Spins Systems, Lattice Gauge Theories and The Toric Code

Başar Deniz Sevinç[†] , Yunus Emre Sargut[†]

[†]*Department of Physics,Middle East Technical University, 06800 Ankara, Turkey*

E-mail: basar.sevinc@metu.edu.tr, yunus.sargut@metu.edu.tr

ABSTRACT: In this work, the spin systems, particularly Ising Model was first studied to learn the nature of phase transitions. In 1D Ising Model, the transfer matrix methods was used to establish that there is no phase transition in this theory where magnetization was our interest. However, 2D Ising Model where magnetization was our interest again, it is far more rich and interesting. Using transfer matrix methods, one can map the phases of the theory by mapping the critical curve of the theory. In addition, it is exactly solvable at the critical point. After this, following Wegner, we learn that we should use gauge invariant correlation functions to map the phases of a theory. Due to Elitzur theorem, a correlation function is non-vanishing if it is gauge invariant. And spins had $\mathbb{Z}_2$ degree of freedom. However, no phase transition was found in this theory. Then, Wilson generalized this idea of discrete gauge invariance to continuous group $U(1)$, where spins are now angular variables. This theory enables us to see confinement and Coulomb phases between quark-antiquark pairs. The non-Abelian lattice gauge theory was inspected however, there is no promising results in low temperature. Another aspect was the XY Model, which exhibited no phase transition using high and low temperature expansions not through spontaneous symmetry breaking but topological means. After that, a gauged limit of the Transverse Field Ising Model, the Toric Code was inspected.

Contents

1 Introduction

The study of lattice systems has taught physicists a variety of new phenomena on the nature of phase transitions, non-perturbative effects, quantum field theories, and confinement. The versatility of lattice systems has revolutionized the study of complex systems. Discretization of continuum systems have enabled physicists to the rigorously perform calculations on the strong coupling regime of theories. Therefore, the lattice formalism is the best candidate to understand the non-perturbative nature of quantum field theories when perturbative techniques are insufficient.

In this work, the spin systems was first studied to get insights and learn useful methods for further lattice systems. After that, it has been investigated whether we can model the confining phase, strong interactions through lattice gauge systems, which is still a hot subject for various branches of physics. The strong coupling regime of quantum chromodynamics was inaccessible via perturbative methods [1]. However, even with the lattice formulations there are still open problems yet to be resolved. For example, the low temperature regime of the non-Abelian Lattice Gauge theory is still an open problem. Finally, the rather interesting relation of lattice formalism with the quantum computing was investigated [2].

The Ising Model have provided effective methods to model the phase transition in magnets. The experiments can be done. The Monte Carlo Simulations for it was done in 1979.([3]) Through phase transition maps by using the partition function methods and transfer matrix methods, one can easily distinguish between ordered and disordered phases, which have important applications in condensed matter systems. The 1-Dimensional version was mapped by [4], which had no phase transitions. Following his work, 2-D Ising Model was inspected by [1] and involved promising results as it had phase transitions. Moreover, it has been also shown in the same article that the theory is self-dual by relating the low temperature to high temperature. This is called Kramers-Wannier duality [5]. After this, the discrete gauge invariant lattice systems were inspected and it was realized that only non-vanishing correlation functions are the ones that are locally gauge invariant. [6] We can classify this topic as Wegner's Ising Lattice Gauge Theory, [7]. The main motivation of his work was the planar lattice models in 2 dimensions and models which could not magnetize but have non trivial phase diagrams.

Another aspect of this project was the XY model. The work is mostly done by Kosterlitz and Thouless[8] in 1973. The XY-model has spins living on links and the aim is to find the correlation between distant spins. The methods are analogous to the case in Abelian lattice theories. For high temperature, one should do the series expansions and for low temperature one should declare that spins is slowly varying and theory resembles the free Maxwell theory then continue the calculations in the framework of Field Theories.

On the Lattice Gauge Theory side, the discrete symmetries of the Ising Models

Z_2 was generalized to Abelian continuous groups $U(1)$ by Wilson where he had accomplished useful calculations on confinement problem. In these theories, in d-dimensions, the spins lived on links but not sites and have circular degree of freedom. In all Abelian lattice gauge theories, the action of the theory resembled the action of free Maxwell theory in some way. However, in this topic the main problem is to map the confining phases of theories by looking at the high temperature and low temperature expansions. High temperature expansions can be done by series expansion since the action is inversely proportional to the temperature, and low temperature expansions was done by using the knowledge of slowly varying spins. Choosing an appropriate order parameter for a system to map the phases, which is the Wilson Line in most cases, the potential between particle-antiparticle was found. These insightful calculations were done by Wilson on his seminal paper in 1974 [9]. The calculations for non-Abelian part is the same with the Abelian one but by considering the non-Commutative nature. Here one can inspect the high-temperature regime but the low-temperature scheme is still an open problem. [1]

The renormalization group (RG) methods are also investigated by in 1972 by 't Hooft, Politzer, Gross, Wilcek [10] [11] to have important results. ın their work, it was shown that non-Abelian gauge theories flow toward strong coupling limit. It is called *the asymptotic freedom*. These notions were invented in the framework of continuum non-Abelian gauge theories but they can also be applied on the lattice in 1973 by Kosterlitz [12]. Thus, the RG methods can provide useful insights on theories. In this project, it is shown that all $O(2)$ spin systems have asymptotic freedom. This means that as we go to higher energies as so temperatures, the theory is going to be still free of divergences.

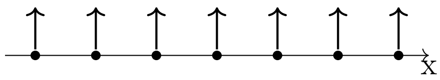
Finally, the usage of lattice systems in fact illuminated not just the confinement problem but also the field of quantum information and topological quantum computation , and the toric code which is explored in the Alexei Kitaev's work in 1977 [2] and also in 2009 with Chris Laumann [13]. It is a Z_2 model living on the lattice. This model exhibits itself with its exactly solvable nature. And the model is shown to be resembling quantum computation via toric code. The system is compactified such that it lives on a torus. The calculations for anyonic excitations were redone and presented in this report.

The main idea of this project was, whether we can use discrete gauge models to gather information on confinement and magnetization.

Furthermore, the ideas in phase transition still captivates the physicist such that it gave birth to many fields such as conformal bootstrap and generalized symmetries.[14]

2 1D Ising Model

In 1D Ising Model, we can consider one spin variable at each site and there is a fixed lattice spacing between them.



We consider the spin-spin interactions. However, one should note that only the nearest site spins have interactions between them. This will also exclude any diagonal spin to interact if the same is done for two-dimensional models. Spins have $\mathbb{Z}_2$ degree of freedom. So, we can label the spins by $\sigma_3(n+1)$ where this is the 3rd Pauli matrix with the components $diag(-1,1)$. We know that the eigenvalues of matrix is 1 or -1, and these can be used to label the spin up and down of the individual spins. The action can be written as:

$$S = -J\sum_{n=0}^{N}\sigma_3(n)\sigma_3(n+1) \tag{2.1}$$

where we choose J to be a positive coefficient. Writing the action in this way we can cover all the lattice. Furthermore, we can see from the action that if the all spins' alignment is the same we get the minimum energy state. This is the ground state of the system. Now, let us put the system in a constant magnetic field B. Action is now:

$$S = -J\sum_{n=0}^{N}\sigma_3(n)\sigma_3(n+1) - B\sum_{n=0}^{N}\sigma_3(n) \tag{2.2}$$

The partition function of the system is then:

$$Z = \sum_{config.} exp(-\beta S) \tag{2.3}$$

$$Z = \sum_{config.} exp(k\sum_{n=0}^{N}\sigma_3(n)\sigma_3(n+1) - h\sum_{n=0}^{N}\sigma_3(n)) \tag{2.4}$$

where $k = \beta J$ and $h = JB$. And the sum is over all possible spin configurations, whether a spin is up or down. By explicitly writing the summation:

$$Z = \sum_{config.} exp(k[\sigma_3(0)\sigma_3(1) + \sigma_3(1)\sigma_3(2) + ...]) + h[\sigma_3(0) + \sigma_3(1) + ...]) \tag{2.5}$$

Now we impose periodic boundary conditions. That is, we identify the endpoint of the lattice and have a circular shape with $\sigma_3(0) = \sigma_3(N)$. Now we can write the equation above as:

$$Z = \sum_{config.} e^{k[\sigma_3(0)\sigma_3(1)+\frac{h}{2}(\sigma_3(0)+\sigma_3(1)])} \cdot .... \cdot e^{k[\sigma_3(N)\sigma_3(0)+\frac{h}{2}(\sigma_3(N)+\sigma_3(0))]} \tag{2.6}$$

2.1 The Transfer Matrix Method

Now we will use the transfer matrix methods. Define a function:

$$V(\sigma_3, \sigma_3') = \exp\left(k\sigma_3\sigma_3' + \frac{h}{2}(\sigma_3 + \sigma_3')\right) \tag{2.7}$$

The partition function now can be written as:

$$Z = \sum_{\sigma_3(0)}\sum_{\sigma_3(1)} .. \sum_{\sigma_3(N)} V(\sigma_3(0), \sigma_3(1))V(\sigma_3(1), \sigma_3(2))...V(\sigma_3(N-1), \sigma_3(N)) \tag{2.8}$$

For spins neighboring, there are 4 scenarios. 1 and 2 can have values from - to -, - to + and + to -, + to +. We can represent it in a matrix:

$$T = \begin{pmatrix} + \rightarrow + & + \rightarrow - \\ - \rightarrow + & - \rightarrow - \end{pmatrix} = \begin{pmatrix} e^{k+h} & e^{-k} \\ e^{-k} & e^{k-h} \end{pmatrix} \tag{2.9}$$

with the entries of V functions. However matrices are all the same so we can consider the partition function as:

$$Z = \sum_{\sigma_3(0)} T^n \tag{2.10}$$

and since the geometry is cyclic:

$$Z = \mathrm{Tr}\{T^n\} \tag{2.11}$$

Now we can diagonalize T to get the eigenvalues.

$$T = PDP^{-1} \tag{2.12}$$

where D is a diagonal matrix with entries $diag(\lambda_1, \lambda_2)$. If we insert this for T into the partition function we get:

$$Z = \mathrm{Tr}\big\{PDP^{-1}PDP^{-1}....PDP^{-1}\big\} = \mathrm{Tr}\big\{PD^NP^{-1}\big\} \tag{2.13}$$

from the cyclic property of trace we get:

$$Z = \mathrm{Tr}\big\{PDP^{-1}PDP^{-1}....PDP^{-1}\big\} = \mathrm{Tr}\big\{D^N\big\} \tag{2.14}$$

D having diagonal entries, partition function can now be written as:

$$Z = \mathrm{Tr}\big\{T^N\big\} = \mathrm{Tr}\big\{D^N\big\} = \lambda_1^N + \lambda_2^N \tag{2.15}$$

We are now ready to calculate the Helmholtz free energy. It can be written as:

$$-(k_BT)^{-1}F = \ln(Z) = \ln\big(\lambda_1^N + \lambda_2^N\big) \tag{2.16}$$

Then we can define the energy per lattice site as $f = \frac{F}{N}$. So, for a lattice site energy is:

$$-(k_B T)^{-1}\frac{F}{N} = \frac{1}{N}\ln\left(\lambda_1^N(1+\frac{\lambda_2^N}{\lambda_1^N})\right) = \frac{1}{N}[\ln\lambda_1^N + \ln\left(1+[\frac{\lambda_2}{\lambda_1}]^N\right)] \tag{2.17}$$

Where we assumed $\lambda_1 > \lambda_2$ without loss of generality.

$$= \ln(\lambda_1) + \frac{1}{N}\ln\left(1+[\frac{\lambda_2}{\lambda_1}]^N\right)$$

We are interested in the limit when N is too many. We see that as $N \to \infty$ the second term goes to 0.

$$= \ln(\lambda_1) \tag{2.18}$$

So that the energy per lattice site is:

$$f = -k_B T\ln(\lambda_1) \tag{2.19}$$

Only left thing to do is calculating λ_1. This can be done by calculating the eigenvalues of the transfer matrix T. They can be calculated easily, but the bigger one, λ_1 is:

$$\lambda_1 = e^k\cosh h + \sqrt{e^{2k}\sinh^2(h) + e^{-2k}} \tag{2.20}$$

So that the energy per lattice is a function of h, T.

$$f(h,T) = -k_B T\ln[e^k\cosh h + \sqrt{e^{2k}\sinh^2(h) + e^{-2k}}] \tag{2.21}$$

The magnetization can be obtained from the Helmholtz free energy:

$$M = -\frac{\partial F}{\partial B} \tag{2.22}$$

where B is the magnetic field. Therefore,

$$f = k_B T\frac{\lambda_1'}{\lambda_1} \tag{2.23}$$

After taking the derivative, one can find the magnetization to be:

$$M(B,T) = \frac{\sinh(\beta B)}{\sqrt{\sinh^2(\beta B) + e^{-4\beta J}}} \tag{2.24}$$

$$M(B,T) = \frac{\sinh\left(\frac{1}{k_B T}B\right)}{\sqrt{\sinh^2(\frac{1}{k_B T}B) + e^{-4\frac{1}{k_B T}}}} \tag{2.25}$$

This is the magnetization per spin. in the limit, $T \to 0$ we see that:

$$M(B,T) = sgn(B) \tag{2.26}$$

and when $T \to 0$:

$$M(B,T) \to 0 \tag{2.27}$$

because the thermal fluctuations dominate. and if we take $B = 0$ we see that:

$$M(0,T) = 0 \tag{2.28}$$

Result 2.1: The 1 Dimensional Ising Model

The last result is important which indicates the system never magnetizes. And there is no spontaneous symmetry breaking.

