Hi :)

Convergence of random variables

Many types of convergence!

in L^p $\Rightarrow$ in probability (*) $\Rightarrow$ in distribution

almost surely $\Rightarrow$ in probability

$$X_n \xrightarrow{P} X \qquad \operatorname*{plim}_{n\to\infty} X_n = X$$

def $\operatorname*{plim}_{n\to\infty} X_n = X$

$$\lim_{n\to\infty} P\big(\underbrace{|X_n - X| > \delta}_{\text{„big" deviation}}\big) = 0 \quad \text{for } \delta > 0$$

In statistics: $\hat{\Theta}_n$ is consistent $\Leftrightarrow$

$$\operatorname*{plim}_{n\to\infty} \hat{\Theta}_n = \Theta$$

sequence of estimators. ($\hat{\Theta}_n$)

unknown value (Θ)

Danger! wrong definition

. . . . for $\delta \geq 0$

Properties.

$X_n \xrightarrow{P} X$

① Properties of our Old friend „lim"

[*] [-] [/]

1.1. $\text{plim}(X_n+Y_n) = \text{plim}\, X_n + \text{plim}\, Y_n$ (if both sides exist)

$\lim_{n\to\infty} \frac{n}{n+10} = 1$

$\text{plim}_{n\to\infty} \frac{n}{n+10} = 1$

1.2. If f is continuous then

$$\text{plim}_{n\to\infty} f(X_n) = f(\text{plim}\, X_n)$$

* ② If $\lim E(X_n) = \mu$, $\text{Var}(X_n) = 0$

then $\text{plim}\, X_n = \mu$

③ LLN (Law of Large Numbers)

If $W_1, W_2, W_3, \dots$ are iid

then $X_n = \overline{W} = \frac{W_1 + W_2 + \dots + W_n}{n}$

$$\text{plim}_{n\to\infty} X_n = E(W_1)$$

External exam. ST2134 Summer 2021

№1 Sample $X_1 X_2 \dots X_n \sim$ iid

$E(X_i) = \alpha\theta$

$\text{Var}(X_i) = \alpha\theta^2$

pdf: $f(x) = \frac{1}{(\alpha-1)!\,\theta^\alpha} x^{\alpha-1} \exp(-x/\theta)$

α is known

a) $\hat{\theta}_{ML}$

$\text{plim}\, \hat{\theta}_n = \theta$ ← b) Check whether $\hat{\theta}_n$ is consistent.

External exam. ST2134 Summer 2021

$E(X_i)=\alpha\theta$
$Var(X_i)=\alpha\theta^2$

N1 Sample $X_1, X_2 \dots X_n \sim iid$

pdf: $f(x) = \frac{1}{(\alpha-1)!\,\theta^\alpha} x^{\alpha-1} \exp(-x/\theta)$

α is known

a) $\hat{\theta}_{ML}$

plim $\hat{\theta}_n = \theta$ ← b) Check whether $\hat{\theta}_n$ is consistent.

a) $\max\limits_{\theta}$? $f(x_1, x_2, \dots x_n)$

$\max\limits_{\theta} f(x_1) \cdot f(x_2) \cdot \dots f(x_n)$

$\max\limits_{\theta} \ln \left(f(x_1) \cdot f(x_2) \cdot \dots f(x_n) \right)$

take ln!

① it's easy to $(f+g)'$

② it's numerically stable

in discrete

$\max\limits_{\theta}$

$P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \dots$

$\max\limits_{\theta} \ln f(x_1) + \ln f(x_2) + \dots + \ln f(x_n)$

$n = 2000$

$f(x_i) \approx 0.5$ $\quad f(x_1) \cdot f(x_2) \cdot \dots \cdot f(x_n) \approx$

$\approx 0.5^{2000} =$

$= 0.000000 \dots\dots\dots 0\,\varepsilon$

≈ 300 zeros

$\ln 0.5^{2000} = 2000 \cdot \ln 0.5 \approx$

≈ -1386.294

$\ln f(x_i) =$

$$= \ln\left(\frac{1}{(\alpha-1)!\,\theta^{\alpha}} \cdot x^{\alpha-1} \cdot \exp(-x/\theta)\right) =$$

$\exp(t) = e^{t}$

$$= -\ln((\alpha-1)!) - \alpha \cdot \ln\theta + (\alpha-1)\ln x_i$$
$$- x_i/\theta$$

