# Time Series and Stochastic Processes exams

Angry Teachers, Folklore

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## Contents

# Description

See updates at https://github.com/bdemeshev/tssp_exams.
Many more problems can be found at https://github.com/bdemeshev/stochastic_pro.
Click on red hyperlinks inside pdf, you can get to the answers and back!
Any comments? Bugs? https://github.com/bdemeshev/tssp_hse_exams/issues/.
The order of topics has changed substantionally after the first course iteration in 2020-21. The interested reader may find relevant exercises by looking through all 2020-21 exams.

## Greatings to the contributors

Here we describe only the style guidelines and typical erros. For more information on tex one may read the book by K. Vorontsov.

1. Use decimal point as a separator: 3.14 — good style, 3,14 — bad style. This goes against russian tradition, but favors copy-pasting numbers in software for computations.
2. Use `\[...\]` for display math formulas. Do not use $$...$$!
3. Use `cases` for systems of equations, `align*` for multiline formulas, `enumerate` for enumerations.
4. Inside formulas use `\text{...}` to write text.
5. Use `\ldots` for ellipsis.
6. You can find useful macros in the preamble, like `\P, \E, \Var, \Cov, \Corr, \cN`.
7. Use backslash before functions: `\ln, \exp, \cos`...
8. Use booktabs style for tables. You may use online tablesgenerator. Choose booktabs table style instead of default table style.
9. Respect the letter ë! :)
10. Start every sentence in tex source from a new line. There will be no additional newlines in final pdf but tex file will be easier to read.
11. For multiplication use `\cdot`. Please never use * :)

# 1 October exam

## 1.1 2025-2026

**FINAL SALE**: Every problem will bring you 10 ~~99.99~~ points!
Scratch paper price: ~~10 points per sheet only~~ **FREE**[1]!

1. Dragon Erik has three towns to kidnap a princess from: $A$, $B$ and $C$. He kidnaps the first princess from town $A$ and chooses every next town according to a Markov chain with transition matrix $T$, where states are written in alphabetical order:

$$T = \begin{pmatrix} 0.1 & 0.2 & ? \\ ? & 0 & 0.7 \\ 0.2 & ? & 0.5 \end{pmatrix}.$$

   (a) [2] Fill in the missing values.

   (b) [2] Draw the graph of the Markov chain.

   (c) [3] What is the probability that the third princess will be from town $B$?

   (d) [3] What is the probability that the second princess was from $A$ given that the third princess was from $B$?

2. Every minute the cat Tikhon says «meow» with probability $1/3$ or «purr» with probability $2/3$ independently of all other words. For every «purr» that follows «meow» you get 1 PEPE, for every «meow» that follows «purr» you pay 3 PEPEs. For «purr» that follows «purr» or «meow» that follows «meow» you get nothing.

   You have just earned 1 PEPE. Let $T$ be the time to get the next positive reward.

   (a) [5] Find $\mathbb{E}(T)$.

   (b) [5] Find $\mathrm{Var}(T)$.

3. [10] Gumbatali starts with zero initial sum $S_0 = 0$. Every minute he either wins $X_t = 1$ dollar with probability $1/3$ or pays 1 dollar, $X_t = -1$, with probability $2/3$. He can't go neither below $S_t = -2$ nor above $S_t = 3$, so that

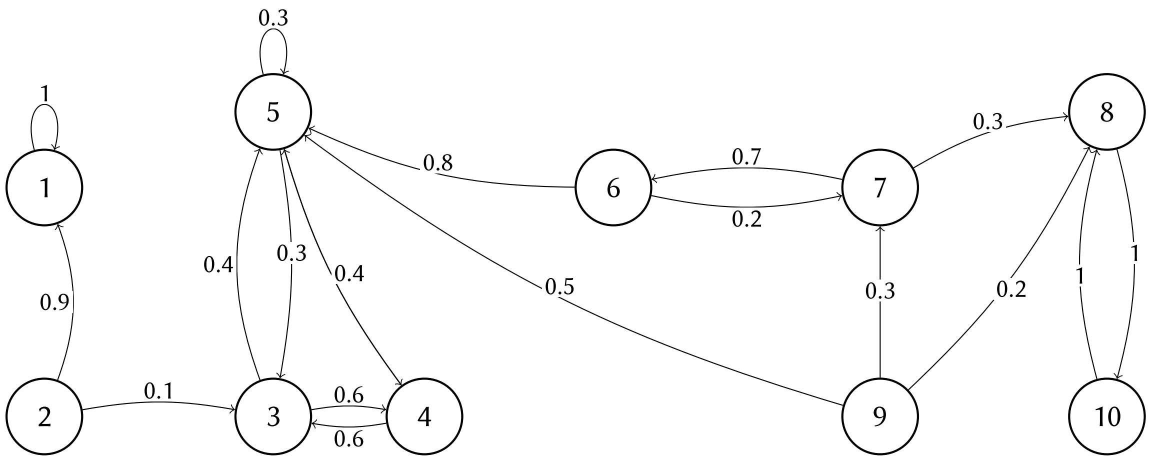
$$S_t = \min\{3, \max\{-2, S_{t-1} + X_t\}\}.$$

   The random variables $(X_t)$ are independent.

   Find the stationary distribution of $S_t$.

[1] Members only deal!

4. Consider the Markov chain:




   (a) [5] Using a chainsaw cut the Markov chain into communicating classes and classify them as transient, positive recurrent or null-recurrent.
   (b) [1] Is the chain irreducible?
   (c) [4] Find the period of every state.

5. Let $X$ be a Poisson distributed random variable with intensity $\lambda = 2$. The probability mass function of a Poisson random variable is:

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots$$

   (a) [4] Derive the moment generating function $M_X(t) = \mathbb{E}(e^{tX})$.
   (b) [4] Using moment generating function find the mean and variance of $X$.
   (c) [2] Let $X_1 \sim \text{Pois}(\lambda = 2)$ and $X_2 \sim \text{Pois}(\lambda = 3)$ be independent random variables. Find the moment generating function of $S = X_1 + X_2$.

6. By the mayor's decree, the city installs $T_n$ New Year's trees each day, $n = 1, 2, \dots$, starting from October 20th. The random variables $T_1, T_2, \dots$ are independent and take any natural value from $4$ to $10$ with equal probabilities.

   The amounts of snow $(W_n)$ are independent and uniformly distributed on $[0; 1/n]$.

   Determine the probability limits for the following sequences:

   (a) [2] mayor's New Year kpi, $X_n = \frac{1}{n}\sum_{i=1}^n T_i$
   (b) [3] the New Year's mood given by $Y_n = \frac{2X_n+1}{X_n+3}$
   (c) [5] The average number of snowmen that can be built each day: $S_n = \frac{1}{n}\sum_{i=1}^n W_i$.

## 1.2 2024-2025

1. [10] Michael stands in a corner of a hexagonal room with white soft walls. Stochastic Processes course has caused him a deep trauma. Michael has insomnia and two independent personal identities. At each iteration each personal identity moves from one vertex of the hexagon to the adjacent one. Two personal identities move independently with equal probabilities in both directions.

   Consider the Markov chain where the state is a relative position of two identities of Michael in a hexagon.

   (a) [4] Draw the diagram of chain states and find the transition matrix.
   (b) [2] Classify the states as transient or recurrent.
   (c) [4] Which proportion of his eternal life Michael spends in a perfect harmony with himself (two personal identities are located at the same vertex)?

2. [10] Compare sigma-algebras and conditional expected values!

(a) [5] Consider sigma-algebras

$$\mathcal{F}_1 = \sigma(X), \quad \mathcal{F}_2 = \sigma(Y), \quad \mathcal{F}_3 = \sigma(X, Y), \quad \mathcal{F}_4 = \sigma(X + Y, X - Y).$$

Which of them are always equal? Which sigma-algebra is always a subset of another one?

(b) [5] Consider random variables

$$R_1 = \mathbb{E}(X \mid Y), \quad R_2 = \mathbb{E}(1/X \mid Y), \quad R_3 = \mathbb{E}(1/X \mid 1/Y), \quad R_4 = \mathbb{E}(X \mid 1/Y).$$

Which of them are always equal provided that they exist?

3. [10] The random variables $(X_n)$ are independent and uniform on $[0; 1]$ and $S_n = X_1 + X_2 + \cdots + X_n$.

(a) [4] Find the moment generating function of $X_1$.

(b) [2] Find the moment generating function of $R = S_{10} - 5$.

(c) [4] Find $\mathbb{E}(X_1 \mid S_3)$.

4. [10] I throw a fair coin. Let $Y$ be the number of throws until I obtain the sequence head-tail-head.

   (a) [4] Find $\mathbb{E}(Y)$.

   (b) [6] Find $\mathbb{E}(Y^2)$ and hence $\text{Var}(Y)$.

5. [10] Consider two non-random sequences, $h_n = 1/n$ and $t_n = 1 + 1/n$. Elon Musk throws a fair coin once and selects the sequence $(h_n)$ if it lands on head and selects the sequence $(t_n)$ otherwise. Hence Elon obtains a random sequence $(X_n)$.

   (a) [2] What is the distribution of $X_5$?

   (b) [4] What is the distribution of $\lim X_n$?

   (c) [4] Write $\text{plim}\, X_n$ explicitely as a function of $X_1$.

6. [10] The random variables $(X_n)$ are independent and they have Poisson distribution with rate $\lambda = 2$. Consider the cumulative sum $S_n = X_1 + X_2 + \cdots + X_n$ with $S_0 = 0$ and the natural filtration $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$.

   (a) [4] Provide two examples of events that belong to $\mathcal{F}_9$ but do not belong to $\mathcal{F}_7$.

   (b) [6] Find all constants $a$ and $b$ such that $M_n = S_n + a + b \cdot n$ is a martingale.

   Hint: if $R \sim \text{Pois}(\lambda)$ then $\mathbb{E}(R) = \lambda$ and $\text{Var}(R) = \lambda$.

### 1.3 2023-2024

Short rules: 90 minutes, offline. You may use one A4 cheat sheet.
Date: 2023-10-21.

1. The hedgehog Melissa starts at the vertex $A$ of a triangle $\Delta ABC$. Each minute she randomly moves to an adjacent vertex with probabilities $\mathbb{P}(A \to B) = 0.7$, $\mathbb{P}(A \to C) = 0.3$, $\mathbb{P}(B \to C) = \mathbb{P}(B \to A) = 0.5$, $\mathbb{P}(C \to B) = \mathbb{P}(C \to A) = 0.5$.

   (a) What is the probability that she will be in vertex $B$ after 3 steps?

   (b) Write down the transition matrix of this Markov chain.

   (c) What proportion of time Melissa will spend in each state in the long run?

2. The number of players $N$ who will win the lottery is a random variable with probability mass function $\mathbb{P}(N = k) = 7 \cdot 0.3^k/3$ for $k \geqslant 1$. Each player will get a random prize $X_i \sim U[0; 1]$. All random variables are independent. Let $S$ be the sum of all the prizes.

   (a) Find $\mathbb{E}(S \mid N)$ and conditional moment generating function $M_{S|N}(u)$.

   (b) Find the unconditional moment generating function $M_S(u)$.

   (c) What is the probabilistic meaning of $M_S''(0) - (M_S'(0))^2$?

   Note: you don't need to calculate the value in (c).

3. Consider the stochastic process $(X_n)$, where $X_0$ is uniform on $[0; 2]$ and $X_n = (1 + X_{n-1})/2$.

   (a) Find $\mathbb{E}(X_n)$ and $\text{Var}(X_n)$.

   (b) Find the probability limit $\text{plim}\, X_n$.

4. Students arrive in the Grusha café according to the Poisson arrival process $(X_t)$ with constant rate $\lambda$.

   The probability of no visitors during 5 minutes is 0.05.

   (a) Find the value of $\lambda$.

   (b) Find the variance and expected number of arrivals between 5 pm and 8 pm.

(c) What is the probability of exactly 5 arrivals between 5 pm and 8 pm?

5. The random variables $X_1$ and $X_2$ are independent and normally distributed, $X_1 \sim \mathcal{N}(1;1)$, $X_2 \sim \mathcal{N}(2;2)$. I choose $X_1$ with probability 0.3 and $X_2$ with probability 0.7 without knowing their values.

   Casino pays me the value $Y$ that is equal to the chosen random variable.

   Let the indicator $I$ be equal to 1 if I choose $X_1$ and 0 otherwise.

   (a) Express $Y$ in terms of $X_1$, $X_2$ and $I$.
   (b) Find $\mathbb{E}(Y \mid I)$, $\text{Var}(Y \mid I)$.
   (c) Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.

6. The joint distribution of $X$ and $Y$ is given in the table

| | $X=-2$ | $X=0$ | $X=2$ |
|---|---|---|---|
| $Y=-1$ | 0.1 | 0.2 | 0.3 |
| $Y=1$ | 0.2 | 0.1 | 0.1 |

   (a) Explicitely find the $\sigma$-algebra $\sigma(X)$.
   (b) How many elements are there in $\sigma(X \cdot Y)$?