3 2D Ising Model

3.1 Phases of the theory

The 2D Ising Model consists of spins living on sites. Through transfer matrix methods, one can map the phases of the theory [15]. This was not done in this work. However, the resultant critical curve of the theory is given by figure 1.

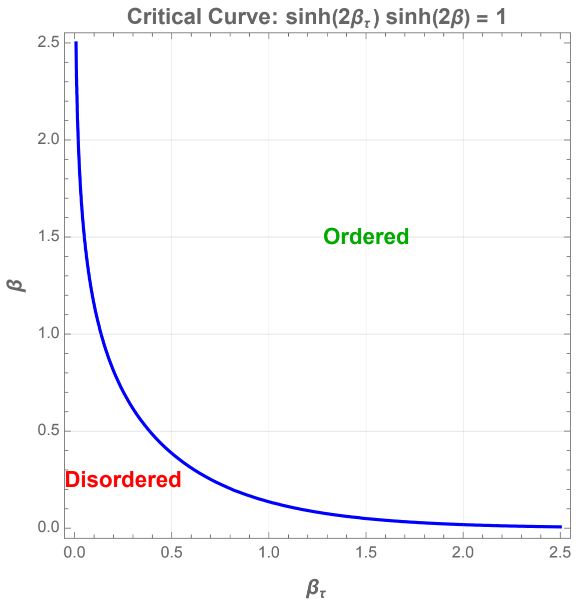


Figure 1. Critical Curve of The 2D Ising Model

The action is given by anisotropic couplings.

$$S = -\sum_n \left[\beta_\tau \sigma_3(n+\hat{\tau})\sigma_3(n) + \beta\sigma_3(n+\hat{x})\sigma_3(n)\right], \tag{3.1}$$

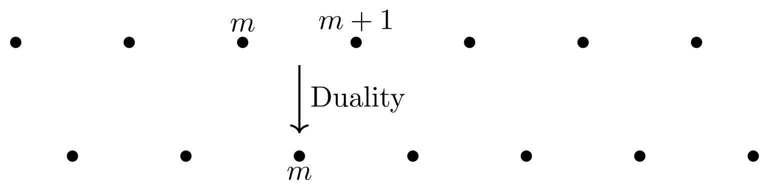
where β_τ is coupling in temporal direction and β is the coupling is spatial direction. And the figure 1 displays the phases of the theory in the (β_τ, β) plane. The ordered phase is magnetized while disordered phase is demagnetized. Thus, the theory have two phases on the contrary to 1D Ising Model.

3.2 Self-Duality

The 2D Ising Model is self-dual. It relates the high temperature and low temperature of the theory. In addition, it relates the version of the thoery where spins on links called the dual lattice and spins on sites. This duality is also called Kramers-Wannier's duality [5]. The work here were learned from the seminal paper of Fradkin and Susskind.[15] The calculation is as follows. Through transfer matrix methods, one can arrive at the Hamiltonian:

$$H = -\sum_n \sigma_1(n) - \lambda \sum_n \sigma_3(n+1)\sigma_3(n) \tag{3.2}$$

This is a 1 dimensional quantum mechanical problem. We first associate a dual lattice with the original spatial lattice such that:



We define the operators on the dual lattice:

$$\begin{aligned} \mu_1(n) &= \sigma_3(n+1)\sigma_3(n) \\ \mu_3(n) &= \prod_{m<n} \sigma_1(m) \end{aligned} \tag{3.3}$$

The first operator checks if all the spins are alligned and the second operator flips all the spins to the left of n. In addition dual operators μ satisfy the same pauli algebra as σ and the Hamiltonian can be rewritten using the dual operators. let us chech the first one: If there are odd numbered of spins, the even numbered spin operators can be made into 1. Since $(\sigma_i)^2 = 1$. So that at the end we have:

$$\sigma_1(n)\sigma_3(n) = -\sigma_3(n)\sigma_1(n) \tag{3.4}$$

So that we have:

$$\begin{aligned} \mu_1(n)\mu_3(n) &= \mu_3(n)\mu_1(n) \ \ if \ \ n \neq m \\ \mu_1(n)\mu_3(n) &= -\mu_3(n)\mu_1(n) \end{aligned} \tag{3.5}$$

For the second one, we need to write:

$$\begin{aligned} \sigma_3(n+1)\sigma_3(n+1) &= \mu_1(n) \\ \sigma_1(n) &= \mu_3(n+1)\mu_3(n) \end{aligned} \tag{3.6}$$

So that the Hamiltonian becomes:

$$\begin{aligned} H &= -\sum_m \mu_3(m)\mu_3(m+1) - \lambda \sum_m \mu_1(m) \\ &= \lambda[-\lambda^{-1} \sum_m \mu_3(m)\mu_3(m+1) - \lambda \sum_m \mu_1(m)] \end{aligned} \tag{3.7}$$

which has the form:

$$H(\sigma;\lambda) = \lambda H(\mu;\lambda^{-1}) \tag{3.8}$$

This exactly is the duality between high temperature and low temperature of the theory. It turns out that by this, we can map them onto together. However, this is a discrete symmetry, not like a scale symmetry. The energy equation is:

$$E(\lambda) = \lambda E(\lambda^{-1}) \tag{3.9}$$

Suppose that energy is 0 somewhere for a particular λ. This equation tells us that the energy should also go to 0 at λ^{-1}. This is only true at the critical point:

$$\lambda = \lambda_c = 1 \tag{3.10}$$

3.3 Exact Solution

Our aim now is to see that at the critical point the theory is exactly solvable and the theory becomes massless Majorana fermions (free, massless, self charge conjugate). This was a really important achievement in the history of physics [16].

First we will consider the Hamiltonian:

$$H = -\sum_n \sigma_3(n) - \lambda \sum_n \sigma_1(n)\sigma_1(n+1) \tag{3.11}$$

To expose the hiding fermion, we do *Jordan-Wigner* transformation. Define the raising and lowering operators:

$$\begin{aligned} \sigma^+(n) &= \frac{1}{2}[\sigma_1(n) + i\sigma_2(n)] \\ \sigma^-(n) &= \frac{1}{2}[\sigma_1(n) - i\sigma_2(n)] \end{aligned} \tag{3.12}$$

Now we can label the lattice sites as $(n = -N, -N+1, ...0, .., N)$. We can now construct the fermion operators:

$$\begin{aligned} c(n) &= \prod_{j=-N}^{n-1} \exp\big[i\pi\sigma^+(j)\sigma^-(j)\big]\sigma^-(n) \\ c^\dagger(n) &= \prod_{j=-N}^{n-1} \sigma^+(n)\exp\big[-i\pi\sigma^+(j)\sigma^-(j)\big] \end{aligned} \tag{3.13}$$

To see whether operator c's are in fact fermions, we can see the algebra. But first for easiness of notation, calculate:

$$\begin{aligned} \sigma^+(n)\sigma^-(n) &= \frac{1}{4}[\sigma_1(n) + i\sigma_2(n)][\sigma_1(n) - i\sigma_2(n)] \\ &= \frac{1}{4}[\sigma_1(n)^2 + \sigma_2(n)^2 - i\sigma_1(n)\sigma_2(n) + i\sigma_2(n)\sigma_1(n)] \\ &= 1 - \frac{i}{2}[\sigma_1(n), \sigma_2(n)] \end{aligned}$$

By using the Pauli matrix algebra $SU(2)$: $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$ we get:

$$
\begin{aligned}
&= 1 - \frac{i}{2}[\sigma_1(n), \sigma_2(n)] \\
&= \frac{1}{2}(1 + \sigma_3(n))
\end{aligned}
$$

We can do this for also the reversed order product. We get:

$$
\begin{aligned}
\sigma^+(n)\sigma^-(n) &= \frac{1}{2}(1 + \sigma_3(n)) \\
\sigma^-(n)\sigma^+(n) &= \frac{1}{2}(1 - \sigma_3(n)) \\
\exp\Big(i\frac{\pi}{2}\sigma_3(n)\Big) &= i\sigma_3(n)
\end{aligned}
$$

Now we can insert these into c's in 3.13:

$$
\begin{aligned}
c(n) &= \prod_{j=-N}^{n-1} \exp\left[\frac{i\pi}{2}(1 - \sigma_3(j))\right]\sigma^-(n) \\
&= \prod_{j=-N}^{n-1} \exp\Big\{(\frac{i\pi}{2})\Big\} \exp\Big\{(\frac{-i\pi}{2}\sigma_3(j))\Big\}\sigma^-(n) \\
&= \prod_{j=-N}^{n-1} -\sigma_3(j)\sigma^-(n)
\end{aligned}
\tag{3.14}
$$

For the hermitian conjugate of this, the process is the same:

$$
c^\dagger(n) = \sigma^+(n) \prod_{j=-N}^{n-1} -\sigma_3(j)
$$

We will make use of the Pauli matrix algebra $\{\sigma_i(n), \sigma_j(n')\} = 2I\delta_{ij}\delta_{nn'}$. This means that if two Pauli matrices are living on different sites, they anti-commute. We will also use:

$$
\{\sigma_3(n), \sigma^-(n)\} = 0 \tag{3.15}
$$

We can calculate the fermionic nature of these variables now:

$$
\begin{aligned}
\{c(n), c^\dagger(n')\} &= c(n)c^\dagger(n') + c^\dagger(n')c(n) \\
&= [\prod_{j=-N}^{n-1} \sigma^-(n)][\sigma^+(n) \prod_{j=-N}^{n'-1} -\sigma_3(j)] + [\sigma^+(n) \prod_{j=-N}^{n'-1} -\sigma_3(j)][\prod_{j=-N}^{n-1} \sigma^-(n)] \\
&= \delta_{nn'}
\end{aligned}
\tag{3.16}
$$

Now we know that with the usage of 3.15 they are indeed fermionic operators.

We can write the Hamiltonian in terms of these operators. First let us build the σ matrices out of the fermionic operators.

$$\sigma_3(n) = 2c^\dagger(n)c(n) - 1 \tag{3.17}$$

We have already established the single term. But what about the spin-spin coupling term. With the usage of 3.12 we can write:

$$\sigma_1(n)\sigma_1(n+1) = [\sigma^+(n) + \sigma^-(n)][\sigma^+(n+1) + \sigma^-(n+1)] \tag{3.18}$$

There are also other products:

$$\begin{aligned} c^\dagger(n)c(n+1) &= \sigma^+(n)[-\sigma_3(n)]\sigma^-(n+1) \\ \sigma^+(n)\sigma_3(n) &= -\sigma^+(n) \\ c^\dagger(n)c(n+1) &= \sigma^+(n)\sigma^-(n+1) \end{aligned} \tag{3.19}$$

And we also have:

$$\begin{aligned} c(n)c^\dagger(n+1) &= -\sigma^-(n)\sigma^+(n+1) \\ c^\dagger(n)c^\dagger(n+1) &= \sigma^+(n)\sigma^+(n+1) \\ c(n)c(n+1) &= -\sigma^-(n)\sigma^-(n+1) \end{aligned} \tag{3.20}$$

After using these we will have for products of σ_1:

$$\sigma_1(n)\sigma_1(n+1) = [c^\dagger(n) - c(n)][c^\dagger(n+1) + c(n+1)] \tag{3.21}$$

We now have all the terms. Upon using 3.17 and 3.21 we can write down the Hamiltonian in terms of fermionic components:

$$H = -2\sum_n c^\dagger(n)c(n) - \lambda\sum_n [c^\dagger(n) - c(n)][c^\dagger(n+1) + c(n+1)] \tag{3.22}$$

The Hamiltonian only involves quadratic terms so we should be able to solve it. However, we will also write down the Hamiltonian in the momentum space. So we should define the Fourier expansion:

$$a_k = \sqrt{\frac{1}{2N+1}} \sum_{n=-N}^{N} e^{ikn} c(n) \tag{3.23}$$

There is also a restriction on the wave-vectors k. This is for appropriate boundary conditions. But we do not need them. One can verify the operators anticommute:

$$\begin{aligned}
\{a_k, a_{k'}^\dagger\} &= \sqrt{\frac{1}{2N+1}}\sqrt{\frac{1}{2M+1}} \sum_{n=-N}^{N} \sum_{m=-M}^{M} e^{ikn} e^{-ik'm} \{c(n), c^\dagger(m)\} \\
&= \sqrt{\frac{1}{2N+1}}\sqrt{\frac{1}{2M+1}} \sum_{n=-N}^{N} \sum_{m=-M}^{M} e^{ikn} e^{-ik'm} \delta_{mn} \\
&= \frac{1}{2N+1}\sqrt{\frac{1}{2N+1}} \sum_{m=-M}^{M} e^{ikn} e^{-ik'n} \\
&= \frac{1}{2N+1} \sum_{n=-N}^{N} e^{in(k-k')} = \delta_{kk'}
\end{aligned} \tag{3.24}$$