α - known

θ - unknown (we should estimate)

$$\max_{\theta} \sum\left(\underline{-\ln((\alpha-1)!)} - \alpha\ln\theta + \underline{(\alpha-1)\ln x_i} - \frac{x_i}{\theta}\right)$$

$$\ell'(\theta) = \sum_{i=1}^{n}\left(-\alpha \cdot \frac{1}{\theta} + \frac{x_i}{\theta^2}\right)$$

θ >> unknown

F.O.C.
$$\sum_{i=1}^{n} -\alpha \cdot \frac{1}{\hat\theta} + \frac{x_i}{\hat\theta^2} = 0$$

$\hat\theta$ >> my estimate of θ

10 times
8 Heads
$\cancel{p = \frac{8}{10}}$ $\hat p = \frac{8}{10}$

$$\sum(-\alpha\hat\theta + x_i) = 0$$
$$-n\alpha\hat\theta + \sum x_i = 0$$
$$\hat\theta = \frac{\sum X_i}{n\alpha}$$

SOC. [you may check!] $\boxed{\ell''(\hat\theta) < 0}$

$$\hat\theta_n = \frac{\sum X_i}{n\alpha}$$

b) $\operatorname{plim} \hat\theta_n = \operatorname{plim} \frac{\sum X_i}{n\alpha}$

b) $\operatorname{plim} \hat{\theta}_n = \operatorname{plim} \frac{\sum X_i}{n\alpha}$

$$E(\hat{\theta}_n) = E\left(\frac{\sum X_i}{n\alpha}\right) = \frac{1}{n\alpha} \cdot \sum E(X_i) =$$

$$= \frac{1}{n\alpha} \cdot n \cdot \alpha \cdot \theta = \theta$$

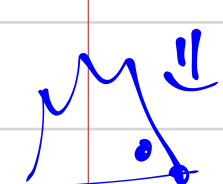
$$Var(\hat{\theta}_n) = Var\left(\frac{\sum X_i}{n\alpha}\right) = \left(\frac{1}{n\alpha}\right)^2 \cdot Var\left(\sum X_i\right) =$$

$$= \left(\frac{1}{n\alpha}\right)^2 \left(Var(X_1) + Var(X_2) + \dots + Var(X_n)\right) =$$

$$= \left(\frac{1}{n\alpha}\right)^2 \cdot n \cdot \alpha \cdot \theta^2 = \frac{\theta^2}{\alpha n} \xrightarrow{n \to \infty} 0$$

$$\lim_{n\to\infty} E(\hat{\theta}_n) = \theta \qquad \lim_{n\to\infty} Var(\hat{\theta}_n) = 0$$

$$\operatorname{plim}_{n\to\infty} \hat{\theta}_n = \theta$$



The sequence $\hat{\theta}_n$ is consistent.

Ex. $X_1, X_2 \dots X_n \sim Unif[2;5]$ independ.

a) $\operatorname{plim}_{n\to\infty} \min\{X_1, X_2 \dots X_n\}$

b) $\operatorname{plim}_{n\to\infty} (X_1 + X_2 + X_3 + \dots X_n)/(n+10)$

c)* $\operatorname{plim}_{n\to\infty} (X_1 \cdot X_2 \cdot X_3 \cdot \dots \cdot X_n)^{\frac{1}{n}}$

---

a)

[Int. guess]

intuition first

$\operatorname{plim}_{n\to\infty} \min\{X_1 \dots X_n\} = 2$

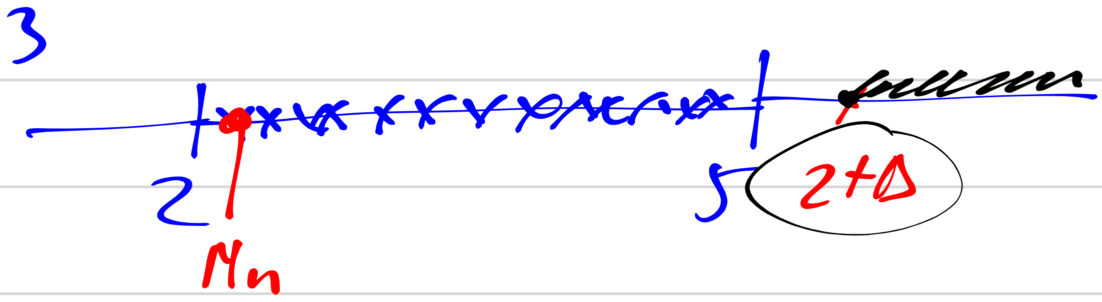
let's use the definition. $M_n = \min(X_1, \ldots, X_n)$

$$\lim P(|M_n - 2| > \Delta) = \qquad \Delta > 0$$

$$X_1 \ldots X_n \geqslant 2 \Rightarrow M_n = \min\{X_1, \ldots X_n\} \geqslant 2$$

