Introduction to Supersymmetry

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Abstract

This introduction aims to give a review of supersymmetry, how it can impact on future theories and give the basic ideas of supersymmetric construction. The paper starts with explaining why SM is not enough and(Murayama,2007) how supersymmetry can be helpful in grand unified theories.(Murayama 2007)(Enno Fischer,2023). Then we move onto a review of Lorentz and Poincare Algebras and show the relation between the associated algebras. And there is the construction of Spinor Algebra and some calculations with Spinors. (Müller-Kirsten and Wiedemann, 1986). After that, we introduce the Coleman-Mandula theorem.(Weinberg,2013)(Bertolini,2024) To rigorously construct Super-Poincare algebra, we introduce the graded algebras.(Müller-Kirsten and Wiedemann, 1986)(Wess,Bager 1992) Then we show how superspace and superfields are constructed with graded Lie algebras.(Nath, 2016)(Dine, 2016) (Wess, Bager 1992)(Bilal, 2001) We then show how supersymmetric actions can be constructed by using Superfields (Wess,Bager ,1992)(Müller-Kirsten, Wiedemann, 1986). After that we arrive break apart some Lagrangian formalism and give introduction to Wess-Zumino action.(Shirmann, 2008)(Haber, Haskins 2017) After some explicit calculations we show breaking of sussy. (Tong, 2024)(Müller-Kirsten and Wiedemann, 1986)(Nath, 2016)(Haber, Haskins 2017)To support breaking of SUSSY, R-symmetry was introduced.(Sun,2022)(Murayama,2007)Lastly, Witten's remarkable paper on his Witten Index was shown to remark the impact on supersymmetry breaking.(Witten, 1982)

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1 Introduction

Supersymmetry is a BSM(Beyond Standard Model Theory) and this symmetry enables us to transform fermions into bosons and visa versa. Historically, it accelereated the progress of string theory, supergravity and more. But, we do not observe it in the nature. The problem is that according to supersymmetry, particles should have superpartners corresponding to each of them, which were not observed yet. Thus, it doubles the degrees of freedom i.e the top quark's partner, stop.

The Supersymmetry has long been promising because it promises many answers to physics big problems. For example hierarchy problem, which is the non-appearent reason why there is a gap between the scales electroweak symmetry breaking (10^2) eV and Planck's scale (10^{19}) eV, where gravity is relevant. There is a whole desert in between. And supersymmetry holds promises to overcome this problem. Secondly, the interactions are governed by gauge parameters. These parameters in Standard Model depend on the energy scale. However, in Standard Model, these gauge couplings at high energies never meet. (We expect them to meet at a high energy scale for grand unified theories.) Luckily, in supersymmetry these parameters meet at the scale 10^{16} GeV. This can be understood as a one unified interaction. Third, Supersymmetry can suggest candidated for Dark Matter problem. We also in this review give a candidate for the WIMP(weakly interacting matter). (Enno Fischer, 2023) Thus, supersymmetry is a strong and promising path for a physicsist.

As we have already mentioned, for supersymmetry the superpartners have not been detected yet. So, supersymmetry should be spontaneously broken at some point. It has been believed that it is difficult to break super symmetry and also they are not generic among supersymmetric theories. Even among them, they are rather difficult to use for constructing phenomenologically acceptable models, so a lot of them are not so useful from the beginning. However, physicist now believe that there are a lot of viable theories that elegantly break supersymmetry in the landscape of supersymmetric theories. And, as we also are going to discuss, the change in the parameters do not affect this. (Murayama,2007) Also, the string theories are supersymmetric theories and a breakthrough paper 'Candidate de Sitter Vacua' was just published.(Liam et. al., 2024). And the theory that is being argued on this paper is a supergravity theory.

These are the reasons we are going to give an introduction to Supersymmetry in this article. The main point of this article is to achieve Supersymmetry Breaking in the most simple terms as possible.

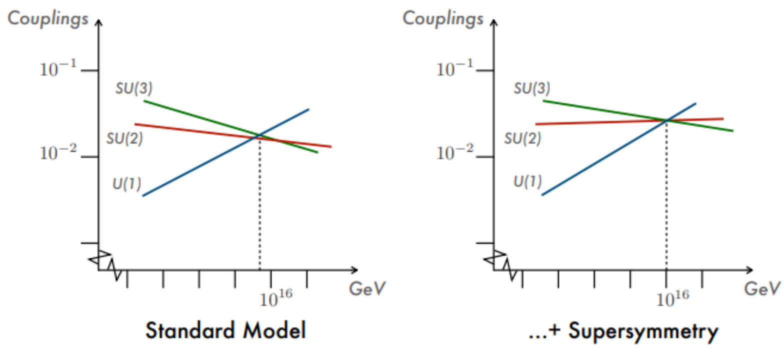

Figure 1: The couplings approximately meet in the Standard Model at $10^{16}GeV$, but they certainly meet in Supersymmetry at energy $10^{16}GeV$.(Bertolini, 2024, p.17)

2 Mathematical Formalism for Supersymmetry

2.1 Restricted Lorentz Group

The General Lorentz Algebra, which is with the metric signature $(+,-)$ is $SO(1,3,\mathbf{R})$. And we can derive it as follows, take the neighbourhood of identity of the group:

$$\Lambda = 1_{4x4} + \omega \tag{1}$$

ω is a matrix with infinitesimal parameters and it is antisymmetric. To show this:

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma} = \eta_{\mu\nu}(\delta^{\mu}_{\rho} + \omega^{\mu}_{\rho})(\delta^{\nu}_{\sigma} + \omega^{\nu}_{\sigma}) \tag{2}$$

$$= \eta_{\mu\nu}\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} + \eta_{\mu\nu}\delta^{\mu}_{\rho}\omega^{\nu}_{\sigma} + \eta_{\mu\nu}\omega^{\mu}_{\rho}\delta^{\nu}_{\sigma} + O(\omega^2) \tag{3}$$

$$= \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} \tag{4}$$

Thus, ω is shown to be an antisymmetric tensor with 6 infinitesimal parameters. By explicit verification and taking the neighborhood of identity again, we will have $M_{\mu\nu}$ as the generator of Lorentz Group. If we take:

$$M_{mn} = \epsilon_{mni}J_i \tag{5}$$

$$M_{0i} = -K_i \tag{6}$$

We shall have the lie algebras:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{7}$$

which are the rotation generators and boost generators are:

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{8}$$

The minus sign between these commutators expresses the difference between non-compact $SO(1, 3, \mathbf{R})$ and compact $SO(4, \mathbf{R})$. These algebras are locally homomorphic because the lie groups are homomorphic locally. We say that this $SO(1, 3, \mathbf{R})$ proper Lorentz group and they are non-compact. They are proper because the determinant of Λ is bigger than 0. Which is already derived from Weinberg's 1st book Quantum Theory of Fields.(Weinberg,2005) But, the algebra is not closed in this way because of equations 7 and 8. So, we could not classify all irreducible, non-unitary rep's in this way. To classify all irreducible, finite dim., non-unitary rep.'s of restricted Lorentz Group $L_+^\uparrow$, we have to change the basis to complex by introducing:

$$S_i := \frac{1}{2}(J_i + iK_i) \tag{9}$$

$$T_i := \frac{1}{2}(J_i - iK_i) \tag{10}$$

One can now derive by explicit verification that:

$$[S_i, S_j] = i\epsilon_{ijk}S_k \tag{11}$$

$$[T_i, T_j] = i\epsilon_{ijk}T_k \tag{12}$$

So far, we've decomposed $SO(1, 3, \mathbf{R})$ to $SU(2, \mathbf{C}) \times SU(2, \mathbf{C})$. These are two non-unitary representations of the Restricted Lorentz Group.

2.2 Poincare Group

Poincare transformations are kind of transformations that leave the spacetime length $(x - y)^2$ invariant. And it is:

$$x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu \tag{13}$$

As it can be seen from above, the Restricted Lorentz group is a subset of the Poincare group. The generators of this group, which are rotation and translation generators' lie algebra is:

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \tag{14}$$

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \tag{15}$$

and

$$[P_\mu, P_\nu] = 0 \tag{16}$$

$M^{\rho\sigma}$ is antisymmetric under $L_+^\uparrow$ and P^μ is a vector operator under $L_+^\uparrow$. And this is the full $SO(1, 3, \mathbf{R})$ Lie algebra. This algebra have two Casimir operators. 1st one is:

$$P^2 = P^\mu P_\mu \tag{17}$$

and the second one is the Pauli-Ljubanski Polarization Vector:

$$W_\mu := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma} \tag{18}$$

One can also show that the Pauli-Ljubanski vector is invariant under translations by taking the commutator with P^μ.