### 1.4 2022-2023

Short rules: 120 minutes, online and offline. You may use one A4 cheat sheet.
Date: 2022-10-29

1. [10] The random variables $X_i$ are independend and uniformly distributed on $[0;2]$. Find

$$\operatorname*{plim}_{n\to\infty} \frac{(X_1-\bar{X})^3 + (X_2-\bar{X})^3 + \ldots + (X_n-\bar{X})^3}{n+2022}.$$

2. A Hedgehog starts at the point $x = 2$ on the real line. Every minute he moves one step left with probability 0.3 or one step right with probability 0.7. There are two exceptions from this rule: the absorbing point $x = 0$ and the reflecting barrier at $x = 3$.

   If the Hedgehog reaches the absorbing point $x = 0$ then he stops moving and stays there. If the Hedgehog reaches the reflecting barrier $x = 3$ then his next move will be one step left with probability 1.

   (a) [2] Write the transition matrix of this Markov chain.
   (b) [3] What is the probability that Hedgehog will be at $x = 1$ after exactly 3 steps?
   (c) [5] What is the expected time to reach the absorbing state?

3. The random variables $X_i$ are independent and they take values $+1$ or $-1$ with equal probability.

   (a) [3] Explicitely list all the events in sigma-algebra $\sigma(X_1 \cdot X_2)$.
   (b) [3] Pavel says that he knows only whether $X_1$ and $X_3$ are equal. How will you describe his knowledge with sigma-algebra?
   (c) [4] How many events are in the sigma-algebra $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3)$?

4. Masha receives on average 10 sms per minute. Sms arrival is well described by the Poisson process.

   (a) [3] What is the probability that Masha receives exactly 10 sms in the next 40 seconds?
   (b) [3] Masha just received an sms. What is the probability that she will wait more that 2.5 seconds before the next one?

(c) [4] Find the covariance between the number of sms in the first 3 minutes and the number of sms in the first 10 minutes.

5. The random variables $X_i$ are independent and they take values $+1$ or $-1$ with equal probability.

   (a) [3] Find $\mathbb{E}(X_3 \mid X_1, X_2)$, $\mathbb{E}(X_3 \mid X_1 + X_3)$.

   (b) [3] Find $\operatorname{Var}(X_3 \mid X_1, X_2, X_3)$, $\operatorname{Var}(X_3 \mid X_1 + X_3)$.

   (c) [4] Let $Y_n$ be equal to $\mathbb{E}(X_1 + \ldots + X_{2022} \mid X_1, X_2, \ldots, X_n)$.
   Is the process $Y_1$, $Y_2$, ..., $Y_{2022}$ a martingale?

6. Consider a Wiener process $(W_t)$.

   (a) [4] Let $Y_t = tW_{2t}$. What is the distribution of $Y_t - Y_s$ for $t \geqslant s$? Is $Y_t$ a Wiener process?

   (b) [6] Find a constant $\alpha$ such that $M_t = W_t^3 + \alpha t W_t$ is a martingale.

### 1.5 2021-2022

Short rules: 120 minutes, online without proctoring. You may use any source you want but don't cheat.
Date: 2021-10-28

1. (10 points) Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0 & 0.1 & 0.9 \\ 0 & 0 & 0.8 & 0.2 \end{pmatrix}.$$

   (a) (3 points) Split the chain in classes and classify them into closed or not closed.

   (b) (2 points) Classify the states into recurrent or transient.

   (c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.
   What is the approximate probability that the Hedgehog will be in the state four after $10^{2021}$ moves?

   Note: state number is the row (or column) number.

2. (10 points) Gleb Zheglov catches one criminal every day. With probability 0.2 the catched criminal is replaced by $w$ new criminals. Initially there are $n$ criminals in the town.

   What is the expected time to the ultimate crime eradication in the town?

   (a) (4 points) Solve the problem for $w = 1$ and $n = 1$.

   (b) (6 points) Solve the problem for arbitrary $w$ and $n$.

3. (10 points) The random variables $X_i$ are independend and uniformly distributed on $[0; 1]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max \left\{ \frac{\sum_{i=1}^n X_i}{n}, \frac{2\sum_{i=1}^n X_i^2}{n} \right\}.$$

4. (10 points) Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes.

   Let $Y_t$ be the number of taxis that will arrive between 0 and $t$ minutes.

   (a) (2 points) Sketch the expected value of $Y_t$ as a function of $t$.

   (b) (8 points) Sketch the probability $\mathbb{P}(Y_t = Y_{60})$ as a function of $t$.

   Note: special points like intercepts or extrema should be explicitely marked.

5. (10 points) Prince Myshkin throws a fair coin until two consecutive heads appear. Let $N$ be the number of throws.

   Find the moment generating function of $N$.

   Hint: you may use the first step approach.

6. (20 points) Vincenzo Peruggia makes attempts to steal the Mona Lisa painting until the first success. Each attempt is successful with probability $0.1$.

   Let $X$ be the number of attempts and $Z = \min\{X, 5\}$.

   (a) (5 points) How many events are in sigma-algebras $\sigma(Z)$ and $\sigma(X)$?

   (b) (5 points) If possible provide an example of events $A$ and $B$ such that: $A \in \sigma(Z)$ but $A \notin \sigma(X)$; $B \in \sigma(X)$ but $B \notin \sigma(Z)$.

   (c) (10 points) Find $\mathbb{E}(Z \mid X)$ and $\mathbb{E}(X \mid Z)$.

## 1.6 2021-2022 retake

Short rules: 120 minutes, online without proctoring. You may use any source you want but don't cheat.

1. (10 points) Consider the Markov chain with the transition matrix

$$P = \begin{pmatrix} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0.3 & 0.3 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) (3 points) Split the chain in classes and classify them into closed or not closed.

   (b) (2 points) Classify the states into recurrent or transient.

   (c) (5 points) A Hedgehog starts in the state one and moves randomly between states according to the transition matrix.
   What is the approximate probability that the Hedgehog will be in the state four after $10^{2021}$ moves?

   Note: state number is the row (or column) number.

2. (10 points) Consider infinite ladder with steps numbered from 0 to infinity. I start at step 0. Every day with probability $u$ I go one step up. With probability $d$ I go one step down. With probability $1 - u - d$ I stay on the same step.

   If I am at step 0 then I stay there with probability $1 - u$ because it's impossible to go down.

   Consider the case $d > u$.

   What is the probability that I will be at step 0 after $10^{1000}$ days?

3. (10 points) The random variables $X_i$ are independend and uniformly distributed on $[0; 2]$. Find the probability limit

$$\operatorname*{plim}_{n\to\infty} \max \left\{ \frac{\sum_{i=1}^{10} X_i}{n}, \frac{\sum_{i=1}^{n} X_i^3}{n+1} \right\}.$$

4. (10 points) Taxis arrive to the station according to the Poisson process with rate 1 per 5 minutes.

   Let $Y_t$ be the number of taxis that will arrive between 0 and $t$ minutes.

   (a) (5 points) Sketch the probability $\mathbb{P}(Y_{t+3} = 1 \mid Y_t = 0)$ as a function of $t$.

   (b) (5 points) Sketch the covariance $\mathrm{Cov}(Y_t, Y_{60})$ as a function of $t$.

   Note: special points like intercepts or extrema should be explicitely marked.

5. (10 points) The moment generating function of a random variable $X$ is $1/(1-2t)$.
   (a) Find the moment generating function of $2X$.
   (b) Find the moment generating function of $X+Y$ where $X$ and $Y$ are independent and identically distributed.
   (c) Do you remember the sum of geometric progression? Find $\mathbb{E}(X^{2021})$.
6. (20 points) Variables $X_1$, $X_2$, ...$X_{100}$ are independent and identically distributed with mean 1 and variance 2. Each $X_i$ has only three possible values: 0, 1, and 2.
   (a) (5 points) How many events are in sigma-algebras $\sigma(X_1, X_2)$ and $\sigma(X_1 - X_2)$?
   (b) (5 points) If possible provide an example of events $A$ and $B$ such that: $A \in \sigma(X_1, X_2)$ but $A \notin \sigma(X_1 - X_2)$; $B \in \sigma(X_1 - X_2)$ but $B \notin \sigma(X_1, X_2)$.
   (c) (10 points) Find $\mathbb{E}(X_1 + \ldots + X_{100} \mid X_1 + \ldots + X_{50})$ and $\mathbb{E}(X_1 + \ldots + X_{50} \mid X_1 + \ldots + X_{100})$.

### 1.7 2020-2021

Here $(W_t)$ denotes the standard Wiener process.
Date: 2020-10-30

1. For $r < s < t < u$ find the following expected values
   (a) $\mathbb{E}((W_u - W_t)^2(W_s - W_r)^2)$;
   (b) $\mathbb{E}((W_u - W_s)(W_t - W_r))$;
   (c) $\mathbb{E}((W_t - W_r)(W_s - W_r)^2)$;
   (d) $\mathbb{E}(W_r W_s W_t)$;
   (e) $\mathbb{E}(W_r W_s W_t \mid W_s)$;
2. Consider Ito process $X_t$
$$dX_t = \exp(t) W_t \, dt + \exp(2W_t) \, dW_t, \quad X_0 = 1.$$
   Consider two processes, $A_t = 1 + t^2 + X_t^3$ and $B_t = 1 + t^2 + X_t^3 W_t^4$.
   (a) Find $dA_t$ and $dB_t$.
   (b) Write the corresponding explicit expressions for $A_t$ and $B_t$:
$$const + \int_0^t \ldots dW_u + \int_0^t \ldots du$$
   (c) Check whether $X_t$ is a martingale.
3. Let $S_0 = 0$, $S_t = X_1 + X_2 + \ldots + X_t$. The increments $X_t$ are independent and identically distributed:

| $x$ | $-1$ | $0$ | $1$ |
|---|---|---|---|
| $\mathbb{P}(X_t = x)$ | 0.2 | 0.2 | 0.6 |

   (a) If possible find all constants $a$ such that $M_t = S_t + at$ is a martingale.
   (b) If possible find all constants $b$ such that $R_t = b^{S_t}$ is a martingale.
4. Consider the process $X_t$
$$X_t = tW_t + \int_0^t u W_u^2 \, dW_u.$$
   (a) Find $\mathbb{E}(X_t)$, $\mathrm{Var}(X_t)$.

(b) Find $dX_t$.

(c) Check whether $X_t$ is a martingale.

5. A Hedgehog in the fog starts in $(0, 0)$ at $t = 0$ and moves randomly with equal probabilities in four directions (north, south, east, west) by one unit every minute.

   Let $X_t$ and $Y_t$ be his coordinates after $t$ minutes and $S_t = X_t + Y_t$.

   (a) Find $\mathbb{E}(X_2 \mid S_2)$;

   (b) Find $\mathrm{Var}(X_2 \mid S_2)$.

   Hint: $\mathrm{Var}(Y \mid X) = \mathbb{E}(Y^2 \mid X) - (\mathbb{E}(Y \mid X))^2$.

6. Vampire Petr and Markov Chains.

   Vampire Petr drinks blood of a new victim every day. Unfortunately 20% of the population are vaccinated against vampires. If more than one victim of the last three victims are vaccinated then Petr will be instantaneously cured and will return to the normal life.

   For simplicity let's assume that the last three victims were not vaccinated.

   (a) What is the probability that vampire Petr will be cured in the next three days?

   (b) How many victims will be bitten by vampire Petr on average?

7. Vampire Boris and Martingales.

   To survive vampire Boris needs to bite 70 talented students.

   These 70 talented students have formed a secret group. They have written their emails on small pieces of paper and have randomly distributed these pieces among them. Each student has exactly one piece of paper with an email[1].

   Initially vampire Boris knows contacts of just two persons from the group. Today he will contact them, drink their blood and get the emails they have. Then vampire Boris will contact new victims and so on.

   (a) For $t \geqslant 1$ consider the process $M_t$, the proportion of non bitten students after the day $t$.
   Is this process a martingale?

   (b) Using martingale stopping theorem or otherwise find the probability that vampire Boris will bite all 70 students.

[1] The group is so secret that it is possible that a student has his own email on his piece of paper

# 2 December exam

## 2.1 2025-2026b

Official chief mistress of King Louis XV Madame de Pompadour was born exactly 204 years earlier, on 29 December 1721. Her original name was Jeanne Antoinette Poisson.

1. Jeanne Antoinette Poisson receives letters from Diderot and d'Alembert. Each of them independently may be in a good mood or bad mood equiprobably. Every person sends letters to Jeanne Antoinette Poisson according to an independent Poisson process with intensity $\lambda_{\text{good}} = 2$ or $\lambda_{\text{bad}} = 1$ [letters per week] depending on his mood.

   Let $(M_t)$ be the number of letters received by Madame de Pompadour in $t$ weeks.

   (a) [2 + 3] Find $\mathbb{E}(M_t)$ and $\text{Var}(M_t)$.