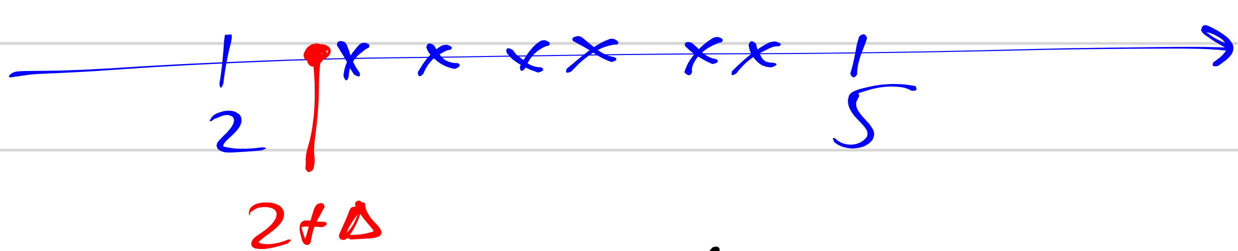
$$= \lim P(M_n - 2 > \Delta) = \lim P(M_n > \underline{2+\Delta})$$

case 1. $\Delta \geqslant 3$



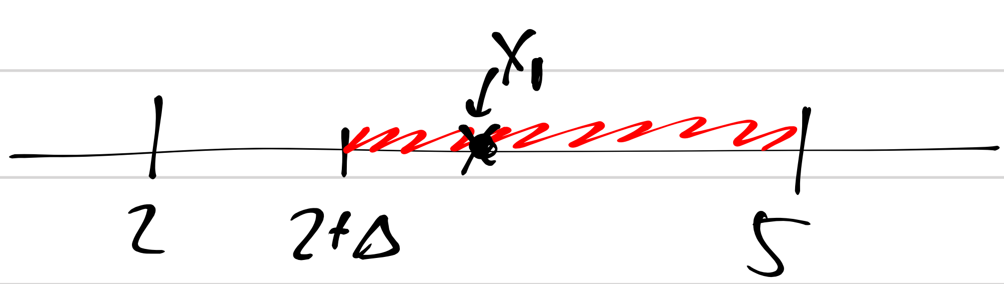
$$P(M_n > 2+\Delta) = 0$$

case 2. $\Delta \in (0;3)$



$$P(M_n > 2+\Delta) = P(X_1 > 2+\Delta, X_2 > 2+\Delta \ldots X_n > 2+\Delta) =$$

$$= P(X_1 > 2+\Delta) \cdot \ldots\ldots \cdot P(X_n > 2+\Delta) =$$



$$= \frac{5-(2+\Delta)}{5-2} \cdot \frac{5-(2+\Delta)}{5-2} \cdot \ldots \cdot \frac{5-(2+\Delta)}{5-2} =$$

$$= \left(\frac{3-\Delta}{3}\right)^n \xrightarrow[\frac{3-\Delta}{3} \in (0;1)]{n \to \infty}$$

$$\lim_{n \to \infty} P(|M_n - 2| > \Delta) \begin{cases} \lim 0 \quad [\Delta \geqslant 3] & = 0 \\ \lim \left(\frac{3-\Delta}{3}\right)^n \quad [\Delta \in (0;3)] & = 0 \end{cases}$$

$\square$

Conv. in Prob

probability of „big" [bigger than Δ] deviations goes to zero when n tends to ∞

another approach:

Step 1. pdf M_n ?

