Time Series and Stochastic Processes exams

Angry Teachers, Folklore

March 6, 2024

Contents

Description

See updates at https://github.com/bdemeshev/tssp_exams.
Click on red hyperlinks inside pdf, you can get to the answers and back!
Any comments? Bugs? https://github.com/bdemeshev/tssp_hse_exams/issues/.
The order of topics has changed after the first course iteration in 2020-21. The interested reader may find relevant exercises by looking through all 2020-21 exams.

Greatings to the contributors

Here we describe only the style guidelines and typical erros. For more information on tex one may read the book by K. Vorontsov.

1. Use decimal point as a separator: 3.14 — good style, $3{,}14$ — bad style. This goes against russian tradition, but favors copy-pasting numbers in software for computations.
2. Use `\[...\]` for display math formulas. Do not use $$...$$!
3. Use `cases` for systems of equations, `align*` for multiline formulas, `enumerate` for enumerations.
4. Inside formulas use `\text{...}` to write text.
5. Use `\ldots` for ellipsis.
6. You can find useful macros in the preamble, like `\P, \E, \Var, \Cov, \Corr, \cN`.
7. Use backslash before functions: `\ln, \exp, \cos`...
8. Use booktabs style for tables. You may use online tablesgenerator. Choose booktabs table style instead of default table style.
9. Respect the letter ë! :)
10. Start every sentence in tex source from a new line. There will be no additional newlines in final pdf but tex file will be easier to read.
11. For multiplication use `\cdot`. Please never use `*` :)

1 October exam

1.1 2023-2024

Short rules: 90 minutes, offline. You may use one A4 cheat sheet.
Date: 2023-10-21.

1. The hedgehog Melissa starts at the vertex A of a triangle ΔABC. Each minute she randomly moves to an adjacent vertex with probabilities $\mathbb{P}(A \to B) = 0.7$, $\mathbb{P}(A \to C) = 0.3$, $\mathbb{P}(B \to C) = \mathbb{P}(B \to A) = 0.5$, $\mathbb{P}(C \to B) = \mathbb{P}(C \to A) = 0.5$.

   (a) What is the probability that she will be in vertex B after 3 steps?

   (b) Write down the transition matrix of this Markov chain.

   (c) What proportion of time Melissa will spend in each state in the long run?

2. The number of players N who will win the lottery is a random variable with probability mass function $\mathbb{P}(N = k) = 7 \cdot 0.3^k/3$ for $k \geqslant 1$. Each player will get a random prize $X_i \sim U[0;1]$. All random variables are independent. Let S be the sum of all the prizes.

   (a) Find $\mathbb{E}(S \mid N)$ and conditional moment generating function $M_{S|N}(u)$.

   (b) Find the unconditional moment generating function $M_S(u)$.

   (c) What is the probabilistic meaning of $M''_S(0) - (M'_S(0))^2$?

   Note: you don't need to calculate the value in (c).

3. Consider the stochastic process (X_n), where X_0 is uniform on $[0;2]$ and $X_n = (1 + X_{n-1})/2$.

   (a) Find $\mathbb{E}(X_n)$ and $\mathrm{Var}(X_n)$.

   (b) Find the probability limit $\operatorname{plim} X_n$.

4. Students arrive in the Grusha café according to the Poisson arrival process (X_t) with constant rate λ.

   The probability of no visitors during 5 minutes is 0.05.

   (a) Find the value of λ.

   (b) Find the variance and expected number of arrivals between 5 pm and 8 pm.

   (c) What is the probability of exactly 5 arrivals between 5 pm and 8 pm?

5. The random variables X_1 and X_2 are independent and normally distributed, $X_1 \sim \mathcal{N}(1;1)$, $X_2 \sim \mathcal{N}(2;2)$. I choose X_1 with probability 0.3 and X_2 with probability 0.7 without knowing their values.

   Casino pays me the value Y that is equal to the chosen random variable.

   Let the indicator I be equal to 1 if I choose X_1 and 0 otherwise.

   (a) Express Y in terms of X_1, X_2 and I.

   (b) Find $\mathbb{E}(Y \mid I)$, $\mathrm{Var}(Y \mid I)$.

   (c) Find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

6. The joint distribution of X and Y is given in the table

	$X = -2$	$X = 0$	$X = 2$
$Y = -1$	0.1	0.2	0.3
$Y = 1$	0.2	0.1	0.1

   (a) Explicitely find the σ-algebra $\sigma(X)$.

   (b) How many elements are there in $\sigma(X \cdot Y)$?

1.2 2022-2023

Short rules: 120 minutes, online and offline. You may use one A4 cheat sheet.
Date: 2022-10-29

1. [10] The random variables X_i are independend and uniformly distributed on $[0;2]$. Find

$$\operatorname*{plim}_{n\to\infty} \frac{(X_1-\bar{X})^3+(X_2-\bar{X})^3+\ldots+(X_n-\bar{X})^3}{n+2022}.$$

2. A Hedgehog starts at the point $x=2$ on the real line. Every minute he moves one step left with probability 0.3 or one step right with probability 0.7. There are two exceptions from this rule: the absorbing point $x=0$ and the reflecting barrier at $x=3$.

   If the Hedgehog reaches the absorbing point $x=0$ then he stops moving and stays there. If the Hedgehog reaches the reflecting barrier $x=3$ then his next move will be one step left with probability 1.

   (a) [2] Write the transition matrix of this Markov chain.
   (b) [3] What is the probability that Hedgehog will be at $x=1$ after exactly 3 steps?
   (c) [5] What is the expected time to reach the absorbing state?

3. The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

   (a) [3] Explicitely list all the events in sigma-algebra $\sigma(X_1\cdot X_2)$.
   (b) [3] Pavel says that he knows only whether X_1 and X_3 are equal. How will you describe his knowledge with sigma-algebra?
   (c) [4] How many events are in the sigma-algebra $\sigma(X_1, X_1+X_2, X_1+X_2+X_3)$?

4. Masha receives on average 10 sms per minute. Sms arrival is well described by the Poisson process.

   (a) [3] What is the probability that Masha receives exactly 10 sms in the next 40 seconds?
   (b) [3] Masha just received an sms. What is the probability that she will wait more that 2.5 seconds before the next one?
   (c) [4] Find the covariance between the number of sms in the first 3 minutes and the number of sms in the first 10 minutes.

5. The random variables X_i are independent and they take values $+1$ or -1 with equal probability.

   (a) [3] Find $\mathbb{E}(X_3 \mid X_1, X_2)$, $\mathbb{E}(X_3 \mid X_1+X_3)$.
   (b) [3] Find $\mathrm{Var}(X_3 \mid X_1, X_2, X_3)$, $\mathrm{Var}(X_3 \mid X_1+X_3)$.
   (c) [4] Let Y_n be equal to $\mathbb{E}(X_1+\ldots+X_{2022} \mid X_1, X_2, \ldots, X_n)$.
   Is the process Y_1, Y_2, ..., Y_{2022} a martingale?

6. Consider a Wiener process (W_t).

   (a) [4] Let $Y_t = tW_{2t}$. What is the distribution of $Y_t - Y_s$ for $t \geqslant s$? Is Y_t a Wiener process?
   (b) [6] Find a constant α such that $M_t = W_t^3 + \alpha t W_t$ is a martingale.