   (b) [2] Is $(M_t)$ a Poisson process? If so, find its intensity.

   (c) [3] Let $T_1$ be the time of the first letter. Find its cumulative distribution function $F(t)$.

2. We select a point $(X, Y)$ randomly uniformly inside the square with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, and $(0, 4)$.

   (a) [2 + 3] Find $\mathbb{E}(Y \mid Y - X)$ and $\text{Var}(Y \mid Y - X)$.

   (b) [2] Consider the $\sigma$-algebra $\mathcal{F} = \sigma(Y - X)$. Provide an example of two non-trivial events from $\mathcal{F}$ and two events not from $\mathcal{F}$.

   (c) [3] Find $\text{Cov}(X, Y \mid Y - X)$.

   Hint: by definition $\text{Cov}(X, Y \mid \mathcal{F}) = \mathbb{E}(XY \mid \mathcal{F}) - \mathbb{E}(X \mid \mathcal{F})\,\mathbb{E}(Y \mid \mathcal{F})$.

3. Let $S_0 = 0$. Each hour independently of previous history the process $S_n$ increases by $1$ with probability $0.25$, decrease by $1$ with probability $0.25$ or does not change. Consider the first moment of time $T$ when $|S_T| = 4$.

   (a) [3] If possible find $\alpha$ such that $Q_t = S_t^2 + \alpha t$ is a martingale.

   (b) [3] If possible find $\beta$ such that $Q_t = \exp(S_t + \beta t)$ is a martingale.

   (c) [4] Find $\mathbb{E}(T)$.

4. Consider the standard Wiener process $(W_t)_{t \geqslant 0}$.

   (a) [4] Find $\text{Corr}(W_4 - W_2, W_3 - W_1)$.

   (b) [6] Find $\mathbb{E}(W_3 - W_1 \mid W_4 - W_2)$ and $\text{Var}(W_3 - W_1 \mid W_4 - W_2)$.

5. Consider three processes, $A_t = \int_0^t u \, dW_u, \quad B_t = \int_0^t u W_u^4 \, dW_u, \quad C_t = \int_0^t u W_u^2 \, du.$

   (a) [2 + 2] Find $\mathbb{E}(A_t)$, $\mathbb{E}(C_t)$.

   (b) [2 + 4] Find $\text{Var}(B_t)$, $\text{Cov}(A_t, C_t)$.

6. Be brave! Consider the stochastic process $X_t = g(t)(\exp(2026 W_t) + \exp(-2026 W_t))$ where $g(t)$ is some non-random function.

   (a) [6] Find $dX_t$ and hence represent $X_t$ as a sum of a constant, an Ito integreal and a Riemann integral.

   (b) [4] If possible find any $g(t) \neq 0$ such that $X_t$ is a martingale.

## 2.2 2025-2026a

Сегодня по народному календарю Евстратиев день. Нельзя в этот день сквернословить — не то ведьмы с неба упадут прямо на голову. Также нельзя веник на крыльце оставлять — не то ведьмы утащат.

1. The Legendary November Midterm Grader grades tests according to Poisson point process with variable intensity rate. The first 5 weeks the rate is equal to $\lambda_1 = 1$ test per week. Later the rate increases to $\lambda_2 = 100$ tests per week. Let $X_t$ be the number of tests graded after $t$ weeks.

   (a) [2] Sketch the curve $v(t) = \text{Var}(X_t)$.

   (b) [4] Using normal approximation find the probability that LNMG will check not less than 120 works at time $t = 6$.

   (c) [4] Now $t = 4.5$ weeks have already passed. Find the expected time for the next test checked.

   Note: in point (b) you may use the standard normal cdf $F()$ in your answer.

2. The joint distribution of $X$ and $Y$ is given by the table:

   | | $Y=-1$ | $Y=0$ | $Y=1$ |
   |---|---|---|---|
   | $X=-1$ | $1/8$ | $1/16$ | $1/8$ |
   | $X=0$ | $1/16$ | $1/4$ | $1/16$ |
   | $X=1$ | $1/8$ | $1/16$ | $1/8$ |

   Three students are building statistical models with different information sets:

   - Roman Bokhyan: $\mathcal{F}_1 = \sigma(X, Y)$, the $\sigma$-algebra generated by $X$ and $Y$.
   - Mikhail N: $\mathcal{F}_2 = \sigma(X + Y, X - Y)$.
   - Andrey L: $\mathcal{F}_3 = \sigma(X^2 + Y^2)$.

   (a) [3] For each pair of $\sigma$-algebras above determine whether one is contained in the other, they are equal, or neither. Justify your claims.

   (b) [2] Andrey and Mikhail decide to share their information. Compute the number of elements in the smallest $\sigma$-algebra containing both $\mathcal{F}_2$ and $\mathcal{F}_3$.

   (c) [5] Help Andrey and compute $\mathbb{E}(X \mid \mathcal{F}_3)$, $\text{Var}(X \mid \mathcal{F}_3)$.

3. The process $S_n$ models the «accumulated wrath of the heavens». Let $S_0 = 0$. Each hour independently of previous history $S_n$ increases by 1 with probability $1/3$, decrease by 1 with probability $1/6$ or does not change.

   The witches fall from the sky when $S_n$ first reaches the level of 4. Let $T = \min\{n : S_n = 4\}$.

   (a) [4] For which $\alpha \neq 1$ is the process $M_n = \alpha^{S_n}$ a martingale?

   (b) [2] Check whether $Q_n = S_n - n/6$ is a martingale.

   (c) [4] Find $\mathbb{E}(T)$.

   Hint: you may use Doob's theorem without checking technical details.

4. Consider three processes,

$$A_t = \int_0^t W_u^2 \, dW_u, \quad B_t = \int_0^t W_u^4 \, dW_u, \quad C_t = \int_0^t W_u^2 \, du.$$

   (a) [2 + 2] Find $\mathbb{E}(A_t)$, $\mathbb{E}(C_t)$.

   (b) [2 + 2 + 2] Find $\text{Var}(A_t)$, $\text{Cov}(A_t, B_t)$, $\text{Var}(C_t)$.

5. Be brave!

(a) [2 + 3] Find $\mathbb{E}(W_3^4)$, $\mathbb{E}(W_3^4 \mid W_2 = 10)$.

(b) [5] Find $\operatorname{Cov}(W_1 W_2, W_5 W_6)$.

6. Consider the process

$$R_t = \exp\left(\int_0^t u^3 dW_u + t^2\right).$$

(a) [6] Write $R_t$ as a sum of a constant, an Ito's integral and a Riemann integral.

(b) [2] Is $(R_t)$ a martingale?

(c) [2] Provide an ordinary (non-stochastic) differential equation for $h(t) = \mathbb{E}(R_t)$. You don't need to solve this equation.

Hint: you may write $R_t$ as $R_t = \exp(Q_t)$ and apply Ito's lemma :)

### 2.3 2024-2025

In this text $(W_t)$ denotes the standard Wiener process[2].

1. [10] We keep our promises!

(a) [3] What is the distribution of $W_7 + 2W_8$?

(b) [5] What is the conditional distribution of $(W_7 + 2W_8 \mid W_1 = 2)$?

(c) [2] Find the probability $\mathbb{P}(W_7 + 2W_8 > 1 \mid W_1 = 2)$ in terms of a standard normal cdf $F()$.

2. [10] Consider processes $X_t = \int_0^t W_u^3 dW_u$ and $Y_t = \int_0^t W_u^4 du$.

(a) [3 + 3] Find $\mathbb{E}(X_t + Y_t)$ and $\operatorname{Var}(X_t)$.

(b) [4] Find $\operatorname{Cov}(X_t, W_t)$.

3. [10] Let $X_t = (W_t + g(t)) \exp(-W_t - t/2)$ with $X_0 = 0$.

(a) [4 + 2] Find $dX_t$ and write $X_t$ as a sum of two integrals.

(b) [4] Find at least one function $g(t)$ such that $X_t$ is a martingale.

4. [10] It's the midterm exam. Five students are sitting in the last row. The student in the middle of the last row is the only one who brought a calculator.

Every minute his calculator moves one seat to the right (+1) or one seat to the left (-1) with equal probabilities.

Let's $X_n$ be the coordinate of the calculator at time $n$ with $X_0 = 0$.

(a) [4] Check whether $M_n = X_n^2 - n$ is a martingale.

(b) [6] Find the average time for the calculator to reach the end of the row (left or right).

5. [10] Consider a two-period binomial tree model with an initial share price $S_0 = 100$. The up and down share price multipliers are $u = 2$ and $d = 0.5$. Risk-free interest rate is $r = 10\%$ in the first period. The central bank will increase the interest rate in the second period exactly to $r = 20\%$.

The option is a European call with strike price $K = 300$ and maturity $T = 2$.

(a) [4] Find the risk neutral probabilities.

(b) [6] Find the arbitrage free price $X_0$ of this option.

6. [10] The assistant has not checked the home assignments in time. Hence teachers are receiving student complaints according to a Poisson process with daily rate $\lambda = 3$ messages per hour from $7:00$ to $23:00$ and nightly rate $\lambda = 1$ message per hour from $23:00$ to $7:00$.

Let $X$ be the number of messages from $6:00$ to $9:00$ and $Y$ — number of messages from $7:00$ to $9:00$.

(a) [2+2+2] Find $\mathbb{E}(X)$, $\operatorname{Var}(X)$ and $\mathbb{P}(X = 2)$.

(b) [4] Find the conditional distribution of $Y$ given that $X = 2$.

[2] Броуновское движение [ещё пока не запрещено на территории РФ]

## 2.4 2023-2024

1. [10 points] Let $(X_t)$ be independent identically distributed random variables with $\mathbb{P}(X_i = -1) = 0.4$ and $\mathbb{P}(X_i = +1) = 0.6$. Consider the sum $S_t = X_1 + X_2 + \ldots + X_t$.

   (a) [3] Is $S_t$ a martingale?

   (b) [7] Find all constants $c$ such that $M_t = c^{S_t}$ is a martingale.

2. [10 points] Let $a(t)$ be a deterministic function, $M_t = a(t)\cos(3W_t)$ and $(W_t)$ is a Wiener process.

   (a) [4] Find $dM_t$.

   (b) [6] Find a non-zero function $a(t)$ such that $(M_t)$ is a martingale.

3. [10 points] You have two correlated Wiener processes, $(A_t)$ and $(B_t)$, with $\text{Corr}(A_t - A_s, B_t - B_s) = \rho$ for all $t > s$.

   Split the time interval $[0; t]$ into $n$ small segments of equal length. Let $\Delta_i^A$ be the increment of the Wiener process $(A_t)$ on the $i$-th small segment, i.e. $\Delta_i^A = A(it/n) - A((i-1)t/n)$. Let $\Delta_i^B$ be the increment of the Wiener process $(B_t)$ on the $i$-th small segment.

   Consider the sum of cross-products, $S_n = \sum_{i=1}^n \Delta_i^A \Delta_i^B$.

   (a) [3] Find $\mathbb{E}(S_n)$.

   (b) [4] Does $\text{Var}(S_n)$ tend to 0 when $n \to \infty$?

   (c) [2] Find the mean square limit of $S_n$.

   (d) [1] How would you write this limit in a short hand notation with $dA_t$ and $dB_t$?

4. [10 points] The process $(X_t)$ has $X_0 = 2024$, $dX_t = W_t^2 dW_t + W_t dt$, where $(W_t)$ is a Wiener process.

   (a) [2] Is $(X_t)$ a martingale?

   (b) [4] Find $d(X_t W_t)$.

   (c) [4] Find $\text{Cov}(X_t, W_t)$.

5. [10 points] Consider two-period binomial model with initial share price $S_0 = 600$. Up and down share price multipliers are $u = 1.2$, $d = 0.9$, risk-free interest rate is $r = 0.05$ per period.

   The option pays you the maximal share price $X_2 = \max\{S_0, S_1, S_2\}$ at $t = 2$.

   (a) [4] Find the risk neutral probabilities.

   (b) [6] Find the current price $X_0$ of this option.

6. [10 points] Consider the framework of Black and Scholes model with riskless rate $r$, volatility $\sigma$ and initial share price $S_0$.

   Find the current price $X_0$ of an option that pays you $X_T = 1$ at fixed time $T$ if $S_T \geqslant 2S_0$.

   Hint: you may use the standard normal cumulative distribution function in your answer.

## 2.5 2022-2023

Short rules: 120 minutes, you may use two A4 cheat-sheets, offline + one online group.

1. Consider $X_t = \int_0^t W_u^3 dW_u + \int_0^t (W_u^3 + 3W_u u) du - W_t^3 \cdot t$.

   (a) Find $dX_t$ and the corresponding full form.

   (b) Is $X_t$ a martingale?

2. Consider $X_t = \exp(-2W_t - 2t)$.

(a) Find $dX_t$. Is $X_t$ a martingale?

(b) Find $\mathbb{E}(X_t)$ and $\mathrm{Var}(X_t)$.

(c) Find $\int_0^t X_u dW_u$.