2.3 Spinor Algebra

$$SL(2,\mathcal{C}) = \{M \in GL(2,\mathcal{C}) \mid detM = +1\} \tag{19}$$

While GL is the general linear group with complex entries 2x2. We will see that apart from vectors or scalars spinors are objects that belong to $SL(2,\mathcal{C})$ and transform under Poincare algebra different. If we look at the linear representation of this group (Kirsten-Müller,2010) into a certain vector space F, through representation theory, we see that $SL(2,\mathcal{C})$ admits only 2 inequivalent self-representation of spinors. Which are:

- Self-representation

$$\psi'_A = M_A^{\ B}\psi^B \to (\frac{1}{2},0) \tag{20}$$

- Complex Conjugate Self Representation

$$\overline{\psi}'_{\dot{A}} = (M^*)_{\dot{A}}^{\dot{B}}\overline{\psi}_{\dot{B}} \to (0,\frac{1}{2}) \tag{21}$$

As we can see, complex representation is dotted while self representation is undotted. The representations are called left handed and right handed Wely-Spinors respectively. And these two representations are inequivalent. We can see this by searching for a 2x2 matrix with: $M = \epsilon M^* \epsilon^{-1}$, $\epsilon \in GL(2,\mathcal{C})$. But, one will see that the final matrix just switches between these two representations. This matrix has the properties:

$$(M^{-1})^T = \epsilon^{AB} \quad M^* = \epsilon^{\dot{A}\dot{B}} \tag{22}$$

$$\epsilon^{AB} = (\epsilon_{AB})^T \tag{23}$$

and

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{24}$$

$$\epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{25}$$

And one can easily check that:

$$\epsilon^{AB}\epsilon_{BC} = \delta^A_C \quad \epsilon^T_{AB}\epsilon^{BC} = -\delta^C_A \tag{26}$$

$$\epsilon_{AB} = \epsilon^{\dot{A}\dot{B}} \tag{27}$$

Transformation of Weyl Spinors under these are as follows:

- Left handed ψ^A contravariant spinor transforms under $(M^{-1})^T$:

$$\psi^A = \epsilon^{AB}\psi_B = -\psi_B\epsilon^{BA} \tag{28}$$

- Left handed Weyl Spinor under $(M^{-1})^T$ transform as:

$$\psi_A := \epsilon_{AB}\psi^B \tag{29}$$

- Right handed contravariant spinor under $SL(2,\mathcal{C})$ transform with M^*:

$$\overline{\psi}^{\dot{A}} := \epsilon^{\dot{A}\dot{B}}\overline{\psi}_{\dot{B}} = -\overline{\psi}_{\dot{B}}\epsilon^{\dot{B}\dot{A}} \tag{30}$$

- Right handed spinor in rep. $(0, \frac{1}{2})$ transforms under $(M^{-1})^T$:

$$\overline{\psi}_{\dot{A}} := \epsilon_{\dot{A}\dot{B}}\overline{\psi}^{\dot{B}} \tag{31}$$

2.4 Calculations with Spinors

In order to write Lagrangians with spinors we will need scalars, to construct them we will:

Let $\psi \in F^*$, $\chi \in F$, F is in the representation space of $SL(2,\mathcal{C})$

$$(\psi\chi) := \psi^A\chi_A \tag{32}$$

is invariant under transformations of $SL(2,\mathcal{C})$. This can also be defined for dotted indices but, nothing changes but the transformation. But, there is a certain prblem here. If spinors commute, the square of a spinor will be 0. This is a problem, so we postulate the Grassmanian numbers.

2.5 Grassmanian Numbers

Postulate

The components of a spinor are Grasmanian. We require $\psi'_A s$ to anticommute. So that square expressions do not vanish.

$$\begin{aligned}(\psi\psi) = \psi^A\psi_A = \psi^A\epsilon_{AB}\psi^B = \psi^A\psi^B\epsilon_{AB} = \psi^1\psi^2\epsilon_{12} + \psi^2\psi^1\epsilon_{21} \\ = \psi^1\psi^2 - \psi^2\psi^1 = \psi_2\psi_1 - \psi_1\psi_2\end{aligned} \tag{33}$$

This is because the half integer spins obey Fermi-Dirac statistics and integer spins obey Bose-Einstein statistics. From the Grassmanian properties, one can show that:

$$(\psi\chi) = (\chi\psi) \quad (\overline{\psi}\overline{\chi}) = (\overline{\chi}\overline{\psi}) \tag{34}$$

Another important property of Grassmanian numbers is the Fierz Reordering Formula for spinors:

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta) = -\frac{1}{2}(\theta\theta)(\phi\psi) \tag{35}$$

Which can be shown as follows:

$$(\theta\phi)(\theta\psi) = (\phi\theta)(\theta\psi) = \phi_A\theta^A\theta_B\psi^B = \phi^A\epsilon_{AC}\theta^C\theta^B\psi_B \tag{36}$$

$$= \phi^A\epsilon_{AC}(-\frac{1}{2}\epsilon^{CB})(\theta\theta)\psi_B = \phi^A\delta_A{}^B(-\frac{1}{2})(\theta\theta)\psi_B \tag{37}$$

$$= -\frac{1}{2}(\phi\psi)(\theta\theta) \tag{38}$$

2.6 Connection between $SL(2,\mathbf{C})$ and $L_+^\uparrow$

We will show that restricted Lorentz group is a subgroup of $SL(2,\mathbf{C})$. We will show this by constructing a homomorphism between these two groups. Now suppose, for any $M \in SL(2,\mathbf{C})$ there is a Lorentz matrix $\Lambda = \Lambda(M) \in L_+^\uparrow$ such that:

$$\Lambda(M_1)\Lambda(M_2) = \Lambda(M_1M_2) \quad and \quad \Lambda(M^{-1}) = \Lambda(M)^{-1} \tag{39}$$

to show this we must first introduce the Pauli matrices:

$$\sigma^\mu = (1_{2\times2}, \vec{\sigma}) \quad \overline{\sigma}^\mu = (1_{2\times2}, -\vec{\sigma}) \tag{40}$$

with the properties:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{41}$$

$$Tr[\sigma_i] = 0 \quad \frac{1}{2}Tr[\sigma_i\sigma_j] = \delta_{ij} \tag{42}$$

$$Tr[\sigma^\mu\overline{\sigma}^\nu] = 2\eta^{\mu\nu} \quad (+,-,-,-) \tag{43}$$

Now, we construct a map: $M_4 \to \mathbf{H}(2,\mathcal{C})$ by:

$$x^\mu \to \rho(x^\mu) = x_\mu\sigma^\mu := X \tag{44}$$

$M_4 \to \lfloor\{H(2,\mathcal{C})$ is the map from Minkowski space to the space of 2x2 complex matrices. The convention is:

$$x^\mu = (x^0, \vec{x}) \quad x_\mu = (x^0, -\vec{x}) \tag{45}$$

The inverse map would be $\rho^{-1} : \mathcal{H} \to M_4$ and it is:

$$X \to \rho^{-1} = \frac{1}{2}Tr[X\sigma^\mu] \tag{46}$$

The proof of this is trivial starting from the RHS of eq. 46. From Pauli matrices, one can easily see that:

$$X = X^\dagger = (X^*)^T \tag{47}$$

Now, consider the action of $SL(2,\mathcal{C})$ on $\mathcal{H}(\in,\mathcal{C})$ in the adjoint representation, defined by:

$$ad : SL(2,\mathcal{C}) \to Aut(\mathcal{H}(2,\mathcal{C})) \tag{48}$$

$$M \to ad(M)X \tag{49}$$

with:

$$M' = adM(X) := MXM^\dagger \quad M, M^\dagger \in SL(2,\mathcal{C}) \tag{50}$$

The automorhism group of $\mathcal{H}(2,\mathcal{C})$ is $GL(\mathcal{H}(2,\mathcal{C}))$ Now, we will verify that $adM(X) \in \mathcal{H}(2,\mathcal{C})$. Proof:

$$[adM(X)]^\dagger = (MXM^\dagger)^\dagger = MX^\dagger M^\dagger = MXM^\dagger \tag{51}$$

because X is constructed with Pauli matrices and it is Hermitian. Now, ρ transforms under $SL(2,\mathcal{C})$ as adjoint as:

$$X' = MXM^\dagger \quad or \quad x'_\mu \sigma^\mu = M\sigma^\nu x_\nu M^\dagger \tag{52}$$