1.3 2021-2022

Short rules: 120 minutes, online without proctoring. You may use any source you want but don't cheat.
Date: 2021-10-28

1. (10 points) Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

   (a) (3 points) Split the chain in classes and classify them into closed or not closed.

   (b) (2 points) Classify the states into recurrent or transient.

   (c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.

   What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

   Note: state number is the row (or column) number.

2. (10 points) Gleb Zheglov catches one criminal every day. With probability 0.2 the catched criminal is replaced by w new criminals. Initially there are n criminals in the town.

   What is the expected time to the ultimate crime eradication in the town?

   (a) (4 points) Solve the problem for $w = 1$ and $n = 1$.

   (b) (6 points) Solve the problem for arbitrary w and n.

3. (10 points) The random variables X_i are independend and uniformly distributed on $[0; 1]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^n X_i}{n}, \frac{2\sum_{i=1}^n X_i^2}{n}\right\}.$$

4. (10 points) Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes.

   Let Y_t be the number of taxis that will arrive between 0 and t minutes.

   (a) (2 points) Sketch the expected value of Y_t as a function of t.

   (b) (8 points) Sketch the probability $\mathbb{P}(Y_t = Y_{60})$ as a function of t.

   Note: special points like intercepts or extrema should be explicitely marked.

5. (10 points) Prince Myshkin throws a fair coin until two consecutive heads appear. Let N be the number of throws.

   Find the moment generating function of N.

   Hint: you may use the first step approach.

6. (20 points) Vincenzo Peruggia makes attempts to steal the Mona Lisa painting until the first success. Each attempt is successful with probability 0.1.

   Let X be the number of attempts and $Z = \min\{X, 5\}$.

   (a) (5 points) How many events are in sigma-algebras $\sigma(Z)$ and $\sigma(X)$?

   (b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(Z)$ but $A \notin \sigma(X)$; $B \in \sigma(X)$ but $B \notin \sigma(Z)$.

   (c) (10 points) Find $\mathbb{E}(Z \mid X)$ and $\mathbb{E}(X \mid Z)$.

1.4 2021-2022 retake

Short rules: 120 minutes, online without proctoring. You may use any source you want but don't cheat.

1. (10 points) Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) (3 points) Split the chain in classes and classify them into closed or not closed.

   (b) (2 points) Classify the states into recurrent or transient.

   (c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.

   What is the approximate probability that the Hedgehog will be in the state four after 10^{2021} moves?

   Note: state number is the row (or column) number.

2. (10 points) Consider infinite ladder with steps numbered from 0 to infinity. I start at step 0. Every day with probability u I go one step up. With probability d I go one step down. With probability $1-u-d$ I stay on the same step.

   If I am at step 0 then I stay there with probability $1-u$ because it's impossible to go down.

   Consider the case $d > u$.

   What is the probability that I will be at step 0 after 10^{1000} days?

3. (10 points) The random variables X_i are independend and uniformly distributed on $[0;2]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max\left\{\frac{\sum_{i=1}^{10} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^3}{n+1}\right\}.$$

4. (10 points) Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes.

   Let Y_t be the number of taxis that will arrive between 0 and t minutes.

   (a) (5 points) Sketch the probability $\mathbb{P}(Y_{t+3} = 1 \mid Y_t = 0)$ as a function of t.

   (b) (5 points) Sketch the covariance $\operatorname{Cov}(Y_t, Y_{60})$ as a function of t.

   Note: special points like intercepts or extrema should be explicitely marked.

5. (10 points) The moment generating function of a random variable X is $1/(1-2t)$.

   (a) Find the moment generating function of $2X$.

   (b) Find the moment generating function of $X+Y$ where X and Y are independent and identically distributed.

   (c) Do you remember the sum of geometric progression? Find $\mathbb{E}(X^{2021})$.

6. (20 points) Variables X_1, X_2, ...X_{100} are independent and identically distributed with mean 1 and variance 2. Each X_i has only three possible values: 0, 1, and 2.

   (a) (5 points) How many events are in sigma-algebras $\sigma(X_1, X_2)$ and $\sigma(X_1 - X_2)$?

   (b) (5 points) If possible provide an example of events A and B such that: $A \in \sigma(X_1, X_2)$ but $A \notin \sigma(X_1 - X_2)$; $B \in \sigma(X_1 - X_2)$ but $B \notin \sigma(X_1, X_2)$.

   (c) (10 points) Find $\mathbb{E}(X_1 + \ldots + X_{100} \mid X_1 + \ldots + X_{50})$ and $\mathbb{E}(X_1 + \ldots + X_{50} \mid X_1 + \ldots + X_{100})$.

1.5 2020-2021

Here (W_t) denotes the standard Wiener process.
Date: 2020-10-30

1. For $r < s < t < u$ find the following expected values

   (a) $\mathbb{E}((W_u - W_t)^2(W_s - W_r)^2)$;

   (b) $\mathbb{E}((W_u - W_s)(W_t - W_r))$;

   (c) $\mathbb{E}((W_t - W_r)(W_s - W_r)^2)$;

   (d) $\mathbb{E}(W_r W_s W_t)$;

   (e) $\mathbb{E}(W_r W_s W_t \mid W_s)$;

2. Consider Ito process X_t

$$dX_t = \exp(t)W_t\,dt + \exp(2W_t)\,dW_t, \quad X_0 = 1.$$

   Consider two processes, $A_t = 1 + t^2 + X_t^3$ and $B_t = 1 + t^2 + X_t^3 W_t^4$.

   (a) Find dA_t and dB_t.

   (b) Write the corresponding explicit expressions for A_t and B_t:

$$const + \int_0^t \ldots dW_u + \int_0^t \ldots du$$

   (c) Check whether X_t is a martingale.

3. Let $S_0 = 0$, $S_t = X_1 + X_2 + \ldots + X_t$. The increments X_t are independent and identically distributed:

x	-1	0	1
$\mathbb{P}(X_t = x)$	0.2	0.2	0.6

   (a) If possible find all constants a such that $M_t = S_t + at$ is a martingale.

   (b) If possible find all constants b such that $R_t = b^{S_t}$ is a martingale.

4. Consider the process X_t

$$X_t = tW_t + \int_0^t uW_u^2\,dW_u.$$

   (a) Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$.

   (b) Find dX_t.

   (c) Check whether X_t is a martingale.

5. A Hedgehog in the fog starts in $(0, 0)$ at $t = 0$ and moves randomly with equal probabilities in four directions (north, south, east, west) by one unit every minute.

   Let X_t and Y_t be his coordinates after t minutes and $S_t = X_t + Y_t$.

   (a) Find $\mathbb{E}(X_2 \mid S_2)$;

   (b) Find $\mathrm{Var}(X_2 \mid S_2)$.

   Hint: $\mathrm{Var}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - (\mathbb{E}(Y \mid X))^2$.

6. Vampire Petr and Markov Chains.

   Vampire Petr drinks blood of a new victim every day. Unfortunately 20% of the population are vaccinated against vampires. If more than one victim of the last three victims are vaccinated then Petr will be instantaneously cured and will return to the normal life.