3. As usual $(W_t)$ is a Wiener process.

   (a) Find $\mathbb{E}(W_5 W_4 \mid W_4)$, $\mathrm{Var}(W_5 W_4 \mid W_4)$.

   (b) Find covariance $\mathrm{Cov}(W_4 W_5, W_5 W_6)$.

4. Let $X_i$ be independent identically distributed with $\mathbb{P}(X_i = 1) = 0.9$, $\mathbb{P}(X_i = -1) = 0.1$.

   Find all constants $a$ and $b$ such that $Y_t = a \exp\left(b \sum_{i=1}^{t} X_i\right)$ is a martingale.

5. Consider two-period binomial model with initial share price $S_0 = 600$, Up and down multipliers are $u = 1.2$, $d = 0.9$, risk-free interest rate is $r = 0.05$ per period.

   Consider an option that pays you $X_2 = 100$ at $T = 2$ if $S_2 > S_1$ and nothing otherwise.

   (a) Find the risk neutral probabilities.

   (b) Find the current price $X_0$ of the asset.

   (c) How much shares should I have at $t = 1$ in the «up» state of the world to replicate the option?

6. Consider Black and Scholes model with riskless rate $r$, volatility $\sigma$ and initial share price $S_0$.

   Find the current price $X_0$ of an option that pays you $X_2 = S_1^3$ at time $T = 2$.

### 2.6 2021-2022

Short rules: 120 minutes, online without proctoring, $(W_t)$ is a standard Wiener process.
Date: 2021-12-25

1. (10 points) Consider an Ito's process $I_t = 2022 + W_t t^2 + \int_0^t W_u^3 dW_u + \int_0^t W_u^2 du$.

   (a) Find $dI_t$ and check whether $I_t$ is a martingale.

   (b) Check whether $J_t = I_t - \mathbb{E}(I_t)$ is a martingale.

2. (10 points) The random variables $(Z_t)$ are independent identically distributed with moment generating function given by $M_Z(u) = 1/(1-5u)^3$.

   We define $X_t$ as $X_t = \exp(Z_1 + 2Z_2 + 3Z_3 + \ldots + tZ_t)$ with $X_0 = 0$.

   If possible find a martingale of the form $Y_t = h(t)X_t$ where $h()$ is a non-random function.

3. (10 points) The process $(Z_t)$ in discrete time is called *stationary* if it has constant expected value and constant covariances $\gamma_k$ that do not depend on $t$.

$$
\begin{cases}
\mathbb{E}(Z_t) = \mu; \\
\mathrm{Cov}(Z_t, Z_t) = \gamma_0; \\
\mathrm{Cov}(Z_t, Z_{t+1}) = \gamma_1; \\
\mathrm{Cov}(Z_t, Z_{t+2}) = \gamma_2; \\
\ldots
\end{cases}
$$

   (a) If possible provide an example of a martingale that is not stationary.

   (b) If possible provide an example of a stationary process that is not a martingale.

4. (10 points) Find $\mathbb{E}(W_1 W_2 W_3)$ and $\mathbb{E}(W_2 W_3 \mid W_1)$.

5. (10 points) Ded Moroz would like to receive $X_T = S_T^{-1}$ at time $T$ if $S_T < 1$ and nothing otherwise.

   Assume the framework of Black and Scholes model, $S_t$ is the share price, $r$ is the risk free rate, $\sigma$ is the volatility.

   How much Ded Moroz should pay now at $t = 0$?

6. (20 points) Martingales are everywhere :)

   Consider the process $Y_t = \exp(-uW_t)$.

   (a) Find a multiplier $h(u,t)$ such that $M_t = h(u,t) \cdot Y_t$ is a martingale.
   (b) Find $dY_t$, $\mathbb{E}(Y_t)$ and $\mathrm{Var}(Y_t)$.
   (c) Consider $M_t$ that you have found as a function of $u$. Find the Taylor approximation of the function $M_t(u)$ up to $u^4$.
   (d) Consider the coefficient before $u^4$ in the Taylor expansion of $M_t(u)$. Is it a martingale?

7. Bonus point. Guess your exam result (out of 70 possible points).

### 2.7 2020-2021

Today we celebrate Christmas Eve and 78 years of the Narkompros (People's Commissariat for Education) order governing the compulsory use of the letter «ë» in education process.
Date: 2020-12-24

1. Ded Moroz would like to receive $S_1^3$ roubles at time $T = 2$, where $S_t$ is the share price. Assume Black-Schëles model is valid, the risk-free rate is $r = 0.1$ and current share price is $S_0 = 100$.

   How much Ded Moroz should pay now at $t = 0$?

2. Consider stationary $AR(2)$ model, $y_t = 2 + 0.3y_{t-1} - 0.02y_{t-2} + u_t$, where $(u_t)$ is a white noise with $\mathrm{Var}(u_t) = 4$.

   The last two observations are $y_{100} = 2$, $y_{99} = 1$.

   (a) Find 95% predictive interval for $y_{102}$.
   (b) Find the first two values of the autocorrelation function, $\rho_1$, $\rho_2$.
   (c) Find the first two values of the partial autocorrelation function, $\phi_{11}$, $\phi_{22}$.

   Hint: you need no more than 10 seconds to find both partial autocorrelations provided (b) is sëlved.

3. The process $y_t$ is described by a simple $GARCH(1,1)$ model:

$$\begin{cases} y_t = \sigma_t \nu_t \\ \sigma_t^2 = 1 + 0.2y_{t-1}^2 + 0.3\sigma_{t-1}^2 \\ \nu_t \sim \mathcal{N}(0;1) \end{cases}$$

   The variables $\nu_t$ are independent of past variables $y_{t-k}$, $\nu_{t-k}$, $\sigma_{t-k}$ for all $k \geqslant 1$. The prëcesses $y_t$, $\sigma_t^2$ are stationary.

   Given $\sigma_{100} = 1$ and $\nu_{100} = 0.5$ find 95% predictive interval for $y_{102}$.

4. Snegurochka studies a stochastic analog of the Fibonacci sequence

$$y_t = y_{t-1} + y_{t-2} + u_t,$$

   where $(u_t)$ is a white noise process.

   (a) How many non-stationary solutions are there?
   (b) What can you say about the number and the structure of the stationary solutions?
   (c) Can Snëgurochka find two starting constants $y_0 = c_0$ and $y_1 = c_1$ in such a way to make a solution stationary?

Be brave! There are two more exercises!

5. The semi-annual $y_t$ is modelled by $ETS(AAA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ b_t = b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + b_{t-1} + s_{t-2} + u_t \end{cases}$$

   (a) Given that $s_{100} = 2$, $s_{99} = -1.9$, $b_{100} = 0.5$, $\ell_{100} = 4$ find 95% prëdictive interval for $y_{102}$.

   (b) In this problem particular values of parameters are specified. And how many parameters are estimated in semi-annual $ETS(AAA)$ model before real forecasting?

6. The variables $x_t$ take values $0$ or $1$ with equal probabilities. The variables $u_t$ are normal $\mathcal{N}(0; 1)$. All variables are independent.

   Consider the process $z_t = x_t(1 - x_{t-2})u_t$.

   (a) Find the covariance $\mathrm{Cov}(z_t, z_s)$. Is the process $z_t$ stationary?

   (b) Given that $z_{100} = 2.3$ find shërtest predictive intervals for $z_{101}$ and $z_{102}$ with probability of coverage at least 95%.

Bënus: How many letters «ë» have you spotted?

# 3 April exam

## 3.1 2024-2025

1. [10] The process $(u_t)$ is a white noise with $\text{Var}(u_t) = \sigma^2$. Consider the process

$$y_t = (2 + (-1)^t)u_1 + (3 + (-1)^t)u_2.$$

   (a) [4] Find $\mathbb{E}(y_t)$ and $\text{Var}(y_t)$.

   (b) [4] Find $\text{Cov}(y_t, y_s)$.

   (c) [2] Is the process $(y_t)$ stationary?

2. [10] Consider the stationary solution of the equation $y_t = 2 + 0.9y_{t-1} + u_t - 0.5u_{t-1}$, where $(u_t)$ is a white noise process with variance 60.

   (a) [4] If possible rewrite this solution as $AR(\infty)$ process.

   (b) [4] If possible rewrite this solution as $MA(\infty)$ process.

   (c) [2] Find $\text{Cov}(u_t, y_s)$ for this solution.

3. [10] Consider the equation $y_t = 7 + 0.4y_{t-1} - 0.13y_{t-2} + u_t + 2u_{t-1}$, where $(u_t)$ is a white noise.

   (a) [1] How many non-stationary solutions does this equation have?

   (b) [4] How many stationary solutions of $MA(\infty)$ form with respect to $(u_t)$ does this equation have?

   (c) [3] Can we rewrite the stationary solution in $AR(\infty)$ form with respect to $(u_t)$?

   (d) [2] Find $\mathbb{E}(y_t)$ for the stationary solution.

4. [10] Let $(y_t)$ be the solution of the equation $y_t = 2y_{t-1} - y_{t-2} + u_t$, where $(u_t)$ are independent and normally distributed $\mathcal{N}(0; 9)$ and $y_0$ is a constant.

   (a) [5] Find 95% confidence for $y_{101}$ given that $y_{100} = 3$ and $y_{99} = 4$.

   (b) [5] Find 95% confidence for $y_{102}$ given that $y_{100} = 3$ and $y_{99} = 4$.

5. [10] For the stationary solution of the equation $y_t = 0.3y_{t-1} - 0.02y_{t-2} + u_t$, where $(u_t)$ is a white noise process.

   (a) [5] Find the first three values of the autocorrelation function $\rho_1$, $\rho_2$, $\rho_3$.

   (b) [5] Find all values of the partial autocorrelation function $\phi_{kk}$.

6. [10] Let $(y_t)$ be $MA(1)$ process.

   (a) [5] What are the possible values of $\rho_1 = \text{Corr}(y_t, y_{t-1})$?

   (b) [5] What are the possible values of the partial correlation $\phi_{22} = \text{pCorr}(y_t, y_{t-2}; y_{t-1})$?

## 3.2 2023-2024

Short rules: 120 minutes, one A4 cheat sheet allowed.
Date: 2024-03-05

1. Consider $MA(2)$ process given by

$$y_t = 5 + u_t + 2u_{t-1} + 4u_{t-2},$$

   where $(u_t)$ is a white noise with $\text{Var}(u_t) = \sigma^2$.

   (a) [1] Find the expected value $\mathbb{E}(y_t)$.

   (b) [7] Find the autocorrelation function $\rho_k = \text{Corr}(y_t, y_{t-k})$.

   (c) [2] Is the process $(y_t)$ stationary?

2. Consider $MA(2)$ process given by
$$y_t = 5 + u_t + 2u_{t-1} + 4u_{t-2},$$
where $u_t$ are normal independent random variables with $\text{Var}(u_t) = 4$.
You know that $u_{100} = 2$ and $u_{99} = -1$.

   (a) [5] Find the 95% predictive interval for $y_{101}$.
   (b) [5] Find the 95% predictive interval for $y_{1000001}$.

3. The stationary process $(y_t)$ has autocorrelation function $\rho_k = 0.2^k$ and expected value 100.

   (a) [7] Find the first two values of the partial autocorrelation function, $\phi_{11}$ and $\phi_{22}$.
   (b) [3] Provide a possible linear recurrence equation for this process. Your equation may include $y_t$, its lags and a white noise process $(u_t)$.

4. Consider the equation $y_t = 5 + 2.5y_{t-1} - y_{t-2} + u_t$, where $(u_t)$ is a white noise process.

   (a) [3] Find the roots of the corresponding characteristic equation.
   (b) [4] Rewrite the process as $A(L)(y_t - \mu) = u_t$. You should explicitely write the lag polynomial $A(L)$ and the value of $\mu$.
   (c) [1] How many non-stationary solutions does the equation have?
   (d) [1] How many stationary solutions does the equation have?
   (e) [1] How many stationary solutions of the $MA(\infty)$ form with respect to $(u_t)$ does the equation have?

5. [10] The semi-annual $(y_t)$ is modelled by $ETS(ANA)$ process:
$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \end{cases}$$
Given that $s_{100} = 3$, $s_{99} = -2$, $\ell_{100} = 100$ find 95% predictive interval for $y_{102}$.

6. [10] The semi-annual $(y_t)$ is modelled by $ETS(ANA)$ process:
$$\begin{cases} u_t \sim \mathcal{N}(0; 4) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \\ \ell_0 = 100, s_0 = -3, s_{-1} = 3 \end{cases}$$
Check whether the process $(y_t)$ is stationary.

### 3.3 2022-2023

Short rules: 90 minutes, one A4 cheat sheet allowed.
Date: 2023-03-25

1. Consider $ETS(AAdN)$ model
$$\begin{cases} u_t \sim \mathcal{N}(0; 20) \\ b_t = 0.9b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + 0.9b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + 0.9b_{t-1} + u_t \end{cases}$$
with $\ell_{100} = 20$ and $b_{100} = 1$.

