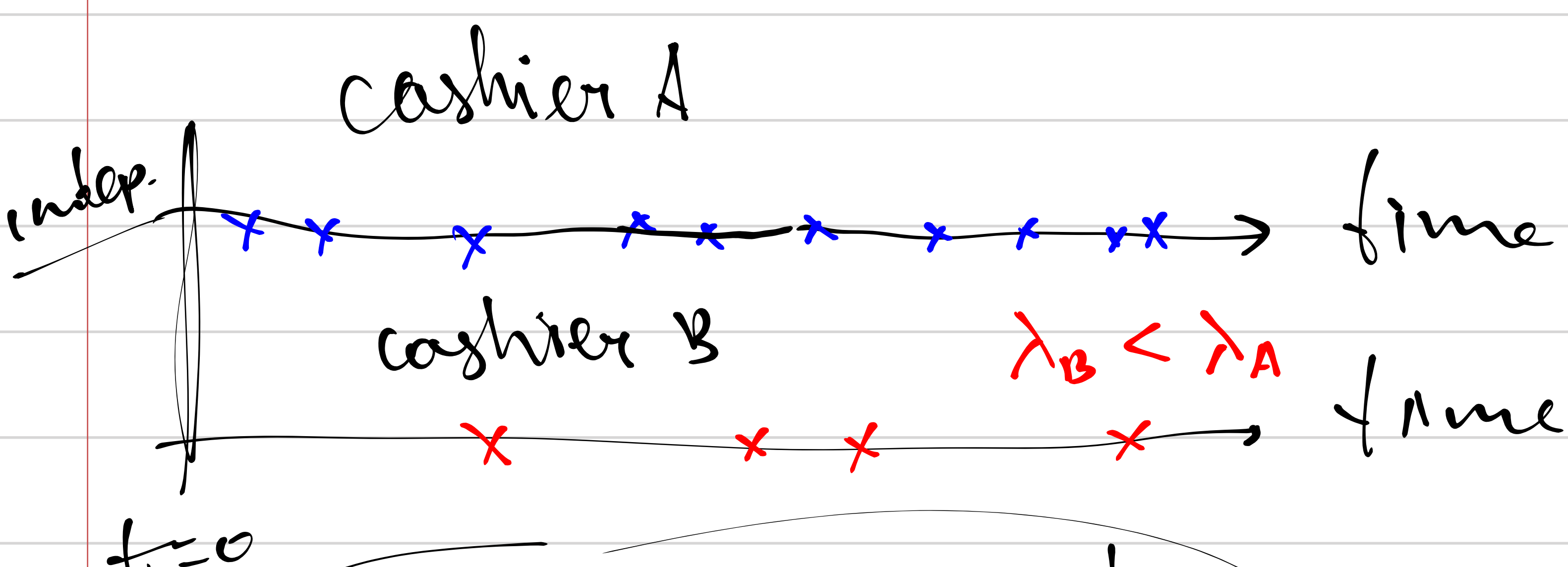
Ex.




$\lambda_A = 5$ per minute

$\lambda_B = 2$ per min.

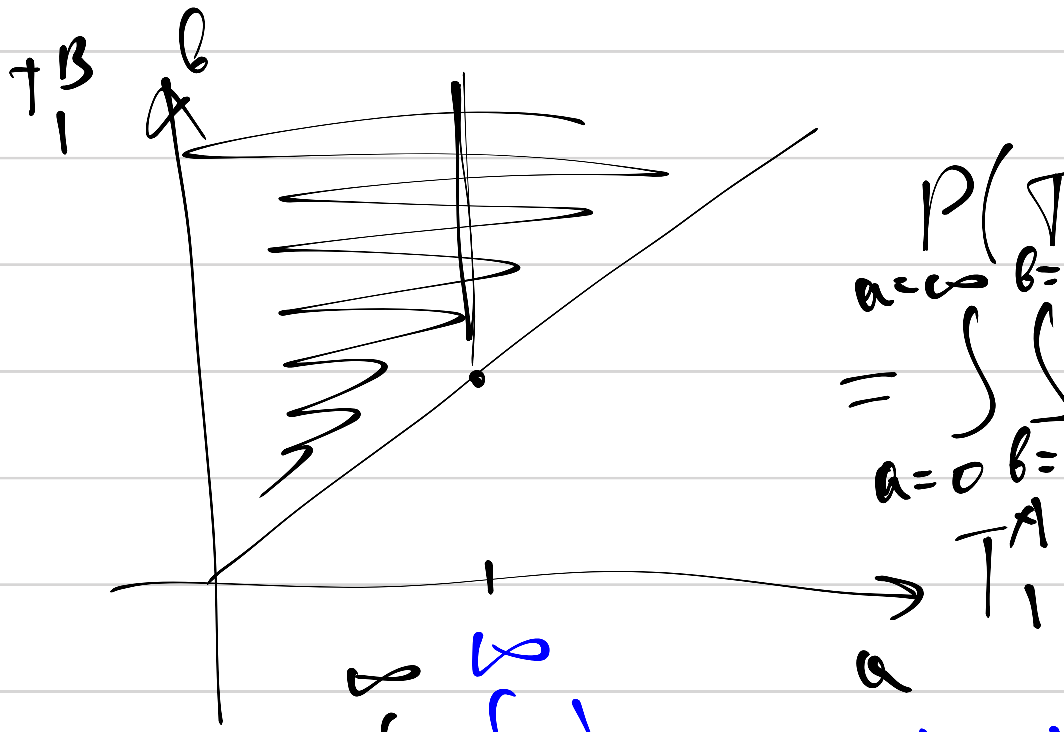
$P(T_1^A < T_1^B)$ ?