   For simplicity let's assume that the last three victims were not vaccinated.

   (a) What is the probability that vampire Petr will be cured in the next three days?
   (b) How many victims will be bitten by vampire Petr on average?

7. Vampire Boris and Martingales.

   To survive vampire Boris needs to bite 70 talented students.

   These 70 talented students have formed a secret group. They have written their emails on small pieces of paper and have randomly distributed these pieces among them. Each student has exactly one piece of paper with an email[1].

   Initially vampire Boris knows contacts of just two persons from the group. Today he will contact them, drink their blood and get the emails they have. Then vampire Boris will contact new victims and so on.

   (a) For $t \geqslant 1$ consider the process M_t, the proportion of non bitten students after the day t. Is this process a martingale?
   (b) Using martingale stopping theorem or otherwise find the probability that vampire Boris will bite all 70 students.

[1] The group is so secret that it is possible that a student has his own email on his piece of paper

2 December exam

2.1 2022-2023

Short rules: 120 minutes, you may use two A4 cheat-sheets, offline + one online group.

1. Consider $X_t = \int_0^t W_u^3 dW_u + \int_0^t (W_u^3 + 3W_u u)du - W_t^3 \cdot t$.
   (a) Find dX_t and the corresponding full form.
   (b) Is X_t a martingale?
2. Consider $X_t = \exp(-2W_t - 2t)$.
   (a) Find dX_t. Is X_t a martingale?
   (b) Find $\mathbb{E}(X_t)$ and $\operatorname{Var}(X_t)$.
   (c) Find $\int_0^t X_u dW_u$.
3. As usual (W_t) is a Wiener process.
   (a) Find $\mathbb{E}(W_5W_4 \mid W_4)$, $\operatorname{Var}(W_5W_4 \mid W_4)$.
   (b) Find covariance $\operatorname{Cov}(W_4W_5, W_5W_6)$.
4. Let X_i be independent identically distributed with $\mathbb{P}(X_i = 1) = 0.9$, $\mathbb{P}(X_i = -1) = 0.1$.

   Find all constants a and b such that $Y_t = a \exp\left(b \sum_{i=1}^t X_i\right)$ is a martingale.
5. Consider two-period binomial model with initial share price $S_0 = 600$, Up and down multipliers are $u = 1.2$, $d = 0.9$, risk-free interest rate is $r = 0.05$ per period.

   Consider an option that pays you $X_2 = 100$ at $T = 2$ if $S_2 > S_1$ and nothing otherwise.
   (a) Find the risk neutral probabilities.
   (b) Find the current price X_0 of the asset.
   (c) How much shares should I have at $t = 1$ in the «up» state of the world to replicate the option?
6. Consider Black and Scholes model with riskless rate r, volatility σ and initial share price S_0.

   Find the current price X_0 of an option that pays you $X_2 = S_1^3$ at time $T = 2$.

2.2 2021-2022

Short rules: 120 minutes, online without proctoring, (W_t) is a standard Wiener process.
Date: 2021-12-25

1. (10 points) Consider an Ito's process $I_t = 2022 + W_t t^2 + \int_0^t W_u^3 dW_u + \int_0^t W_u^2 du$.
   (a) Find dI_t and check whether I_t is a martingale.
   (b) Check whether $J_t = I_t - \mathbb{E}(I_t)$ is a martingale.
2. (10 points) The random variables (Z_t) are independent identically distributed with moment generating function given by $M_Z(u) = 1/(1-5u)^3$.

   We define X_t as $X_t = \exp(Z_1 + 2Z_2 + 3Z_3 + \ldots + tZ_t)$ with $X_0 = 0$.

   If possible find a martingale of the form $Y_t = h(t)X_t$ where $h()$ is a non-random function.

3. (10 points) The process (Z_t) in discrete time is called *stationary* if it has constant expected value and constant covariances γ_k that do not depend on t.

$$\begin{cases} \mathbb{E}(Z_t) = \mu; \\ \mathrm{Cov}(Z_t, Z_t) = \gamma_0; \\ \mathrm{Cov}(Z_t, Z_{t+1}) = \gamma_1; \\ \mathrm{Cov}(Z_t, Z_{t+2}) = \gamma_2; \\ \ldots \end{cases}$$

   (a) If possible provide an example of a martingale that is not stationary.
   (b) If possible provide an example of a stationary process that is not a martingale.

4. (10 points) Find $\mathbb{E}(W_1 W_2 W_3)$ and $\mathbb{E}(W_2 W_3 \mid W_1)$.

5. (10 points) Ded Moroz would like to receive $X_T = S_T^{-1}$ at time T if $S_T < 1$ and nothing otherwise.

   Assume the framework of Black and Scholes model, S_t is the share price, r is the risk free rate, σ is the volatility.

   How much Ded Moroz should pay now at $t = 0$?

6. (20 points) Martingales are everywhere :)

   Consider the process $Y_t = \exp(-uW_t)$.

   (a) Find a multiplier $h(u, t)$ such that $M_t = h(u, t) \cdot Y_t$ is a martingale.
   (b) Find dY_t, $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.
   (c) Consider M_t that you have found as a function of u. Find the Taylor approximation of the function $M_t(u)$ up to u^4.
   (d) Consider the coefficient before u^4 in the Taylor expansion of $M_t(u)$. Is it a martingale?

7. Bonus point. Guess your exam result (out of 70 possible points).

2.3 2020-2021

Today we celebrate Christmas Eve and 78 years of the Narkompros (People's Commissariat for Education) order governing the compulsory use of the letter «ё» in education process.
Date: 2020-12-24

1. Ded Moroz would like to receive S_1^3 roubles at time $T = 2$, where S_t is the share price. Assume Black-Schёles model is valid, the risk-free rate is $r = 0.1$ and current share price is $S_0 = 100$.

   How much Ded Moroz should pay now at $t = 0$?

2. Consider stationary $AR(2)$ model, $y_t = 2 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$, where (u_t) is a white noise with $\mathrm{Var}(u_t) = 4$.

   The last two observations are $y_{100} = 2$, $y_{99} = 1$.

   (a) Find 95% predictive interval for y_{102}.
   (b) Find the first two values of the autocorrelation function, ρ_1, ρ_2.
   (c) Find the first two values of the partial autocorrelation function, ϕ_{11}, ϕ_{22}.

   Hint: you need no more than 10 seconds to find both partial autocorrelations provided (b) is sёlved.

3. The process y_t is described by a simple $GARCH(1, 1)$ model:

$$\begin{cases} y_t = \sigma_t \nu_t \\ \sigma_t^2 = 1 + 0.2y_{t-1}^2 + 0.3\sigma_{t-1}^2 \\ \nu_t \sim \mathcal{N}(0; 1) \end{cases}$$

The variables ν_t are independent of past variables y_{t-k}, ν_{t-k}, σ_{t-k} for all $k \geqslant 1$. The prëcesses y_t, σ_t^2 are stationary.

Given $\sigma_{100} = 1$ and $\nu_{100} = 0.5$ find 95% predictive interval for y_{102}.