Switching to index notation:

$$x'_\mu (\sigma^\mu)_{A\dot{A}} = M_A^{\ B} (\sigma^\nu)_{B\dot{B}} x_\nu (M^*)_{\dot{A}}^{\dot{B}} \tag{53}$$

from here, we see that Pauli Matrices should have 1 undotted and 1 dotted index.

$$\sigma^\mu : \dot{F} \to F \tag{54}$$

$$\overline{\sigma}^\mu : F \to \dot{F} \tag{55}$$

We know that because $M \in SL(2,\mathcal{C})$, $detM = 1$. And so:

$$det(X') = x'^\mu x'_\mu = det(MXM^\dagger) = det(M)det(X)det(M^\dagger) = x^2 \tag{56}$$

Thus the quadratic form $x^\mu x_\mu$ is left invariant under adjoint representation of $SL(2,\mathcal{C})$. We schematically now able to do:

$$\mathcal{M}_4 \xrightarrow{\rho} \mathcal{H}(2,\mathcal{C}) \xrightarrow{adM} Aut(\mathcal{H}(2,\mathcal{C})) \xrightarrow{\rho^{-1}} \mathcal{M}_4 \tag{57}$$

What we have done is rather remarkable because now we have a transformation from M_4 to M_4 which is a Lorentz transformation. Which transforms 4-vectors into 4-vectors which should leave the line element invariant.

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{58}$$

$$x'^{\mu} = \frac{1}{2}Tr[MXM^{\dagger}\overline{\sigma}^{\mu}] = \frac{1}{2}Tr[Mx_{\nu}\sigma^{\nu}M^{\dagger}\overline{\sigma}^{\mu}] \tag{59}$$

$$= \frac{1}{2}Tr[M\sigma^{\nu}M^{\dagger}\overline{\sigma}^{\mu}]x_{\nu} = \frac{1}{2}Tr[x_{\nu}\sigma^{\nu}\overline{\sigma}^{\mu}] = \frac{1}{2}x^{\nu}Tr[\overline{\sigma^{\mu}}M\sigma_{\nu}M^{\dagger}] \tag{60}$$

Therefore:

$$\Lambda^{\mu}_{\nu} = \frac{1}{2}Tr[\overline{\sigma^{\mu}}M\sigma_{\nu}M^{\dagger}] \tag{61}$$

This is the explicit form of group homomorphism:

$$SL(2,\mathcal{C}) \to L^{\uparrow}_{+} \tag{62}$$

Also, M as a function of generators of Lorentz group is:

$$M(\Lambda) = \pm\frac{1}{(det[\Lambda^{\mu}_{\nu}\sigma_{\mu}\overline{\sigma}^{\nu}])^{\frac{1}{2}}}\Lambda^{\mu}_{\nu}\sigma_{\mu}\overline{\sigma}^{\nu} \tag{63}$$

A careful reader will certainly realize that this expression has a certain ambiguity. Because both $SL(2,\mathcal{C})$ matrices M and $-M$ will lead to same transformation. This gives us a two valued representation of $L^{\uparrow}_{+}$. Which is called the spinor representation. This leads to the important identification:

$$L^{\uparrow}_{+} \cong SL(2,\mathcal{C})/\mathbf{Z}_2 \tag{64}$$

So, $SL(2,\mathcal{C})$ is a double cover of $L^{\uparrow}_{+}$. And it is the universal covering group of $L^{\uparrow}_{+}$.

3 Coleman-Mandula Theorem

Weinberg's book on Supersymmetry (Weinberg, 2013) gives a proof of the Coleman-Mandula theorem, which we will not be bothered with in this review. It states that: only possible Lie Algebra of symmetry generators is of translations and lorentz transformations, which are: P_{μ} and $J_{\mu\nu}$ respectively, together with any possible internal symmetry generators which commute with P_{μ} and $J_{\mu\nu}$ and act on physical states by multiplying them with spin-independent, momentum independent Hermitian matrices. Of course, besides C, P, T. These are the only symmetries of the S matrix under the assumptions of locality, finiteness of initial particle states, causality, and positivity of energy. And the internal symmetry group can be U(1). Thus, the full symmetry algebra of the S-matrix is:

$$[P_{\mu}, P_{\nu}] = 0 \tag{65}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \tag{66}$$

$$[M_{\mu\nu}, P_{\rho}] = -i\eta_{\rho\mu}P_{\nu} + i\eta_{\rho\nu}P_{\mu} \tag{67}$$

$$[B_l, B_m] = i\epsilon_{ijk}B_k \tag{68}$$

$$[P_{\mu}, B_l] = 0 \tag{69}$$

$$[M_{\mu\nu}, B_l] = 0 \tag{70}$$

These last two equations say that the total algebra is the direct sum of:

$$ISO(1,3) \times G \tag{71}$$

Where G is the symmetry group such as in the Gauge theories. The theorem assumes that this set of commutators is the sole thing we need for a full symmetry group. But, this assumption can be weakened to also get anticommutators as we will do in supersymmetry. Haag, Lopuzsanzski and Sohnius generalized this notion of a Lie Algebra to get graded Lie Algebras. e.g. Z_2 graded lie algebra.

4 Graded Algebras

In the notion of an ordinary Lie Algebra, there is a vector space over a field ($\mathcal{R}$ or $\mathcal{C}$) involving an antisymmetric product as:

$$\circ : L \times L \to L \tag{72}$$

The elements of this algebra obey linearity, closure and Jacobi Identity. Such as the Lie Algebra of $SU(2, \mathbf{C})$ which we've also talked about. It was defined as follows with the Pauli Matrices:

$$\tau_i := \frac{i}{2}\sigma_i \tag{73}$$

One can easily show that the elements of this algebra satisfied the rules of a Lie Algebra.

4.1 Graded Lie Algebra

Now we shall consider a vector space $\mathbf{L}$ which is the direct sum of two subspaces $\mathbf{L}_0$ and $\mathbf{L}_2$. And of course a product $\circ$.

$$\mathbf{L} = \mathbf{L}_0 \oplus \mathbf{L}_1 \tag{74}$$

with

$$\circ : \mathbf{L} \times \mathbf{L} \to \mathbf{L} \tag{75}$$

I will write the rules here because they are rather non-ordinary:

- Grading: $\forall\ x_i \in \mathbf{L}_i$, i = 0,1

$$x_i \circ x_j \in \mathbf{L}_{i+j \ \ mod2} \tag{76}$$

- Supersymmetrization: $\forall\ x_i \in \mathbf{L}_i$, $x_j \in \mathbf{L}_j$, i,j = 0,1

$$x_i \circ x_j = (-1)^{ij} x_j \circ x_i \tag{77}$$

- Generalized Jacobi Identity: $\forall\, x_k \in \mathbf{L}_k$, $\forall\, x_l \in \mathbf{L}_l$, $\forall\, x_m \in \mathbf{L}_m$, k,l,m $\in \mathbf{Z}_2$

$$x_k \circ (x_l \circ x_m)(-1)^{km} + x_l \circ (x_m \circ x_k)(-1)^{lk} + x_m \circ (x_k \circ x_l)(-1)^{ml} = 0 \quad (78)$$

With these definitions, $\mathbf{L}$ as a vector space becomes a graded Lie Algebra. It is sometimes antisymmetric and sometimes symmetric due to 77. This is a grade 2 algebra because the equation 76 is mod2. L_0 is the Poincare Algebra and $L_1 = (Q^I_\alpha, \overline{Q}^I_\alpha)$ is the anticommuting fermionic generators. I runs through 1 to N and L_1 has members of N+N=2N number. In the Lorentz representations of $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ respectively. As one would notice, this is a direct consequence of the double covering of $SL(2, C)$ that we have talked about. These Q's are simply the Majorana spinor charges. Now, let us generalize Poincare algebra as follows:

$$[P_\mu, Q^I_\alpha] = 0 \quad (79)$$

$$[P_\mu, \overline{Q_\alpha}^I] = 0 \quad (80)$$

$$[M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha^{\ \beta} Q^I_\beta \quad (81)$$

$$[M_{\mu\nu}, \overline{Q}^{I\dot{\alpha}}] = i(\overline{\sigma_{\mu\nu}})_{\dot{\alpha}\dot{\beta}} \overline{Q}^{I\dot{\beta}} \quad (82)$$

$$\{Q^I_\alpha, \overline{Q}^J_{\dot{\beta}}\} = 2\sigma^\mu_{\ \alpha\dot{\beta}} \delta^{IJ} P_\mu \quad (83)$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \quad (84)$$

$$\{Q^I_{\dot{\alpha}}, Q^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})* \quad (85)$$