Step 2. $E(M_n)$? $\qquad \int \dots dt$

Step 3. $Var(M_n)$? $\qquad \int \dots dt$

Step 4. $\lim E(M_n) \dots = 2$

Step 5. $\lim Var(M_n) \dots = 0$

$\Rightarrow \operatorname{plim} M_n = 2$

b) $$\operatorname{plim} \frac{X_1 + X_2 + \dots + X_n}{n+10} =$$

$$= \operatorname{plim} \frac{X_1 + X_2 + \dots + X_n}{n} \cdot \frac{n}{n+10} =$$

$$= \underbrace{\operatorname{plim} \frac{X_1 + X_2 + \dots X_n}{n}}_{\text{by LLN}} \cdot \underbrace{\operatorname{plim} \frac{n}{n+10}}_{=1} =$$

old friend !!

$$= E(X_1) \cdot 1 = \frac{2+5}{2} \cdot 1 = 3.5$$

c) $$\operatorname*{plim}_{n\to\infty} (X_1 \cdot X_2 \cdot X_3 \cdot \dots \cdot X_n)^{1/n} =$$

continuous

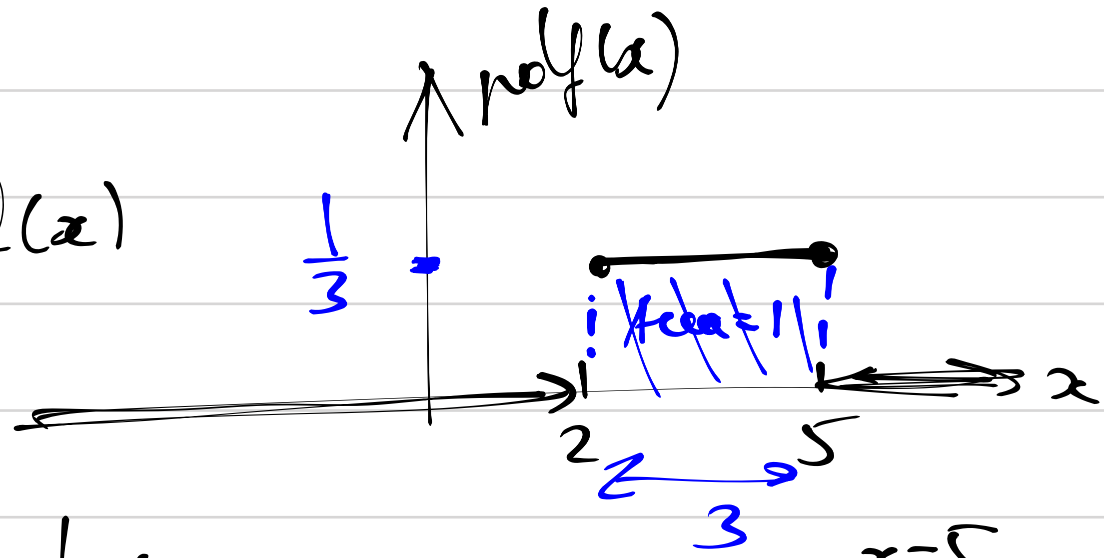
$$= \operatorname*{plim}_{n\to\infty} \exp\left(\ln\left((X_1 \cdot \dots \cdot X_n)^{\frac{1}{n}} \right) \right) =$$

$$= \exp\left[\operatorname{plim} \frac{1}{n} (\ln X_1 + \ln X_2 + \dots + \ln X_n) \right] =$$

$$= \exp\left[\operatorname{plim} \frac{\ln X_1 + \ln X_2 + \dots + \ln X_n}{n} \right] = \exp(E(\ln X_1))$$

$$E(\ln X_1) = \int_2^5 \ln x \cdot \mathrm{pdf}(x)\,dx =$$

pdf(x)



$$= \int_2^5 \ln x \cdot \frac{1}{3}\,dx = \left[x \ln x - x\right]_{x=2}^{x=5} \cdot \frac{1}{3} =$$

$$(x \ln x)' = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$(x \ln x - x)' = \ln x$$

$$= \left[5 \ln 5 - 5 - 2\ln 2 + 2\right] \cdot \frac{1}{3} = \frac{5\ln 5 - 2\ln 2}{3} - 1$$

$$E(\ln X_1) = \frac{5\ln 5 - 2\ln 2}{3} - 1 = \ln\left[\left(5^5/2^2\right)^{\frac{1}{3}}/e\right]$$

$$\operatorname*{plim}_{n\to\infty} \left(X_1 \cdot X_2 \cdot \ldots \cdot X_n\right)^{\frac{1}{n}} = \exp\left(E(\ln X_1)\right) =$$

$$= \exp\left(\ln\left[\left(5^5/2^2\right)^{\frac{1}{3}}/e\right]\right) =$$

$$= \frac{(25/4)^{\frac{1}{3}}}{e}$$

!!