4. Snegurochka studies a stochastic analog of the Fibonacci sequence

$$y_t = y_{t-1} + y_{t-2} + u_t,$$

where (u_t) is a white noise process.

(a) How many non-stationary solutions are there?

(b) What can you say about the number and the structure of the stationary solutions?

(c) Can Snëgurochka find two starting constants $y_0 = c_0$ and $y_1 = c_1$ in such a way to make a solution stationary?

Be brave! There are two more exercises!

5. The semi-annual y_t is modelled by $ETS(AAA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ b_t = b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-2} + u_t \end{cases}$$

(a) Given that $s_{100} = 2$, $s_{99} = -1.9$, $b_{100} = 0.5$, $\ell_{100} = 4$ find 95% prëdictive interval for y_{102}.

(b) In this problem particular values of parameters are specified. And how many parameters are estimated in semi-annual $ETS(AAA)$ model before real forecasting?

6. The variables x_t take values 0 or 1 with equal probabilities. The variables u_t are normal $\mathcal{N}(0; 1)$. All variables are independent.

Consider the process $z_t = x_t(1 - x_{t-2})u_t$.

(a) Find the covariance $\operatorname{Cov}(z_t, z_s)$. Is the process z_t stationary?

(b) Given that $z_{100} = 2.3$ find shërtest predictive intervals for z_{101} and z_{102} with probability of coverage at least 95%.

Bënus: How many letters «ё» have you spotted?

3 April exam

3.1 2023-2024

Short rules: 120 minutes, one A4 cheat sheet allowed.
Date: 2024-03-05

1. Consider $MA(2)$ process given by
$$y_t = 5 + u_t + 2u_{t-1} + 4u_{t-2},$$
where (u_t) is a white noise with $\mathrm{Var}(u_t) = \sigma^2$.

    (a) [1] Find the expected value $\mathbb{E}(y_t)$.
    (b) [7] Find the autocorrelation function $\rho_k = \mathrm{Corr}(y_t, y_{t-k})$.
    (c) [2] Is the process (y_t) stationary?

2. Consider $MA(2)$ process given by
$$y_t = 5 + u_t + 2u_{t-1} + 4u_{t-2},$$
where u_t are normal independent random variables with $\mathrm{Var}(u_t) = 4$.
You know that $u_{100} = 2$ and $u_{99} = -1$.

    (a) [5] Find the 95% predictive interval for y_{101}.
    (b) [5] Find the 95% predictive interval for $y_{1000001}$.

3. The stationary process (y_t) has autocorrelation function $\rho_k = 0.2^k$ and expected value 100.

    (a) [7] Find the first two values of the partial autocorrelation function, ϕ_{11} and ϕ_{22}.
    (b) [3] Provide a possible linear recurrence equation for this process. Your equation may include y_t, its lags and a white noise process (u_t).

4. Consider the equation $y_t = 5 + 2.5y_{t-1} - y_{t-2} + u_t$, where (u_t) is a white noise process.

    (a) [3] Find the roots of the corresponding characteristic equation.
    (b) [4] Rewrite the process as $A(L)(y_t - \mu) = u_t$. You should explicitely write the lag polynomial $A(L)$ and the value of μ.
    (c) [1] How many non-stationary solutions does the equation have?
    (d) [1] How many stationary solutions does the equation have?
    (e) [1] How many stationary solutions of the $MA(\infty)$ form with respect to (u_t) does the equation have?

5. [10] The semi-annual (y_t) is modelled by $ETS(ANA)$ process:
$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \end{cases}$$
Given that $s_{100} = 3$, $s_{99} = -2$, $\ell_{100} = 100$ find 95% predictive interval for y_{102}.

6. [10] The semi-annual (y_t) is modelled by $ETS(ANA)$ process:
$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \\ \ell_0 = 100, s_0 = -3, s_{-1} = 3 \end{cases}$$
Check whether the process (y_t) is stationary.

3.2 2022-2023

Short rules: 90 minutes, one A4 cheat sheet allowed.
Date: 2023-03-25

1. Consider $ETS(AAdN)$ model

$$\begin{cases} u_t \sim \mathcal{N}(0; 20) \\ b_t = 0.9b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + 0.9b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + 0.9b_{t-1} + u_t \end{cases}$$

with $\ell_{100} = 20$ and $b_{100} = 1$.

   (a) Find 95% prediction interval for y_{102}.

   (b) Approximately find the best point forecast for y_{10000}.

2. Consider the difference equation:

$$y_t = 0.7y_{t-1} - 0.12y_{t-2} + u_t,$$

where (u_t) is a white noise.

   (a) How many stationary and non-stationary solutions does the difference equation have?

Consider stationary $AR(2)$ process that satisfies the difference equation.

   (b) Find first two values of autocorrelation function.

   (c) Find α_1 and α_2 in $MA(\infty)$ representation

$$y_t = u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \alpha_3 u_{t-3} + \ldots$$

3. The strictly stationary white noise (u_t) follows $ARCH(1)$ model $\sigma_t^2 = 3 + 0.5u_{t-1}^2$ where $u_t = \sigma_t \nu_t$ and $\nu_t \sim \mathcal{N}(0; 1)$.

   (a) Find 95% prediction interval for u_{101} given that $u_{100} = -1$.

   (b) Find $\mathbb{E}(u_t)$, $\mathrm{Var}(u_t)$.

   (c) Find $\mathrm{Corr}(u_t, u_{t-1})$, $\mathrm{Corr}(u_t^2, u_{t-1}^2)$.

4. The weight of a fish Y_i is a discrete random variables with distribution and observed frequencies given in the table

Weight [kg]	1	2	4
Probability	$0.2 + a$	$0.3 - a$	0.5
Observed frequency	N_1	N_2	N_4

Fish weights Y_i are independent.

   (a) Find the maximum likelihood estimator of the parameter a.

   (b) Find the method of moments estimator of the parameter a.

5. You observe time between taxi arrivals on a stop, $Y_1, Y_2, \ldots, Y_n$. Assume that Y_i are independent and exponentially distributed with $\mathbb{E}(Y_i) = \theta$, that means the density of each Y_i is $f(y) = \exp(-y/\theta)/\theta$ for $y \geqslant 0$. Consider the following estimator of expected value

$$\hat{\theta} = n \cdot \min\{Y_1, Y_2, \ldots, Y_n\}$$

   (a) Find the probability density function of $\hat{\theta}$.

   (b) Is $\hat{\theta}$ unbiased?

   (c) Is $\hat{\theta}$ consistent?

3.3 2021-2022

Short rules: 120 minutes, one A4 cheat sheet allowed.
Date: 2022-04-04

1. Consider $ETS(AAN)$ model, $\begin{cases} y_t = \ell_{t-1} + b_{t-1} + u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ b_t = b_{t-1} + \beta u_t \\ u_t \sim \mathcal{N}(0; \sigma^2). \end{cases}$

   Let $\ell_{100} = 50$, $b_{100} = 2$, $\alpha = 0.4$, $\beta = 0.5$, $\sigma^2 = 16$.

   Calculate one step and two steps ahead 95% predictive intervals.