And of course, Z is antisymmetric. And Z is the central charge of the theory. So, they commute with all of the other group generators B_l. Let's now stop for a moment and look at the equations 81 and 82 in terms of angular momentum. Recall: $M_{12} = J_3$ the z component of the angular momenta.

$$[J_3, Q^I_1] = i(\sigma_{12})_\alpha^{\ \beta} Q^I_1 = \frac{1}{2} Q^I_1 \quad (86)$$

$$[J_3, Q^I_2] = i(\sigma_{12})_\alpha^{\ \beta} Q^I_2 = -\frac{1}{2} Q^I_2 \quad (87)$$

We can also take the complex conjugate of these to get 82. But, it is easy to see that Q^I_1 and $\overline{Q}^I_2$ raises the z-component of the spin by half unit, while other two lowers it bu half unit. Another comment one could make is that since fermionic generators form an irreducible representation of the Lorentz group in Z_2 sense, one can conclude that their representation is: $(\frac{1}{2}, \frac{1}{2})$. And make an ansatz that :

$$\{Q^I_\alpha, \overline{Q}_{\dot{\beta} J}\} = \sigma^\mu_{\ \alpha\dot{\beta}} P_\mu K^I_J \quad (88)$$

D is Hermitian and can be unitarily diagonalized (Wess, Bager 1992). Thus K should be a delta function. And for equation 84, let us calculate:

$$\{Q^I_\alpha, Q^J_\beta\} \tag{89}$$

This can have an antisymmetric part and a symmetric part:

$$= \epsilon_{\alpha\beta} Z^{IJ} + M_{\alpha\beta} C^{IJ} \tag{90}$$

From the Coleman-Mandula theorem, the only candidate for the symmetric part is the symmetric spin-1 tensor $M_{\alpha\beta}$. (Bertolini, 2024)

Having made the ansatz of the algebra, we can verify the truthness of the algebra by Z_2 graded algebra as we have defined it above. It is rather straightforward, we put Q's instead of x's. And also, many properties and anticommutators that we have shown for supersymmetry, can be proven by using the generalized Jacobi identity. By using the triples:(P,P,Q),(Q,$\overline{Q}$,P),($M_{\mu\nu}$,Q,$\overline{Q}$). The calculations of these are long enough to not be written in this small review. So, C^{IJ} comes out as 0. So we have:

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \tag{91}$$

However, by explicit calculation and to stay consistent with Majarona formalism of fermions.(Kirsten-Müller, Weiedemann 1987). One can show that the Super Poincare algebra for N=1 is fully:

$$[P_\mu, Q_\alpha] = 0 \tag{92}$$
$$[P_\mu, P_\nu] = 0 \tag{93}$$
$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho})P_\nu - \eta_{\nu\rho} P_\mu) \tag{94}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \tag{95}$$
$$[M_{\mu\nu}, Q^\alpha] = -(\sigma^4_{\mu\nu})_{ab} Q_b \tag{96}$$
$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\gamma^\mu)_{ab} P_\mu \tag{97}$$
$$\{Q_\alpha, Q_\beta\} = -2(\gamma^\mu C)_{ab} P_\mu \tag{98}$$
$$\{\overline{Q}_a, \overline{Q}_b\} = 2(C^{-1}\gamma^\mu_{ab}) P_\mu \tag{99}$$

In Majarona formalism.

5 Going super

5.1 Superspace

The idea here in superspace is to enlarge the idea of space labels (x^μ) in QFT. In the sense of supersymmetry, we introduce two more coordinates θ_α and $\overline{\theta}_{\dot\alpha}$ to stay consistent with the supersymmetric QFT. However, this is only due to we have restricted ourselves with N=1. These labels are Grasmaninan and they behave according to the Fierz reordering formulas as we have stated in the beginning. Note that the square of these coordinates vanishes. Also, derivatives anticommute:(Nath, 2016)

$$\{\frac{\partial}{\partial\theta_\alpha}, \frac{\partial}{\partial\overline{\theta}_{\dot{\beta}}}\} = 0 \tag{100}$$

In the Poincare invariance discussion of QFT, one should have:

$$\int dx\ f(x+a) = \int dx\ f(x) \tag{101}$$

The same should also hold for the Grassman integration: (Dine, 2016)

$$\int d\theta\ \theta(x+\epsilon) = \int d\theta\ \theta(x) \tag{102}$$

and this is satisfied by the integration rule:

$$\int d\theta\ (1,\theta) = (0,1) \tag{103}$$

By using this construction, a simple integral table is:

$$\int d^2\theta\ \theta^2 = 1 \quad and \quad \int d^2\overline{\theta}\ \overline{\theta}^2 = 1 \tag{104}$$

The Grassmann version of FTC is:

$$\int d\theta \frac{df}{d\theta} = 0 \tag{105}$$

And, if we have a nice function f(x, θ), Grassmann integration picks θ out:

$$\int d\theta f(x,\theta) = \int d\theta (f(x) + \theta f(x)_1) = f_1(x) \tag{106}$$

The δ can be obtained by:

$$\int d^2\theta(\theta\theta) = \int d^2\theta\delta^2(\theta) = 1 \tag{107}$$

Another important property is that, as we have seen in the beginning, when we were playing with Grassmanian numbers θ, on a superspace, more than two same genre θ's vanish due to anticommutation. So, the most general scalar function one could write is:(Bilal, 2001)

$$\begin{aligned} F(x,\theta,\overline{\theta}) = f(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta m(x) + \overline{\theta}\overline{\theta}n(x) + \theta\sigma^\mu\overline{\theta}A_\mu(x) \\ + \theta\theta\overline{\theta\lambda(x)} + \overline{\theta}\overline{\theta}\theta\rho(x) + \theta\theta\overline{\theta}\overline{\theta}K(x) \end{aligned} \tag{108}$$

Another interesting property in this space is:

$$\int d^2\theta\ \theta^2 = \frac{1}{4}\epsilon^{ab}\frac{\partial}{\partial\theta_a}\frac{\partial}{\partial\theta_b} \quad and \quad \int d^2\theta\ \overline{\theta}^2 = -\frac{1}{4}\epsilon^{\dot{a}\dot{b}}\frac{\partial}{\partial\overline{\theta}^{\dot{a}}}\frac{\partial}{\partial\overline{\theta}^{\dot{b}}} \tag{109}$$

Now we should realize that these Q's are generators of SUSSY in the superspace. And they are induced by a Noether current. So, we are physicists, we should take a look at the infinitesimal expansion of Q and see what transformation it yields. We want $i\epsilon^a\theta_a$ to generate n infinitesimal translation in θ by a constant ϵ and also some translation in x. We can infer right now that, the spacetime transformation on x, is determined by the anticommutator 83. Because as we can see from the RHS, it generates the spacetime transformations with P. So:

$$(1 + i\epsilon Q)F(x, \theta, \overline{\theta}) = F(x + \delta x, \theta + \delta\theta, \overline{\theta}) \tag{110}$$

Now, we know that the iQ_α must be of the form:

$$iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} + K_\alpha \tag{111}$$

To determine K_α, we make the following ansatz: $K_\alpha = c(\sigma^\mu\overline{\theta})_\alpha P_\mu = -ic(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$. Cause we know from ordinary QFT that $(P_\mu \to -i\partial_\mu)$. This ansatz make total sense because it contains the elements we have talked so far and the generators of spacetime transformations. Choose c=1.

$$-\{Q_A, \overline{Q}_{\dot{B}}\} = \{\partial_A - i\sigma^\mu_{A\dot{C}}\overline{\theta}^{\dot{C}}, -\overline{\partial}_{\dot{B}} + i\theta^C\sigma^\nu_{C\dot{B}}\partial_\nu\} \tag{112}$$

the indeces easily label the coordinates.

