Name, group no:

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1. [10 points] Let (X_t) be independent identically distributed random variables with $\mathbb{P}(X_i = -1) = 0.4$ and $\mathbb{P}(X_i = +1) = 0.6$. Consider the sum $S_t = X_1 + X_2 + \ldots + X_t$.

   (a) [3] Is S_t a martingale?

   (b) [7] Find all constants c such that $M_t = c^{S_t}$ is a martingale.

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2. [10 points] Let $a(t)$ be a deterministic function, $M_t = a(t)\cos(3W_t)$ and (W_t) is a Wiener process.

   (a) [4] Find dM_t.

   (b) [6] Find a non-zero function $a(t)$ such that (M_t) is a martingale.

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3. [10 points] You have two correlated Wiener processes, (A_t) and (B_t), with $\mathrm{Corr}(A_t - A_s, B_t - B_s) = \rho$ for all $t > s$.

   Split the time interval $[0; t]$ into n small segments of equal length. Let Δ_i^A be the increment of the Wiener process (A_t) on the i-th small segment, i.e. $\Delta_i^A = A(it/n) - A((i-1)t/n)$. Let Δ_i^B be the increment of the Wiener process (B_t) on the i-th small segment.

   Consider the sum of cross-products, $S_n = \sum_{i=1}^n \Delta_i^A \Delta_i^B$.

   (a) [3] Find $\mathbb{E}(S_n)$.

   (b) [4] Does $\mathrm{Var}(S_n)$ tend to 0 when $n \to \infty$?

   (c) [2] Find the mean square limit of S_n.

   (d) [1] How would you write this limit in a short hand notation with dA_t and dB_t?

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4. [10 points] The process (X_t) has $X_0 = 2024$, $dX_t = W_t^2 dW_t + W_t dt$, where (W_t) is a Wiener process.
   (a) [2] Is (X_t) a martingale?
   (b) [4] Find $d(X_t W_t)$.
   (c) [4] Find $\mathrm{Cov}(X_t, W_t)$.

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5. [10 points] Consider two-period binomial model with initial share price $S_0 = 600$. Up and down share price multipliers are $u = 1.2$, $d = 0.9$, risk-free interest rate is $r = 0.05$ per period.

   The option pays you the maximal share price $X_2 = \max\{S_0, S_1, S_2\}$ at $t = 2$.

   (a) [4] Find the risk neutral probabilities.

   (b) [6] Find the current price X_0 of this option.

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6. [10 points] Consider the framework of Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

   Find the current price X_0 of an option that pays you $X_T = 1$ at fixed time T if $S_T \geq 2S_0$.

   Hint: you may use the standard normal cumulative distribution function in your answer.