Where we used the discrete Dirac-delta and sometimes called completeness relations. [1] Other commutation relations are readily seen:

$$\begin{aligned}
\{a_k, a_{k'}\} &= 0 \\
\{a_k^\dagger, a_{k'}^\dagger\} &= 0
\end{aligned} \tag{3.25}$$

Now we can invert the 3.23 to write fermionic operators in terms of our new variables.

$$c(n) = (\frac{1}{2N+1})^{\frac{1}{2}} \sum_k e^{ikn} a_k \tag{3.26}$$

Now let us evaluate the terms in the Hamiltonian:

$$\begin{aligned}
\sum_n c^\dagger(n) c^\dagger(n+1) &= \frac{1}{2N+1} \sum_n \sum_k \sum_{k'} e^{ikn} e^{ik'(n+1)} a_k^\dagger a_{k'}^\dagger \\
&= \sum_k \sum_{k'} e^{ik'} \delta_{k,-k'} a_k^\dagger a_{k'}^\dagger \\
&= \sum_k e^{-ik} a_k^\dagger a_{-k'}^\dagger
\end{aligned} \tag{3.27}$$

Doing the same for other terms in the Hamiltonian yields:

$$H = -2 \sum_k (1 + \lambda \cos k) a_k a_{k'}^\dagger - \lambda \sum_k e^{-ik} (a_k^\dagger a_{-k'}^\dagger - e^{ik} a_k a_{-k'}) \tag{3.28}$$

It will be more convenient to write the sum over positive k modes:

$$H = -2 \sum_{k>0} (1 + \lambda \cos k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + 2i\lambda \sum_{k>0} \sin k (a_k^\dagger a_{-k'}^\dagger + a_k a_{-k'}) \tag{3.29}$$

But right now, the ground state of H is not the same with ground state of a_k because of the second term: $a_k^\dagger a_{-k'}^\dagger + a_k a_{-k'}$. We want to write the Hamiltonian in the form:

$$H = \sum_k \Lambda_k \eta_k^\dagger \eta_k + constants \tag{3.30}$$

so that we can $(\eta_k |0\rangle = 0)$. We can do the following transformations:

$$\eta_k = u_k a_k + i v_k a^\dagger_{-k}, \quad \eta_{-k} = u_k a_{-k} - i v_k a^\dagger_k, \tag{3.31}$$
$$\eta^\dagger_k = u_k a^\dagger_k - i v_k a_{-k}, \quad \eta^\dagger_{-k} = u_k a^\dagger_{-k} + i v_k a_k. \tag{3.32}$$

The choice seems rather arbitrary. However, the following criteria can be met if u_k and v_k are real, even functions of k. (1), η_k and η_{-k} should be fermion operators. (2) Hamiltonian should be diagonalized when written as 3.30. The fermionic nature of η's can be easily verified through anti-commutation relations. But these lead to a relation:

$$u_k^2 + v_k^2 = 1 \tag{3.33}$$

We need the inverted formulas for η to write the Hamiltonian:

$$\begin{aligned} a_k &= u_k \eta_k - i v_k \eta^\dagger_{-k}, \quad a_{-k} = u_k \eta_{-k} + i v_k \eta^\dagger_k, \\ a^\dagger_k &= u_k \eta^\dagger_k + i v_k \eta_{-k}, \quad a^\dagger_{-k} = u_k \eta^\dagger_{-k} - i v_k \eta_k. \end{aligned} \tag{3.34}$$

Substituting into the Hamiltonian yields:

$$\begin{aligned} H = \sum_{k>0} \Big[-2(1+\lambda\cos k)(u_k^2 - v_k^2) + 4\lambda \sin k\, u_k v_k \Big] \Big(\eta^\dagger_k \eta_k + \eta^\dagger_{-k}\eta_{-k} \Big) \\ + \sum_{k>0} \Big[4i(1+\lambda\cos k) u_k v_k + 2i\lambda \sin k (u_k^2 - v_k^2) \Big] \Big(\eta^\dagger_k \eta^\dagger_{-k} + \eta_{-k}\eta_k \Big). \end{aligned} \tag{3.35}$$

We force that H have the same form as 3.30:

$$4(1+\lambda\cos k) u_k v_k + 2\lambda \sin k (u_k^2 - v_k^2) = 0. \tag{3.36}$$

Due to the condition 3.33, we can write:

$$u_k = \cos\theta_k, \quad v_k = \sin\theta_k. \tag{3.37}$$

Then

$$2u_k v_k = \sin 2\theta_k, \quad u_k^2 - v_k^2 = \cos 2\theta_k, \tag{3.38}$$

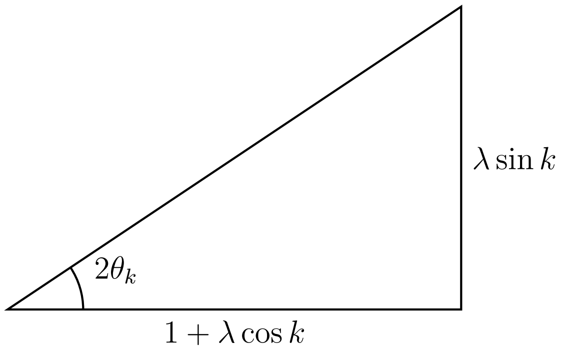
so equation 3.36 becomes:

$$2(1+\lambda\cos k)\sin 2\theta_k + 2\lambda \sin k \cos 2\theta_k = 0. \tag{3.39}$$

Or

$$\tan 2\theta_k = -\frac{\lambda \sin k}{1+\lambda\cos k}. \tag{3.40}$$

This represents a triangle:



The hypothenus is:

$$\sqrt{\lambda^2 + 2\lambda \cos k + 1} \tag{3.41}$$

And we chose the sign convention:

$$\begin{aligned} \sin(2\theta_k) &= \frac{\lambda \sin k}{\sqrt{\lambda^2 + 2\lambda \cos k + 1}} \\ \cos(2\theta_k) &= -\frac{1 + \lambda \cos k}{\sqrt{\lambda^2 + 2\lambda \cos k + 1}} \end{aligned} \tag{3.42}$$

After inserting these into the Hamiltonian, we get:

$$H = 2\sum_k \sqrt{1 + 2\lambda \cos k + \lambda^2}\eta_k^\dagger \eta_k + const. \tag{3.43}$$

So that the coefficient in 3.30:

$$\Lambda_k = 2\sqrt{1 + 2\lambda \cos k + \lambda^2} \tag{3.44}$$

Since Λ_k has minimum and maximum at $k = \pm\pi$, we measure momentum from π. Let

$$k = \pi + k'a \tag{3.45}$$

where a is the lattice spacing. Also now, k has the correct dimensions for momentum. We can define the energy:

$$E(k') = \frac{\Lambda_k}{2a} \tag{3.46}$$

But we want to consider finite k' values as the lattice spacing $a \to 0$. So that the k is forced to be π. Because in this way we can ensure that the energy is not divergent. So that Λ is:

$$\Lambda = 2\sqrt{1 + 2\lambda \cos(\pi + k'a) + \lambda^2} \approx 2[(1-\lambda)^2 + \lambda(k'a)^2] \tag{3.47}$$

It is plotted in 2.

Furthermore, if λ is not in the critical region, we do not have a nontrivial continuum limit. Becuase,

$$E(k') = \sqrt{\frac{(1-\lambda)^2}{a^2} + \lambda k'^2} \tag{3.48}$$

But if $\lambda = 1$ we get Majorana fermions:

$$E(k') = |k'| \tag{3.49}$$

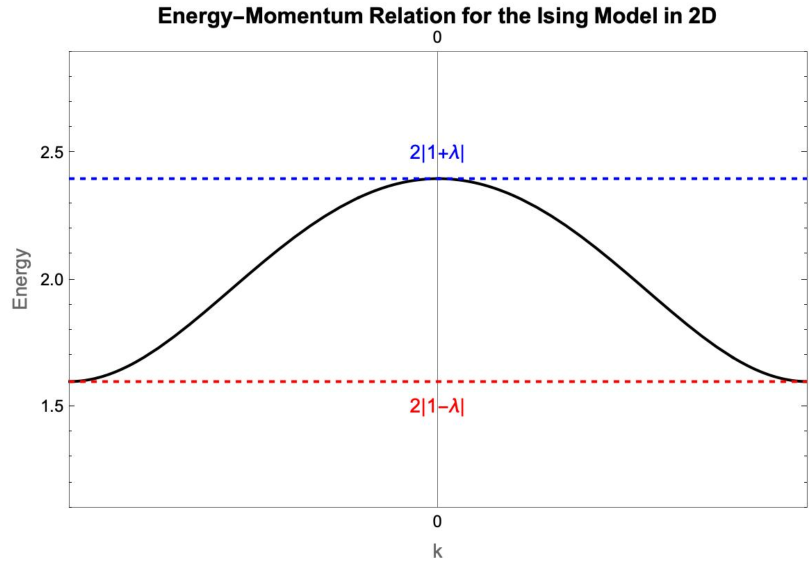


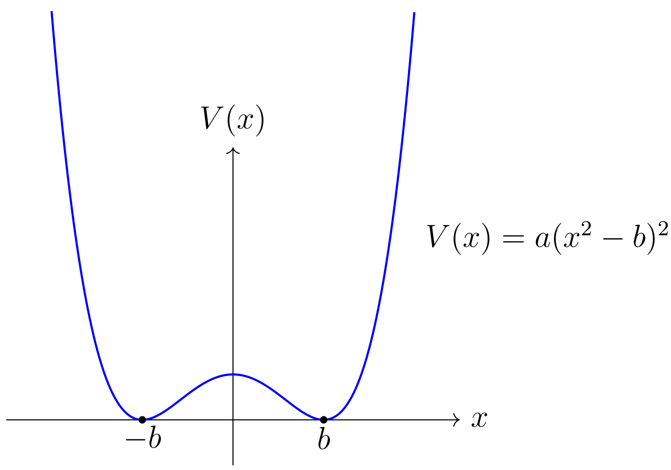
Figure 2. The graph for equation 3.44

4 Wegner's Ising Lattice Gauge Theory

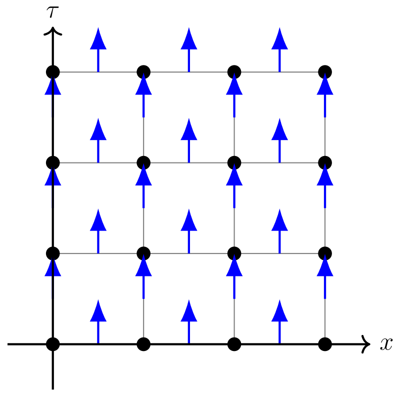
A phase transition can be via spontaneous symmetry breaking of a symmetry group. In 1971, [7] Wegner came up with the idea that one could models which could not magnetize but have non-trivial phase transitions. His inspiration was from the planar model in two dimensions. He constructed a Z^2 symmetric modeli spins are living on links and made the correlation function such that it is invariant under local the transformation. Before diving into the topic, first look at the Hydrogen atom to understand the symmetry concept.

The non relativistic Hydrogen atom is invariant under the rotation group $SO(3)$ and its ground state is also spherically symmetric and unique.

Another example is the particle in one dimension in a spherically symmetric, double well potential.



