Practice Problems

Imperfect competition

1. Herfindahl-Hirschman Index in Cournot competition

An industry consists of three firms. All three have identical variable costs $\text{VC}(q_i) = 5\,q_i + q_1{}^2$. Firms 2 and 3 have fixed costs of \$2,000. Firm 1's fixed costs are \$3,000. market demand is $Q = 335 - P$ and the firms play Cournot.

A. What is the Herfindahl-Hirschman Index of the industry?

Cournot equilibrium

To solve for the Cournot equilibrium, we need to derive each firm's profit functions and maximize profits with knowledge that profit depends on the other firms playing their best strategy. Starting with firm 1.

Generally profit π_i = Revenue – Total variable costs – total fixed costs. Each firm faces the following profit function

$$\pi_1 = \text{Pq}_1 - \left(5\,q_1 + q_1{}^2\right) - 3000$$
$$\pi_2 = \text{Pq}_2 - \left(5\,q_1 + q_1{}^2\right) - 2000$$
$$\pi_3 = \text{Pq}_3 - \left(5\,q_1 + q_1{}^2\right) - 2000$$

We need to start eliminating variables. P can be eliminating but plugging in the right-hand side of the inverse demand function, $P = 335 - Q$.

$$\pi_1 = (335 - Q)\,q_1 - \left(5\,q_1 + q_1{}^2\right) - 3000$$
$$\pi_2 = (335 - Q)\,q_2 - \left(5\,q_1 + q_1{}^2\right) - 2000$$
$$\pi_3 = (335 - Q)\,q_3 - \left(5\,q_1 + q_1{}^2\right) - 2000$$

Next we can eliminate Q with the fact that the total market quantity, Q, is the sum of the quantity produced by the three comprising firms $Q = q_1 + q_2 + q_3$.

$$\pi_1 = (335 - q_1 - q_2 - q_3)\,q_1 - \left(5\,q_1 + q_1{}^2\right) - 3000$$
$$\pi_2 = (335 - q_1 - q_2 - q_3)\,q_2 - \left(5\,q_1 + q_1{}^2\right) - 2000$$
$$\pi_3 = (335 - q_1 - q_2 - q_3)\,q_3 - \left(5\,q_1 + q_1{}^2\right) - 2000$$

Each firm seeks to optimize profits but does not know the quantities produced by the other two firms in the market. Thus, the profit of firm 1 is best written as $\pi_1(q_1;\, q_2,\, q_3)$ in order to highlight the fact that firm 1 only has control of firm 1's quantity.

Each firm maximizes their profit by solving the first order condition of their profit functions.

$$\frac{\partial \pi_1}{\partial q_1} = 335 - 2\,q_1 - q_2 - q_3 - 5 - 2\,q_1 = 330 - 4\,q_1 - q_2 - q_3 = 0$$
$$\frac{\partial \pi_1}{\partial q_2} = 330 - 4\,q_2 - q_1 - q_3 = 0$$
$$\frac{\partial \pi_1}{\partial q_1} = 330 - 4\,q_3 - q_2 - q_1 = 0$$

In Cournot competition each firm will assume that the others are producing at their profit maximizing quantity and that each firm has full knowledge over each others profit functions. Thus the reaction functions are common knowledge.

$q_1^* = 82.5 - \frac{1}{4}\, q_2 - \frac{1}{4}\, q_3$
$q_2^* = 82.5 - \frac{1}{4}\, q_1 - \frac{1}{4}\, q_3$
$q_3^* = 82.5 - \frac{1}{4}\, q_2 - \frac{1}{4}\, q_1$

Each of the firms has the exact same reaction function. Adding the three reaction functions (or inspecting the demand function) reveals that

$(q_1^* + q_2^* + q_3^*) = 247.5 - \frac{2}{4}\,(q_1^* + q_2^* + q_3^*)$
$1.5\ Q^* = 247.5$

$Q^* = 165$ and $q_1^* = q_2^* = q_3^* = 55$. Price is $335 - 165 = 170$. Firm's 1 earns \$3,050 and firms 2 and 3 earn\$4,050 each. Market profit is \$11,150

Herfindahl-Hirschman Index

The Herfindahl-Hirschman Index is a measure of market concentration where scores closer to 0 indicate a perfectly competitive industry and scores near to 1 indicate a monopoly. Mathematically, the Herfindahl-Hirschman Index is calculated as

$$\mathrm{HHI} = \sum_{i=1}^{N} s_i^{\,2} \tag{1}$$

where s_i is the market share of firm i. Market share is defined as a firms production, q_i divided by total market production.