(a) Find 95% prediction interval for $y_{102}$.

(b) Approximately find the best point forecast for $y_{10000}$.

2. Consider the difference equation:

$$y_t = 0.7y_{t-1} - 0.12y_{t-2} + u_t,$$

where $(u_t)$ is a white noise.

(a) How many stationary and non-stationary solutions does the difference equation have?

Consider stationary $AR(2)$ process that satisfies the difference equation.

(b) Find first two values of autocorrelation function.

(c) Find $\alpha_1$ and $\alpha_2$ in $MA(\infty)$ representation

$$y_t = u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \alpha_3 u_{t-3} + \ldots$$

3. The strictly stationary white noise $(u_t)$ follows $ARCH(1)$ model $\sigma_t^2 = 3 + 0.5u_{t-1}^2$ where $u_t = \sigma_t \nu_t$ and $\nu_t \sim \mathcal{N}(0; 1)$.

(a) Find 95% prediction interval for $u_{101}$ given that $u_{100} = -1$.

(b) Find $\mathbb{E}(u_t)$, $\mathrm{Var}(u_t)$.

(c) Find $\mathrm{Corr}(u_t, u_{t-1})$, $\mathrm{Corr}(u_t^2, u_{t-1}^2)$.

4. The weight of a fish $Y_i$ is a discrete random variables with distribution and observed frequencies given in the table

| Weight [kg] | 1 | 2 | 4 |
|---|---|---|---|
| Probability | $0.2 + a$ | $0.3 - a$ | $0.5$ |
| Observed frequency | $N_1$ | $N_2$ | $N_4$ |

Fish weights $Y_i$ are independent.

(a) Find the maximum likelihood estimator of the parameter $a$.

(b) Find the method of moments estimator of the parameter $a$.

5. You observe time between taxi arrivals on a stop, $Y_1, Y_2, \ldots, Y_n$. Assume that $Y_i$ are independent and exponentially distributed with $\mathbb{E}(Y_i) = \theta$, that means the density of each $Y_i$ is $f(y) = \exp(-y/\theta)/\theta$ for $y \geqslant 0$. Consider the following estimator of expected value

$$\hat{\theta} = n \cdot \min\{Y_1, Y_2, \ldots, Y_n\}$$

(a) Find the probability density function of $\hat{\theta}$.

(b) Is $\hat{\theta}$ unbiased?

(c) Is $\hat{\theta}$ consistent?

### 3.4 2021-2022

Short rules: 120 minutes, one A4 cheat sheet allowed.
Date: 2022-04-04

1. Consider $ETS(AAN)$ model, $\begin{cases} y_t = \ell_{t-1} + b_{t-1} + u_t \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t \\ b_t = b_{t-1} + \beta u_t \\ u_t \sim \mathcal{N}(0; \sigma^2). \end{cases}$

Let $\ell_{100} = 50$, $b_{100} = 2$, $\alpha = 0.4$, $\beta = 0.5$, $\sigma^2 = 16$.

Calculate one step and two steps ahead 95% predictive intervals.

2. Consider the process $y_t = 4 + u_t + u_{t-1} + 2u_{t-2}$, where $(u_t)$ is a white noise with variance 16.

   (a) Is this process stationary? Explain.

   (b) Find the autocorrelation function of this process. Explain the meaning of $\rho_2$.

   (c) Consider the process $d_t = \Delta y_t$. Is it $ARIMA(p, d, q)$? If yes, then find $p$, $d$ and $q$.

3. Consider the stationary $AR(2)$ process $y_t = 5 - 0.9y_{t-1} - 0.2y_{t-2} + u_t$, where $(u_t)$ is a white noise.

   (a) Find the first value of autocorrelation function $\rho_1$.

   (b) Find the partial autocorrelation function of this process. Explain the meaning of $\phi_{22}$.

   (c) What is the relationship between values of autocorrelation function $\rho_{100}$, $\rho_{99}$ and $\rho_{98}$.

   Hint: values $\phi_{22}$, $\phi_{33}$ etc may be calculated almost effortlessly :)

4. Consider iid sample from bivariate normal distribution, $\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \theta \\ 2\theta \end{pmatrix}; \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}\right)$.

   Calculate Fischer information for the following cases:

   (a) You observe $X_1$ only.

   (b) You observe $X_1$, ..., $X_n$.

   (c) You observe $X_1$, ..., $X_n$, $Y_1$, ..., $Y_n$.

   Hint: the multivariate normal density is $f(u) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp(-\frac{1}{2}(u-\mu)^T\Sigma^{-1}(u-\mu))$.

5. Random variables $X_1$, ..., $X_n$ are independent with density $f(x) = \begin{cases} -\ln(a) \cdot a^x, & \text{if } x \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$

   (a) Estimate $a$ using maximum likelihood.

   (b) Check whether the estimator is unbiased and consistent.

   (c) Check whether the corresponding Cramer-Rao lower bound is attained.

6. Consider the $ARCH(1)$ model, $u_t = \sigma_t \nu_t$, where $\nu_t$ are iid $\mathcal{N}(0; 1)$ and $\sigma_t^2 = 1 + 0.3u_{t-1}^2$.

   (a) Find 95% predictive interval for $u_{101}$ if $u_{100} = -2$.

   (b) Find the autocorrelation function of $r_t = u_t^2$.

### 3.5 2020-2021

Date: 2021-04-13, Rock 'N' Roll day

**Estimation questions**

1. To go to the mountain top I use a gondola lift in the morning. I go back from the top using the same gondola lift in the evening. Cabins are numbered from 1 to $a$.

   I have noticed that the absolute difference of cabin numbers of my two trips was 10.

   (a) Estimate $a$ using maximum likelihood.

   (b) Estimate $a$ using method of moments.

2. Random variables $X_1$, $X_2$, ..., $X_n$ are independent identically distributed with density

$$f(x_i \mid \lambda, a) = \frac{\lambda}{2} \exp(-\lambda|x_i - a|).$$

   Observed values for $n = 3$ are $-3$, 1, 11.

(a) Estimate $\lambda$ using method of moments for fixed $a = 1$.

(b) Estimate $\lambda$ and $a$ using maximum likelihood.

3. Random variables $X_1$, ..., $X_n$ are independent and normally distributed $\mathcal{N}(1, 1/b)$.

   (a) Estimate $b$ using maximum likelihood.

   (b) Does the estimator achive the Cramer-Rao lower bound?

   (c) Is the estimator consistent?

   (d) Is the estimator unbiased?

4. Random variables $X_1$, $X_2$, ..., $X_n$ are independent identically distributed with density

$$f(x_i \mid \lambda) = \frac{\lambda}{2} \exp(-\lambda |x_i|).$$

   For $n = 100$ I have 40 negative values with sum equal to $-300$ and 60 positive values with sum equal to $500$.

   (a) Test the hypothesis $\lambda = 1$ using LR approach at significance level $\alpha = 0.01$.

   (b) Test the hypothesis $\lambda = 1$ using LM approach at significance level $\alpha = 0.01$.

**Distribution questions**

5. I have three problems in the home assignment. Time spent on each problem is modelled by independend exponentially distributed random variables with rate $\lambda$: $X_1$, $X_2$, $X_3$.

   (a) Find the moment generating function of $X_i$ and hence the moment generating function of $S = X_1 + X_2 + X_3$.

   (b) Find $\mathbb{E}(S^3)$.

   (c) Find the joint density of $R = X_1/(X_1 + X_2 + X_3)$ and $S$.

6. I have 100 numbers written on small sheets of paper: $x_1$, $x_2$, ..., $x_{100}$. The sum of these numbers is 1.

   Find the possible values of the sum

$$\frac{x_1}{\sqrt{1 - x_1}} + \frac{x_2}{\sqrt{1 - x_2}} + \ldots + \frac{x_{100}}{\sqrt{1 - x_{100}}}.$$

   Hint: consider a randomly selected number $X$ and apply the Jensen's inequality.

# 4 Final exam

## 4.1 2024-2025

1. Unobserved random variables $(y_i)$ are independent and uniform on $[-2a, a]$ with unknown $a > 0$.

   (a) [4] Find $\mathbb{E}(y_i^2)$ and $\mathbb{E}(|y_i|)$.

   (b) [3] Find the method of moments estimator of $a$ if you know the value of $\sum y_i^2$.

   (c) [3] Find the method of moments estimator of $a$ if you know the value of $\sum |y_i|$.

2. Dragon Erik receives gifts from two kingdoms: $x_1$, $x_2$, ..., $x_{10}$ from Ex-Kingdom and $y_1$, $y_2$, ..., $y_{20}$ from Why-Kingdom. Variables $(x_i)$ are exponentially distributed with rate $\lambda_x$ and variables $(y_i)$ — with rate $\lambda_y$. All variables are independent. Erik has two hypothesis, $H_0$: $\lambda_x = \lambda_y$ and $H_1$: $\lambda_x \neq \lambda_y$. In the observed sample $\sum x_i = 300$, $\sum y_i = 500$.

   (a) [4] Find the unrestricted maximum likelihood estimates of $\lambda_x$ and $\lambda_y$.

   (b) [3] Find the maximum likelihood estimate of the rate under $H_0$.

   (c) [3] Test $H_0$ against $H_1$ at 0.05 significance level using likelihood ratio test.

   Note: $\mathbb{P}(\chi_1^2 \geqslant 3.84) = 0.05$, $\mathbb{P}(\chi_2^2 \geqslant 5.99) = 0.05$, $\mathbb{P}(\chi_3^2 \geqslant 7.81) = 0.05$.

3. The variables $(X_i)$ are independent and have exponential distribution with unknown rate $\lambda$.

   Yuki-the-Hedgehog would like to estimate the unknown parameter $a = 1/\lambda^2$. He uses the estimator

$$\hat{a} = \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{2n + 1}$$

   (a) [5] Is the estimator unbiased?

   (b) [5] Is the estimator consistent?

4. Princess Lika Pankova would like to test the hypothesis $H_0$ that $y_1$ and $y_2$ are independent and uniform on $[0, 1]$ against the alternative that they have the joint density $f(y_1, y_2) = 2 - y_1 - y_2$ for $y_1, y_2 \in [0, 1]$.

   Lika accepts only the best!

   (a) [7] Construct the test with 0.1 probability of the first type error and lowest probability of the second type error.

   (b) [3] What is the minimal achieved second type error probability?

5. Arina loves all kinds of raisin-studded Easter bread, but her grandmother's is her absolute favorite. Grandmother's Easter bread sizes $(y_i)$ are independent identically distributed with density $f(y) = ab^a/y^{a+1}$ for $y \geqslant b$ and zero otherwise.

   (a) [2] Draw a typical density for some values of parameters $a$ and $b$.

   (b) [4] Find a sufficient statistic for the unknown $a$ given that $b = 1$ kilo.

   (c) [4] Find a sufficient statistic for the unknown $a$ and $b$.

6. The semi-annual $(y_t)$ is modelled by $ETS(ANA)$ process:

$$\begin{cases} u_t \sim \mathcal{N}(0; 9) \\ s_t = s_{t-2} + 0.1u_t \\ \ell_t = \ell_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + s_{t-2} + u_t \end{cases}$$

   Let $s_{100} = 2$, $s_{99} = -3$, $\ell_{100} = 200$.

   (a) [6] Find 95% predictive interval for $y_{102}$.

   (b) [4] Write this model in the form $A(L)y_t = B(L)u_t$, where $A(L)$ and $B(L)$ are lag polynomials.

### 4.2 2023-2024

Short rules: 120 minutes, you may use one A4 cheat-sheet, offline
Date: 2024-04-27

1. The variables $X_1$, ..., $X_n$ are independent identically distributed with density

$$f(x) = \begin{cases} \lambda \exp(-\lambda(x - \theta)), & \text{if } x \geqslant \theta \\ 0, & \text{otherwise.} \end{cases}$$

   (a) [5] Find the method of moments estimator of $\lambda$ for known value $\theta = 1$ using the first moment.

   (b) [5] Find the method of moments estimator of $\lambda$ for unknown value $\theta$ using the first two moments.

2. [10] The variables $X_1$, ..., $X_n$ are independent and normally distributed $\mathcal{N}(a, 2a)$.

   Find the maximum likelihood estimator of $a$.

   Hint: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x - \mu)^2/2\sigma^2)$.

3. The variables $X_1$, ..., $X_n$ are independent and uniformly distributed $\mathrm{U}[0; a]$ with $a > 1$. We do not observe $X_i$ directly but we know whether each $X_i$ is larger than 1. Hence we observe the indicators $Y_i = I(X_i > 1)$.

   Consider the estimator $\hat{a} = 1/(1 - \bar{Y})$.

   (a) [5] Is $\hat{a}$ consistent?

   (b) [5] Is $\hat{a}$ unbiased for $n = 2$?

4. The variables $X_1$, ..., $X_n$ are independent and have Poisson distribution with intensity rate $\lambda$. In other words the probability mass function is given by $\mathbb{P}(X_i = k) = \exp(-\lambda)\lambda^k/k!$.