2. Consider the process $y_t = 4 + u_t + u_{t-1} + 2u_{t-2}$, where (u_t) is a white noise with variance 16.
   (a) Is this process stationary? Explain.
   (b) Find the autocorrelation function of this process. Explain the meaning of ρ_2.
   (c) Consider the process $d_t = \Delta y_t$. Is it $ARIMA(p, d, q)$? If yes, then find p, d and q.

3. Consider the stationary $AR(2)$ process $y_t = 5 - 0.9y_{t-1} - 0.2y_{t-2} + u_t$, where (u_t) is a white noise.
   (a) Find the first value of autocorrelation function ρ_1.
   (b) Find the partial autocorrelation function of this process. Explain the meaning of ϕ_{22}.
   (c) What is the relationship between values of autocorrelation function ρ_{100}, ρ_{99} and ρ_{98}.

   Hint: values ϕ_{22}, ϕ_{33} etc may be calculated almost effortlessly :)

4. Consider iid sample from bivariate normal distribution, $\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \theta \\ 2\theta \end{pmatrix}; \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}\right)$.

   Calculate Fischer information for the following cases:
   (a) You observe X_1 only.
   (b) You observe $X_1, \ldots, X_n$.
   (c) You observe $X_1, \ldots, X_n, Y_1, \ldots, Y_n$.

   Hint: the multivariate normal density is $f(u) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(u-\mu)^T \Sigma^{-1}(u-\mu)\right)$.

5. Random variables $X_1, \ldots, X_n$ are independent with density $f(x) = \begin{cases} -\ln(a) \cdot a^x, \text{ if } x \geqslant 0, \\ 0, \text{ otherwise.} \end{cases}$
   (a) Estimate a using maximum likelihood.
   (b) Check whether the estimator is unbiased and consistent.
   (c) Check whether the corresponding Cramer-Rao lower bound is attained.

6. Consider the $ARCH(1)$ model, $u_t = \sigma_t \nu_t$, where ν_t are iid $\mathcal{N}(0; 1)$ and $\sigma_t^2 = 1 + 0.3u_{t-1}^2$.
   (a) Find 95% predictive interval for u_{101} if $u_{100} = -2$.
   (b) Find the autocorrelation function of $r_t = u_t^2$.

3.4 2020-2021

Date: 2021-04-13, Rock 'N' Roll day

Estimation questions

1. To go to the mountain top I use a gondola lift in the morning. I go back from the top using the same gondola lift in the evening. Cabins are numbered from 1 to a.

   I have noticed that the absolute difference of cabin numbers of my two trips was 10.

   (a) Estimate a using maximum likelihood.

   (b) Estimate a using method of moments.

2. Random variables $X_1, X_2, \ldots, X_n$ are independent identically distributed with density

$$f(x_i \mid \lambda, a) = \frac{\lambda}{2}\exp(-\lambda|x_i - a|).$$

   Observed values for $n = 3$ are -3, 1, 11.

   (a) Estimate λ using method of moments for fixed $a = 1$.

   (b) Estimate λ and a using maximum likelihood.

3. Random variables $X_1, \ldots, X_n$ are independent and normally distributed $\mathcal{N}(1, 1/b)$.

   (a) Estimate b using maximum likelihood.

   (b) Does the estimator achive the Cramer-Rao lower bound?

   (c) Is the estimator consistent?

   (d) Is the estimator unbiased?

4. Random variables $X_1, X_2, \ldots, X_n$ are independent identically distributed with density

$$f(x_i \mid \lambda) = \frac{\lambda}{2}\exp(-\lambda|x_i|).$$

   For $n = 100$ I have 40 negative values with sum equal to -300 and 60 positive values with sum equal to 500.

   (a) Test the hypothesis $\lambda = 1$ using LR approach at significance level $\alpha = 0.01$.

   (b) Test the hypothesis $\lambda = 1$ using LM approach at significance level $\alpha = 0.01$.

Distribution questions

5. I have three problems in the home assignment. Time spent on each problem is modelled by independend exponentially distributed random variables with rate λ: X_1, X_2, X_3.

   (a) Find the moment generating function of X_i and hence the moment generating function of $S = X_1 + X_2 + X_3$.

   (b) Find $\mathbb{E}(S^3)$.

   (c) Find the joint density of $R = X_1/(X_1 + X_2 + X_3)$ and S.

6. I have 100 numbers written on small sheets of paper: $x_1, x_2, \ldots, x_{100}$. The sum of these numbers is 1.

   Find the possible values of the sum

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \ldots + \frac{x_{100}}{\sqrt{1-x_{100}}}.$$

   Hint: consider a randomly selected number X and apply the Jensen's inequality.

4 Final exam

4.1 2021-2022

Short rules: 120 minutes, offline, one A4 cheat sheet allowed.
Date: 2022-06-25

1. Consider $ETS(ANN)$ model, $\begin{cases} y_t = \ell_{t-1} + u_t \\ \ell_t = \ell_{t-1} + \alpha u_t \\ u_t \sim \mathcal{N}(0; \sigma^2). \end{cases}$ Let $\ell_{99} = 50$, $\alpha = 1/2$, $\sigma^2 = 16$, $y_{98} = 48$, $y_{99} = 52$, $y_{100} = 55$. Calculate 95% predictive interval for y_{101}.

2. Young investor Winnie-the-Crypto compares two trading strategies: buying bitcoins from good bees and from bad bees. Let d_t be the price difference at day t (bad minus good). Winnie-the-Crypto would like to test H_0: $\mathbb{E}(d_t) = 0$ against H_a: $\mathbb{E}(d_t) \neq 0$ at 5% significance level.

   Winnie assumed that (d_t) can be approximated by a $MA(1)$ process and estimated the parameters using $T = 400$ observations, $\hat{d}_t = 2 + u_t + 0.7u_{t-1}$ with $\hat{\sigma}_u^2 = 4$.

   (a) Estimate $\mathbb{E}(d_t)$, $\operatorname{Var}(d_t)$ and $\operatorname{Cov}(d_t, d_{t-1})$.

   (b) Estimate $\mathbb{E}(\bar{d})$, $\operatorname{Var}(\bar{d})$ and help Winnie by considering $Z = \frac{\bar{d}-0}{se(\bar{d})}$.

3. The variables X_1, ..., X_n are independent and uniformly distributed on $[0; 2a]$ for some positive a.

   (a) Find any sufficient statistic for a.

   (b) How the answer will change if $X_i \sim U[-a; 2a]$?

4. Consider an estimator $\hat{a}$ with $\mathbb{E}(\hat{a}) = 0.5a + 3$. For the given sample size the Fisher information is $I_F(a) = 400/a^2$.

   (a) What is the theoretical minimal variance of $\hat{a}$?

   (b) Assume that $\hat{a}$ attains the minimal variance boundary and is asymptotically normal. Given that $\hat{a} = 2022$ provide 95% CI for a.

5. You observe X_1, ..., X_{400} and Y_1, ..., Y_{400}, $\bar{X} = 5$, $\bar{Y} = 6$. All variables are independent.

   Consider the null hypothesis that all random variables are exponentially distributed with common parameter λ against alternative that parameter is λ_X for every X_i and λ_Y for every Y_j.

   (a) Estimate common λ using maximum likelihood for the restricted model.