$$-\{\partial_A, \overline{\partial}_{\dot{B}}\} + i\{\partial_A, \theta^C\sigma^\nu_{C\dot{B}}\partial_\nu\} + i\{\sigma^\mu_{A\dot{C}}\overline{\theta}^{\dot{C}}\partial_\mu, \overline{\partial}_{\dot{B}}\} + \{\sigma^\mu_{A\dot{c}}\overline{\theta}^{\dot{C}}\partial_\mu, \theta^B\sigma^\nu_{C\dot{B}}\partial_\nu\} \tag{113}$$

The first term is zero due to Grassmanian nature of partial derivatives. The last term is 0.

$$i\theta^C\sigma^\nu_{C\dot{B}}\partial_\nu\partial_A + i\sigma^\mu_{A\dot{C}}\overline{\theta}^{\dot{C}}\partial_\mu\overline{\partial}_{\dot{B}} - i\sigma^\mu_{A\dot{B}}\partial_\mu - i\sigma^\nu_{A\dot{B}}\partial_\nu \tag{114}$$

$$= 2i\sigma^\nu_{A\dot{B}}\partial_\nu = -2\sigma^\mu_{A\dot{B}}P_\mu \tag{115}$$

Thus we conclude with our choices this is consistent.

$$\{Q_A, \overline{Q}_{\dot{B}}\} = 2\sigma^\mu_{A\dot{B}}P_\mu \tag{116}$$

There is additional content to be talked about here. Q's turn fermionic states into bosons and bosons into fermions. (Kirsten-Müller, Wiedemann 1987) And there is this important theorem: Every representation of the supersymmetry algebra as we have written down, contains as many bosonic states as fermionic states.

Proof: We introduce the fermion number operator N_F, which has the property $(-1)^{N_F}$ as eigenvalue +1 when hits on bosonic states.(states containing an even number of fermions n_F). Thus,

$$(-1)^{N_F}|\ \rangle = (-1)^{n_F}|\ \rangle \tag{117}$$

the state here contains n_F fermions. Q_A and $\overline{Q}^{\dot{A}}$ annihilate or create fermions, we have:

$$Q_A(-1)^{N_F}|\ \rangle = (-1)^{N_F-1}Q_A|\ \rangle \tag{118}$$

or

$$(-1)^{N_F}Q_A = -Q_A(-1)^{N_F} \tag{119}$$

In our definition of the Super-Poincare algebra, the trace is well-defined.

$$Tr[(-1)^{N_F}\{Q_A, \overline{Q}_{\dot{B}}\}] = Tr[(-1)^{N_F}Q_A\overline{Q}_{\dot{B}}] + Tr[Q_A(-1)^{N_F}\overline{Q}_{\dot{B}}] = 0 \tag{120}$$

by using the above equation. However, if we use the explicit anticommutation:

$$Tr[(-1)^{N_F}\{Q_A, \overline{Q}_{\dot{B}}\}] = Tr[(-1)^{N_F}2(\sigma^\mu)_{A\dot{B}}P_\mu] = 2(\sigma^\mu)_{A\dot{B}}P_\mu Tr[(-1)^{N_F}] \tag{121}$$

Then in a theory where Hamiltonian or 3 momentum is conserved, or if we simply fix the momentum:

$$= Tr[(-1)^{N_F}] = 0 \tag{122}$$

This results in:

$$\sum_B \langle B|(-1)^{N_F}|B\rangle + \sum_F \langle F|(-1)^{N_F}|F\rangle = N_B - N_F = 0 \tag{123}$$

Where B is for bosonic and F is for fermionic. This will be the case for supersymmetry breaking.

5.2 Superfields

A general superfield, such as the function in equation 108, can contain many terms. But the most general superfield is:

$$\begin{aligned}\Phi(x,\theta,\overline{\theta}) = f(x) + \theta\phi(x) + \overline{\theta}\chi(x) + (\theta\theta)n(x) + (\theta\sigma^\mu\overline{\theta})V_\mu(x)\\ + (\theta\theta)\overline{\theta\lambda} + (\overline{\theta\theta})(\theta\psi(x)) + (\theta\theta)(\overline{\theta\theta})d(x)\end{aligned} \tag{124}$$

One can convince themselves to believe that this is the most general superfield one can get. The fields in this expansion are called component fields. Under Lorentz group the components are:(Kirsten-Müller,Wiedemann 1987)

- $f(x), m(x), n(x)$ are complex scalar/pseudo-scalar fields
- $\psi(x)$,$\phi(x)$ are left-handed Weyl Spinors
- $\overline{\chi}(x)$,$\overline{\lambda}$ are right-handed Weyl Spinors
- $V_\mu(x)$ is a Lorentz 4-vector field.
- $d(x)$ is a scalar field.

The field has too many terms. That is because it is does not have a non-reducible representation. So, to reduce the terms, we employ the notion of covariant derivative:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu \tag{125}$$

and with the dotted indices:

$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{126}$$

Now, a chiral superfield is a field that satisfies:

$$\overline{D}_{\dot{\alpha}}\phi = 0 \tag{127}$$

and an anti-chiral field is:

$$D_\alpha\overline{\phi} = 0 \tag{128}$$

It is easier to solve this by:(Wess, Bager 1992)

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta} \tag{129}$$

for:

$$\overline{D}_{\dot{\alpha}}(x^\mu + i\theta\sigma^\mu\overline{\theta}) = 0 \tag{130}$$

because obviously,

$$\begin{aligned} &= (-\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu)(x^\mu + i\theta\sigma^\mu\overline{\theta}) \\ &= i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}} - i\theta^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\delta^\mu_\nu = 0 \end{aligned} \tag{131}$$

and

$$\overline{D}_{\dot{\alpha}}(\theta) = 0 \tag{132}$$

any function of these variables will satisfy the equation. However, we should have a limited field which can solve this because certain combinations are 0. Now, as we also did in equation 108:

$$\Phi = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{133}$$

$$= A(x)+i\theta\sigma^\mu\overline{\theta}\partial_\mu A(x)+\frac{1}{4}\theta\theta\overline{\theta\theta}\partial^2 A(x)+\sqrt{2}\theta\psi(x)-\frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\overline{\theta}+\theta\theta F(x) \tag{134}$$

Terms cancel out due to the nature of Grassmanian numbers. And eq. 134 is just the Taylor expansion of equation 133. This is the full exact solution of the chirality condition. Thus, any chiral field should involve these. Physically, such a super chiral field describes a scalar field, A fermionic field ψ , and an auxilary field F. We can also now see how the component fields transform by explicit SUSSY transformations. We have already seen the infinitesimal transformation of SUSSY. We will have the same approach here. First lets change the variables from $(x, \theta, \overline{\theta})$ to $(y, \theta, \overline{\theta})$. By doing this we now have:

$$Q_\alpha = -i\frac{\partial}{\partial_\theta^\alpha} \tag{135}$$

$$\overline{Q}_\alpha = i\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + 2\theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \frac{\partial}{\partial y^\mu} \tag{136}$$

So the variation is:

$$\delta\phi(y,\theta) = (i\epsilon Q + i\overline{\epsilon}\overline{Q})\phi(y,\theta) = (\epsilon^\alpha \partial_\alpha + 2i\theta\sigma^\mu\overline{\epsilon}\frac{\partial}{\partial y^\mu})\phi(y,\theta) \tag{137}$$

$$= \sqrt{2}\epsilon\psi - 2\epsilon\theta f + 2i\theta\sigma^\mu\overline{\epsilon}(\partial_\mu z + \sqrt{2}\theta\partial_\mu\psi) \tag{138}$$

$$= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(-\sqrt{2}\epsilon f + \sqrt{2}i\sigma^\mu\overline{\epsilon}\partial_\mu z) - \theta\theta(-i\sqrt{2}\overline{\epsilon\sigma^\mu}\partial_\mu\psi) \tag{139}$$

Thus the changes in the fields are:

$$\delta z = \sqrt{2}\epsilon\psi \tag{140}$$

$$\delta\psi = -\sqrt{2}\epsilon f + \sqrt{2}i\sigma^\mu\overline{\epsilon}\partial_\mu z \tag{141}$$

$$\delta f = i\sqrt{2}\overline{\epsilon\sigma^\mu}\partial_\mu\psi \tag{142}$$

The Fierz identities that we have used in137 are:

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{143}$$

$$\overline{\theta}^{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\overline{\theta\theta} \tag{144}$$

$$\theta\sigma^\mu\overline{\theta}\theta\sigma^\nu\overline{\theta} = -\frac{1}{2}\eta^{\mu\nu}\theta\theta\overline{\theta\theta} \tag{145}$$

Let us do a small summary here. Up to this point, we evaded the Coleman-Mandula theorem and showed that there should be other symmetry such as supersymmetry. We have defined Graded Lie Algebra, namely Z_2 graded Lie algebra, which enabled us to define fermionic charges. Then, we have made the ansatz that ordinary Poincare algebra can be extended with the help of supersymmetry and gauge theory. We have defined superspace and superfields which will enable us to use the Lagrangian formalism for SUSSY. We have also shown how fields transform under SUSSY transformations. We have explicitly seen that scalars change into fermions, fermions going into scalars and auxiliary fields, and auxiliary fields going into fermions.