(a) [5] Find theoretical Fisher information for $\lambda$ contained in the sample.

(b) [2] Derive the maximum likelihood estimator for $\lambda$.

(c) [3] Does the maximum likelihood estimator attain the Cramer-Rao lower bound for variance?

5. [10] The variables $X_1$, ..., $X_n$ are independent and gamma distributed with density

$$f(x) = \begin{cases} \lambda^\alpha x^{\alpha-1} \exp(-\lambda x)/\Gamma(\alpha), & \text{if } x \geqslant 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) [5] Find a sufficient statistic for $\alpha$ if we know that $\lambda = 1$.

(b) [5] Find a two dimensional sufficient statistic for unknown $\alpha$ and $\lambda$.

6. We have two independent random samples $X_1$, $X_2$, ..., $X_{n_x}$ and $Y_1$, $Y_2$, ..., $Y_{n_y}$. The random variables $X_i$ follow Poisson distribution with intensity rate $\lambda_x$, random variables $Y_i$ follow Poisson distribution with intensity rate $\lambda_y$.

We would like to test $H_0$: $\lambda_x = \lambda_y$ against $H_1$: $\lambda_x \neq \lambda_y$.

(a) [3] Find the maximal value of log-likelihood under $H_0$.

(b) [3] Find the maximal value of log-likelihood under unrestricted model.

(c) [2] Construct the likelihood ratio test.

(d) [2] Do you reject $H_0$ if $n_x = 100$, $n_y = 200$, $\sum x_i = 500$, $\sum y_i = 900$ at significance level 5%?

Hint: chi-squared critical values for $\alpha = 0.05$ are $\chi^2_{df=1} = 3.84$, $\chi^2_{df=2} = 5.99$.

### 4.3 2022-2023

Short rules: 120 minutes, you may use one A4 cheat-sheet, offline + online.
Notes: $W_t$ denotes the standard Wiener process, you may use standard normal cumulative distribution function in your answers.
Date: Balalayka day, 2023-06-23.

1. The weight of a fish $Y_i$ is a discrete random variables with distribution and observed frequencies given in the table

| Weight [kg] | 1 | 2 | $a$ |
|---|---|---|---|
| Probability | $0.2 + 0.1a$ | $0.3 - 0.1a$ | 0.5 |
| Observed frequency | $N_1$ | $N_2$ | $N_a$ |

Fish weights $Y_i$ are independent, $a > 10$ is unknown.

(a) Find the method of moments estimator of the parameter $a$.

(b) Find the maximum likelihood estimator of the parameter $a$.

2. The $ETS(AAdN)$ model is given by the system

$$\begin{cases} u_t \sim \mathcal{N}(0; 20) \\ b_t = 0.9b_{t-1} + 0.2u_t \\ \ell_t = \ell_{t-1} + 0.9b_{t-1} + 0.3u_t \\ y_t = \ell_{t-1} + 0.9b_{t-1} + u_t \end{cases}$$

with $\ell_{100} = 20$ and $b_{100} = 2$.

(a) Find conditional probability $\mathbb{P}(y_{102} > 30 \mid \ell_{100}, b_{100})$.

(b) Approximately find the best point forecast for $y_{10000}$.

3. Stochastic process $X_t$ is defined by $X_t = 7 + u_t + 0.3u_{t-1}$, where $(u_t)$ is a white noise with variance $\sigma^2$.
    (a) Is $(X_t)$ stationary?
    (b) Find the autocorrelation function of $(X_t)$.
    (c) Find $\mathbb{E}(X_{t+2} \mid X_t, X_{t-1}, \ldots)$.
4. Consider the process $X_t = \int_0^t W_u^2 dW_u + \int_0^t (W_u^2 + 2W_u u)du - W_t^2 \cdot t$.
    (a) Find $dX_t$ and the corresponding full form.
    (b) Is $X_t$ a martingale?
    (c) Find $\mathbb{E}(X_t)$.
5. Consider the Black and Scholes model with riskless rate $r$, volatility $\sigma$ and initial share price $S_0$.

    Find the current price $X_0$ of an option that pays you one dollar at time $T = 2$ only if $S_2 > \exp(3r)S_0$.
6. A hedgehog moves at random on the vertices $A$, $B$, $C$ and $D$ of a regular tetrahedron (тетраэдр). She start at the vertex $A$ and every minute changes her position to one of the adjacent vertices with probability $1/3$ independently of past moves.
    (a) Write down the transition matrix of this Markov chain.
    (b) What is the expected time of the first return to the starting vertex $A$?

### 4.4 2021-2022

Short rules: 120 minutes, offline, one A4 cheat sheet allowed.
Date: 2022-06-25

1. Consider $ETS(ANN)$ model, $\begin{cases} y_t = \ell_{t-1} + u_t \\ \ell_t = \ell_{t-1} + \alpha u_t \\ u_t \sim \mathcal{N}(0; \sigma^2). \end{cases}$ Let $\ell_{99} = 50$, $\alpha = 1/2$, $\sigma^2 = 16$, $y_{98} = 48$, $y_{99} = 52$, $y_{100} = 55$. Calculate 95% predictive interval for $y_{101}$.
2. Young investor Winnie-the-Crypto compares two trading strategies: buying bitcoins from good bees and from bad bees. Let $d_t$ be the price difference at day $t$ (bad minus good). Winnie-the-Crypto would like to test $H_0$: $\mathbb{E}(d_t) = 0$ against $H_a$: $\mathbb{E}(d_t) \neq 0$ at 5% significance level.

    Winnie assumed that $(d_t)$ can be approximated by a $MA(1)$ process and estimated the parameters using $T = 400$ observations, $\hat{d}_t = 2 + u_t + 0.7u_{t-1}$ with $\hat{\sigma}_u^2 = 4$.
    (a) Estimate $\mathbb{E}(d_t)$, $\mathrm{Var}(d_t)$ and $\mathrm{Cov}(d_t, d_{t-1})$.
    (b) Estimate $\mathbb{E}(\bar{d})$, $\mathrm{Var}(\bar{d})$ and help Winnie by considering $Z = \frac{\bar{d}-0}{se(\bar{d})}$.
3. The variables $X_1$, ..., $X_n$ are independent and uniformly distributed on $[0; 2a]$ for some positive $a$.
    (a) Find any sufficient statistic for $a$.
    (b) How the answer will change if $X_i \sim U[-a; 2a]$?
4. Consider an estimator $\hat{a}$ with $\mathbb{E}(\hat{a}) = 0.5a + 3$. For the given sample size the Fisher information is $I_F(a) = 400/a^2$.
    (a) What is the theoretical minimal variance of $\hat{a}$?
    (b) Assume that $\hat{a}$ attains the minimal variance boundary and is asymptotically normal. Given that $\hat{a} = 2022$ provide 95% CI for $a$.

5. You observe $X_1, ..., X_{400}$ and $Y_1, ..., Y_{400}$, $\bar{X} = 5$, $\bar{Y} = 6$. All variables are independent.

   Consider the null hypothesis that all random variables are exponentially distributed with common parameter $\lambda$ against alternative that parameter is $\lambda_X$ for every $X_i$ and $\lambda_Y$ for every $Y_j$.

   (a) Estimate common $\lambda$ using maximum likelihood for the restricted model.

   (b) Estimate both $\lambda_X$ and $\lambda_Y$ using maximum likelihood in the unrestricted model.

   (c) Use LR-test to test the null hyphotesis at 5% significance level.

6. The ultimate goal of this exercise is to prove the good upper bound for tail probability of a normal distribution: if $X \sim \mathcal{N}(0; \sigma^2)$ then $\mathbb{P}(X > c) \leqslant \exp(-c^2/2\sigma^2)$.

   Here are the guiding hints (you free to use not use them):

   (a) State the MGF of $X$. You may derive it or simply write it if you remember.

   (b) Consider $Y = \exp(uX)$. Using Markov inequality provide the upper bound for $\mathbb{P}(Y > \exp(uc))$.

   (c) Prove that $\mathbb{P}(X > c) \leqslant MGF_X(u)\exp(-uc)$ for any $u$.

   (d) Find the value of $u$ that makes the upper bound as tight as possible.

7. (bonus) Draw good bees and bad bees selling crypto. Any funny statistics/math joke is also ok!

### 4.5 2020-2021

Today: $+31°$, World Refrigiration Day :)
You have 100 minutes. You can use A4 cheat sheet and calculator. Be brave!
Date: 2021-06-26

1. I throw a fair die until the sequence 626 appears. Let $N$ be the number of throws.

   (a) What is the expected value $\mathbb{E}(N)$?

   (b) Write down the system of linear equations for the moment generating function of $N$. You don't need to solve it!

2. Consider the following stationary process

$$y_t = 1 + 0.5y_{t-2} + u_t + u_{t-1},$$

   where random variables $u_t$ are independent $\mathcal{N}(0; 4)$.

   (a) Find the 95% predictive interval for $y_{101}$ given that $y_{100} = 2$, $y_{99} = 3$, $y_{98} = 1$, $u_{99} = -1$.

   (b) Find the point forecast for $y_{101}$ given that $y_{100} = 2$.

3. I have an unfair coin with probability of heads equal to $h \in (0; 1)$.

   (a) Let $N$ be the number of tails before the first head. Find the MGF of $N$.

   (b) Let $S$ be the number of tails before $k$ heads (not necessary consecutive). Find the MGF of $S$.

   (c) What is the limit of $MGF_S(t)$ when $k \to \infty$ and $k \times h \to 0.5$? What is the name of the corresponding distribution?

4. Consider the stochastic process $X_t = f(t)\cos(2021W_t)$.

   (a) Find $dX_t$.

   (b) Find any $f(t) \neq 0$ such that $X_t$ is a martingale.

   (c) Using $f(t)$ from the previous point find $\mathbb{E}(\cos(2021W_t))$.

# 5 October exam solutions

## 5.1 2025-2026

## 5.2 2024-2025

## 5.3 2023-2024

1. (a) $\mathbb{P}(S_3 = B) = \mathbb{P}(A \to B \to A \to B) + \mathbb{P}(A \to C \to A \to B) + \mathbb{P}(A \to B \to C \to B)$

   (b)
   $$\begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

   (c)
   $$\begin{cases} a = 0.5b + 0.5c \\ b = 0.7a + 0.5c \\ c = 0.3a + 0.5b \\ a + b + c = 1 \end{cases}$$

   The solution is $a = 15/45$, $b = 17/45$, $c = 13/45$.

2. (a) Consider $N$ as known fixed value, $\mathbb{E}(S \mid N) = N\,\mathbb{E}(X_1) = N \cdot 0.5$. First, let's find moment generating funciton for $X_i$:
   $$M_X(u) = \int_0^1 \exp(xu) \cdot 1\,dx = \frac{\exp(u) - 1}{u};$$
   Hence $M_{S|N}(u) = (M_X(u))^N$ as $S$ is the sum of $N$ independent variables.

   (b) Random variable $N$ is discrete, $M_S(u) = \mathbb{P}(N = 1)(M_X(u))^1 + \mathbb{P}(N = 2)(M_X(u))^2 + \ldots = \frac{0.7M_X(u)}{1-0.3M_X(u)}$.

   (c) Moment generating function is used to calculate moments, $M_S''(0) - (M_S'(0))^2 = \operatorname{Var}(S)$.

3. Start with $X_0$: $\mathbb{E}(X_0) = 1$, $\operatorname{Var}(X_0) = 4/12 = 1/3$.

   (a) Expected value is constant, $\mathbb{E}(X_n) = 0.5 + 0.5\,\mathbb{E}(X_{n-1})$, hence $\mathbb{E}(X_n) = 1$. Variance goes to zero, $\operatorname{Var}(X_n) = 0.25\operatorname{Var}(X_{n-1})$.

   (b) $\operatorname{plim} X_n = 1$

4. Let's measure time in minutes.

   (a) $\mathbb{P}(X_5 = 0) = \exp(-5\lambda) = 0.05$, so $\lambda = \ln(0.05)/-5 = \ln(20)/5$.

   (b) $\mathbb{E}(X_{180}) = 180\lambda$, $\operatorname{Var}(X_{180}) = 180\lambda$

   (c) $\mathbb{P}(X_{180} = 5) = \exp(-180\lambda)(180\lambda)^5/5!$

5. (a) $Y = IX_1 + (1 - I)X_2$

   (b) Consider $I$ as known or fixed variable, $\mathbb{E}(Y \mid I) = I\,\mathbb{E}(X_1) + (1 - I)\,\mathbb{E}(X_2)$. Note that $I^2 = I$ and $(1 - I)^2 = 1 - I$, hence $\operatorname{Var}(Y \mid I) = I\operatorname{Var}(X_1) + (1 - I)\operatorname{Var}(X_2)$.

