Conformal Bootstrap

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1 Basic Aspects of CFT & AdS

Conformal symmetry is an extension of Poincare symmetry imposing that transformations should preserve the angles.

$$x \to x'$$

such that it leaves the metric invariant up to a factor. In Euclidean signature,

$$\delta_{\mu\nu} \to C(x)\delta_{\mu\nu} = g_{\mu\nu}(x') \tag{1}$$

The generators are:

- Poincare Symmetry: $M_{\mu\nu}, P_\mu$
- Dilatation (D) : $x'_\mu = \lambda x_\mu$
- Special Conformal Symmetry: (K_μ) $x'_\mu = \frac{x_\mu - a_\mu x^2}{(1-2(ax)+a^2x^2}$

The Conformal Algebra is:

$$\begin{aligned}
[D, K_\mu] &= -K_\mu \\
[D, P_\mu] &= P_\mu \\
[K_\mu, P_\nu] &= \delta_{\mu\nu} D - 2iM_{\mu\nu}
\end{aligned}$$

In any CFT, we have the state operator correspondence. In other words, there is a one-to-one correspondence between local operators and states.

$$\theta(x) \longleftrightarrow |\theta\rangle = \theta(0)\,|0\rangle \tag{2}$$

and those states are the eigenvectors of Dilatation operator.

$$D\,|\theta_\Delta\rangle = \Delta\,|\theta_\Delta\rangle \tag{3}$$

where Δ is the conformal dimension of the operator. So that,

$$[D, \theta_\Delta(0)] = \Delta\theta_\Delta(0)$$

and these operators and states can be seperated in two types: Primary and descendants.

Definition 1.1 (*Primary Operator*)**.** The primary operators are operators such that they cannot be written as derivatives acting on other operators.

Such examples are:

$$K_\mu |\theta_\Delta\rangle = 0$$
$$D |\theta_\Delta\rangle = \Delta |\theta_\Delta\rangle$$

or in operator language, commutation relation of K and local field vanishes at the origin:

$$[K_\mu, \theta_\Delta(0)] = 0$$

In this sense, as can also be seen from the Conformal algebra, K_μ is like the annihilation operator that reduces the conformal dimension by 1. And P_μ can be seen as the creation operator. Diagrammatically:

$$\theta_{\Delta+n} \xrightarrow{K_\mu} \theta_{\Delta+(n-1)} \xrightarrow{K_\mu} \theta_{\Delta+(n-2)} \xrightarrow{K_\mu} \dots \theta_\Delta \xrightarrow{K_\mu} 0 \tag{4}$$

And while P_μ reverses the arrows.

$$\theta_{\Delta+n} \xleftarrow{P_\mu} \theta_{\Delta+(n-1)} \xleftarrow{P_\mu} \theta_{\Delta+(n-2)} \xleftarrow{P_\mu} \dots \theta_\Delta \xleftarrow{P_\mu} 0 \tag{5}$$

The physics of CFT are correlation functions. This is due to seminal papers of Wilson Kogut, Polyakov(1977),... And it can be calculated that conformal symmetry fixes 2& 3 point correlation functions of primary operators. The correlation function of two primary operators are given as:

$$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{(x_{12}^2)^{\Delta_1}} \tag{6}$$

upto normalization factors. Where, $x_{12}^\mu = x_1^\mu - x_2^\mu$. Similarly, at 3 point level,

$$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\rangle = \frac{c_{123}(g, N, ...)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \tag{7}$$

where c can be due to the gauge coupling, the dimension of the continuous group in non-Abelian gauge theories, such as $SU(N)$. But, it does not contain any space dependence because it is fixed by the conformal symmetry. Also for, 4-point function:

$$\langle \theta_{\Delta_1}(x_1)\theta_{\Delta_2}(x_2)\theta_{\Delta_3}(x_3)\theta_{\Delta_4}(x_4)\rangle = \tag{8}$$

AdS^{d+1} Spacetime

AdS spacetime is a solution to Einstein's field equations with negative cosmological constant. Notes from written compare ads and ds.... It describes a hyperboloid with signature:

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + ... + (X^{d+1})^2 = -R^2 \tag{9}$$

where R is the AdS radius. We will keep R unity here. Embedding in $R^{d+1,1}$ goes as:

$$X = \{X^0 = \frac{1 + x^2 + z^2}{2z}, \S^\mu = \frac{x^\mu}{z}, x^{d+1} = \frac{1 - x^2 - z^2}{z^2}\} \tag{10}$$

we have the Poincare coordinates with:

$$ds^2 = \frac{dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu}{z^2} \tag{11}$$

and the conformal boundary is as $z \to 0$. x goes p.

A beautiful remark here is that AdS^{d+1} has the isometry group $SO(d+1,1)$, which is the conformal group in the space R^d.

So, what are the Feynman diagrams in AdS space? They should be important due to the correlation function calculations. Clearly, the metric is different; we now have a boundary and the interior, which is the bulk. They are now called Witten Diagrams. In this formalism, particles can travel from bulk to bulk, bulk to boundary and visa versa.