6 Supersymmetric Actions' construction

6.1 Change Induced by Supersymmetry

The infinitesimal change in the superfield is:

$$\delta\phi = i[\epsilon Q + \bar{\epsilon}\overline{Q}, \phi] = i(\epsilon Q + \bar{\epsilon}\overline{Q})\phi \tag{146}$$

So an action should be invariant under this transformation. We have:

$$\delta S = \int d^4\theta \int d^4x \delta K \tag{147}$$

Where K is a superfield satisfying the reality condition. We have already seen that a super field under SUSSY transformations changes as:

$$\delta K = \epsilon^\alpha(\partial_\alpha K - i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu K) + (-\overline{\partial}_{\dot{\alpha}} K + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu K)\epsilon^{\dot{\alpha}} \tag{148}$$

We can expand K:

$$K(x, \theta, \overline{\theta}) = K_{first}(x) + ... + \theta^2\overline{\theta}^2 K_{last}(x) \tag{149}$$

However, the 4-dim integration measure $d^4\theta$ will only pick the last term.

$$S = \int d^4x K_{last}(x) \tag{150}$$

The terms that come from integrating out all the superspace is D terms.(Kinetic terms) This is great. Because any term in Lagrangian that is a D-term transforms as a total derivative and integrates out thus invariant.

6.2 Possible Actions

Now we are in a position to try to construct Actions of supersymmetry and determine the potentials and define vector superfields. A supersymmetric action of course involve integrals over grassmanian space labels θ and $\overline{\theta}$. As we did above, we can write products of chiral and antichiral fields as polynomials of θ and $\overline{\theta}$. The simplest case would be the product of $\Phi(z, \overline{\theta})$ and $\Phi(y, \theta)$ then integrating them over superspace components. The most general supersymmetric and renormalizable Lagrangian is: (Kirsten-Müller,Wiedemann 1987)

$$\mathcal{L} = \Phi_i^\dagger \Phi + (g_i \Phi_i + \frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k)\delta^2(\overline{\theta}) + \tag{151}$$

$$(g_i^* \Phi_i + \frac{1}{2} m_{ij}^* \Phi_i^\dagger \Phi_j^\dagger + \frac{1}{3}\lambda_{ijk}^* \Phi_i^\dagger \Phi_j^\dagger \Phi_k^\dagger)\delta^2(\theta) \tag{152}$$

The renormalizability condition for this Lagrangian forbids terms of higher than degree three. (Wess, Bager 1992) From equation 133 we know that the power of two terms has the expansion:

$$\Phi_i\Phi_k = A_i(y)A_k(y) + \sqrt{2}\theta[A_i(y)\psi_k(y) + \psi(y)A_k(y)] \tag{153}$$
$$+(\theta\theta)[A_i(y)F_k(y) + F_i(y)A_k(y) - \psi_i(y)\psi_k(y)] \tag{154}$$

Which is also a left handed chiral field. The dagger one can be found easily:

$$\Phi_i^\dagger\Phi_k^\dagger = A_i^*(z)A_k^*(z) + \sqrt{2}\theta[A_i^*(z)\overline{\psi}_k(z) + \overline{\psi}(z)A_k^*(z)] + \tag{155}$$
$$(\overline{\theta\theta})[A_i^*(z)F_k^*(z) + F_i^*(z)A_k^*(z) - \overline{\psi}_i(z)\overline{\psi}_k(z)] \tag{156}$$

Which is also a right handed chiral field. The power of three left handed and right handed chiral fields can be found by explicit computation. However, they should be easy because powers of 3 θ's will vanish. The measure is:

$$\int d^4x \int d^4\theta = \int d^4x \int d^2\theta d^2\overline{\theta} \tag{157}$$

If one does to build an action with just the kinetic term (D-term):

$$S = \int d^4x \int d^4\theta \ \Phi\Phi^\dagger \tag{158}$$

After explicit calculation, integration by parts, and remembering that any $n > 4$ θ^n configuration vanishes, also $n = 3$, one gets:

$$S = \int d^4x \ [\partial_\mu\phi^\dagger\partial^\mu\phi - i\overline{\psi}\overline{\sigma^\mu}\partial_\mu\psi + F^\dagger F] \tag{159}$$

One should notive that F, the auxilary field does not have kinetic term. SO, it should be just potential.

6.3 Dimensions

From the covariant derivative one can easily see that θ has dimension $\frac{1}{2}$ in length. From Berezin integration, we deduce that $d\theta$ has dimension $-\frac{1}{2}$. Therefore, the measure $\int d^4\theta$'s dimension is $-\frac{1}{2}$. The spinor has usual dimension $-\frac{3}{2}$. For the lagrangian to stay dimensionless, we can now assign superfield dimension -1. By this method, the mass term, m_{ij} has dimension -1. And λ_{ijk} is dimensionless. Which confirms that it is an interaction term.

6.4 Superpotential

A function of chiral superfields, W, is also a chiral superfield. It's components are:

$$W(\Phi) = W(\phi) + \sqrt{2}\frac{\partial W}{\partial\phi}\theta\psi + \theta^2(\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial^2\phi}\psi\psi + ...) \tag{160}$$

However, if we look at equation 134 there are 4 terms that are total derivatives. (The ones that contain $\overline{\theta}$). So they will not contribute to the action. So, W cannot be a function of $\overline{\theta}$. Thus, a function of superfield should be integrated on just the respected half of the superspace. The $W(\phi)$ should be integrated on $d^2\theta$ and $W(\phi^\dagger)$ should be integrated on $d^2\overline{\theta}$. Otherwise, they will contribute nothing. Thus,

$$S_W = \int d^4x[\int d^2\,\theta W(\Phi) + \int d\overline{\theta}^2\; W(\Phi^\dagger)] \tag{161}$$

The term, θ^2 will give us contribution as an F-term. This is the auxiliary field as we mentioned. And we also see from equation 142, that F under SUSSY transformations transforms as a total derivative. So, now we know that S_W term in action is invariant. Let us now put, D-term and F-term together.(The D-term here is usually called the Kahler potential, and has a huge role in non-linear sigma models.)

$$S = \int d^4x\, [\partial_\mu\phi^\dagger\partial^\mu\phi - i\overline{\psi}\overline{\sigma^\mu}\partial_\mu\psi + F^\dagger F + (\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial^2\phi}\psi\psi + h.c.)] \tag{162}$$

Thus, we have finally arrived at the famous Wess-Zumino action. The W is the superpotential. One can chose it cubic for renormalizability. We can eliminate F from the action because it has no dynamics. The EoM of $F^\dagger$ is:

$$F + \frac{\partial W^\dagger}{\partial\phi^\dagger} = 0 \tag{163}$$

For F :

$$F^\dagger + \frac{\partial W}{\partial\phi} = 0 \tag{164}$$

So that we can rewrite the action with the scalar potential:

$$S = \int d^4x\, [\partial_\mu\phi^\dagger\partial^\mu\phi - i\overline{\psi}\overline{\sigma^\mu}\partial_\mu\psi + |\frac{\partial W}{\partial\phi}|^2 + (-\frac{1}{2}\frac{\partial^2 W}{\partial^2\phi}\psi\psi + h.c.)] \tag{165}$$

Where scalar potential is:

$$V = |\frac{\partial W}{\partial\phi}|^2 \tag{166}$$

For a theory with more than one scalar field with a canonical Kahler Potential:

$$V = \sum_i |\frac{\partial W}{\partial\phi_i}|^2 \tag{167}$$

The symmetry transformations for this model are:

$$\delta\phi = \sqrt{2}\epsilon\psi \tag{168}$$

$$\delta\psi = -\sqrt{2}i\overline{\epsilon}\sigma^\mu\partial_\mu\phi^\dagger + \sqrt{2}\overline{\epsilon}F \tag{169}$$

$$\delta F = \sqrt{2}i\overline{\epsilon}\overline{\sigma}^\mu\partial_\mu\overline{\psi} \tag{170}$$

Remembering that $\psi\sigma^\mu\overline{\epsilon} = -\overline{\epsilon}\overline{\sigma}^\mu\psi$.