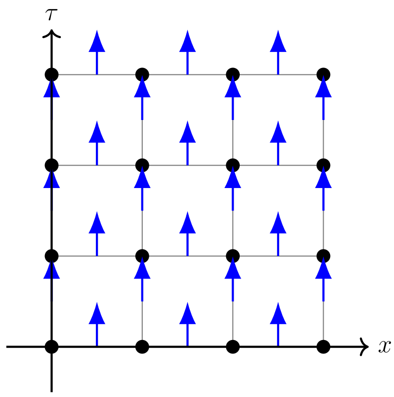
It has two minima and the ground state is doubly-degenerate. However, The ground state is unique again due to the tunneling property. We say that it is $\mathbb{Z}_2$ symmetric. This is also the case in the 1D, 2D Ising Model and also the Wegner's Ising Lattice Gauge Theory. The ground state is usually globally $\mathbb{Z}_2$ symmetric, which corresponds to the all spins's allignment is the same case. Whether up or down. An example is provided below:

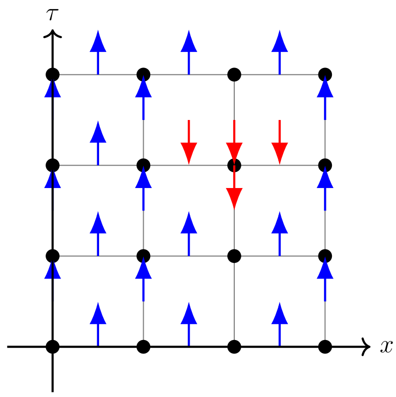


We can write the action of the Wegner's Ising Lattice Gauge Theory as:

$$S = -\beta \sum_{n,\mu\nu} \sigma_3(n,\mu)\sigma_3(n+\mu,\nu)\sigma_3(n+\mu+\nu,-\mu)\sigma_3(n+\nu,-\nu) \tag{4.1}$$

where $\beta \propto \frac{1}{k_B T}$. T is the temperature and k_B is the Boltzmann constant. The action adds all the plaquette's energy together. Now, if we choose a spin variable, $\sigma_3(n,x)$ as our correlation function, one will see that it is vanishing. This is a consequence of the Elitzur [6] theorem. The main question was how to label the phases of a theory having a local symmetry group. The answer of Wegner was that we should consider the spatial dependence of correalation functions. So, we should choose a gauge invariant, correlation function. A possible choice is: $\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle$. Where C is for the contour and the product is over the 3rd Pauli matrices. The size of the contour can be arbitrary. The gauge transformation considered here can be observed in the figure below:





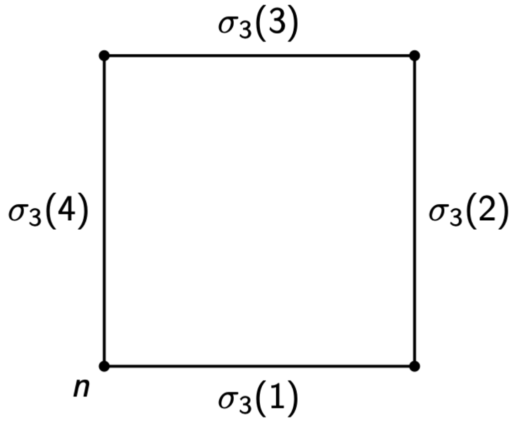


Figure 3. Plaquette

We applied the gauge transformation on the $(2,2)$ lattice site. And the spins living on links connected to the site reversed their direction. For example, if we had chosen the contour as the plaquette in the middle of the figure, where it has 4 spins on it, only 2 of them reversed their direction. Generalizing this to arbitrary contours, we see that the number of the spins flipped will always be in even number. So we say that our correlation function $\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle$ is gauge invariant and naturally one considers closed path links. One can see from equation 4.1 that the action is also gauge invariant since it adds up the plaquettes.

Now we can move on with the evaluation of the correlations function:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = \frac{1}{Z}[\prod_{l\in C} \sigma_3(l)]e^{-S} \tag{4.2}$$

This was the definition of correlation function.

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = \frac{1}{Z}[\prod_{l\in C} \sigma_3(l)]e^{\beta \sum_{n,\mu\nu} \prod_{p\in u} \sigma_3(u)} \tag{4.3}$$

This is a short-hand notation for the action. We know that β is inversely proportional to temperature. So at the high temperatures, we can do power expansions. First, we will use the identity:

$$\exp\left(\beta \prod_{l\in C} \sigma_3(u)\right) = \cosh\beta + \prod_{p\in u} \sigma_3(u) \sinh\beta = (1 + \prod_{p\in u} \sigma_3(u) \tanh\beta) \cosh\beta \tag{4.4}$$

So that the correlation function becomes:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = \frac{\sum_{conf}[\prod_{l\in C} \sigma_3(l)] \prod_p (1 + \prod_{p\in u} \sigma_3(u) \tanh\beta)}{\sum_{conf} \prod_p (1 + \prod_{p\in u} \sigma_3(u) \tanh\beta)} \tag{4.5}$$

We could do this since the sum in the exponential could be written as product of seperate exponentials. We can even power expand the tanh because β is small. It is: $\tanh\beta \approx \beta$. Again, since β is small, we expect the first orders to have a huge contribution in constrast to higher orders. In lowest order we should know the spin sum results. Those are:

$$\sum_{\sigma_3=\pm 1} \sigma_3 = 0 \;\; and \;\; \sum_{\sigma_3=\pm 1} (\sigma_3)^2 = 2 \tag{4.6}$$

In order to obtain a non-zero result, we should have all the spin sums in the plaquette. Because the products will make the result zero otherwise. Thus, at the first order in β the first non-zero contribution will be spanned by the plaquettes residing inside the contour. So now we know that:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle \propto (\tanh\beta)^{N_P} \tag{4.7}$$

Where N_P is the number of the contours insider the plaquette. Put in another way:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle \propto e^{N_P \ln(\tanh\beta)} \tag{4.8}$$

But we know that the number of the plaquette's inside will be proportional to the area of our contour with the lattice spacing. So we conclude that at high temperatures the theory exhibits an *area law*:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = e^{-f(\beta)A} \tag{4.9}$$

where $f(\beta) = -\ln(\tanh\beta)$. Note that one can also prove that this result lies in the radius of convergence. [1]

The low temperature expansions was not done in this work. However, from Wegner's article [7] we know that it obeys a perimeter law.

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = e^{-g(\beta)P} \tag{4.10}$$

The perimeter is the perimeter of the contour. We can conclude that theory has 2 phases. However we should be careful. In 2 dimensions we cannot do these calculations for low temperature due to radius of convergence reasons but the area low for temperature is independent of dimensions. But we should again be careful.

Result 4.1: Wegner's Ising Lattice Gauge Theory

In 4 dimensions, theory exhibits 2 phases which can be given by:

- At high temperature:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = e^{-f(\beta)A}$$

- At low temperature:

$$\left\langle \prod_{l\in C} \sigma_3(l) \right\rangle = e^{-g(\beta)P}$$

However, due to the formulas provided above for high temperature and low temperature expansions we can say that the theory has no phase transition. Furthermore, we can put this in another way.

We will now show that this theory by choosing a particular gauge makes the model equivalent to one-dimensional Ising Model. Thus showcasing no phase transition. Choose the temporal gauge:

$$\sigma_3(n, \hat{\tau}) = 1 \tag{4.11}$$

Remembering the action in 4.1, and the ν direction shows us the temporal side, we can reduce the action to the form:

$$S = -J \sum_{n,\mu\nu} \sigma_3(n, \hat{x})\sigma_3(n + \hat{\tau}, \hat{x}) \tag{4.12}$$

which resembles the Ising Model in 1D. But since we will do a summation over all these indices, the result will not be so different. We can conclude that the theory in 2D has no phase transition because:

$$\text{Wegner's Ising Lattice Gauge Theory} = \text{1D Ising model}$$

5 Abelian Lattice Gauge Theory

Now we will consider spins living on lattice links again but this time the spins will have $U(1)$ degree of freedom. That is, the spins are planar:

$$\vec{S}(n) = \begin{pmatrix} \cos\theta(n) \\ \sin\theta(n) \end{pmatrix} \tag{5.1}$$

We want to write a Hamiltonian for the system. The spin-spin coupling can be given by:

$$\begin{aligned} S &= -J \sum_{n,\mu} \vec{S}(n) \cdot \vec{S}(n+\mu) \\ &= -J \sum_{n,\mu} \cos\theta(n)\cos\theta(n+\mu) + \sin\theta(n)\sin\theta(n+\mu) \\ &= -J \sum_{n,\mu} \cos[\theta(n) - \theta(n+\mu)] \\ &= -J \sum_{n,\mu} cos[\Delta_\mu \theta(n)] \end{aligned} \tag{5.2}$$

In the third line we have used addition rules for trigonometric functions and in the last line we have defined the difference operator: $\Delta_\mu\theta(n) = \theta(n+\mu) - \theta(n)$. The action has the global $U(1)$ symmetry. Which can be seen as sending $\theta \to \theta + \alpha$. Since this is a new angle, call it χ and the final form will be the same.

Now, we place planar spins onto links as in Ising Lattice Gauge theory and suppose at every site n there is a local symmetry operator $\chi(n)$, which rotates the spins living on the links connected to it. More precisely, let $\theta_\mu(n)$ be the angular variable on the (n,μ) link. However, $(n,\mu) = (n+\mu,-\mu)$ up to a sign with this labeling. So we define: $\theta_{-\mu}(n+\mu) = -\theta_\mu(n)$. We associate a discrete curl:

$$\begin{aligned} \theta_{\mu\nu}(n) &= \Delta_\mu\theta_\nu(n) - \Delta_\nu\theta_\mu(n) \\ &= \theta_\nu(n+\mu) - \theta_\nu(n) - \theta_\mu(n+\nu) + \theta_\mu(n) \\ &= \theta_\mu(n) + \theta_\nu(n+\mu) + \theta_{-\nu}(n+\nu) + \theta_{-\mu}(n+\mu+\nu) \end{aligned} \tag{5.3}$$

This curl is an oriented sum of the angular spins living on the links. This can be seen in 4.

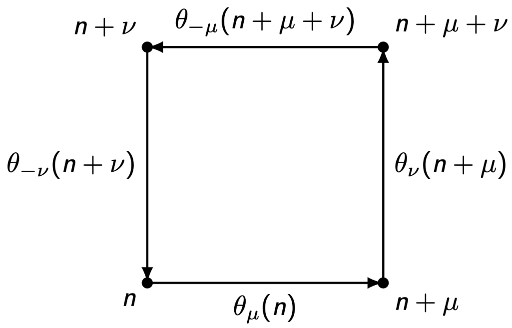


Figure 4. Plaquette

This curl is invariant under local gauge transformations seperately:

$$\begin{aligned} \theta_\mu(n) &\to \theta_\mu(n) - \chi(n) \\ \theta_{-\mu}(n+\mu) &\to \theta_{-\mu}(n+\mu) + \chi(n) \end{aligned} \tag{5.4}$$

But what happens if we do them combined:

$$\begin{aligned}\theta_\mu(n) &\to \theta_\mu(n) - \chi(n) + \chi(n+\mu) \\ &\to \theta_\mu(n) + \Delta_\mu\chi(n)\end{aligned} \tag{5.5}$$

This is, we do two gauge transformations on diagonal sites on a plaquette. So that the curl:

$$\begin{aligned}\theta_{\mu\nu} \to\ & \theta_\mu(n) + \theta_\nu(n+\mu) + \theta_{-\mu}(n+\mu+\nu) + \theta_{-\nu}(n+\nu) \\ &+ \Delta_\mu\chi(n) + \Delta_\nu\chi(n) - \Delta_\mu\chi(n) - \Delta_\nu\chi(n)\end{aligned} \tag{5.6}$$

The terms on the 2^{nd} line cancels so that we have two transformations that leave the curl $\theta_{\mu\nu}$ invariant. First is we do 1 gauge transformation on 1 site. The second is we do 2 gauge transformations those are to the diagonal sites of the plaquette.

These all resemble a discrete version of the Maxwell thoery where:

$$A_\mu \to A_\mu + \partial_\mu\chi \ \ , \ \ F_{\mu\nu} \to F_{\mu\nu} \tag{5.7}$$

So, we can write:

$$S = J\sum_{n,\mu\nu}[1 - \cos\theta_{\mu\nu}(n)] \tag{5.8}$$

This action 5.8has some nice features coming with.

- It is locally gauge invariant because it was constructed from the discrete curl.
- $\theta_{\mu\nu}$ is periodic.
- It is globally $\mathbb{Z}_2$ symmetric.
- It resembles the discrete Maxwell action.

Now we will see in detail how the last feature is fulfilled. At the weak coupling limit we can Taylor expand the terms:

$$1 - \cos\theta_{\mu\nu}(n) \approx 1 - [1 - \frac{1}{2}\theta_{\mu\nu}^2 + ...] \approx \frac{1}{2}\theta_{\mu\nu}^2 \tag{5.9}$$

Then we can replace the sum with integral because we can interpret the low temperature limit as going to the continuum. In $D = 4$ we have:

$$\sum_{n,\mu\nu} \to \frac{1}{a^4}\int d^4x \tag{5.10}$$

The factor $\frac{1}{a^4}$ is to match the dimensions and a is the lattice spacing. The action **??** now becomes:

$$S \approx J\int \frac{d^4x}{a^4}\frac{1}{2}\theta_{\mu\nu}^2 \tag{5.11}$$

If we now identify:

$$\theta_{\mu\nu} = a^2 g F_{\mu\nu} \quad , \quad J = \frac{1}{2g^2} \tag{5.12}$$

The action becomes:

$$S \approx \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.13}$$

We see that this relates our spins variable to vector potential:

$$\theta_\mu(n) = agA_\mu(n) \tag{5.14}$$

where g is the lattice coupling constant.