   (c) $\mathbb{E}(Y) = p\mu_1 + (1-p)\mu_2$ and $\operatorname{Var}(Y) = p(1-p)(\mu_1 - \mu_2)^2 + p\sigma_1^2 + (1-p)\sigma_2^2$, where $p = 0.3$, $\mu_1 = \sigma_1^2 = 1$, $\mu_2 = \sigma_2^2 = 2$.

6. (a) Sigma-algebra: $\sigma(X) = \{\emptyset, \Omega, \{X = -2\}, \{X = 0\}, \{X = 2\}, \{X \neq -2\}, \{X \neq 0\}, \{X \neq 2\}\}$. Other descriptions are possible, for example, one may replace $\{X = -2\}$ by $\{X < 0\}$.

   (b) Random variable $XY$ takes 3 distinct values, hence $\operatorname{card} \sigma(X \cdot Y) = 2^3 = 8$.

## 5.4 2022-2023

1.

$$\operatorname{plim} \frac{\sum (X_i - \bar{X})^3}{n + 2022} = \operatorname{plim} \frac{\sum X_i^3 - 3\bar{X}\sum X_i^2 + 3\bar{X}^2 \sum X_i - \sum \bar{X}^3}{n+2022} = \\ = \mathbb{E}(X_1^3) - 3\,\mathbb{E}(X_1^2) + 3\,\mathbb{E}(X_1) - 1 = 0;$$

Note that $\mathbb{E}(X_1^2) = 4/3$, $\mathbb{E}(X_1^3) = 2$.

2. $\mathbb{P}(X_3 = 1) = 0.3 \cdot 0.7 \cdot 0.3 + 0.7 \cdot 1 \cdot 0.3 = 0.21 \cdot 1.3$ Let's denote $\tau_j = \min\{t \mid X_t = 0, X_0 = j\}$, $\mu_j = \mathbb{E}(\tau_j)$.

$$\begin{cases} \mu_0 = 0 \\ \mu_1 = 1 + 0.7\mu_2 \\ \mu_2 = 1 + 0.3\mu_1 + 0.7\mu_3 \\ \mu_3 = \mu_2 + 1 \end{cases}$$

We get $\mu_2 = 200/9$.

3. (a) $\sigma(X_1 \cdot X_2) = \{\emptyset, \Omega, \{X_1X_2 = 1\}, \{X_1X_2 = -1\}\}$;
   (b) Many answers are ok, for example $\sigma(X_1 = X_3)$.
   (c) Note that $\sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3) = \sigma(X_1, X_2, X_3)$, the number of events in sigma-algebra is $\operatorname{card} \sigma(X_1, X_1 + X_2, X_1 + X_2 + X_3) = 2^8 = 256$.

4. $\operatorname{Cov}(N_3, N_{10}) = \operatorname{Cov}(N_3, N_3 + (N_{10} - N_3)) = \operatorname{Var}(N_3) = 3\lambda$.

5. $\mathbb{E}(X_3 \mid X_1, X_2) = \mathbb{E}(X_3) = 0$, $\mathbb{E}(X_3 \mid X_1 + X_3) = (X_1 + X_3)/2$, $\operatorname{Var}(X_3 \mid X_1, X_3) = 0$, $\operatorname{Var}(X_3 \mid X_1 + X_3) = 1 - (X_1 + X_3)^2/4$.

   Посчитаем ожидание и получим $Y_n = X_1 + X_2 + \ldots + X_n$, the process $(Y_n)$ is a martingale.

6. $\operatorname{Var}(Y_t - Y_s) = \operatorname{Var}(tW_{2t} - sW_{2s}) = 2t^3 + 2s^3 - 4ts^2$. We get $\mathbb{E}(M_{t+u} \mid \mathcal{F}_t) = W_t^3 + 3W_t u + \alpha(t+u)W_t$. From $\mathbb{E}(M_{t+u} \mid \mathcal{F}_t) = W_t^3 + \alpha t W_t$ it follows that $\alpha = -3$.

## 5.5 2021-2022

1.

## 5.6 2021-2022 retake

1.

## 5.7 2020-2021

1.

# 6 December exam solutions

## 6.1 2025-2026b

1.
2.
3.
4.
5.
6.

## 6.2 2025-2026a

1.
2.
3.
4.
5.
6.

## 6.3 2024-2025

## 6.4 2023-2024

1. (a) [3] $S_t$ is not a martingale, $\mathbb{E}(S_{t+1} \mid \mathcal{F}_t) = S_t + 0.6 - 0.4 \neq S_t$.
   (b) [7] Two solutions, one is trivial $c = 1$, the other...

2. [10 points]

   (a) [4] $dM_t = -3a(t)\sin(3W_t)dW_t + a'(t)\cos(3W_t)dt - 0.5 \cdot 9a(t)\cos(3W_t)dt$
   (b) [6] If $(M_t)$ is a martingale then $a'(t) - 4.5a(t) = 0$, hence $a(t) = c \cdot \exp(4.5t)$.

3. [10 points]

   (a) [3] $\mathbb{E}(S_n) = \rho t$.
   (b) [4] $\mathrm{Var}(S_n) \to 0$
   (c) [2] $S_n \to \rho t$ in mean squared sense.
   (d) [1] $dA_t dB_t = \rho dt$

4. (a) [2] $(X_t)$ is not a martingale, as we have $W_t dt$ term;
   (b) [4]

   $$d(X_tW_t) = W_t dX_t + X_t dW_t + dX_t dW_t = (W_t^3 dW_t + W_t^2 dt) + X_t dW_t + W_t^2 dt = (W_t^3 + X_t)dW_t + 2W_t^2 dt$$

   (c) [4] $\mathrm{Cov}(X_t, W_t) = \mathbb{E}(X_tW_t) - 0$,

   $$X_tW_t = 0 + \int_0^t (W_u^3 + X_u)dW_u + \int_0^t 2W_u^2 du$$

   $$\mathbb{E}(X_tW_t) = \int_0^t \mathbb{E}(2W_u^2)du = \int_0^t 2u du = t^2.$$

5. [10 points]

6. [10 points]

   $$X_0 = \exp(-rT)\,\mathbb{E}^*(X_T) = \exp(-rT)\,\mathbb{P}^*(S_T \geqslant 2S_0) = \dots$$

### 6.5 2022-2023

1. (a) $dX_t = (W_t^3 - 3W_t^2 \cdot t)dW_t$ (4 points), 1 point for comment how you get the answer (definition and Ito's lemma), 2 points for full form
$$X_t = X_0 + \int_0^t W_u^3 - 3W_u^2 \cdot u \, dWu$$
   (b) A process is a martingale as in short form $A_t dt = 0$ (3 points)

2. (a) $dX_t = -2X_t dW_t$ (2 points), this process is a martingale (1 point)
   (b) $\mathbb{E}(X) = 1$ (2 points), $\mathrm{Var}(X) = \exp(4t) - 1$ (2 points)
   (c)
$$\int_0^t X_u \, dWu = \frac{1 - X_t}{2}$$
   (3 points)

3. (a) 2 points for $\mathbb{E}(W_5W_4 \mid W_4) = W_4^2$, 3 points for $\mathrm{Var}(W_5W_4 \mid W_4) = W_4^2$
   (b) i. 1-2 points for clever ideas
      ii. 3 points for solution with serious mistakes
      iii. 4 points for solutions with arithmetic errors
      iv. 5 points for $\mathrm{Cov}(W_5W_4, W_5W_6) = 40$

4. 1-3 points depending on the cleverness of ideas.
   5 points if one got correct martingale:
$$\mathbb{E}(Y_{t+1} \mid Y_t) = Y_t \, \mathbb{E}(e^{bX_{t+1}})$$
   10 points if one solved equation correctly:
$$\mathbb{E}(e^{bX_{t+1}}) = 1 \to b = 0 or b = \ln(1/9)$$
   Minus 1 point if one forgot trivial solution $a = 0$ and any $b$.

5. (a) $p_u^* = p_d^* = 1/2$ (3 points)
   (b) $X_1^u = X_1^d = (0.5 \cdot 100 + 0.5 \cdot 0)/1.05$, hence $X_0 = 50/1.05^2 \approx 45.35$ (3 points)
   (c) $\alpha = X_2^{uu} - X_2^{ud}/(S_2^{uu} - S_2^{ud}) = 100/216 \approx 0.46$ (4 points)

6. You get 2 points almost for nothing:
$$X_0 = \exp(-2r) \, \mathbb{E}_*(X_2)$$
   Correct formula for $X_2$ in terms of $W_1^*$ gives your 4 points:
$$X_2 = S_1^3 = S_0^3 \exp(3r) \exp(3\sigma W_1^* - 9\sigma^2/2).$$
   Calculations of expected value (4 points more):
$$X_0 = S_0^3 \exp(r) \exp(3\sigma^2).$$

### 6.6 2021-2022

1.

### 6.7 2020-2021

1.

# 7 April exam solutions

## 7.1 2024-2025

## 7.2 2023-2024

1. (a) $\mathbb{E}(y_t) = 5$
   (b) $\rho_3 = \rho_4 = \ldots = 0$
   (c) The process is stationary.

2.

3. (a) $\phi_{11} = \rho_1 = 0.2$, $\phi_{22} = 0$
   (b) Possible equation is $y_t = 0.2y_{t-1} + u_t$. Another possibility is $y_t = 5y_{t-1} + u_t$. In the second case the stationary solution will be forward-looking and not $MA(\infty)$ with respect to $(u_t)$.

4. (a) $\lambda_1 = 2$, $\lambda_2 = 0.5$, here the roots of the lag polynomial are exactly the same.
   (b) $(1 - 2L)(1 - 0.5L)(y_t + 10) = u_t$
   (c) The equation has infinitely many non-stationary solutions.
   (d) The equation has unique stationary solution.
   (e) The equation has no stationary solutions that are $MA(\infty)$ with respect to $(u_t)$.

5.

6. The process is not stationary as $\mathbb{E}(y_1) = 3$ and $\mathbb{E}(y_2) = -3$.

## 7.3 2022-2023

1. (a) [6 points]
$$y_{102} = \ell_{100} + (0.9 + 0.9^2)b_{100} + (0.3 + 0.18)u_{101} + u_{102}$$
$$(y_{102} \mid y_1, \ldots, y_{100}) \sim \mathcal{N}(21.71, 24.608)$$
   The interval
$$[21.71 - 1.96 \cdot 4.96; 21.71 + 1.96 \cdot 4.96]$$
   (b) [4 points]
$$\lim_{h\to\infty} \mathbb{E}(y_{100+h} \mid y_1, \ldots, y_{100}) = \ell_{100} + (0.9 + 0.9^2 + \ldots)b_{100} = 20 + 9 \cdot 1$$

2. (a) [2 points] $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, one stationary solution, infinitely many non-stationary solutions.
   (b) [6 points]: [2 points] for the system + [2 points] for $\rho_1$ + [2 points] for $\rho_2$.
$$\begin{cases} \gamma_1 = 0.7\gamma_0 - 0.12\gamma_1 \\ \gamma_2 = 0.7\gamma_1 - 0.12\gamma_0. \end{cases}$$
$$\rho_1 = 70/112 = 0.625, \quad \rho_2 = 49/112 - 0.12 = 0.3175$$
   (c) [2 points]
$$\alpha_1 = 0.7, \quad \alpha_2 = 0.37$$

3. (a) [4 points]
$$\sigma_{101}^2 = 3 + 0.5(-1)^2 = 3.5$$
$$(u_{101} \mid \sigma_{101}) \sim \mathcal{N}(0; \sigma_{101}^2)$$
$$[-1.96\sqrt{3.5}; +1.96\sqrt{3.5}]$$

(b) [3 points] [1 point] for $\mathbb{E}(u_t)$ and [2 points] for $\text{Var}(u_t)$ The process $(u_t)$ is a white noise, hence

$$\mathbb{E}(u_t) = 0.$$

$$\sigma_u^2 = 3 + 0.5 \cdot \sigma_u^2$$

(c) [3 points]: [1 point] for $\text{Corr}(u_t, u_{t-1})$ and [2 points] for $\text{Corr}(u_t^2, u_{t-1}^2)$ The process $(u_t)$ is a white noise, hence

$$\text{Corr}(u_t, u_{t-1}) = 0.$$

$$u_t^2 = 3 + 0.5u_{t-1}^2 + (u_t^2 - \sigma_t^2)$$

We notice that $r_t = u_t^2 - \sigma_t^2$ is a white noise, hence $u_t^2$ is an $AR(1)$ process. Hence, $\text{Corr}(u_t^2, u_{t-1}^2) = 0.5$.

4. (a) [5 points]

$$L = const(0.2 + a)^{N_1}(0.3 - a)^{N_2}0.5^{N_3}$$

$$\ell = const + N_1 \ln(0.2 + a) + N_2 \ln(0.3 - a) + N_3 \ln 0.5$$

$$\frac{\partial \ell}{\partial a} = \frac{N_1}{0.2 + a} - \frac{N_2}{0.3 - a}$$

$$\hat{a}_{ML} = \frac{0.3N_1 - 0.2N_2}{N_1 + N_2}$$

We see that $\partial\ell/\partial a$ decreases as $a$ increases, so $\hat{a}_{ML}$ is indeed the point of maximum.

