

Home Assignment 1

1. Consider two identical hedgehogs starting at the vertices A and B of a polygon $ABCDE$. Each minute they simultaneously and independently choose to go clockwise or counter-clockwise in the next vertex. The brotherhood of two brave hedgehogs can be in three states: in one vertex, in two adjacent vertices, in two non-adjacent vertices.
 - (a) What is the probability that they will be in one vertex after 3 steps?
 - (b) Write down the transition matrix of the brotherhood Markov chain.
 - (c) What proportion of time the brotherhood will spend in each state in the long run?
 - (d) Find the expected time until the hedgehogs meet in one vertex.

2. Each day the Random Restaurant is independently closed with probability p . If the restaurant is open then the number of clients has Poisson distribution with mean μ .

After N days (working or non-working) the Random Restaurant will permanently close and you are right, N is random and has Poisson distribution with mean n .

- (a) Find the moment generating function of the number of clients during day 1, assuming $N \geq 1$.
 - (b) Find the moment generating function of the total number of clients served in the Random Restaurant.
3. Find the probability limit $\text{plim } X_n$, where

$$X_n = \frac{Y_1 + 2Y_2 + 3Y_3 + \dots + nY_n}{n^2}$$

and Y_1, Y_2, \dots are independent uniform on $[0; 1]$.

Hint: try to calculate $\mathbb{E}(X_n), \text{Var}(X_n)$. You may google the formulas for $1+2+\dots+n$ and $1^2+2^2+\dots+n^2$ or ask ChatGPT.

4. Consider the Poisson arrival process X_t with constant rate λ .
Now let's scale the time in a non-linear fashion, $Y_t = X_{t^2}$.
 - (a) Find $\mathbb{E}(Y_t), \text{Var}(Y_t), \mathbb{P}(Y_t = 0)$.
 - (b) Find $\mathbb{E}(Y_{t+5} | Y_t)$ and $\text{Var}(Y_{t+5} | Y_t)$.
5. Let's toss a dice until the first six appears. Let X be the result of the first toss and Y — the total number of tosses.
 - (a) Find $\mathbb{E}(X | Y), \mathbb{E}(Y | X)$.
 - (b) Find $\text{Var}(X | Y), \text{Var}(Y | X)$.

6. The joint distribution of X and Y is given in the table

	$X = -1$	$X = 0$	$X = 1$
$Y = 0$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

- (a) Explicitely find the σ -algebras $\sigma(X), \sigma(Y), \sigma(X \cdot Y)$.
 - (b) How many elements are there in $\sigma(X, Y), \sigma(X + Y), \sigma(X, Y, X + Y)$?