   (b) Estimate both λ_X and λ_Y using maximum likelihood in the unrestricted model.

   (c) Use LR-test to test the null hyphotesis at 5% significance level.

6. The ultimate goal of this exercise is to prove the good upper bound for tail probability of a normal distribution: if $X \sim \mathcal{N}(0; \sigma^2)$ then $\mathbb{P}(X > c) \leqslant \exp(-c^2/2\sigma^2)$.

   Here are the guiding hints (you free to use not use them):

   (a) State the MGF of X. You may derive it or simply write it if you remember.

   (b) Consider $Y = \exp(uX)$. Using Markov inequality provide the upper bound for $\mathbb{P}(Y > \exp(uc))$.

   (c) Prove that $\mathbb{P}(X > c) \leqslant MGF_X(u)\exp(-uc)$ for any u.

   (d) Find the value of u that makes the upper bound as tight as possible.

7. (bonus) Draw good bees and bad bees selling crypto. Any funny statistics/math joke is also ok!

4.2 2020-2021

Today: $+31^{\circ}$, World Refrigiration Day :)
You have 100 minutes. You can use A4 cheat sheet and calculator. Be brave!
Date: 2021-06-26

1. I throw a fair die until the sequence 626 appears. Let N be the number of throws.
   (a) What is the expected value $\mathbb{E}(N)$?
   (b) Write down the system of linear equations for the moment generating function of N. You don't need to solve it!

2. Consider the following stationary process
$$y_t = 1 + 0.5y_{t-2} + u_t + u_{t-1},$$
   where random variables u_t are independent $\mathcal{N}(0; 4)$.
   (a) Find the 95% predictive interval for y_{101} given that $y_{100} = 2$, $y_{99} = 3$, $y_{98} = 1$, $u_{99} = -1$.
   (b) Find the point forecast for y_{101} given that $y_{100} = 2$.

3. I have an unfair coin with probability of heads equal to $h \in (0; 1)$.
   (a) Let N be the number of tails before the first head. Find the MGF of N.
   (b) Let S be the number of tails before k heads (not necessary consecutive). Find the MGF of S.
   (c) What is the limit of $MGF_S(t)$ when $k \to \infty$ and $k \times h \to 0.5$? What is the name of the corresponding distribution?

4. Consider the stochastic process $X_t = f(t)\cos(2021W_t)$.
   (a) Find dX_t.
   (b) Find any $f(t) \neq 0$ such that X_t is a martingale.
   (c) Using $f(t)$ from the previous point find $\mathbb{E}(\cos(2021W_t))$.

5 October exam solutions

5.1 2023-2024

1. (a) $\mathbb{P}(S_3 = B) = \mathbb{P}(A \to B \to A \to B) + \mathbb{P}(A \to C \to A \to B) + \mathbb{P}(A \to B \to C \to B)$

   (b)
$$\begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

   (c)
$$\begin{cases} a = 0.5b + 0.5c \\ b = 0.7a + 0.5c \\ c = 0.3a + 0.5b \\ a + b + c = 1 \end{cases}$$

   The solution is $a = 15/45$, $b = 17/45$, $c = 13/45$.

2. (a) Consider N as known fixed value, $\mathbb{E}(S \mid N) = N\,\mathbb{E}(X_1) = N \cdot 0.5$. First, let's find moment generating funciton for X_i:
$$M_X(u) = \int_0^1 \exp(xu) \cdot 1\,dx = \frac{\exp(u) - 1}{u};$$
   Hence $M_{S|N}(u) = (M_X(u))^N$ as S is the sum of N independent variables.

   (b) Random variable N is discrete, $M_S(u) = \mathbb{P}(N = 1)(M_X(u))^1 + \mathbb{P}(N = 2)(M_X(u))^2 + \ldots = \frac{0.7M_X(u)}{1-0.3M_X(u)}$.

   (c) Moment generating function is used to calculate moments, $M_S''(0) - (M_S'(0))^2 = \operatorname{Var}(S)$.

3. Start with X_0: $\mathbb{E}(X_0) = 1$, $\operatorname{Var}(X_0) = 4/12 = 1/3$.

   (a) Expected value is constant, $\mathbb{E}(X_n) = 0.5 + 0.5\,\mathbb{E}(X_{n-1})$, hence $\mathbb{E}(X_n) = 1$. Variance goes to zero, $\operatorname{Var}(X_n) = 0.25\operatorname{Var}(X_{n-1})$.

   (b) $\operatorname{plim} X_n = 1$

4. Let's measure time in minutes.

   (a) $\mathbb{P}(X_5 = 0) = \exp(-5\lambda) = 0.05$, so $\lambda = \ln(0.05)/-5 = \ln(20)/5$.

   (b) $\mathbb{E}(X_{180}) = 180\lambda$, $\operatorname{Var}(X_{180}) = 180\lambda$

   (c) $\mathbb{P}(X_{180} = 5) = \exp(-180\lambda)(180\lambda)^5/5!$

5. (a) $Y = IX_1 + (1 - I)X_2$

   (b) Consider I as known or fixed variable, $\mathbb{E}(Y \mid I) = I\,\mathbb{E}(X_1) + (1 - I)\,\mathbb{E}(X_2)$. Note that $I^2 = I$ and $(1 - I)^2 = 1 - I$, hence $\operatorname{Var}(Y \mid I) = I\operatorname{Var}(X_1) + (1 - I)\operatorname{Var}(X_2)$.

   (c) $\mathbb{E}(Y) = p\mu_1 + (1-p)\mu_2$ and $\operatorname{Var}(Y) = p(1-p)(\mu_1 - \mu_2)^2 + p\sigma_1^2 + (1-p)\sigma_2^2$, where $p = 0.3$, $\mu_1 = \sigma_1^2 = 1$, $\mu_2 = \sigma_2^2 = 2$.

6. (a) Sigma-algebra: $\sigma(X) = \{\emptyset, \Omega, \{X = -2\}, \{X = 0\}, \{X = 2\}, \{X \neq -2\}, \{X \neq 0\}, \{X \neq 2\}\}$. Other descriptions are possible, for example, one may replace $\{X = -2\}$ by $\{X < 0\}$.

   (b) Random variable XY takes 3 distinct values, hence $\operatorname{card}\sigma(X \cdot Y) = 2^3 = 8$.

5.2 2022-2023

1.

$$\operatorname{plim} \frac{\sum (X_i - \bar{X})^3}{n + 2022} = \operatorname{plim} \frac{\sum X_i^3 - 3\bar{X} \sum X_i^2 + 3\bar{X}^2 \sum X_i - \bar{X}^3}{n + 2022} =$$
$$= \mathbb{E}(X_1^3) - 3\,\mathbb{E}(X_1^2) + 3\,\mathbb{E}(X_1) - 1 = 0;$$

Note that $\mathbb{E}(X_1^2) = 4/3, \mathbb{E}(X_1^3) = 2$.