6.5 Wess-Zumino Action

One usually chooses the superpotential for the Wess-Zumino action ϕ^3, due to renormalizability issues(Haber, Haskins 2017) making it the supersymmetric extension of ϕ^3 theory.

$$W = a\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 \tag{171}$$

one should note that any other theory without canonical Kahler potential is nonrenormalizable.(Tong, 2024) We will interpret this model as an effective low energy description of a more fundamental theory in which parameters m and λ arise as vacuum expectation values of heavy superfields. This interpretation enhances the apparent symmetries of the theory. (Shirman,2008)

7 Symmetry Breaking

To this day, physicists were unable to detect any superpartners. So, it has to be broken somewhere. And (Murayama,2007) physicists believe that there is a large landscape of theories where supersymmetry is broken. The Hilbert

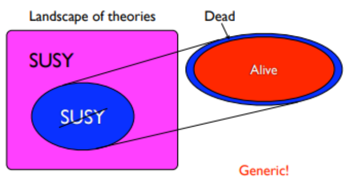

Figure 2: There are a lot of generic theories with supersymmetry breaking that are phenomenological viable, easy and generic. (Murayama, 2007 p.2)

Space in the quantum level of supersymmetry is decomposed into two parts:

$$\mathcal{H}_B \oplus \mathcal{H}_F \tag{172}$$

which is Z_2 grading of Hilbert Space. If Q's act on ground states and not giving 0 eigenvalue. Then, the theory is said to be spontaneously broken. As in gauge theory, then the theory has multiple ground states. Thus giving away goldstone bosons. And, the supersymmetric partner of the goldstone boson is the goldstino fermion. Also, the ground energy can also be introduced in this way: We define a new operator:

$$Q' = \{Q_1, \overline{Q}_1\} + \{Q_2, \overline{Q}_2\} = 4P_0 \tag{173}$$

Which is the Hamiltonian.

$$H = \frac{1}{4}\{Q_1, \overline{Q}_1\} + \frac{1}{4}\{Q_2, \overline{Q}_2\} \tag{174}$$

Take the ground state:

$$\langle 0|\frac{1}{4}\{Q_1, \overline{Q}_1\} + \frac{1}{4}\{Q_2, \overline{Q}_2\}|0\rangle = E_0 \tag{175}$$

It is known that all supersymmetric theories have non-negative energy. (Tong, 2024). This reflects on the fact that E = 0 must be the ground state. These states will obey

$$Q|0\rangle = 0 \tag{176}$$

And conversely, the supersymmetry is broken if the energy of the ground state is nonvanishing. To look for these, we should look at the potential as we do in gauge theory. We have seen the theories with canonical Kahler potential, the potential energy is:

$$V = |\frac{\partial W}{\partial \phi}|^2 = F^2 \tag{177}$$

Now, the ground state is 0 only if V is vanishing. That is, F^2 is vanishing. So theories with:

$$F \neq 0 \tag{178}$$

This is known as F-term supersymmetry breaking. As an example, consider the potential in equation 171 :

$$W = \mu^2\Phi + \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 \tag{179}$$

Scalar potential is:(Nath, 2016)

$$V(\phi) = |\mu^2 + m\phi + \lambda\phi^2|^2 \tag{180}$$

Thus the condition that supersymmetry is not broken at the quantum level is:$\langle V\rangle = 0$

$$\mu^2 + m\overline{\phi} + \lambda\overline{\phi}^2 = 0 \tag{181}$$

which gives:

$$\overline{\phi} = \frac{-1}{2\lambda}[m \pm \sqrt{m^2 - 4\lambda\mu^2}] \tag{182}$$

So, there is a root. SUSSY is not spontaneously broken. However, U(1) is still broken thus we have a goldstone.

After O'Raifeartaigh, we will see the golstino fermion. The O'Raifeartaigh potential is:

$$W[\Phi_i] = \lambda\Phi_1(\Phi_3^2 - \mu^2) + m\Phi_2\Phi_3 \tag{183}$$

Where Φ has the usual expansion as in equation 133. And we have again the canonical Kahler potential in kinetic term Then define,

$$F_i = -\frac{\partial W^*}{\partial \phi_i^*} \quad and \quad F_i^* = -\frac{\partial W}{\partial \phi_i} \tag{184}$$

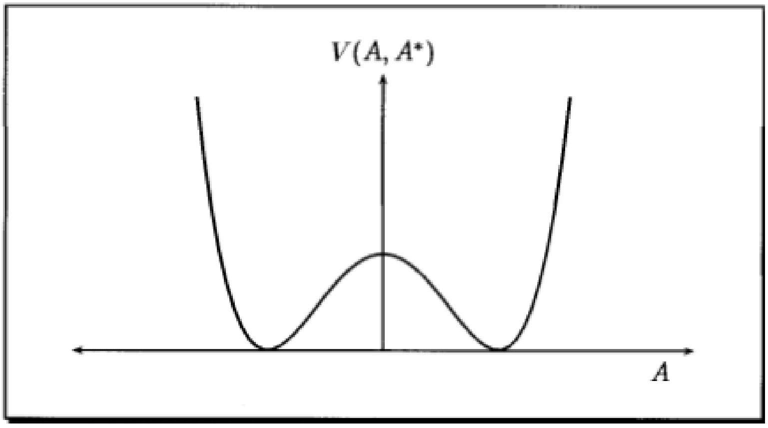

Figure 3: The Potential Diagram for the Φ^3 potential (Müller-Kirsten, Wiedemann, p.367)

Then F terms are:

$$F_1^* = -\lambda(\phi_3^2 - \mu^2) \tag{185}$$

$$F_2^* = -m\phi_3 \tag{186}$$

$$F_3^* = -(2\lambda\phi_1\phi_3 + m\phi_2) \tag{187}$$

The Scalar potential is:

$$V = \sum_i F_i F_i^* \tag{188}$$

One can now easily see that supersymmetry is spontaneously broken because $V \neq 0$ anywhere. But, the model becomes interesting after one imposes one of F is 0, to get the correct mass sign. One can inspect the symmetries in there. Becuase in this theory, there are actually two minimas so that the internal $U(1)$ symmetry is also broken. The potential diagram for this is in figure 4.

7.1 Goldstino

We know that a vacuum-breaking sussy has the following properties:

- if it is a vacuum
$$\frac{\partial V}{\partial z_i}(\langle z^i\rangle, \langle z_i^\dagger\rangle) = 0 \tag{189}$$

- $\langle f_i\rangle \neq 0$ or $\langle D^\alpha\rangle \neq 0$

Then we have:(Bilal,2001)

$$\frac{\partial V}{\partial z^i} = f^j \frac{\partial W}{\partial z^i \partial z^j} - g^a D^a z_j^\dagger (T^a)_i^j \tag{190}$$

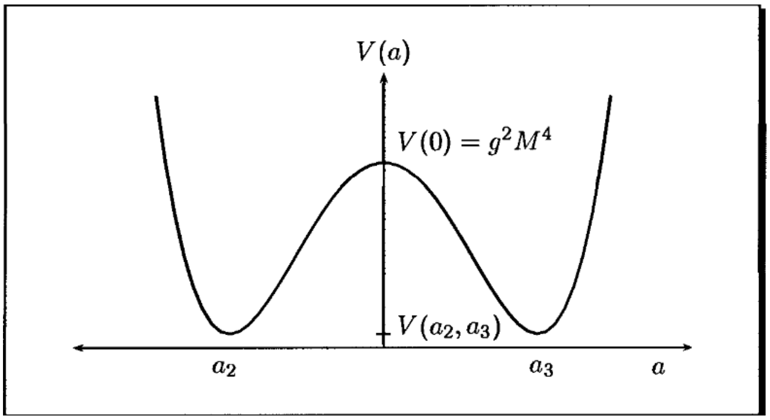

Figure 4: The Potential Diagram for O'Raifeartaigh (Müller-Kirsten, Wiedemann, p.376)