(b) [5 points]

$$\mathbb{E}(Y_i) = (0.2 + a) + 2(0.3 - a) + 4 \cdot 0.5 = 2.8 - a$$

$$\bar{Y} = \frac{N_1 + 2N_2 + 4N_4}{N_1 + N_2 + N_4}$$

$$\hat{a}_{MM} = 2.8 - \frac{N_1 + 2N_2 + 4N_4}{N_1 + N_2 + N_4}$$

5. (a) [6 points]

$$\mathbb{P}(\hat{\theta} > y) = \mathbb{P}(Y_1 > y/n)^n = (\exp(-y/n\theta))^n = \exp(-y/\theta)$$

Hence $\hat{\theta}$ has exponential distribution with rate $1/\theta$ and probability density function

$$f(t) = \begin{cases} \exp(-t/\theta)/\theta, & \text{if } t \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) [2 points]
The estimator is unbiased as

$$\mathbb{E}(\hat{\theta}) = 1/(1/\theta) = \theta.$$

(c) [2 points]
The estimator is non consistent as its distribution does not depend on $n$.

## 7.4 2021-2022

1.

2. (a) Yes, the process is stationary, that is $MA(2)$ process.
   (b) $\rho_3 = \rho_4 = \ldots = 0$
   (c) $d_t = u_t + u_{t-1} + 2u_{t-2} - u_{t-1} - u_{t-2} - 2u_{t-3}$, hence $d_t \sim ARIMA(0, 0, 3)$.

3. (a)
   (b) $\phi_{11} = \rho_1$, $\phi_{22} = -0.2$, $\phi_{33} = \phi_{44} = \ldots = 0$. The partial correlation $\phi_{22}$ measures how will $y_t$ on average react to the unit change of $y_{t-2}$ given fixed $y_{t-1}$.
   (c) $\rho_{100} = -0.9\rho_{99} - 0.2\rho_{98}$

## 7.5 2020-2021

1.

# 8 Final exam solutions

## 8.1 2024-2025

## 8.2 2023-2024

1. Let's observe that we may decompose $X_i$ as a sum $X_i = Y_i + \theta$, where $Y_i \sim \text{Exp}(\lambda)$.
   Hence, $\mathbb{E}(X_i) = 1/\lambda + \theta$, $\text{Var}(X_i) = \text{Var}(Y_i) = 1/\lambda^2$ and $\mathbb{E}(X_i^2) = 1/\lambda^2 + (1/\lambda + \theta)^2$.
   There is an alternative solution with direct integration:
   $$\mathbb{E}(X_i) = \int_\theta^{+\infty} xf(x)\,dx, \quad \mathbb{E}(X_i^2) = \int_\theta^{+\infty} x^2 f(x)\,dx.$$
   (a) Solving $1/\hat{\lambda} + 1 = \bar{X}$ we obtain $\hat{\lambda} = 1/(\bar{X} - 1)$.
   (b) Solving for $\hat{\lambda}$ and $\hat{\theta}$ the system
   $$\begin{cases} 1/\hat{\lambda} + \hat{\theta} = \bar{X} \\ 1/\hat{\lambda}^2 + (1/\hat{\lambda} + \hat{\theta})^2 = M_2 \text{ with } M_2 = \sum X_i^2/n \end{cases}$$
   we obtain
   $$\hat{\lambda} = \frac{1}{\sqrt{M_2 - \bar{X}^2}}, \quad \hat{\theta} = \bar{X} - \sqrt{M_2 - \bar{X}^2}$$

2. The log-likelihood function is equal to
   $$\ell(a) = \sum_{i=1}^n \left((-0.5)\ln(4\pi) - 0.5\ln a - (x_i - a)^2/4a\right).$$
   The equation $\ell'(a) = 0$ may be simplified to
   $$n\hat{a}^2 + 2n\hat{a} - \sum X_i^2 = 0$$
   Hence,
   $$\hat{a} = \frac{-2n \pm \sqrt{4n^2 + 4n\sum X_i^2}}{2n}$$
   We choose the root $\hat{a} > 0$ as $\text{Var}(X_i) = 2a > 0$.
   $$\hat{a} = \sqrt{1 + \sum X_i^2/n} - 1$$
   Just for fun. In the case $X_i \sim \mathcal{N}(a, ka)$ the equation would be
   $$n\hat{a}^2 + kn\hat{a} - \sum X_i^2 = 0$$
   And
   $$\hat{a} = \frac{-nk + \sqrt{k^2n^2 + 4n\sum X_i^2}}{2n}.$$

3. $\mathbb{E}(Y_i) = \mathbb{P}(X_i > 1) = (a-1)/a = p$.
   (a) The estimator is consistent as
   $$\text{plim}\,\hat{a} = \frac{1}{1 - \text{plim}\,\bar{Y}} = \frac{1}{1 - \frac{a-1}{a}} = a$$
   (b) For $n = 2$ we have the positive probability $p^2$ that $\bar{Y} = 1$. Hence with positive probability $\hat{a}$ is not defined. The value $\mathbb{E}(\hat{a})$ does not exist for $n = 2$.

4. (a) The log-likelihood function is equal to

$$\ell(\lambda) = \sum_{i=1}^{n} (-\lambda + X_i \ln \lambda - \ln(X_i!))$$

The score function is

$$\text{score}(\lambda) = \ell'(\lambda) = \sum_{i=1}^{n} (-1 + X_i/\lambda) .$$

And

$$\ell''(\lambda) = \sum_{i=1}^{n} \left(-X_i/\lambda^2\right) .$$

Fisher information is

$$I_F = -\mathbb{E}(\ell''(\lambda)) = \sum \mathbb{E}(X_i)/\lambda^2 = n\lambda/\lambda^2 = n/\lambda.$$

(b) Solving $\ell' = 0$ we obtain

$$\hat{\lambda} = \bar{X}$$

(c) Rewrite $\ell'(\lambda)$ using $\hat{\lambda}$. Be careful! Do not confound $\lambda$ and $\hat{\lambda}$.

$$\text{score}(\lambda) = \ell'(\lambda) = -n + n\hat{\lambda}/\lambda.$$

Hence the score function is linear function of $\hat{\lambda}$, $\text{Corr}(\text{score}(\lambda), \hat{\lambda}) = 1$ and the Cramer-Rao bound is attained.

One may also find $\mathbb{E}\,\hat{\lambda} = \lambda$, $\text{Var}(\hat{\lambda}) = \lambda/n$ and explicitly check that the general bound

$$\text{Var}(\hat{\lambda}) \geqslant 1/I_F$$

is attained as equality in our case

$$\lambda/n = 1/(n/\lambda).$$

5. We do not need the formula for $\Gamma(\alpha)$ here.

(a) For known $\lambda = 1$ the likelihood is

$$L = \left(\prod X_i\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \cdot \exp(-\sum X_i).$$

If we optimize this function for $\alpha$ the optimal $\hat{\alpha}$ will depend only on $\prod X_i$. Hence $\prod X_i$ is a sufficient statistic for $\alpha$. There are many other sufficient statistics, $\sum \ln X_i$ is another example.

(b) Now the likelihood is

$$L = \left(\prod X_i\right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} \exp(-\lambda \sum X_i).$$

If we optimize this function for $\alpha$ and $\lambda$ the optimal point will depend only on $\prod X_i$ and $\sum X_i$. Hence $\left(\prod X_i \quad \sum X_i\right)$ is a two dimensional sufficient statistic for $(\alpha, \lambda)$.

6. (a) Under $H_0$ we have $X_i \sim \text{Pois}(\lambda)$, $Y_i \sim \text{Pois}(\lambda)$.

$$\ell(\lambda) = \sum_{i=1}^{n_x} (-\lambda + X_i \ln \lambda - \ln(X_i!)) + \sum_{i=1}^{n_y} (-\lambda + Y_i \ln \lambda - \ln(Y_i!))$$

The score function is

$$\text{score}(\lambda) = \ell'(\lambda) = \sum_{i=1}^{n_x} (-1 + X_i/\lambda) + \sum_{i=1}^{n_y} (-1 + Y_i/\lambda) .$$

The estimator is $\hat{\lambda} = (\sum X_i + \sum Y_i)/(n_x + n_y)$.

$$\max \ell_R = -\hat{\lambda}(n_x + n_y) + \left(\sum X_i + \sum Y_i\right) \ln \hat{\lambda} - \sum \ln X_i! - \sum \ln Y_i!$$

(b) In unrestricted model we have two independent estimators,

$$\hat\lambda_x = \bar X, \quad \hat\lambda_y = \bar Y$$

$$\max \ell_{UR} = -\hat\lambda_x n_x + \sum X_i \ln \hat\lambda_x + \sum Y_i \ln \hat\lambda_y - \sum \ln X_i! - \sum \ln Y_i!$$

(c)

$$LR = 2(\max \ell_{UR} - \max \ell_R) = 2\sum X_i(\ln \hat\lambda_x - \ln \hat\lambda) + 2\sum Y_i(\ln \hat\lambda_y - \ln \hat\lambda)$$

(d) Unrestricted model has two parameters, restricted model has one parameter, hence we use chi-squared disribution with $2-1=1$ degree of freedom, $LR_{\text{crit}} = 3.84$. We calculate estimates, $\hat\lambda_x = 5$, $\hat\lambda_y = 4.5$, $\hat\lambda = 14/3$.

$$LR = 1000(\ln 5 - \ln(14/3)) + 1800(\ln 4.5 - \ln(14/3)) \approx 3.5$$

We do not reject $H_0$.

## 8.3 2022-2023

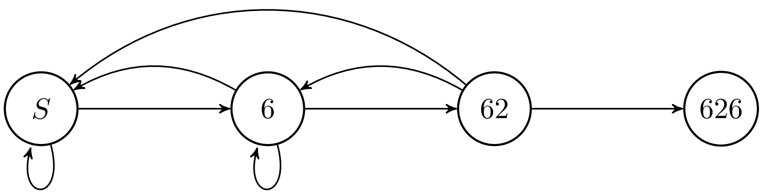
1.

## 8.4 2021-2022

1.

## 8.5 2020-2021

1. Let's draw the chain




The system of equations for expected values:

$$\begin{cases} x_s = 1 + \frac{1}{6}x_6 + \frac{5}{6}x_s \\ x_6 = 1 + \frac{1}{6}x_6 + \frac{1}{6}x_{62} + \frac{4}{6}x_s \\ x_{62} = 1 + \frac{1}{6}\cdot 0 + \frac{1}{6}x_6 + \frac{4}{6}x_s \end{cases}$$

The system of equations for moment generating functions:

$$\begin{cases} m_s(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{5}{6}m_s(t)\right) \\ m_6(t) = \exp(t)\left(\frac{1}{6}m_6(t) + \frac{1}{6}m_{62}(t) + \frac{4}{6}m_s(t)\right) \\ m_{62}(t) = \exp(t)\left(\frac{1}{6}\cdot 1 + \frac{1}{6}m_6(t) + \frac{4}{6}m_s(t)\right) \end{cases}$$

2. (a) Let's denote by $x$ all available information,

$$x = \begin{pmatrix} y_{100} \\ y_{99} \\ y_{98} \\ u_{99} \end{pmatrix}$$

Let's use $t = 100$:

$$y_{100} = 1 + 0.5y_{98} + u_{100} + u_{99}$$

Using all available information we obtain $u_{100} = 1.5$ and hence

$$y_{101} \mid x \sim \mathcal{N}(1 + 0.5y_{99} + u_{100}; 4)$$

(b) Here we work with true betas:

$$\mathbb{E}(y_{101} \mid y_{100}) = \mu_y + \frac{\mathrm{Cov}(y_{100}, y_{101})}{\mathrm{Var}(y_{100})}(y_{100} - \mu_y)$$

3. (a) Moment generating function

$$m_N(t) = \sum_{j=0} \exp(tj)(1-h)^j h = h \sum_{j=0} (\exp(t)(1-h))^j = \frac{h}{1 - \exp(t)(1-h)}$$

(b) As $S = N_1 + N_2 + \ldots + N_k$:

$$m_S(t) = \left(\frac{h}{1 - \exp(t)(1-h)}\right)^k$$

(c) Due to my mistake the limit is easy, 0.
In my dream it was $k \to \infty$, $k \cdot (1-h) \to 0.5$ and that would be fun!

4. (a) Let's use Ito's lemma

$$dX_t = f'(t)\cos(2021W_t)dt - 2021 f(t)\sin(2021W_t)dW_t + \frac{1}{2}2021^2 f(t)\cos(2021W_t)dt$$

(b) To make $X_t$ a martingale we should kill $dt$ term.

(c) As $X_t$ is martingale $\mathbb{E}(X_t) = \mathbb{E}(X_0) = f(0)$. So $\mathbb{E}(\cos(2021W_t)) = f(0)/f(t)$.