2. $\mathbb{P}(X_3 = 1) = 0.3 \cdot 0.7 \cdot 0.3 + 0.7 \cdot 1 \cdot 0.3 = 0.21 \cdot 1.3$ Let's denote $\tau_j = \min\{t \mid X_t = 0, X_0 = j\}$, $\mu_j = \mathbb{E}(\tau_j)$.

$$\begin{cases} \mu_0 = 0 \\ \mu_1 = 1 + 0.7\mu_2 \\ \mu_2 = 1 + 0.3\mu_1 + 0.7\mu_3 \\ \mu_3 = \mu_2 + 1 \end{cases}$$

We get $\mu_2 = 200/9$.

3. (a) $\sigma(X_1 \cdot X_2) = \{\emptyset, \Omega, \{X_1 X_2 = 1\}, \{X_1 X_2 = -1\}\}$;
   (b) Many answers are ok, for example $\sigma(X_1 X_3)$.
   (c) Note that $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3) = \sigma(X_1, X_2, X_3)$, the number of events in sigma-algebra is $\operatorname{card} \sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3) = 2^8 = 256$.

4. $\operatorname{Cov}(N_3, N_{10}) = \operatorname{Cov}(N_3, N_3 + (N_{10} - N_3)) = \operatorname{Var}(N_3) = 3\lambda$.

5. $\mathbb{E}(X_3 \mid X_1, X_2) = \mathbb{E}(X_3) = 0$, $\mathbb{E}(X_3 \mid X_1 + X_3) = (X_1 + X_3)/2$, $\operatorname{Var}(X_3 \mid X_1, X_3) = 0$, $\operatorname{Var}(X_3 \mid X_1 + X_3) = 1 - (X_1 + X_3)^2/4$.

6. Посчитаем ожидание и получим $Y_n = X_1 + X_2 + \ldots + X_n$, the process (Y_n) is a martingale.

7. $\operatorname{Var}(Y_t - Y_s) = \operatorname{Var}(tW_{2t} - sW_{2s}) = 2t^3 + 2s^3 - 4ts^2$. We get $\mathbb{E}(M_{t+u} \mid \mathcal{F}_t) = W_t^3 + 3W_t u + \alpha(t + u)W_t$. From $\mathbb{E}(M_{t+u} \mid \mathcal{F}_t) = W_t^3 + \alpha t W_t$ it follows that $\alpha = -3$.

5.3 2021-2022

1.

5.4 2021-2022 retake

1.

5.5 2020-2021

1.

6 December exam solutions

6.1 2022-2023

1. (a) $dX_t = (W_t^3 - 3W_t^2 \cdot t)dW_t$ (4 points), 1 point for comment how you get the answer (definition and Ito's lemma), 2 points for full form
$$X_t = X_0 + \int_0^t W_u^3 - 3W_u^2 \cdot u \, dWu$$
   (b) A process is a martingale as in short form $A_t dt = 0$ (3 points)
2. (a) $dX_t = -2X_t dW_t$ (2 points), this process is a martingale (1 point)
   (b) $\mathbb{E}(X) = 1$ (2 points), $\text{Var}(X) = \exp(4t) - 1$ (2 points)
   (c)
$$\int_0^t X_u \, dWu = \frac{1 - X_t}{2}$$
   (3 points)
3. (a) 2 points for $\mathbb{E}(W_5 W_4 | W_4) = W_4^2$, 3 points for $\text{Var}(W_5 W_4 | W_4) = W_4^2$
   (b) i. 1-2 points for clever ideas
      ii. 3 points for solution with serious mistakes
      iii. 4 points for solutions with arithmetic errors
      iv. 5 points for $\text{Cov}(W_5 W_4, W_5 W_6) = 40$
4. 1-3 points depending on the cleverness of ideas.
   5 points if one got correct martingale:
$$\mathbb{E}(Y_{t+1} | Y_t) = Y_t \, \mathbb{E}(e^{bX_{t+1}})$$
   10 points if one solved equation correctly:
$$\mathbb{E}(e^{bX_{t+1}}) = 1 \to b = 0 or b = \ln(1/9)$$
   Minus 1 point if one forgot trivial solution a = 0 and b - any
5. (a) $p_u^* = p_d^* = 1/2$ (3 points)
   (b) $X_1^u = X_1^d = (0.5 \cdot 100 + 0.5 \cdot 0)/1.05$, hence $X_0 = 50/1.05^2 \approx 45.35$ (3 points)
   (c) $\alpha = X_2^{uu} - X_2^{ud}/(S_2^{uu} - S_2^{ud}) = 100/216 \approx 0.46$ (4 points)
6. You get 2 points almost for nothing:
$$X_0 = \exp(-2r) \, \mathbb{E}_*(X_2)$$
   Correct formula for X_2 in terms of W_1^* gives your 4 points:
$$X_2 = S_1^3 = S_0^3 \exp(3r) \exp\big(3\sigma W_1^* - 9\sigma^2/2\big).$$
   Calculations of expected value (4 points more):
$$X_0 = S_0^3 \exp(r) \exp\big(3\sigma^2\big).$$

6.2 2021-2022

1.

6.3 2020-2021

1.

7 April exam solutions

7.1 2023-2024

1. (a) $\mathbb{E}(y_t) = 5$
   (b) $\rho_3 = \rho_4 = \ldots = 0$
   (c) The process is stationary.

2.

3. (a) $\phi_{11} = \rho_1 = 0.2$, $\phi_{22} = 0$
   (b) Possible equation is $y_t = 0.2y_{t-1} + u_t$. Another possibility is $y_t = 5y_{t-1} + u_t$. In the second case the stationary solution will be forward-looking and not $MA(\infty)$ with respect to (u_t).

4. (a) $\lambda_1 = 2$, $\lambda_2 = 0.5$, here the roots of the lag polynomial are exactly the same.
   (b) $(1 - 2L)(1 - 0.5L)(y_t + 10) = u_t$
   (c) The equation has infinitely many non-stationary solutions.
   (d) The equation has unique stationary solution.
   (e) The equation has no stationary solutions that are $MA(\infty)$ with respect to (u_t).

5.

6. The process is not stationary as $\mathbb{E}(y_1) = 3$ and $\mathbb{E}(y_2) = -3$.

7.2 2022-2023

1. (a) [6 points]

$$y_{102} = \ell_{100} + (0.9 + 0.9^2)b_{100} + (0.3 + 0.18)u_{101} + u_{102}$$

$$(y_{102} \mid y_1, \ldots, y_{100}) \sim \mathcal{N}(21.71, 24.608)$$

   The interval

$$[21.71 - 1.96 \cdot 4.96; 21.71 + 1.96 \cdot 4.96]$$

   (b) [4 points]

$$\lim_{h\to\infty} \mathbb{E}(y_{100+h} \mid y_1, \ldots, y_{100}) = \ell_{100} + (0.9 + 0.9^2 + \ldots)b_{100} = 20 + 9 \cdot 1$$

2. (a) [2 points] $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, one stationary solution, infinitely many non-stationary solutions.
   (b) [6 points]: [2 points] for the system + [2 points] for ρ_1 + [2 points] for ρ_2.