This must vanish for any vacuum. Combining the former equation with:

$$0 = \delta^a_{gauge} W = \frac{\partial W}{\partial z^i} \delta^a_{gauge} z^i = f^\dagger_i (T^a)^i_j z^j \tag{191}$$

Then we take vacuum expectation value of equation 190 and with equation 191 then tun it into a matrix equation:(Bilal, 2001)

$$M = \begin{pmatrix} \langle \frac{\partial W}{\partial z^i \partial z^j} \rangle & -g^a \langle z^\dagger_l (T^a)^l_i \rangle \\ -g^b \langle z^\dagger_l (T^b)^l_j & 0 \end{pmatrix} \tag{192}$$

so that:

$$M \begin{pmatrix} \langle f_j \rangle \\ \langle D^a \rangle \end{pmatrix} = 0 \tag{193}$$

Where $\langle f_i \rangle$ stands for $f^i(\langle z^\dagger \rangle)$ and the same goes for D^a. So now we know that M has a zero eigenvalue. However, M is the same as the fermion mass matrix in full $N = 1$ with gauge, matter, Lagrangian. One can look at Adel Bilal's Supersymmetry review for this Lagrangian. The related terms are:

$$i\sqrt{2} g^a \langle z^\dagger_j \rangle (T^a)^j_i \lambda^a \psi^i - \frac{1}{2} \langle \frac{\partial W}{\partial z^i \partial z^j} \rangle \psi^i \psi^j + h.c. \tag{194}$$

$$= -\frac{1}{2} \begin{pmatrix} \psi^i i & \sqrt{2} i \lambda^b \end{pmatrix} M \begin{pmatrix} \psi^i i \\ \sqrt{2} i \lambda^b \end{pmatrix} \tag{195}$$

However, this makes this matrix, we put M in the middle so it has a zero eigenvalue. Which means that there is a massless fermion at the vacua. Which is the goldstino or goldstone fermion. An important remark is that Goldstino Boson is a candidate for dark matter.

7.2 R-Symmetry

In supersymmetric theory, the proton lifetime is 10^{-12} sec. Which is really short from the real-time of 10^{34} years! So, one defines the R-partiy to get rid of this absurdly short time. (Murayama, 2007) The Super Poincare algebra can also be extended with gauge symmetries in superspace. For example defining $U(1)_R$ generator such that:(Haber, Haskins, 2017)

$$[R, Q] = -Q \ \ and \ \ [R, \overline{Q}] = \overline{Q} \tag{196}$$

Also, R should commute with other operators in our algebra. This generator can also be written in differential form such that:(Nath, 2016)

$$[\Phi, R] = \hat{R}\Phi \tag{197}$$

where $\hat{R} = \theta^{\alpha}\partial_{\alpha} - \overline{\theta}_{\dot{\alpha}}\partial^{\dot{\alpha}} - n$. Where n is the R-charge. Acting on a superfield it has:

$$R\Phi = e^{ina}\Phi(x, e^{-ia}\theta, e^{ia}\overline{\theta}) \tag{198}$$

The component fields under R transform as:

$$A \rightarrow e^{ina}A \tag{199}$$

$$\psi \rightarrow e^{i(n-1)a}\psi \tag{200}$$

$$F \rightarrow e^{i(n-2)a}F \tag{201}$$

We can make some observations here. The canonical Kahler potential is automatically invariant under R. However, the F term is invariant under R only if W has charge n=2 with canonical Kahler potential.

An important theorem to mention here is the Nelson-Seiberg theorem about R-Symmetry. Which is: (Sun,2022,p.15)

> In a Wess-Zumino model with a generic superpotential, an R-symmetry is a necessary condition, and a spontaneously broken R-symmetry is a sufficient condition for SUSY breaking at the vacuum of a global minimum.

7.3 Witten Index

We must note that all theories cannot spontaneously break supersymmetry. A topological quantity Witten Index is the perfect condition for it. It is:

$$Tr[(-1)^F]e^{-\beta H} \tag{202}$$

Where H is the Hamiltonian and $(-1)^F$ is the fermion number operator that we introduced before. In Witten's 1982 paper, the operator $(-1)^F$ is denoted as: $exp(2\pi i J_z)$. This makes total sense because for bosonic states the eigenvalue is 1 and for fermionic states, it is -1. So that,

$$Tr[(-1)^F] = n_B^{E=0} - n_F^{E=0} \tag{203}$$

β is a regulator on high-energy states. As a result, high-energy states they do not contribute to the Witten Index. Also, the formula is independent of β because $E \neq 0$ states do not contribute. We will only count at the ground state, the difference between fermion and boson number. And the Witten index counts exactly this. (Bertolini,2024)

Let us do a simple example here. Take the Wess-Zumino model and the superspace potential:

$$W(\phi) = \frac{1}{3} g\phi^3 - \frac{m^2}{4g}\phi \tag{204}$$

The ordinary potential is now:

$$V = |\frac{\partial W}{\partial \phi}|^2 = g^2|\phi^2 - \frac{m^2}{4g^2}|^2 \tag{205}$$

There is also the fermionic field that is involved in the Yukawa coupling: $g\phi\psi_L^\alpha\psi_L^\beta\epsilon_{\alpha\beta} + h.c.$ But, we will not be interested in it. In the weak coupling, the quantum level has two vacuum states:

$$\langle\phi\rangle = \pm\frac{m}{2g} \tag{206}$$

So that there is the Z_2 symmetry for scalar field minimum. Then, expanding around the minimum one finds that ϕ and ψ is massive for weak coupling.

$$m_\phi = m_\psi = m(1 + O(2)) \tag{207}$$

And in each minimum there is only one zero energy state. Which is the vacuum, the vacuum has spin zero therefore bosonic. All other states are acquired by adding ϕ or ψ into the vacuum. However, they have energy $E \geq m$. Since the ϕ and ψ's mass are apprximately m. Hence don't contribute to the $Tr[(-1)^F]$. Each of the two vacuum states contributes 1. So:

$$Tr[(-1)^F] = 2 \tag{208}$$

SUSSY is not spontaneously broken for this model. Because the Witten index is not 0. (Witten, 1982) There is an example figure (5) for the energy states directly from Witten's 1982 paper.

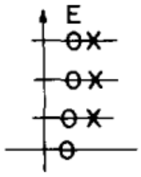

Figure 5: Energy Level Diagram for a Potential (Witten,1982, p. 256)

8 Discussion and Conclusion

We have worked on a comprehensive introduction to Supersymmetry by first considering the group theory, explicitly showing the relation between $SO(1,3,R)$ and $SL(2.C)$. Then we briefly explain the Coleman-Mandula theorem and give the representation of N=1 Super-Poincare algebra. Giving the notion of superspaces, and how to do integrals and derivatives in it, we explain how superfields are constructed by using Taylor expansions. In this chapter, we see that there are only a limited generic terms of a superfield, in terms of spinors, scalars and vectors. Moving on to the construction of supersymmetric action, we also see that there are only a few renormalizable actions. That limits the terms we can write in the Lagrangian. Supersymmetry, in fact, looks complicated to learn, however after dwelling into the basics of it, one can see that the terms cancel out considering the superspace components with θ^n, $n \geq 5$ vanish so you cannot get an unlimited expansion. Also, considering the renormalizability of the theory and the covariant derivative to get the reducible representations of supersymmetric fields, it gets much less. Then we give the action with canonical Kahler potential, giving the D-term $\Phi\Phi^\dagger$ and considering the potential with Φ^3, which makes the Wess-Zumino action the natural embodiment of the extension of ϕ^3 theory in ordinary field theories. After showing how superpotential is defined $\sum_i |\frac{\partial W}{\partial \phi_i}|$ (where i is the index of the scalar field because there can be a number of them.) and showing the symmetry transformations we move on to the most brilliant topic of this review, the supersymmetry breaking. We show that only theories with ground state $E > 0$ can be spontaneously broken with F-term breaking. We also illustrate the breaking with explicit potential diagrams. Then we briefly give an introduction to R-symmetries, which enables us to get rid of the absurdly short proton lifetime. Finally, we inspect the topological quantity Witten Index which is a must for supersymmetry breaking.

To sum up, the strengths of this review lie in its path to supersymmetry breaking. Supersymmetry breaking is the most crucial notion of a grand unified model because if supersymmetry is real, then it should be broken because we cannot see it. Hence, we inspected the F-term breaking here by giving explicit calculations. In addition, this was a comprehensive study for those who want to dive into the world of supersymmetry.

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