In the case of the above example, HHI $= 3 \times \left(\frac{55}{165}\right)^2 = 0.12$

B. What happens to it if Firms 1 and 2 merge?

We need to refine some of the global variables. First, lets call the quantity produced by the merged firm $q_m = q_1 + q_2$. Total market quantity is now the sum of the merged firm and firm 3, $Q' = q_m + q_3$.

The new firm decides to use firm 1 and 2's accumulated physical capital to jointly maximize a profit function. But before profit optimization, it must decide how much to produce using firm 1's old capacity and how much to produce using firm 2's old capacity. It will allocation production to the cost minimizing combination.

$C_m = C_1 + C_2$
$C_m = \left(5\ q_1 + q_1^{\,2} + 3000\right) + \left(5\ q_2 + q_2^{\,2} + 2000\right)$

Lets eliminate q_2 from the cost function by substituting $q_m - q_1$

$C_m = \left(5\ q_1 + q_1^{\,2} + 3000\right) + \left(5\ (q_m - q_1) + (q_m - q_1)^2 + 2000\right)$
$\frac{\partial C_m}{\partial q_1} = 5 + 2\ q_1 - 5 - 2\ (q_m - q_1) = 0$
$q_1 = \frac{1}{2}\ q_m$

Solving the FOC confirms that the cost minimizing production schemes is to produce half of total output with firm 1's technology (facing its cost function) and half with firm 2's technology (facing its cost function). Solving for the SOC will confirm that the solution is a minima.

We can use our optimal production scheme for the merged firm to eliminate $q_1$from its cost function.

$$C_m = 5000 + \frac{5}{2}\, q_m + \frac{1}{4}\, {q_m}^2 + 5\left(q_m - \frac{1}{2}\, q_m\right) + \left(q_m - \frac{1}{2}\, q_m\right)^2$$
$$C_m = 5000 + \frac{5}{2}\, q_m + \frac{1}{4}\, {q_m}^2 + 5\, q_m - \frac{5}{2}\, q_m + {q_m}^2 - 2\left(\frac{1}{2}\, {q_m}^2\right) + \frac{1}{4}\, {q_m}^2$$
$$C_m = 5000 + \frac{1}{2}\, {q_m}^2 + 5\, q_m$$

Using this new cost function, we can calculate the profit and reaction function for the merged firm

$$\pi_m = (335 - q_m - q_3)\, q_m - 5000 - \frac{1}{2}\, {q_m}^2 - 5\, q_m$$
$$\frac{\partial \pi_m}{C_m} = 335 - 2\, q_m - q_3 - q_m - 5 = 0$$
$$q_m{}^* = 110 - \frac{1}{3}\, q_3$$

Adapting firm 3's reaction function from above

$$q_3{}^* = 82.5 - \frac{1}{4}\, q_m$$
$$q_3{}^* = 82.5 - \frac{1}{4}\left(110 - \frac{1}{3}\, q_m\right)$$
$$q_3{}^* = 55 * \frac{12}{11} = 60$$
$$q_m{}^* = 110 - 20 = 90$$

Q' is now 90 + 60 = 150 and price is 185. The merged firm earns \$11,650 and firm 3 earns \$5,200. Market profits are \$16,850. Notice that the merged also increased firm 3's profits.

C. Herfindahl-Hirschman Index in merged market

Using the merged market we can recalculate the HHI as $\left(\frac{90}{150}\right)^2 + \left(\frac{60}{150}\right)^2 = 0.52$. The dramatic increase in market concentration from 0.12 to 0.52 helps explain why firm 3’s profits increased. It benefited from the decrease in competition.