$$\begin{cases} \gamma_1 = 0.7\gamma_0 - 0.12\gamma_1 \\ \gamma_2 = 0.7\gamma_1 - 0.12\gamma_0. \end{cases}$$

$$\rho_1 = 70/112 = 0.625, \quad \rho_2 = 49/112 - 0.12 = 0.3175$$

   (c) [2 points]

$$\alpha_1 = 0.7, \quad \alpha_2 = 0.37$$

3. (a) [4 points]

$$\sigma_{101}^2 = 3 + 0.5(-1)^2 = 3.5$$

$$(u_{101} \mid \sigma_{101}) \sim \mathcal{N}(0; \sigma_{101}^2)$$

$$[-1.96\sqrt{3.5}; +1.96\sqrt{3.5}]$$

   (b) [3 points] [1 point] for $\mathbb{E}(u_t)$ and [2 points] for $\mathrm{Var}(u_t)$ The process (u_t) is a white noise, hence

$$\mathbb{E}(u_t) = 0.$$

$$\sigma_u^2 = 3 + 0.5 \cdot \sigma_u^2$$

(c) [3 points]: [1 point] for $\text{Corr}(u_t, u_{t-1})$ and [2 points] for $\text{Corr}(u_t^2, u_{t-1}^2)$ The process (u_t) is a white noise, hence

$$\text{Corr}(u_t, u_{t-1}) = 0.$$

$$u_t^2 = 3 + 0.5u_{t-1}^2 + (u_t^2 - \sigma_t^2)$$

We notice that $r_t = u_t^2 - \sigma_t^2$ is a white noise, hence u_t^2 is an $AR(1)$ process. Hence, $\text{Corr}(u_t^2, u_{t-1}^2) = 0.5$.

4. (a) [5 points]

$$L = const(0.2 + a)^{N_1}(0.3 - a)^{N_2}0.5^{N_3}$$

$$\ell = const + N_1 \ln(0.2 + a) + N_2 \ln(0.3 - a) + N_3 \ln 0.5$$

$$\frac{\partial \ell}{\partial a} = \frac{N_1}{0.2 + a} - \frac{N_2}{0.3 - a}$$

$$\hat{a}_{ML} = \frac{0.3N_1 - 0.2N_2}{N_1 + N_2}$$

We see that $\partial\ell/\partial a$ decreases as a increases, so $\hat{a}_{ML}$ is indeed the point of maximum.

(b) [5 points]

$$\mathbb{E}(Y_i) = (0.2 + a) + 2(0.3 - a) + 4 \cdot 0.5 = 2.8 - a$$

$$\bar{Y} = \frac{N_1 + 2N_2 + 4N_4}{N_1 + N_2 + N_4}$$

$$\hat{a}_{MM} = 2.8 - \frac{N_1 + 2N_2 + 4N_4}{N_1 + N_2 + N_4}$$

5. (a) [6 points]

$$\mathbb{P}(\hat{\theta} > y) = \mathbb{P}(Y_1 > y/n)^n = (\exp(-y/n\theta))^n = \exp(-y/\theta)$$

Hence $\hat{\theta}$ has exponential distribution with rate $1/\theta$ and probability density function

$$f(t) = \begin{cases} \exp(-t/\theta)/\theta, & \text{if } t \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) [2 points]

The estimator is unbiased as

$$\mathbb{E}(\hat{\theta}) = 1/(1/\theta) = \theta.$$

(c) [2 points]

The estimator is non consistent as its distribution does not depend on n.

7.3 2021-2022

1.

2. (a) Yes, the process is stationary, that is $MA(2)$ process.

(b) $\rho_3 = \rho_4 = \ldots = 0$

(c) $d_t = u_t + u_{t-1} + 2u_{t-2} - u_{t-1} - u_{t-2} - 2u_{t-3}$, hence $d_t \sim ARIMA(0, 0, 3)$.

3. (a)

(b) $\phi_{11} = \rho_1$, $\phi_{22} = -0.2$, $\phi_{33} = \phi_{44} = \ldots = 0$. The partial correlation ϕ_{22} measures how will y_t on average react to the unit change of y_{t-2} given fixed y_{t-1}.

(c) $\rho_{100} = -0.9\rho_{99} - 0.2\rho_{98}$

7.4 2020-2021

1.

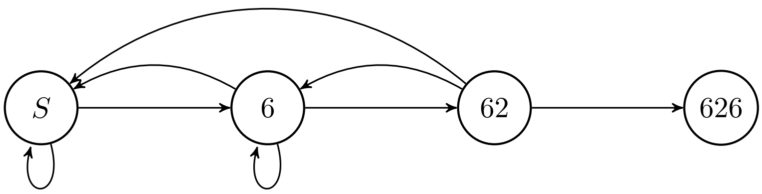
8 Final exam solutions

8.1 2021-2022

1.

8.2 2020-2021

1. Let's draw the chain



The system of equations for expected values:

$$\begin{cases} x_s = 1 + \frac{1}{6}x_6 + \frac{5}{6}x_s \\ x_6 = 1 + \frac{1}{6}x_6 + \frac{1}{6}x_{62} + \frac{4}{6}x_s \\ x_{62} = 1 + \frac{1}{6}\cdot 0 + \frac{1}{6}x_6 + \frac{4}{6}x_s \end{cases}$$

The system of equations for moment generating functions:

$$\begin{cases} m_s(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{5}{6}m_s(t)\right) \\ m_6(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{1}{6}m_{62}(t) + \frac{4}{6}m_s(t)\right) \\ m_{62}(t) = \exp(t)\left(\frac{1}{6}\cdot 1 + \frac{1}{6}m_6(t) + \frac{4}{6}m_s(t)\right) \end{cases}$$

2. (a) Let's denote by x all available information,

$$x = \begin{pmatrix} y_{100} \\ y_{99} \\ y_{98} \\ u_{99} \end{pmatrix}$$

Let's use $t = 100$:

$$y_{100} = 1 + 0.5y_{98} + u_{100} + u_{99}$$

Using all available information we obtain $u_{100} = 1.5$ and hence

$$y_{101} \mid x \sim \mathcal{N}(1 + 0.5y_{99} + u_{100}; 4)$$

(b) Here we work with true betas:

$$\mathbb{E}(y_{101} \mid y_{100}) = \mu_y + \frac{\operatorname{Cov}(y_{100}, y_{101})}{\operatorname{Var}(y_{100})}(y_{100} - \mu_y)$$

3. (a) Moment generating function

$$m_N(t) = \sum_{j=0} \exp(tj)(1-h)^j h = h\sum_{j=0}(\exp(t)(1-h))^j = \frac{h}{1-\exp(t)(1-h)}$$

(b) As $S = N_1 + N_2 + \ldots + N_k$:

$$m_S(t) = \left(\frac{h}{1-\exp(t)(1-h)}\right)^k$$

(c) Due to my mistake the limit is easy, 0.
In my dream it was $k \to \infty$, $k \cdot (1 - h) \to 0.5$ and that would be fun!

4. (a) Let's use Ito's lemma

$$dX_t = f'(t)\cos(2021W_t)dt - 2021f(t)\sin(2021W_t)dW_t + \frac{1}{2}2021^2 f(t)\cos(2021W_t)dt$$

(b) To make X_t a martingale we should kill dt term.

(c) As X_t is martingale $\mathbb{E}(X_t) = \mathbb{E}(X_0) = f(0)$. So $\mathbb{E}(\cos(2021W_t)) = f(0)/f(t)$.