5.1 Wilson Loop, Phases of 4 Dimensional Theory

The partition function can be written as:

$$Z = \int \prod_{r,\mu} d\theta_\mu(r) \exp\left\{ -\frac{1}{2g^2} \sum_{n,\mu} [1 - cos(\theta_{\mu\nu}(n))] \right\} \tag{5.15}$$

The Elitzur Theorem is still valid, a local continuos symmetry cannot break down spontaneously. Therefore a correlation function like

$$\langle \cos\theta_\mu(n) \rangle = 0. \tag{5.16}$$

will certainly vanish. Consider a different gauge invariant operator. Following [9], consider the Wilson line:

$$\left\langle \exp\left\{ i \sum_C \theta_\mu(r) \right\} \right\rangle = \frac{1}{Z} \prod_{r,\mu} \int_0^{2\pi} d\theta_\mu(r) \exp\left\{ i \sum_C \theta_\mu(r) \right\} \exp(-S) \tag{5.17}$$

The sum is over a closed curve. And it can be shown that the Wilson loop is gauge invariant under usual transformations. The different thing here from the other discrete group theories is that we now have an integration. Now suppose $g^2 \gg 1$ and we have large loops so that the expression inside the correlation function is huge. The correlation function 5.17becomes for the action in 5.15:

$$= \prod_{r,\mu} \int_0^{2\pi} d\theta_\mu(r) \exp\left\{ i \sum_C \theta_\mu(r) \right\} \tag{5.18}$$

However not establishing this, for large loops C, low orders in the expansion becomes:

$$\int_0^{2\pi} d\theta_\mu(r) e^{i\theta_\mu(r)} = 0 \tag{5.19}$$

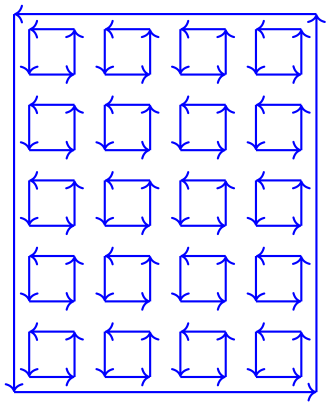
So that the any phase factor within the integrand yields 0. However, if we can cancel all the phases then we have:

$$\int_0^{2\pi} d\theta_\mu(r) = 2\pi \tag{5.20}$$

which give a finite contribution. Now we learned that we should consider the exponential of the second part of the action:

$$\begin{aligned}
\exp\left\{\frac{1}{2g^2}\sum cos\theta_{\mu\nu}\right\} &= \prod_{r,\mu\nu}\exp\left\{\frac{1}{2g^2}\cos\theta_{\mu\nu}(r)\right\} \\
&= \prod_{r,\mu\nu}\exp\left\{\frac{1}{4g^2}[e^{i\theta_{\mu\nu}(r)}+e^{-i\theta_{\mu\nu}(r)}]\right\} \\
&= \prod_{r,\mu\nu}\sum_{n}\frac{1}{n!}\left\{\frac{1}{4g^2}[e^{i\sum_p\theta_\mu}+e^{-i\sum_p\theta_\mu}]\right\}^n
\end{aligned} \tag{5.21}$$

where $\sum_p$ is the sum over the links of the plaquette $(r, \mu\nu)$ and in the last line we have used discrete Stoke's thoerem such that:



Right now we know that we should have the fewest possible powers of $\frac{1}{4g^2}$. Because g is huge. So take $n = 1$. We also have not established this in the project but the number of plaquettes residing inside C should cancel the phase factors on C and the interior region [17]. However the result is given by the minimal area A enclosed by contour C:

$$\left\langle \exp\left\{i\sum_C\theta_\mu(r)\right\}\right\rangle \approx (\frac{1}{4g^2})^A = e^{-\ln\left(4g^2\right)A} \tag{5.22}$$

The same with the Wegner's Ising Lattice Gauge theory. In general, assuming the strong-coupling, which is high-temperature expansion has a finite radius of convergence, we conclude that:

$$\left\langle \exp\left\{i\sum_C\theta_\mu(r)\right\}\right\rangle = e^{-f(g^2)A} \tag{5.23}$$

where f, in the leading order is $\ln(4g^2)$.

The weak coupling was not established in this project however, the answer is, for $g \ll 1$ and at $D = 4$:

$$\langle W[C]\rangle = exp[\frac{-1}{2}g^2cP + \frac{g^2}{8\pi}\frac{T}{R} + \frac{g^2}{2\pi^2}\ln\left(\frac{R}{A}\right)] \tag{5.24}$$

where c: arbitrary constant, P: Perimeter of W Wilson line, T: length in time direction, R: length in spatial direction, A: Area of the Wilson loop. The second term is of importance here as we will see in the next section. However, this result is found by going to the field theory limit of the model, which is the Maxwell action. This is justified by at really low temperatures, there is no fluctuation so we can treat the theory as a field theory.

5.2 Potential Between Static Charges

Coupling of electromagnetic fields to currents is done by:

$$e \int A_\mu(x) J^\mu(x) d^4x, \tag{5.25}$$

adding the term to the action 5.25 since $d * J = 0$, we can consider the current as a closed loop. We can approximate this as J current having an infinitesimal cross-section. This results in that we are considering not a current density but a current. Let us parametrize the current. For a given time, τ, $J_0 = -1$ at $x = 0$ and $J_0 = 1$ at $x = R$. Hence we have a static charge at $x = R$ and its antiparticle at $x = 0$. In addition, we can consider the **??** as like a potential added to the action. This enables us to write the Wilson loop $\langle W[C] \rangle$ as the ratio:

$$\left\langle \left\{ ig \int_C A_\mu dx^\mu \right\} \right\rangle = \frac{Z[J]}{Z[0]} \tag{5.26}$$

Let us express these in terms of free energy as in the 1D Ising Model $\mathcal{F} = -\ln Z$:

$$\frac{Z[J]}{Z[0]} = \exp\{\{-[\mathcal{F}(J) - \mathcal{F}(0)]\}\} \tag{5.27}$$

In the limit where $T \to \infty$. T is the temporal side of the Wilson loop. We have:

$$\mathcal{F}(J) - \mathcal{F}(0) \propto T \tag{5.28}$$

And since the charges are static, the energy difference between these are potential with purely spatial dependence:

$$\mathcal{F}(J) - \mathcal{F}(0) = V(R)T \tag{5.29}$$

Thus the correlation function becomes:

$$\left\langle \left\{ ig \int_C A_\mu dx^\mu \right\} \right\rangle = \exp[-V(R)T] \tag{5.30}$$

So that we can write the potential as:

$$V = -\lim_{T\to\infty} \frac{1}{T} \ln \langle W[C] \rangle \tag{5.31}$$

Now returning back to the results we had in Abelian lattice gauge theory, the area can be given by $A = RT$. This means, at the high-temperature expansion:

$$V(R) \sim |R| \tag{5.32}$$

while the perimeter law produces a constant potential between quark-antiquark pairs. Considering the middle term in the 5.24, produces a potential:

$$V(R) \sim const.(-\frac{e^2}{2})(\frac{1}{R}) \tag{5.33}$$

which is the Coulomb's law. Thus as a result of Abelian Lattice Gauge theory, we have:

Result 5.1: Abelian Lattice Gauge Theory

In 4 dimensions, theory exhibits 2 phases which can be given by:

- At high temperature, $g \gg 1$:

$$\langle W[C] \rangle = e^{-f(\beta)A}$$
$$V(R) \sim |R|$$

  This is the confining phase.

- At low temperature, $g \ll 1$:

$$\langle W[C] \rangle = exp[\frac{-1}{2}g^2 cP + \frac{g^2}{8\pi}\frac{T}{R} + \frac{g^2}{2\pi^2}\ln\left(\frac{R}{A}\right)]$$
$$V(R) \sim const.(-\frac{e^2}{2})(\frac{1}{R})$$

  This is the Coulomb phase.

5.3 2D Abelian System

Let us only consider an Abelian-two dimensional system both at the high temperature and low temperature. Now we choose temporal gauge:

$$\theta_0(n) = 0 \tag{5.34}$$

so that on a contour, only spatial directions contribute. The correlation function we will consider is:

$$\left\langle \exp\left\{ i \sum_C \theta_\mu(r) \right\} \right\rangle = \frac{1}{Z} \int \prod d\theta_\mu(n) \exp\left[\beta \sum_{n,\mu\nu} \cos\theta_{\mu\nu} + i \sum_C \theta_\mu \right] \tag{5.35}$$

where $\beta = \frac{1}{2g^2}$. We notice that,

$$\theta_{10} = -\theta_1(n+\tau) + \theta_1(n) \tag{5.36}$$

leading to:

$$\begin{aligned}\theta_1(n) &= -\theta_{10}(n-\tau) + \theta_1(n-\tau) \\ &= -\theta_{10}(n-\tau) - \theta_{10}(n-2\tau) + \theta_1(n-2\tau) \\ &= -\sum_{\tau'<\tau} \theta_{\mu\nu}(\tau', x)\end{aligned} \tag{5.37}$$

Just like a recursion relation. And we can now use the Stoke's law:

$$\sum_C \theta_\mu = \sum_{P_C} \theta_{\mu\nu}(n) \tag{5.38}$$

where P_C is for the plaquettes within closed loop C. Thus the explicit correlation function is:

$$\left\langle \exp\left(i\sum_C \theta_\mu\right)\right\rangle = \frac{\int \prod_{[P_c]} d\theta_{\mu\nu}(n) \exp\left\{\beta \sum_{n,\mu\nu} \cos\theta_{\mu\nu} + i\sum_{P_s} \theta_{\mu\nu}\right\}}{\int \prod_{[P_c]} d\theta_{\mu\nu}(n) \exp\left\{\beta \sum_{n,\mu\nu} \cos\theta_{\mu\nu}\right\}}, \tag{5.39}$$

This is the product of independent integrations over decoupled plaquettes,however we could not understand this.

$$\left\langle \exp\left(i\sum_C \theta_\mu\right)\right\rangle = \left\{\frac{\int_0^{2\pi} d\theta_{\mu\nu} \exp\left(\beta\cos\theta_{\mu\nu} + i\theta_{\mu\nu}\right)}{\int_0^{2\pi} d\theta_{\mu\nu} \exp\left(\beta\cos\theta_{\mu\nu}\right)}\right\}^A, \tag{5.40}$$

where A is the number of enclosed plaquettes. The integrals in 5.40 are just Bessel functions of the 1st kind and 0th kind with imaginary argument. The 1st Bessel function is on the numerator while the 0th Bessel function is on the denominator.

$$\left\langle \exp\left(i\sum_C \theta_\mu\right)\right\rangle = \left(\frac{I_1(\beta)}{I_0(\beta)}\right)^A, \tag{5.41}$$

which gives us the expected area law for all coupling. Consider 5.41 in the limiting cases of strong and weak coupling. For $g^2 \gg 1$ where $\beta = 1/2g^2 \ll 1$, these constitute to the low and high temperature respectively.

$$(\frac{I_1(\beta)}{I_0(\beta)})^A \approx (\frac{\beta}{2})^A = (\frac{1}{4g^2})A = e^{-\ln(4g^2)A}, \tag{5.42}$$

And if $g^2 \ll 1$, we have the approximation:

$$(\frac{I_1(\beta)}{I_0(\beta)})^A \approx (\frac{1}{2\beta})^A = (1-g^2)A = e^{-g^2 A} \tag{5.43}$$

Thus, the 2 dimensional model confines for all coupling g.

Result 5.2: 2D Abelian System

In 2 dimensions, theory exhibits no phase transition which can be given by:

- At high temperature, $g \gg 1$:

$$\langle W[C] \rangle = e^{-g^2 A}$$
$$V(R) \sim |R|$$

  This is the confining phase.

- At low temperature, $g \ll 1$:

$$\langle W[C] \rangle = exp[-g^2 A]$$
$$V(R) \sim |R|$$

  This is also the confining phase.

The theory, exhibiting always confining phases leads to say that [1]:

2D Abelian Lattice Gauge Thoery = 1D Planar Spin Model

However, we did not establish this in the work.