guess $\boxed{= \frac{\lambda_A}{\lambda_A + \lambda_B}}$

First calculation

$T_1^A \sim Exp(\lambda_A) \qquad T_1^B \sim Exp(\lambda_B)$

$f(a,b) = f_{T_1^A}(a) \cdot f_{T_1^B}(b)$




$$P(T_1^A < T_1^B) = \int_{a=0}^{a<\infty} \int_{b=a}^{b=\infty} f(a,b)\, db\, da =$$

$$= \int_0^\infty \int_a^\infty \lambda_a \cdot \exp(-\lambda_a \cdot a) \cdot \lambda_b \cdot \exp(-\lambda_b b) \cdot da\, db$$

$= (\ldots$

# Intuitive solution

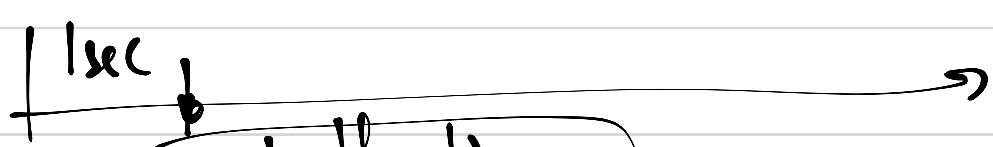

$\boxed{h = 1\,\text{sec}}$

$P(\text{cash A will service the client}) \approx$

$\approx 1\,\text{sec} \cdot \lambda_A$

$P(\text{cash B will service the client}) \approx$

$\approx 1\,\text{sec} \cdot \lambda_B$

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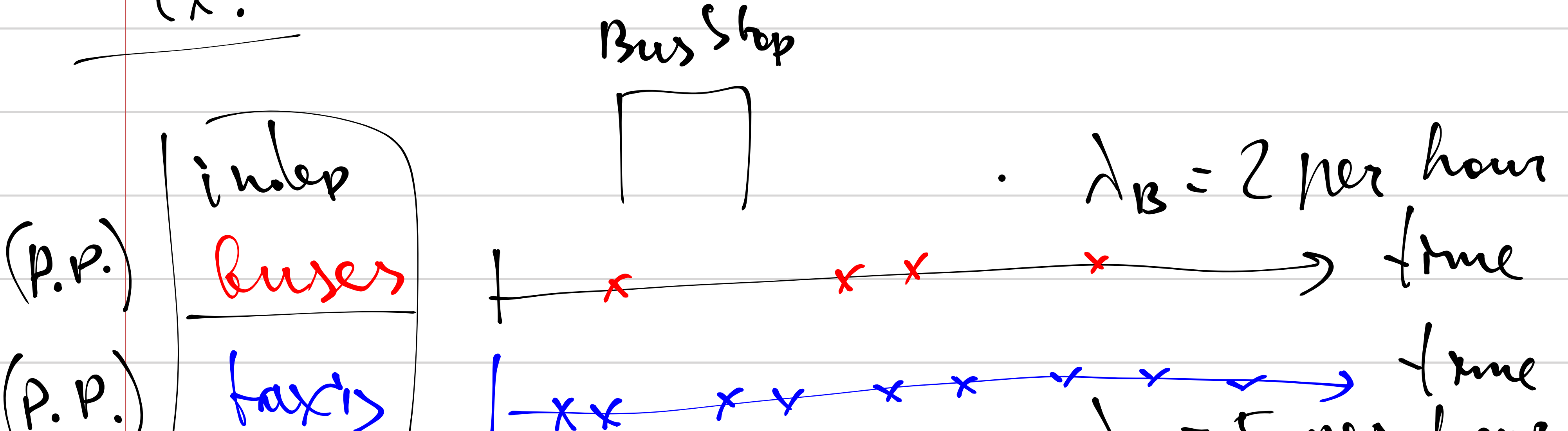
Ex.




a) P(I will observe exactly 2 taxis before a bus)?

two

b) P( —//— at least 2 taxis before a bus)? $\left(\frac{5}{7}\right)^2$

$1 - \frac{2}{7} - \frac{5}{7}\cdot\frac{2}{7}$

---

P(I will observe a taxi before a bus) =

$= \frac{5}{2+5} = \frac{5}{7}$ !!

$\frac{5}{7}\cdot\frac{5}{7}\cdot\frac{2}{7}$ P(ex-ly two taxis before a bus)

Way 3: $\left(\frac{5}{7}\right)^2 \cdot \frac{2}{7} + \left(\frac{5}{7}\right)^3 \cdot \frac{2}{7} + \dots\dots = \frac{b_1}{1-q} =$

$$= \frac{\left(\frac{5}{7}\right)^2 \cdot \frac{2}{7}}{1 - \frac{5}{7}} = \left(\frac{5}{7}\right)^2$$

Recap.

5 assumptions.

(1) $X_0 = 0$

$X_t$ - number of arrivals in $[0; t]$

(3) $X_{t+h} - X_t \sim \text{law}(h)$
Stat. increments

(2) Indep. Increm.

(4) $P(X_{t+h} - X_t = 1) = \lambda \cdot h + o(h)$

(5) $P(X_{t+h} - X_t \geq 2) = o(h)$

$\hookrightarrow$ $T_k$ = time between arr-l $N k$ and $N(k-1)$.

$T_k \sim \text{Exp}(\lambda)$

$X_{t+h} - X_t \sim \text{Poiss}(\text{rate} = \lambda h)$

Property 1.

If $(X_t)$ and $(Y_t)$ are indep then
P.P. P.P

$Z_t = X_t + Y_t \sim PP$
$(\lambda_Z = \lambda_X + \lambda_Y)$

Property 2.

$$P(T_1^X < T_1^Y) = \frac{\lambda_X}{\lambda_X + \lambda_Y}$$

Insp paradox