5.4 Quantum Hamiltonian of the model

In this topic, we saw the relation between the lattice model and the quantum Hamiltonian model. This was especially very instructive because we see in here that the electric charge is already quantized. Consider the action of the Abelian lattice gauge theory 5.2 with anisotropic couplings, we also had the $\tau-$continuum approach here because as also [1] suggests it is more accessible.

$$S = \beta_\tau \sum_{n,k} [1 - \cos\theta_{0k}(n)] - \beta \sum_{n,ik} \cos\theta_{ik} \tag{5.44}$$

Choosing the temporal gauge $\theta_0 = 0$

$$\begin{aligned} \theta_{0k} &= \theta_0(n) + \theta_k(n+\mu) - \theta_0(n+\mu+\nu) - \theta_k(n+\nu) \\ &= \theta_k(n+\tau) - \theta_k(n) \end{aligned} \tag{5.45}$$

We are interested in the thermodynamic limit. So that $\beta_\tau \to \infty$. This will force θ_{0k} to be small and slowly varying. With the usage of lattice spacing to match the dimensions, we could write:

$$1 - \cos\theta_{0k} \approx \frac{1}{2}\theta_{0k} \approx \frac{1}{2} a_\tau^2 (\frac{\partial\theta_k}{\partial\tau})^2 \tag{5.46}$$

Then also, the sums over lattice can be replaced by the integrals over τ:

$$\sum_{n,k} \to \int \frac{1}{a_\tau} d\tau \sum_{n,k} \tag{5.47}$$

k now has just spatial dependence. Action in now becomes5.44:

$$S \to \int d\tau \left\{ \frac{1}{2}\beta_\tau a_\tau \sum_{n,k} [\dot{\theta}_k(\tau, n)]^2 - \frac{1}{a_\tau}\beta \sum_{n,k} \cos\theta_{ik}(\tau, n) \right\} \tag{5.48}$$

The limits:

$$\begin{aligned} \beta_\tau &\to \infty \\ \beta &\to 0 \end{aligned} \tag{5.49}$$

So we identify:

$$\begin{aligned} \beta_\tau &= \frac{g^2}{a} \\ \beta &= \frac{a_\tau}{g^2} \end{aligned} \tag{5.50}$$

Since τ is a continuous parameter now, we will use equal-time commutation relations to set-up a Hilbert Space for each τ surface. Our quantum field is $\theta_k(n)$. Defining the conjugate momentum density $L_i(n)$:

$$[\theta_k(n'), L_i(n)] = i\delta_{ik}\delta_{n',n} \tag{5.51}$$

So that with the identification $\dot{\theta}_k(\tau, n)g = L_k$ we have the new Hamiltonian:

$$H = \frac{1}{2}\frac{g^2}{a} \sum_{n,k} L_k^2(n) - \frac{1}{ag^2} \sum_{n,ik} \cos\theta_{ik}(\tau, n) \tag{5.52}$$

There is a rigorous way on achieving this. However it is not included in this project. But one can see the lecture notes of Muramatsu [17]. To match the appropriate unit, we can write:

$$aH = \frac{1}{2}g^2 \sum_{n,k} L_k^2(n) - \frac{1}{g^2} \sum_{n,ik} \cos\theta_{ik}(\tau, n) \tag{5.53}$$

where a is lattice constant. Now we should learn what the L_k is. We should look at the symmetries. There is also the symmetry of the action that adds a common angle to all the angular variables to the spins coming out of nth labeled site. We know that under this, Hamiltonian is invariant because it is constructed from $\theta_{\mu\nu}$ and we have seen that $\theta_{\mu\nu}$ is invariant under this. The operator is:

$$G_\chi(n) = \exp\left\{ i \sum_{\pm j} L_j(n)\chi \right\} \tag{5.54}$$

This is the rotation operator. We know that because it induces a change on $\theta_i(n) \to \theta_i(n) + \chi$ for links coming out of site n. Furthermore, we now know that the L_k's are angular momentum operators from experience in quantum mechanics. In quantum mechanics, we can also build a rotation operator with exponentiating the angular momentum operator. For arbitrary n, we can construct an operator that acts on all sites 5.55.

$$G(\chi) = \exp\left[i \sum_{n,l} L_l(n)\chi(n)\right] \tag{5.55}$$

because there is a sum over n. We can see the action of the θ_k by:

$$G_\chi \theta_k G_\chi^{-1} = \theta_k + \chi(n) - \chi(n+k) = \theta_k - \Delta_k \chi \tag{5.56}$$

We could no establish this result. So that any term with discrete curl $\theta_{\mu\nu}$ is invariant. Since we know that Hamiltonian is invariant, we have:

$$G_\chi H G_\chi^{-1} = H \tag{5.57}$$

Now let us look into the canonical quantization perspective in more detail. What should L_i be in electromagnetic theory? If we do the identification:

$$\theta_k(n) = agA_k(n) \tag{5.58}$$

So that we have, for the equal-time commutation relation 5.51:

$$(\frac{g^2}{a})[A_i(n), L_j(n')] = i\delta_{ij}\frac{1}{a^3}\delta_{nn'} \tag{5.59}$$

now we should identify $\delta_i j \frac{1}{a^3}$ as the discrete Dirac-delta. And use the canonical commutation relations from electromagnetism:

$$[E_i(\vec{r}), A_j(\vec{r}')] = i\delta_{ij}\delta(\vec{r} - \vec{r}') \tag{5.60}$$

Therefore after inspection we have:

$$E_i(n) = \frac{g}{a^2} L_i(n) \tag{5.61}$$

This implies electric flux on a link. This is a really important result of the lattice gauge theory. Since L_i is the angular momentum operator, we know that its eigenvalues are discrete. This implies that electric field is also discrete. Thus, electric charge can be quantized in lattice gauge theory naturally. Furthermore, now Hamiltonian is:

$$H = a^3 \sum_{n,k} \frac{1}{2} E_k^2(n) - \frac{1}{g^2} a \sum_{n,ik} \cos\theta_{ik}(n) \tag{5.62}$$

The first term involves the square of the electric field. So we expect the second term to be magnetic field squared as we are aware of the relation between abelian lattice gauge theory and electromagnetism.

$$\begin{aligned} B_i &= \epsilon_{ijk}\partial_j A_k \\ \epsilon_{ijk}\theta_{jk} &:= a^2 g B_i \end{aligned} \tag{5.63}$$

where in the last equation we defined the magnetic field. In this way, even the quantum theory resembles the Maxwell theory. Further into [1] says that the there is an operator that produces quark pairs on the lattice and the distance between them should be the minimum because the angular momentum is quantized. However, we could not establish this in this work.

6 XY Model

The discussion of the abelian model involves more. The XY Model was also studied in this work. Let us start with the *Mermin-Wagner theorem*[18]. It states that a continuous global symmetry cannot be spontaneously broken in $d \leq 2$, this means that for models with continuous symmetry group like $U(1)$ in our case, the correlation function will vanish in the appropriate dimension. However, we also see phase transition in the globally $U(1)$ symmetric 2D XY model. That is becuase the phase transition is not due to spontaneous symmetry breaking mechanism but the topological order [8]. In this case, it is called *BKT (Berezinskii-Kosterlitz-Thouless)* transition. This time we will not look at expectation value of the local order parameter but the expectation value of spin-spin correlation function. The Hamiltonian for the model is:

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) = -J \frac{1}{2} \sum_{\langle ij \rangle} [e^{\theta_i - \theta_j}, h.c] \tag{6.1}$$

First let us do the high temperature expansion:

6.1 High Temperature

The spin-spin correlation function is:

$$\begin{aligned} \langle \vec{s}_0 \cdot \vec{s}_r \rangle &= \langle \cos(\theta_0 - \theta_r) \rangle \\ &= \frac{1}{Z} \frac{1}{(2\pi)^N} \int_0^{2\pi} d\theta_1 d\theta_2 ....d\theta_N \cos(\theta_0 - \theta_r) \prod_{\langle ij \rangle} e^{\beta J \sum_{ij} \cos(\theta_i - \theta_j)} \end{aligned} \tag{6.2}$$

At high temperatures, we can taylor expand the exponential. Let us take only the first order:

$$= \frac{1}{Z} \frac{1}{(2\pi)^N} \int_0^{2\pi} d\theta_1 d\theta_2 ....d\theta_N \cos(\theta_0 - \theta_r) \prod_{\langle ij \rangle} [1 + \beta J \cos(\theta_i - \theta_j)] \tag{6.3}$$

we also have the integral identities:

$$\begin{gathered} \int_0^{2\pi} \frac{d\theta_1}{2\pi} cos(\theta_1 - \theta_2) = 0 \\ \int_0^{2\pi} \frac{d\theta_2}{2\pi} cos(\theta_1 - \theta_2) cos(\theta_2 - \theta_3) = \frac{1}{2} cos(\theta_1 - \theta_3) \end{gathered} \tag{6.4}$$

Therefore the 1st term inside the parantheses 6.3 vanish. Let us explicitly write down the integral:

$$= \frac{1}{Z}\frac{1}{(2\pi)^N}\int_0^{2\pi} d\theta_1....d\theta_N \cos(\theta_0-\theta_r)\cos(\theta_0-\theta_1)\cos(\theta_1-\theta_2) \\ ....\cos(\theta_{r-1}-\theta_r)\cos(\theta_r-\theta_{r+1})...\cos(\theta_{n-1}-\theta_n) \tag{6.5}$$

Here are the steps after this, taking the integrals expect r, then taking the integrals of r in a non-vanishing way yields:

$$\int_0^{2\pi}\frac{d\theta_0 d\theta_r}{(2\pi)^2}\cos(\theta_0-\theta_r)^2 = \frac{1}{2} \tag{6.6}$$

By this integral and with the appropriate coefficients, we get for the corelation function:

$$\langle \vec{s}_0\cdot\vec{s}_r\rangle = (\frac{J}{2})^r \tag{6.7}$$

upon defining the correlation length $\xi = \frac{1}{\ln(2/K)}$, we can write this as:

$$\langle \vec{s}_0\cdot\vec{s}_r\rangle = e^{-\frac{r}{\xi}} \tag{6.8}$$

this is the result for the high temperature. Now let us do low temperature:

6.2 Low Temperature

In doing the low temperature expansion we will use the form 6.1. In the low temperature limit, the difference between neighbouring spins are small and they slowly vary. Thus we can do the approximation:

$$\theta_i - \theta_j \approx \nabla\theta_i\cdot a_{ij} \tag{6.9}$$

Now we evaluate the correlation function:

$$\left\langle e^{i(\theta_0-\theta_n)}\right\rangle = \frac{1}{Z}\int d\theta_0.....d\theta_n e^{i(\theta_0-\theta_n)}\exp\left\{\beta J\sum_i a^2(\nabla\theta_i)^2\right\} \tag{6.10}$$

Convert the sum to an integral since we are in low temperature limit:

$$\begin{aligned}&= \frac{1}{Z}\int d\theta_0.....d\theta_n e^{i(\theta(x_0)-\theta(x_n))}\exp\left\{\left\{\beta J\int d^2x(\nabla\theta)^2\right\}\right\} \\ &= \frac{1}{Z}\int d\theta_0.....d\theta_n \exp\left\{\left\{i\int d^2x(\delta(x-x_0)\theta(x)-\delta(x-x_n)\theta(x))\right\}\right\} \\ &\exp\left\{-\beta J\int d^2x\theta(x)\nabla^2\theta(x)\right\}\end{aligned} \tag{6.11}$$

Where we have done integration by parts on the last step. This was to see the propogator:

$$\int d^2x\theta(x)\nabla^2\theta(x) = \int d^2xd^2x'\theta(x)G^{-1}(x-x')\theta(x) \tag{6.12}$$

The G^{-1} is the inverse propogator. The remaining terms are just Gaussian integral now:

$$\left\langle e^{i(\theta_0-\theta_n)}\right\rangle = \exp\left(\frac{1}{\beta J}G(x_0-x_n)\right) \tag{6.13}$$

Propogator in 2 dimensions explicitly is:

$$G(x-x') = \int \frac{d^2k}{(2\pi)^2}\frac{e^{-i\vec{k}(\vec{x}-\vec{x}')}}{\vec{k}^2+m^2} = \frac{1}{2\pi}K_0(m|x-x'|) \tag{6.14}$$

where K_0 is the modified Bessel function. We can send $m \to 0$ since it was just for regulating the divergences in the integral. In the limit

$$\lim_{m\to 0} K_0(m|x-x'|) = -\ln(m|x-x'|) \tag{6.15}$$

putting this into the result we have:

$$\left\langle e^{i(\theta_0-\theta_n)}\right\rangle = \exp\left(-\frac{1}{\beta J}\ln(m|x-x'|)\right) = (\frac{1}{|x_0-x_n|})^{\frac{k_BT}{2\pi J}} \tag{6.16}$$

this is a decaying power law. We find the results as:

Result 6.1: XY Model

In 2 dimensions, theory exhibits 2 phases which can be given by:

- At high temperature:

$$\langle \vec{s}_0 \cdot \vec{s}_r\rangle = e^{-\frac{r}{\xi}}$$

  This is exponentially decaying.

- At low temperature:

$$\langle \vec{s}_0 \cdot \vec{s}_r\rangle = e^{-\frac{r}{\xi}}$$

  This is decaying power law which is algebraic rather than exponential.

7 Non-Abelian Lattice Gauge Theory

The ideas of the Abelian Lattice Gauge Theory with the group $U(1)$ is generalized to group $\mathcal{G} = SU(2)$ here. On the sites there are reference frames which corresponds to the internal symmetry of $\mathcal{G}$. In the case of $SU(2)$, the frame of reference is 3-dimensional because there are 3 different generators of the group $\mathcal{G} = SU(2)$. We want to construct an action such that , in the continuum limit, it gives the Yang-Mills action because we know that in Abelian lattice gauge theory we had Maxwell action. Second is that it is gauge invariant under the change of reference frames on sites.

To achieve that we define a rotation matrix between two sites n and $n + \mu$.

$$U_\mu(n) = e^{iB_\mu(n)} \;\; and \;\; U_{-\mu}(n+\mu) = U_\mu^{-1}(n) \tag{7.1}$$

where $B_\mu(n)$ is $SU(2)$ operator valued variable and it is defined as

$$B_\mu(n) = \frac{1}{2} a g \tau_i A_\mu^i(n). \tag{7.2}$$

The U transforms as adjointly on site n and $n + \mu$:

$$[U_\mu(n)]_{ij} = \sum_{k,l} \left\{ \exp\left[-\frac{i}{2}\tau^\alpha \chi^\alpha(n) \right] \right\}_{ik} \{U_\mu(n)\}_{jl} \left\{ \exp\left[\frac{i}{2}\tau^\alpha \chi^\alpha(n+\mu) \right] \right\}_{kl}, \tag{7.3}$$

Using this transformation property we can write the following gauge invariant action:

$$S = -\frac{1}{2g^2} \sum_{n,\mu\nu} \mathrm{Tr}\big[U_\mu(n) U_\nu(n+\mu) U_{-\mu}(n+\mu+\nu) U_{-\nu}(n+\nu) \big] + h.c. \tag{7.4}$$

Let us see now how this action relates to the Yang-Mills Action. First since we are dealing with this in the low temperature limit, we will Taylor expand our fields:

$$B_\nu(n+\mu) \simeq B_\nu(n) + a\partial_\mu B_\nu(n) \tag{7.5}$$

$$B_{-\mu}(n+\nu) = -B_\mu(n+\nu) - [B_\mu(n) + a\partial_\nu B_\mu(n)] \tag{7.6}$$

$$B_{-\nu}(n+\nu) = -B_\nu(n) \tag{7.7}$$

Then, we can write the product of Us in 7.4 as

$$\exp\{iB_\nu\} \exp\{i(B_\mu + a\partial_\nu B_\mu)\} \exp\{-i(B_\nu + a\partial_\mu B_\nu)\} \exp\{-iB_\mu\}. \tag{7.8}$$

Here we must use the BCH(Baker-Campbell-Hausdorff) formula to exploit the nature of lie algebra valued functions B.

$$\exp\left\{ i(B_\mu + B_\nu + a\partial_\nu B_\mu) - \frac{1}{2}[B_\nu, B_\nu + a\partial_\nu B_\mu] \right\} \tag{7.9}$$

$$\times \exp\left\{ -i(B_\mu + B_\nu + a\partial_\mu B_\nu) - \frac{1}{2}[B_\mu, B_\nu + a\partial_\nu B_\mu] \right\}.$$

We neglect the $\mathcal{O}(B^3)$ terms. Therefore, the terms $[B_\mu, a\partial_\nu B_\mu] = [B_\nu, a\partial_\mu B_\nu] = 0$ vanish. Upon using the BCH formula again:

$$\simeq \exp\{ia(\partial_\mu B_\nu - \partial_\nu B_\mu) - [B_\mu, B_\nu]\} = \exp\{ia^2 g F_{\mu\nu}\}, \tag{7.10}$$

where we have defined

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig[A_\mu, A_\nu], \tag{7.11}$$

with $A_\mu = \frac{1}{2}\tau_i A^i_\mu$. The action has now become:

$$S \simeq -\frac{1}{2g^2} \sum_{n,\mu\nu} \mathrm{Tr}\big[\exp\{ia^2 g F_{\mu\nu}\}\big]. \tag{7.12}$$

We can do a Taylor expansion again $a^2 g F_{\mu\nu}$:

$$\mathrm{Tr}\big[\exp\{ia^2 g F_{\mu\nu}\}\big] \simeq \mathrm{Tr}\left[1 + a^2 g F_{\mu\nu} + \frac{i}{2} a^4 g^2 F^2_{\mu\nu}\right]. \tag{7.13}$$

The first term is constant and does not contribute to the dynamics of the action. The second term is linear in $F_{\mu\nu}$, which is traceless because we know that antisymmetric two-form F is traceless.

$$[\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k,$$

We find

$$\mathrm{Tr}F^2_{\mu\nu} = \frac{1}{2}\big(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu - g\epsilon_{kij} A^i_\mu A^j_\nu\big)^2. \tag{7.14}$$

Now the Action becomes

$$S \simeq \frac{1}{2g^2} \int \frac{d^4x}{a^4} a^4 g^2 \frac{1}{2}\big(\partial_\mu A^k_\nu - \partial_\nu A^k_\mu - g\epsilon_{kij} A^i_\mu A^j_\nu\big)^2 + \mathcal{O}(a^2). \tag{7.15}$$

So

$$S = \frac{1}{4} \int d^4x (F^i_{\mu\nu})^2, \tag{7.16}$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - g\epsilon^{ijk} A^j_\mu A^k_\nu. \tag{7.17}$$

And this is exactly the Yang-Mills term. So, now we know that this theory resembles the Yang-Mills. From here. we should say that there are no promising results for the low temperature expansion of the theory. So, it is still an open problem. However, through renormalization group methods we can say something. Let us do the strong coupling limit.

7.1 Strong Coupling Limit

The notions are same as the Abelian lattice gauge theory. However this time we will have non-commutative nature. Also this time, we have our fields on the exponentials. So, we should have for the Wilson loop:

$$W[C] = \prod_C e^{iB_\mu(n)} \tag{7.18}$$

for the expectation value of this we have:

$$\langle W[C]\rangle = \int \prod_{n,\mu\nu} dB_\mu(n) e^{-S} \prod_C e^{iB_\mu(n)} \tag{7.19}$$

This will lead to area law as before.

$$\langle W[C]\rangle = e^{-F(g^2)A} \tag{7.20}$$

From section 5.2, we know the potential between quark-antiquark pairs, recalling 5.31. The potential is again confining:

$$V(R) \sim |R| \tag{7.21}$$

Result 7.1: Non-Abelian Lattice Gauge Theory

The nature of the phases are still unknown because low temperature is still an open problem.

- At high temperature, $g \gg 1$:

$$\langle W[C]\rangle = e^{-g^2 A}$$
$$V(R) \sim |R|$$

  This is the confining phase.

- At low temperature, $g \ll 1$: open problem.

7.2 Asymptotic Freedom of $O(n)$ Models in 2D

Let us start with the condition of spins on the sphere:

$$\left|\vec{S}_i\right|^2 = 1 \tag{7.22}$$

From this one can hace the lagrangian:

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu \vec{S})^2 \tag{7.23}$$

We will see that, all $O(n)$ Models in 2D are asymptotically free. That is in another way, local curvature of the sphere implies asymptotic freedom. [1] We expect that the action can be written as:

$$\mathcal{S}' = \frac{1}{2g'}(\partial_\mu \vec{S})(\partial_\mu \vec{S}) + \frac{1}{2f'}[(\partial_\mu \vec{S})\partial_\mu \vec{S}]^2 \tag{7.24}$$

Now we shall parameterize each component of $\vec{S}$ to high and low frequency components. We demand that:

- S_1, S_2 to be slowly varying and large
- S_3 to be rapidly varying and small

Paramtrization is:

$$\begin{aligned} S_1 &= \sqrt{1-S_3^2}\sin\theta \\ S_2 &= \sqrt{1-S_3^2}\cos\theta \\ S_3 &= S_3 \end{aligned} \tag{7.25}$$

Under these transformations the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2g}[(1-S_3^2)(\partial_\mu\theta)^2 + \frac{(\partial_\mu S_3)^2}{1-S_3^2}] \tag{7.26}$$

We were not able to derive this Lagrangian. Now moving on, since S_3 is small we can:

$$\frac{1}{1-S_3^2} = 1 + S_3^2 + ... \tag{7.27}$$

Our approximations lead us to a Lagrangian:

$$\mathcal{L} = \frac{1}{2g}[(1-S_3^2)(\partial_\mu\theta)^2 + (\partial_\mu S_3)^2 S_3^2)] \tag{7.28}$$

Now we can rescale S_3 to eliminate g by $S_3 = h\sqrt{g}$. Under this:

$$= \frac{1}{2}[(\frac{1}{g} - h^2)(\partial_\mu\theta)^2 + (\partial_\mu h)^2 + gh^2(\partial_\mu h)^2] \tag{7.29}$$

Now under these circumstances, our demands on componenets of spin become:

- h is rapidly varying but small
- θ is slowly varying but large

Let us introduce momentum cut-off, and partition function is:

$$\begin{aligned} Z = &\int_{0<|p|<\lambda} D\theta(p)\ \exp\left\{(-\frac{1}{2g}\int d^2x(\partial_\mu\theta)^2)\right\} \\ &\times \int_{\Lambda'<|p|<\Lambda} Dh(p) \exp\left\{(-\frac{1}{2}\int d^2x(\partial_\mu h)^2)\right\} \\ &\times \exp\left\{(\frac{1}{2}\int d^2x h^2(\partial_\mu\theta)^2)\right\} \end{aligned} \tag{7.30}$$

We have dropped the last term in the 7.30 because g is small. We could not establish the next step but the integral over $h(p)$, meaning the last two terms can be rewritten as:

$$
\begin{aligned}
\int_{\Lambda'<|p|<\Lambda} Dh(p)\exp\Big\{(-\frac{1}{2}\int d^2x(\partial_\mu h)^2)\Big\} &\times \exp\Big\{(\frac{1}{2}\int d^2x h^2(\partial_\mu\theta)^2)\Big\} \\
&= N\exp\Big\{\frac{1}{2}\int d^2x \left\langle h^2\right\rangle_h (\partial_\mu\theta)^2\Big\} \\
&= N\exp\Big\{\frac{1}{2}\int d^2x \int_{\Lambda'}^{\Lambda}\frac{d^2p}{(2\pi)^2}\frac{1}{p^2}(\partial_\mu\theta)^2\Big\}
\end{aligned}
\tag{7.31}
$$

where N is a constant and

$$
\left\langle h^2\right\rangle_h = \int_{\Lambda'}^{\Lambda}\frac{d^2p}{(2\pi)^2}\frac{1}{p^2} = \frac{1}{2\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right) \tag{7.32}
$$

So that 7.31 becomes:

$$
N\exp\Big\{\frac{1}{4\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right)\int d^2x(\partial_\mu\theta)^2\Big\} \tag{7.33}
$$

Let us put this back into the partition function:

$$
\begin{aligned}
Z = \int_{0<|p|<\lambda} D\theta(p)\ \exp&\Big\{(-\frac{1}{2g}\int d^2x(\partial_\mu\theta)^2)\Big\} \\
\times N\exp&\Big\{\frac{1}{4\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right)\int d^2x(\partial_\mu\theta)^2\Big\}
\end{aligned}
\tag{7.34}
$$

Now we have a term in the effective $\mathcal{L}$ in the form:

$$
\mathcal{L}' = \frac{1}{2}\Big\{[\frac{1}{g} - \frac{1}{2\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right)](\partial_\mu\theta)^2\Big\} \tag{7.35}
$$

Recall the form of the Lagrangian with spins 7.26. We can identify:

$$
\frac{1}{g} = \frac{1}{g'} - \frac{1}{2\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right) \tag{7.36}
$$

We will convert this into a differential equation:

$$
\frac{1}{g} - \frac{1}{g'} = d(\frac{1}{g}) = -\frac{1}{g^2}dg \tag{7.37}
$$

Therefore:

$$
\frac{1}{g^2}dg = -\frac{1}{2\pi}\ln\left(\frac{\Lambda}{\Lambda'}\right) \tag{7.38}
$$

Then impose that $\Lambda' = \Lambda + \delta\Lambda$, which shows that $\ln\left(1+\frac{\delta\Lambda}{\Lambda}\right) \approx \frac{\delta\Lambda}{\Lambda}$ at the first order. Now we can:

$$
\begin{aligned}
\frac{1}{g^2}dg &= \frac{1}{2\pi}\frac{\delta\Lambda'}{\Lambda'} \\
\Lambda\frac{dg}{d\Lambda} &= -\frac{g^2}{2\pi}
\end{aligned}
\tag{7.39}
$$

Writing this in terms of real space time cut-off:

$$a\frac{dg}{da} = \frac{g^2}{2\pi} \tag{7.40}$$

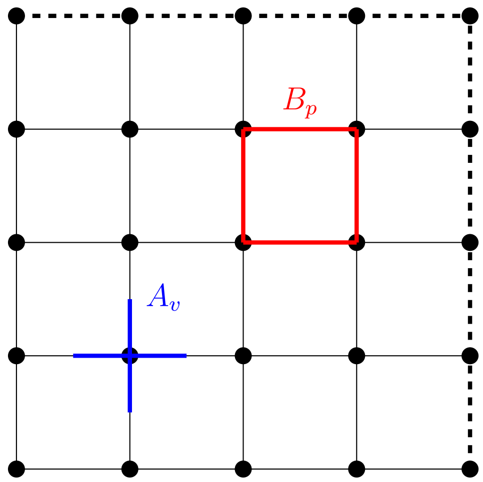
the last equation here means that the theory is asymptotically free.[1] This would mean, that confinement should be expected as we lower the energy scales. However this result is different for 4 dimensions. The key point here was the integral in 2 dimensions in 7.31. Generalizing the RG equation we have [1]:

$$a\frac{dg}{da} = (N-2)\frac{g^2}{2\pi} \tag{7.41}$$

where N is for dimension of the group $O(N)$.

8 The Toric Code

The Toric Code is an exactly solvable model which was introduced by Kitaev [2]. The model is familiar because it is defined on a 2D lattice and spins have $\mathbb{Z}_2$ degree of freedom. This time, there will be different operators defined on the theory and the parallel sides of the lattice sheet will be identified so that the spins are now living on the torus.



In the figure, dashed lines indicate the identification of parallel sides so that the geometry is a genus 1 torus. There are two types of operators:

$$A_v = \prod_{i \in \text{vertex}} \tau_i^z \quad \text{and} \quad B_p = \prod_{j \in \text{plaquette}} \tau_j^x \tag{201}$$

A_v are products of 3rd Pauli Matrix with diagonal entries $(-1, 1)$. And B_p are products of 1st Pauli Matrices. They obey the usual commutation and anti-commutation relations

$$[\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k \quad \text{and} \quad \{\tau_i, \tau_j\} = 2\delta_{ij}I \tag{8.1}$$

In the τ_z basis, τ_x behaves as a spin flip operator. We have also seen this in the 1 dimensional Ising model. The Hamiltonian is the sum of all vertex and plaquette operators:

$$H_{TC} = -\sum_v A_v - \sum_p B_p \tag{8.2}$$

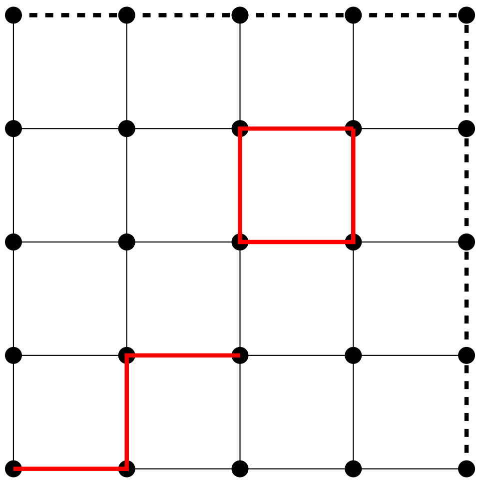
The first term is like the kinetic term while the other is the potential term. The model is exactly solvable because we can diagonalize the opeartors in the Hamiltonian at the same time. That is:

$$\begin{aligned} [A_v, A_{v'}] &= 0 \\ [B_p, B_{p'}] &= 0 \\ [A_v, B_p] &= 0 \end{aligned} \tag{8.3}$$

The last commutation relation is the only non-trivial one. Because the first two of them represent the non-intersecting matrices, which already commute. The reason behind the third is one can see that if the two operators coincide, they coincide on a number of two links. Thus, this makes the commutator vanish. We can easily see this using the anticommutation relations of Pauli matrices. Since we have two crossings, there will be an even number of matrices thet should anticommute. After that, they will give as $\tau_i^2 = 1$. Now let us see the ground state of this system.

8.1 Energy Gap and The Ground State Degeneracy

Since we know the relation $\tau_i^2 = 1$,their eigenvalues are ± 1. Since there exists an overall minus sign in the Hamiltonian 8.2, the ground state will be achieved when the eigenvalues are $+1$. However, if all the spins are spin up, we have the ground state as in the 2D Ising model. In addition, if we have all spins down, we still have a ground state. If we want to do perturbations on the ground state, they will be done with string operators. For convenience, they will represent the spin down state:



These red lines are *string operators* and they make the spins up state down. So that the lattice is excited, deviated from the ground state energy. However, upon using the B_p operators on loops such as on the figure, we see that it does not change the energy. This means that there is no energy difference between a loop and a lattice free of string operators with spins all up. As a result, in the ground state of the toric code, all loops are in a ground state. But there is something very important here. Since the model lives on a torus there are both contractible and non-contractible loops. Non-contractible loops cannot be changed with any of the operators. Then we know that there are 2 non-contractible loops on the torus. So we conclude that there are 4 ground states of the toric code. The ground state is 4-fold degenerate.

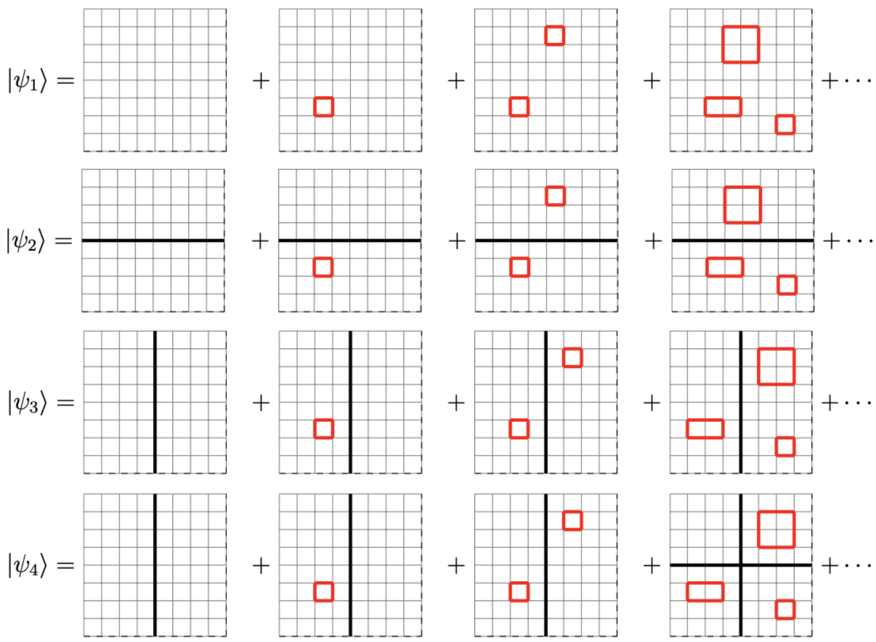
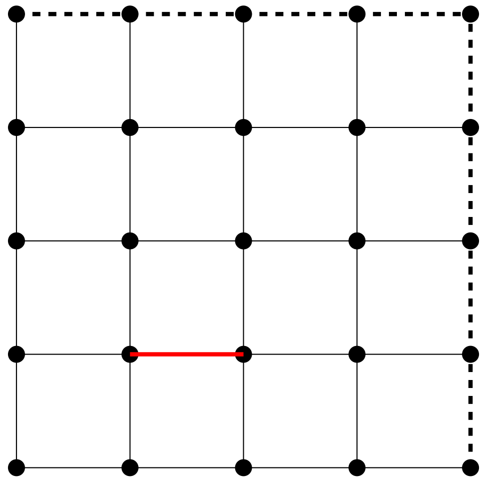


Figure 5. The 4 Ground States

I took this figure from Yunus Emre Sargut. We can also see from here that if we apply the operator B_p on these we can transform ground states onto each other. So we see that ground state does not change. We are now ready to calculate the mass gap. The first excited state will obviously be 1 link string operator. Suppose there is a single excitation.



The figure above describes a single excitation. As we calculate the energy, there are only 2 A_v: operators that will contribute the energy. By using the eigenvalue equation:

$$H\psi = E\psi \tag{8.4}$$

this amounts to having the gap between 1st excited state and the ground state:

$$-2N - (-2(N-2)) = -4 \tag{8.5}$$

So we now know that:

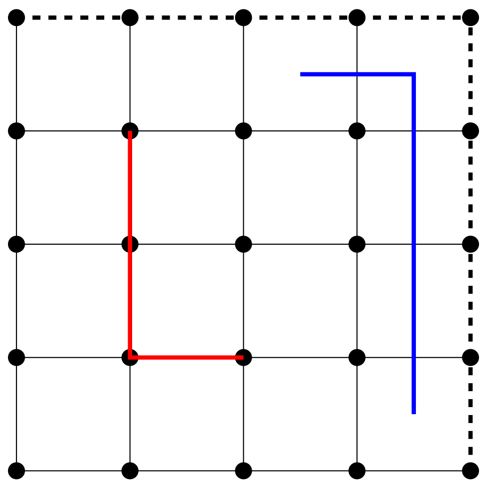
$$\Delta E = 4 \tag{8.6}$$

8.2 Anyonic Excitations

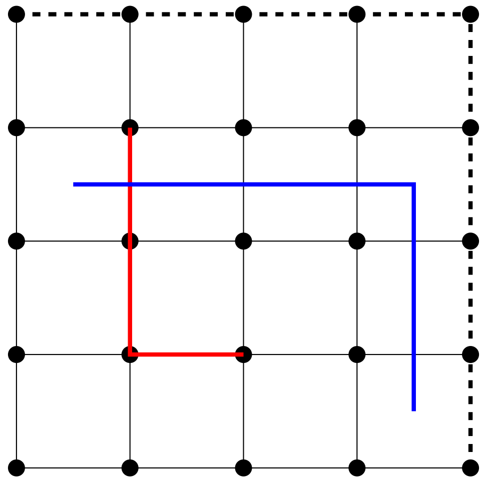
We define two string operators:

$$S_l^x = \prod_{i \in l} \tau_i^x \quad \text{and} \quad S_{l'}^z = \prod_{j \in l'} \tau_j^z \tag{8.7}$$

where l and l' are paths on the lattice and the dual lattice respectively. The excitations created by S_l^x are electric charge and denote with e, similarly, the excitations created by $S_{l'}^z$ magnetic charge and denote with m.



at the ends of the red line there are electric charges. And at the end of the blue lines there are magnetic charges. Now we will see the self statistics. If both operators are self crossing, they are bosonic. Becuase the only interesting part is where they cross themselves [19] and at the point σ_z or σ_x commute among themselves. However, if they cross each other:



at the crossing point we know they anticommute. As a result:

Result 8.1: Excitations

The electric charge is bosonic inside. And magnetic charges are bosonic among themselves. However, if they cross each other they obey fermionic statistics. We shall call this ϵ.

8.3 Fusion Rules

There are 4 types of charges:

- 1: vacuum
- e: electric charge
- m: magnetic charge
- ϵ : fermion

The fusion rules are:

$$\begin{aligned} e \times e = 1, \quad m \times m = 1, \quad \epsilon \times \epsilon = 1 \\ e \times m = \epsilon, \quad e \times \epsilon = m, \quad m \times \epsilon = e \end{aligned} \tag{8.8}$$

8.4 Transverse Field Ising Model

We have seen the transverse field Ising Model before. This time we will take the Hamiltonian:

$$H = -J\sum_{\langle ij\rangle}\sigma_i^x\sigma_j^x - \sum_i \sigma_i^z \tag{8.9}$$

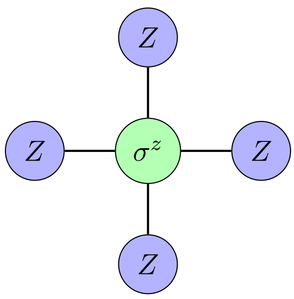
We have used τ as our gauge field living on links. Now, $\sigma's$ will represent the matter fields. And they will live on sites instead of links. We know that the model has a global $\mathbb{Z}_2$ symmetry. It can be given by the operator:

$$U = \prod_i \sigma_i^z \tag{8.10}$$

There were two phases of the model in this case. The first term breakes this symmetry. We can look at the scenario $J \gg 1$ and decide that second term does not conrtibute. And it is easily seen that this violates the symmetry U. It is called the symmetry breaking phase. We will now try to gauge it.

8.5 Coupling the Transverse Field Ising Model to a Gauge Field

There are couple of ways to make an expression invariant. One is the Noetherian way where we add the non-gauge invariant piece into the Lagrangian. Or we can try to minimally couple the action to a gauge field. The latter procedure resembles this case the most.



Note that τ transforms under two local operations. $\sum_i \sigma_i^z$ term is already locally symmetric. To make the first term locally symmetric we couple it with the gauge fields:

$$-\sum_{\langle ij\rangle}\sigma_i^x\sigma_j^x \rightarrow -\sum_{\langle ij\rangle}\sigma_i^x\tau_{\langle ij\rangle}\sigma_j^x \tag{8.11}$$

However, adding this piece has did a bad thing by adding a huge degeneracy to the model. We can impose the *zero-flux condition* by introducing using plaquette terms. This will remove the degenracy as in the case of Toric code. Our new Hamiltonian is:

$$H' = -J\sum_{\langle ij\rangle}\sigma_i^x\tau_{\langle ij\rangle}\sigma_j^x - \sum_i \sigma_i^z - \sum_p B_p \tag{8.12}$$

Let us see what $J = 0$ limit brings.

$$H' = -\sum_i \sigma_i^z - \sum_p B_p \tag{8.13}$$

By choosing a gauge, particularly $U_v = 1$, we can write the matter field as:

$$\sigma^z = \prod \tau^z \tag{8.14}$$

We then have:

$$\begin{aligned} H' &= \sum_i \prod \tau^z - \sum_p B_p \\ &= -\sum_v A_v - \sum_p B_p = H_{ToricCode} \end{aligned} \tag{8.15}$$

Which is equivalent to the Hamiltonian of the Toric Code. However, in order to get this, we did a coupling to get the local symmetry then we ditched it. As a result, we learned that, the Toric Code is a limit of the gauged Transverse Field Ising Model.

As a conclusion, we have learned a lot about the nature of the phase transitions and I personally found the application Toric Code beautiful.

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